Grid Homology in Lens spaces

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Introduction

Knot Floer Homology $HFK^\circ(Y^3, L)$

Heegaard Floer Homology: homology theory for 3-manifolds developed around 2001 by Peter Ozsváth and Zoltán Szabó as a variant of Lagrangian Floer theory.

In 2003 (O-S + J. Rasmussen) discover that links induce filtrations on $HFK^\circ(Y^3) \rightsquigarrow$ Knot Floer Homology $HFK^\circ(Y^3, L)$

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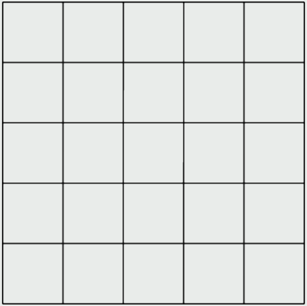
(toroidal) *Grid diagrams*:



- $n \times n$ grid in $\mathbb{R}^2$
- curves $\underline{\alpha} = \{\alpha_i\}$ $\underline{\beta} = \{\beta_i\}$
- $\mathbb{X} = \{\mathbb{X}_i\}$ $\mathbb{O} = \{\mathbb{O}_i\}$ $i = 1, \ldots, n$ markings (no two on same row/col)
- top-bottom and left-right identifications (multipointed genus 1 H.D. for S^3)

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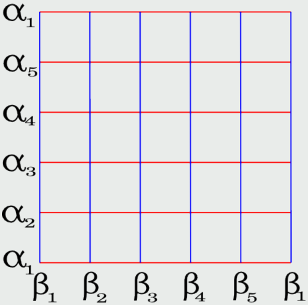

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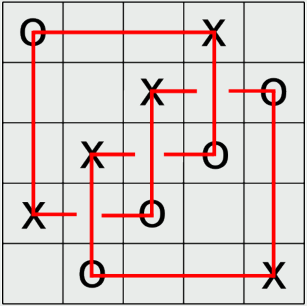

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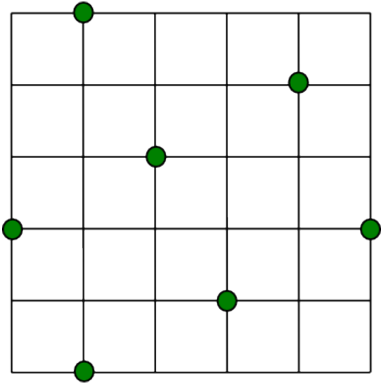
Every oriented link in S^3 can be encoded in this way

Reidemeister Moves $\rightsquigarrow$ Cromwell Moves

The complex $GC^-(G)$:

Take a grid diagram G for the knot $K \subset S^3$

$$S(G) = \left\{\text{bijections } \underline{\alpha} \leftrightarrow \underline{\beta}\right\} \cong \mathfrak{S}_n$$



$$\sigma(x) = \binom{12345}{24135}$$

Bigrading

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Bigrading

the generating set $S(G)$ can be bigraded:

- Maslov degree $M : S(G) \longrightarrow \mathbb{Z}$ (homological)
- Alexander degree $A : S(G) \longrightarrow \mathbb{Z}$ (filtration)

A, B sets of finite points in $\mathbb{R}^2$:
$I(A,B) = \{((a_1,a_2),(b_1,b_2)) \subset A \times B \mid a_1 < b_1 \text{ and } a_2 < b_2\}$

$$M(x) = M_{\mathbb{O}}(x) = I(x,x) - I(x,\mathbb{O}) - I(\mathbb{O},x) + I(\mathbb{O},\mathbb{O}) + 1$$

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The complex $GC^-(G)$ is the $\mathbb{F}_2[U_1, \ldots, U_n]$-module freely generated over $S(G)$.

We need to extend the degrees to this coefficient ring:

- M(Ux) = M(x) -2
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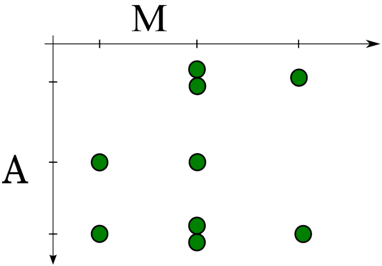

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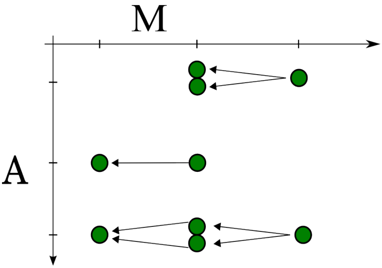

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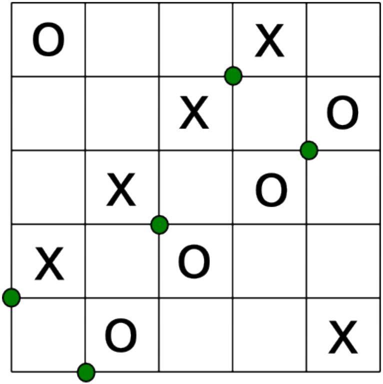

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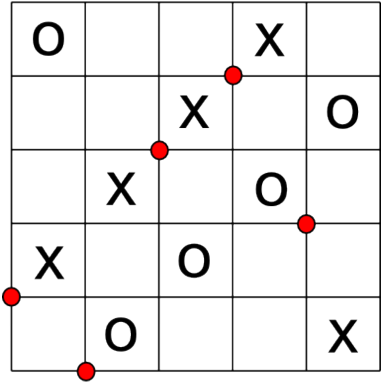

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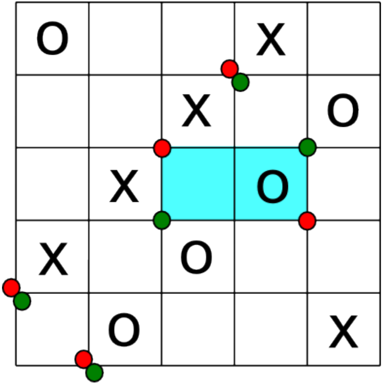

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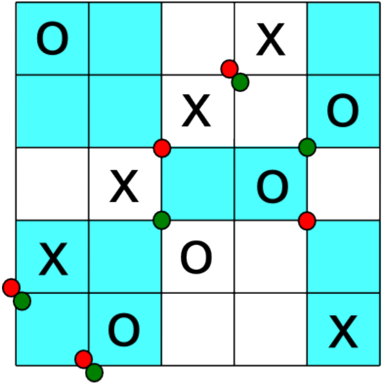

If two elements $x, y \in S(G)$ differ by a transposition,they can be connected by an oriented rectangle on the grid:

$$\partial^-(x) = \sum_{y \in S(G)} \sum_{\substack{r \in Rect^\circ(x,y) \\ r \cap \mathbb{X} = \emptyset}} \left(\prod_{i=1}^{n} U_i^{O_i(r)} \right) y$$

- $Rect^\circ(x, y)$ are the *empty* rectangles conneting x to y, *i.e.* $\mathring{r} \cap x = \mathring{r} \cap y = \emptyset$
- $O_i(r) = \#\{\mathbb{O}_i \cap r\}$

Basic properties of GH^-

- $(\partial^-)^2 = 0$
- $A(\partial^-(x)) = A(x)$
- $M(\partial^-(x)) = M(x) - 1$

Theorem (Ozváth - Rasmussen - Szabó - Stipsicz...)

$$H_*(GC^-(G), \partial^-) = GH^-(G; \mathbb{F}_2) \cong HFK^-(K; \mathbb{F}_2)$$

is an invariant of K, which categorifies the Alexander polynomial:

$$\chi_t(GH^-(G)) = \sum_{a,m \in \mathbb{Z}} (-1)^m rk(GH^-_m(G,a))t^a = \triangle_K(t)$$

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$$(\partial^-)^2(x) = \sum_{z\in S(G)} \sum_{\substack{\psi_j\in Poly^\circ(x,z)\\ \psi_j\cap\mathbb{X}=\emptyset}} \left(\sum_j N(\psi_j) \left(\prod_{i=1}^{n} U_i^{O_i(\psi_j)} \right) \right) z$$

Remark:

This differential is only well defined over $\mathbb{F}_2$!

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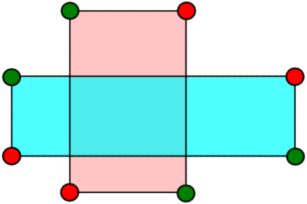

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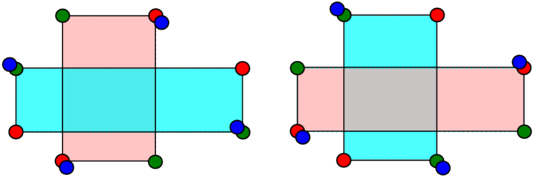

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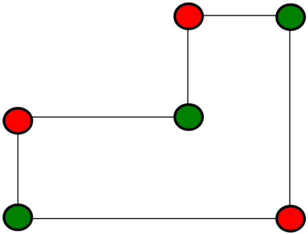

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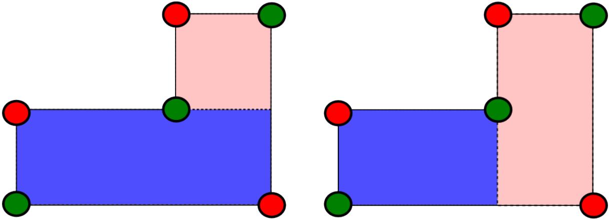

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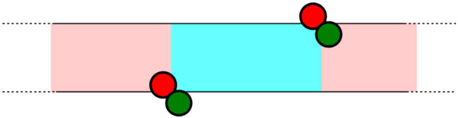

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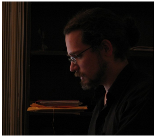

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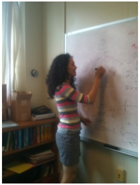

J. Elisenda Grigsby

Matt Hedden

"Grid Diagrams for Lens Spaces and Combinatorial Knot Floer Homology"(2007)

(p, q) coprime integers, define $L(p, q) = S^3_{-\frac{p}{q}}(\bigcirc)$

$$H_1(L(p, q); \mathbb{Z}) = \mathbb{Z}/_{p\mathbb{Z}} \leftrightarrow Spin^c(L(p, q))$$

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We can develop the same approach used with toroidal grids in S^3!

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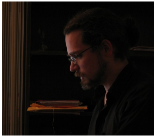

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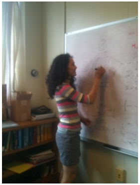

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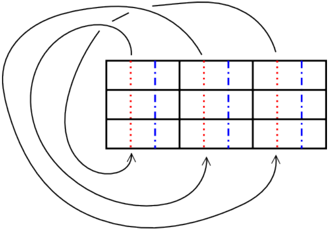
A *toroidal* **twisted** *grid diagram* for a link $L \subset L(p,q)$:



$n \times np$ grid, with $n\mathbb{X}$ and $n\mathbb{O}$ markings (no two on the same row/column). As before

$$S(G) = \{\text{bijections between } \underline{\alpha} \leftrightarrow \underline{\beta}\}$$

But this time $|\alpha_i \cap \beta_j| = p \ \forall i, j = 1, \ldots, n$, so

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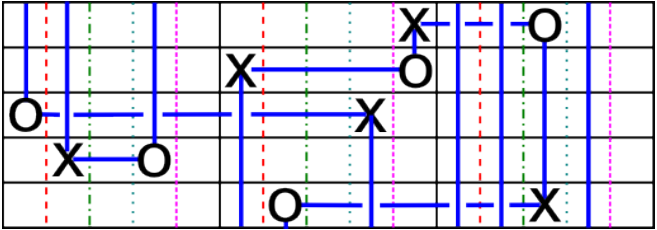

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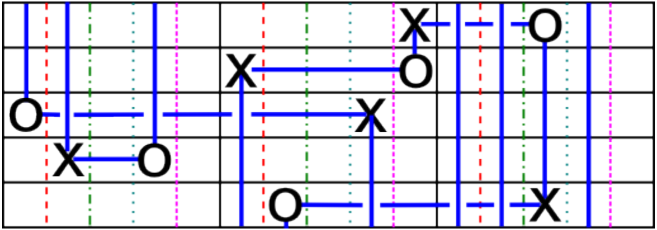

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We lift markings and generators to S^3 to define the gradings.

A	B	C

B	C	A
C	A	B
A	B	C

C	A	B
B	C	A
A	B	C

Lifts for $L(3,1)$ and $L(3,2)$

and obtain analogous formulas:

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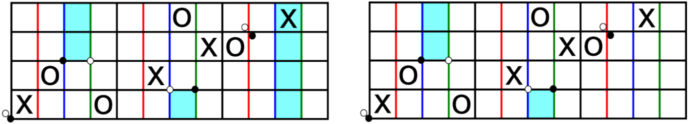

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The differential respects the $Spin^c$ grading: $S(x) = S(\partial^-(x))$

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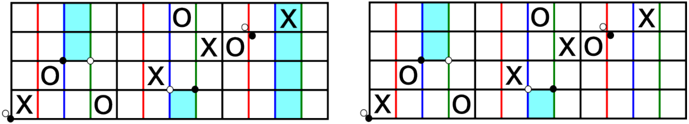

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Signs

Lift of the coefficients from $\mathbb{F}_2[U_1, \dots, U_n]$ to $\mathbb{Z}[U_1, \dots, U_n]$:

Sign assignment:

$$\mathcal{S} : Rect(G) \longrightarrow \{\pm 1\}$$

such that the following conditions hold:

1. If $r_1 * r_2 = r_3 * r_4$ then $\mathcal{S}(r_1)\mathcal{S}(r_2) = -\mathcal{S}(r_3)\mathcal{S}(r_4)$.
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Differential over $\mathbb{Z}$:

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How can we construct such an $\mathcal{S}$ on S^3 or $L(p,q)$? Gallais (2008)

$Spin$ extension of $\mathfrak{S}_n = \langle \tau_{i,j} \mid 1 \leq i < j \leq n \rangle$

It's the group $\widetilde{\mathfrak{S}}_n$ generated by

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We can associate a generalized transposition to each rectangle:

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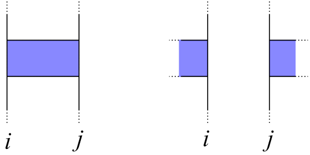

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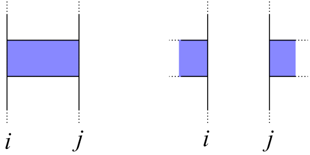

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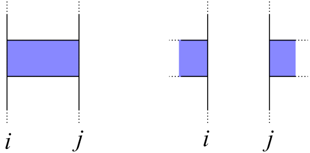

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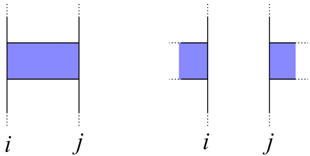

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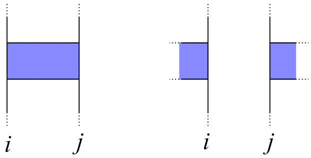

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$$\mathcal{G}(G) = \left\{ v : S(G) \longrightarrow \mathbb{Z}/_{2\mathbb{Z}} \right\} \textit{ Gauge group.}$$

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Coincides with the Alexander polynomial from Cornwell's HOMFLYPT specialization?

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