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Corso di Laurea in Matematica

Thurston's Earthquake Theorem via Anti-de Sitter Geometry

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Introduction

In Riemannian geometry the *most interesting* geometry of constant sectional curvature is the hyperbolic one; the analogous in Lorentzian geometry (as in with constant negative curvature) is *Anti-de Sitter* geometry. The thesis presents Anti-de Sitter (AdS) geometry in dimension 3 and its relationship with the theory of earthquakes on hyperbolic surfaces.
After a brief introduction to Lorentzian geometry, we will introduce the most common models of AdS spaces in any dimension following [2]. The focus will rapidly shift to dimension 3 (2+1) where the model space $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ (known as the Klein model in the literature) can be identified with the Lie group $\mathrm{PSL}(2,\mathbb{R})$ which moreover is isomorphic to $\mathrm{Isom}_0(\mathbb{H}^2)$.
Following the pioneering work of Geoffrey Mess in 1990 [11], we will develop the classification of maximal globally hyperbolic (MGH) AdS spacetimes of genus $r > 2$, 3-manifolds locally isometric to $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ characterized by the existance of a *Cauchy surface* of genus r, namely a surface Σ of genus r that intersects every inextensible timelike curve exactly once and a property of *maximal* inclusion that we will investigate in the thesis.
A first result due to Geroch [7] states that such spacetimes have to be diffeomorphic to $\Sigma \times \mathbb{R}$. Even when the topological data of the surface Σ is fixed the geometry of the resulting spacetime can vary significantly. If Σ_r is a closed surface of genus r, we denote the *deformation space* of MGH spacetimes of genus r by:

$$\mathcal{MGH}(\Sigma_r) = \{g \text{ MGH AdS metric on } \Sigma_r \times \mathbb{R}\}/\mathrm{Diff}_0(\Sigma_r \times \mathbb{R})$$

and with $\mathcal{T}(\Sigma_r)$ the Teichmüller space of the surface Σ_r. The main result of the classification (in Chapter 4) will be the following:

Theorem 0.1 (Mess [11])**.** *The holonomy map $\rho : \mathcal{MGH}(\Sigma_r) \to \mathcal{T}(\Sigma_r) \times \mathcal{T}(\Sigma_r)$ is a homeomorphism.*

To obtain such a result we will develop the theory of *achronal surfaces*, surfaces whose points are not connected via timelike curves, and the related theory of *achronal meridians*. The observation that the graph Λ_φ of any orientation-preserving homeomorphism φ of the circle can be identified with an achronal meridian in the universal cover of $\mathbb{AdS}^{2,1}$ will help us relating AdS geometry to the theory of earthquakes on hyperbolic surfaces developed by Thurston in [15].

Following again Mess, we will explain the example of pleated surfaces in $\mathbb{AdS}^{2,1}$, relating bending laminations to geodesic laminations and the earthquake map to projections from the boundary of the convex hull of Λ_φ. With the assist of the Gauss map we will be able to recover Thurston's earthquakes theorem:

Theorem 0.2 ("Geology is transitive", Thurston [15])**.** *Given any orientation-preserving homeomorphism $\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, there exists a left earthquake map of $\mathbb{H}^2$, and a right earthquake map, that extends continuously to φ on $\partial\mathbb{H}^2$.*

As a corollary of the main theorem we will recover the classical Kerchoff's formulation of the earthquakes theorem for hyperbolic surfaces:

Corollary 0.3. *Let S be a closed oriented surface and let $\rho, \varrho : \pi_1(S) \to PSL(2, \mathbb{R})$ be two Fuchsian representations. Then there exists a $(\rho, \varrho)-$equivariant left earthquake map of $\mathbb{H}^2$, and a $(\rho, \varrho)-$equivariant right earthquake map.*

CHAPTER 1

Lorentzian geometry

In this first chapter we introduce Lorentzian manifolds of constant curvature and observe that, as in the Riemannian case, two manifolds of constant sectional curvature K are locally isometric. We start by recalling some basic definitions of Lorentzian geometry to set notation. We will then define what we mean by a manifold with maximal isometry group as spaces with such property provide *models* of manifolds with constant curvature: if we have M a Lorentzian manifold with constant sectional curvature K and maximal isometry group, then any Lorentzian manifold with constant sectional curvature K carries a natural $(\mathrm{Isom}(M), M)$-atlas made of local isometries.
Simply connected spaces have maximal isometry group, but in general the converse is false. In particular, in the course of the dissertation, we will focus on $K = -1$ and in that case it will be convenient to use non-simply connected models.

Basic Definitions

By a *Lorentzian metric* on a $n+1$ manifold we mean a non degenerate 2-tensor g of signature $(n, 1)$. A *Lorentzian manifold* is a connected manifold M equipped with a Lorentzian metric g.
In a Lorentzian manifold M we say that a non-zero vector $v \in T_pM$ is *time-like* if $g(v, v) < 0$, *space-like* if $g(v, v) > 0$ and *light-like* if $g(v, v) = 0$. More generally, we say that a linear subspace $V \subset T_xM$ is *spacelike, lightlike, timelike* if the restriction of g to V is positive definite, degenerate or negative definite respectively.

The set of lightlike vectors, together with the null vector, disconnects T_xM into 3 regions: two convex open cones formed by timelike vectors, one opposite to the other, and the region of spacelike vectors. As a consequence the set of timelike vectors in the total space TM is either connected or is made by two connected components. In the latter case M is said to be *time-orientable*, and a time orientation is the choice of one of those components. Vectors in the chosen component are said to be *future-directed*, vectors in the other component are said to be *past-directed*. A spacetime is a Lorentzian manifold (M, g) with the choice of a time orientation.

A differentiable curve is said to be *timelike, spacelike, lightlike* if its tangent vector at every point is timelike, spacelike, lightlike respectively. The curve is called *causal* if the tangent vector is either timelike or lightlike.

Given a subset S in a time-oriented Lorentzian manifold M, the *future* of S is the set $I^+(S)$ of points which are connected to points of S by a future-directed causal curve. The *past* of S, $I^-(S)$, is defined in an analogous way for past-directed causal curves.

As in the Riemannian setting, on a Lorentzian manifold M there is a unique linear connection ∇ which is symmetric and compatible with the Lorentzian metric g. We refer to it as the *Levi-Civita connection* of (M, g).

Following the analogy with Riemannian geometry, the Levi-Civita connection determines the Riemann curvature tensor defined by:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

We then say that a Lorentzian manifold has constant sectional curvature K if:

$$g(R(u, v)v, u) = K(g(u, u)g(v, v) - g(u, v)^2) \tag{1.1}$$

for every pair of vectors $u, v \in T_xM$ and every $x \in M$. Even though the definition is analogous to the one given in the Riemannian realm, we recall that in the Lorentzian setting the sectional curvature can be defined only for planes in T_xM where g is non-degenerate. The manifold M is said to be *geodesically complete* if every geodesic is defined for all times, equivalently the exponential map is defined everywhere.

Maximal isometry group and geodesic completeness

Two Riemannian manifolds M and N of constant curvature K are locally isometric, the same statement holds for Lorentzian manifolds. The proof is analogous to the one in the Riemannian setting, and it is based on the use of the Cartan-Ambrose-Hicks theorem (the pseudo-Riemannian case is treated in [13]). More precisely we have the following:

Lemma 1.1. Let M and N be Lorentzian manifolds of constant curvature K. Then every linear isometry $L : T_xM \to T_yN$ extends to an isometry $f : U \to V$ where U, V are open neighbourhoods of x, y respectively. Any two extensions $f : U \to V$ and $f' : U' \to V'$ of L coincide on $U \cap U'$. Moreover L extends to a local isometry $f : M \to N$ provided that M is simply connected and N is geodesically complete.

As a direct consequence of the aforementioned lemma we have:

Corollary 1.2. *Let M and N be simply connected, geodesically complete Lorentzian manifolds of constant curvature K. Then any linear isometry $L : T_xM \to T_yN$ extends to a global isometry $f : M \to N$.*

In particular, there is a unique simply connected geodesically complete Lorentzian manifold of constant curvature K up to isometries. For instance for $K = 0$ a model is the Minkowski space $\mathbb{R}^{n,1}$, that is $\mathbb{R}^{n+1}$ provided with the standard metric

$$g = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

Another consequence of Lemma 1.1 is that, fixing a point $x_0 \in M$, the set of isometries of M, which we will denote by $\mathrm{Isom}(M)$, can be realized as a subset of $\mathrm{ISO}(T_{x_0}M, TM)$ namely the fiber bundle over M whose fiber over $x \in M$ is the space of linear isometries of $T_{x_0}M$ into T_xM. Now $\mathrm{Isom}(M)$ considered with the operation of composition is a Lie group and the inclusion $\mathrm{Isom}(M) \to \mathrm{ISO}(T_{x_0}M, TM)$ is a differentiable proper embedding ([12], Theorem 4.1). It follows that the maximal dimension of $\mathrm{Isom}(M)$ is $\dim(O(n,1)) + n + 1 = (n+1)(n+2)/2$.

Definition 1.3. A Lorentzian manifold M has *maximal isometry group* if the action of $\mathrm{Isom}(M)$ is transitive and, for every point $x \in M$, every linear isometry $L : T_xM \to T_xM$ extends to an isometry of M.

Equivalently M has maximal isometry group if the above inclusion of $\mathrm{Isom}(M)$ into $\mathrm{ISO}(T_{x_0}M, TM)$ is a bijection. Hence,if M has maximal isometry group, its isometry group has maximal dimension, as a justification for the name used. From Corollary 1.2, every simply connected Lorentzian manifold M has maximal isometry group if it has constant sectional curvature and is geodesically complete. The converse holds even without the simple connectedness assumption. Namely:

Lemma 1.4. If M is a Lorentzian manifold with maximal isometry group, then M has constant sectional curvature and is geodesically complete.

Proof. Fix a point $x \in M$. As the identity component of $O(T_xM) \simeq O(n,1)$ acts transitively on spacelike planes, there exists a constant K such that Equation 1.1 holds for every pair (u,v) of vectors tangent at x which generate a spacelike plane. Now, for every point $x \in M$ both sides of Equation 1.1 are polynomial in $u, v \in T_xM$. Since the set of pairs (u,v) which generate spacelike planes is open in $T_xM \times T_xM$, Equation 1.1 must hold for every pair of vectors $u, v \in T_xM$.
Since $\mathrm{Isom}(M)$ acts transitively on M, it follows that M has constant sectional curvature K.
Let us now show that the manifold is geodesically complete. Suppose γ is a parametrized geodesic with $\gamma(0) = x$ and $\gamma'(0) = v \in T_xM$, which is defined for a finite maximal time $T > 0$. Let $T_0 = T - \epsilon > 0$. By assumption one can find an isometry $f : M \to M$ such that $f(x) = \gamma(T_0)$ and $df_x(v) = \gamma'(T_0)$. Then $t \to f \circ \gamma(t - T_0)$ is a parametrized geodesic which provides a continuation of γ beyond T, leading us to a contradiction. □

We conclude these preliminaries on Lorentzian geometry with a result of classification which will be useful in the following, explaining our interest in spaces with maximal isometry group.

Proposition 1.5. *Let M_K be a simply connected Lorentzian manifold of constant sectional curvature K with maximal isometry group, and let M be a Lorentzian manifold of constant sectional curvature K. Then:*

- *M is geodesically complete if and only if there is a local isometry $p : M_K \to M$ which is a universal covering.*
- *M has maximal isometry group if and only if Aut($p : M_K \to M$) is normal in Isom(M_K).*

Proof. Suppose M is geodesically complete, then by lifting the metric to the universal cover $\widetilde{M}$ one gets a simply connected geodesically complete Lorentzian manifold of constant sectional curvature K which by Corollary 1.2 is isometric to M_K. The covering map $p : M_K \to M$ is then a local isometry by construction. The converse is straightforward.
Now let Γ be $\mathrm{Aut}(p : M_K \to M)$, which is a discrete subgroup of $\mathrm{Isom}(M_K)$. Thus M is obtained as the quotient $M = M_K/\Gamma$, where Γ acts freely and properly discontinuously on M_K. The isometry group of M is isomorphic to $N(\Gamma)/\Gamma$, where by $N(\Gamma)$ we denote the normalizer of Γ in $\mathrm{Isom}(M_K)$. The

isomorphism is based on the observation that any isometry of $\widetilde{M}$ which normalizes Γ descends to an isometry of M, and conversely the lifting of any isometry of M must be in $N(\Gamma)$.
Hence the condition that M has maximal isometry group is equivalent to the condition that every element f of $\mathrm{Isom}(M_K)$ descends to the quotient to an isometry of M. This is in turn equivalent to the condition that $f\Gamma f^{-1} = \Gamma$ for every $f \in \mathrm{Isom}(M_K)$, which is the same as saying that Γ is normal in $\mathrm{Isom}(M_K)$. $\square$

Finally, any isometry between connected open subsets of a Lorentzian manifold M with maximal isometry group extends to a global isometry. In particular if M_K is a Lorentzian manifold of constant sectional curvature K with maximal isometry group, then any Lorentzian manifold M of constant sectional curvature K admits a natural $(\mathrm{Isom}(M_K), M_K)$-structure whose charts are isometries between open subsets of M and open subsets of M_K. We will refer to Lorentzian manifolds of constant sectional curvature K with maximal isometry group as *models* of constant sectional curvature K (following Klein's terminology).

CHAPTER 2

Models of Anti-de Sitter (n+1)-space

The aim of this chapter is to construct models of Lorentzian manifolds with constant sectional curvature -1 and maximal isometry group in any dimension. We are also interested in *stressing* the analogies between these manifolds with the models of hyperbolic space in the Riemannian setting. We will show that hyperbolic space is *naturally* embedded in Anti-de Sitter space, and we will later develop this topic in our study of earthquakes theory. After introducing the models, we are interested in studying the geometry of such manifolds: we will *give* a conformal (visual) boundary to Anti-de Sitter space and we will characterize geodesics and totally geodesic subspaces. We will also introduce the notion of polarity in Anti-de Sitter space (in some sense the *correct* Lorentzian correspondence between points and hyperplane, analogous to Euclidean orthogonality) and study its properties.

2.1 The quadric model

We want to introduce the analogue of the hyperboloid model of hyperbolic space. Denote by $\mathbb{R}^{n,2}$ the real vector space $\mathbb{R}^{n+2}$ equipped with the quadratic form

$$q_{n,2}(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2.$$

and by $\langle v, w\rangle_{n,2}$ the associated symmetric form. Finally, let $O(n,2)$ be the group of linear transformations of $\mathbb{R}^{n+2}$ that preserve $q_{n,2}$. Then we define:

$$\mathbb{H}^{n,1} = \{x \in \mathbb{R}^{n,2} \mid q_{n,2}(x) = -1\}.$$

It is immediate to check that $\mathbb{H}^{n,1}$, as the pre-image of a regular value of $q_{n,2}$, is a smooth connected submanifold of $\mathbb{R}^{n,2}$ of dimension $n+1$. The tangent space $T_x\mathbb{H}^{n,1}$, regarded as a subspace of $\mathbb{R}^{n+2}$, coincides with the orthogonal space $x^{\perp} = \{y \in \mathbb{R}^{n+2} \mid \langle x, y\rangle_{n,2} = 0\}$. A simple signature argument (along with the fact that $q_{n,2}(x) = -1$ for every $x \in \mathbb{H}^{n,1}$) shows that the restriction of the symmetric form $\langle ., .\rangle_{n,2}$ to $T_x\mathbb{H}^{n,1}$ has Lorentzian signature, hence it makes $\mathbb{H}^{n,1}$ a Lorentzian manifold. We remark that this model is the analogue of the hyperboloid model of hyperbolic space, in fact $\mathbb{H}^n$ is isometrically embedded in $\mathbb{H}^{n,1}$ as the submanifold defined by $x_{n+2} = 0$, $x_{n+1} > 0$. The natural action of $O(n,2)$ on $\mathbb{R}^{n,2}$ preserves $\mathbb{H}^{n,1}$, and in facts $O(n,2)$ acts by isometries on $\mathbb{H}^{n,1}$. We remark that $O(n,2)$ acts transitively on $\mathbb{H}^{n,1}$ and that the stabilizer of a point x acts transitively on the space of orthonormal bases of $T_x\mathbb{H}^{n,1}$. Hence $\mathbb{H}^{n,1}$ has maximal isometry group and $\mathrm{Isom}(\mathbb{H}^{n,1}) \simeq O(n,2)$.
By Lemma 1.4, $\mathbb{H}^{n,1}$ has constant sectional curvature. Let us now check that the sectional curvature is negative (in particular we will find $K = -1$). For this purpose, observe that the normal line in $\mathbb{R}^{n,2}$ to $\mathbb{H}^{n,1}$ at x is identified with the line generated by x itself. It follows that, if v, w are tangent vector fields along $\mathbb{H}^{n,1}$, we have the orthogonal decomposition:

$$(D_v w)(x) = (\nabla_v w)(x) + \langle v, w\rangle x,$$

where D is the flat connection of $\mathbb{R}^{n+2}$ and ∇ is the Levi-Civita connection of $\mathbb{H}^{n,1}$. Using the flatness of D we get:

$$R(u, v)w = \langle u, w\rangle v - \langle v, w\rangle u,$$

so that

$$\langle R(u, v)v, u\rangle = -(\langle u, u\rangle\langle v, v\rangle - \langle v, u\rangle^2),$$

and this shows that $\mathbb{H}^{n,1}$ has constant sectional curvature -1. We also remark that $\mathbb{H}^{n,1}$ is not simply connected, being homeomorphic to $\mathbb{R}^n \times \mathbb{S}^1$.

2.2 The "Klein model" and its boundary

Let us introduce a projective model, also known as the "Klein model", for Anti-de Sitter geometry. Let us define:

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = \mathbb{H}^{n,1}/\{\pm \mathrm{Id}\}.$$

Since $\{\pm \mathrm{Id}\}$ is the center of $O(n,2)$ (hence normal), $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ (when endowed with the Lorentzian metric induced by the quotient) has maximal isometry group by Proposition 1.5 and is therefore a model of constant sectional curvature -1. It can also be shown that the center of the isometry group of the Klein model is trivial, hence $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is the *minimal* model of AdS geometry, in the sense that any other model is a covering of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.
By definition, $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is naturally identified with a subspace of real projective space $\mathbb{R}\mathrm{P}^{n+1}$, more explicitly with the subset of timelike directions of $\mathbb{R}^{n,2}$:

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = \{[x] \in \mathbb{R}\mathrm{P}^{n+1} \mid q_{n,2}(x) < 0\}.$$

Like in hyperbolic geometry, the boundary of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ in projective space is a quadric, that is the projectivization of lightlike vectors in $\mathbb{R}^{n,2}$. We denote this quadric by $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = \{[x] \in \mathbb{R}\mathrm{P}^{n+1} \mid q_{n,2}(x) = 0\}$.
We observe that isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ induce projective transformations which preserve $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

The conformal Lorentzian structure of the boundary

We want to continue to develop the analogy with hyperbolic geometry and equip $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ with a conformal Lorentzian structure that extends the one on $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, similar to the conformal *visual boundary* in hyperbolic geometry.
A point $\ell \in \mathbb{R}\mathrm{P}^{n+1}$ is identified with $\mathrm{Span}(x)$ for some $x \in \mathbb{R}^{n,2}$, and the tangent space of real projective space has the canonical identification

$$T_\ell \mathbb{R}\mathrm{P}^{n+1} \simeq \mathrm{Hom}(\ell, \mathbb{R}^{n+2}/\ell).$$

Now, if ℓ is timelike, we can identify $\mathbb{R}^{n+2}/\ell$ with $\ell^\perp$. For any given local section $\sigma : \mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \to \mathbb{R}^{n,2}$ of the projection $\mathbb{R}^{n,2} \to \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, we can define a Lorentzian metric on $T\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ setting:

$$\langle\langle f, g\rangle\rangle_\sigma = \langle f(\sigma[x]), g(\sigma[x])\rangle_{n,2}$$

for $f, g \in T_x\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \simeq \mathrm{Hom}(\ell, \ell^\perp)$. If the section σ has image in $\mathbb{H}^{n,1}$, then the aforementioned metric coincides with the pull-back of the metric over $\mathbb{R}^{n,2}$, since the differential of σ identifies $T_{[x]}\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = T_x\mathbb{H}^{n,1} = x^\perp$. For a general section the identification does not hold, but we can recover a conformal metric via the formula:

$$\langle\langle f, g\rangle\rangle_{\lambda\sigma} = \lambda^2 \langle\langle f, g\rangle\rangle_\sigma \tag{2.1}$$

for any function λ.
We consider now the case where $\ell = \mathrm{Span}(x)$ is lightlike, i.e. $q_{n,2}(x) = 0$.

In this case we can not induce any natural metric on $\mathbb{R}^{n,2}/\ell$. However, if we let

$$\mathbb{L} = \{x \in \mathbb{R}^{n,2} \mid q_{n,2}(x) = 0\}$$

be the space of lightlike vectors, then $T_x\mathbb{L}$ is precisely $\ell^\perp$ and contains ℓ itself. We have recovered a canonical identification: $T_\ell \partial \mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \simeq \mathrm{Hom}(\ell, \ell^\perp/\ell)$. The bilinear form of $\mathbb{R}^{n,2}$, when restricted to $\ell^\perp$, induces a non degenerate bilinear form (of signature $(n-1,1)$) on $\ell^\perp/\ell$. We will denote such a restriction as $\langle v, w\rangle_{\ell^\perp/\ell}$.

We can now define a metric on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ for any section $\sigma : \partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \to \mathbb{L}$ of the canonical projection by the formula:

$$((f,g))_\sigma = \langle f(\sigma[x]), g(\sigma[x])\rangle_{\ell^\perp/\ell}, \tag{2.2}$$

for all $f, g \in \mathrm{Hom}(\ell, \ell^\perp/\ell)$. This metric can be viewed as the pull-back of the metric:

$$((f,g))_\sigma = \langle \sigma_*(f), \sigma_*(g)\rangle_{n,2}, \tag{2.3}$$

since the degenerate metric on $T_x\mathbb{L} = \ell^\perp$ is, by construction, the pull-back of the metric of $\ell^\perp/\ell$ by the projection along the degenerate direction ℓ.

The relation valid for the metric on $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ also holds for the metric on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, that is:

$$((f,g))_{\lambda\sigma} = \lambda^2((f,g))_\sigma, \tag{2.4}$$

and therefore the induced conformal class over $T\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is well defined and independent of the choice of σ and equips the tangent space of the boundary with a conformal Lorentzian metric. Let σ be a section of the projection $\pi : \mathbb{R}^{n,2} \to \mathbb{R}\mathrm{P}^{n+1}$ defined in a neighborhood U of a point $x \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. By construction the metric $((\cdot,\cdot))_\sigma$ over $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap U$ is the limit, as $y \to x$ for $y \in \mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap U$, of the conformal metric associated to σ defined over $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap U$.

Remark 2.1. *Let us make some observations about the light cone in the case of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. If $[y] \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ Equation* 2.3 *implies that the lightlike vectors in $T_{[y]}\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ are exactly the projection of vectors $x \in \mathbb{R}^{n,2}$ such that $\langle x, y\rangle_{n,2} = 0$ and $q_{n,2}(x) = 0$. These vectors are such that Span(x, y) are totally degenerate planes in $\mathbb{R}^{n,2}$, equivalently they are projective lines contained in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. Therefore the light cone in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ through $[y]$ is the union of all the projective lines through $[y]$ that are contained in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.*

The "Poincaré model" for the universal cover

We have already observed that $\mathbb{H}^{n,1}$, and its quotient $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, are not simply connected. We want to construct a simply connected model for AdS geometry.

For this purpose we introduce the universal cover of $\mathbb{H}^{n,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.
Let $\mathbb{H}^n$ be the hyperboloid model of hyperbolic space. Then:

$$\pi(y,t) = (y_1, \dots, y_n, y_{n+1}\cos t, y_{n+2}\sin t) \tag{2.5}$$

defines a map $\pi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^{n,1}$ which is a covering with deck transformations of the form $(y,t) \mapsto (y, t+2k\pi)$ for $k \in \mathbb{Z}$. We denote the covering space by $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ and we observe that it is also the universal cover of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, where the covering map is the composition of π and the quotient by $\{\pm \mathrm{Id}\}$. Pulling back the Lorentzian metric over $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, we get a simply connected Lorentzian manifold of constant curvature -1. The metric on $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ is a warped product of the form:

$$\pi^* g_{\mathbb{H}^{n,1}} = g_{\mathbb{H}^n} - y_{n+1}^2 dt^2. \tag{2.6}$$

Moreover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ has maximal isometry group, hence we have obtained a simply connected model for AdS geometry. More precisely we have a central extension, that is a (non split) short exact sequence:

$$0 \to \mathbb{Z} \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}) \to O(n,2) \to 1.$$

It is sometimes convenient to express the metric (2.6) using the Poincaré model of hyperbolic space. Recall that the disk model of the hyperbolic space is the unit disk $\mathbb{D}^n$ endowed with the conformal metric: $\frac{4}{(1-r^2)^2}\sum dx_i^2$, where $r^2 = |x|^2$. In our setting the isometry between the disk and the hyperboloid model is given by:

$$(x_1, \dots, x_n) \mapsto \Big(y_1 = \frac{2x_1}{1-r^2}, \dots, y_n = \frac{2x_n}{1-r^2}, y_{n+1} = \frac{1+r^2}{1-r^2}\Big). \tag{2.7}$$

The Poincaré model of AdS geometry is then the cylinder $\mathbb{D}^n \times \mathbb{R}$ endowed with the metric

$$\frac{4}{(1-r^2)^2}(dx_1^2 + \cdots + dx_n^2) - \Big(\frac{1+r^2}{1-r^2}\Big)^2 dt^2. \tag{2.8}$$

It follows from the definition that each slice $\{t = c\}$ is a totally geodesic copy of $\mathbb{H}^n$. The metric defined in (2.8) also shows that the vector field $\partial/\partial t$ is a timelike non-vanishing vector field on $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, giving it the structure of a time-orientable manifold. Any choice of time orientation is preserved by the action of the deck transformations of the covering from the Poincaré to the Klein model, hence both $\mathbb{H}^{n,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ are time-orientable.

2.3 Geodesics

We have presented our various models as manifolds and now we would like to improve our knowledge of their geometry. As always we start by studying and characterizing geodesics.

In the quadric model

Let us start with the exponential map in the hyperboloid model. Given a point $x \in \mathbb{H}^{n,1}$ and a vector $v \in T_x\mathbb{H}^{n,1}$, we want to determine the geodesic through x with speed v. We will distinguish several cases according to the sign of $q_{n,2}(v)$. If v is lightlike, then:

$$\gamma(t) = x + tv$$

is a geodesic of $\mathbb{R}^{n,2}$ and is contained in $\mathbb{H}^{n,1}$, hence γ is a geodesic of $\mathbb{H}^{n,1}$. If v is either timelike or spacelike, we claim that the geodesic $\gamma(t) = \exp_x(tv)$ is contained in the linear plane $W = \mathrm{Span}(x, v)$. In fact, the linear transformation T that fixes pointwise W and whose restriction to $W^\perp$ is $-\mathrm{Id}_{W^\perp}$ is in $O(n, 2)$. By the uniqueness of the geodesic, $T \circ \gamma = \gamma$ hence γ is contained in $\mathbb{H}^{n,1} \cap W$. We can easily derive the expressions

$$\gamma(t) = \cosh(t)x + \sinh(t)v \tag{2.9}$$

if $q_{n,2}(v) = 1$ and

$$\gamma(t) = \cos(t)x + \sin(t)v \tag{2.10}$$

if $q_{n,2}(v) = -1$.

In the Klein model

In analogy with the Riemannian case, in the Klein model $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ geodesics are intersections of projective lines with the domain of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \subset \mathbb{R}\mathrm{P}^{n+1}$. From what we have already said:

- Timelike geodesics correspond to projective lines that are entirely contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, are closed non-trivial loops and have length π.
- Spacelike geodesics correspond to lines that meet $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ transversally in two points. They have infinite length.
- Lightlike geodesics correspond to lines tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

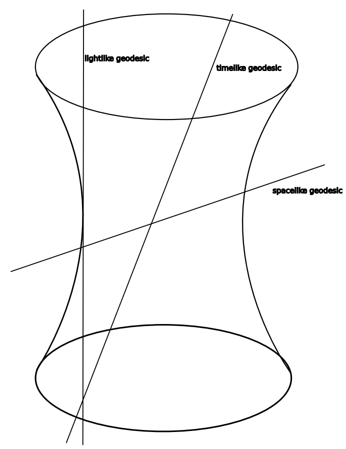

Figure 2.1: Geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

In particular the light cone through a point $[x] \in \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ coincides with the cone of lines through $[x]$ tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.
For instance in the affine chart $\mathbb{A}_{n+2} = \{x_{n+2} \neq 0\}$, where in coordinates $(y_1, \ldots, y_{n+1}) = (x_1/x_{n+2}, \ldots, x_{n+1}/x_{n+2})$, the intersection $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap \mathbb{A}_{n+2}$ is the interior of a one sheeted hyperboloid, that is:

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap \mathbb{A}_{n+2} = \{y_1^2 + \cdots + y_n^2 - y_{n+1}^2 < 1\},$$

while the boundary is the one-sheeted hyperboloid itself:

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap \mathbb{A}_{n+2} = \{y_1^2 + \cdots + y_n^2 - y_{n+1}^2 = 1\}.$$

In an affine chart, timelike geodesics correspond to affine lines which are entirely contained in the Anti-de Sitter space, and which are not asymptotic to its boundary; lightlike geodesics are tangent to the one sheeted hyperboloid, or are asymptotic to it (tangent at infinity). Spacelike geodesics are the last case, they are the intersection of two spacelike planes and meet the boundary transversally in two points.

Totally geodesic subspaces

Totally geodesic subspaces of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ of dimension k are obtained as the intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ with the projectivisation $\mathrm{P}(W)$ of a linear subspace W of $\mathbb{R}^{n,2}$ of dimension $k+1$. The negative index of W can be $1, 2$, for otherwise the intersection $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap \mathrm{P}(W)$ would be empty. We have several cases:

- If W has signature $(k-1,\ 2)$, then $\mathrm{P}(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is isometric to $\mathbb{A}\mathrm{d}\mathbb{S}^{k-1,1}$.
- If W has signature $(k-2, 1)$, then it is a copy of Minkowski space $\mathbb{R}^{k-2,1}$, hence $\mathrm{P}(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is a copy of the Klein model of hyperbolic space.
- If W is degenerate, then $\mathrm{P}(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$, is a lightlike subspace foliated by lightlike geodesics tangent to the same point of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

A particular case of the last point is when W is degenerate and $\dim(W) = n+1$. Then $\mathrm{P}(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is a projective hyperplane tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ at a point $[x]$ and $\mathrm{P}(W) \cap \partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is the lightlike cone of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ through $[x]$.

In the universal cover. In the universal cover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, geodesics are just the lifts of geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ or $\mathbb{H}^{n,1}$. Hence, every spacelike or lightlike geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ and $\mathbb{H}^{n,1}$, which is topologically a line, has a countable number of lifts to $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$. Timelike geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ and $\mathbb{H}^{n,1}$ are topologically circles and are in bijections with timelike geodesics in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, as the covering map restricted to a timelike geodesic induces a covering map onto the circle. Using the Poincaré model we can give an explicit description of lightlike geodesics. In fact, in Lorentzian geometry not only the nature of a vector is conformally invariant but also unparametrized lightlike geodesics are a conformal property ([14], Proposition 2.131):

Theorem 2.2. *If two Lorentzian metrics g, g' on a manifold M are conformal, then they have the same unparametrized lightlike geodesics.*

Because of Theorem 2.2 we can replace the Poincaré metric by the conformally equivalent -and often more easy to manage in calculation- metric given by:

$$\frac{4}{(1+r^2)^2}(dx_1^2 + \cdots + dx_n^2) - dt^2 \tag{2.11}$$

Now we observe that the first term in Equation 2.11 is exactly the form of the spherical metric on a hemisphere, pulled-back to the unit disk by the stereographic projection. We will call such a metric hemispherical and will

denote it by $g_{\mathbb{S}^n}$. Notice that the boundary of $\mathbb{D}^n$ is an equator for the hemispherical metric, and in fact it is the only equator completely contained in $(\mathbb{D}^n \cup \partial\mathbb{D}^n, g_{\mathbb{S}^n})$, a justification to the fact that we will refer to it as *the* equator.
As a consequence, unparametrized lightlike geodesics of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, going through a point (p_0, t_0) are characterized by the condition that they are mapped to spherical geodesic under the vertical projection $(p,t) \to p$ and moreover:

$$t - t_0 = d_{\mathbb{S}^n}(p, p_0)$$

on the geodesic. In particular, these lightlike geodesics meet the boundary of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ at the point that satisfies the above conditions such that p is on the equator of the hemisphere: as an example, if p_0 is the center of the hemisphere, then the points at infinity of the lightcone over (p_0, t_0) are the horizontal slice $t = t_0 + \pi/2$. This sphere is also the boundary of a hyperplane dual to (p_0, t_0), in a sense that we will explain in the following section.
By an analogous reasoning we can give an explicit description of a lightlike hyperplane in the Poincaré model: the lightlike plane having (p_0, t_0) as a past endpoint, (where now p_0 is on the equator) is precisely $\{(p,t) \mid t - t_0 = d_{\mathbb{S}^n}(p, p_0)\}$ and its future endpoint is $(-p_0, t + \pi.)$

2.4 Polarity in AdS

The quadratic form $q_{n,2}$ induces a polarity on the projective space $\mathbb{R}\mathrm{P}^{n+1}$, explicitly the correspondence associates to a projective subspace $\mathrm{P}(W)$ the subspace $\mathrm{P}(W^\perp)$. In particular, we have an induced duality between spacelike totally geodesic subspaces of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ where the dual of a spacelike $k-$dimensional subspace is an $n-k+1$ subspace.
For instance, if we consider the dual of a point $[x] \in \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ it will be an $n-$dimensional spacelike hyperplane $P_{[x]} = \mathrm{P}(x^\perp)$. Projectively, $P_{[x]}$ is characterized as the hyperplane spanned by the intersection of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n-1,1}$ with the lightcone from $[x]$. More geometrically, it can be checked that $P_{[x]}$ is the set of antipodal points to $[x]$ along timelike geodesics through $[x]$. Also, every timelike geodesic through $[x]$ meets $P_{[x]}$ orthogonally at time $\pi/2$. Conversely, given a totally geodesic spacelike hyperplane H, all the timelike geodesics that meet H orthogonally intersect in a single point, which is the dual point of H.

In the quadric model

We would like to *lift* the idea of duality to coverings of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. Observe that in $\mathbb{H}^{n,1}$ there are two dual planes associated to any point x, namely the sets:

$$P_x^{\pm} = \{\exp_x(\pm(\pi/2)v) \mid q_{n,2}(v) = -1,\ v \text{ future-directed}\}.$$

Now the points P_x^+ and P_x^- are antipodal and $P_{-x}^{\pm} = P_x^{\mp}$. The planes $P_x^{\pm}$ disconnect $\mathbb{H}^{n,1}$ in two regions U_x and U_{-x}, where U_x is the connected component containing x. They can be characterized as:

$$U_x = \{y \in \mathbb{H}^{n,1} \mid \langle x, y\rangle_{n,1} < 0\}.$$

Spacelike and lightlike geodesics through x do not exit U_x, while all the timelike geodesics through x meet orthogonally $P_x^{\pm}$ and they all pass through the point $-x$. More precisely a point $y \neq x$ is connected to x:

- by a spacelike geodesic if and only if $\langle x, y\rangle_{n,1} < -1$,
- by a lightlike geodesic if and only if $\langle x, y\rangle_{n,1} = -1$,
- by a timelike geodesic if and only if $|\langle x, y\rangle_{n,1}| < 1$.

An immediate consequence is that if y is connected to x by a spacelike geodesic, there is no geodesic joining y to $-x$. Hence the exponential map of $\mathbb{H}^{n,1}$ is not surjective. But as any point $y \in \mathbb{H}^{n,1}$ can be connected through a geodesic either to x or $-x$ it follows that the exponential over $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is surjective.

CHAPTER 3

Anti-de Sitter space in dimension (2+1)

We will now restrict our attention to dimension three Anti-de Sitter geometry, as it will be the specific model geometry of the manifolds of our next interest. In this chapter we will highlight the structure of Lie group that $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ has in this specific dimension and we will study its geometry using tools of Lie group theory. Most of the results are just a particular case of the theory developed in the previous chapter, but the Lie structure permits to give a more explicit description of the geometry of the phenomena we are interested in.

3.1 The PSL$(2, \mathbb{R})$ model

The fundamental observation is the existence of a special model in dimension three which endows Anti-de Sitter space with a Lie group structure. To construct such a model we observe that $q = -\det$ is a quadratic form with signature (2,2) over the real vector space $\mathcal{M}(2, \mathbb{R})$ (the signature is evident when we consider the basis consisting of elementary matrices). The associated bilinear form is expressed by the formula:

$$\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}(A \cdot \mathrm{adj}(B)) \tag{3.1}$$

for $A, B \in \mathcal{M}(2,\mathbb{R})$, where adj denotes the adjugate matrix, namely:

$$\operatorname{adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

hence, via Sylvester's theorem, there is an identification between $(\mathcal{M}(2,\mathbb{R}), q)$ and $(\mathbb{R}^{2,2}, q_{2,2})$, unique up to composition by elements in $O(2,2)$. Under this isomorphism $\mathbb{H}^{2,1}$ is identified with the Lie group $\mathrm{SL}(2,\mathbb{R})$.
Let us observe that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts linearly on $\mathcal{M}(2,\mathbb{R})$ by left and right multiplication:

$$(A, B) \cdot X = AXB^{-1}.$$

As a direct consequence of the Binet formula, the action preserves the quadratic form q and thus induces a representation:

$$\rho : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to O(\mathcal{M}(2,\mathbb{R}), q).$$

Since the center of $\mathrm{SL}(2,\mathbb{R})$ is $\{\pm \mathrm{Id}\}$, the kernel of ρ is $K = \{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$, and by a dimensional argument (and connectedness of $\mathrm{SL}(2,\mathbb{R})$) it turns out that the image of the representation is the connected component of the identity:

$$\mathrm{Isom}_0(\mathbb{H}^{2,1}) = \mathrm{SO}_0(\mathcal{M}(2,\mathbb{R}), q) \simeq (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/K$$

Using this model, one has a natural identification between $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and the Lie group $\mathrm{PSL}(2,\mathbb{R})$, in such a way that:

$$\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \simeq \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \tag{3.2}$$

acting by left and right multiplication on $\mathrm{PSL}(2,\mathbb{R})$.
We are mostly interested in orientation-preserving and time-preserving notions that do not depend on a chosen orientation, nevertheless we will fix here an orientation and a time-orientation of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathrm{PSL}(2,\mathbb{R})$. As we are dealing with a Lie group it is sufficient to define an orientation of the Lie algebra, namely the tangent at the identity Id. We declare as (positive) oriented basis of $\mathfrak{sl}(2,\mathbb{R})$:

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

The first two vectors V, W are spacelike, while U is timelike. U is the tangent vector to the one-parameter group of elliptic isometries of $\mathbb{H}^2$ fixing $i \in \mathbb{H}^2$, parametrized by the angle of clockwise rotations; V and W are vectors tangent to the one-parameter groups of loxodromic isometries fixing the geodesic with endpoints $(-1, 1)$ and $(0, \infty)$ respectively. Time-orientation can also be inherited by the Lie algebra, we declare that U is a future-pointing timelike vector.

The stabilizer of the identity in $\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ is the diagonal subgroup $\Delta < \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Under the obvious identification of $\mathrm{PSL}(2,\mathbb{R})$ and Δ, the action of the identity stabilizer on the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) = T_{\mathrm{Id}}\mathrm{PSL}(2,\mathbb{R})$ is the adjoint action of $\mathrm{PSL}(2,\mathbb{R})$. A direct consequence of this construction is the bi-invariance of the quadratic form q. Indeed, denoting by q_{Id} the restriction of q to $T_{\mathrm{Id}}\mathrm{SL}(2,\mathbb{R})$, a direct computation shows that q_{Id} equals $(1/8)\kappa$ where $\kappa(X,Y) = 4tr(XY)$ is the Killing form of $\mathfrak{sl}(2,\mathbb{R})$.

Remark 3.1. *The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ equipped with the quadratic form q_{Id} is then a copy of the 3-dimensional Minkowski space, hence the adjoint action yields a representation*

$$PSL(2,\mathbb{R}) \to O(\mathfrak{sl}(2,\mathbb{R}), q_{Id})$$

which in turn induces the well-known isomorphism:

$$SO_0(2,1) \simeq SO_0(\mathfrak{sl}(2,\mathbb{R}), q_{Id}) \simeq PSL(2,\mathbb{R}),$$

which is nothing but the restriction of the isomorphism of Equation 3.2 to the stabilizer of the identity in the left-hand side ($Isom(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$), and to the diagonal subgroup Δ in the right-hand side ($PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$).

3.2 The boundary of $\mathbf{PSL}(2,\mathbb{R})$

From the aforementioned identification we obtain a new one between $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the boundary of $\mathrm{PSL}(2,\mathbb{R})$, inside $\mathrm{P}(\mathcal{M}(2,\mathbb{R}))$ as the projectivization of the cone of rank 1 matrices. Therefore from now on we shall consider

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} = \{[X] \in \mathrm{P}(\mathcal{M}(2,\mathbb{R})) |\ \mathrm{rank}(X) = 1\}.$$

We endow $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}} := \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the topology induced by seeing both as subsets of the real projective space $\mathrm{P}(\mathcal{M}(2,\mathbb{R}))$. We want to observe that we have the following homeomorphism:

$$\begin{aligned} \delta : \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} &\to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \\ [X] &\mapsto (\mathrm{Im}(X), \mathrm{Ker}(X)) \end{aligned}$$

where we are considering $\mathbb{R}\mathrm{P}^1$ as the space of one-dimensional subspaces of $\mathbb{R}^2$. Since we have that $\mathrm{Im}(AXB^{-1}) = A \cdot \mathrm{Im}(X)$ and $\mathrm{Ker}(AXB^{-1}) =$

$B \cdot \mathrm{Ker}(X)$, the map δ is equivariant with respect to the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, acting on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ as the natural extension of the group of isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and on $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ by the obvious product action.
In this setting our choice of a time-orientation *can be modified* according to the following Lemma:

Lemma 3.2. The inversion map $\iota[X] = [X]^{-1}$ is a time-reversing isometry of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ which induces the homeomorphism $(x, y) \to (y, x)$ on the boundary $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Proof. It follows from definition that ι is equivariant with respect to the isomorphism of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ which switches left and right factors, more explicitly we have that for every $A, B \in \mathrm{PSL}(2,\mathbb{R})$ the following holds: $\iota[(A, B)X] = (B, A) \cdot [\iota X] = BX^{-1}A^{-1}$. As $\mathrm{d}_{\mathrm{Id}}\iota = -\mathrm{Id}$ is a linear isometry, ι is an isometry, the differential being minus the identity also shows that it revers time-orientation.
The second part of the statement can be checked via the following. For a 2×2 matrix the Cayley-Hamilton theorem implies the equality $(\det X)X^{-1} = (\mathrm{tr}X)\mathrm{Id} - X$, so that projectively we have $[X^{-1}] = [\mathrm{tr}X\mathrm{Id} - X]$. Hence ι extends to the transformation $[X] \to [\mathrm{tr}X\mathrm{Id} - X]$ on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Now if X is a rank 1 matrix, it is traceless if and only if $X^2 = 0$, hence $\mathrm{Ker}(X) = \mathrm{Im}(X)$. If $\mathrm{tr}(X) \neq 0$, then X is diagonalizable with eigenvalues 0 and $\mathrm{tr}(X)$. Moreover $\mathrm{Ker}(X)$ and $\mathrm{Im}(X)$ are the corresponding eigenspaces. It follows then that $\mathrm{Ker}(\mathrm{tr}X\mathrm{Id} - X) = \mathrm{Im}(X)$ and $\mathrm{Im}(\mathrm{tr}X\mathrm{Id} - X) = \mathrm{Ker}X$ □

Consider the hyperbolic model of the upper half-plane $\mathbb{H}^2$. $\mathbb{R}\mathrm{P}^1$ corresponds to the boundary at infinity $\partial\mathbb{H}^2$ via the identification mapping the line spanned by (a, b) to $\frac{a}{b}$ and $\mathrm{PSL}(2,\mathbb{R})$ is identified to $\mathrm{Isom}_0(\mathbb{H}^2)$. From this perspective we can consider $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ as $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. We can then interpret the convergence to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ in this setting:

Lemma 3.3. A sequence $[X_n] \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ converges to $(x, y) \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ if and only if for every $p \in \mathbb{H}^2$, $X_n(p) \to x$ and $X_n^{-1}(p) \to y$.

Proof. Since $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ via isometries, if the conditions holds for some p then it holds for all $q \in \mathbb{H}^2$, as the distance from p to any other point q is bounded. Without loss of generality we can assume $p = i$ in the upper half-plane. Assuming $[X_n] = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ converges projectively to a rank 1 matrix means that there exists a sequence of real numbers $\lambda_n \to 0$ such that $\lambda_n X_n \to X$. As the limit matrix has rank one at least one of the successions of coefficient (when multiplied by λ_n) does not converge to zero. We can assume that $\lambda_n a_n \to a$ does not converge to 0, the other cases are all analogous. The assumption of $a \neq 0$ and and $\operatorname{rank}(X) = 1$, leave us with the following possibilities for X:

- Assume $\lambda_n c_n \to 0$ and $\lambda_n d_n \to 0$, then we have that $[X_n]$ converges to a $[X]$ of the form:

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \; X(i) = \lim_{n\to\infty} \lambda_n \frac{a_n i + b_n}{c_n i + d_n} = \frac{a}{0}$$

- If $b = 0$ and $d = 0$, then $[X_n]$ converges to $[X]$ with

$$X = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}, \; X(i) = \lim_{n\to\infty} \lambda_n \frac{a_n i + b_n}{c_n i + d_n} = \frac{a}{c}$$

- If $b, c \neq 0$ it follows from the rank one condition that $d = bc$, then $[X_n]$ converges to $[X]$ with

$$X = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}, \; X(i) = \lim_{n\to\infty} \lambda_n \frac{a_n i + b_n}{c_n i + d_n} = \frac{a^2 i + ab}{aci + bc} = \frac{a(ai+b)}{c(ai+b)} = \frac{a}{c}.$$

$\square$

In dimension three, $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a double ruled quadric. We shall describe such rulings in a more geometric way. Given any $(x_0, y_0) \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the set

$$\lambda_{y_0} := \{(x, y_0) | x \in \mathbb{R}\mathrm{P}^1\} \tag{3.4}$$

describes a projective line in $\mathbb{R}\mathrm{P}^3$ which is contained in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, hence lightlike for the conformal Lorentzian structure of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, as seen in Remark 2.1. In fact λ_{y_0} is the orbit of (x_0, y_0) by the action of $\mathrm{PSL}(2,\mathbb{R}) \times \{\mathrm{Id}\}$, or by the (now free) action of $\mathrm{PSO}(2,\mathbb{R}) \times \{\mathrm{Id}\}$. Here $\mathrm{PSO}(2,\mathbb{R})$ corresponds to a 1-parameter elliptic subgroup in $\mathrm{PSL}(2,\mathbb{R})$. In short:

$$\lambda_{y_0} = \mathrm{PSL}(2,\mathbb{R}) \cdot (x_0, y_0) = \mathrm{PSO}(2,\mathbb{R}) \cdot (x_0, y_0).$$

We refer to λ_{y_0} as the *left ruling* through (x_0, y_0), and similarly the *right ruling* is the set:

$$\mu_{x_0} := \{(x_0, y) | y \in \mathbb{R}\mathrm{P}^1\}.$$

We can express the conformal Lorentzian structure of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the rulings as shown in Figure 3.1. The action of $\mathrm{PSO}(2,\mathbb{R}) \times \{\mathrm{Id}\}$ on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ yields a flow on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ generated by a right-invariant vector field, which at Id is the *positive*

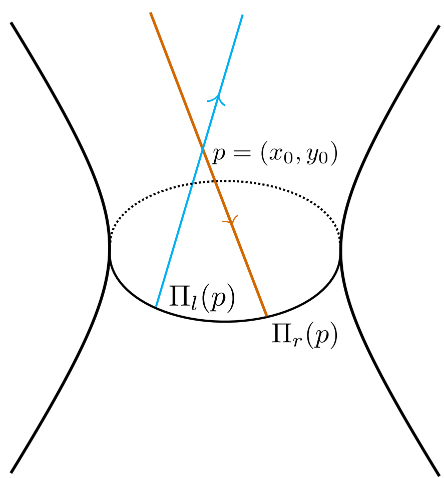

Figure 3.1: Left and right projection from a point $p \in \partial \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ to the plane $P = \{x_3 = 0\}$. The rulings induce a time-orientation on the boundary.

tangent vector of $\mathrm{PSO}(2,\mathbb{R})$. So orbits are all timelike and *future directed*. In similar fashion the action of $\{\mathrm{Id}\}\times\mathrm{PSO}(2,\mathbb{R})$ yields a flow generated by a left-invariant vector field, which at Id is the *negative* tangent vector at $\mathrm{PSO}(2,\mathbb{R})$, and its orbits are all timelike and *past directed*.

Proposition 3.4. *Let $\pi_l,\pi_r : \mathbb{RP}^1\times\mathbb{RP}^1\to\mathbb{RP}^1$ be the canonical projection and $d\theta$ be the angular form on $\mathbb{RP}^1\simeq\partial\mathbb{H}^2$. Then the symmetric product $\pi_l^*(d\theta)\pi_r^*(d\theta)$ is in the conformal class of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.*

Proof. Since we already know that the left and right rulings are lightlike for the conformal class of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, it only remains to check the sign of time orientation. We observe that λ_{y_0} is the orbit of the action of $\mathrm{PSO}(2,\mathbb{R})\times\{\mathrm{Id}\}$, while μ_{x_0} is the orbit of the action of $\{\mathrm{Id}\}\times\mathrm{PSO}(2,\mathbb{R})$, the induced orientation agrees with the one given by the pullback of the metrics. □

It follows, from the conformal structure that $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ inherits from the rulings, that a C^1 curve in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is spacelike when it is locally the graph of an orientation-preserving function (the product of the pullbacks is different from zero and we can use the implicit function theorem), and timelike when it is locally the graph of an orientation-reversing function. Given two intervals I_1, I_2 in $\partial\mathbb{H}^2$ and assuming that θ_1,θ_2 are angle determination over the two intervals, the future $I^+_{I_1\times I_2}(p_0,q_0)$ of a point (p_0,q_0) in $I_1\times I_2$ is the region made up of points (p,q) where $\theta_1(p)-\theta_2(p_0)>0$ and $\theta_2(q)-\theta_2(q_0)<0$. The past is determined by reversing both inequalities. In conclusion:

$$I^+_{I_1\times I_2}(p_0,q_0)\cup I^-_{I_1\times I_2}(p_0,q_0)=\{(p,q)\in I_1\times I_2 \mid (\theta_1(p)-\theta_1(p_0))(\theta_2(q)-\theta_2(q_0))<0\}. \tag{3.5}$$

We want to stress that our interest in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is mainly justified by the following: when we will deal with earthquake theory we will often consider $\varphi:\mathbb{RP}^1\to\mathbb{RP}^1$ an orientation-preserving homeomorphism of the circle. The associated graph Λ_φ via the identification given by δ is a subset of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We observe that from equivariance of δ the following holds for every $(\alpha,\beta)\in\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$:

$$(\alpha,\beta)\cdot\Lambda_\varphi=\Lambda_{\beta\circ\varphi\circ\alpha^{-1}}. \tag{3.6}$$

We remark one last time that we will consider $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ as always implicitly identified with $\mathbb{RP}^1\times\mathbb{RP}^1$ via δ.

3.3 Geodesics in PSL$(2,\mathbb{R})$

We have already seen geodesics in the general Anti-de Sitter space, we would like to specialize here using the model of $\mathrm{PSL}(2,\mathbb{R})$ and tools from general Lie groups theory. In particular we recall the following (the necessary tools in Lie groups theory are introduced in [2]):

Lemma 3.5. Given left-invariant vector fields V and W on a Lie group G, the Levi-Civita connection of a bi-invariant metric has the expression:

$$\nabla_V W = \frac{1}{2}[V, W].$$

In particular, the Lie group exponential coincides with the pseudo-Riemannian exponential map.

We would like to start by considering geodesics through the identity. The Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$ is isometrically identified with Minkowski space, as seen in Remark 3.1, where under such an isometry the stabilizer of a point corresponds to the group of linear isometries of Minkowski space. Moreover, by Lemma 3.5 it suffices to understand the one-parameter group for the Lie group structure of $\mathrm{PSL}(2,\mathbb{R})$. We immediately get the following:

- Timelike geodesics are, up to conjugacy, of the form:

  $$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

  namely, under the identification of $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^2)$, they are elliptic one-parameter groups fixing a point in $\mathbb{H}^2$. In this example, the tangent vector is the matrix

  $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

  These are closed geodesics, parametrized by arclenght, of total length π.

- Spacelike geodesics are, again up to conjugacy, of the form:

  $$\begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$

  with initial velocity:

  $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

  In the hyperbolic settings, these are loxodromic one-parameter groups, fixing two points in the boundary of $\mathbb{H}^2$ (in this particular case, ± 1).

- Finally, lightlike geodesics are the parabolic one-parameter groups conjugate to:
$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$
whose initial vector has indeed zero length.

Timelike geodesics

When looking for a complete description of timelike geodesics it suffices to let (the identity component of) the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ (namely $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$) act on $\mathrm{PSL}(2,\mathbb{R})$ via left and right multiplication. In particular, we can describe the whole space of timelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ as follows:

Proposition 3.6. *There is a homeomorphism between the space of (unparametrized) timelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and $\mathbb{H}^2 \times \mathbb{H}^2$. The homeomorphism is equivariant for the action of $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \simeq PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.*

Proof. The homeomorphism is defined as follows. Given a point $(p,q) \in \mathbb{H}^2 \times \mathbb{H}^2$, we associate to it the subset:
$$L_{p,q} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \mid X \cdot q = p\}$$
By the previous discussion, timelike geodesics through the identity are precisely of the form $L_{p,p}$ for some $p \in \mathbb{H}^2$. The map $(p,q) \mapsto L_{p,q}$ is equivariant for the natural actions of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, namely $(A,B) \cdot L_{p,q} = L_{A\cdot p, B\cdot q}$, which also implies that $L_{p,q}$ is an unparameterized timelike geodesic and that all the unparameterized timelike geodesics are of this form, namely the map we defined is surjective. It remains to show injectivity; if $L_{p,q} = L_{p',q'}$ for $(p,q) \neq (p',q')$ then in particular there exists an isometry X_1 of $\mathbb{H}^2$ sending p to q and p' to q', but such an isometry is necessarily unique. Suppose the existence of an $X_2 \neq X_1$ isometry of $\mathbb{H}^2$ with the same property, then $X_2^{-1} \circ X_1$ fixes p, p', an absurd since the identity is the only isometry of $\mathbb{H}^2$ fixing two different points. $\square$

Spacelike geodesics

Let us consider ℓ an oriented geodesics of the hyperbolic plane $\mathbb{H}^2$. From what we have already discussed the one-parameter group of loxodromic transformations fixing ℓ as an *oriented* geodesic constitutes a spacelike geodesic through the origin. By an argument analogous to the one given in Proposition 3.6, relying on the equivariance of the construction by the action of

$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, one proves that every spacelike geodesic is of the form:

$$L_{\ell,\jmath} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \mid X \cdot \jmath = \ell \text{ as oriented geodesics}\},$$

where ℓ and $\jmath$ denote oriented geodesics of $\mathbb{H}^2$. We want to emphasize that every (unparameterized, unoriented) spacelike geodesic can be expressed in the above form in two ways, as we could change the orientation of both ℓ and $\jmath$. Rephrasing we can state:

Proposition 3.7. *There is a homeomorphism between the space of (unparametrized) oriented spacelike geodesics of* $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ *and the product of two copies of* $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \backslash \Delta$*, the space of oriented geodesics of* $\mathbb{H}^2$*. The homeomorphism is equivariant for the action of* $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \simeq PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$*.*

However we will not be really interested in oriented geodesics, hence we will have identification $L_{\ell,\jmath} = L_{\ell',\jmath'}$ where with ℓ' we denote ℓ endowed with the opposite orientation.
Given a spacelike geodesic, there is a natural notion of *dual* spacelike geodesic (3.2), which we define using the projectivity duality between points and planes discussed in section 2.4:

Definition 3.8. Given a spacelike geodesic $L_{\ell,\jmath}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the *dual spacelike geodesic* is the intersection of all the spacelike planes dual to points of $L_{\ell,\jmath}$.

Let us see it with an explicit. Consider the geodesic $L_{\ell,\ell}$ through the origin, which consists of the one-parameter loxodromic group of $\mathrm{PSL}(2,\mathbb{R})$ translating along ℓ. It can be checked that the dual geodesic consists of all elliptic order-two elements whose fixed point lies in ℓ. To see this we can suppose (as a consequence of the transitivity of the action of $\mathrm{PSL}(2,\mathbb{R})$ on geodesics of $\mathbb{H}^2$) that ℓ is the imaginary axis. In this case $L_{\ell,\ell} = \{M_k | k \in \mathbb{R}\}$ where

$$M_k = \begin{bmatrix} e^{-k/2} & 0 \\ 0 & e^{k/2} \end{bmatrix}.$$

We observe now, and will develop more thoroughly the theory of totally geodesic planes in Section 3.4, that given an element $M \in \mathrm{PSL}(2,\mathbb{R})$ the set:

$$P_{[M]} = \{[X] \in \mathrm{PSL}(2,\mathbb{R}) \mid \langle X, M\rangle = 0\} \tag{3.7}$$

defines the intersection of a projective subspace of $\mathrm{P}\mathcal{M}(2,\mathbb{R})$ with $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, hence a totally geodesic subspace. Given a generic matrix X:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

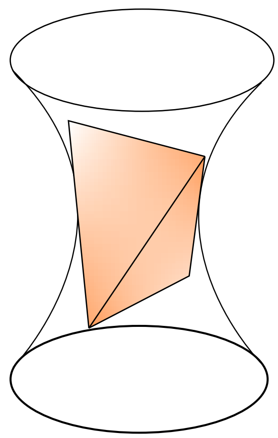

Figure 3.2: Top and bottom edges are spacelike lines, dual to one another. The tetrahedron is obtained connecting the vertexes of the spacelike geodesics via lightlike segments.

the property of being a point in the plane can be restated as the condition

$$ae^{\frac{k}{2}} + de^{-\frac{k}{2}} = 0. \tag{3.8}$$

As we are asking that Equation 3.8 holds for every k it must be $a = d = 0$, hence X (seen as a Möbius transformation) is of the form $X(z) = -\frac{b^2}{z}$ an elliptic transformation of order two with fixed point $ib \in \mathbb{H}^2$.
In other words, the dual spacelike geodesic of $L_{\ell,\ell}$ is $L_{\ell,\ell'}$.
We can explicitly describe the points at infinity in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ of these geodesics. Using Lemma 3.3, if x and y are the endpoints at infinity of ℓ in $\partial\mathbb{H}^2$, then any sequence diverging towards an end of $L_{\ell,\ell'} \subset \mathrm{PSL}(2,\mathbb{R})$ maps an interior point towards x, and the sequence of inverses towards y (up to switching the two points). In other words, under the identification given by δ between $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$, the endpoints of $L_{\ell,\ell}$ are (x,y) and (y,x). A similar argument applied to the geodesic $L_{\ell,\ell'}$, which consists of order-two elliptic isometries with fixed point in ℓ shows that its endpoints are (x,x) and (y,y).

Recalling the description of the left and right rulings of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ given in (3.4), we can conclude that the endpoints of a spacelike geodesic and its dual are mutually connected by lightlike segments in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.
The transitivity of the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ let us state:

Proposition 3.9. *Given a spacelike geodesic $L_{\ell,\jmath}$ of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, its endpoints in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ are (x_1, y_2) and (y_1, x_2), where x_1 and y_1 are the final and initial endpoints of ℓ in $\partial\mathbb{H}^2$, and x_2 and y_2 are the final and initial endpoints of $\jmath$. The dual geodesic is $L_{\ell,\jmath'}$ and has endpoints (x_1, x_2) and (y_1, y_2).*

3.4 Spacelike planes

Now we want to study totally geodesic spacelike planes in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. They are all obtained as the intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with a projective subspace in the projective space $\mathrm{P}\mathcal{M}(2,\mathbb{R})$. Hence they are all of the the following form:

$$P_{[A]} = \{[X] \in \mathrm{PSL}(2,\mathbb{R}) \mid \langle X, A\rangle = 0\} \tag{3.9}$$

for some non-zero matrix A. The notation is justified by the observation that the plane defined by P_A depends only on the projective class of A. The totally geodesic plane is spacelike if and only if $q(A) = -\det A$ is negative. We will call such a plane *dual plane* of A, in particular the dual plane P_γ of an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ is a spacelike totally geodesic plane.

Example 3.10: Before the general treatment we want to focus on a concrete example. Consider $\gamma = \mathrm{Id} \in \mathrm{PSL}(2,\mathbb{R})$. Now because of Equation 3.1, P_{Id} is the subset of $\mathrm{PSL}(2,\mathbb{R})$ consisting of projective classes of X with $\mathrm{tr}(X) = 0$. By the Cayley-Hamilton theorem, $X^2 = -\mathrm{Id}$, hence the elements of P_{Id} are order-two isometries of $\mathbb{H}^2$, that is, elliptic elements with rotation angle π. We can also observe that P_{Id} is invariant under the action of $\mathrm{PSL}(2,\mathbb{R})$ by conjugation, which corresponds to the diagonal in the isometry group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Using Lemma 3.3, we can see that the boundary of P_{Id} in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ is the diagonal; more precisely:

$$\partial P_{\mathrm{Id}} = \mathrm{graph}(\mathrm{Id}) \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1. \tag{3.10}$$

Now consider a point $z \in \mathbb{H}^2$, and let us denote by $\mathcal{R}_z$ the order-two elliptic isometry with fixed point z. We claim that the map

$$\iota : \mathbb{H}^2 \to P_{\mathrm{Id}}, \quad \iota(z) = \mathcal{R}_z$$

is an isometry with respect to the hyperbolic metric of $\mathbb{H}^2$ and the induced metric on P_{Id}. First, the inverse of ι is simply the fixed-point map $\mathrm{Fix} : P_{\mathrm{Id}} \to \mathbb{H}^2$ sending an elliptic isometry to its fixed point, which also shows that ι is equivariant with respect to the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$ by homographies and on P_{Id} by conjugation, since $\mathrm{Fix}(\alpha\gamma\alpha^{-1}) = \alpha \mathrm{Fix}(\gamma)$. That is, the following holds:

$$\iota(\alpha \cdot p) = \alpha \circ \iota(p) \circ \alpha^{-1}. \tag{3.11}$$

A direct consequence is that ι is isometric, since the pull-back of the metric of P_{Id} is necessarily $\mathrm{PSL}(2,\mathbb{R})$-invariant and has constant curvature -1, hence it coincides with the standard hyperbolic metric on the upper half-plane.

This simple example is actually the key case to understand general spacelike totally geodesic planes as every spacelike totally geodesic plane is of the form P_γ for some $\gamma \in \mathrm{PSL}(2,\mathbb{R})$. To see this, observe that the action of the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ on spacelike totally geodetic planes is transitive, and that $P_\gamma = (\gamma, \mathrm{Id}) \cdot P_{\mathrm{Id}}$ as the isometry (γ, Id) maps $\mathrm{Id} \to \gamma$, and therefore the dual plane of Id to the dual plane of γ. In view of the observations given in (3.6) and (3.10), we can conclude the following:

Lemma 3.11. Every spacelike totally geodesic plane of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is of the form P_γ for some orientation-preserving isometry $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, and

$$\partial P_\gamma = \mathrm{graph}(\gamma^{-1}) \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1.$$

3.5 Timelike planes

Let us now consider a matrix $A \in \mathcal{M}(2,\mathbb{R})$ such that $\det(A) = -1$. The corresponding plane P_A defined by Equation 3.9 is a timelike totally geodesic plane. Associated with $[A]$ is an orientation-reversing isometry η of $\mathbb{H}^2$. We will thus denote $P_{[A]}$ by P_η.
The totally geodesic timelike plane P_η can now be parametrized as follows. We have a map:

$$\mathcal{I} \to \mathcal{I} \circ \eta \tag{3.12}$$

from the spaces of reflections $\mathcal{I}$ along geodesic of $\mathbb{H}^2$, with values in $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. As seen in the proof of Lemma 3.2 we have that an X with determinant -1 is an inversion if and only if traceless. As $\det(A) = -1$ we have $\mathrm{adj}(A) = -A^{-1}$, therefore $\langle XA, A\rangle = 0$ if and only if $\mathrm{tr}(X) = 0$, that is X is traceless hence an involution. This shows that the image of the map defined in Equation 3.12 is the entire plane P_η.

In similar fashion to the spacelike case, using the transitivity of the group of isometries on timelike planes, every timelike plane is of the form above. We can show the following:

Lemma 3.12. Every timelike totally geodesic plane of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is of the form P_η for some orientation-reversing isometry $\eta \in \mathrm{PSL}(2,\mathbb{R})$, and

$$\partial P_\eta = \mathrm{graph}(\eta^{-1}) \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1.$$

Proof. We want to use Lemma 3.3 and the parametrization given in Equation 3.12. Suppose we have a sequence $\mathcal{I}_n$ such that $\mathcal{I}_n\eta(z) \to x \in \partial\mathbb{H}^2$, for any $z \in \mathbb{H}^2$. Then, using that $\mathcal{I}_n$ is an involution and the continuity of the action of η on $\overline{\mathbb{H}}^2$, $(\mathcal{I}_n\eta)^{-1}(z) = \eta^{-1}\mathcal{I}_n^{-1}(z) = \eta^{-1}\mathcal{I}_n(z) = \eta^{-1}(x)$. $\square$

3.6 Lightlike planes

We are only left with case of lightlike totally geodesic planes. Those are the form $P_{[A]}$ for a nonzero matrix A with $\det(A) = 0$. Before giving an explicit description of such planes, we want to observe that their boundary will not be a graph in $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$, unlike the case of spacelike and timelike planes.

Lemma 3.13. Every lightlike totally geodesic plane of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is of the form $P_{[A]}$ for some rank one matrix A, and:

$$\partial P_{[A]} = (\mathrm{Im}(A) \times \mathbb{R}\mathrm{P}^1) \cup (\mathbb{R}\mathrm{P}^1 \times \mathrm{Ker}(A)).$$

Proof. The point in $\partial P_{[A]}$ are projective class of rank one matrices satisfying $\langle X, A\rangle = 0$, that is, such that $\mathrm{tr}(X\mathrm{adj}(A)) = 0$. Given that $X\mathrm{adj}(A)$ has vanishing determinant, by the Cayley-Hamilton theorem $X\mathrm{adj}(A)$ is traceless if and only if it is nilpotent, that is, if and only if $X\mathrm{adj}(A)X\mathrm{adj}(A) = 0$. Now, given that image and kernel of both X and $\mathrm{adj}(A)$ have dimension 1, this happens if and only if:

$$\mathrm{Im}(X) = \mathrm{Ker}(\mathrm{adj}(A)) \text{ or } \mathrm{Im}(\mathrm{adj}(A)) = \mathrm{Ker}(X). \tag{3.13}$$

Now, since $\det(A) = 0$ implies $\mathrm{adj}(A)A = A\mathrm{adj}(A) = 0$, the relations $\mathrm{Ker}(\mathrm{adj}(A)) = \mathrm{Im}(A)$ and $\mathrm{Im}(\mathrm{adj}(A)) = \mathrm{Ker}(A)$ hold. Hence $X \in P_{[A]}$ if and only if $\mathrm{Im}(X) = \mathrm{Im}(A)$ or $\mathrm{Ker}(X) = \mathrm{Ker}(A)$, which concludes the proof because of the definition of the homeomorphism δ. $\square$

Geometrically $\partial P_{[A]}$ is the union of two circles in $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$, one horizontal and one vertical, which intersect exactly at the point in $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ corresponding to $[A] \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ via δ.

CHAPTER 4

Mess' Work

In his 1990 paper "Lorentz Spacetimes of Constant Curvature" [11], Geoffrey Mess offered a completely new approach to the study of spacetimes in 2+1-dimension by employing tools and techniques from low-dimensional geometry and topology. The aim of this chapter is to give a brief introduction to Mess' ideas, with a special attention to AdS geometry. Our treatment will follow the setting introduced by Bonsante and Seppi in [2]. We will give definitions of general Lorentzian sets such as *achronal subsets, invisible domains, domains of dependence* and how they relate to the graphs of circle homeomorphisms (and their convex hull) in our specialized Anti-de Sitter setting. We will show that those graphs are *proper achronal sets* in the projective model and always lift to achronal sets in the Poincaré model.

4.1 Causality and Convexity properties

We begin by giving some definitions:

Definition 4.1. A subset X of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is achronal (respectively acausal) if no pair of points in X is connected by timelike (resp. causal) lines in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.

Since acausality and achronality are conformally invariant notions, it will be often convenient to consider the metric $g_{\mathbb{S}^2} - dt^2$ on $\mathbb{D} \times \mathbb{R}$ introduced in 2.11 which is conformal to the Poincaré model. We give now a first useful characterization of achronal and acausal sets.

Lemma 4.2. A subset X of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is achronal (respectively acausal) if and only if it is the graph of a function $f : D \to \mathbb{R}$ which is 1-Lipschitz, (resp. strictly 1-Lipschitz) with respect to the distance induced by the hemispherical metric $g_{\mathbb{S}^2}$, where we have denoted $D = \pi_{\mathbb{D}}(X)$.

Proof. Let us assume X is achronal. Now, since vertical lines in the Poincaré model are of timelike type, the restriction of the projection $\pi_{\mathbb{D}} : \mathbb{D} \times \mathbb{R} \to \mathbb{D}$ to X is injective. But then, X can be interpreted as the graph of a function $f : D \to \mathbb{R}$. By imposing that $(x, f(x))$ and $(y, f(y))$ are not connected by a timelike curve we deduce that:

$$|f(x) - f(y)| \leqslant d_{\mathbb{S}^2}(x, y) \tag{4.1}$$

where $d_{\mathbb{S}^2}$ is the distance induced by the hemispherical metric. By the same reasoning we show that a 1-Lipschitz graph over $\mathbb{D}$ is achronal. Moreover two points (x, t) and (y, s) are on the same lightlike geodesic if and only if $d_{\mathbb{S}^2}(x, y) = |t - s|$. Hence X is acausal if and only if the inequality of Equation 4.1 is strict. $\square$

Now a 1-Lipschitz function on a region $D \subset \mathbb{D}$ extends uniquely to the boundary of D. As a simple consequence of the previous lemma, we thus have:

Lemma 4.3. An achronal subset X in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is properly embedded if and only if it is a global graph over $\mathbb{D}$, and in this case it extends uniquely to the global graph of a 1-Lipschitz function over $\mathbb{D} \cup \partial\mathbb{D}$.

Because of Lemma 4.3 we will often refer to an *achronal surface* as an achronal subset $X \subset \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ which is the graph of a 1-Lipschitz function defined on a domain in $\mathbb{D}$. Before moving over to the study of properties of achronal sets we shall remark how achronality and acausality are global conditions.

Definition 4.4. Given a surface S and a Lorentzian manifold (M, g), a $\mathcal{C}^1$ immersion $\sigma : S \to M$ is *spacelike* if the pull-back metric $\sigma^* g$ is Riemannian. If σ is an embedding, we refer to its image as a *spacelike surface.*

A spacelike surface S is locally acausal, but there are examples of spacelike surfaces which are not achronal (hence a fortiori not acausal), a fact that highlights the global character of the achronality condition. On the other hand the following is true:

Lemma 4.5. Any properly embedded spacelike surface in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is acausal.

Proof. By Lemma 4.3, any properly embedded spacelike surface S in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ disconnects the spaces in two regions U, V whose common boundary is S, and we can assume that the outward pointing normal from U (resp. V) is past-directed (resp. future directed). It turns out that any future oriented causal path that meets S passes from V towards U. This implies that any causal path meets S at most once. □

We have remarked in Theorem 2.2 that unparametrized lightlike geodesics only depend on the conformal class of the Lorentzian metric, hence we will just refer to lightlike geodesics in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, even when we are considering it endowed with the hemispherical metric of 2.11.

Lemma 4.6. Let S be a properly embedded achronal surface of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ and assume that a lightlike geodesic segment γ joins two points of S. Then γ is entirely contained in S.

Proof. We want to exploit Lemma 4.3, let $f^S : \overline{\mathbb{D}} \to \mathbb{R}$ be the function that defines S, which is 1-Lipschitz with respect to the hemispherical metric. Now if our segment γ joins $(x, f^S(x))$ to $(y, f^S(y))$, then (up to switching the role of x and y) it holds: $f^S(y) = f^S(x) + d_{\mathbb{S}^2}(x, y)$. Now, γ is lightlike and hence consists of points of the form $(z, f^S(x) + d_{\mathbb{S}^2}(x, z))$, for the points z on the $g_{\mathbb{S}^2}-$geodesic segment joining x to y. By achronality of S we deduce:

$$f^S(z)-f^S(x) \leqslant d_{\mathbb{S}^2}(x,z) \text{ and } f^S(y)-f^S(z) \leqslant d_{\mathbb{S}^2}(z,y) = d_{\mathbb{S}^2}(x,y)-d_{\mathbb{S}^2}(x,z).$$

The second inequality implies that $f^S(z) \geqslant f^S(x) + d_{\mathbb{S}^2}(x, z)$, it follows that $f^S(z) = f(x) + d_{\mathbb{S}^2}(x, z)$ and hence γ is entirely contained in S. □

Invisible domains

Invisible domains were not treated in the original work of Mess but were later introduced in the literature by Barbot in [1]. We give the general definition and properties for a generic X subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, later we will focus on X entirely contained in the boundary.

Definition 4.7. Given an achronal domain X in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, the *invisible domain* of X is the subset, that we will denote by $\Omega(X)$, of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ defined as the set of points which are connected to X by no causal path.

Roughly speaking, $\Omega(X)$ is the union of all acausal subset containing X (hence the *maximal* for such a property). We recall that any 1-Lipschitz function on a subset of a metric space admits a 1-Lipschitz extension everywhere (Mc Shane's theorem [10]). In our setting this allows us show that any

achronal set X is a subset of a properly embedded achronal surface.
We consider two particular extensions $f_\pm^X : \mathbb{D} \cup \partial\mathbb{D}$, which we will refer to as *extremal* extensions:

$$f_-^X(y) = \sup\{f^X(x) - d_{\mathbb{S}^2}(x,y) \mid x \in \pi_{\mathbb{D}}(X)\}$$
$$f_+^X(y) = \inf\{f^X(x) + d_{\mathbb{S}^2}(x,y) \mid x \in \pi_{\mathbb{D}}(X)\}.$$

Lemma 4.8. Let X be any closed achronal subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, and let $S_\pm(X)$ be the graphs of the extremal extensions $f_\pm^X$.

- The properly embedded surfaces $S_-(X)$ and $S_+(X)$ are achronal with $S_-(X) \subset \overline{I^-(S_+(X))}$, and $\Omega(X) = I^+(S_-(X)) \cap I^-(S_+(X))$.
- Every achronal subset containing X is contained in $S_-(X) \cup \Omega(X) \cup S_+(X)$.
- Every point of $S_\pm(X)$ is connected to X by at least one lightlike geodesic segment, which is contained in $S_\pm(X)$. Finally $S_+(X) \cap S_-(X)$ is the union of X and all lightlike geodesic segments joining points of X.

Proof. We begin by showing that $S_-(X) \subset \overline{\mathrm{I}^-(S_+(X))}$. Given a point (y,t), $t \leqslant f_+^X(y)$ if and only if $t \leqslant f^X(x) + d_{\mathbb{S}^2}(x,y)$ for every $x \in \pi_{\mathbb{D}}(X)$, that is, if and only if (y,t) lies outside $\mathrm{I}^+(X)$. Similary (y,t) lies outside $\mathrm{I}^-(X)$ if and only if $t \geqslant f_-^X(y)$. By achronality, $S_+(X)$ does not meet the past of X, so we deduce that $f_+^X(y) \geqslant f_-^X(y)$ for all $y \in \overline{\mathbb{D}}$. Hence $S_-(X) \subset \overline{\mathrm{I}^-(S_+(X))}$.
As a similar observation, given a point (y,t) we have that $\{(y,t)\} \cup X$ is achronal if and only if $f_-^X(t) \leqslant t \leqslant f_+^X(y)$. Moreover (y,t) is connected to X by no causal curve if and only if $f_-^X(y) < t < f_+^X(y)$. This shows that

$$\Omega(X) = \{(y,t) \mid f_-^X(y) < t < f_+^X(y)\}.$$

and also the second item, by applying the previous observation to any point of an achronal set containing X which is not in X itself.
To prove the third item, fix a point $(y,t) \in S_+(X)$. As we are assuming that X is closed in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, the fact that f^X is 1-Lipschitz implies that $\pi_{\mathbb{D}}(X)$ is closed in $\mathbb{D} \cup \partial\mathbb{D}$, so it is also compact. In particular, there exists $x \in \partial\mathbb{D}$ such that $t = f_+^X(y) = f^X(x) + d_{\mathbb{S}^2}(x,y)$. Thus (y,t) is connected to $(x, f^X(x))$ by a lightlike geodesic segment. By Lemma 4.6 this geodesic is entirely contained in $S_+(X)$. Clearly the same proof works for $S_-(X)$.
We are left with the computation of $S_-(X) \cap S_+(X)$. For this purpose, notice that if two points of X are connected by a lightlike goedesic segment, again

via Lemma 4.6, we deduce that $\gamma \subset S_-(X) \cap S_+(X)$. Conversely let $(y,t) \in S_-(X) \cap S_+(X)$ so that $f_-^X(y) = f_+^X(y)$. There exist x and x' in $\pi_{\mathbb{D}}(X)$ such that

$$f_+^X(y) = f^X(x) + d_{\mathbb{S}^2}(x,y) \text{ and } f_-^X(y) = f^X(x') - d_{\mathbb{S}^2}(x',y).$$

Using the equality $f_-^X(y) = f_+^X(y)$, the triangle inequality and the fact that f^X is 1-Lipschitz we deduce that

$$f^X(x) - f^X(x') = d_{\mathbb{S}^2}(x,x') = d_{\mathbb{S}^2}(x,y) + d_{\mathbb{S}^2}(y,x'). \tag{4.2}$$

Hence the points $(x, f^X(x))$ and $(x', f^X(x'))$ are joined by a lightlike segment. If x, x' are not antipodal points on $\partial\mathbb{D}$, there is a unique hemispherical geodesic η in $\overline{\mathbb{D}}$ joining x to x', which must pass through y by Equation 4.2, and which we may assume parametrized by arc length. In this case the geodesic segment joining $(x, f^X(x))$ to $(x', f^X(x'))$ takes the form $t \mapsto (\eta(t), f^X(x') + t)$, so it passes through $(y, f_+^X(y)) = (y, f_-^X(y))$.
If x and x' are antipodal, then there are infinitely many geodesics joining x to x', and we can pick one going through y. Then the same argument as above applies. □

Remark 4.9. *Given a point (y,t) the set of points (x,s) satisfying $|s-t| < d_{\mathbb{S}^2}(x,y)$ coincides with the region of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ which is connected to (y,t) by a spacelike geodesic for the Anti-de Sitter metric. It coincides also with the region of points connected to (y,t) by a spacelike geodesic for the conformal emispherical metric (although in general spacelike geodesics for the two metrics do not coincide). Now, since $f_-^X(y) \leqslant t \leqslant f_+^X(y)$ is equivalent to the condition that $|s-t| < d_{\mathbb{S}^2}(x,y)$ for all $(x,t) \in X$, the region*

$$S_+(X) \cup \Omega(X) \cup S_-(X) = \{(y,t) \mid f_-^X(y) \leqslant t \leqslant f_+^X(y)\}$$

consist of all the points that are connected to any point of X by spacelike or lightlike geodesics. Moreover $\Omega(X)$ consist of points connected to any point of X by a spacelike geodesic. We observe that $\Omega(X)$ could be empty, for instance when X is a global graph then $S_-(X) = S_+(X) = X$ and $\Omega(X)$ is empty.

Remark 4.10. *Since any point of $S_\pm(X)$ is connected to X by a lightlike geodesic, it follows from Lemma 4.6 that the intersection of any properly embedded achronal surface containing X with $S_\pm(X)$ is a union of lightlike geodesic segments with an endpoint in X. In particular any properly embedded acausal surface containing X is contained in $\Omega(X)$.*

We will need the following technical definition:

Definition 4.11. Given a function $f : \overline{\mathbb{D}} \to \mathbb{R}$, we define its oscillation as:

$$\operatorname{osc}(f) := \max_{y \in \overline{\mathbb{D}}} f(y) - \min_{y \in \overline{\mathbb{D}}} f(y).$$

We observe that such a quantity is *not* invariant under the action of the isometry group of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.

With this definition we can state the following lemma that we will soon use:

Lemma 4.12. Let S be a properly embedded achronal surface, defined as the graph of $f^S : \overline{\mathbb{D}} \to \mathbb{R}$. Then $\operatorname{osc}(f^S) \leqslant \pi$ moreover $\operatorname{osc}(f^S) = \pi$ if and only if S is a lightlike plane.

Proof. We observe that f^S is $1-$Lipschitz for the hemispherical metric, and the diameter of $\mathbb{D}$ for $g_{\mathbb{S}^2}$ is π, hence we easily obtain that $\operatorname{osc}(f^S)$ is bounded by π. Moreover if the value is attained it follows that there are two antipodal points $y, y' \in \partial\mathbb{D}$ such that: $f^S(y') = f^S(y) + \pi$. We recall from the remark following Equation 2.11 that the lightlike plane with past and future points $(y, f^S(y))$ and $(y', f^S(y) + \pi)$ is:

$$P = \{(x,t) \mid t = f^S(x) + d_{\mathbb{S}^2}(x, y)\}$$

which moreover is foliated by lightlike geodesic joining $(y, f^S(y))$ to $(y', f^S(y) + \pi)$. By Lemma 4.6, P is included in S. Since both are global graphs over $\overline{\mathbb{D}}$ we have $S = P$. □

Achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$

We will be interested in the study of invisible domains of achronal meridians Λ in the boundary of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, that are graphs of 1-Lipschitz functions $f : \partial\mathbb{D} \to \mathbb{R}$.

Lemma 4.13. Let Λ be an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. Then either Λ is the boundary of a lightlike plane, or $S_+(\Lambda) \cap S_-(\Lambda) = \Lambda$. In the latter case there is an achronal properly embedded surface in $\Omega(\Lambda)$ whose boundary in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is Λ.

Proof. Let $f : \partial\mathbb{D} \to \mathbb{R}$ be the function whose graph is Λ. We recall from Lemma 4.12 that $\operatorname{osc}(f) \leqslant \pi$. If there are points x_0, x_0' such that $f(x_0') = f(x_0) + \pi$ then combining both Lemma 4.12 and Lemma 4.8 we deduce that Λ is the boundary of a lightlike plane, and this lightlike plane coincides with $S_+(\Lambda) \cap S_-(\Lambda)$.

Assume now that the maximal oscillation of f is smaller than π, and let us show that $S_+(\Lambda) \cap S_-(\Lambda) = \Lambda$. By the assumption, if a lightlike geodesic connects $(x_0, f(x_0))$ to $(x_0', f(x_0'))$ then x_0 and x_0' are not antipodal. But then x_0, x_0' are connected by a unique length-minimizing geodesic in $\overline{\mathbb{D}}$ for the hemispherical metric, which lies in $\partial\mathbb{D}$. So the lightlike line connecting $(x_0, f(x_0))$ to $(x_0', f(x_0'))$ is contained in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. By Lemma 4.8 we conclude that $S_-(\Lambda)$ and $S_+(\Lambda)$ do not meet in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ and therefore $S_+(\Lambda) \cap S_-(\Lambda) = \Lambda$.
Finally, in this latter case the function $F = (f_-^\Lambda + f_+^\Lambda)/2$ is $1-$Lipschitz and defines an achronal properly embedded surface contained in $\Omega(\Lambda)$, whose boundary is Λ. □

We remark that in fact for any achronal meridian there is a spacelike surface whose boundary at infinity is Λ; we will expand on this in Remark 4.23. Now, given a point $x \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, we recall that the Dirichlet domain of x is the region R_x containing x and bounded by two spacelike plane "dual" to x. Namely the planes that we denote (with a slight abuse of notation) P_x^+ and P_x^-, consisting of points at timelike distance $\pi/2$ in the future (resp. past) along timelike geodesic with initial point x.

Proposition 4.14. *Let Λ be an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. Then:*

- *A point $x \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ lies in $\Omega(\Lambda)$ if and only if Λ is contained in the interior of the Dirichlet region R_x.*
- *For any $z \in \Lambda$, let $L_-(z)$ and $L_+(z)$ be the two lightlike planes such that z is the past vertex of $L_+(z)$ and the future vertex of $L_-(z)$. Then*
$$\Omega(\Lambda) = \bigcap_{z\in\Lambda} I^+(L_-(z)) \cap I^-(L_+(z)).$$
- *The length of the intersection of $\Omega(\Lambda)$ with any timelike geodesic of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is at most π. Moreover, there exist a timelike geodesic whose intersection with $\Omega(\Lambda)$ has length π if and only if Λ is the boundary at infinity of a spacelike plane.*

Proof. By Remark 4.9 a point x lies in $\Omega(\Lambda)$ if and only if it is connected to any point of Λ by a spacelike geodesic. The region of points connected to x by a spacelike geodesic has boundary the lightcone from x, whose intersection with $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ coincides with $P_x^\pm \cap \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.
Moving on the second item we observe that the region bounded by $L_+(z)$ and $L_-(z)$ contains exactly points connected to z by a spacelike geodesic. Using

the characterization of $\Omega(\Lambda)$ as above, we have the second statement.
Lastly, if a timelike geodesic γ meets $\Omega(\Lambda)$ at a point x, then $\Omega(\Lambda) \subset R_x$, so that the length of $\gamma \cap \Omega(\Lambda)$ is smaller than the length of $\gamma \cap R_x$. But the latter is π. Now assume the existence of a geodesic γ such that the length of $\gamma \cap R_x$ equals π. Up to applying an isometry of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ we may assume that γ is vertical in the Poincaré model of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ and the mid-point of $\gamma \cap \Omega(\Lambda)$ is $(0,0)$. Thus $(0,-\pi/2)$ and $(0,\pi/2)$ lie on $S_-(\Lambda)$ and $S_+(\Lambda)$ respectively.
Now, again by 4.9 points of Λ are connected to $(0,-\pi/2)$ by a spacelike or lightlike geodesic, hence $s \leqslant 0$ for all $(\xi, s) \in \Lambda$. Analogously using the point $(0,\pi/2)$ we deduce that $s \geqslant 0$ for all $(\xi, s) \in \Lambda$, so that $\Lambda = \partial\mathbb{D} \times \{0\}$. ☐

Arguing in similar fashion, we obtain that the invisible domain of an achronal meridian which is not the boundary of a lightlike plane is always contained in a Dirichlet region.

Proposition 4.15. *Given an achronal meridian Λ in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane, the invisible domain $\Omega(\Lambda)$ is contained in a Dirichlet region unless Λ is the boundary of a spacelike plane.*

Proof. We start by defining $a_+ = \sup f_+^\Lambda$ and $a_- = \inf f_-^\Lambda$, and we consider the planes:

$$P_{a_+} = \{(x,t) \mid t = a_+\} \text{ and } P_{a_-} = \{(x,t) \mid t = a_-\}$$

in the Poincaré model. Since $\Omega(\Lambda)$ lies in the open region bounded by those planes, it is sufficient to show that $a_+ - a_- \leqslant \pi$. Assume by contradiction the converse. Notice tht P_{a_+} meets $S_+(\Lambda)$ at some point $p_+ = (x_+, a_+)$ and P_{a_-} meets $S_-(\Lambda)$ at some point $p_- = (x_-, a_-)$ where x_+ and x_- are points on $\overline{\mathbb{D}}$. For $\epsilon = (a_+ + a_- - \pi)/2$ we can find x'_+ and x'_- in $\mathbb{D}$ such that $p'_+ = (x'_+, a_+ - \epsilon)$ and $p'_- = (x'_-, a_- + \epsilon)$ lie in $\Omega(\Lambda)$ (clearly if $x_\pm$ lies in $\mathbb{D}$ we can take $x'_\pm = x_\pm$). As $(a_+ - \epsilon) - (a_- - \epsilon) = \pi$, the geodesic segment γ joining p'_+ and p'_- is timelike of length π. Its end-points are in $\mathrm{I}^+(S_-(\Lambda)) \cap \mathrm{I}^-(S_+(\Lambda))$, so γ is entirely contained in $\Omega(\Lambda)$. As end-points of γ are contained in $\Omega(\Lambda)$, γ can be extended within $\Omega(\Lambda)$ but this contradicts the third point of Proposition 4.14. The third point of Proposition 4.14 then shows that if $a_+ - a_- = \pi$ then Λ is the boundary of a spacelike plane. Hence, apart from this case, one has $a_+ - a_- < \pi$, so the closure of $\Omega(\Lambda)$ is contained in a Dirichlet region.

☐

Remark 4.16. *When Λ is the boundary of a spacelike plane P, then there are two points x_- and x_+ such that $P = P^+_{x_-} = P^-_{x_+}$. The previous argument shows that $\Omega(\Lambda)$ is the union of all timelike geodesics joining x_- to x_+ In this* *case $S_-(\Lambda)$ is the union of the future directed lightlike geodesic rays emanating from x_-, whereas $S_+(\Lambda)$ is the union of future directed lightlike rays ending at x_+.*

Domain of dependence

We want to define and study properties of Cauchy surfaces and domains of dependence, due to a theorem of Geroch [7] the study of this particular surfaces will impose strict conditions on the topology of the spacetimes.

Definition 4.17. Given an achronal subset X in a Lorentzian manifold (M, g) the *domain of dependence* of X is the set:

$$\mathcal{D}(X) = \{p \in M |\ \text{every inextensible causal curve through } p \text{ meets } X\}.$$

We say that X is a *Cauchy surface* of M if $\mathcal{D}(X) = M$. A spacetime M is called *globally hyperbolic* if it admits a Cauchy surface.

The theory of globally hyperbolic spacetimes is a well-developed topic in Lorentzian geometry, we will just state the facts that we will need in the thesis. The following result is due to Geroch [7].

Theorem 4.18. *Let M be a globally hyperbolic spacetime. Then:*

1. *Any two Cauchy surfaces in M are diffeomorphic.*
2. *There exists a submersion $\tau : M \to \mathbb{R}$ whose fibers are Cauchy surfaces.*
3. *M is diffeomorphic to $\Sigma \times \mathbb{R}$ where Σ is any Cauchy surface in M.*

Remark 4.19. *The spacetime $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is not globally hyperbolic. In fact if X is achronal, it is contained in the graph of a 1-Lipschitz function $f : (\mathbb{D} \cup \partial\mathbb{D}, g_{\mathbb{S}^2}) \to \mathbb{R}$. If $t_0 > \sup f$ and $\xi \in \partial\mathbb{D}$, then any lightlike ray with past end-point (ξ, t_0) does not intersect X.*

Remark 4.20. *As causality notions are invariant under conformal change of metrics, we observe that causal paths in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ are the graphs of 1-Lipschitz functions from (intervals in) $\mathbb{R}$ to $\mathbb{D}$ with respect to the hemispherical metric in the image. Hence an inextesible causal curve in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is either the graph of a global 1-Lipschitz function from $\mathbb{R}$, or it is defined on a proper interval and has endpoint(s) in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.*

Lemma 4.21. Given an achronal meridian Λ in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ any Cauchy surface in $\Omega(\Lambda)$ is properly embedded with boundary at infinity Λ.

Proof. Let S be a Cauchy surface in $\Omega(\Lambda)$. For every $x \in \mathbb{D}$, the vertical line through x in the Poincaré model meets $\Omega(\Lambda)$, and its intersection with $\Omega(\Lambda)$ must meet S by definition of Cauchy surface. This shows that S is a graph over $\mathbb{D}$, hence properly embedded, and clearly $\partial S = \Lambda$. □

Proposition 4.22. *Let Λ be an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. Let S be a properly embedded achronal surface in $\Omega(\Lambda)$. Then $\mathcal{D}(S) = \Omega(\Lambda)$. In particular $\Omega(\Lambda)$ is a globally hyperbolic spacetime.*

Proof. Let x be any point in $\Omega(\Lambda)$ and take any inextensible causal path through x. A priori its future endpoint might be either in $S_+(\Lambda)$ or in Λ, but by definition of $\Omega(\Lambda)$, x cannot be connected by any causal path to Λ, hence the latter case is excluded. The same argument shows that the past endpoint is in $S_-(\Lambda)$. Since the inextensible causal path meets both $S_+(\Lambda)$ and $S_-(\Lambda)$, it must meet S by Lemma 4.21, hence $x \in \mathcal{D}(S)$.
Conversely, consider a x that is not in $\Omega(\Lambda)$, then one can find a causal path joining x to Λ, which is necessarily inextensible. Hence x is not in $\mathcal{D}(S)$. □

Remark 4.23. *As a direct consequence of Theorem 4.18 and Proposition 4.22 we have that Λ is the boundary of a spacelike surface in $\Omega(\Lambda)$, namely a Cauchy surface in $\Omega(\Lambda)$. By Lemma 4.21, the surface is properly embedded, hence the graph of a global 1-Lipschitz function. This shows that any proper achronal meridian Λ is the boundary at infinity of a properly embedded spacelike surface, we remark that we have improved the statement of Lemma 4.13.*

The most remarkable property of the domain of dependence of a properly embedded surface in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, which is a direct consequence of Proposition 4.22, is that it only depends on the boundary at infinity. In detail:

Corollary 4.24. *If S and S' are properly embedded spacelike surfaces in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, then $\mathcal{D}(S) = \mathcal{D}(S')$ if and only if $\partial S = \partial S'$.*

Properly achronal sets in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

For our interest will be important to consider the model $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. As $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ contains closed timelike lines, it does not contain any achronal subset. However if P is a spacelike plane in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ then $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \setminus P$ does not contain closed causal curves as we have discussed in Section 2.4. Indeed it is simply connected, so it admits an isometric embedding into $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, given by a section of the covering map $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, and whose image is a Dirichlet region.

Definition 4.25. A subset X of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a *proper achronal subset* if there exists a spacelike plane P such that X is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \backslash \overline{P}$ and is achronal as a subset of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \backslash \overline{P}$.

It follows from the definition that if X is a proper achronal subset of $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$, then it admits a section to $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, and the image remains achronal after the lifting. Conversely, let $\widetilde{X}$ be an achronal subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from a lightlike plane, then it is contained in a Dirichlet region as a consequence of Lemma 4.12, and the fact that any achronal subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is contained in properly embedded one. As Dirichlet regions are projected in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ to the complement of a spacelike plane, the image of $\widetilde{X}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ is a proper achronal subset. The following lemma will be key in our *path* to prove the earthquake theorem:

Lemma 4.26. Let $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be an orientation preserving homeomorphism. Then the graph of φ, say $\Lambda_\varphi \subset \mathbb{RP}^1 \times \mathbb{RP}^1 \simeq \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a proper achronal subset and any lift, denoted as $\widetilde{\Lambda_\varphi}$, is an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.

Proof. We start by proving that Λ_φ is locally achronal. Consider U and V intervals around x and $\varphi(x)$ and let θ_1 and θ_2 be positive coordinates on U and V respectively. Then the timelike curves $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $U \times V$ are characterized by the property that $\theta_1'(t)\theta_2'(t) < 0$, where $\theta_i(t) = \theta_i(\gamma_i(t))$ as we saw in Equation 3.5.
In particular points on $\Lambda_\varphi \cap U \times V$ are not related by a timelike curve contained in $U \times V$, by the assumption that φ is orientation-preserving.
Let us prove the existence of a spacelike plane P such that $\overline{P} \cap \Lambda_\varphi = \varnothing$. Let us consider the identification $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$, and take $\varphi_0 \in \mathrm{PSL}(2, \mathbb{R})$ so that $\varphi_0^{-1}\varphi(0) = 1, \varphi_0^{-1}\varphi(1) = \infty$ and $\varphi_0^{-1}\varphi(\infty) = 0$. It follows that $\varphi_0^{-1}\varphi$ sends the intervals $(\infty, 0), (0, 1)$ and $(1, \infty)$ respectively to $(0, 1), (1, \infty), (\infty, 0)$. Thus $\varphi_0^{-1}\varphi$ has no fixed points, that is, the graph of φ does not intersect the graph of φ_0, which is the asymptotic boundary of the spacelike plane P_{φ_0}.
Let us consider now the lift $\widetilde{\Lambda}_\varphi$ to the boundary of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. As Λ_φ is contained in a simply connected region of $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$, its lift is a closed locally achronal curve contained in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. In particular the projection $\widetilde{\Lambda}_\varphi \to \partial\mathbb{D}$ is locally injective. As $\widetilde{\Lambda}_\varphi$ is compact, the map is a covering. On the other hand, since Λ_φ is homotopic to the boundary of a plane in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$, it turns out that $\widetilde{\Lambda}_\varphi$ is homotopic to $\partial\mathbb{D}$ in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, so that the projections $\widetilde{\Lambda}_\varphi \to \partial\mathbb{D}$ is bijective. It follows that $\widetilde{\Lambda}_\varphi$ is achronal. $\square$

What we have shown until now for achronal sets in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ can be rephrased for proper achronal sets of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. For example, any proper achronal set X can be extended to a properly embedded proper achronal surface and there are two extremal extensions, just as in Lemma 4.8
Now we would like to focus on proper achronal meridians in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. They lift to achronal meridians in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of lightlike planes. Indeed the boundary of a lightlike plane is not contained in a Dirichlet region. Conversely, any achronal meridian on $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane projects to an achronal meridian of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Proposition 4.27. *Let Λ be a proper achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and denote by $\widetilde{\Lambda}$ any lift to the universal covering. Then the universal covering map of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ maps $\Omega(\widetilde{\Lambda})$ injectively to the domain:*

$$\Omega(\Lambda) := \{x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \mid P_x \cap \Lambda = \varnothing\}.$$

Proof. Consider the covering $p : \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, by Proposition 4.15 the invisible domain $\Omega(\widetilde{\Lambda})$ is contained in a Dirichlet region $R_{\widetilde{x}}$, hence the restriction of p to $\Omega(\widetilde{\Lambda})$ is injective and its image is contained in $p(R_{\widetilde{x}})$, namely the complement in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$ of the spacelike plane P_x dual to $x = p(\widetilde{x})$. Moreover, by the first item of Proposition 4.14, one can actually pick for $\widetilde{x}$ any point in $\Omega(\widetilde{\Lambda})$, which shows that the image $p(\Omega(\widetilde{\Lambda}))$ is contained in $\Omega(\Lambda)$ defined as in the proposition.
For the converse inclusion, let $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ be a point whose dual plane P_x does not meet Λ. The preimage $p^{-1}(P_x)$ is a countable disjoint union of planes which disconnect $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ in a disjoint union of Dirichlet regions centered at preimages of x. The lift $\widetilde{\Lambda}$ is then contained in exactly one such region, say $R_{\widetilde{x}}$. By the first item of Proposition 4.14, $\widetilde{x} \in \Omega(\widetilde{\Lambda})$ which implies that $x = p(\widetilde{x})$ lies in $p(\Omega(\widetilde{\Lambda}))$. ☐

When Λ_φ is the graph of an orientation-preserving homomorphism $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$, there is a fairly simple characterization of $\Omega(\Lambda_\varphi)$ exploiting the identification $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathrm{PSL}(2,\mathbb{R})$.

Corollary 4.28. *Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation-preserving homeomorphism. Then $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ lies in $\Omega(\Lambda_\varphi)$ if and only if $x \circ \varphi$ has no fixed point as a homeomorphism of $\mathbb{R}\mathrm{P}^1$.*

Proof. The dual plane of $x \in \mathrm{PSL}(2,\mathbb{R})$, meets $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ along the graph of x^{-1}, namely $\Lambda_{x^{-1}}$.
With this remark in hand, we have that $x \in \Omega(\Lambda_\varphi)$ if and only if $\Lambda_{x^{-1}} \cap \Lambda_\varphi = \varnothing$, that is, if and only if $x^{-1} \circ \varphi$ has no fixed point on $\mathbb{R}\mathrm{P}^1$.

□

Proposition 4.29. *Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a proper spacelike immersion. Then*

- *σ is a proper embedding.*
- *σ lifts to a proper embedding $\widetilde{\sigma} : S \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.*
- *The boundary at infinity of $\sigma(S)$ is a proper achronal meridian Λ in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.*
- *$\mathcal{D}(\sigma(S)) = \Omega(\Lambda)$.*

Proof. Denote by $\widehat{S}$ the covering of S admitting a lift $\widehat{\sigma} : \widehat{S} \to \mathbb{H}^{2,1}$. In general either $\widehat{S} = S$ or it is a $2:1$ covering. Since the covering is finite, $\widehat{\sigma}$ is a proper immersion. Let us consider the identification $\mathbb{H}^2 \times \mathbb{S}^1$ that follows from the map given in Equation (2.5). The induced projection: $\mathrm{pr} : \mathbb{H}^{2,1} \to \mathbb{H}^2$ is a proper fibration with timelike fibers. In particular $\widehat{\sigma}$ is transverse to the fibers of pr. It follows that the composition $\mathrm{pr} \circ \sigma : \widehat{S} \to \mathbb{H}^2$ is a proper local diffeomorphism, hence a covering map. Since $\mathbb{H}^2$ is simply connected, we deduce that the covering is actually a homeomorphism, $\widehat{\sigma}$ is an embedding, and $\widehat{S}$ is homeomorphic to the hyperbolic plane.
More is true; we can lift $\widehat{\sigma}$ to the universal covering, say $\widehat{\sigma} : \widehat{S} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, which is still a proper spacelike embedding $\widehat{S} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. By Lemma 4.3 and Lemma 4.5 we know that the image is an achronal meridian, and it is contained in a Dirichlet domain by Lemma 4.12. It follows that $\widehat{\sigma}(\widehat{S})$ is contained in a Dirichlet domain of the covering map $\mathbb{H}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, on which we know that the covering map is injective. In particular σ is injective, hence $S = \widehat{S}$ as desired. □

We therefore have an analogous version of Lemma 4.24 in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Corollary 4.30. *If S and S' are properly embedded spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then $\mathcal{D}(S) = \mathcal{D}(S')$ if and only if $\partial S = \partial S'$.*

Convexity properties

Let Λ be a proper achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We would like to investigate convexity properties of $\Omega(\Lambda)$. We briefly recall that $X \subset \mathbb{RP}^n$ is convex if it is contained in an affine chart, and it is convex in the affine chart. The notion does not depend on the affine chart containing X. We would say that it is a proper convex set if it compactly contained in an affine chart.

Proposition 4.31. *Given a proper achronal meridian Λ in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the invisible domain $\Omega(\Lambda)$ is convex. If Λ is different from the boundary of a spacelike plane then $\Omega(\Lambda)$ is a proper convex set.*

Proof. By Proposition 4.15, there exists a spacelike plane P such that $\Omega(\Lambda)$ is contained in the affine chart V of $\mathbb{R}\mathrm{P}^3$ obtained by removing the projective plane containing P. The domain $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cap V = \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \backslash P$ is isometric to a Dirichlet region R of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, by an isometry that sends Λ to a lifting $\widetilde{\Lambda}$ and $\Omega(\Lambda)$ to $\Omega(\widetilde{\Lambda})$. By the second point of Proposition 4.14 we have

$$\Omega(\widetilde{\Lambda}) = \bigcap_{\widetilde{z}\in\widetilde{\Lambda}} I^+(L_-(\widetilde{z})) \cap I^-(L_+(\widetilde{z})).$$

Now if $\widetilde{z}$ projects to z, then the images of $L_-(\widetilde{z})$ and $L_+(\widetilde{z})$ in V are the two components of $L(z) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, where $L(z)$ is the affine tangent plane of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cap V$ at z. It turns out that the image of the region $I^+(L_-(\widetilde{z})) \cap I^-(L_+(\widetilde{z}))$ is the intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the half-space $U(z)$ bounded by $L(z)$ and whose closure contains Λ. This shows:

$$\Omega(\Lambda) = \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \bigcap_{z\in\Lambda} U(z).$$

Actually we claim that:

$$\Omega(\Lambda) = \bigcap_{z\in\Lambda} U(z) \subset \mathbb{A}\mathrm{d}\mathbb{S}^{2,1},$$

and this will conclude. As $\bigcap_{z\in\Lambda} U(z)$ is connected and meets $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, to show that is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ it suffices to show that it does not meet the boundary. For any $w \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ let us consider the leaf of the left ruling through w, which intersects Λ at a point z. It turns out that $L(z)$ contains the leaf of the left ruling through z, hence $w \notin U(z)$.
Now assume that Λ is not the boundary of a spacelike plane. Then by Proposition 4.15 on the universal covering the compact set $\Omega(\widetilde{\Lambda}) \cup S_+(\widetilde{\Lambda}) \cup S_-(\widetilde{\Lambda})$ is contained in a Dirichlet domain, so its image is a compact set contained in an affine chart. □

As a consequence, we have that Λ is contained in an affine chart whose complement in $\mathbb{R}\mathrm{P}^3$ is a projective plane containing a spacelike plane of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ Hence it makes sense to give the following definition:

Definition 4.32. Given a proper achronal meridian Λ in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, we define $\mathcal{C}(\Lambda)$ to be the convex hull of Λ, which can be taken in an affine chart containing Λ

What we have implicitly proved is that given Λ an achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then $\mathcal{C}(\Lambda)$ is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, which is not immediately obvious as $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is not convex in $\mathbb{RP}^3$.

Example 4.33: Consider $\sigma \in \mathrm{PSL}(2,\mathbb{R})$, the convex hull of the graph Λ_σ is the closure of the totally geodesic spacelike plane $P_{\sigma^{-1}}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$. In particular, following Lemma 3.11, we have that the boundary at infinity of $P_{\sigma^{-1}}$ equals Λ_σ, and moreover $P_{\sigma^{-1}}$ is convex, since spacelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ are lines in an affine chart, and any two points in $\partial\mathbb{H}^2$ are connected by a geodesic. Hence $P_{\sigma^{-1}}$ is clearly the smallest convex set containing Λ_σ.

Remark 4.34. *Since $\Omega(\Lambda)$ is convex, $\mathcal{C}(\Lambda)$ is contained in $\Omega(\Lambda)$. Moreover, if K is any convex set contained in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$ and containing Λ, then $\mathcal{C}(\Lambda) \subset K \subset \overline{\Omega(\Lambda)}$.*
To see this, let V be an affine chart such that $\Lambda \subset V$ is obtained removing a spacelike projective plane. Now, if $z \in \Lambda$ then for any $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cap V$ the segment connecting z and x in V is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ if and only if $x \in U(z)$, the half-space containing Λ and bounded by the tangent space of Λ at z, as in the proof of Proposition 4.31.
It follows, from the characterization of $\Omega(\Lambda)$ as the intersection of the $U(z)$ given in Proposition 4.31, that if x is not in $\overline{\Omega(\Lambda)}$ it cannot be in K. Hence $\overline{\Omega(\Lambda)}$ is the biggest convex set of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ containing Λ.

4.2 Support planes

We still need to borrow some notions and notation from convex analysis. Given a convex body K in an affine space of dimension three, a *support plane* of K is an affine plane Q such that K is contained in a closed half-space bounded by Q, and $\partial K \cap Q \neq \varnothing$. If p is in the intersection of ∂K with Q we will say that Q is a *support plane* at p. As a consequence of the Hahn-Banach theorem there exist a support plane at every point $p \in \partial K$.
We will adapt the terminology to the Anti-de Sitter setting. Given a convex hull $\mathcal{C}(\Lambda_\varphi)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, we say that a totally geodesic plane P is a support plane of $\mathcal{C}(\Lambda_\varphi)$ (at $p \in \partial\mathcal{C}(\Lambda_\varphi)$) is $p \in \mathcal{C}(\Lambda_\varphi) \cap \overline{P} \subset \overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$ and, in an affine chart containing Λ_φ, $\mathcal{C}(\Lambda_\varphi)$ lies in a closed half-space bounded by the affine plane that contains P. Even this definition does not depend on the choice of the affine chart as long this one contains Λ_φ.

Remark 4.35. *We could rephrase the condition of being a support plane as the following: a totally geodesic plane P is support plane for $\mathcal{C}(\Lambda_\varphi)$ if there exists a continuous family $\{P_t\}_{t\in[0,\epsilon)}$ of totally geodesic planes, pairwise disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$, such that $P_0 = P$ and $P_t \cap \mathcal{C}(\Lambda_\varphi) = \varnothing$ for $t > 0$.*

We observe that if X is a set, $\mathcal{C}(X)$ its convex hull and Q an affine support plane for $\mathcal{C}(X)$, then $Q \cap \mathcal{C}(X) = \mathcal{C}(Q \cap X)$. Applying this identity in our setting, we get that for any totally geodesic support plane P:

$$P \cap \mathcal{C}(\Lambda_\varphi) = \mathcal{C}(\partial P \cap \Lambda_\varphi). \tag{4.3}$$

With this new *borrowed tool* of support planes we can make study more carefully the structure of the convex body of our interest. Suppose that Λ is not the boundary of a spacelike plane. Now the topological frontiers in $\mathbb{RP}^3$ of $\Omega(\Lambda)$ and of $\mathcal{C}(\Lambda)$ are Lipschitz surfaces homeomorphic to a sphere. This sphere is disconnected by Λ in two regions, homeomorphic to disks, which form the boundary of $\Omega(\Lambda)$ and of $\mathcal{C}(\Lambda)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. For $\Omega(\Lambda)$ those components are the image of $S_\pm(\widetilde{\Lambda})$ and will be denoted by $S_\pm(\Lambda)$.
For $\mathcal{C}(\Lambda)$, let $\mathcal{C}(\widetilde{\Lambda})$ be a lifting, which is necessarily contained in a Dirichlet region, say R. Let P be a support plane for $\mathcal{C}(\Lambda)$, which is necessarily either spacelike or lightlike, and let $\widetilde{P}$ be its lift which touches $\mathcal{C}(\widetilde{\Lambda})$. Hence either $\widetilde{\Lambda}$ is in $I^+(\widetilde{P}) \cup \widetilde{P}$ or in $I^-(\widetilde{P}) \cup \widetilde{P}$. This permits to distinguish the components of $\partial\mathcal{C}(\Lambda)\backslash\Lambda$: the *past boundary component* $\partial\mathcal{C}_-(\Lambda)$ has the property that $\widetilde{\Lambda}$ is contained in $I^+(\widetilde{P}) \cup \widetilde{P}$ for all support planes which touch $\partial_-\mathcal{C}(\Lambda)$, and analogously for the *future boundary component* $\partial_+\mathcal{C}(\Lambda)$.
We want to show a kind of duality between boundary components $\partial_\pm\mathcal{C}(\Lambda)$ and $S_\pm(\Lambda)$.

Proposition 4.36. *Let Λ be a proper achronal meridian in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and let us denote by P_x the dual plane to x. Then:*

- *$x \in \Omega(\Lambda)$ if and only if $P_x \cap \mathcal{C}(\Lambda) = \varnothing$.*
- *$x \in \mathcal{C}(\Lambda)$ if and only if $P_x \cap \Omega(\Lambda) = \varnothing$.*

In particular if Λ is not the boundary of a spacelike plane, then:

- *$x \in \partial_\pm\Omega(\Lambda)$ if and only if P_x is a support plane for $\partial_\mp\mathcal{C}(\Lambda)$.*
- *$x \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$ if and only if P_x is a support plane for $S_\mp(\Lambda)$.*

Proof. From Proposition 4.27, points in $\Omega(\Lambda)$ are dual to planes disjoint from Λ, which are precisely those which do not intersect $\mathcal{C}(\Lambda)$, by the definition of

convex hull. For the second statement, fix x and observe that $z \in P_x$ if and only if $x \in P_z$. Hence there exists a point z in the intersection $P_x \cap \Omega(\Lambda)$ if and only if x is in a plane P_z which is disjoint from Λ, namely when x is not in $\mathcal{C}(\Lambda)$.
As a consequence $\partial\mathcal{C}(\Lambda)$ consists of points dual to support planes of $\Omega(\Lambda)$. Take a support plane P_x of $S_+(\Lambda)$ (hence dual to a point x) which meets $S_+(\Lambda)$ at z. If $\widetilde{z}$ denotes the corresponding point of $S_+(\widetilde{\Lambda})$, then $\widetilde{\Lambda} \subset I^+(P^-_{\widetilde{z}})$, and $P^-_{\widetilde{z}} \cap \widetilde{\Lambda} = \varnothing$. Thus P_z, which is the projection of $P^-_{\widetilde{z}}$, is a support plane of $\mathcal{C}(\Lambda)$ touching the past boundary. As $x \in P_z$ we conclude that x lies in the past boundary. Similarly points of the future boundary of $\mathcal{C}(\Lambda)$ correspond to support planes for $S_-(\Lambda)$. □

Proposition 4.37. *The past and future boundary components of $\mathcal{C}(\Lambda)$ are achronal surfaces.*

Proof. We will show it for $\partial_+\mathcal{C}(\Lambda)$. Take $x, y \in \partial_+\mathcal{C}(\Lambda)$ and consider the segment joining x to y in an affine chart containing Λ. If this segment was timelike then the dual planes P_x and P_y would be disjoint. Then (up to switching the roles of x and y) we may assume that, in the universal cover $P^1_{\widetilde{x}} \subset I^+(P^1_{\widetilde{y}})$, where $\widetilde{x}$ and $\widetilde{y}$ are the lifting of x and y in the same Dirichlet region mapping to the fixed affine chart. But then $S_+(\widetilde{\Lambda})$ would be contained in $\mathrm{I}^-(P^1_{\widetilde{y}})$ and could not meet $P^1_{\widetilde{x}}$ thus a contradiction to Proposition 4.36 □

We can also characterize what *kind* of support planes of $\mathcal{C}(\Lambda_\varphi)$ are allowed by considering how such planes touch $\mathcal{C}(\Lambda_\varphi)$ at a boundary point.

Proposition 4.38. *Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation-preserving homeomorphism, and let P be a support plane of $\mathcal{C}(\Lambda_\varphi)$ at a point $p \in \partial\mathcal{C}(\Lambda_\varphi)$. Then:*

- *If $p \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then P is a spacelike plane.*
- *If $p \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ then P is either spacelike or lightlike.*

Proof. The key observation is that if P is a spacelike plane, then ∂P and $\Lambda_\varphi = \mathcal{C}(\Lambda_\varphi) \cap \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ do not intersect transversely. From Lemma 3.12 if P is timelike then ∂P is the graph of an orientating-reversing homeomorphism of $\mathbb{R}\mathrm{P}^1$, hence it intersects Λ_φ transversally. From Lemma 3.13, if P is lightlike, then its boundary is the union of the two circles: $\{x\} \times \mathbb{R}\mathrm{P}^1$ and $\mathbb{R}\mathrm{P}^1 \times y$. So if $p \in \partial P \cap \Lambda_\varphi$ and p is not the point $p_0 = (x, y)$, then ∂P and Λ_φ intersect transversally. Hence the only case where P can be a lightlike support plane is when it intersects Λ_φ only at the point p_0. We are left with the task to show that $P \cap \mathcal{C}(\Lambda_\varphi) = \{p_0\}$ and it does not contain any point of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. By contradiction suppose the existence of a $q \in P \cap \mathcal{C}(\Lambda_\varphi)$ with q different from p_0, then by Equation (4.3), $\partial P \cap \Lambda_\varphi$ would also contain another point different from p_0, a contradiction. $\square$

Now, given a spacelike support plane P of $\mathcal{C}(\Lambda_\varphi)$ at a point p, we say that P is a *future* (resp. *past*) support plane if in a small simply connected neighbourhood of $p \in \overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}, \mathcal{C}(\Lambda_\varphi)$ is contained in the closure of the connected components of $U \backslash P$ which is the past (resp. future) of P. This means that there exists a future-oriented (resp. past-oriented) timelike curve leaving $\mathcal{C}(\Lambda_\varphi) \cap U$ and reaching $P \cap U$.

We observe that $\mathcal{C}(\Lambda_\varphi)$ cannot have both a future and past support plane at p unless $\mathcal{C}(\Lambda_\varphi)$ has empty interior, a situation that happens exactly when φ is an element of $\mathrm{PSL}(2,\mathbb{R})$, see Example 4.33.

We can now state the following lemma related to convergence of support planes:

Lemma 4.39. Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation-preserving homeomorphism which is not in $\mathrm{PSL}(2,\mathbb{R})$, p_n a sequence of points in $\partial\mathcal{C}(\Lambda_\varphi)$, and P_{γ_n} a sequence of future (resp. past) spacelike supports planes at p_n, for $\gamma_n \in \mathrm{PSL}(2,\mathbb{R})$. Up to extracting a subsequence, we can assume $p_n \to p$ and $P_{\gamma_n} \to P$. Then:

- If $p \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then $P = P_\gamma$ is a future (resp. past) support plane of $\mathcal{C}(\Lambda_\varphi)$, for $\gamma_n \to \gamma \in \mathrm{PSL}(2,\mathbb{R})$.

- If $p \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then either P is a lightlike plane whose boundary is the union of two circle meeting at p, or it is a future (resp. past) support plane as in the previous point.

Proof. We can extract a converging subsequence from p_n and P_{γ_n}, by compactness of $\mathcal{C}(\Lambda_\varphi)$ and of the space of planes in projective space. Also, the limit of the sequence of support planes P_{γ_n} at p_n is a support plane of P at p, since both conditions that $p_n \in \mathcal{C}(\Lambda_\varphi)$ and that $\mathcal{C}(\Lambda_\varphi)$ is contained in a closed half-space bounded by P_{γ_n} are closed conditions. Now, by Proposition 4.38, if the limit p is in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then P is a spacelike support plane, which is future (resp. past) if all the P_{γ_n} are future (resp. past). The situation could also occur analogously if $p \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$; the other possibility being that P is lightlike, and in this case the proof of Proposition 4.38 shows $P = P_{[A]}$ if p is represented by the projective class of the one-rank matrix A. $\square$

Corollary 4.40. *Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation-preserving homeomorphism which is not in $PSL(2,\mathbb{R})$. Then $\partial\mathcal{C}(\Lambda_\varphi)$ is the disjoint union of graph$(f) = \mathcal{C}(\Lambda_\varphi) \cap \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and of two topological discs, of which one admits future support planes, and the other only admits past support planes.*

Proof. We have already hinted at this after Remark 4.34, we will now give a more detailed proof. It is a basic result in convex analysis that $\partial\mathcal{C}(\Lambda_\varphi)$ is homeomorphic to $\mathbb{S}^2$; by Proposition 4.31 its intersection with $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ equals Λ_φ and is therefore a simple closed curve. By the Jordan curve theorem, the complement of Λ_φ is the disjoint union of two topological discs, each of which is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.
Now by Lemma 4.39, the set of point $p \in \partial\mathcal{C}(\Lambda_\varphi)$ admitting a future support plane is close. But is also open because its complement is the set of points admitting a past support plane, for which the same argument applies. Hence each connected component of the complement of Λ_φ admits only future support planes, or only past support planes. Finally, $\partial\mathcal{C}(\Lambda_\varphi)$ necessarily admits both a past and a future support plane, otherwise it would not be compact in an affine chart. $\square$

According to Corollary 4.40 we will call the connected component of $\partial\mathcal{C}(\Lambda_\varphi)\backslash\Lambda_\varphi$ that only admits future support planes the *future boundary component*, and denote it by $\partial_+\mathcal{C}(\Lambda_\varphi)$. Similarly, we will call the connected component of $\partial\mathcal{C}(\Lambda_\varphi)\backslash\Lambda_\varphi$ that only admits past support planes the *past boundary component*, and denote it by $\partial_-\mathcal{C}(\Lambda_\varphi)$.

4.3 Globally Hyperbolic three-manifolds

We want now to focus our attention on maximal globally hyperbolic (MGH) Anti-de Sitter spacetimes containing a compact Cauchy surfaces of genus r (we will, with a slight abuse of notation, say that the spacetime has genus r.) We will show that there are no such manifolds when $r = 0$. We will then focus on the most interesting case, $r \geqslant 2$, that will lead to a complete classification.

General facts

We will now state some general facts that will be useful in our classification.

Lemma 4.41. Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a spacelike immersion. If $\sigma^*(g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}})$ is a complete Riemannian metric, then σ is a proper embedding and S is diffeomorphic to $\mathbb{R}^2$

Proof. Thanks to Proposition 4.29 it suffices to prove that σ is a proper immersion. We keep the notation of the proof of Proposition 4.29, consider a lift $\widehat{\sigma} : \widehat{S} \to \mathbb{H}^{2,1}$ We need to show that the map $\widehat{\sigma}$ is proper. We will show that if $\gamma : [0, 1) \to \widehat{S}$ is a path such taht the limit $\lim_{t\to 1} \widehat{\sigma}(\gamma(t))$ exists, then also $\lim_{t\to 1} \gamma(t)$ exists.
Using the expression for the metric on $\mathbb{H}^{2,1}$ given by Equation 2.8 (under the identification of $\mathbb{H}^{2,1}$ with $\mathbb{H}^2 \times \mathbb{S}^1$ given by the composition of the projection from the universal cover with the map π given in Equation 2.5), we see that the lenght of γ for the pull-back metric is smaller than the lenght of the projection of γ to the $\mathbb{H}^2$ factor, with respect to the hyperbolic metric on $\mathbb{H}^2$. The latter hyperbolic lenght is finite by the assumption, hence the lenght of γ is also finite for the pull-back metric. Now the assumption of completeness on the pull-back metric implies the existence of the limit point for $\gamma(t)$. $\square$

As an immediate consequence there cannot be any globally hyperbolic spacetime with genus 0. In fact, suppose such a spacetime exists and denote by Σ a Cauchy surface diffeomorphic to $\mathbb{S}^2$, the developing map restricted to Σ would be a spacelike immersion, and the pull-back metric would be complete by compactness. But this contradicts Lemma 4.41. Hence:

Corollary 4.42. *There exists no globally hyperbolic Anti-de Sitter spacetimes of genus 0.*

The following is a fundamental result on the structure of globally hyperbolic AdS spacetimes.

Proposition 4.43. *Let M be a globally hyperbolic Anti-de Sitter spacetime of genus $r \geqslant 1$. Then:*

1. *The developing map $dev : \widetilde{M} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is injective.*
2. *If Σ is a Cauchy surface of M, then the image of dev is contained in $\Omega(\Lambda)$, where Λ is boundary at infinity of $dev(\widetilde{\Sigma})$.*
3. *If $\rho : \pi_1(M) \to Isom(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1,})$ is the holonomy representation, $\rho(\pi_1(M))$ acts freely and properly discontinuously on $\Omega(\Lambda)$, and $\Omega(\Lambda)$ is a globally hyperbolic spacetime containing M.*

Proof. Let $\widetilde{\mathrm{dev}} : \widetilde{M} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ be a lift of dev to the universal cover. By Theorem 4.18, the spacetime M admits a foliation by smooth spacelike surfaces $(\Sigma_t)_{t\in\mathbb{R}}$ of genus $r \geqslant 1$, such that $\Sigma_t \subset I^+(\Sigma_{t'})$ for $t > t'$. Let $\widetilde{\Sigma}_t$ be the lift of the foliation on $\widetilde{M}$. Since Σ_t is closed, the induced metric on Σ_t is complete, and so is the induced metric on $\widetilde{\Sigma}_t$. As $\widetilde{\mathrm{dev}}$ is a local isometry, we deduce by Lemma 4.41 that the restriction of $\widetilde{\mathrm{dev}}$ is a proper embedding. Assume now by contradiction that $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t) \cap \widetilde{\mathrm{dev}}(\widetilde{\Sigma}_{t'}) \neq \varnothing$ for some $t \geqslant t'$. Then there is a point $x \in \widetilde{\Sigma}_t$ such that $\widetilde{\mathrm{dev}}(x) \in \mathrm{dev}(\widetilde{\Sigma}'_t)$. By assumption, x is connected to $\widetilde{\Sigma}_{t'}$ by a timelike arc η in $\widetilde{M}$. Then $\widetilde{\mathrm{dev}}(\eta)$ is a timelike arc in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ with endpoints in $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_{t'})$ and this contradicts the achronality of $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_{t'})$. This shows that $\widetilde{\mathrm{dev}}$ is injective, and moreover we conclude that $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)$ is a Cauchy surface of $\widetilde{\mathrm{dev}}(\widetilde{M})$.

It follows from Proposition 4.29 that $\widetilde{\mathrm{dev}}(M) \subset \mathcal{D}(\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)) = \Omega(\widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is the boundary at infinity of $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)$, and this proves the second point. Now, the map $\widetilde{\mathrm{dev}}$ is $\widetilde{\rho}-$equivariant, for a representation $\widetilde{\rho} : \pi_1(M) \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1})$ which is a lift of the holonomy of M. As $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)$ is $\widetilde{\rho}$-invariant, the so are $\widetilde{\Lambda}$ and $\Omega(\widetilde{\Lambda})$. We shall prove that the action of $\pi_1(M)$ on $\Omega(\widetilde{\Lambda})$ given by $\widetilde{\rho}$ is proper. This will also show that the action is free since $\pi_1(M)$ is isomorphic to $\pi_1(\Sigma_r)$ and therefore has no torsion.

For this purpose, let us notice that if K is relatively compact in $\Omega(\widetilde{\Lambda})$ then

$$X_K := (I^+(K) \cup I^-(K)) \cap \widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)$$

is relatively compact as well. As the action of $\pi_1(M)$ on $\widetilde{\Sigma}_t$, and thus on $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)$, is proper and $X_{\gamma K} = \gamma(X_K)$, we deduce that the set of γ such that $X_{\gamma K} \cap X_K \neq \varnothing$ is finite. On the other hand if $K \cap \gamma K \neq \varnothing$ then $X_K \cap X_{\gamma K} \neq \varnothing$. We thus conclude that the action is proper. By applying the path lifting property, one sees that the quotient $\widetilde{\mathrm{dev}}(\widetilde{\Sigma}_t)/\pi_1(M)$ is a Cauchy surface of $\Omega(\widetilde{\Lambda})/\pi_1(M)$, which is therefore globally hyperbolic. This proves the third point as by Proposition 4.27 the restriction of the covering map $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ to $\Omega(\widetilde{\Lambda}) \cup \widetilde{\Lambda}$ is injective. $\square$

A remarkable difference between Lorentzian and Riemannian geometry is that in Lorentzian geometry completeness is a very strong assumption, and there is not any counterpart to the Hopf-Rinow theorem. In fact, interesting classification results are obtained removing such condition. However, it is necessary to impose some maximality condition to compensate for non-completeness. Following [2] we restrict to a less general setting, but one could give similar definitions in the larger class of Einstein spacetimes.

Definition 4.44. A globally hyperbolic Anti-de Sitter manifold (M, g) is *maximal* if any isometric embedding of (M, g) into a globally hyperbolic Anti-de Sitter manifold (M', g') which sends a Cauchy surface of M to a Cauchy surface of M' is surjective.

As a product of this definition and as a direct consequence of Proposition 4.43 we have:

Corollary 4.45. *An Anti-de Sitter globally hyperbolic spacetime M is maximal if and only if $\widetilde{M}$ is isometric to the invisible domain of a proper achronal meridian in* $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Examples of genus $r \geqslant 2$

Let Σ_r be an oriented surface of genus $r \geqslant 2$. We recall the definition of Fuchsian representation.

Definition 4.46. A representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is called positive *Fuchsian* if there is a ρ-equivariant orientation preserving homeomorphism $\delta : \widetilde{\Sigma}_r \to \mathbb{H}^2$.

The definition is invariant under conjugation in $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Isom}_0(\mathbb{H}^2)$, but not under conjugation in $\mathrm{Isom}(\mathbb{H}^2)$.

Another useful tool for the classification of genus $r \geqslant 2$ is the following fact in Teichmüller theory [5]:

Lemma 4.47. Given two positive Fuchsian representations $\rho, \varrho : \pi_1(\Sigma_r) \to \mathrm{PSL}(2,\mathbb{R})$, any $(\rho,\varrho)-$equivariant orientation-preserving homeomorphism of $\mathbb{H}^2$, extends continuously to an orientation-preserving homeomorphism of $\mathbb{H}^2 \cup \mathbb{R}\mathrm{P}^1$ (we can consider $\delta_\varrho^{-1} \circ \delta_\rho$ given by Definition 4.46). Moreover, its extension $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ is the unique $(\rho,\varrho)-$equivariant orientation homeomorphism of $\mathbb{R}\mathrm{P}^1$.

Here by (ρ,ϱ)-equivariant, we mean that for every $\gamma \in \pi_1(S)$:

$$\varphi \circ \rho(\gamma) = \varrho(\gamma) \circ \varphi.$$

Now let ρ, ϱ be two positive Fuchsian representations of Σ_r. We will be interested in the representation:

$$\omega = (\rho,\varrho) : \pi_1(\Sigma_r) \to \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \simeq \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}).$$

Definition 4.48. Given a pair of positive Fuchsian representations $\rho, \varrho : \pi_1(\Sigma_r) \to \mathrm{PSL}(2,\mathbb{R})$ we define Λ_ω to be the graph in $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ of the unique $(\rho,\varrho)-$equivariant orientation-preserving homeomorphism of $\mathbb{R}\mathrm{P}^1$, and $\Omega_\omega := \Omega(\Lambda_\omega)$ its invisible domain in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

Using the above construction, we can build examples of MGH spacetimes having holonomy any $\omega = (\rho, \varrho)$ of this form.

Proposition 4.49. *The domain Ω_ω is invariant under the isometric action of $\pi_1(\Sigma_r)$ on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ induced by ω. Moreover $\pi_1(\Sigma_r)$ acts freely and properly discontinuosly on Ω_ω and the quotient is a MGH spacetime of genus r and holonomy ω.*

Proof. By the definition of φ and the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ we have that Λ_ω is invariant under the action of $(\rho(\gamma), \varrho(\gamma))$, for every $\gamma \in \pi_1(\Sigma_r)$. We recall from Corollary 4.28 that Ω_ω is the set of elements $x \in \mathrm{PSL}(2,\mathbb{R})$, such that $x \circ \varphi$ has no fixed point on $\mathbb{R}\mathrm{P}^1$. The invariance of Ω_ω also follows immediately, indeed:

$$(\rho(\gamma) \circ x \circ \varrho(\gamma)^{-1}) \circ \varphi = \rho(\gamma) \circ (x \circ \varphi) \circ \rho_2(\gamma)$$

acts freely on $\mathbb{R}\mathrm{P}^1$ if $x \circ \varphi$ does.
Let us show that for a compact set K in Ω_ω, $\rho(\gamma) \cdot K$ stays in a compact region of Ω_ρ only for finitely many $\gamma \in \pi_1(\Sigma_r)$. This will also show that the action is free, since $\pi_1(\Sigma_r)$ has no torsion. For this purpose, take a sequence $x_n \in K$ and a sequence $\gamma_n \in \pi_1(\Sigma_r)$ not definitely constant. We claim that up to a subsequence, $(\rho(\gamma_n) \cdot x_n)$ converges to some $(\xi_+, \varphi(\xi_+))$ in $\Lambda(\rho)$. We will apply the criterion of convergence in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ seen in Lemma 3.3.
Since Fuchsian representations act cocompactly on $\mathbb{H}^2$, the sequence $\rho(\gamma_n)$ has no converging subsequence in $\mathrm{PSL}(2,\mathbb{R})$. By a known dynamical property of $\mathrm{PSL}(2,\mathbb{R})$ (for more information view [BZ06]), up to taking a subsequence, there exist $\xi_-, \xi_+ \in \mathbb{R}\mathrm{P}^1$ such that $\rho(\gamma_n)^{\pm 1}(\xi) \to \xi_\pm$ for all $\xi \neq \xi_\mp$ and that the convergence is uniform on compact sets of $(H^2 \cup \mathbb{R}\mathrm{P}^1)\backslash\{\xi_\mp\}$. By the equivariance condition, the same holds for $\varrho(\gamma_n)$ where now $\xi_\pm$ are replaced by $\varphi(\xi_\pm)$.
To apply the criterion of Lemma 3.3, pick any $p \in \mathbb{H}^2$, and recall that $\rho(\gamma_n) \cdot x_n = \rho(\gamma_n) \circ x_n \circ \varrho(\gamma_n)^{-1}$. By the dynamical property above, for any $\delta > 0$ one can find n_0 such that $\varrho(\gamma_n)^{-1}(p)$ is in the δ-neighborhood of $\varphi(\xi_-)$ (for the Euclidean metric on the closed disc), say U_δ. Since x_n lies in a compact region of Ω_ω, we can assume that is converges to $x_\infty \in \Omega_\omega$, hence $x_\infty \circ \varphi$ has no fixed point, and in particular $x_\infty \circ \varphi(\xi_-) \neq \xi_-$.
Up to taking δ sufficiently small and n_0 large, $x_n(U_\delta)$ lies in a neighborhood V_ϵ of $x_\infty \circ \varphi(\xi_-)$ such that the closure of V_ϵ is disjoint from ξ_-. By construction $x_n \circ \varrho(\gamma_n)^{-1}(p)$ converges to ξ_+ for n large. The very same argument then shows that $(\rho(\gamma_n) \cdot x_n)^{-1}(p) = \varrho(\gamma_n) \circ x_n^{-1} \circ \rho(\gamma_n)^{-1}(p)$ converges to $\varphi(\xi_+)$. And this concludes the claim.
Finally, past and future boundary components $\partial_\pm\mathcal{C}(\Lambda(\rho))$ are contained in Ω_ρ,

since $\Lambda(\rho)$ is the graph of an orientation-preserving homeomorphism. Hence they are $\rho-$ invariant properly embedded Cauchy surfaces in Ω_ω and project to Cauchy surfaces of the quotient of the action of $\rho(\pi_1(\Sigma_r))$, which are homeomorphic to Σ_r. This shows that the quotient is a globally hyperbolic spacetime of genus r, which is maximal by Proposition 4.43. $\square$

Classification of genus $r \geqslant 2$

We will now show that the examples of Proposition 4.49 are *all* the MGH spacetimes of genus r.

Lemma 4.50. Let $\omega = (\rho, \varrho)$ be a pair of positive Fuchsian representations, and $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be the unique $(\rho, \varrho)-$equivariant orientation-preserving homeomorphism of $\mathbb{R}\mathrm{P}^1$. Then $\Lambda(\rho)$ is the unique proper achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ invariant under the action of $\pi_1(\Sigma_r)$ induced by ρ.

Proof. Let Λ be a proper achronal meridian invariant under the action of $\pi_1(\Sigma_r)$. We claim that the intersection $\Lambda \cap \Lambda_\varphi$ is not empty. Once the claim will be showed, the proof is concluded in the following way. If $(\xi_0, \varphi(\xi_0)) \in \Lambda$, then:

$$(\rho(\gamma)\cdot\xi_0, \varphi(\rho(\gamma)\cdot\xi_0)) = (\rho(\gamma)\cdot\xi_0, \varrho(\gamma)\cdot\varphi(\xi_0)) \in \Lambda.$$

However the $\rho(\pi_1(\Sigma_r))$-orbit of ξ_0 is dense in $\mathbb{R}\mathrm{P}^1$, hence we deduce that Λ contains Λ_φ. But both Λ and Λ_φ are homeomorphic to $\mathbb{S}^1$, which necessarily implies $\Lambda = \Lambda_\varphi$.
Let us then show the claim. Let γ be a non-trivial element in $\pi_1(\Sigma_r)$. It is known that $\rho(\gamma)$ and $\varrho(\gamma)$ are necessarily loxodromic elements in $\mathrm{PSL}(2,\mathbb{R})$, hence we denote by $\xi_l^+(\gamma)$, and $\xi_r^+(\gamma)$ their attractive fixed points respectively. Notice that $\xi_r^+(\gamma) = \varphi(\xi_l^+(\gamma))$, hence:

$$(\xi_l^+(\gamma), \xi_r^+(\gamma)) \in \Lambda_\varphi.$$

By homological reasoning the curve Λ must meet the leaf of the left ruling of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$:

$$\lambda_{\xi_r^+(\gamma)} = \{(\eta, \xi_r^+(\gamma)) \mid \eta \in \mathbb{R}\mathrm{P}^1\}.$$

That is, there exists $\eta_0 \in S$ such that $(\eta_0, \xi_r^+(\gamma))$ lies in Λ. But then we have that the point $(\rho(\gamma)^k\eta_0, \xi_r^+(\gamma))$ lies in Λ for $k > 0$. If $\eta_0 \neq \xi_l^-(\gamma)$ we can pass to the limit on k and deduce that $(\xi_l^+(\gamma), \xi_r^+(\gamma))$ lies in Λ.
So far, the choice of γ was arbitrary. To conclude, assume now by contradiction that for every $\gamma \in \pi_1(\Sigma_r)$ the point $(\xi_l^-(\gamma), \xi_r^+(\gamma))$ lies in Λ. Take $\alpha, \beta \in \pi_1(S)$

so that the axes of $\rho(\alpha)$ and $\rho(\beta)$ do not intersect. We may assume that the cyclic order of end-points of those axes is

$$\xi_l^+(\alpha) < \xi_l^+(\beta) < \xi_l^-(\beta) < \xi_l^-(\alpha). \tag{4.4}$$

Now given that $\xi_r^\pm(\alpha) = \varphi(\xi_l^\pm(\alpha))$ and the same holds for β (because φ is an orientation-preserving homeomorphism) we have:

$$\xi_r^+(\alpha) < \xi_r^+(\beta) < \xi_r^-(\beta) < \xi_r^-(\alpha). \tag{4.5}$$

On the other hand, by assumption (applied to α, β and their inverses) the curve Λ contains $(\xi_l^+(\alpha), \xi_r^-(\alpha)), (\xi_l^+(\beta), \xi_r^-(\beta)), (\xi_l^-(\beta), \xi_r^+(\beta))$ and $(\xi_l^-(\alpha), \xi_r^+(\alpha))$. By achronality of Λ, the cyclic order to the second components must be the same as that of the first components, hence from Equation 4.4 we obtain:

$$\xi_r^-(\alpha) < \xi_r^-(\beta) < \xi_r^+(\beta) < \xi_r^+(\alpha),$$

which contradicts Equation 4.5. □

Given a pair $\rho = (\rho, \varrho)$ of positive Fuchsian representations of $\pi_1(\Sigma_r)$, we denote by M_ρ the MGH spacetime $\Omega_\omega/\rho(\pi_1(\Sigma_r))$ constructed in Proposition 4.49.

Corollary 4.51. *For any pair $\rho = (\rho, \varrho)$ of positive Fuchsian representations of $\pi_1(\Sigma_r)$, M_ρ is the unique MGH spacetime with holonomy ρ.*

We are only left with one last step for the classification result, we want to show that the left and right holonomies are necessarily positive Fuchsian.

Proposition 4.52. *Let M be an oriented, time-oriented, globally hyperbolic spacetime of genus $r \geqslant 2$ and let us endow a Cauchy surface Σ with the orientation induced by the future normal vector. Then the left and right components of the holonomy $\rho = (\rho, \varrho) : \pi_1(\Sigma) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ are positive Fuchsian representations.*

Remark 4.53. *We refer to the holonomy ρ with respect to an orientation-preserving developing map. Therefore ρ is well-defined up to conjugacy in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$*

Proof. We will prove that the $\mathbb{R}\mathrm{P}^1$-flat bundles with holonomy ρ and ϱ are isomorphic to the unit tangent bundle of Σ. The condition is equivalent to the representation being positive Fuchsian due to a result of Goldman [6]. For the sake of definiteness, let us focus on ρ. We will construct an isomorphism:

$$\Phi_l : T^1\widetilde{\Sigma} \to \widetilde{\Sigma} \times \mathbb{R}\mathrm{P}^1$$

equivariant with the respect to the action on $T^1\widetilde{\Sigma}$ by the actions by deck transformation and the diagonal action given by ρ on $\widetilde{\Sigma} \times \mathbb{R}\mathrm{P}^1$.
We define Φ_l as follows: for an element $(x, v) \in T^1\widetilde{\Sigma}$, let

$$\xi(x, v) = (\xi^l(x, v), \xi^r(x, v)) \in \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$$

be the end-point of the spacelike geodesic ray $\exp_x(tv)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ for positive t, Then we define $\Phi_l(x, v) = (x, \xi^l(x, v))$. This map is clearly continuous, proper, equivariant and fiber preserving.
In order to prove that it is a bijection it suffices to notice that for any $x \in \widetilde{\Sigma}$ the map $\xi_x : T^1_x(\widetilde{\Sigma}) \to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ is an embedding with image the boundary of the totally geodesic plane tangent to $\widetilde{\Sigma}$ at x. This boundary is the graph of an orientation-preserving map of $\mathbb{R}\mathrm{P}^1$, so the projection $v \to \xi^l(x, v)$ is bijective. Moreover, by our choice of the orientation on Σ, the orientation on $T^1_x\widetilde{\Sigma}$ corresponds to the orientation induced on $\xi_x(T^1_x\widetilde{\Sigma})$ as graph of an orientation-preserving homeomorphism. The proof for ϱ is analogous. □

We can now state the classification result. We denote the *deformation space* of MGH spacetimes of genus r by:

$$\mathcal{MGH}(\Sigma_r) = \{g \text{ MGH AdS metric on } \Sigma_r \times \mathbb{R}\}/\mathrm{Diff}_0(\Sigma_r \times \mathbb{R})$$

where the group of diffeomorphisms isotopic to the identity is acting by pull-back. The holonomy map takes value in the space of representation of $\pi_1(\Sigma_r)$ into $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and is well-defined on the quotient $\mathcal{MGH}(\Sigma_r)$. As a consequence of Proposition 4.52, the left and right components of the holonomy of elements of $\mathcal{MGH}(\Sigma_r)$ are positive Fuchsian representations. The space of these representations up to conjugacy is identified with the Teichmüller space of Σ_r by the aforementioned work of Goldman:

$$\mathcal{T}(\Sigma_r) \simeq \{\rho : \pi_1(\Sigma_r) \to \mathrm{PSL}(2, \mathbb{R}) \text{ positive Fuchsian representation}\}/\mathrm{PSL}(2, \mathbb{R})$$

Therefore the holonomy map can be considered as a map from $\mathcal{MGH}(\Sigma_r)$ with values in $\mathcal{T}(\Sigma_r) \times \mathcal{T}(\Sigma_r)$. Restating the classification in the original set of Mess [11]:

Theorem 4.54. *The holonomy map $\rho : \mathcal{MGH}(\Sigma_r) \to \mathcal{T}(\Sigma_r) \times \mathcal{T}(\Sigma_r)$ is a homeomorphism.*

4.4 Gauss map of spacelike surfaces

We will now introduce the *Gauss map* associated to a spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. This tool will be useful in the study of the relation between Anti-de Sitter geometry and the Teichmüller theory of hyperbolic surfaces. We will start by fixing notation and recalling some general facts about immersed spacelike surfaces in Anti-de Sitter space. The theory is a pseudo-Riemannian equivalent to surfaces embedded in $\mathbb{R}^3$ hence we will use the adapted standard notation. For the moment we will assume that all our immersed surfaces are of class C^1.

Given a regular immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ we recall that we refer to it as *spacelike* if the pull-back of the ambient metric $g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$, namely $I := \sigma^* g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$ is Riemannian. We call I the first fundamental form of σ.

The tangent bundle TS is naturally identified with a subbundle of $T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ by means of $d\sigma$. The normal bundle N_σ is then defined as the $g_{\mathbb{A}\mathrm{d}\mathbb{S}}$-orthogonal complement of TS in $T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, and the restriction of $g_{\mathbb{A}\mathrm{d}\mathbb{S}}$ to N_σ is negative definite. Using the $g_{\mathbb{A}\mathrm{d}\mathbb{S}}-$orthogonal decomposition:

$$\sigma^* T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} = TS \oplus N_\sigma,$$

the pull-back of the ambient Levi-Civita connection ∇, restricted to sections tangent to S, splits as the sum of the Levi-Civita connection ∇^I of the first fundamental form I and a symmetric 2-form with value in N_σ. We recall that $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a time-orientable manifold therefore the normal bundle admits a natural trivialization, which is the same as a choice of a continuous unit normal vector field for σ. We will denote by $\nu : S \to N_\sigma$ the future-directed unit normal section, and consider the decomposition for all vector field V, W tangent to S:

$$\nabla_V W = \nabla^I_V W + II(V, W)\nu,$$

where the symmetric $(2, 0)$ tensor II is called *second fundamental form*. To conclude we introduce the $I-$symmetric $(1, 1)$-tensor $B \in (TS)^* \otimes TS$ defined by $II(V, W) = I(B(V), W)$ which is called the *shape operator* of σ. As in the Riemannian case, it turns out that $\sigma_*(B(v)) = \nabla_v \nu$.

The first and second fundamental form of an immersion σ satisfy constraint equations, known as *Gauss-Codazzi equations*. More precisely the Gauss equation consists of the identity:

$$K_I = -1 - \det_I II \tag{4.6}$$

where K_I is the curvature of I and $\det_I II$ is $\det B$ by definition. On the other hand the Codazzi equation states that $\nabla^I II$ is a totally symmetric $3-$form:

$$(\nabla^I_V II)(W, U) = (\nabla^I_W II)(V, U). \tag{4.7}$$

What can be shown is that the Gauss-Codazzi equations relating first and second fundamental forms are *necessary* but also *sufficient*, more explicitly we have:

Theorem 4.55. *Let S be a simply connected surface, let I be a Riemannian metric on S and II be a symmetric $(2,0)-$tensor on S. If I and II satisfy the Gauss-Codazzi equations* 4.6 *and* 4.7*, then there exists a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ having I and II as first and second fundamental form. Moreover if σ, σ' are two such immersions, then there exists a time-preserving isometry f such that $\sigma' = f \circ \sigma$.*

Germs of spacelike immersions in AdS manifolds

Let us now consider the case of an oriented surface Σ, not necessarily simply connected. Given a spacelike immersion $\sigma : \Sigma \to (M, g)$ where (M, g) is an oriented Anti-de Sitter manifold, we can associate to σ the pair (I, II) as done in the previous section, where II is computed with respect to the future unit normal vector ν of σ. We will always assume that the orientation of Σ and ν are compatible with the orientation on M.
The pair (I, II) is made up of local operators and only depends on the *germ* of the immersion σ.
Given a pair (I, II) on a surface Σ, we can perform the following construction: let $\pi : \widetilde{\Sigma} \to \Sigma$ be a universal cover, it follows that the pair $(\pi^* I, \pi^* II)$ satisfies the Gauss-Codazzi equations on $\widetilde{\Sigma}$, hence by the existence part of Theorem 4.55, there exists a spacelike immersion $\widetilde{\sigma} : \widetilde{\Sigma} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ having *immersion data* $(\pi^* I, \pi^* II)$. The uniqueness part of Theorem 4.55 has two main consequences:

- Any two such immersions differ by post-composition by a global isometry of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.
- Given any such $\widetilde{\sigma}$, there exists a map $\rho : \pi_1(\Sigma) \to \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ such that, for every $\gamma \in \pi_1(\Sigma)$, $f \circ \gamma = \rho(\gamma) \circ f$.

More can be proved: ρ is a group representation and changing $\widetilde{\sigma}$ by post-composition with an isometry f has the effect of conjugating ρ by f. The immersion σ can then be extended to an immersion of U, an open neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$, into $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, by mapping (x, t) to the point $\gamma(t)$ on the timelike geodesic γ such that $\gamma(0) = \sigma(p)$ and $\gamma'(0)$ is the future normal vector of σ at x. We want to explicit the expressions of the Anti-de Sitter metric in such a tubular neighborhood of σ as it will be useful in the following:

Lemma 4.56. Given a spacelike immersion $\sigma : \Sigma \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the pull-back of the ambient metric by means of the map $(p,t) \to \exp_{\sigma(p)}(t\nu(x))$ has the expression:

$$-dt^2 + \cos^2(t)I + 2\cos(t)\sin(t)II + \sin^2(t)III, \tag{4.8}$$

where I, II, III are the first, second and third fundamental form of σ respectively. We recall that the third fundamental form can be expressed as $III(\cdot,\cdot) = I(B(\cdot), B(\cdot))$ where B is the shape operator.

Proof. For ease, we will use the quadric model introduced in Section 2.1. We fix $(x,t) \in \Sigma \times \mathbb{R}$, and a vector $(v, \frac{\partial}{\partial t}) \in T_{(x,t)}\Sigma \times \mathbb{R}$. By Equation (2.10), we have $\exp_{\sigma(x)}(t\nu(x)) = \cos(t)\sigma(x) + \sin(t)\nu(x)$. The differential in t gives the vector $u = -\sin(t)\sigma(x) + \cos(t)\nu(x)$, while the differential in the space direction gives the vector $w = \cos(t)d\sigma_x(v) + \sin(t)d\nu_x(v)$. The two vectors are orthogonal, it follows from the observation $T_x\mathbb{H}^{n,1} = x^\perp$ in the quadric model.
Recall that $I(\cdot,\cdot) = \langle d\sigma(\cdot), d\sigma(\cdot)\rangle$ by definition and that the differential of σ identifies $B(v)$ and $\nabla_v \nu$, namely the tangential component of $d\nu(v)$. Consider now $z = u + w$, we have the following expression:

$$\begin{aligned}\langle z, z\rangle_{g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}} &= \sin^2(t)\langle\sigma(x),\sigma(x)\rangle + \cos^2(t)\langle\nu(x),\nu(x)\rangle - 2\cos(t)\sin(t)\langle\sigma(x),\nu(x)\rangle + \\ &\quad + \cos^2(t)\langle d\sigma_x(v), d\sigma_x(v)\rangle + 2\cos(t)\sin(t)\langle d\sigma_x(v), d\nu_x(v)\rangle + \\ &\quad + \sin^2(t)\langle d\nu_x(v), d\nu_x(v)\rangle \\ &= -1^2 + \cos^2(t)I + 2\cos(t)\sin(t)II + \sin^2(t)III.\end{aligned}$$

Where in the last equality we have use the aforementioned properties of I, II, III and the orthogonality of σ and $\nu(x)$. $\square$

In short, given a pair (I, II), Expression 4.8 provides a Lorentzian metric of constant curvature -1 on an open set U in $\Sigma \times \mathbb{R}$ containing the slice $\Sigma \times \{0\}$, and thus a germ of immersion of Σ into an Anti-de Sitter three-manifold with immersion data (I, II). We summarize what we have accomplished so far:

Proposition 4.57. *Given a surface Σ, there are natural identifications between the following spaces:*

- *The space of pairs (I, II) on Σ which are the solution of the Gauss-Codazzi equations.*
- *The space of germs of spacelike immersion of Σ into Anti-de Sitter manifolds.*

- *The space of spacelike immersion of* $\widetilde{\Sigma}$ *into* $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$*, equivariant with respect to a representation* $\rho : \pi_1\Sigma \to Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$*, up to the action of* $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ *via post-composition.*

All the identifications are equivariant to the actions of Diff(Σ)*, by pull-back in the first item and by pre-composition in the second and third item.*

Now we want to focus on the case where Σ is a closed surface. In such instance, the equivariant immersion $\widetilde{\sigma}$ via the representation as in Proposition 4.57 is necessarily an embedding, which can be extended to an embedding of $\widetilde{\Sigma} \times \mathbb{R}$ onto a domain of dependence in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. The representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ coincides with the holonomy of a maximal globally hyperbolic Anti-de Sitter manifold (M, g) (after the identification between $\pi_1(\Sigma)$ with $\pi_1(M)$ via embedding of $\Sigma \to M \simeq \Sigma \times \mathbb{R}$), therefore ρ consists of a pair of positive Fuchsian representations by Proposition 4.52.
What is really remarkable is that the embedding data, namely the pair (I, II), permits to recover explicitly the pair of elements in the space $\mathcal{T}(S) \times \mathcal{T}(S)$ which parametrizes maximal globally hyperbolic Anti-de Sitter manifolds with compact Cauchy surfaces.

Gauss map and projections

We can finally define our map for spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We have seen in Proposition 3.6 that the space of timelike geodesics of $\mathrm{PSL}(2,\mathbb{R})$ is naturally identified with $\mathbb{H}^2 \times \mathbb{H}^2$, where the identification maps a geodesic of the form

$$L_{p,q} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \mid X \cdot q = p\}$$

to the pair $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^2$. We still suppose that the spacelike immersion is C^1 for the moment, we will deal later with the case of weaker regularity.

Definition 4.58. Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a spacelike immersion. The *Gauss map* of σ is the map $G_\sigma : S \to \mathbb{H}^2 \times \mathbb{H}^2$ defined by $G_\sigma(x) = (p, q)$ such that $L_{p,q}$ is the timelike geodesic orthogonal to $\mathrm{Im}(d_x\sigma)$ at $\sigma(x)$.

It follows from the equivariance shown in Proposition 3.6, that the Gauss map G_σ is natural with respect to the action of the isometry group, namely:

$$G_{f\circ\sigma} = f \cdot G_\sigma$$

for every $f \in \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Example 4.59: In Example 3.10 we gave an isometric embedding of $\mathbb{H}^2$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with image the plane P_{Id} dual to the identity. This isometric embedding was defined by sending $p \in \mathbb{H}^2$ to the unique order-two element in $\mathrm{PSL}(2,\mathbb{R})$ fixing p, which by definition lies on the geodesic $L_{p,p}$. Moreover the geodesic $L_{p,p}$ is orthogonal to P_{Id}. Hence the Gauss map associated to this isometric embedding of $\mathbb{H}^2$ is just $p \mapsto (p,p)$.

By construction, the Gauss map of a spacelike immersion σ is invariant by reparametrization, in the sense that $G_{\sigma\circ\phi} = G_\sigma \circ \phi$ for a diffeomorphism $\phi : S' \to S$. Hence it makes sense to talk about *the* Gauss map of a spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. For example, as we have just remarked, for the plane dual to the identity, the Gauss map sends order-two element of $\mathrm{PSL}(2,\mathbb{R})$ to the pair (p,p) where p is the fixed point of the isometry.

Lemma 4.60. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, with future unit normal vector field ν, if $\sigma(p) = \mathrm{Id}$, then

$$G_\sigma(p) = G_{P_{\mathrm{Id}}}(\exp(\frac{\pi}{2}\nu(p))). \tag{4.9}$$

Proof. It is a consequence of Example 4.59. We need to observe that the geodesic leaving from Id with velocity $\nu(p)$ meets orthogonally P_{Id} at $\exp((\pi/2)\nu(p))$. □

Now we denote with $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ the hyperboloid of future unit timelike vectors in $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and consider the following map:

$$\mathrm{Fix} : T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1} \to \mathbb{H}^2$$

defined so that $\mathrm{Fix}(\nu)$ is the fixed point of the one-parameter elliptic group $\{\exp(t\nu) \mid t \in \mathbb{R}\}$. This map is equivariant for the action of $\mathrm{PSL}(2,\mathbb{R})$, which acts on the hyperboloid $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ by the adjoint representation and on $\mathbb{H}^2$ by the obvious action. Since both $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ and $\mathbb{H}^2$ have constant curvature -1, it follows from equivariance that Fix is an isometry.
In terms of the map Fix, Equation 4.9 reads as:

$$G_\sigma(p) = (\mathrm{Fix}(\nu(p)), \mathrm{Fix}(\nu(p))), \tag{4.10}$$

provided that $\sigma(p) = \mathrm{Id}$.
Via Lemma 4.60 and the naturality, we can recover a different description of the Gauss map. Recalling the structure of Lie group of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathrm{PSL}(2,\mathbb{R})$ we will denote by R_γ and L_γ the right and left multiplication by $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ respectively.

Lemma 4.61. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ with future unit normal vector field ν,

$$G_\sigma(p) = (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((L_{\sigma(p)^{-1}})_*(\nu(p)))).$$

Proof. If $\sigma(p) = \mathrm{Id}$, then the equality holds true by Equation 4.10. For the general case, the immersion $\sigma' = (\mathrm{Id}, \sigma(p)) \circ \sigma$ has the property that $\sigma'(p) = \mathrm{Id}$, and the future normal vector at $\sigma'(p)$ equals $\nu'(p) = (R_{\sigma(p)^{-1}}))_*(\nu(p))$. Therefore:

$$G_{\sigma'}(p) = (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p))))).$$

Now by the naturality of the Gauss map it follows,

$$\begin{aligned} G_\sigma(p) &= (\mathrm{Id}, \sigma(p)^{-1}) \cdot G_{\sigma'}(p) \\ &= (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \sigma(p)^{-1} \circ \mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p))))) \\ &= (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((L_{\sigma(p)^{-1}})_*(\nu(p))))). \end{aligned}$$

here in the last line we used that Fix is equivariant with respect to the adjoint action on the hyperboloid $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$.

□

We will refer to the components of the Gauss map as the *left* and *right projections,* and will denote them accordingly: $\Pi_l, \Pi_r : S \to \mathbb{H}^2$.

Remark 4.62. *In the work of Mess, we can find yet a different interpretation of the Gauss map. Given $p \in S$, one can find a unique left isometry $f_l(p)$ and a unique right isometry $f_r(p)$, sending the tangent plane P to the image of σ at $\sigma(p)$ to P_{Id}. Indeed the isometries $f_l(p)$ and $f_r(p)$ are simply obtained by left and right multiplication by the inverse of dual point of the tangent plane P, namely $\zeta(p) = \exp_{\sigma(p)}((\pi/2)\nu(p))$. Using the identification of the dual plane P_{Id} with $\mathbb{H}^2$, provided by Example 3.10, $\Pi_l(p)$ and $\Pi_r(p)$ are the image of $\sigma(p)$ under the right and left isometries, respectively:*

$$\Pi_l(p) = f_r(p) \circ \sigma(p) = (Id, \zeta(p)) \cdot \sigma(p) \text{ and } \Pi_r(p) = f_l \circ \sigma(p) = (\zeta(p)^{-1}, Id) \cdot \sigma(p).$$

Non-smooth surfaces. The construction of the Gauss map can be extended (as we will see in more detail in Section 5.2) in the non-smooth setting, for instance for convex spacelike surfaces $S \subset \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Then one defines the set-valued Gauss map as the map sending each $x \in S$ to the set of future unit vectors in $T_x^{1,+}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ orthogonal to support planes of S at x. Hence the image of x is a convex subset of $T_x\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, and it reduces to a single point if and only if S is differentiable at x. The image of G in $T_x^{1,+}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a $C^{1,1}$ surface.

CHAPTER 5

Thurston's earthquake theorem

We pause for the moment with our exploration of the Anti-de Sitter realm. We want to recall basic definitions of geodesic laminations and earthquakes, with the introduction of a first *basic* example. We will then give an outline of the deep underlying relation between pleated surfaces and earthquakes discovered by Mess and then finally move on the proof of the earthquake theorem.

5.1 Earthquake theory

The theory of earthquakes was introduced by Thurston as a tool to study the Teichmüller space of closed surfaces and is treated in detail in [8]. We will just summarize the main results and definitions that we need to set up our proof.

Definition 5.1. A geodesic lamination λ of $\mathbb{H}^2$ is a collection of disjoint geodesics that foliate a closed subset X of $\mathbb{H}^2$. The set X is called the *support* of λ. The geodesics in λ are called *leaves* (as in classic foliation terminology). The connected components of $\mathbb{H}^2 \setminus X$ are called *gaps*. The *strata* of λ are the leaves and the gaps.

Consider γ a loxodromic isometry of $\mathbb{H}^2$. The *axis* of the isometry is the geodesic ℓ of $\mathbb{H}^2$ connecting the two fixed points of γ in $\partial\mathbb{H}^2$. It follows from the classification of Möbius transformations that the geodesic is preserved

by the isometry, and when restricted to such a curve $\gamma|_\ell : \ell \to \ell$ acts as a translation with respect to any constant speed parametrization of ℓ.
Given A, B subsets of $\mathbb{H}^2$, we say that a geodesic ℓ *weakly separates* A and B if A and B are contained in the closure of different connected components of $\mathbb{H}^2 \setminus \ell$.

Definition 5.2. A *left* (resp. *right*) *earthquake* of $\mathbb{H}^2$ is a bijective map $E : \mathbb{H}^2 \to \mathbb{H}^2$ such that there exists a geodesic lamination λ for which the restriction $E|_S$ to any stratum S of λ is equal to the restriction of an isometry of $\mathbb{H}^2$, and for any two strata S and S' of λ the comparison isometry

$$\mathrm{Comp}(S, S') := (E|_S)^{-1} \circ E|_{S'}$$

is the restriction of an isometry γ of $\mathbb{H}^2$ such that:

- γ is different from the identity, unless possibly where one of the two strata S and S' is contained in the closure of the other;
- when it is not the identity, γ is a loxodromic transformation whose axis ℓ weakly separates S and S';
- moreover, γ translates to the left (resp right), seen from S to S'.

Let us explain more carefully what we mean by the last condition. Suppose $f : [0, 1] \to \mathbb{H}^2$ is a smooth path such that $f(0) \in S$, $f(1) \in S'$ and the image of f intersects ℓ transversally and exactly at one point $z_0 = f(t_0) \in \ell$. Let $v = f'(t_0) \in T_{z_0}\mathbb{H}^2$ be the tangent vector at the intersection point. Let $w \in T_{z_0}\mathbb{H}^2$ be a vector tangent to the geodesic ℓ pointing towards $\gamma(z_0)$. Then we say that γ translates to the left seen from S to S' if v, w is a positive basis of $T_{z_0}\mathbb{H}^2$, for the standard orientation of $\mathbb{H}^2$.
We observe that such a condition is independent of the order in which we choose S and S'. If $\mathrm{Comp}(S, S')$ translates to the left seen from S to S', then $\mathrm{Comp}(S', S)$ translates to the left seen from S' to S.

Let us consider a first basic example:

Example 5.3: The map

$$E : \mathbb{H}^2 \to \mathbb{H}^2$$

defined by:

$$E(z) = \begin{cases} z & \text{if } \mathrm{Re}(z) < 0 \\ az & \text{if } \mathrm{Re}(z) = 0 \\ bz & \text{if } \mathrm{Re}(z) > 0 \end{cases}$$

is a left earthquake if $1 < a < b$, and a right earthquake if $0 < b < a < 1$. The lamination λ that satisfies the definition consists of a unique geodesic, namely the geodesic ℓ corresponding to the imaginary axis.
Such a map is clearly not continuous along ℓ. Thurston proved in [15] that any earthquake map extends continuously to an orientation-preserving homeomorphism of $\partial\mathbb{H}^2$ meaning that there exists a (unique) orientation-preserving homeomorphism $\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ such that the map:

$$\overline{E}(z) = \begin{cases} E(z), & \text{if } z \in \mathbb{H}^2 \\ \varphi(z), & \text{if } z \in \partial\mathbb{H}^2; \end{cases}$$

is continuous at any point of $\partial\mathbb{H}^2$.
Then Thurston proved the following (in some sense *dual*) theorem, that he called *"geology is transitive"*:

Theorem 5.4 ("Geology is transitive"). *Given any orientation-preserving homeomorphism $\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, there exists a left earthquake map of $\mathbb{H}^2$, and a right earthquake map, that extends continuously to φ on $\partial\mathbb{H}^2$.*

Having all the needed definition we would now like to give a different proof of the statement of Theorem 5.4 using the tools of Anti-de Sitter geometry developed in the previous chapters. $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ will be the *correct* ambient space to study *pleated surfaces.* We adapt to the case of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ the classic definition for pleated surfaces in $\mathbb{H}^3$ given in [3].

Definition 5.5. A *pleated surfaces* in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a complete hyperbolic surface S together with an isometric map $f : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ such that every point $s \in S$ is in the interior of some geodesic arc which is mapped by f to a geodesic arc in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Definition 5.6. If (S, f) is a pleated surface, then we define its *pleating locus* to be those points of S contained in the interior of one and only one geodesic arc which is mapped by f to a geodesics arc.

More in detail, the key observation that we will use is due to Mess' work [11], that highlighted the relation between pleated surfaces and earthquake maps. Recall that given an achronal meridian $\Lambda \subset \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the upper and lower boundary components $\partial_\pm\mathcal{C}(\Lambda_\varphi)$ of the convex hull of Λ are a convex and a concave pleated surface, see Proposition 4.37.
We give a brief sketch of the idea of the proof and then will fill in the details. Consider left and right projections from $\partial_+\mathcal{C}(\Lambda_\varphi)$ to $\mathbb{H}^2$, now the composition

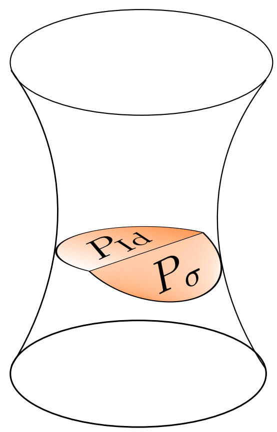

Figure 5.1: A bending surface in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, consisting of two half-planes meeting along a common geodesic. The bending geodesic is the axis of the loxodromic isometry σ.

$\Pi_r \circ \Pi_l^{-1}$ is a left earthquake map defined in the complement of the simplicial leaves of the lamination, and its earthquake lamination is identified to the bending lamination of $\partial_+\mathcal{C}(\Lambda_\varphi)$. A completely analogous statement holds for $\partial_-\mathcal{C}(\Lambda_\varphi)$ by reversing the roles of left and right.
Now, when the curve Λ is the graph of an orientation-preserving homeomorphism of $\mathbb{R}\mathrm{P}^1$, one obtains as a result earthquakes maps of $\mathbb{H}^2$. When moreover φ is the homeomorphism which conjugates left and right representations $\rho_l, \rho_r : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ of the holonomy of a MGH Cauchy compact manifold, the naturality of the construction implies that the earthquake map descends to an earthquake map from the left to the right hyperbolic surfaces, namely $\mathbb{H}^2/\rho_l(\pi_1(\Sigma))$ and $\mathbb{H}^2/\rho_r(\pi_1(\Sigma))$, and we can recover the earthquake theorem as in Kerchoff's (weaker) original formulation [9].
Given the previous discussion, we can now start with the details.

5.2 Earthquake Theorem

We consider $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ an orientation-preserving homeomorphism of the circle, and by Λ_φ we denote its graph as a subset of $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ identified as $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We recall that with the notation of the previous chapter, Λ_φ is a properly achronal meridian. By means of the Gauss map we had defined left and right projections for $\mathcal{C}^1$ embeddings. Now we would like to extend the Gauss map even when we have weaker regularity condition. Consider a point $p \in \partial_\pm \mathcal{C}(\Lambda_\varphi)$ and let P be a support plane of $\mathcal{C}(\Lambda_\varphi)$ at p. By Proposition 4.38 the support plane is necessarily spacelike, hence of the form $P = P_\gamma$ for some $\gamma \in \mathrm{PSL}(2,\mathbb{R})$. What happens when $\partial_\pm \mathcal{C}(\Lambda_\varphi)$ is not C^1 at p is that we do not have a unique support plane. Hence we *choose* a support plane P_γ at p, requiring that the choice of support planes is made so that the support plane is constant on any connected component of the subset of $\partial_\pm \mathcal{C}(\Lambda_\varphi)$ consisting of those points that admit more than one support plane. The definition of the Gauss map then *depends* on the choice of P_γ (see Corollary 5.13 for more details on how the choice influences the map). Once we have chosen the support planes we can just follow *verbatim* the construction of the Gauss map given in the smooth case.

Example 5.7: Let us consider a toy case where $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ so that $\mathcal{C}(\Lambda_\varphi) = P_{\varphi^{-1}}$ as in the previous Example 4.33. This is in some sense a degenerate case, as $\mathcal{C}(\Lambda_\varphi)$ has empty interior, hence Corollary 4.40 does not apply and it does not *really* make sense to talk about the future and the past component boundary. However, we can still define a left and right projections. Since $P_{\varphi^{-1}}$ itself is the unique support plane at any of its points, from the definition of the Gauss map we have the following expressions for the left and right projection $\Pi_l, \Pi_r : P_{\varphi^{-1}} \to \mathbb{H}^2$:

$$\Pi_l(p) = \mathrm{Fix}(p \circ \varphi) \;\; \Pi_r(p) = \mathrm{Fix}(\varphi \circ p). \tag{5.1}$$

We can also extend the two maps to the boundary of $P_{\varphi^{-1}}$: recalling that its boundary coincides with the graph of φ (Lemma 3.11) we have:

$$\Pi_l(x, \varphi(x)) = x \quad \Pi_r(x, \varphi(x)) = \varphi(x). \tag{5.2}$$

Equation 5.2 is immediately checked when $\varphi = \mathrm{Id}$, because in that case we have that Π_l, Π_r simply coincide with the fixed point map $\mathrm{Fix} : P_{\mathrm{Id}} \to \mathbb{H}^2$, and we observed previously that Fix extends to the map $(x,x) \to x$ from ∂P_{Id} to $\partial\mathbb{H}^2$. The general case of Equation 5.2 is then consequence of the equivariance of the Gauss map, with the additional observation that the isometry (Id, φ)

maps graph(Id) to graph(φ) and P_{Id} to $P_{\varphi^{-1}}$.
We can now compute the map of $\mathbb{H}^2$ obtained by composing the inverse of the left projection with the right projection. Indeed, this is induced by the map $P_{\mathrm{Id}} \to P_{\mathrm{Id}}$ sending an order-two elliptic element $\mathcal{R} = p \circ \varphi \in P_{\mathrm{Id}}$ to $\varphi \circ p = \varphi \circ \mathcal{R} \circ \varphi^{-1}$. Hence we have

$$\Pi_r \circ \Pi_l^{-1} = \varphi : \mathbb{H}^2 \to \mathbb{H}^2. \tag{5.3}$$

In conclusion, we have that the composition of the maps $\Pi_r \circ \Pi_l^{-1}$ is an isometry and its extension to the boundary of $\mathbb{H}^2$ is precisely the map $f = \varphi$ of which $\partial P_{\varphi^{-1}}$ is the graph. In what follows we will observe that this is what happens in the general case, that is, given an orientation-preserving homeomorphism of the circle φ, the composition $\Pi_r^{\pm} \circ (\Pi_l^{\pm})^{-1}$ associated with $\partial_{\pm}\mathcal{C}(\Lambda_\varphi)$ will be the left and right earthquake extending φ.

5.3 The fundamental example

We want to move one more intermediate step towards the final theorem. After the simple case, this time we will describe what we can consider *the fundamental example*. Consider piecewise totally geodesic surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, which are obtained as the union of two connected subsets, each contained in a totally geodesic spacelike plane, meeting along a common geodesic.
Let us formalize this idea in a more precise way. Consider the union of two half-planes, each contained in a totally geodesic spacelike plane $P_{\gamma_1}, P_{\gamma_2}$. The first key fact is the following:

Lemma 5.8. Let $\gamma_1 \neq \gamma_2 \in \mathrm{PSL}(2,\mathbb{R})$. Then P_{γ_1} and P_{γ_2} intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ if and only if $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry.

Proof. As in Example 4.33, P_{γ_i} is the convex hull of $\partial P_{\gamma_i} = \mathrm{graph}(\gamma_i^{-1})$, the closures $\overline{P}_{\gamma_i}$ intersect in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$ if and only if $\mathrm{graph}(\gamma_1) \cap \mathrm{graph}(\gamma_2) \neq \varnothing$. Moreover, by Equation 4.3, P_{γ_1} and P_{γ_2} intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ if and only if $\mathrm{graph}(\gamma_1) \cap \mathrm{graph}(\gamma_2)$ contains at least two different points.
We know that $(x,y) \in \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ is in $\mathrm{graph}(\gamma_1) \cap \mathrm{graph}(\gamma_2)$ if and only if $y = \gamma_1^{-1}(x) = \gamma_2^{-1}(x)$, which is equivalent to asking that $x \in \mathrm{Fix}(\gamma_2 \circ \gamma_1^{-1})$. But the composition $\gamma_2 \circ \gamma_1^{-1}$ is an element of $\mathrm{PSL}(2,\mathbb{R})$, hence it has two fixed points in $\partial\mathbb{H}^2 \simeq \mathbb{R}\mathrm{P}^1$ if and only if it is a loxodromic isometry. $\square$

Now consider $\mathbb{R}\mathrm{P}^1 = I_1 \cup I_2$ where I_1, I_2 are two closed intervals such that $I_1 \cap I_2$ consists exactly of the two fixed points of $\gamma_2 \circ \gamma_1^{-1}$. There are two possibilities to produce a homeomorphism of $\mathbb{RP}^1$ by composing the restriction of γ_1^{-1} and γ_2^{-1} to the intervals I_j's, namely:

$$\varphi^+_{\gamma_1,\gamma_2}(x) = \begin{cases} \gamma_1^{-1}, & \text{if } x \in I_1 \\ \gamma_2^{-1}, & \text{if } x \in I_2 \end{cases} \text{ and } \varphi^-_{\gamma_1,\gamma_2}(x) = \begin{cases} \gamma_2^{-1}, & \text{if } x \in I_1 \\ \gamma_1^{-1}, & \text{if } x \in I_2 \end{cases}. \tag{5.4}$$

Both $\varphi^\pm_{\gamma_1,\gamma_2}$ are orientation-preserving homeomorphism, since γ_1^{-1} and γ_2^{-1} map homeomorphically the intervals I_1 and I_2 to the same intervals $J_1 := \gamma_1^{-1}(I_1) = \gamma_2^{-1}(I_1)$ and $J_2 := \gamma_1^{-1}(I_2) = \gamma_2^{-1}(I_2)$ which intersects only at their endpoints.
We denote by D_i the convex hull of I_i in $\mathbb{H}^2$, and by $\ell = D_1 \cap D_2$ the axis of $\gamma_2 \circ \gamma_1^{-1}$.

Proposition 5.9. *Suppose that $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry that translates along ℓ to the left, as seen from D_1 to D_2. Then:*

- *The future boundary component $\partial_+\mathcal{C}(\varphi^+_{\gamma^+,\gamma_2})$ coincides with the union of the convex envelope of graph$(\gamma_1^{-1}|_{I_1})$ and the convex envelope of graph$(\gamma_2^{-1}|_{I_2})$.*
- *The past boundary component $\partial_-\mathcal{C}(\varphi^-_{\gamma_1,\gamma_2})$ coincides with the union of the convex envelope of graph$(\gamma_1^{-1}|_{I_2})$ and of the convex envelope of graph$(\gamma_2^{-1}|_{I_1})$.*

If instead $\gamma_2 \circ \gamma_1^{-1}$ translates along ℓ to the right as seen from D_1 to D_2, then:

- *The past boundary component of $\partial_-\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ coincides with the union of the convex envelope of graph$(\gamma_1^{-1}|_{I_1})$ and of the convex envelope of graph$(\gamma_2^{-1}|_{I_2})$.*
- *The future boundary component $\partial_+\mathcal{C}(\varphi^-_{\gamma_1,\gamma_2})$ is the union of the convex envelope of graph$(\gamma_1^{-1}|_{I_2})$ and of the convex envelope of graph$(\gamma_2^{-1}|_{I_1})$.*

Proof. Let us consider the case where $\gamma_2 \circ \gamma_1^{-1}$ translates to the left along ℓ, and let us prove the first item. Let x, x' be the fixed points of $\gamma_2 \circ \gamma_1^{-1}$, let $y = \gamma_1^{-1}(x) = \gamma_2^{-1}(x)$ and $y' = \gamma_1^{-1}(x') = \gamma_2^{-1}(x')$. Then the convex envelope of graph$(\gamma_i^{-1}|_{I_i})$ is a half-plane A_i in P_{γ_i} bounded by the geodesic $P_{\gamma_1} \cap P_{\gamma_2}$, which has endpoints (x, y) and (x', y'), Clearly both the convex envelope of graph$(\gamma_i^{-1}|_{I_i})$ are contained in $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$.
We could be even more precise. We claim that P_{γ_1} and P_{γ_2} are future support planes for $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$. The claim will imply that the union of A_1 and A_2 is contained in the future boundary component $\partial_+\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$, because every point

$p \in A_1 \cup A_2$ admits a future support plane through p which is either P_{γ_1} or P_{γ_2}. However $A_1 \cup A_2$ is a topological disc in $\partial_+\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$, whose boundary is precisely the curve graph$(\varphi^+_{\gamma_1,\gamma_2})$ by construction. Hence the claim will imply that $A_1 \cup A_2 = \partial_+(\varphi^+_{\gamma_1,\gamma_2})$.
We prove the claim for P_{γ_1}, proof for P_{γ_2} is analogous. For convenience we set $\gamma_1 = \mathrm{Id}$ and $\gamma_2 = \gamma$ is a loxodromic isometry with fixed points x, x', translating to the left as seen from D_1 to D_2. Indeed, we can reach such a configuration by applying (Id, γ_1), which sends P_{γ_1} to P_1, P_{γ_2} to $P_{\gamma_2\gamma_1^{-1}}$, and graph$(\varphi^+_{\gamma_1,\gamma_2})$ to graph$(\varphi^+_{\mathrm{Id},\gamma_2\gamma_1^{-1}})$.
We can now consider a path σ_t, for $t \in [0, \epsilon]$ of elliptic elements fixing a given point $z_0 \in \mathbb{H}^2$, that rotate clockwise by an angle t. As in the proof of Lemma 5.8 the planes P_{σ_t} are pairwise disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}$, because $\sigma_{t_2} \circ \sigma_{t_1}^{-1}$ is still an elliptic element fixing z_0 for $t_1 \neq t_2$, hence it has no fixed points in $\mathbb{RP}^1$. Moreover recall that γ^{-1} has a fixed axis ℓ and translates along ℓ to the right as seen from D_1 to D_2. Now $\varphi^+_{\mathrm{Id},\gamma}$ equals the identity on I_1 and γ_2 on I_2, it fixes I_1 pointwise and moves points on I_2 clockwise. It follows that the equation $\varphi^+_{\mathrm{Id},\gamma}(x) = \gamma_t^{-1}(x)$ has no solution for $t > 0$, because $\sigma_t^{-1} = \sigma_{-t}$ moves all the points counterclockwise if t is positive. This shows that $P_{\sigma_t} \cap \mathcal{C}(\varphi^+_{\mathrm{Id},\gamma}) = \varnothing$ for $t > 0$, and thus P_{Id} is a support plane for $\mathcal{C}(\varphi^+_{\mathrm{Id},\gamma})$ by Remark 4.35.
Moreover if it is a future support plane: indeed one can check that $\sigma_{t+\pi/2} = \mathcal{R}_{z_0} \circ \sigma_t \in P_{\sigma_t}$, and the path $t \mapsto \sigma_t$ is future-directed because, from the discussion after 3.3, its tangent vector is future-directed, hence $\mathcal{C}(\varphi^+_{\mathrm{Id},\gamma})$ is locally in the past of P_{Id}.
We have shown the first item of the proposition, all the others follow with completely analogous arguments. □

We want to put more focus on some elements of the proof that will be exploited in the following:

Corollary 5.10. *Suppose that $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry that translates along ℓ to the left (resp. right), as seen from D_1 to D_2, and write $\gamma_2 \circ \gamma_1^{-1} = \exp(\mathfrak{a})$ for some $\mathfrak{a} \in \mathfrak{sl}(2, \mathbb{R})$. Let p be a point in the future (resp. past) boundary components of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$. Then:*

- *If $p \in int(A_1)$, then P_{γ_1} is the unique support plane of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ at p.*
- *If $p \in int(A_2)$, then P_{γ_2} is the unique support plane of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ at p.*
- *If $p \in A_1 \cap A_2 = P_{\gamma_1} \cap P_{\gamma_2}$, then the support planes of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ at p are precisely those of the form $P_{\sigma\gamma_1}$ where $\sigma = exp(t\mathfrak{a})$ for $t \in [0, 1]$.*

We are still using the notation introduced in the fundamental example: $A_i \subset P_{\gamma_i}$ is the convex envelope of $\mathrm{graph}(\gamma_i^{-1}\big|_{I_i})$, an half-plane bounded by the geodesic $P_{\gamma_1} \cap P_{\gamma_2}$. As expected a completely analogous statement could be formulated for $\mathcal{C}(\varphi^-_{\gamma_1,\gamma_2})$ but we restrict to the study of $\varphi^+_{\gamma_1,\gamma_2}$ for simplicity.

Proof. The pleated surface that we obtained as the union of $A_1 \subset P_{\gamma_1}$ and $A_2 \subset P_{\gamma_2}$ coincides with $\partial_+\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ if $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry that translates along ℓ to the left, and with $\partial_-\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ if it translates to the right, as we saw in the previous example.
The first two items are then clear, since P_{γ_i} are smooth surfaces, hence A_i is smooth at any interior point, and therefore has a unique support plane at the point. For the last item we can assume $\gamma_1 = \mathrm{Id}$ and $\gamma_2 = \gamma$ is a loxodromic isometry translating on the left (resp. right) along ℓ. By Equation 4.3, if P_σ is a support plane at p, then p is in the convex hull of the pairs $(y, \sigma^{-1}(y))$ where y satisfies the relation $\sigma^{-1}(y) = \varphi^{\pm}_{\mathrm{Id},\gamma}(y)$. The only possibility is then for p to lie in the geodesic connecting the points (x, x) and (x', x') in $\mathbb{RP}^1 \times \mathbb{RP}^1$, where x, x' are the fixed points of γ. Hence σ must have the same fixed point as γ. It follows that σ is then a loxodromic isometry with axis ℓ (or the identity). Moreover, P_σ is in the future (resp. past) of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ if and only if σ translates on the left (resp. right), and its translation length is less than that of γ. Hence γ is of the form $\exp(t\mathfrak{a})$ for $t \in [0, 1]$. $\square$

We are finally arrived to the case of considering orientation-preserving homeomorphism obtained by combining two elements of $\mathrm{PSL}(2, \mathbb{R})$. We want to show that in a similar setting the composition of the projections $\Pi_l^{\pm}$ and $\Pi_r^{\pm}$ provide the earthquake map as in Example 5.3. At first glance this does not seem like a huge achievement as we are just recovering a simple earthquake map that we were already able to define explicitly. Nevertheless, the following proposition will be a key step to complete the proof of the earthquake theorem.

Proposition 5.11. *Let $\gamma_1, \gamma_2 \in PSL(2, \mathbb{R})$ be such that $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry, and let $\Pi_l^{\pm}, \Pi_r^{\pm}$ be the projections associated with the convex envelope of $\varphi^+_{\gamma_1,\gamma_2}$. Then:*

1. *$\Pi_l^{\pm}, \Pi_r^{\pm} : \partial_{\pm}\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2}) \to \mathbb{H}^2$ are bijections.*

2. *Assume that $\gamma_2 \circ \gamma_1^{-1}$ translates along ℓ to the right (resp. left), as seen from D_1 to D_2. Then the composition $\Pi_r^- \circ (\Pi_l^-)^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ (resp. $\Pi_r^+ \circ (\Pi_l^+)^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$) is a left (resp. right) earthquake map extending $\varphi^+_{\gamma_1,\gamma_2}$.*

We remark that we are limiting to the case of $\varphi^+_{\gamma_1,\gamma_2}$ for simplicity and completely analogous results could be formulated in terms of $\varphi^-_{\gamma_1,\gamma_2}$. Before proving it we remark that proposition 5.11 holds for *any choice* of support planes that is needed to define the projections.

Proof. For the first point, recall that $A_i \subset P_{\gamma_i}$, and that the union $A_1 \cup A_2$ is the past (resp. future) boundary component for $\varphi^+_{\gamma_1,\gamma_2}$ if $\gamma_2 \circ \gamma_1^{-1}$ translates along ℓ to the right (resp. left).
Hence $(\Pi^\pm_l)\big|_{\mathrm{int}(A_i)}$ and $(\Pi^\pm_r)|_{\mathrm{int}(A_i)}$ are the restrictions of the projections associated with the totally geodesic plane P_{γ_i} just as seen in Example 5.7. In particular, $(\Pi^\pm_l)\big|_{\mathrm{int}(A_i)}$ and $(\Pi^\pm_r)|_{\mathrm{int}(A_i)}$ are the restriction to $\mathrm{int}(A_i)$ of global isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ (those defined by multiplication on the left or on the right by γ_i^{-1}) sending P_{γ_i} to P_{Id}, post-composed with the usual isometry $\mathrm{Fix}: P_{\mathrm{Id}} \to \mathbb{H}^2$. It follows that the restrictions of projections map geodesic of P_{γ_i} to geodesic of $\mathbb{H}^2$. More is true, due to Equation 5.2, $(\Pi^\pm_l)\big|_{\mathrm{int}(A_i)}$ maps $\mathrm{int}(\partial(A_i)) = \mathrm{graph}(\gamma_i^{-1}\big|_{\mathrm{int}(I_i)})$ to $\mathrm{int}(I_i)$. Hence $\Pi^\pm_l(\mathrm{int}(A_i)) = \mathrm{int}(D_i)$. In similar fashion, $\Pi^\pm_r(\mathrm{int}(A_i)) = \gamma_1^{-1}(\mathrm{int}(D_1)) = \gamma_2^{-1}(\mathrm{int}(D_2))$.
We want to show that the projections are bijective. To do so we will show that the image of the geodesic $A_1 \cap A_2 = P_{\gamma_1} \cap P_{\gamma_2}$, via $\Pi^\pm_l$ is the geodesic $\ell = D_1 \cap D_2$, while the image via $\Pi^\pm_r$ is the geodesic $\gamma_1^{-1}(\ell) = \gamma_2^{-1}(\ell)$. The definition of $\Pi^\pm_l$ and $\Pi^\pm_r$ on $A_1 \cap A_2$ *depends* on the choice of a support plane. We recall that we must choose the *same* support plane at any point $p \in A_1 \cap A_2$. Now, because of Corollary 5.10, the possible choices of support planes at p are all of the form $P_{\sigma\gamma_1}$, for some σ that has the same fixed points as $\gamma_2 \circ \gamma_1^{-1}$, which are precisely the common endpoint of I_1 and I_2.
We stay consistent with the notation of Lemma 5.8, thus the endpoints at infinity of $A_1 \cap A_2$ are the points (x, y) and (x', y') where x, x' are the fixed point of $\gamma_2 \circ \gamma_1^{-1}$ (and of σ). Again from Equation 5.2 we have (for any choice of σ as in the third item of Corollary 5.10) $\Pi^\pm_l(x, y) = x$ and $\Pi^\pm_l(x', y') = x'$. Since $\Pi^\pm_l$ is, as before, the restriction of an isometry between $P_{\sigma\gamma_1}$ and $\mathbb{H}^2$, it maps geodesics to geodesics, hence $\Pi^\pm_l(A_1 \cap A_2) = \ell$. Analogously $\Pi^\pm_r(x, y) = y$ and $\Pi^\pm_r(x', y') = y'$, from which it follows that $\Pi^\pm_l(A_1 \cap A_2) = \gamma_1^{-1}(\ell) = \gamma_2^{-1}(\ell)$.
We move now on item number two. Define $E := \Pi^-_r \circ (\Pi^-_l)^{-1}$, which is a bijection of $\mathbb{H}^2$. Consider the geodesic lamination of $\mathbb{H}^2$ composed by the sole geodesic ℓ. Hence the strata of ℓ are $\mathrm{int}D_1$, $\mathrm{int}(D_2)$ and ℓ. We will show that the comparison isometries $\mathrm{Comp}(S, S') := (E|_S)^{-1} \circ E|_{S'}$ translate to the right or to the left seen from one stratum to another, according to as $\gamma_2 \circ \gamma_1^{-1}$ translates to the left or to the right seen from D_1 to D_2.
Let us consider $S = \mathrm{int}(D_1)$ and $S' = \mathrm{int}(D_2)$. Then, by Example 5.7, E

equals γ_i^{-1} on $\mathrm{int}(D_i)$, because $(\Pi_l^{\pm})^{-1}(\mathrm{int}(D_i)) = \mathrm{int}(A_i) \subset P_{\gamma_i^{-1}}$.
Hence the comparison isometry $\mathrm{Comp}(\mathrm{int}(D_1), \mathrm{int}(D_2))$ equals $\gamma_1 \circ \gamma_2^{-1}$, and it translates to the left (resp. right) seen from $\mathrm{int}(D_1)$ to $\mathrm{int}(D_2)$ exactly when $\gamma_2 \circ \gamma_1^{-1}$, which is its inverse, translates to the right (resp. left). The proof when one of the two strata S or S' is ℓ is completely analogous, by using the third item of Corollary 5.10. Indeed (via Remark 4.11), by any choice of σ of the form $\sigma = \exp(t\mathfrak{a})$ with $t \in (0,1)$, $\mathrm{Comp}(\ell, \mathrm{int}(D_2)) = \sigma \circ \gamma_2^{-1}$ translates to the left (resp. right) seen from ℓ to $\mathrm{int}(D_2)$, and $\mathrm{Comp}(\mathrm{int}(D_1),\ell) = \gamma_1 \circ \sigma^{-1}$ translates to the left (resp. right) seen from $\mathrm{int}(D_1)$ to ℓ. If instead $\sigma = \exp(t\mathfrak{a})$ with $t \in \{0,1\}$, then σ coincides with γ_1 or with γ_2, hence on of the comparison isometries $\mathrm{Comp}(\mathrm{int}(D_1), \ell)$ and $\mathrm{Comp}(\mathrm{int}(D,2), \ell)$ translates to the left, while the other it the identity, which is still *allowed* in the definition of earthquake because ℓ is in the boundary on $\mathrm{int}(D_i)$. □

5.4 The example is prototypical

We have just treated what seems to be a very *special and convenient* simple earthquake. What we want to show now is that it is actually the prototypical example, that will serve to treat the general case of the earthquake theorem. The following lemma explains *how* the situation of two intersecting planes is actually pretty common.

Lemma 5.12. Let $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be an orientation-preserving homeomorphism which is not in $\mathrm{PSL}(2,\mathbb{R})$. Then:

- Any two support planes of $\mathcal{C}(\Lambda_\varphi)$ at points of $\partial_+\mathcal{C}(\Lambda_\varphi)$ intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Analogously, any two support planes of $\mathcal{C}(\Lambda_\varphi)$ at points of $\partial_-\mathcal{C}(\Lambda_\varphi)$ intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.
- Given a point $p \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$, if there exist two support planes at p, then their intersection (which is a spacelike geodesic) is contained in $\partial_\pm\mathcal{C}(\Lambda_\varphi)$. As a consequence, any other support plane at p contains this spacelike geodesic.

Proof. Let us consider future support planes, the other case being analogous. For the first item, let P and Q be support planes intersecting $\partial_+\mathcal{C}(\Lambda_\varphi)$, which are spacelike by Proposition 4.38, and suppose by contradiction that P and Q are disjoint. Then we can slightly move them in the future to get spacelike planes, P', Q' such that P, Q, P' and Q' are mutually disjoint and $P' \cap \partial_+\mathcal{C}(\Lambda_\varphi) = Q' \cap \partial_+\mathcal{C}(\Lambda_\varphi) = \varnothing$.
Now notice that the complement of $P' \cup Q'$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is the disjoint union of

two cylinders and P and Q lie in different connected components of this complement. However, $\partial_+\mathcal{C}(\Lambda_\varphi)$ is connected, and has empty intersection with P and Q, leading to a contradiction.
Let us move on to the second item. Let $P = P_{\gamma_1}$ and $Q = P_{\gamma_2}$ be support planes such that $p \in \partial_+\mathcal{C}(\Lambda_\varphi) \cap P \cap Q$. By Lemma 5.8, $\gamma_2 \circ \gamma_1^{-1}$ is loxodromic. Up to switching the roles of γ_1 and γ_2 we can assume that $\gamma_2 \circ \gamma_1^{-1}$ translates to the left seen from D_1 to D_2, where as usual D_i is the convex hull of the interval I_i, and the common endpoints x, x' of I_1 and I_2 are the fixed points of $\gamma_2 \circ \gamma_1^{-1}$. Hence $\partial P_{\gamma_1} \cap \partial P_{\gamma_2} = \{(x, y), (x', y')\}$ where $y = \gamma_1^{-1}(x) = \gamma_2^{-1}(x)$ and $y' = \gamma_1^{-1}(x') = \gamma_2^{-1}(x')$.
Now, via Equation 4.3, $P_{\gamma_i} \cap \mathrm{graph}(f)$ consists of at least two points for $i = 1, 2$. We claim that the aforementioned intersections contains at least (x, y) and (x', y'). Indeed, since P_{γ_2} is a support plane, $\mathcal{C}(\Lambda_\varphi) \cap P_{\gamma_1}$ is contained in the half-plane $A_1 \subset P_{\gamma_1}$. If $\mathrm{graph}(f) \cap P_{\gamma_1}$ had not contained (x, y) and (x', y'), then $\mathcal{C}(\Lambda_\varphi) \cap P_{\gamma_1}$ would not contain the boundary geodesic $A_1 \cap A_2$, and thus would not contain p. A *verbatim* argument holds also for P_{γ_2}. This shows that both (x, y) and (x', y') are in $\mathcal{C}(\Lambda_\varphi)$, and therefore the spacelike geodesic $P_{\gamma_1} \cap P_{\gamma_2}$ is in $\partial_\pm\mathcal{C}(\Lambda_\varphi)$.

$\square$

Recall that we have defined the left and right projections $\Pi_l^\pm, \Pi_r^\pm$, and they depended on the choice of a support plane at all points p that admit more than one support plane. Moreover, we require that this support plane is chosen to be constant on any connected component of the subset of $\partial_\pm\mathcal{C}(\Lambda_\varphi)$ consisting of points that admit more than one support plane. We want to show how the projections are related to this choice:

Corollary 5.13. *Let $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be an orientation-preserving homeomorphism which is not in $PSL(2, \mathbb{R})$, and suppose $p \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$ has at least two support planes. Then there exist $\gamma_1, \gamma_2 \in PSL(2, \mathbb{R})$ such that $\gamma_2 \circ \gamma_1^{-1} = \exp(\mathfrak{a})$ is a loxodromic element, such that all support planes at p are precisely those of the form $P_{\sigma\gamma_1}$ where $\sigma = \exp(t\mathfrak{a})$ for $t \in [0, 1]$. The same conclusion holds for all other point $p' \in P_{\gamma_1} \cap P_{\gamma_2}$.*
In particular, the image of the spacelike geodesic $P_{\gamma_1} \cap P_{\gamma_2}$ under the projections $\Pi_l^\pm$ and $\Pi_r^\pm$ is a geodesic in $\mathbb{H}^2$ that does not depend on the choice of the support plane as in the definition of the projections.

Proof. Suppose $P_{\widetilde{\gamma}_1}$ and $P_{\widetilde{\gamma}_2}$ are (say, future) distinct support planes at p. Write $\widetilde{\gamma}_2 \circ \widetilde{\gamma}_1^{-1} = \exp(\widetilde{\mathfrak{a}})$, which is a loxodromic element by Lemma 5.8 and the first item of Lemma 5.12. By the second item of Lemma 5.12, any other support plane at p must be of the form $P_{\sigma\widetilde{\gamma}_1}$ for σ an element having the same

fixed points as $\widetilde{\gamma}_2 \circ \widetilde{\gamma}_1^{-1}$. That is, σ is the form $\exp(s\widetilde{\mathfrak{a}})$ for some $s \in \mathbb{R}$.
We claim now that the set:

$$I = \{s \in \mathbb{R} \mid \exp(s\widetilde{\mathfrak{a}}) \text{ is a support plane of } \mathcal{C}(\Lambda_\varphi) \text{ at } p\}$$

is a compact interval. This would conclude the proof, up to applying an affine change of variable mapping the interval $I = [s_1, s_2]$ to $[0,1]$, and defining $\gamma_i = \exp(s_i\widetilde{\mathfrak{a}})$.
Let us prove the compactness of I, suppose $s, s' \in I$. Now $\mathcal{C}(\Lambda_\varphi)$ is contained in the past of a pleated surface obtained as the union of two half-spaces, one contained in $P_{\exp(s\widetilde{\mathfrak{a}})\widetilde{\gamma}_1}$ and the other in $P_{\exp(s'\widetilde{\mathfrak{a}})\widetilde{\gamma}_1}$, meeting along the spacelike geodesic $P_{\widetilde{\gamma}_1} \cap P_{\widetilde{\gamma}_2}$. Then every support plane for this pleated surface is a support plane for $\mathcal{C}(\Lambda_\varphi)$ as well. That is, by the last item of Corollary 5.10, $[s, s'] \subset I$. This shows that I is an interval. It is compact by Lemma 4.39, applied to the constant sequence $p_n = p$ and to $\gamma_n = \exp(s_n\widetilde{\mathfrak{a}})$, showing that s_n must by converging (up to subsequences) and its limit is in I. This concludes the proof.

□

5.5 Proof of the earthquake theorem

We state now two lemmas about the actions on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ of sequences of elements in $\mathrm{PSL}(2,\mathbb{R})$. We are in particular interested in the case of sequences of order-two elliptic isometries. We denote by $\mathcal{R}_w$ the order-two elliptic isometry of $\mathbb{H}^2$ that fixes $w \in \mathbb{H}^2$. The proofs of the lemmas are straightforward computations and can be found in the appendix of [4].

Lemma 5.14. Let w_n be a sequence in $\mathbb{H}^2$ converging to $w \in \mathbb{H}^2$. Then $\mathcal{R}_{w_n}$ converges to $\mathcal{R}_w$ uniformly on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Lemma 5.15. Let w_n be a sequence in $\mathbb{H}^2$ converging to $w \in \partial\mathbb{H}^2$. Then, for every neighbourhood U of w, there exists n_0 such that $\mathcal{R}_{w_n}((\mathbb{H}^2 \cup \partial\mathbb{H}^2)\setminus U) \subset U$ for $n \geqslant n_0$.

We have now all the tools that are required for the proof of the earthquake theorem. We outline the strategy that we will follow: given an orientation-preserving homeomorphism $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ (we can assume that it is not in $\mathrm{PSL}(2,\mathbb{R})$), we consider the projections $\Pi_l^\pm, \Pi_r^\mp : \partial_\pm\mathcal{C}(\Lambda_\varphi) \to \mathbb{H}^2$, and we want to show that the composition $\Pi_r^\pm \circ (\Pi_l^\pm)^{-1}$ is well-defined and is a (left or right) earthquake map extending φ. We are going to divide the proof in the following steps: Proposition 5.17, Corollary 5.18 and then Proposition 5.19 below.

5.6 Extension to the boundary

We study the extension of the projections $\Pi_l^{\pm}, \Pi_r^{\pm}$ to the boundary.

Proposition 5.16. *The projections $\Pi_l^{\pm}, \Pi_r^{\pm}$ extend to Λ_φ. More precisely, if $p_n \in \partial_{\pm}\mathcal{C}(\Lambda_\varphi) \to (x, y) \in \Lambda_\varphi$, then $\Pi_l^{\pm}(p_n) \to x$ and $\Pi_r^{\pm}(p_n) \to y$.*

Proof. Let $p_n \in \partial_{\pm}\mathcal{C}(\Lambda_\varphi)$ be a sequence converging to $(x, y) \in \Lambda_\varphi$, and let P_{γ_n} be a sequence of support planes of $\mathcal{C}(\Lambda_\varphi)$ at p_n, which are necessarily spacelike because of the results of Proposition 4.38. By Lemma 4.39, up to extracting a subsequence, there are two possibilities: either $\gamma_n \to \gamma$ and P_{γ_n} converges to the spacelike support plane P_γ, or γ_n diverges in $\mathrm{PSL}(2, \mathbb{R})$ and P_{γ_n} converges to the lightlike plane whose boundary is $(\{x\} \times \mathbb{RP}^1) \cup (\mathbb{RP}^1 \times \{y\})$. We treat the two cases separately, and we remand to Lemma 3.3 for our characterization of convergence to the boundary. We start by supposing the convergence $\gamma_n \to \gamma$. By hypothesis:

$$p_n(z_0) \to x \quad p_n^{-1}(z_0) \to y \tag{5.5}$$

for any $z_0 \in \mathbb{H}^2$. It also follows from the definition of the projections:

$$\Pi_l^{\pm}(p_n) = \mathrm{Fix}(p_n\gamma_n^{-1}) \text{ and } \Pi_r^{\pm}(p_n) = \mathrm{Fix}(\gamma_n^{-1}p_n). \tag{5.6}$$

We consider the identification of ∂P_{Id} with $\mathbb{RP}^1$ via $(x, x) \to x$. We thus have shown (choosing for instance the point $z_0 = i$) that: $p_n\gamma_n^{-1}(i) \to x$ and $\gamma_n^{-1}p_n(i) \to y$.
However, since $\gamma_n \to \gamma$, $p_n\gamma_n^{-1}(i)$ is at bounded distance from $p_n\gamma^{-1}(i)$. Applying the hypothesis, namely Equation 5.5, to $z_0 = \gamma^{-1}(i)$, we have $p_n\gamma^{-1}(i) \to x$ and therefore $p_n\gamma_n^{-1}(i) \to x$ as well.
The argument is analogous to show that $\gamma_n^{-1}p_n(i) \to y$, except that it is useful to observe that $\gamma_n^{-1}p_n = p_n^{-1}\gamma_n$ since p_n is an order-two isometry. Now $p_n^{-1}\gamma_n(i)$ is at bounded distance from $p_n^{-1}\gamma(i)$, which converges to y by hypothesis. Hence $p_n^{-1}\gamma_n(i) \to y$ as well.
Let us move to the latter case, namely γ_n diverges in $\mathrm{PSL}(2, \mathbb{R})$. Here we will use not only the assumption of Equation 5.5, but also:

$$\gamma_n(z_0) \to x \text{ and } \gamma_n^{-1}(z_0) \to y, \tag{5.7}$$

for any $z_0 \in \mathbb{H}^2$. The condition (5.7) holds because γ_n converges to the projective class of a rank one matrix A, such that $P_{[A]}$ is a lightlike support plane. We have already observed that the boundary at infinity of $P_{[A]}$ must be equal to $(\{x\} \times \mathbb{RP}^1) \cup (\mathbb{RP}^1 \times \{y\})$. Combining Lemma 3.3 and Lemma 3.13, we

deduce that $\gamma_n(z_0) \to x$ and $\gamma_n^{-1}(z_0) \to y$ as claimed.
With this observation, we can rewrite Equation 5.6 as the identities:

$$p_n = \mathcal{R}_{\Pi_l^{\pm}(p_n)} \circ \gamma_n \quad \text{and} \quad p_n^{-1} = \mathcal{R}_{\Pi_r^{\pm}(p_n)} \circ \gamma_n^{-1}, \tag{5.8}$$

where we recall that for us $\mathcal{R}_w$ denotes the order two elliptic isometry with fixed point $w \in \mathbb{H}^2$.
Up to extracting a subsequence, we can assume that $\Pi_l^{\pm}(p_n) \to \widehat{x}_{\pm}$ and $\Pi_r^{\pm}(p_n) \to \widehat{y}_{\pm}$, for some points $\widehat{x}_{\pm}, \widehat{y}_{\pm} \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$. We need to show that $\widehat{x}_{\pm} = x$ and $\widehat{y}_{\pm} = y$.
In this direction, suppose by contradiction $\widehat{x}_{\pm} \neq x$. Suppose first that $\widehat{x}_{\pm} \in \mathbb{H}^2$. We will use the statement of Lemma 5.14 that if $w_n \to w \in \mathbb{H}^2$, then $\mathcal{R}_{w_n}$ converges to $\mathcal{R}_{w_n}$ uniformly on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. From Equation 5.8, and the fact that, from Equations 5.5 and 5.7, both $p_n(z_0)$ and $\gamma_n(z_0)$ converge to x, we would then have:

$$x = \lim_n p_n(z_0) = \lim_n \mathcal{R}_{\Pi_l^{\pm}(p_n)}(\gamma_n(z_0)) = \mathcal{R}_{\widehat{x}_{\pm}}(x) \neq x$$

since $\mathcal{R}_{\widehat{x}_{\pm}}$ does not have fixed points on $\partial\mathbb{H}^2$, thus giving a contradiction. If $\widehat{y}_{\pm} \in \mathbb{H}^2$, the same argument works flawlessly.
Lastly suppose $\widehat{x}_{\pm} \in \partial\mathbb{H}^2$, in this case, because of Lemma 5.15, we can find a neighbourhood U of $\widehat{x}_{\pm}$ not containing x, such that for n large $\mathcal{R}_{\Pi_l^{\pm}(p_n)}$ maps the complement of U inside U. This gives rise to a contradiction with the condition (5.8) because $p_n(z_0)$ and $\gamma_n(z_0)$ are in the complement of U for n large, but at the same time $\mathcal{R}_{\Pi_l^{\pm}(p_n)}(\gamma_n(z_0))$ should be in U for n large. The argument for $\widehat{y}$ is the same *verbatim*. $\square$

We want to remark that the conclusion of Proposition 5.16 holds for *any choice* of the projections, regardless of the chosen support planes where several choices are available.

Proposition 5.17. *The projections $\Pi_l^{\pm}, \Pi_r^{\pm} : \partial_{\pm}\mathcal{C}(\Lambda_\varphi) \to \mathbb{H}^2$ are bijections.*

Proof. We will prove the statement for $\Pi_l^{\pm}$, the proof for $\Pi_r^{\pm}$ being analogous. We start by showing that $\Pi_l^{\pm}$ is injective. Given $p_1, p_2 \in \partial_{\pm}\mathcal{C}(\Lambda_\varphi)$, let P_{γ_1} and P_{γ_2} be support planes at p_1 and p_2 respectively. Now by Lemma 5.8 and Lemma 5.12, $\gamma_2 \circ \gamma_1^{-1}$ is loxodromic; let D_1, D_2 be the convex envelopes in $\mathbb{H}^2$ of the two intervals I_1 and I_2 with endpoints the fixed points of $\gamma_2 \circ \gamma_1^{-1}$. Up to switching γ_1 and γ_2, we can assume that $\gamma_2 \circ \gamma_1^{-1}$ translates to the left seen from D_1 to D_2.
Now, we will refer to the fundamental example of the previous section. Let

$\varphi^+_{\gamma_1,\gamma_2}$ be defined as in (5.4). By Corollary 5.10, P_{γ_i} is the support plane of $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ at the point $p_i \in \partial_\pm\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$ for $i = 1, 2$.
It follows that $\Pi^\pm_l(p_i) = \hat{\Pi}^\pm_l(p_i)$, where we denote by $\hat{\Pi}^\pm_l$ the left projection associated with $\mathcal{C}(\varphi^+_{\gamma_1,\gamma_2})$. Now both $\hat{\Pi}^\pm_l(p_i)$ are bijective by Proposition 5.11, $\Pi^\pm_l(p_1) \neq \Pi_\pm(p_2)$. This shows injectivity.
We move on now on surjectivity, we will first show that the image is closed. Suppose $z_n = \Pi^\pm_l(p_n) \to z \in \mathbb{H}^2$. Up to extracting a subsequence, we can assume $p_n \to p \in \partial_\pm\mathcal{C}(\Lambda_\varphi) \cup \Lambda_\varphi$. From Proposition 5.16, we have that $p \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$, because if $p = (x, y) \in \Lambda_\varphi$, then $\Pi_l(p_n) \to x \in \partial\mathbb{H}^2$, which is a contradiction with the convergence of $z_n \to z \in \mathbb{H}^2$. Now, let P_{γ_n} be a support plane at the point p_n, which is spacelike by Proposition 4.38. By Lemma 4.39, up to extracting a subsequence, $\gamma_n \to \gamma \in \mathrm{PSL}(2,\mathbb{R})$ and P_γ is a spacelike support plane at p. We remark that the convergent plane might not be the one decided in the definition of the projections, but this does not change the image by Corollary 5.13. Hence we can assume that P_γ is the support plane chosen at p. It means that from Equation 5.1, $\Pi^\pm_l(p) = \mathrm{Fix}(p \circ \gamma^{-1})$. We can now conclude that z is in the image of $\Pi^\pm_l$: on one hand $z_n = \Pi^\pm_l(p_n) = \mathrm{Fix}(p_n \circ \gamma_n^{-1})$ converges to z by hypothesism and on the other hand it converges to $\Pi^\pm_l(p) = \mathrm{Fix}(p \circ \gamma^{-1})$ because $p_n \to p$ and $\gamma_n \to \gamma$ and Fix is continuous. This shows that $z \in \Pi^\pm_l(\partial_\pm\mathcal{C}(\Lambda_\varphi))$, and thus the image is closed.
We are ready for the surjectivity. Suppose by contradiction that there is a point $w \in \mathbb{H}^2$ which is not in the image of the projections $\Pi^\pm_l$. Let $r_0 = \inf\{r \mid B(w, r) \cap \Pi^\pm_l(\partial_\pm\mathcal{C}(\varphi)) \neq \varnothing\}$, where $B(w, r)$ is an open ball centered at w of radius r with respect to the hyperbolic metric on $\mathbb{H}^2$. Since the image of $\Pi^\pm_l$ is closed, we have that $r_0 > 0$, $B(w, r_0)$ is disjoint from the image of $\Pi^\pm_l$, and there exists a point $z \in \partial B(w, r_0)$ which is in the image of $\Pi^\pm_l$. Say that $z = \Pi^\pm_l(p)$. We will obtain a contradiction by finding points close to p which are mapped by $\Pi^\pm_l$ inside $B(w, r_0)$.

Let P_γ be a support plane of $\mathcal{C}(\Lambda_\varphi)$ at p. By (4.3), $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$ is the convex hull of $\partial_\infty P_\gamma \cap \Lambda_\varphi$, which contains at least two points. If p is in the interior of $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$ (which is non-empty if and only if $\partial_\infty P_\gamma \cap \Lambda_\varphi$ contains at least three points), then the restriction of $\Pi^\pm_l$ to the interior of $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$ is an isometry onto its image in $\mathbb{H}^2$, because P_γ is the unique support plane at interior points p', and $\Pi^\pm_l(p') = \mathrm{Fix}(p' \circ \gamma^{-1})$. Hence $\Pi^\pm_l$ maps a small neighbourhood of p to a neighbourhood of z, which intersects $B(w, r_0)$, giving a contradiction. We are only left with the case where p is not in the interior of $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$. In this case, there is a geodesic L contained in $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$ such that $p \in L$. (The geodesic L might be equal to $P_\gamma \cap \mathcal{C}(\Lambda_\varphi)$ or not.) As before, the image of L is a geodesic ℓ in $\mathbb{H}^2$ because $(\Pi^\pm_l)|_L$ is an isometry onto its image, and $z \in \ell$. We claim that in the image of $\Pi_l^\pm$ there are two sequences of geodesics $\ell_n \subset \mathrm{Im}(\Pi_l^\pm)$ converging to ℓ (in other words, such that the endpoints of ℓ_n converge to the endpoint of ℓ); moreover the two sequences are contained in different connected components of $\mathbb{H}^2 \setminus \ell$. This will give a contradiction, because for one of these two sequences, ℓ_n must intersect $B(w, r_0)$ for n large.

To show the claim, and thus conclude the proof, observe that L disconnects $\partial_\pm \mathcal{C}(\Lambda_\varphi)$ in two connected components, and let p_n be a sequence converging to p contained in one connected component of $\partial_\pm \mathcal{C}(\Lambda_\varphi) \setminus L$. Let P_{γ_n} be the support plane for $\mathcal{C}(\Lambda_\varphi)$ at p_n which has been chosen to define $\Pi_l^\pm$. By Lemma 4.39, P_{γ_n} converges to a support plane P_γ at p, which as before we can assume is the support plane that defined $\Pi_l^\pm$ at p, since the image does not depend on this choice by Corollary 5.13. Also, we can assume that each p_n is contained in a geodesic L_n in $P_{\gamma_n} \cap \partial_\pm \mathcal{C}(\Lambda_\varphi)$: indeed, it suffices to replace p_n by the point in $P_{\gamma_n} \cap \partial_\pm \mathcal{C}(\Lambda_\varphi)$ which is closest to p (where closest is with respect to the induced metric on $\partial_\pm \mathcal{C}(\Lambda_\varphi)$, or to any auxiliary Riemannian metric). If $P_{\gamma_n} \cap \partial_\pm \mathcal{C}(\Lambda_\varphi)$ is not already a geodesic, with this assumption p_n now belongs to a boundary component which is the geodesic L_n. As observed before, $\Pi_l^\pm$ maps the geodesic L_n to a geodesic $\ell_n = \Pi_l^\pm(L_n)$ in $\mathbb{H}^2$, and (as in the argument that showed that $\mathrm{Im}(\Pi_l^\pm)$ is closed), the limit of $\Pi_l^\pm(p_n)$ is a point in $\ell = \Pi_l^\pm(L)$.

Moreover $\ell_n \cap \ell = \varnothing$, and the ℓ_n are all contained in the same connected component of $\mathbb{H}^2 \setminus \ell$: this follows from observing again (compare with the injectivity at the beginning of this proof) that $(\Pi_l^\pm)|_{L_n \cup L}$ equals the left projection associated with the surface $\partial_\pm \mathcal{C}(\varphi^+_{\gamma_n,\gamma})$ studied in Section 5.3, where $\varphi^+_{\gamma_1,\gamma_2}$ is defined as in (5.4), and thus maps $\partial_\pm \mathcal{C}(\Lambda_\varphi) \cap P_{\gamma_n}$ (which in particular contains L_n) to a subset (containing ℓ_n) disjoint from ℓ and included in a connected component of $\mathbb{H}^2 \setminus \ell$ which does not depend on n.

This implies that ℓ_n converges to ℓ as $n \to +\infty$. Clearly if we had chosen p_n in the other connected component of $\partial_\pm \mathcal{C}(\Lambda_\varphi) \setminus L$, then the ℓ_n would be contained in the other connected component of $\mathbb{H}^2 \setminus \ell$. This concludes the claim and thus the proof. $\square$

As a direct consequence of Proposition 5.17, the composition $\Pi_r^\pm \circ (\Pi_l^\pm)^{-1}$ is well-defined and is a bijection of $\mathbb{H}^2$ to itself. This and Proposition 5.16 let us show the following:

Corollary 5.18. *The composition $\Pi_r^\pm \circ (\Pi_l^\pm)^{-1}$ extends to a bijection from $\mathbb{H}^2 \cup \partial \mathbb{H}^2$ to itself, which equals φ on $\partial \mathbb{H}^2$ and is continuous at any point of $\partial \mathbb{H}^2$.*

Proof. Since $\Pi_l^\pm$ and $\Pi_r^\pm$ are bijective and extend to the bijections from Λ_φ to $\partial\mathbb{H}^2$ sending $(x,y) \to x$ and $(x,y) \to \varphi(x)$ respectively, the composition $\Pi_r^\pm \circ (\Pi_l^\pm)^{-1}$ extends to a bijection of $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ to itself sending $x \to \varphi(x)$.
We need to check continuity. Proposition 5.16 shows that the extensions $\Pi_l^\pm$ and $\Pi_r^\pm$ to $\partial_\pm\mathcal{C}(\Lambda_\varphi) \cup \Lambda_\varphi$ are continuous at any point of Λ_φ. Hence it remains to check that $(\Pi_l^\pm)^{-1}$ is continuous at any point of $\partial\mathbb{H}^2$.
This is pretty straightforward: let z_n be a sequence in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ converging to $x \in \partial\mathbb{H}^2$, and let $p_n = (\Pi_l^\pm)^{-1}(z_n)$. Up to extracting a subsequence, $p_n \to p$. The limit p must be in Λ_φ because if $p \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$, although $\Pi_l^\pm$ might not be continuous there, we have already seen in Proposition 5.17 (in the passage about closeness of the image of $\Pi_l^\pm$) that $\lim_n \Pi_l^\pm(p_n) = \lim_n z_n$ is a point of $\mathbb{H}^2$, thus giving a contradiction with $\lim_n z_n = x \in \partial\mathbb{H}^2$.
If $p \in \Lambda_\varphi$, then we can use the continuity and injectivity of $\Pi_l^\pm$ on Λ_φ to infer that $p = (\Pi_l^\pm)^{-1}(x)$.

□

We are only left with verification that $\Pi_r^\pm \circ (\Pi_l^\pm)^{-1}$ satisfies the defining properties of an earthquake map.

Proposition 5.19. *The composition $\Pi_r^- \circ (\Pi_l^-)^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is a left earthquake map. Analogously, $\Pi_r^+ \circ (\Pi_l^+)^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is a right earthquake map.*

Proof. Let us start by defining a geodesic lamination λ. Let us consider all the support planes P_γ of $\mathcal{C}(\Lambda_\varphi)$ at points of $\partial_\pm\mathcal{C}(\Lambda_\varphi)$ (which we recall must be spacelike by Proposition 4.38). Define $\mathcal{L}$ to be the collection of all the connected components of $(P_\gamma \cap \partial_\pm\mathcal{C}(\Lambda_\varphi))\backslash\mathrm{int}(P_\gamma \cap \partial_\pm\mathcal{C}(\Lambda_\varphi))$, as P_γ varies over all support planes. As observed before, by Equation 4.3, $P_\gamma \cap \partial_\pm\mathcal{C}(\Lambda_\varphi)$ is the convex hull in P_γ of $\partial P_\gamma \cap \Lambda_\varphi$, which consists of at least two points. If it consists of exactly two points, then $P_\gamma \cap \partial_\pm\mathcal{C}(\Lambda_\varphi)$ is a spacelike geodesic L; otherwise $P_\gamma \cap \partial_\pm\mathcal{C}(\Lambda_\varphi)$ has nonempty interior and each connected component of its boundary is a spacelike geodesic. Now, $\Pi_l^\pm$ is an isometry onto its image when restricted to any $L \in \mathcal{L}$. Hence we define λ to be the collection of all the $\Pi_l^\pm(L)$ as L varies over $\mathcal{L}$.
To show that λ is a geodesic lamination, we first observe that the geodesic $\ell \in \lambda$ are pairwise disjoint, because the spacelike geodesics L in $\mathcal{L}$ are pairwise disjoint and $\Pi_l^\pm$ is injective. It remains to show that their union is a closed subset of $\mathbb{H}^2$. But this follows immediately from the proof of Proposition 5.17. Indeed, suppose that $\ell_n = \Pi_l^\pm(L_n)$ converges to $\ell = \Pi_l^\pm(L)$, and let $z_n = \Pi_l^\pm(p_n) \in \ell_n$ be a sequence converging to $z \in \ell$. Since $\mathrm{Im}(\Pi_l^\pm)$ is closed, $z \in \mathrm{Im}(\Pi_l^\pm)$, and given the injectivity of $\Pi_l^\pm$, $z = \Pi_l^\pm(p)$ for some $p \in L$. In the last part of the proof of Proposition 5.17 we have shown that in this situation

ℓ_n converges to ℓ.
Having shown that λ is a geodesic lamination, we are ready to check that $\Pi_r^- \circ (\Pi_l^-)^{-1}$ is an earthquake map. Observe that the gaps of λ are precisely the images under $\Pi_l^\pm$ of the interior of the sets $P_\gamma \cap \partial_\pm \mathcal{C}(\Lambda_\varphi)$ (when this intersection is not reduced to a geodesic), as P_γ varies among all the support planes.
Let S_1 and S_2 be two strata of λ, and let $\Sigma_i = (\Pi_l^\pm)^{-1}(S_i)$. Hence $\Sigma_i \subset P_{\gamma_i} \cap \partial_\pm \mathcal{C}(\Lambda_\varphi)$, where P_{γ_i} is a support plane. As usual, there could be several support planes at points of Σ_i, and this can occur only if Σ_i is reduced to a geodesic by Lemma 5.12. Recalling that the chosen support plane is assumed to be constant along Σ_i, we can suppose that P_{γ_i} is the support plane chosen in the definition of $\Pi_l^\pm$ and $\Pi_r^\pm$.
Now we proceed as in the proof of injectivity in Proposition 5.17. Consider first the case that $\gamma_1 \neq \gamma_2$. By Lemma 5.8, $\gamma_2 \circ \gamma_1^{-1}$ is a loxodromic isometry; let D_1 and D_2 be the convex envelopes in $\mathbb{H}^2$ of the two intervals I_1 and I_2 with endpoints the fixed points of $\gamma_2 \circ \gamma_1^{-1}$. Up to switching γ_1 with γ_2, we assume that $\gamma_2 \circ \gamma_1^{-1}$ translates to the left seen from D_1 to D_2. Then $\Pi_l^\pm|_{\Sigma_i} = (\widehat{\Pi})_l^\pm\Big|_{\Sigma_i}$ and $\Pi_r^\pm|_{\Sigma_i} = (\widehat{\Pi})_r^\pm\Big|_{\Sigma_i}$, where $\widehat{\Pi}_l^\pm$ and $\widehat{\Pi}_r^\pm$ are the left and right projections associated with $\mathcal{C}(\Lambda_{\varphi^+_{\gamma_1,\gamma_2}})$ and moreover $S_i \subset D_i$.
Just as the case we treated in Section 5.3, the comparison isometry $\widehat{\mathrm{Comp}}(D_1, D_2)$ of the map $\widehat{\Pi}_r^\pm \circ (\widehat{\Pi}_l^\pm)^{-1}$ translates to the left (for Π_r^- and Π_l^-) or right (for Π_r^+ and Π_l^+) seen from D_1 to D_2. Then $\mathrm{Comp}(S_1, S_2)$, which is indeed equal to $\widehat{\mathrm{Comp}}(D_1, D_2)$, translates to the left (or right) seen from S_1 to S_2.
Finally, we consider the case $\gamma_1 = \gamma_2$, which can only happen either if $\Sigma_1 = \Sigma_2$ (hence $S_1 = S_2$) or if Σ_1 has nonempty interior and Σ_2 is one of its boundary components (or vice versa exchanging Σ_1 with Σ_2). In this case we already have that $\mathrm{Comp}(S_1, S_2) = \mathrm{Id}$. But the comparison isometry is allowed to be the identity, when one of the two strata is contained in the closure of the other. This concludes the proof.

□

Recovering earthquakes of closed surfaces

We have proved Thurston's theorem. We would like to recover the *original* earthquake theorem due to Nielsen and recover the existence of earthquake maps between two homeomorphic closed hyperbolic surfaces.
We recall briefly the definition of equivariance given in Section 4.3.

Corollary 5.20. *Let S be a closed oriented surface and let $\rho, \varrho : \pi_1(S) \to$*

$PSL(2,\mathbb{R})$ be two Fuchsian representations. Then there exists a $(\rho,\varrho)-$equivariant left earthquake map of $\mathbb{H}^2$, and a $(\rho,\varrho)-$equivariant right earthquake map.

Proof. Let $\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ be the unique (ρ,ϱ)-equivariant orientation-preserving homeomorphism. We claim that there exists a left (resp. right) earthquake which extends φ and is itself (ρ,ϱ)-equivariant.
Observe that for any $g \in \pi_1(S)$, the pair $(\rho(g),\varrho(g)) \in \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ acts on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ preserving Λ_φ, in fact the definition of $(\rho,\varrho)-$equivariancy and Equation 3.6:

$$(\rho(g),\varrho(g)) \cdot \Lambda_\varphi = \Lambda_{\varrho(g)\circ f\circ\rho^{-1}(g)} = \Lambda_\varphi.$$

Hence, the convex hull of Λ_φ is preserved by the action of $(\rho(g),\varrho(g))$ for every loop $g \in \pi_1(S)$. In order to conclude the proof we need that we can choose support planes at every point of both boundary components of $\mathcal{C}(\Lambda_\varphi)\backslash\Lambda_\varphi$ in such a way that the choice of support planes is preserved by the action of $(\rho(g),\varrho(g))$ (we need to verify the condition only for those points that admits more than one support plane, because if $p \in \partial_\pm\mathcal{C}(\Lambda_\varphi)$ has a unique support plane P, then $(\rho(g),\varrho(g)) \cdot P$ is the unique support plane at $(\rho(g),\varrho(g)) \cdot p$.
After having proven the last statement we will just consider left and right projection Π_l, Π_r defined via this invariant choice of support planes. By equivariance of the Gauss map, we will then deduce that the left projection is equivariant with respect to the action of $(\rho(g),\varrho(g))$ on $\partial_\pm\mathcal{C}(\Lambda_\varphi)$ and the action of $\rho(g)$ on $\mathbb{H}^2$. An analogous statement holds for the right projection where $\varrho(g)$ is acting on $\mathbb{H}^2$. Following the proof of the earthquake theorem, left and right earthquake will be obtained considering the composition of left and right projection (choosing to invert the respective projection), and such a composition will be (ρ,ϱ)-equivariant, holding the statement of the corollary.
Suppose $p \in \mathcal{C}(\Lambda_\varphi)$ admits several support planes. By Lemma 5.12, there is a spacelike geodesic $L \subset \partial_\pm\mathcal{C}(\Lambda_\varphi)$ containing p. Let $g \in \pi_1(S)$ be such that $(\rho(g),\varrho(g)) \cdot L = L$. Then we claim that $(\rho(g),\varrho(g))$ maps every support plane at p to itself. To prove this claim, we use Corollary 5.13 and suppose up to an isometry (so that, following notation of Corollary 5.13 we have $\gamma_1 = \mathrm{Id}$) that all the support planes at p are of the form $P_{\exp(t\mathfrak{a})}$ for $t \in [0,1]$ and $\gamma := \exp(\mathfrak{a})$ is a loxodromic element of $\mathrm{PSL}(2,\mathbb{R})$. Now the action of $(\rho(g),\varrho(g))$ must preserve the pair of *extreme* support planes P_{Id} and P_γ. Therefore there are only two possibilities: either $(\rho(g),\varrho(g))$ maps Id to Id and γ to γ or the two points get switched. However the latter scenario is not possible, since the identities $\rho(g)\varrho\mathrm{Id}(g)^{-1} = \gamma$ and $\rho(g)\varrho(g)^{-1} = \mathrm{Id}$ would imply that γ has order two, a contradiction for a loxodormic element of $\mathrm{PSL}(2,\mathbb{R})$. We thus have $(\rho(g),\varrho(g)) \cdot \mathrm{Id} = \mathrm{Id}$ and $(\rho(g),\varrho(g)) \cdot \gamma = \gamma$. This

implies $\rho(g) = \varrho(g)$ and $\rho(g)\gamma\rho(g)^{-1}$, hence $\rho(g) = \varrho(g) = \exp(s\mathfrak{a})$ for some $s \in \mathbb{R}$. It follows that $\rho(g)\exp(t\mathfrak{a})\rho(g)^{-1} = \exp(t\mathfrak{a})$ for all t, that is the action of $(\rho(g), \varrho(g)) = (\rho(g), \rho(g))$ maps every support plane $P_{\exp(t\mathfrak{a})}$ to itself.
Having shown the claim, we can conclude as follows. Observe that the set of points $p \in \partial_\pm \mathcal{C}(\Lambda_\varphi)$ that admit several support planes form a disjoint union of spacelike geodesics in $\partial_\pm \mathcal{C}(\Lambda_\varphi)$, and that this set (say X) is invariant under the action of $(\rho(g), \varrho(g))$ for all $g \in \pi_1(S)$. Pick a subset $\{L_i\}_{i\in I}$ of this family of geodesics such that its $\pi_1(S)$-orbit is X, and that the orbits of L_i and L_j are disjoint if $i \neq j$. Pick a support plane P_i at $p \in L_i$, and then we declare that $(\rho(g_0), \varrho(g_0)) \cdot P_i$ is the chosen support plane at every point of $(\rho(g_0), \varrho(g_0)) \cdot L_i$. This choice is well-defined by the above claim, which showed that if $(\rho(g), \varrho(g))$ leaves L_i invariant, then it also leaves every support plane at L_i invariant. Moreover, this choice of support planes is invariant by the action of $\pi_1(S)$ by construction. This concludes the proof. □

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