Some concordance invariants from knot Floer homology

DANIELE CELORIA
Topology seminars
Mathematical Institute
University of Oxford

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

There is really a overabundance of concordance invariants one can extract from $CFK^\infty(K)$:

- τ-invariant (O-Sz/Rasmussen)
- d-invariant of surgeries (O-Sz/Rasmussen/Peters)
- δ-invariant of branched cover (Manolescu-Owens)
- ν^+-invariant (Hom-Wu)
- ε-invariant (Hom)
- $\Upsilon(t)$ invariants (O-Stipsicz-Sz)
- $\Upsilon^2(t)$ secondary Upsilon invariants (Kim-Livingston)
- Twisted correction terms (Rubermann-Levine/Behrens-Golla)
- Involutive correction terms (Hendricks-Manolescu)
- ...

There is really a plethora of concordance invariants one can extract from $CFK^\infty(K)$:

- τ-invariant (O-Sz/Rasmussen)
- d-invariant of surgeries (O-Sz/Rasmussen/Peters)
- δ-invariant of branched cover (Manolescu-Owens)
- ν^+-invariant (Hom-Wu)
- ε-invariant (Hom)
- $\Upsilon(t)$ invariants (O-Stipsicz-Sz)
- $\Upsilon^2(t)$ secondary Upsilon invariants (Kim-Livingston)
- Twisted correction terms (Rubermann-Levine/Behrens-Golla)
- Involutive correction terms (Hendricks-Manolescu)
- ...

Slice genus and concordances

Denote by $\mathcal{K}$ the set of knots in S^3; there are two integers we can associate to $K \in \mathcal{K}$:

- **Seifert genus** $g(K)$ (3-dimensional),

  Minimal genus among orientable surfaces bounding K in S^3.

- **Smooth slice genus** $g_*(K)$ (4-dimensional).

  Minimal genus of smooth, orientable surfaces bounding K, properly embedded in $(\mathbb{D}^4, S^3)$.

Clearly $g_*(K) \leq g(K)$, and $g(K)$ is additive under connected sum. Instead g_* induces an equivalence relation on $\mathcal{K}$:

$$K_0 \sim K_1 \iff g_*(K_0 \# \overline{K}_1) = 0$$

i.e. if the connected sum bounds a smooth disk in $\mathbb{D}^4$.

Slice genus and concordances

Denote by $\mathcal{K}$ the set of knots in S^3; there are two integers we can associate to $K \in \mathcal{K}$:

- **Seifert genus** $g(K)$ (3-dimensional),

  Minimal genus among orientable surfaces bounding K in S^3.

- **Smooth slice genus** $g_*(K)$ (4-dimensional).

  Minimal genus of smooth, orientable surfaces bounding K, properly embedded in $(\mathbb{D}^4, S^3)$.

Clearly $g_*(K) \leq g(K)$, and $g(K)$ is additive under connected sum. Instead g_* induces an equivalence relation on $\mathcal{K}$:

$$K_0 \sim K_1 \iff g_*(K_0 \# \overline{K}_1) = 0$$

i.e. if the connected sum bounds a smooth disk in $\mathbb{D}^4$.

Slice genus and concordances

Denote by $\mathcal{K}$ the set of knots in S^3; there are two integers we can associate to $K \in \mathcal{K}$:

- **Seifert genus** $g(K)$ (3-dimensional),

  Minimal genus among orientable surfaces bounding K in S^3.
- **Smooth slice genus** $g_*(K)$ (4-dimensional).

  Minimal genus of smooth, orientable surfaces bounding K, properly embedded in $(\mathbb{D}^4, S^3)$.

Clearly $g_*(K) \leq g(K)$, and $g(K)$ is additive under connected sum. Instead g_* induces an equivalence relation on $\mathcal{K}$:

$$K_0 \sim K_1 \iff g_*(K_0 \# \overline{K}_1) = 0$$

i.e. if the connected sum bounds a smooth disk in $\mathbb{D}^4$.

Slice genus and concordances

Denote by $\mathcal{K}$ the set of knots in S^3; there are two integers we can associate to $K \in \mathcal{K}$:

- **Seifert genus** $g(K)$ (3-dimensional),

  Minimal genus among orientable surfaces bounding K in S^3.
- **Smooth slice genus** $g_*(K)$ (4-dimensional).

  Minimal genus of smooth, orientable surfaces bounding K, properly embedded in $(\mathbb{D}^4, S^3)$.

Clearly $g_*(K) \leq g(K)$, and $g(K)$ is additive under connected sum. Instead g_* induces an equivalence relation on $\mathcal{K}$:

$$K_0 \sim K_1 \iff g_*(K_0 \# \overline{K}_1) = 0$$

i.e. if the connected sum bounds a smooth disk in $\mathbb{D}^4$.

Slice genus and concordances

Denote by $\mathcal{K}$ the set of knots in S^3; there are two integers we can associate to $K \in \mathcal{K}$:

- **Seifert genus** $g(K)$ (3-dimensional),

  Minimal genus among orientable surfaces bounding K in S^3.
- **Smooth slice genus** $g_*(K)$ (4-dimensional).

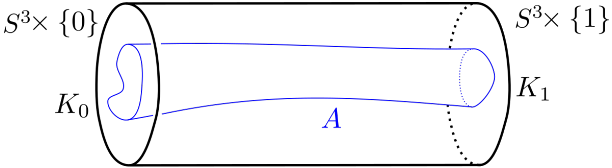
  Minimal genus of smooth, orientable surfaces bounding K, properly embedded in $(\mathbb{D}^4, S^3)$.

Clearly $g_*(K) \leq g(K)$, and $g(K)$ is additive under connected sum. Instead g_* induces an equivalence relation on $\mathcal{K}$:

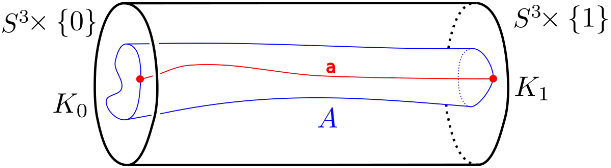
$$K_0 \sim K_1 \iff g_*(K_0 \# \overline{K}_1) = 0$$

i.e. if the connected sum bounds a smooth disk in $\mathbb{D}^4$.

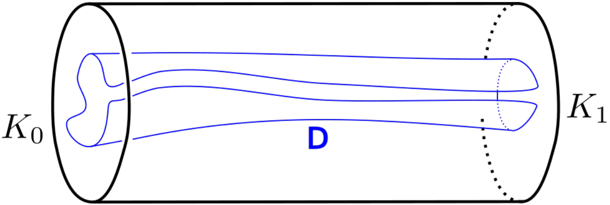
Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:



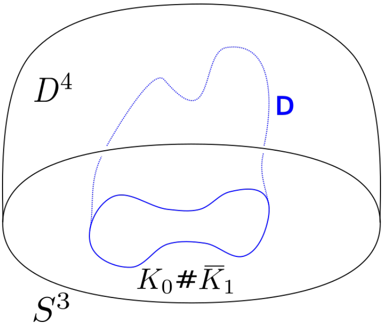
Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:



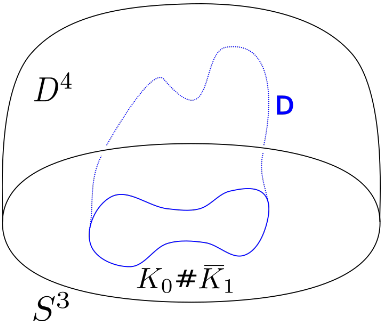
Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:



Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:

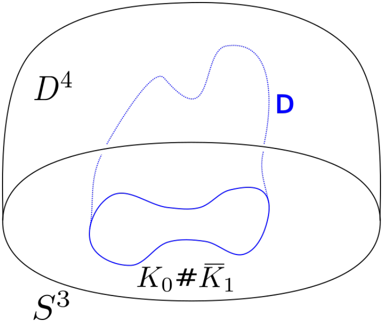


Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:



Connected sum endows $\mathcal{C} = {}^{\mathcal{K}}\!/\!_{\sim}$ with a group operation, and $(\mathcal{C}, \#)$ is known as the (smooth) **concordance group**.

Equivalently $K_0 \sim K_1$ (are **concordant**) *iff* there exists a smooth properly embedded annulus $A \subset S^3 \times [0,1]$ cobounding $K_0 \sqcup \overline{K}_1$:



Connected sum endows $\mathcal{C} = {}^{\mathcal{K}}\!/_{\sim}$ with a group operation, and $(\mathcal{C}, \#)$ is known as the (smooth) **concordance group**.

$\mathcal{C}$ is **big** (contains $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$), and its structure is still misterious.

Knot Floer homology

In 2001 Ozsváth-Szabó introduce **Heegaard Floer homology** for closed 3-manifolds. They associate to a Spin^c 3-manifold $(Y, \mathfrak{s})$ a collection of complexes:

$$(CF^\circ(Y, \mathfrak{s}), \partial^\circ) \text{ where } \circ = \wedge, +, -, \infty.$$

The homologies $HF^\circ(Y, \mathfrak{s})$ are invariants of the pairs, and contain many useful information on the manifold.

We will focus on the **minus** and **infinity** flavours. These are a finitely generated $\mathbb{F}[U]$ and $\mathbb{F}[U^{\pm 1}]$ complexes respectively.
Shortly after the definition of HF, O-Sz and Rasmussen realised that a (nullhomologous) knot K in Y induces a **filtration** on the complexes computing the Heegaard Floer homology of Y.
The resulting filtered quasi-isomorphism type is a knot invariant, the **knot Floer homology** $HFK^\circ(Y, K, \mathfrak{s})$.

Knot Floer homology

In 2001 Ozsváth-Szabó introduce **Heegaard Floer homology** for closed 3-manifolds. They associate to a Spin^c 3-manifold $(Y, \mathfrak{s})$ a collection of complexes:

$$(CF^\circ(Y, \mathfrak{s}), \partial^\circ) \text{ where } \circ = \wedge, +, -, \infty.$$

The homologies $HF^\circ(Y, \mathfrak{s})$ are invariants of the pairs, and contain many useful information on the manifold.

We will focus on the **minus** and **infinity** flavours. These are a finitely generated $\mathbb{F}[U]$ and $\mathbb{F}[U^{\pm 1}]$ complexes respectively.
Shortly after the definition of HF, O-Sz and Rasmussen realised that a (nullhomologous) knot K in Y induces a **filtration** on the complexes computing the Heegaard Floer homology of Y.
The resulting filtered quasi-isomorphism type is a knot invariant, the **knot Floer homology** $HFK^\circ(Y, K, \mathfrak{s})$.

Knot Floer homology

In 2001 Ozsváth-Szabó introduce **Heegaard Floer homology** for closed 3-manifolds. They associate to a Spin^c 3-manifold $(Y, \mathfrak{s})$ a collection of complexes:

$$(CF^\circ(Y, \mathfrak{s}), \partial^\circ) \text{ where } \circ = \wedge, +, -, \infty.$$

The homologies $HF^\circ(Y, \mathfrak{s})$ are invariants of the pairs, and contain many useful information on the manifold.

We will focus on the **minus** and **infinity** flavours. These are a finitely generated $\mathbb{F}[U]$ and $\mathbb{F}[U^{\pm 1}]$ complexes respectively.
Shortly after the definition of HF, O-Sz and Rasmussen realised that a (nullhomologous) knot K in Y induces a **filtration** on the complexes computing the Heegaard Floer homology of Y.
The resulting filtered quasi-isomorphism type is a knot invariant, the **knot Floer homology** $HFK^\circ(Y, K, \mathfrak{s})$.

Properties:

We are going to work with knots in S^3 (and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$).

$$HFK^-(K) = \bigoplus_{a,m \in \mathbb{Z}} HFK^-_m(K,a)$$

where each $HFK^-_m(K,a)$ is a finite $\mathbb{F}$ module.
$HFK^-(K)$ is **graded** (Maslov degree), and **bi-filtered** (Alexander and algebraic filtrations) $\mathbb{F}[U]$ complex.
U is an endomorphism commuting with ∂^-, decreases the degree by 2 and filtration levels by 1.
In each flavour there is a spectral sequence

$$HFK^\circ(K) \Longrightarrow HF^\circ(S^3) \cong \begin{cases} \mathbb{F}[U,U^{-1}] & \text{if } \circ = \infty \\ \mathbb{F}[U] & \text{if } \circ = - \end{cases}$$

where $1 \in \mathbb{F}[U]$ or $\mathbb{F}[U^{\pm 1}]$ has degree 0.

Properties:

We are going to work with knots in S^3 (and $\mathbb{F} = \mathbb{Z}/_{2\mathbb{Z}}$).

$$HFK^-(K) = \bigoplus_{a,m\in\mathbb{Z}} HFK^-_m(K,a)$$

where each $HFK^-_m(K,a)$ is a finite $\mathbb{F}$ module.
$HFK^-(K)$ is **graded** (Maslov degree), and **bi-filtered** (Alexander and algebraic filtrations) $\mathbb{F}[U]$ complex.
U is an endomorphism commuting with ∂^-, decreases the degree by 2 and filtration levels by 1.
In each flavour there is a spectral sequence

$$HFK^\circ(K) \Longrightarrow HF^\circ(S^3) \cong \begin{cases} \mathbb{F}[U,U^{-1}] & \text{if } \circ = \infty \\ \mathbb{F}[U] & \text{if } \circ = - \end{cases}$$

where $1 \in \mathbb{F}[U]$ or $\mathbb{F}[U^{\pm 1}]$ has degree 0.

Properties:

We are going to work with knots in S^3 (and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$).

$$HFK^-(K) = \bigoplus_{a,m\in\mathbb{Z}} HFK^-_m(K,a)$$

where each $HFK^-_m(K,a)$ is a finite $\mathbb{F}$ module.
$HFK^-(K)$ is **graded** (Maslov degree), and **bi-filtered** (Alexander and algebraic filtrations) $\mathbb{F}[U]$ complex.
U is an endomorphism commuting with ∂^-, decreases the degree by 2 and filtration levels by 1.
In each flavour there is a spectral sequence

$$HFK^\circ(K) \Longrightarrow HF^\circ(S^3) \cong \begin{cases} \mathbb{F}[U,U^{-1}] & \text{if } \circ = \infty \\ \mathbb{F}[U] & \text{if } \circ = - \end{cases}$$

where $1 \in \mathbb{F}[U]$ or $\mathbb{F}[U^{\pm 1}]$ has degree 0.

- Behaviour under connected sum

$$HFK^{\circ}(K_0 \# K_1) = HFK^{\circ}(K_0) \otimes HFK^{\circ}(K_1)$$

- Categorification of the Alexander polynomial:

$$\chi_t(HFK^{-}(k)) = \sum_{a,m \in \mathbb{Z}} (-1)^m rk(HFK^{-}_m(K,a)) \cdot t^a = \frac{\triangle_K(t)}{1-t^{-1}}$$

- **Detects** the Seifert genus and fiberdeness $\rightsquigarrow$ distinguishes the unknot, 3_1 and 4_1.

- Behaviour under connected sum

$$HFK^{\circ}(K_0 \# K_1) = HFK^{\circ}(K_0) \otimes HFK^{\circ}(K_1)$$

- Categorification of the Alexander polynomial:

$$\chi_t(HFK^{-}(k)) = \sum_{a,m \in \mathbb{Z}} (-1)^m rk(HFK^{-}_m(K,a)) \cdot t^a = \frac{\triangle_K(t)}{1 - t^{-1}}$$

- **Detects** the Seifert genus and fiberdeness $\rightsquigarrow$ distinguishes the unknot, 3_1 and 4_1.

- Behaviour under connected sum

$$HFK^{\circ}(K_0 \# K_1) = HFK^{\circ}(K_0) \otimes HFK^{\circ}(K_1)$$

- Categorification of the Alexander polynomial:

$$\chi_t(HFK^{-}(k)) = \sum_{a,m \in \mathbb{Z}} (-1)^m rk(HFK^{-}_m(K,a)) \cdot t^a = \frac{\triangle_K(t)}{1 - t^{-1}}$$

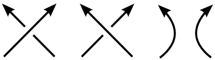
- **Detects** the Seifert genus and fiberdeness $\rightsquigarrow$ distinguishes the unknot, 3_1 and 4_1.

- Behaviour under connected sum

$$HFK^\circ(K_0\#K_1) = HFK^\circ(K_0) \otimes HFK^\circ(K_1)$$

- Categorification of the Alexander polynomial:

$$\chi_t(HFK^-(k)) = \sum_{a,m\in\mathbb{Z}} (-1)^m rk(HFK^-_m(K,a))\cdot t^a = \frac{\triangle_K(t)}{1-t^{-1}}$$

- **Detects** the Seifert genus and fiberdeness $\rightsquigarrow$ distinguishes the unknot, 3_1 and 4_1.
- Skein exact triangles: 

L_+ L_- L_0

The homology of HFK^∞ for knots is boring:

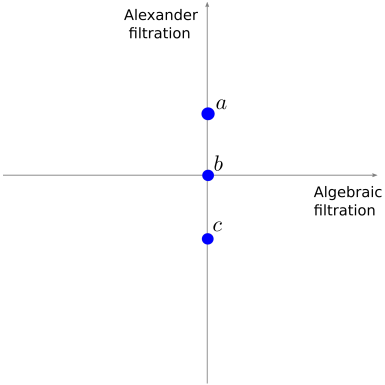
$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

(but the complex contains a huge amount of information).

The homology of HFK^{∞} for knots is boring:

$$HFK^{\infty}(K) \cong \mathbb{F}[U, U^{-1}]$$

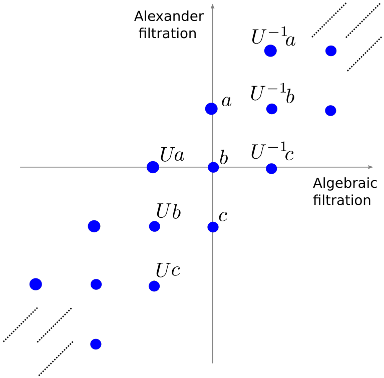
(but the complex contains a huge amount of information).



The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

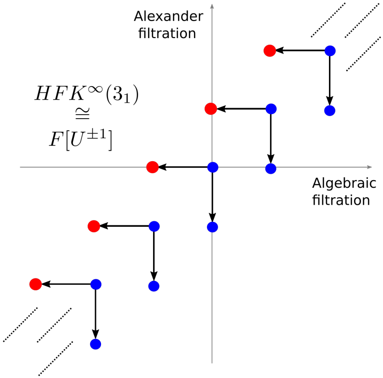
(but the complex contains a huge amount of information).



The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

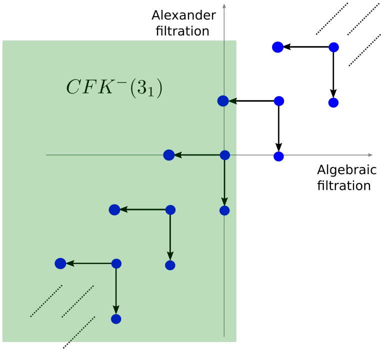
(but the complex contains a huge amount of information).



The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

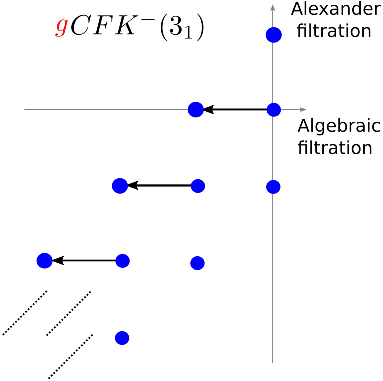
(but the complex contains a huge amount of information).



The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

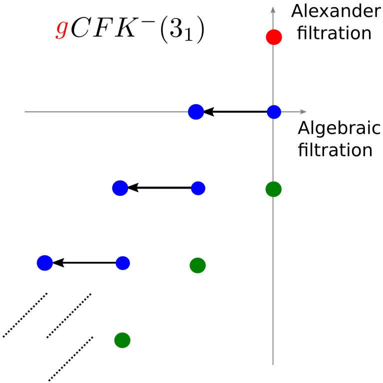
(but the complex contains a huge amount of information).



The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

(but the complex contains a huge amount of information).



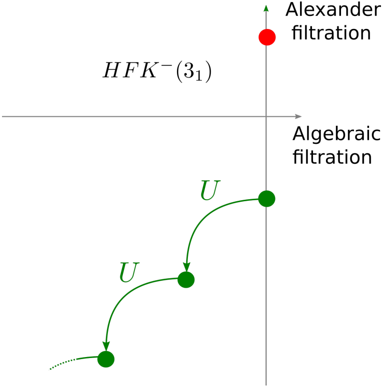
For the (graded) minus flavour:

$$HFK^-(K) \cong \mathbb{F}[U]_{(-2\tau(K), -\tau(K))} \oplus U\text{-torsion part.}$$

The homology of HFK^∞ for knots is boring:

$$HFK^\infty(K) \cong \mathbb{F}[U, U^{-1}]$$

(but the complex contains a huge amount of information).



For the (graded) minus flavour:

$$HFK^-(K) \cong \mathbb{F}[U]_{(-2\tau(K), -\tau(K))} \oplus U\text{-torsion part.}$$

τ-invariant

The first invariant one can extract from HFK^- is the τ-invariant:

$\tau(K) = -\max\{A(x) \mid x \in HFK^-(K)$ is not U-torsion$\}$

Theorem (O-Sz/Rasmussen/Sarkar)

τ is a concordance invariant, and induces an homomorphism $\tau : \mathcal{C} \longrightarrow \mathbb{Z}$. Moreover if $K_0, K_1 \in \mathcal{K}$ are related by a cobordism $\Sigma \subset S^3 \times [0,1]$:

$$|\tau(K_0) - \tau(K_1)| \leq g(\Sigma).$$

In particular $|\tau(K)| \leq g_*(K)$.

- Combinatorial proof of the Milnor conjecture.
- Bennequin inequality: $tb(K) + |rot(K)| \leq 2\tau(K) - 1$.
- Results above hold in more general contexts.

τ-invariant

The first invariant one can extract from HFK^- is the τ-invariant:

$\tau(K) = -\max\{A(x) \mid x \in HFK^-(K) \text{ is not } U\text{-torsion}\}$

Theorem (O-Sz/Rasmussen/Sarkar)

τ is a concordance invariant, and induces an homomorphism $\tau : \mathcal{C} \longrightarrow \mathbb{Z}$. Moreover if $K_0, K_1 \in \mathcal{K}$ are related by a cobordism $\Sigma \subset S^3 \times [0,1]$:

$$|\tau(K_0) - \tau(K_1)| \leq g(\Sigma).$$

In particular $|\tau(K)| \leq g_*(K)$.

- Combinatorial proof of the Milnor conjecture.
- Bennequin inequality: $tb(K) + |rot(K)| \leq 2\tau(K) - 1$.
- Results above hold in more general contexts.

τ-invariant

The first invariant one can extract from HFK^- is the τ-invariant:

$\tau(K) = -\max\{A(x) \mid x \in HFK^-(K) \text{ is not } U\text{-torsion}\}$

Theorem (O-Sz/Rasmussen/Sarkar)

τ is a concordance invariant, and induces an homomorphism $\tau : \mathcal{C} \longrightarrow \mathbb{Z}$. Moreover if $K_0, K_1 \in \mathcal{K}$ are related by a cobordism $\Sigma \subset S^3 \times [0,1]$:

$$|\tau(K_0) - \tau(K_1)| \leq g(\Sigma).$$

In particular $|\tau(K)| \leq g_*(K)$.

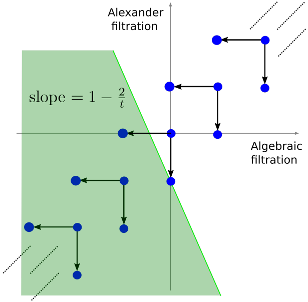
- Combinatorial proof of the Milnor conjecture.
- Bennequin inequality: $tb(K) + |rot(K)| \leq 2\tau(K) - 1$.
- Results above hold in more general contexts.

More recently, O-Sz-Stipsicz defined a concordance invariant $\Upsilon_K(t) \in PL([0,2])$:

$$|\Upsilon_K(t)| \leq t \cdot g_*(K)$$

and $\Upsilon : \mathcal{C} \to PL([0,2])$ is an homomorphism.

Idea: collapse the bi-filtration to obtain a family of singly filtered complexes: $t \in [0,2] \rightsquigarrow \mathcal{F}_t(x) = \frac{t}{2}Alex(x) + (1-\frac{t}{2})Alg(x)$.



$\Upsilon_K(t) = -2\min\{s \mid Im(H(C(K,\mathcal{F}_t)_s)) \to H(C(K))$ is surj.$\}$.

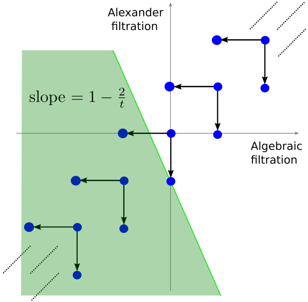
Υ is well suited to prove independence of families in $\mathcal{C}$.

More recently, O-Sz-Stipsicz defined a concordance invariant $\Upsilon_K(t) \in PL([0,2])$:

$$|\Upsilon_K(t)| \leq t \cdot g_*(K)$$

and $\Upsilon : \mathcal{C} \to PL([0,2])$ is an homomorphism.

Idea: collapse the bi-filtration to obtain a family of singly filtered complexes: $t \in [0,2] \rightsquigarrow \mathcal{F}_t(x) = \frac{t}{2}Alex(x) + (1-\frac{t}{2})Alg(x)$.



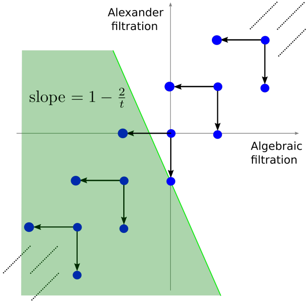
$\Upsilon_K(t) = -2\min\{s \mid Im(H(C(K,\mathcal{F}_t)_s)) \to H(C(K)) \text{ is surj.}\}$.

Υ is well suited to prove independence of families in $\mathcal{C}$.

More recently, O-Sz-Stipsicz defined a concordance invariant $\Upsilon_K(t) \in PL([0,2])$:

$$|\Upsilon_K(t)| \leq t \cdot g_*(K)$$

and $\Upsilon : \mathcal{C} \to PL([0,2])$ is an homomorphism.
Idea: collapse the bi-filtration to obtain a family of singly filtered complexes: $t \in [0,2] \rightsquigarrow \mathcal{F}_t(x) = \frac{t}{2} Alex(x) + (1 - \frac{t}{2}) Alg(x)$.



$$\Upsilon_K(t) = -2 \min\{s \mid Im(H(C(K, \mathcal{F}_t)_s)) \to H(C(K)) \text{ is surj.}\}.$$

Υ is well suited to prove independence of families in $\mathcal{C}$.

If Y is a $\mathbb{Q}HS^3$, then the groups $HF^-(Y,\mathfrak{s})$ admit a $\mathbb{Q}$-grading, and the **correction term** $d(Y,\mathfrak{s}) \in \mathbb{Q}$ is the grading of $1 \in \mathbb{F}[U]$.
The homology

$$HF^-(S^3_p(K)) = \bigoplus_{\mathfrak{s}\in Spin^c(Y)} HF^-(S^3_p(K),\mathfrak{s})$$

can be extracted from the complex $CFK^-(K)$ through a mapping cone construction.
The maps involved provide a sequence $\{V_i(K)\}_{i\in\mathbb{Z}}$ such that: $V_i(K) \geq V_{i+1}(K) \geq 0$ and $V_i(K)$ is eventually 0.

$$d(S^3_p(K),i) = d(L(p,1),i) - 2\max\{V_i, V_{p-i}\}$$

The minimal i such that $V_i(K) = 0$ is called $\nu^+(K)$.
ν^+ is a concordance invariant, and $\nu^+(K) \leq g_*(K)$ (Hom-Wu).

If Y is a $\mathbb{Q}HS^3$, then the groups $HF^-(Y,\mathfrak{s})$ admit a $\mathbb{Q}$-grading, and the **correction term** $d(Y,\mathfrak{s}) \in \mathbb{Q}$ is the grading of $1 \in \mathbb{F}[U]$. The homology

$$HF^-(S^3_p(K)) = \bigoplus_{\mathfrak{s}\in Spin^c(Y)} HF^-(S^3_p(K),\mathfrak{s})$$

can be extracted from the complex $CFK^-(K)$ through a mapping cone construction.
The maps involved provide a sequence $\{V_i(K)\}_{i\in\mathbb{Z}}$ such that: $V_i(K) \geq V_{i+1}(K) \geq 0$ and $V_i(K)$ is eventually 0.

$$d(S^3_p(K),i) = d(L(p,1),i) - 2\max\{V_i, V_{p-i}\}$$

The minimal i such that $V_i(K) = 0$ is called $\nu^+(K)$.
ν^+ is a concordance invariant, and $\nu^+(K) \leq g_*(K)$ (Hom-Wu).

If Y is a $\mathbb{Q}HS^3$, then the groups $HF^-(Y,\mathfrak{s})$ admit a $\mathbb{Q}$-grading, and the **correction term** $d(Y,\mathfrak{s}) \in \mathbb{Q}$ is the grading of $1 \in \mathbb{F}[U]$.
The homology

$$HF^-(S^3_p(K)) = \bigoplus_{\mathfrak{s}\in Spin^c(Y)} HF^-(S^3_p(K),\mathfrak{s})$$

can be extracted from the complex $CFK^-(K)$ through a mapping cone construction.
The maps involved provide a sequence $\{V_i(K)\}_{i\in\mathbb{Z}}$ such that:
$V_i(K) \geq V_{i+1}(K) \geq 0$ and $V_i(K)$ is eventually 0.

$$d(S^3_p(K), i) = d(L(p,1), i) - 2\max\{V_i, V_{p-i}\}$$

The minimal i such that $V_i(K) = 0$ is called $\nu^+(K)$.
ν^+ is a concordance invariant, and $\nu^+(K) \leq g_*(K)$ (Hom-Wu).

Algebraic knots and ν^+

Consider an irreducible polynomial $F \in \mathbb{C}[x, y]$ with a singularity in the origin. (locally $t \mapsto (t^p, t^{q_1} + ... + t^{q_m})$).
Then $K_F = \{F(x, y) = 0\} \cap \partial B_\varepsilon$ is an **algebraic knot** in S^3.
Associated to an algebraic knot there is a *semigroup* $\Gamma(K)$. For torus knots $(p, q) = 1$ and $p > q$ we have

$$\Gamma_{T_{p,q}} =< p, q >= \{0, q, 2q, \ldots, p, p + q, \ldots\},$$

and $|\mathbb{N} \setminus \Gamma_{T_{p,q}}| = \frac{1}{2}(p - 1)(q - 1)$.
Call $\Gamma_K(\cdot)$ the associated counting function, so $\Gamma_K(n)$ is the n-*th* element in Γ_K.

Algebraic knots and ν^+

Consider an irreducible polynomial $F \in \mathbb{C}[x, y]$ with a singularity in the origin. (locally $t \mapsto (t^p, t^{q_1} + ... + t^{q_m})$).
Then $K_F = \{F(x, y) = 0\} \cap \partial B_\varepsilon$ is an **algebraic knot** in S^3.
Associated to an algebraic knot there is a *semigroup* $\Gamma(K)$. For torus knots $(p, q) = 1$ and $p > q$ we have

$$\Gamma_{T_{p,q}} =< p, q >= \{0, q, 2q, \ldots, p, p+q, \ldots\},$$

and $|\mathbb{N} \setminus \Gamma_{T_{p,q}}| = \frac{1}{2}(p-1)(q-1)$.
Call $\Gamma_K(\cdot)$ the associated counting function, so $\Gamma_K(n)$ is the n-*th* element in Γ_K.

Theorem (Bodnár-C.-Golla)

If K and L are algebraic knots with enumerating functions $\Gamma_K(\cdot)$ and $\Gamma_L(\cdot)$ respectively, then:

$$\nu^+(K\#\overline{L}) = \max\left\{g(K) - g(L) + \max_{n\geq 0}\{\Gamma_L(n) - \Gamma_K(n)\}, 0\right\}.$$

Remark

The same result holds for L-space knots (with a suitable definition of the "semigroup" Γ_K).

Idea of the proof: both knots are L-space knots, hence their HFK^∞ has a rather simple form (staircase complex).
Use the **reduced knot Floer complex** of Krcatovich to compute the V_is.

Theorem (Bodnár-C.-Golla)

If K and L are algebraic knots with enumerating functions $\Gamma_K(\cdot)$ and $\Gamma_L(\cdot)$ respectively, then:

$$\nu^+(K\#\overline{L}) = \max\left\{g(K) - g(L) + \max_{n\geq 0}\{\Gamma_L(n) - \Gamma_K(n)\}, 0\right\}.$$

Remark

The same result holds for L-space knots (with a suitable definition of the "semigroup" Γ_K).

Idea of the proof: both knots are L-space knots, hence their HFK^∞ has a rather simple form (staircase complex).
Use the **reduced knot Floer complex** of Krcatovich to compute the V_is.

Theorem (Bodnár-C.-Golla)

- If K_+ is obtained from K_- by changing a negative crossing into a positive one, then:

$$\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1.$$

- The unknotting number, concordance unknotting number, and slicing number of K are bounded from below by

$$\nu^+(K) + \nu^+(\overline{K}).$$

- The invariant is sub-additive:

$$\nu^+(K_0 \# K_1) \leq \nu^+(K_0) + \nu^+(K_1)$$

Theorem (Bodnár-C.-Golla)

- If K_+ is obtained from K_- by changing a negative crossing into a positive one, then:

$$\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1.$$

- The unknotting number, concordance unknotting number, and slicing number of K are bounded from below by

$$\nu^+(K) + \nu^+(\overline{K}).$$

- The invariant is sub-additive:

$$\nu^+(K_0 \# K_1) \leq \nu^+(K_0) + \nu^+(K_1)$$

Theorem (Bodnár-C.-Golla)

- If K_+ is obtained from K_- by changing a negative crossing into a positive one, then:

$$\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1.$$

- The unknotting number, concordance unknotting number, and slicing number of K are bounded from below by

$$\nu^+(K) + \nu^+(\overline{K}).$$

- The invariant is sub-additive:

$$\nu^+(K_0 \# K_1) \leq \nu^+(K_0) + \nu^+(K_1)$$

An infinite family of optimal cobordisms

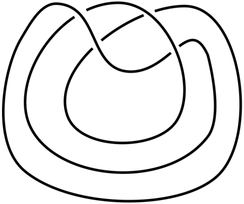
Since $\nu^+(K) \leq g_*(K)$, the quantities $\nu^+(K\#\overline{L})$ and $\nu^+(L\#\overline{K})$ provide (often different) lower bounds for the genus of cobordisms between K and L.

It is possible to construct infinite families where the bounds on cobordisms provided by ν^+ is sharp, while the ones given by τ, Υ, s and Tristram-Levine signatures are arbitrarily "bad":

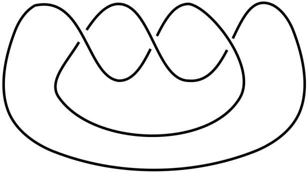
An infinite family of optimal cobordisms

Since $\nu^+(K) \leq g_*(K)$, the quantities $\nu^+(K\#\overline{L})$ and $\nu^+(L\#\overline{K})$ provide (often different) lower bounds for the genus of cobordisms between K and L.

It is possible to construct infinite families where the bounds on cobordisms provided by ν^+ is sharp, while the ones given by τ, Υ, s and Tristram-Levine signatures are arbitrarily "bad":

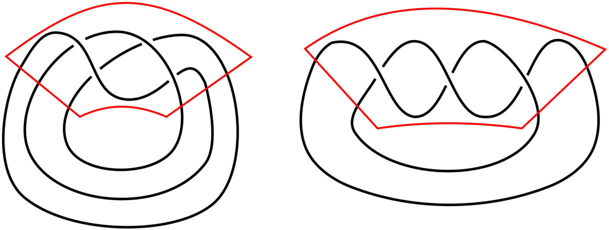


3_1



An infinite family of optimal cobordisms

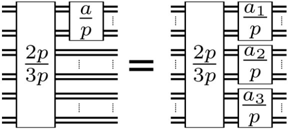
Since $\nu^+(K) \leq g_*(K)$, the quantities $\nu^+(K\#\overline{L})$ and $\nu^+(L\#\overline{K})$ provide (often different) lower bounds for the genus of cobordisms between K and L.
It is possible to construct infinite families where the bounds on cobordisms provided by ν^+ is sharp, while the ones given by τ, Υ, s and Tristram-Levine signatures are arbitrarily "bad":



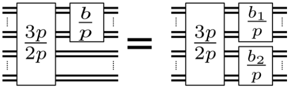
An infinite family of optimal cobordisms

Since $\nu^+(K) \leq g_*(K)$, the quantities $\nu^+(K\#\overline{L})$ and $\nu^+(L\#\overline{K})$ provide (often different) lower bounds for the genus of cobordisms between K and L.

It is possible to construct infinite families where the bounds on cobordisms provided by ν^+ is sharp, while the ones given by τ, Υ, s and Tristram-Levine signatures are arbitrarily "bad":



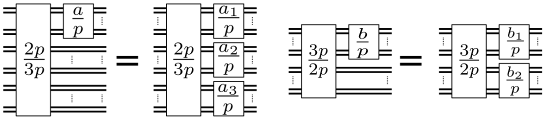
$a = a_1 + a_2 + a_3$



$b = b_1 + b2$

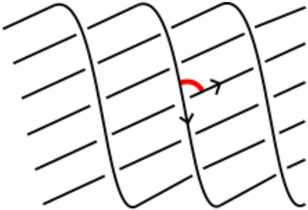
An infinite family of optimal cobordisms

Since $\nu^+(K) \leq g_*(K)$, the quantities $\nu^+(K\#\overline{L})$ and $\nu^+(L\#\overline{K})$ provide (often different) lower bounds for the genus of cobordisms between K and L.
It is possible to construct infinite families where the bounds on cobordisms provided by ν^+ is sharp, while the ones given by τ, Υ, s and Tristram-Levine signatures are arbitrarily "bad":



$K_{(a,p)}$= $(p, 6p + a)$cable of 3_1 $\quad$ $K'_{(b,p)}$=$(p, 6p + b)$ cable of 3_1

Now consider $a = b = 12$, and $(p, a) = 1$: it is possible to connect by a genus 4 cobordism $K_{(12,p)}$ with $T_{2p+4,3p}$, and $K'_{(12,p)}$ to $T_{2p,3p+6}$ with a genus 3 cobordism.

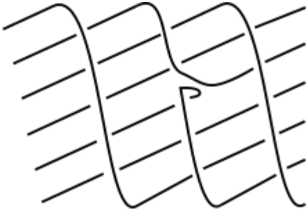


A computation yields:

$$\nu^+(T_{2p+4,3p} \# \overline{T}_{2p,3p+6}) = \nu^+(T_{2p,3p+6} \# \overline{T}_{2p+4,3p}) = 7,$$

so the cobordism has minimal genus (and $d(T_{2p+4,3p}, T_{2p,3p+6}) = 14$). However the bounds given by Υ, s, τ and Tristram-Levine signatures are at most 5.

Now consider $a = b = 12$, and $(p, a) = 1$: it is possible to connect by a genus 4 cobordism $K_{(12,p)}$ with $T_{2p+4,3p}$, and $K'_{(12,p)}$ to $T_{2p,3p+6}$ with a genus 3 cobordism.



A computation yields:

$$\nu^+(T_{2p+4,3p}\#\overline{T}_{2p,3p+6}) = \nu^+(T_{2p,3p+6}\#\overline{T}_{2p+4,3p}) = 7,$$

so the cobordism has minimal genus (and $d(T_{2p+4,3p}, T_{2p,3p+6}) = 14$). However the bounds given by Υ, s, τ and Tristram-Levine signatures are at most 5.

Thanks for your attention!