North meets South Colloquium: "Categorification of knot polynomials"

Daniele Celoria
Mathematical Institute
University of Oxford

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

Plan of the talk:

- Knots
- Polynomials
- Categorification
- Categorification of knot polynomials

A knot is a smooth embedding of S^1 in S^3, considered up to **ambient isotopy**.

A knot is a smooth embedding of S^1 in S^3, considered up to **ambient isotopy**.

The goal of knot theory is to distinguish non-equivalent knots; this can be done by computing some **invariants**.

A knot is a smooth embedding of S^1 in S^3, considered up to **ambient isotopy**.

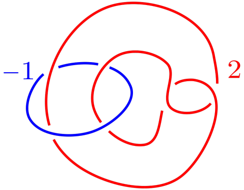
The goal of knot theory is to distinguish non-equivalent knots; this can be done by computing some **invariants**.

$$\text{Invariant : “Knots”} \longrightarrow \text{“something algebraic”}$$

such that if two knots have different images, then they are distinct.

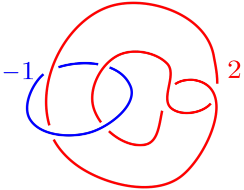
Why do we care?

Why do we care?

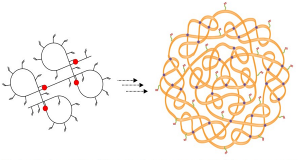


Surgery presentation of 3 and 4-manifolds

Why do we care?

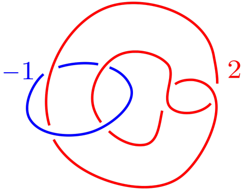


Surgery presentation of 3 and 4-manifolds

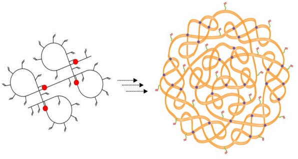


Properties of "random polymers"

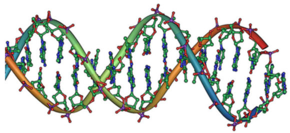
Why do we care?



Surgery presentation of 3 and 4-manifolds

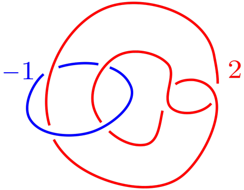


Properties of "random polymers"

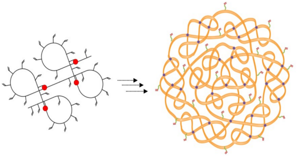


Modelling of DNA and topoisomerase actions

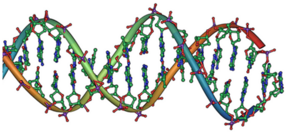
Why do we care?



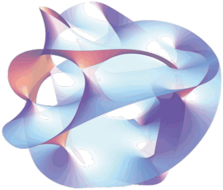
Surgery presentation of 3 and 4-manifolds



Properties of "random polymers"



Modelling of DNA and topoisomerase actions



String theory and Statistical Mechanics

Some of the most versatile knot invariants take the form of polynomials. The two main examples are:

Some of the most versatile knot invariants take the form of polynomials. The two main examples are:

- The **Alexander-Conway polynomial**: $\Delta_K(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$
  - Discovered by J.W. Alexander in 1923 (and reinterpreted by J. Conway in 1967).
  - Tightly connected to the topology of knot complement and branched covers.

Some of the most versatile knot invariants take the form of polynomials. The two main examples are:

- The **Alexander-Conway polynomial**: $\Delta_K(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$
  - Discovered by J.W. Alexander in 1923 (and reinterpreted by J. Conway in 1967).
  - Tightly connected to the topology of knot complement and branched covers.
- The **Jones polynomial**: $V_K(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$
  - Discovered by V.Jones (Fields medalist 1990).
  - Arises as a trace of representations of the braid group, related to QFTs (Witten), quantum invariants and statistical mechanics.

Skein relations

Both these knot polynomials admit a simple recursive definition, in terms of **skein relations**. These are equations relating the invariants of knot diagrams differing only locally.

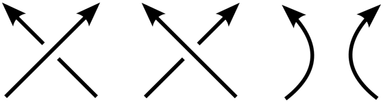
Skein relations

Both these knot polynomials admit a simple recursive definition, in terms of **skein relations**. These are equations relating the invariants of knot diagrams differing only locally.
Normalization on the trivial knot $V_{\bigcirc}(t) = \Delta_{\bigcirc}(t) = 1$

Skein relations

Both these knot polynomials admit a simple recursive definition, in terms of **skein relations**. These are equations relating the invariants of knot diagrams differing only locally.

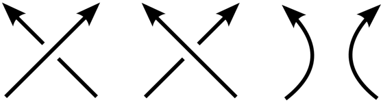
Normalization on the trivial knot $V_{\bigcirc}(t) = \Delta_{\bigcirc}(t) = 1$



L_+ L_- L_0

Skein relations

Both these knot polynomials admit a simple recursive definition, in terms of **skein relations**. These are equations relating the invariants of knot diagrams differing only locally.
Normalization on the trivial knot $V_{\bigcirc}(t) = \Delta_{\bigcirc}(t) = 1$



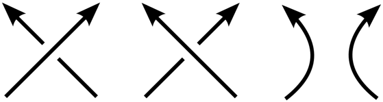
L_+ L_- L_0

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{L_0}(t)$$

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \Delta_{L_0}(t)$$

Skein relations

Both these knot polynomials admit a simple recursive definition, in terms of **skein relations**. These are equations relating the invariants of knot diagrams differing only locally.
Normalization on the trivial knot $V_{\bigcirc}(t) = \Delta_{\bigcirc}(t) = 1$



L_+ L_- L_0

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{L_0}(t)$$

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \Delta_{L_0}(t)$$

The two polynomials have different properties, but can be defined as specializations of a 2-variable polynomial.

Categorification

Term coined by L.Crane in the '90s; informally it means to replace problems in set theory with analogues dealing with categories instead.

Categorification

Term coined by L.Crane in the '90s; informally it means to replace problems in set theory with analogues dealing with categories instead.

Examples:

- $(\mathbb{N}, +, \cdot) \rightsquigarrow (Vect, \oplus, \otimes)$ The dimension function provides the decategorification of the category of vector spaces.
- Betti numbers, Euler characteristic $\rightsquigarrow$ Homology

Categorification

Term coined by L.Crane in the '90s; informally it means to replace problems in set theory with analogues dealing with categories instead.

Examples:

- $(\mathbb{N}, +, \cdot) \rightsquigarrow (Vect, \oplus, \otimes)$ The dimension function provides the decategorification of the category of vector spaces.
- Betti numbers, Euler characteristic $\rightsquigarrow$ Homology

Categorifications of polynomials

Given a finite bi-graded vector space $V = \bigoplus_{i,j \in \mathbb{Z}} V_{i,j}$, define its graded Euler characteristic as:

$$\chi_t(V) = \sum_{i,j \in \mathbb{Z}} (-1)^i \mathrm{dim}(V_{i,j}) \cdot t^j$$

Khovanov homology

M. Khovanov (1999), introduced what is now known as **Khovanov homology**; it assigns to a knot $K \subset S^3$ a bi-graded complex $(CKh^{*,*}(K), \partial)$, such that the homology

$$Kh^{*,*}(K) = H_*(CKh^{*,*}(K), \partial)$$

is a knot invariant.

Khovanov homology

M. Khovanov (1999), introduced what is now known as **Khovanov homology**; it assigns to a knot $K \subset S^3$ a bi-graded complex $(CKh^{*,*}(K), \partial)$, such that the homology

$$Kh^{*,*}(K) = H_*(CKh^{*,*}(K), \partial)$$

is a knot invariant. Its Euler characteristic coincides with the Jones polynomial:

$$\chi_t(Kh^{*,*}(K)) = V_K(t)$$

It can be defined in a purely combinatorial setting (Bar Natan) or from a Gauge theoretic perspective (Witten), and has been used *e.g.* to give a combinatorial proof of the Milnor conjecture (Kronheimer-Mrowka/Rasmussen).

Knot Floer homology

In 2001 Ozsváth and Szabó produce an extremely powerful functorial gauge-theoretic 3-manifold invariant, the **Heegaard-Floer homology**.

Knot Floer homology

In 2001 Ozsváth and Szabó produce an extremely powerful functorial gauge-theoretic 3-manifold invariant, the **Heegaard-Floer homology**.
To a closed 3-manifold Y they associate a collection of graded groups $HF^{\circ}(Y)$.

Knot Floer homology

In 2001 Ozsváth and Szabó produce an extremely powerful functorial gauge-theoretic 3-manifold invariant, the **Heegaard-Floer homology**.
To a closed 3-manifold Y they associate a collection of graded groups $HF^{\circ}(Y)$.
Shortly after, Ozsváth-Szabó/Rasmussen discover that a knot $K \subset Y$ induces a filtration of the complexes computing $HF^{\circ}(Y)$.
From this filtration one can extract a bi-graded homology theory for knots, the **knot Floer homology** $HFK^{\circ}(Y, K)$.

Knot Floer homology

In 2001 Ozsváth and Szabó produce an extremely powerful functorial gauge-theoretic 3-manifold invariant, the **Heegaard-Floer homology**.
To a closed 3-manifold Y they associate a collection of graded groups $HF^\circ(Y)$.
Shortly after, Ozsváth-Szabó/Rasmussen discover that a knot $K \subset Y$ induces a filtration of the complexes computing $HF^\circ(Y)$.
From this filtration one can extract a bi-graded homology theory for knots, the **knot Floer homology** $HFK^\circ(Y, K)$.
These homology groups provide a categorification of the Alexander-Conway polynomial:

$$\chi_t(HFK^{*,*}(K)) = \Delta_K(t)$$

A spectral sequence?

Despite the many formal similarities of these two knot polynomials and their categorified counterparts, there is still no clear connection between $Kh(K)$ and $HFK(K)$.

A spectral sequence?

Despite the many formal similarities of these two knot polynomials and their categorified counterparts, there is still no clear connection between $Kh(K)$ and $HFK(K)$.

Many authors have found spectral sequences of the form:

$$CKh(K) \Longrightarrow I(K)$$

where I is a (usually Floer-theoretic) homology invariant of knots. These sort of spectral sequences are collectively known as **Khovanov-Floer theories**.

A spectral sequence?

Despite the many formal similarities of these two knot polynomials and their categorified counterparts, there is still no clear connection between $Kh(K)$ and $HFK(K)$.

Many authors have found spectral sequences of the form:

$$CKh(K) \Longrightarrow I(K)$$

where I is a (usually Floer-theoretic) homology invariant of knots. These sort of spectral sequences are collectively known as **Khovanov-Floer theories**.

Conjecture (Rasmussen)

There exists a spectral sequence:

$$CKh(K) \Longrightarrow \widehat{HFK}(K)$$

Thanks for your attention!