1 Behaviour of the Schrödinger evolution for initial data near $H^{\frac{1}{4}}$

after L. Carleson [1] and after B. Dahlberg, and C. Kenig [2]
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Abstract

We study pointwise convergence of solutions of the Schrödinger equation on $\mathbb{R}$ as $t \to 0$. For initial data in the Sobolev space $H^s(\mathbb{R})$, Carleson showed that we have almost everywhere convergence when $s \geq \frac{1}{4}$. Dahlberg and Kenig proved that this result is also sharp.

1.1 Introduction

We consider the initial value problem for the Schrödinger equation in $\mathbb{R}$:

$$\begin{cases} i\partial_t \Psi(x,t) + \Delta\Psi(x,t) = 0 \\ \Psi(x,0) = f(x) \end{cases}$$

The solution to this problem is given by

$$e^{it\Delta} f(x) = \int_{\mathbb{R}} e^{ix\xi + it\xi^2} \hat{f}(\xi) \frac{d\xi}{2\pi}.$$

The operator $e^{it\Delta}$ is bounded on L^2, so it is continuous; in particular $\lim_{t\to 0} e^{it\Delta} f = f$ in L^2, or equivalently

$$\lim_{t\to 0} \| e^{it\Delta} f - f \|_{L^2} = 0.$$

But what can we say about the pointwise limit of $e^{it\Delta} f(x)$ as $t \to 0$? For which class of initial data does it hold that

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}?$$

In the 1980's Lennart Carleson gave an answer when the initial data f is compactly supported and α-Hölder continuous with $\alpha > \frac{1}{4}$. Here we state and prove this result for f belonging to the Sobolev space $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$.

Theorem 1 (Carleson)**.** *If $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ then*

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \textit{for almost every } x \in \mathbb{R}.$$

The key of the proof is the bound of the maximal Schrödinger operator for some $p > 1$

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^p} \leq C \|f\|_{H^s(\mathbb{R})}.$$

One year later, Dahlberg and Kenig proved that the above result is *sharp*. They proved the following

Theorem 2 (Dahlberg & Kenig)**.** *Let $s \in \left[0, \frac{1}{4}\right)$. There exists a function $f \in H^s(\mathbb{R})$ and a set E with positive measure such that, for every $x \in E$*

$$\limsup_{t \to 0} |e^{it\Delta} f(x)| = +\infty.$$

1.2 Positive result

In order to prove Theorem 1, we will use an a priori estimate for the maximal operator $\sup_{t>0} |e^{it\Delta} f|$.

Proposition 3 (A priori estimate)**.** *Let $f \in \mathcal{S}(\mathbb{R})$ Schwartz function. Then there exists a constant $C > 0$ such that*

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4(\mathbb{R})} \leq C \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}. \tag{1}$$

Proof. First we aim to prove a *local* estimate, namely

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4([-R,R])} \leq C \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}$$

where the constant C is independent of R. The estimate (1) will follow by taking the limit as $R \to \infty$. We split the proof in steps.

Step 1 We would like to get rid of the supremum. Fix $x \in \mathbb{R}$. There exists a time $t(x) > 0$ such that

$$|e^{it(x)\Delta} f(x)| \geq \frac{1}{2} \sup_{t>0} |e^{it\Delta} f(x)|.$$

Step 2 Then we use *duality*. There exists a function $w \in L^{\frac{4}{3}} \cong (L^4)'$, with $\|w\|_{\frac{4}{3}} = 1$, with $\operatorname{supp}(w) \subset [-R, R]$, such that

$$\|e^{it\Delta} f\|_{L^4([-R,R])} = \int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x) dx.$$

Step 3 Expand the integral, use Fubini[1] and Cauchy-Schwarz.

$$\begin{aligned}
\int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x) &= \iint_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} d\xi \, w(x) \, dx \\
&= \int_{\mathbb{R}} \hat{f}(\xi) |\xi|^{\frac{1}{4}} \int_{\mathbb{R}} e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}} dx \, d\xi \\
&\leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}} dx \right|^2 d\xi \right)^{\frac{1}{2}} = \mathrm{I} \cdot \mathrm{II}.
\end{aligned}$$

Step 4 We bound the two factors separately.

$$\mathrm{I} \leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{\frac{1}{4}} d\xi \right)^{\frac{1}{2}} = \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

For II, a careful estimate of the oscillatory integral inside leads to

$$\mathrm{II}^2 \leq C \int_{\mathbb{R}^2} \frac{w(x) w(y)}{|x-y|^{\frac{1}{2}}} dx dy.$$

Use Hölder and Hardy-Littlewood-Sobolev inequalities to conclude

$$\mathrm{II}^2 \leq C \|w\|_{L^{\frac{4}{3}}} \left\| \int_{\mathbb{R}} \frac{w(y)}{|x-y|^{\frac{1}{2}}} dy \right\|_{L^4} \leq C \|w\|^2_{L^{\frac{4}{3}}(\mathbb{R})}.$$

To sum up:

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4([-R,R])} \leq 2 \left\| e^{it(\,\cdot\,)\Delta} f \right\|_{L^4([-R,R])} \leq C \|w\|_{L^{\frac{4}{3}}(\mathbb{R})} \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

By taking the limit as $R \to \infty$, we conclude. □

Idea of the proof of Theorem 1. By density of Schwartz functions $\mathcal{S}(\mathbb{R})$ in the Sobolev space $H^{\frac{1}{4}}(\mathbb{R})$, the bound (1) holds true for functions in $H^{\frac{1}{4}}(\mathbb{R})$, and also in any $H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$, since they are all contained in $H^{\frac{1}{4}}$.

[1] The function $w \in L^{\frac{4}{3}}([-R,R]) \subset L^1([-R,R])$. In particular w is integrable and we can use Fubini.

Thus the maximal function $\sup_{t>0}|e^{it\Delta}f|$ is bounded from $H^s(\mathbb{R})$ to $L^4(\mathbb{R})$ for $s \geq \frac{1}{4}$. This bound implies pointwise almost everywhere convergence for the family of operators $\{e^{it\Delta}\}_{t\in[0,1]}$, in particular we have

$$\lim_{t\to t_0} e^{it\Delta}f(x) = e^{it_0\Delta}f(x) \qquad \text{for almost every } x \in \mathbb{R},$$

and when $t_0 = 0$, when we get back $f(x)$. $\square$

1.3 Negative result

In his work, Carleson already proved that the convergence to $f \in H^s(\mathbb{R})$ might fail for $s < \frac{1}{8}$. For the proof of the Theorem 2 Björn Dahlberg and Carlos Kenig exploited a theorem by Nikišin, published the same year in [3]. We recall first some notations from [4].

Let (X, μ) and (Y, ν) two σ-finite measure spaces. Let $L^0(Y, \nu)$ the space of a.e. finite real-values measurable functions on Y endowed with the metric of the convergence in measure.

We say that $T\colon L^p(X, \mu) \to L^0(Y, \nu)$ is *linearizable*[2] if for each $f_0 \in L^p(X)$ there exist a *linear* operator H_{f_0} such that

1. $|H_{f_0}f_0| = |Tf_0|$ ν- a.e. and
2. $|H_{f_0}f| \leq |Tf|$ ν- a.e. for all $f \in L^p(X)$.

Remark 4. *For an operator T being linearizable means that there is a family $\{H_{f_0}\}_{f_0\in L^p(X)}$ of linear operators such that T majorizes each one of them and coincides in absolute value with H_{f_0} in f_0.*

Example 5. *Given a sequence of operators $(T_n)_n\colon L^p(X, \mu) \to L^0(Y, \nu)$. The truncated maximal operator of the family T_N^* is linearizable.*

We are ready to state the theorem.

Theorem 6 (Nikišin)**.** *Let $1 \leq p < \infty$, and let $T\colon L^p(X, \mu) \to L^0(Y, \nu)$ linearizable and continuous in measure at 0. Then for every $\epsilon > 0$ there exists a set $E_\epsilon \subset Y$ with $|E_\epsilon| \geq |Y| - \epsilon$ such that*

$$|\{y \in E_\epsilon : Tf(y) > \lambda\}| \leq C_\epsilon \left(\frac{\|f\|_{L^p}}{\lambda}\right)^q,$$

for all $\lambda > 0$, $f \in L^p(X)$, and $q = \min\{p, 2\}$.

[2] or *hyperlinear* in Nikišin's terminology

To show that pointwise convergence a.e. fails, it is enough to show that it fails on an finite interval $I \subset \mathbb{R}$. Aiming to a contradiction, assume that we have convergence a.e. for every $f \in H^s(\mathbb{R})$ with $s < \frac{1}{4}$, then

$$\limsup_{t \to 0} |e^{it\Delta} f(x)| < +\infty \quad \text{for almost every } x \in I.$$

Consider an even function $f \in C_c^\infty(\mathbb{R})$ supported in $I = [-1, 1]$.
For $0 < t < 1$ we rescale and modulate f

$$f_t(x) = f\left(\frac{x}{t}\right) e^{2ix/t^2},$$

such that its Sobolev norm is

$$\|f_t\|_{H^s}^2 \leq C t^{1-4s}.$$

Then let $t(x) = t^2 x$ for $x > 0$. Moreover, we have that

$$|e^{it(x)\Delta} f_t| = \left| \frac{1}{\sqrt{x}} \int_{\mathbb{R}} f(y) e^{iy^2/x} dy \right| =: g(x).$$

Notice that g is a continuous function independent of t.
We can view $e^{it\Delta}$ as an operator acting on the Fourier side and mapping to measurable functions:

$$\begin{aligned} e^{it\Delta} \colon L^2(\mathbb{R}, \langle \xi \rangle^s \, d\xi) &\to L^0(I) \\ \hat{f} &\mapsto \mathcal{F}^{-1}(e^{it\xi^2} \hat{f}) \end{aligned}$$

By our previous assumption, this is a bounded operator from a (weighted) L^2 to measurable functions on an interval. We apply Theorem 6 with $p = 2$, $X = \mathbb{R}$ with the measure $\mu = (1 + |\xi|^2)^{s/2} \, d\xi$, so that $L^2(\mathbb{R}, \mu) = H^s(\mathbb{R})$, and $Tf = \sup_{0<t<1} |e^{it\Delta} f|$.
Then there exists a closed set $E \subset [-1, 1]$ with positive Lebesgue measure[3], and $C > 0$, such that

$$\left| \{ y \in E \, : \, \sup_{0<t<1} |e^{it\Delta} f(y)| > \lambda \} \right| \leq C \left(\frac{\|\hat{f}\|_{L^2(\mathbb{R}, \langle \xi \rangle^s)}}{\lambda} \right)^2 \quad \text{for all } \lambda > 0. \quad (2)$$

[3] actually, the set E is an arbitrarily large subset of $[-1, 1]$

The restriction $g \restriction E$ is continuous. Let $\lambda_0 := \min_{x \in E} g(x)$. Using (2) we have that

$$|E| = |\{x \in E \,:\, g(x) > \lambda_0\}| = |\{x \in E \,:\, |e^{it(x)\Delta} f_t| > \lambda_0\}|$$
$$\leq |\{x \in E \,:\, \sup_{t \in [0,1]} |e^{it(x)\Delta} f_t| > \lambda_0\}| \leq \frac{C}{\lambda_0^2} \|f_t\|^2_{H^s(\mathbb{R})} \lesssim t^{1-4s}.$$

This is a contradiction as long $s < \frac{1}{4}$, since one has

$$0 < |E| \lesssim t^{1-4s} \to 0 \quad \text{ as } t \to 0.$$

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