



Distribution theory done right

or how I lost my mind dealing with locally convex topologies

Notes of course given in the year 2025/2026 by professor Pietro Majer on
distribution theory from a functional analysis point of view, reorganized
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Introduction and notation

One of the most challenging problems a mathematician can face is giving a meaning to “physics math”. To describe natural phenomena physicists use the so called *Dirac delta*: a “function” $\delta : \mathbb{R} \rightarrow \mathbb{R}$ defined as $\delta(0) = +\infty$, $\delta(x) = 0$ on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} \delta(x) f(x) dx = f(0) \text{ for every physically reasonable } f : \mathbb{R} \rightarrow \mathbb{R}.$$

To make sense of notions like this Laurent Schwartz introduced the concept of distributions: we can think the Dirac delta as an operator acting on an appropriate set of *test functions*. The most common choice for test functions is the space $C_0^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set. Of course we don’t want to look at all linear maps $C_0^\infty(\Omega) \rightarrow \mathbb{R}$ but only to a subset of reasonable operators: the ones who are continuous with respect to a certain topology. It turns out the right topology is not easy to define: it is an inductive limit of metrizable topological vector spaces. This topology is not metrizable so it is necessary to invoke the general theory of locally convex topological vector spaces, in fact much of the mathematical research on this topic was driven by distributions. Not many people know that even Alexander Grothendieck, the famous algebraic geometer, did his PHD under the supervision of Schwartz on functional analysis. Nowadays in many fields of analysis turns out one can work with distributions without warring on the topological aspect: usually continuity for an operator $C_0^\infty(\Omega) \rightarrow \mathbb{R}$ is defined in terms of a “notion of convergence”. Still, if someone wants to achieve a full understanding of the theory they should study the topological details. Also I personally believe that this is a really elegant and beautiful theory that is worth studying regarding the applications. Especially if you are like me and cannot stand when things are not clearly defined, I really recommend looking into it.

To give an insight of the power of distributions consider the problem of determining if a partial differential equation admits a solution. Take a linear partial differential operator $P \in \mathbb{R}[\partial_1, \dots, \partial_N]$ (P is a polynomial in the “variables” $\partial_1, \dots, \partial_N$ with **constants** real coefficients) who acts on C^∞ functions, given $g \in C^\infty(\mathbb{R}^N)$ we seek a solution to the partial differential equation

$$Pf = g \text{ for } f \in C^\infty(\mathbb{R}^N).$$

Now we will state three facts about distributions:

- $C^\infty(\mathbb{R}^n)$ functions can be embedded in the space of distributions,
- it is possible to define the notion of the derivative of a distribution, and all distributions are infinitely differentiable,
- we can extend the convolution operation to distributions.

In view of this facts the problem $Pf = g$ makes sense when f and g are distributions. Denote by δ_0 the Dirac delta centered in 0, suppose we find a distribution u that satisfies

$$Pu = \delta_0$$

Such u is called a *fundamental solution*. Then we can convolve both sides with g :

$$(Pu) * g = \delta_0 * g \Rightarrow P(u * g) = g$$

were we used the facts that convolution commutes with P (wich is a classical result for C^∞ functions) and convolving with δ_0 does not change a distribution. Therefore we can easily find solutions with arbitrary known data given a fundamental solution. In general $u * g$ is just a distribution, so the problem of finding a solution turned into the problem of determining the regularity of $u * g$. It is worth knowing that this method can be extended to the case of complex coefficients. Inspired by this and other techniques Lars Hörmander used distribution theory to give the first solid treatment of linear

partial differential equations [1]. Before his contributions there were result focusing on particular equations but mathematics was lacking of a general theory.

I decided not to include in the notes the general theory of topological vector spaces since it is already covered in many textbooks, in the appendix there are some facts used later on topological vector spaces and functional analysis (mostly without proofs).

For the sake of clarity we fix some notation:

- Ω will always denote an open set of $\mathbb{R}^N$
- $\mathcal{K}(\Omega)$ is the set of compact subsets of Ω
- $C^0(\Omega)$ is the space of continuous functions $\Omega \rightarrow \mathbb{R}$
- $C^k(\Omega)$ with $0 < k \leq \infty$ is the space of k -times differentiable functions $\Omega \rightarrow \mathbb{R}$
- $C_c^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions with compact support
- $C_0^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions f such that for every $\varepsilon > 0$ there exist a compact set $K \subset \Omega$ with $|f| < \varepsilon$ on $\Omega \setminus K$
- $C_b^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions f such that $\partial^\alpha f$ is bounded for every multindex $|\alpha| \leq k$
- $L^p(\Omega)$ is the set of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that $\int |f|^p < \infty$
- $L_{loc}^1(\Omega)$ is the set of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that for all $x \in \Omega$ there exist an open neighborhood $U \subseteq \Omega$ with $f \in L^1(U)$.

1. Distributions

1.1. Topology on test spaces

Definition 1.1. For a fixed Ω we define the LCTVS

$$\mathcal{D}_K := \{f \in C_c^\infty(\Omega) \mid \text{supp}(f) \subseteq K\}$$

with the topology induced by the family of seminorms

$$p_m(f) := \max_{|\alpha| \leq m} \|\partial^\alpha f\|_\infty, m \in \mathbb{Z}_{\geq 0}$$

Since $\mathcal{D}_K$ has a countable family of seminorms in indeed metrizable with a translation-invariant metric. Also note that the family of seminorms is filtered ($p_m(f) \leq p_n(f)$ for $m \leq n$).

Proposition 1.2. $\mathcal{D}_K$ is complete, therefore is a Frechét space.

Proof. Let (φ_n) be a Cauchy sequence in $\mathcal{D}_K$, (almost) by definition we have

$$p_K(\varphi_n - \varphi_m) \xrightarrow{n, m \rightarrow \infty} 0 \text{ for all } K \in \mathcal{K}(\Omega).$$

In other words for every multindex α the sequence $(\partial^\alpha \varphi_n)$ is Cauchy with respect to the supremum norm. Since the supremum norm induces a complete topology all partial derivatives converge uniformly, say $\partial^\alpha \varphi_n \xrightarrow{\|\cdot\|_\infty} \eta_\alpha$. It is then a standard fact that $\eta_0 \in C_c^\infty(\Omega)$, it is supported in K and for every multindex $\partial^\alpha \eta_0 = \eta_\alpha$. We conclude that $\varphi_n \xrightarrow{\mathcal{D}_K} \eta_0$. $\square$

Recall the following definition:

Definition 1.3. A partially ordered set I is said to be *directed* if for all $x, y \in I$ there exist $z \in I$ such that $x \leq z$ and $y \leq z$.

It is obvious that $\mathcal{K}(\Omega)$ is a directed set ordered by inclusion. As such, we can form directed limits with objects indexed by $\mathcal{K}(\Omega)$.

Definition 1.4. We define the space of test functions as the direct limit (in the category of topological vector spaces) of the spaces $\mathcal{D}_K$:

$$\mathcal{D}(\Omega) := \lim_{\substack{\longrightarrow \\ \mathcal{K}(\Omega)}} \mathcal{D}_K.$$

Note that if $K_1 \subset K_2$ are two compact sets the topology of $\mathcal{D}_{K_1}$ is the same as the subspace topology of $\mathcal{D}_{K_1} \subset \mathcal{D}_{K_2}$. We conclude that the limit in the definition is strict. Also $\mathcal{D}_{K_1}$ is closed in $\mathcal{D}_{K_2}$. In particular keep in mind that a subset $A \subset \mathcal{D}(\Omega)$ is limited if and only if A is contained and limited in some $\mathcal{D}_K$. Also if (K_n) is any exhaustion of Ω then $\mathcal{D}(\Omega) \cong \lim \mathcal{D}_{K_n}$, and note that as vector spaces $\mathcal{D}(\Omega)$ is just $C_c^\infty(\Omega)$.

Proposition 1.5. The topology on $\mathcal{D}(\Omega)$ is not metrizable.

Proof. Fix an exhaustion (K_n) of Ω . Then $\mathcal{D}(\Omega)$ is the union of a countable family of closed sets

$$\mathcal{D}(\Omega) = \bigcup_n \mathcal{D}_{K_n}.$$

Since $\mathcal{D}_{K_n}$ is a proper subspace it has empty interior. If $\mathcal{D}(\Omega)$ were metrizable then it would be complete by the properties of strict limits, but then Baire's Theorem would imply that also $\mathcal{D}(\Omega)$ has empty interior, a contradiction. $\square$

Definition 1.6. We define the LCTVS of smooth functions $\mathcal{E}(\Omega)$ as the space $C^\infty(\Omega)$ equipped with the seminorms

$$p_{K,m}(f) = \max_{|\alpha| \leq m} \|\partial^\alpha f\|_{K,\infty}, K \in \mathcal{K}(\Omega), m \in \mathbb{Z}_{\geq 0}$$

Observation 1.7. A sequence $(f_n) \subset \mathcal{E}(\Omega)$ converges if and only if every derivative is convergent on compact subsets.

Proposition 1.8. $\mathcal{E}(\Omega)$ is a complete metrizable LCTVS, and therefore Frechét.

Proof. Choose an exhaustion (K_n) of Ω , then is it easy to see that the countable family of seminorms $\{p_{K_n,m}\}_{n,m \geq 0}$ induce the topology on $\mathcal{E}(\Omega)$ and therefore is metrizable. For completeness the proof is similar to Proposition 1.2. $\square$

Proposition 1.9. We have a continuous inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$. Moreover the space $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ is dense with respect to the topology on $\mathcal{E}(\Omega)$.

Proof. By universal property, fix a compact $K \in \mathcal{K}(\Omega)$ and consider $\mathcal{D}_K \hookrightarrow \mathcal{E}(\Omega)$. This is continuous since for every $m \geq 0$ and $K' \in \mathcal{K}(\Omega)$ we have

$$p_{K',m}(f) \leq p_m(f) \text{ for all } f \in \mathcal{D}_K$$

Fix an exhaustion (K_n) and a family of smooth bump functions (η_n) , with $\eta_n = 1$ on K_n . Then for every $f \in \mathcal{E}(\Omega)$ the sequence $(\eta_n f)$ is contained in $\mathcal{D}(\Omega)$ and $\eta_n f \xrightarrow{\mathcal{E}(\Omega)} f$. $\square$

Notation. Given $\alpha = (\alpha_1, \dots, \alpha_N), \beta = (\beta_1, \dots, \beta_N)$ multindices we denote

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \dots \cdot \binom{\alpha_N}{\beta_N}.$$

We say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for every $i = 1, \dots, N$.

Lemma 1.10. Let $\alpha = (\alpha_1, \dots, \alpha_N)$ be a multindex and $f, g \in C^{|\alpha|}(\Omega)$, then

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} g.$$

Moreover we have

$$\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|}.$$

Proof. A boring induction. $\square$

Lemma 1.11. (Leibniz estimate) Let $f, g \in C^m(\Omega)$, for every $|\alpha| \leq m$ we have the following estimate:

$$|\partial^\alpha(fg)(x)| \leq 2^{|\alpha|} \max_{\beta \leq \alpha} |\partial^\beta f(x)| \max_{\beta \leq \alpha} |\partial^\beta g(x)|$$

Proof. We use the previous lemma:

$$\begin{aligned} |\partial^\alpha(fg)(x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta f(x)| |\partial^{\alpha-\beta} g(x)| \leq \max_{\beta \leq \alpha} |\partial^\beta f(x)| \max_{\beta \leq \alpha} |\partial^\beta g(x)| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \\ &= 2^{|\alpha|} \max_{\beta \leq \alpha} |\partial^\beta f(x)| \max_{\beta \leq \alpha} |\partial^\beta g(x)| \end{aligned}$$

$\square$

Definition 1.12. Let $\{\Omega_i\}_{i \in I}$ be an open covering of Ω , we say that a family $\{\eta_i\}_{i \in I}$ is a *smooth partition of unity subordinate to $\{\Omega_i\}_{i \in I}$* if

1. each $\eta_i \in C^\infty(\Omega)$
2. $\text{supp } \eta_i \subset \Omega_i$ for all i
3. the family of supports is locally finite, i.e. every point $x \in \Omega$ has a neighborhood U such that the set

$$\{i \in I \mid \text{supp } \eta_i \cap U \neq \emptyset\} \subset I$$

is finite

4. $\sum_{i \in I} \eta_i(x) = 1$ for all $x \in \Omega$

Theorem 1.13. Smooth partitions of unity always exists.

Proof. Omitted since is a classical result. The main ingredient in the proof is the paracompactness of Ω . Note that in our use cases usually one can construct by hand a partition of unity. $\square$

The following theorem gives a family of seminorms that generates a topology equal to the limit one.

Theorem 1.14. (Gårding-Lions seminorms) For every pair $\sigma, \mu : \Omega \rightarrow \mathbb{R}_{>0}$ of continuous functions we define the seminorm on $\mathcal{D}(\Omega)$

$$q_{\sigma, \mu}(f) = \sup_{\substack{x \in \Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)|$$

by varying σ, μ we obtain a family of seminorms that generates the topology on $\mathcal{D}(\Omega)$.

Proof. We denote by τ_∞ the standard topology on $\mathcal{D}(\Omega)$ and with τ_q the topology generated by the seminorms. To show $\tau_q \subset \tau_\infty$ we use the universal property of inductive limits and show that for all $K \in \mathcal{K}(\Omega)$ the inclusion $\mathcal{D}_K \hookrightarrow (\mathcal{D}(\Omega), \tau_q)$ is continuous. Fix a seminorm $q_{\sigma, \mu}$ on $\mathcal{D}(\Omega)$, for every $f \in \mathcal{D}_K$ we have

$$q_{\sigma, \mu}(f) = \sup_{\substack{x \in \Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)| \leq \left(\max_K \sigma \right) \max_{|\alpha| \leq [\max_K \mu]} \|\partial^\alpha f\|_\infty = \left(\max_K \sigma \right) p_{[\max_K \mu]}(f).$$

For the converse let $\varphi : \Omega \rightarrow [0, +\infty)$ be a continuous proper map (the preimage of compact sets is compact). We define

$$K_n := \varphi^{-1}([n-1, n+1]) \text{ for } n \geq 0$$

$$U_n := \varphi^{-1}((n-1, n+1)) \text{ for } n \geq 0$$

and note that $\bigcup_n U_n = \bigcup_n K_n = \Omega$, $U_n \subset K_n \subset \Omega$ for every n and $U_n \cap U_m \neq \emptyset$ if and only if $|n-m| \leq 1$. Fix an open, convex and balanced neighborhood of zero $V \in \tau_\infty$, for all n we have that $V \cap \mathcal{D}_{K_n}$ is an open set in $\mathcal{D}_{K_n}$ and therefore there exists m_n, δ_n such that

$$\{\varphi \in \mathcal{D}_{K_n} \mid p_{m_n}(\varphi) < \delta_n\} \subset V \cap \mathcal{D}_{K_n}.$$

We wanna patch the constants m_n, δ_n together, in order to achieve this take a smooth partition of unity (η_n) subordinate to (U_n) . Now define:

$$\begin{aligned} \mu(x) &:= \sum_{n \geq 0} \eta_n(x) \cdot \max_{|j-n| \leq 1} (m_j) \\ \sigma(x) &:= \sum_{n \geq 0} \eta_n(x) \cdot \max_{|j-n| \leq 1} \left(\frac{2^{m_j+j+1} p_{m_j, K_j}(\eta_j)}{\delta_j} \right) \end{aligned}$$

Note that if $x \in K_n$ then

$$\begin{aligned}\mu(x) &\geq m_n \\ \sigma(x) &\geq \frac{2^{m_n+n+1}p_{m_n,K_n}(\eta_n)}{\delta_n}\end{aligned}$$

Now take any $f \in \mathcal{D}(\Omega)$ with $q_{\sigma,\mu}(f) < 1$, there exists $\bar{n}$ such that $\text{supp } f \subset U_0 \cup \dots \cup U_{\bar{n}}$ and

$$f = \eta_0 f + \dots + \eta_{\bar{n}} f$$

with $\eta_n f \in \mathcal{D}_{K_n}$ for every $n = 0, \dots, \bar{n}$. Then by Leibniz estimate (Lemma 1.11)

$$p_{m_n}(\eta_n f) = \sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha(\eta_n f)(x)| \leq 2^{|m_n|} p_{m_n,K_n}(\eta_n) p_{m_n,K_n}(f)$$

And also

$$1 > q_{\sigma,\mu}(f) = \sup_{\substack{x \in \Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)| \geq \sup_{\substack{x \in K_n \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)| \geq \frac{2^{m_n+n+1}p_{m_n,K_n}(\eta_n)}{\delta_n} \sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha f(x)|$$

where $\sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha f(x)| = p_{m_n}(f)$. By the last two inequalities:

$$p_{m_n}(\eta_n f) \leq 2^{|m_n|} p_{m_n,K_n}(\eta_n) p_{m_n}(f) \leq \frac{2^{|m_n|} p_{m_n,K_n}(\eta_n) \delta_n}{2^{m_n+n+1} p_{m_n,K_n}(\eta_n)} = \frac{\delta_n}{2^{n+1}}$$

this implies that $\eta_n f \in 2^{-(n+1)}V \cap \mathcal{D}_{K_n}$. Finally note that:

$$f = \eta_0 f + \dots + \eta_{\bar{n}} f \in \frac{1}{2}V + \dots + \frac{1}{2^{\bar{n}+1}}V \subset \left(\frac{1}{2} + \dots + \frac{1}{2^{\bar{n}+1}}\right)V = \left(1 - \frac{1}{2^{\bar{n}+1}}\right)V \subset V$$

Where we used the fact that, since V is convex, $aV + bV \subseteq (a+b)V$ for every $a, b \in \mathbb{R}_{\geq 0}$. This implies $\{f \in \mathcal{D}(\Omega) \mid q_{\sigma,\mu}(f) < 1\} \subset V$ and we conclude $\tau_\infty \subset \tau_q$. $\square$

Definition 1.15. We define the space of distributions $\mathcal{D}'(\Omega)$ as the topological dual space of $\mathcal{D}(\Omega)$.

$$\mathcal{D}'(\Omega) := \{u : \mathcal{D}(\Omega) \rightarrow \mathbb{R} \text{ where } u \text{ is linear and continuous}\}$$

We also define the space of compactly supported distributions (the name will be clear afterword)

$$\mathcal{E}'(\Omega) := \{u : \mathcal{E}(\Omega) \rightarrow \mathbb{R} \text{ where } u \text{ is linear and continuous}\}$$

Notation. We adopt the duality notation: for $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ we write $\langle u, \varphi \rangle$ instead of $u(\varphi)$.

Proposition 1.16. For a linear map $u : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ the following are equivalent:

1. u is continuous, i.e. is a distribution
2. for all $K \in \mathcal{K}(\Omega)$, $u|_{\mathcal{D}_K}$ is continuous
3. for all $K \in \mathcal{K}(\Omega)$, there exists $m = m(K)$, $C = C(K)$, such that $|\langle u, \varphi \rangle| \leq C p_{m(\varphi)}$ for all $\varphi \in \mathcal{D}_K$
4. there exists $\sigma, \mu \in C^0(\Omega)_+$ such that $\langle u, \varphi \rangle \leq q_{\sigma,\mu}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$
5. for all sequences (φ_n) converging to zero in $\mathcal{D}(\Omega)$ we have $\langle u, \varphi_n \rangle \rightarrow 0$
6. for all compact sets $K \in \mathcal{K}(\Omega)$ and all sequences (φ_n) in $\mathcal{D}_K$ converging to zero we have $\langle u, \varphi_n \rangle \rightarrow 0$

Proof. (1) is equivalent to (2) by the universal property of inductive limits. Statement (3) is just (2) stated in terms of seminorms. Statement (4) is continuity stated in terms of Gårding-Lions seminorms.

Statement (6) is equivalent to (2) because $\mathcal{D}_K$ is a metric space and is also equivalent to (5) since a convergin sequence (φ_n) in $\mathcal{D}(\Omega)$ is bounded, and so (φ_n) must lie $\mathcal{D}_K$ for some $K \in \mathcal{K}(\Omega)$. $\square$

It is natural to choose a topology for the space of distributions. Recall:

Definition 1.17. Let X be a vector space, let X^* be the dual of X and $F \subset X^*$ a subspace. We define the weak topology associated to F , denoted $\sigma(X, F)$, as the smallest topology on X making all the maps in F continuos.

Clearly with this topology all maps in F are continuos. It is remarkable that a simple excercise in linear algebra shows:

Exercise 1.18. The topological dual of $(X, \sigma(X, F))$ is indeed F .

We can then define a natural topology on $\mathcal{D}'(\Omega)$:

Definition 1.19. We define the topology on $\mathcal{D}'(\Omega)$ as the weak topology associated with the family of functionals $\mathcal{D}(\Omega)$, seen as embedded into the bidual space of $\mathcal{D}(\Omega)$. More explicitly it is the weak topology associated to the functionals

$$\begin{aligned}\tilde{\varphi} : \mathcal{D}'(\Omega) &\rightarrow \mathbb{R} \\ u &\mapsto \langle u, \varphi \rangle\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Now we will make a simple, but powerful, observation.

Proposition 1.20. Let $T : \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$ be a linear continuos map. Then the transpose $T^* : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is continuos.

Proof. We shall use the universal property of weak topology. Fix $\varphi \in \mathcal{D}(\Omega)$, we prove that the map

$$\begin{aligned}\tilde{\varphi} \circ T^* : \mathcal{D}'(\Omega) &\rightarrow \mathbb{R} \\ u &\mapsto \langle u, T(\varphi) \rangle\end{aligned}$$

is continuos. But this is just the functional on $\mathcal{D}(\Omega)$ represented by $T(\varphi)$ and is therefore continuos by the definition of the weak topology on $\mathcal{D}'(\Omega)$. $\square$

Proposition 1.21. By transposing the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ we obtain a (TODO continuos) injection

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$$

Proof. Suppose $u \in \mathcal{E}'(\Omega)$ has the property that $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, the we must show that $u = 0$. But this follows from the density of test functions in $\mathcal{E}(\Omega)$, Proposition 1.9. $\square$

We can embed all “reasonable” functions in $\mathcal{D}'(\Omega)$, this is why distributions are called generalized functions. Recall that the space $L^1_{\text{loc}}(\Omega)$ is topologized via the seminorms

$$\{p_K(f) = \|f\|_{L^1(K)} \mid K \in \mathcal{K}(\Omega)\}.$$

Theorem 1.22. There is a continuos embedding

$$\begin{aligned}L^1_{\text{loc}}(\Omega) &\hookrightarrow \mathcal{D}'(\Omega) \\ f &\mapsto \int_{\Omega} f \cdot -\end{aligned}$$

Proof. To show continuity we use the universal property. Fix $\varphi \in \mathcal{D}(\Omega)$ and consider

$$L_{\text{loc}}^1 \rightarrow \mathbb{R}$$

$$f \mapsto \int_{\Omega} f \varphi$$

We have

$$\left| \int_{\Omega} f \varphi \right| \leq \|f\|_{L^1(\text{supp } \varphi)} \|\varphi\|_{\infty} = p_{\text{supp } \varphi}(f) \|\varphi\|_{\infty}.$$

and this shows that the functional is continuous over L_{loc}^1 . To show injectiveness we shall prove that

$$\int_{\Omega} f \varphi = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) \Rightarrow f = 0 \text{ a.e.}$$

but this is famous result known as “fundamental Lemma of calculus of variations”. □

Exercise 1.23. Show that the following maps are continuous:

$$\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$$

$$(f, g) \mapsto fg$$

$$\mathcal{E}(\Omega) \times \mathcal{E}(\Omega) \rightarrow \mathcal{E}(\Omega)$$

$$(f, g) \mapsto fg$$

$$\mathcal{E}(\Omega) \times \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$$

$$(f, g) \mapsto fg$$

Exercise 1.24. Recall that for a compact set $K \in \mathcal{K}(\Omega)$ the space $C^0(K)$ is a Banach space with the supremum norm, and we can define a topology on $C_c^0(\Omega)$ by

$$C_c^0(\Omega) := \varinjlim_{K \in \mathcal{K}(\Omega)} C^0(K)$$

where the limits are intended in the **category of topological spaces** (for the sake of the exercise). Show that the map

$$\mathbb{R} \times C_c^0(\Omega) \rightarrow C_c^0(\Omega)$$

$$(\lambda, f) \mapsto \lambda f$$

is continuous, and determine if the summation map is continuous:

$$C_c^0(\Omega) \times C_c^0(\Omega) \rightarrow C_c^0(\Omega)$$

$$(f, g) \mapsto f + g$$

1.2. Operations on distributions

Thanks to Proposition 1.20 we can extend an uncountable amount of operations on $\mathcal{D}(\Omega)$ to the distributions. It is left as an exercise to verify that such operations on $\mathcal{D}(\Omega)$ are continuous.

Definition 1.25. (multiplication by smooth functions) Fix $f \in \mathcal{E}(\Omega)$, we define multiplication by f as the transpose of

$$M_f : \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$$

$$\varphi \mapsto \varphi f$$

Explicitly we have $\langle fu, \varphi \rangle = \langle u, f\varphi \rangle$.

Definition 1.26. (derivative) Fix a multindex α . The derivative operator is the transpose of

$$\begin{aligned}\partial^\alpha : \mathcal{D}(\Omega) &\rightarrow \mathcal{D}(\Omega) \\ \varphi &\mapsto \partial^\alpha \varphi\end{aligned}$$

multiplied by $(-1)^{|\alpha|}$. Explicitly we have $\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle$.

The reason behind the change of sign is to generalize the known formula of integration by parts, in fact for every $f \in C^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$:

$$\int_{\Omega} f \partial_i \varphi = - \int_{\Omega} \partial_i f \varphi$$

where the integral on $\partial\Omega$ is zero because φ has compact support. Note how all distributions have infinite derivatives. As such we can derive all L^1_{loc} functions. Also we can generalize known constructions:

Example 1.27. Sobolev functions $W^{1,2}(\Omega)$ are just L^2 functions with distributional derivative in L^2 . A non trivial result shows that we can define the space of bounded variation functions $BV(\Omega)$ as the space of L^1 functions with a Borel measure as distributional derivative.

1.3. Local nature of distributions

Sheaves are general objects used to track data on topological spaces that can be restricted and glued. Their use was popularized by Grothendieck who used them systematically in algebraic geometry. We shall prove that distributions form a sheaf.

Definition 1.28. (Presheaves) A *presheaf* $\mathcal{F}$ of vector spaces on a topological space X is the datum of:

- a vector space $\mathcal{F}(U)$ for every open set U
- a linear restriction map $\rho_V^U : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for all open sets $V \subset U$

Such that for all triples $W \subset V \subset U$ we have $\rho_W^U = \rho_W^V \rho_V^U$, in other words we have a commutative triangle

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_W^U} & \mathcal{F}(W) \\ & \searrow \rho_V^U & \nearrow \rho_W^V \\ & \mathcal{F}(V) & \end{array}$$

Notation. If $\varphi \in \mathcal{F}(U)$ we write $\varphi|_V$ for $\rho_V^U(\varphi)$.

Definition 1.29. (Sheaves) A presheaf $\mathcal{F}$ is called *separated* if for every open set U and every open covering $U = \bigcup_{i \in I} U_i$ we have

- if $\varphi \in \mathcal{F}(U)$ and $\varphi|_{U_i} = 0$ for all i then $\varphi = 0$.

If $\mathcal{F}$ satisfies also

- for a datum $\{\varphi_i\}_{i \in I}$ with $\varphi_i \in \mathcal{F}(U_i)$, such that $\varphi_i|_{U_j} = \varphi_j|_{U_i}$ for all $i, j \in I$, there exist a (unique by the previous property) $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$;

then $\mathcal{F}$ is called a *sheaf*.

Example 1.30. For all $0 \leq m \leq \infty$, C^m is a sheaf on $\mathbb{R}^n$. On the other hand bounded continuous functions are a separated presheaf but not a sheaf. Similarly L^1 is not a sheaf but L^1_{loc} is. In a certain

sense, we can define L_{loc}^1 as the smallest sheaf containing L^1 . This process is standard and is known as *sheafification*.

What distinguishes presheaves from sheaves is the local nature of the objects encoded.

Proposition 1.31. $\mathcal{D}'$ is a presheaf on $\mathbb{R}^n$.

Proof. Given $V \subset U$ and $u \in \mathcal{D}'(U)$, u is naturally an element of $\mathcal{D}'(V)$ since $\mathcal{D}(V) \subset \mathcal{D}(U)$. $\square$

Lemma 1.32. Let $\Omega = \bigcup \Omega_i$ be an open covering. Then

$$\mathcal{D}(\Omega) = \bigoplus_i \mathcal{D}(\Omega_i),$$

where the direct sum is intended as sum of vector subspaces.

Proof. Let $\varphi \in \mathcal{D}(\Omega)$, by compactness there exist $\Omega_1, \dots, \Omega_n$ elements of the covering such that $\text{supp}(\varphi) \subset \Omega_1 \cup \dots \cup \Omega_n$. Now let $\eta_1, \dots, \eta_n$ be a smooth partition of unity subordinate to $\Omega_1 \cup \dots \cup \Omega_n$, clearly

$$\varphi = \eta_1 \varphi + \dots + \eta_n \varphi$$

and each $\eta_k \varphi$ is an element of $\mathcal{D}(\Omega_k)$. $\square$

Corollary 1.33. $\mathcal{D}'$ is a separated presheaf.

Proposition 1.34. $\mathcal{D}'$ is a sheaf.

Proof. Let $\Omega = \bigcup_i \Omega_i$ be an open covering. Let $\{u_i\}$ be a collection of distributions with $u_i \in \mathcal{D}'(\Omega_i)$ that satisfies the compatibility condition of sheaf definition. Given $\varphi \in \mathcal{D}(\Omega)$ we can decompose φ as in the lemma as a sum $\varphi = \varphi_{i_1} + \dots + \varphi_{i_n}$ with each $\varphi_{i_r} \in \mathcal{D}'(\Omega_{i_r})$. Now we define $\langle u, \varphi \rangle := \langle u_{i_1}, \varphi_{i_1} \rangle + \dots + \langle u_{i_n}, \varphi_{i_n} \rangle$, we just need to check that it is well defined. Let $\varphi = \varphi_{j_1} + \dots + \varphi_{j_n}$ be another decomposition of φ , we should have

$$\sum_{r=1}^n \langle u_{i_r}, \varphi_{i_r} \rangle = \sum_{r=1}^n \langle u_{j_r}, \varphi_{j_r} \rangle.$$

The case $n = 2$ is easy because $\varphi_{i_1} = \varphi_{j_1} = 1$ on $\Omega_1 \setminus \Omega_2$, similarly $\varphi_{i_2} = \varphi_{j_2} = 1$ on $\Omega_2 \setminus \Omega_1$ and on $\Omega_1 \cap \Omega_2$ the distributions u_1, u_2 coincide. The general case follows by induction.

Let now $\Omega_{j_1}, \dots, \Omega_{j_m}$ another covering of $\text{supp } \varphi$ and $\varphi = \varphi_{k_1} + \dots + \varphi_{k_m}$ a decomposition. We look for a common refinement. Let $\eta_{r,s}$ be a smooth partition of unity associated with the covering $\bigcup_{r,s} \Omega_{i_r} \cap \Omega_{k_s}$, then

$$\begin{aligned} \sum_{r=1}^n \langle u_{i_r}, \varphi_{i_r} \rangle &= \sum_{r=1}^n \langle u_{i_r}, \sum_{s=1}^m \varphi_{i_r} \eta_{r,s} \rangle = \sum_{r,s=1}^{n,m} \langle u_{i_r}, \varphi_{i_r} \eta_{r,s} \rangle \\ \sum_{s=1}^m \langle u_{k_s}, \varphi_{k_s} \rangle &= \sum_{s=1}^m \langle u_{k_s}, \sum_{r=1}^n \varphi_{k_s} \eta_{r,s} \rangle = \sum_{r,s=1}^{n,m} \langle u_{k_s}, \varphi_{k_s} \eta_{r,s} \rangle \end{aligned}$$

Now note that $\varphi_{i_r} \eta_{r,s}, \varphi_{k_s} \eta_{r,s}$ are both supported in $\Omega_{i_r} \cap \Omega_{k_s}$ and by compatibility of distributions $u_{i_r} = u_{k_s}$ on $\Omega_{i_r} \cap \Omega_{k_s}$. With the help of the previous part we deduce the right expressions are equal by invariance of the chosen decomposition of φ (for a fixed open covering). $\square$

Observation 1.35. The fact that $\mathcal{D}'$ is a sheaf let us define easily the notion of distributions on manifolds. Let M be a smooth real manifold, we define canonically $\mathcal{D}(U)$ for every affine chart U . Since affine charts form a base for the topology on M the sheaf axioms imply there exists a unique sheaf $\mathcal{D}'$ defined on the topological space M . The reader can check that elements of $\mathcal{D}'(V)$ acts naturally on

elements of $C_0^\infty(V)$ for every open set $V \subset M$ (not necessarily affine). This process is analogous to the construction of the structure sheaf on an affine scheme in algebraic geometry.

1.4. Order of a distribution

Definition 1.36. We say that a distribution $u \in \mathcal{D}'(\Omega)$ has order equal or less than m if for all $K \in \mathcal{K}(\Omega)$ there exist $C = C(K)$ such that

$$|u(f)| \leq Cp_m(f) \text{ for every } f \in \mathcal{D}_K.$$

The minimum of such integers is called the *order* of u . If there is not such m we say that u has infinite order.

Example 1.37. Let $x_0 \in \Omega$, the Dirac delta $\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0)$ is a distribution of order 0. In general $\varphi \mapsto \partial^\alpha \varphi(x_0)$ has order $|\alpha|$.

Example 1.38. Let (x_n) be a sequence in Ω that escapes from all compact sets and let α be a non-zero multindex. For $\varphi \in \mathcal{D}(\Omega)$ define

$$\langle u, \varphi \rangle := \sum_{n=1}^{\infty} \partial^{n \cdot \alpha} \varphi(x_n).$$

It is well defined because for every φ the sum is actually finite. It is easy to see that u is a distribution and has infinite order.

Theorem 1.39. Let $u \in \mathcal{D}'(\Omega)$ be a positive distribution, that is $\langle u, \varphi \rangle \geq 0$ for all $\varphi \geq 0$. If u has order zero then it is a measure.

Proof. Fix $K \in \mathcal{K}(\Omega)$, by hypothesis there exists a $C > 0$ such that $|\langle u, \varphi \rangle| \leq C\|\varphi\|_\infty$ for all $\varphi \in \mathcal{D}_K$. Take now $\eta \in \mathcal{D}(\Omega)$ such that:

- $\eta|_K = 1$
- $0 \leq \eta \leq 1$ in Ω

and by the positivity hypothesis

$$\langle u, \varphi + \|\varphi\|_\infty \eta \rangle \geq 0 \text{ and } \langle u, \varphi - \|\varphi\|_\infty \eta \rangle \leq 0 \Rightarrow |\langle u, \varphi \rangle| \leq \langle u, \eta \rangle \|\varphi\|_\infty$$

– TODO

This implies that $u : \mathcal{D}_K \rightarrow \mathbb{R}$ is continuous with respect to the supremum norm by Hahn-Banach we can extend u on $C^0(K)$. By the characterization of the dual space of $C^0(K)$ we conclude that there exists a measure μ on K such that

$$\langle u, \varphi \rangle = \int_K \varphi d\mu \text{ for all } \varphi \in \mathcal{D}_K$$

□

1.5. Sequences of distributions

Proposition 1.40. Let (u_n) be a sequence in $\mathcal{D}'(\Omega)$ converging pointwise: for all $\varphi \in \mathcal{D}(\Omega)$ the sequence $\langle u_n, \varphi \rangle$ is convergent in $\mathbb{R}$. Then the map

$$u : \mathcal{D}(\Omega) \rightarrow \mathbb{R} \\ \varphi \mapsto \lim_{n \rightarrow \infty} \langle u_n, \varphi \rangle$$

is linear and continuous and so defines a distribution.

Proof. The linearity is clear. To prove continuity fix a compact set $K \in \mathcal{K}(\Omega)$ and consider $u|_{\mathcal{D}_K}$. Since the set $\{\langle u_n, \varphi \rangle \mid \varphi \in \mathcal{D}_K\}$ is bounded and $\mathcal{D}_K$ is a complete metric space we can apply the Banach-Steinhaus Theorem and conclude that the sequence (u_n) is uniformly continuous on $\mathcal{D}_K$. This implies that exists $C = C(K)$, $m = m(K)$ such that:

$$|\langle u_n, \varphi \rangle| \leq Cp_m(\varphi) \text{ for all } \varphi \in \mathcal{D}_K \text{ and all } n \in \mathbb{Z}_{>0} \quad (1)$$

and by passing to the limit on the left we find the required inequality. $\square$

Corollary 1.41. In the setting of the preceding proposition, if (φ_n) is a sequence converging to φ in $\mathcal{D}(\Omega)$ then $\langle u_n, \varphi_n \rangle \rightarrow \langle u, \varphi \rangle$.

Proof. Since (φ_n) is convergent there exist a $K \in \mathcal{K}(\Omega)$ with $\text{supp } \varphi_n \subseteq K$ for all n . We have

$$|\langle u_n, \varphi_n \rangle - \langle u, \varphi \rangle| \leq |\langle u_n, \varphi_n \rangle - \langle u_n, \varphi \rangle| + |\langle u_n, \varphi \rangle - \langle u, \varphi \rangle|.$$

The second term goes to zero by definition of u , while for the first we apply the inequality 1

$$|\langle u_n, \varphi - \varphi_n \rangle| \leq Cp_m(\varphi - \varphi_n) \xrightarrow{n \rightarrow \infty} 0$$

where we used the fact that $\varphi_n \xrightarrow{\mathcal{D}(\Omega)} \varphi$. $\square$

1.6. Support

We can extend the notion of support to distribution. Recall that the support is defined for L^1_{loc} functions:

Definition 1.42. Let $f \in L^1_{\text{loc}}(\Omega)$, we define the support as the complement of the maximal open set $\Omega_0 \subset \Omega$ with $f|_{\Omega_0} = 0$ a.e., that is Ω_0 is the union of all open sets $U \subset \Omega$ with $f|_U = 0$ a.e.

It is slightly non trivial that $f|_{\Omega_0} = 0$ a.e. because Ω_0 can be the union of an uncountable family of open sets. For a proof we refer to [2]. Clearly if f is continuous then this new definition generalizes the classical one. Now we move onto distributions:

Definition 1.43. Let $u \in \mathcal{D}'(\Omega)$, we define the support as the complement of the maximal open set Ω_0 where $u|_{\Omega_0} = 0$. That is, let $\{U_i\}$ be the family of open subsets $U_i \subset \Omega$ with $u|_{U_i} = 0$, then $\Omega_0 := \bigcup_i U_i$ and $\text{supp } u := \Omega \setminus \Omega_0$.

The fact that $u|_{\Omega_0} = 0$ follows from the fact that distributions form a separated presheaf by Corollary 1.33.

Example 1.44. The support of δ_{x_0} is just $\{x_0\}$.

Proposition 1.45. Let $u \in \mathcal{D}'(\Omega)$, $K \in \mathcal{K}(\Omega)$ and consider the following statements:

1. $\text{supp } u \subset \text{Int}(K)$
2. there exists $C > 0$, $m \in \mathbb{Z}_{\geq 0}$ such that $|\langle u, \varphi \rangle| \leq Cp_{m,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$
3. $\text{supp } u \subset K$

then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$ Since $\text{supp } u$ is closed and $\text{supp } u \subset K$ then it is compact. We can find an open set U with $\text{supp } u \subset U \subset K$ and a bump function ψ with

- $0 \leq \psi \leq 1$
- $\psi = 1$ on U
- $\text{supp } \psi \subset K$

This implies that for any $\varphi \in \mathcal{D}(\Omega)$

$$\{x \in \Omega \mid (1 - \psi(x))\varphi(x) \neq 0\} \subset \Omega \setminus U \subset \Omega \setminus \text{supp } u$$

and therefore

$$0 = \langle u, (1 - \psi)\varphi \rangle \Rightarrow \langle u, \varphi \rangle = \langle u, \psi\varphi \rangle.$$

Since $\psi\varphi \in \mathcal{D}_K$, by continuity of u there exists $C > 0, m \in \mathbb{Z}_{\geq 0}$ with

$$|\langle u, \varphi \rangle| = |\langle u, \varphi\psi \rangle| \leq Cp_{m,K}(\psi\varphi)$$

and by Leibniz estimate (Lemma 1.11)

$$Cp_{m,K}(\psi\varphi) \leq C2^m p_{m,K}(\psi)p_{m,K}(\varphi)$$

and this is true for all $\varphi \in \mathcal{D}(\Omega)$.

(2) $\Rightarrow$ (3) Let $U \subset \Omega$ be an open set with $U \cap K = \emptyset$, we must show that $u|_U = 0$. Take any $\varphi \in \mathcal{D}(U) \subset \mathcal{D}(\Omega)$, we have $p_{m,K}(\varphi) = 0$ and $|\langle u, \varphi \rangle| \leq Cp_{m,K}(\varphi) = 0$. $\square$

Remark 1.46. In the hypothesis of the first statement of the last proposition we can extend u on a continuous functional defined over $C^m(\Omega)$. Recall that $C^m(\Omega)$ is topologized via the seminorms $\{p_{m,K}\}_{K \in \mathcal{K}(\Omega)}$, we can define for every $\varphi \in C^\infty(\Omega)$:

$$\langle \tilde{u}, \varphi \rangle := \langle u, \psi\varphi \rangle$$

where ψ is the bump function used in the proof. The inequality remains valid:

$$|\langle \tilde{u}, \varphi \rangle| \leq Cp_{m,K}(\varphi) \text{ for all } \varphi \in C^\infty(\Omega),$$

this implies $\tilde{u}$ is continuous on $C^\infty(\Omega) \subset C^m(\Omega)$. Then extend $\tilde{u}$ on all $C^m(\Omega)$ by density of $C^\infty(\Omega)$.

Now we can give a full characterization of distributions with compact support (and justify the name of $\mathcal{E}'(\Omega)$). Recall that we have an embedding $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ (Proposition 1.21).

Theorem 1.47. Let $u \in \mathcal{D}'(\Omega)$, then the following are equivalent:

1. $\text{supp } u$ is compact
2. $u \in \mathcal{E}'(\Omega)$

Proof. (1) $\Rightarrow$ (2) Find a compact set $K' \in \mathcal{K}(\Omega)$ such that $\text{supp } u \subset \text{Int}(K')$, then by the previous remark u extends on a continuous functional defined on $\mathcal{E}(\Omega)$.

(2) $\Rightarrow$ (1) By continuity of u with respect to the topology on $\mathcal{E}(\Omega)$ there must exist $C > 0, m \in \mathbb{Z}_{\geq 0}$ and $K \in \mathcal{K}(\Omega)$ such that

$$|\langle u, \varphi \rangle| \leq Cp_{m,K}(\varphi) \text{ for all } \varphi \in \mathcal{E}(\Omega).$$

By implication (2) $\Rightarrow$ (3) of the last proposition we get that u is supported in K . $\square$

Example 1.48. The implication (2) $\Rightarrow$ (1) is false, to see it just take any distribution with support a single point.

Example 1.49. The implication (3) $\Rightarrow$ (2) is false, for a counterexample take $\Omega = \mathbb{R}$, $K = \{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{Z}_{>0}}$ and

$$\langle u, \varphi \rangle := \sum_{n \geq 1} \frac{1}{n} \left(\varphi\left(\frac{1}{n}\right) - \varphi(0) \right)$$

then $\text{supp } \varphi = K$, take now a sequence (φ_n) such that:

- $\varphi_n \in \mathcal{D}(\Omega)$
- $\varphi_n(x) = 0$ for $x \leq \frac{1}{n+1}$

- $\varphi_n(x) = 1$ for $x \in [\frac{1}{n}, 1]$
- $0 \leq \varphi_n \leq 1$

then we have $\varphi_n^{(k)}(x) = 0$ for all $k > 0$ and $x \in K$ so the only hope is to bound $|\langle u, \varphi_n \rangle|$ with $\|\varphi_n\|_{\infty, K}$. But:

$$\langle u, \varphi_n \rangle = \sum_{j \geq 1} \frac{1}{j} \xrightarrow{n \rightarrow +\infty} +\infty$$

and $\|\varphi_n\|_{\infty, K} = 1$ for all n .

Exercise 1.50. Find some regularity hypothesis on K in order to make the implication (3) $\Rightarrow$ (2) true.

We can also characterize distributions with support equal to a point.

Proposition 1.51. Let $u \in \mathcal{D}'(\Omega)$ with $\text{supp } u = \{x_0\} \subset \Omega$, then there exists $m \in \mathbb{Z}_{\geq 0}$ and real constants $\{c_\alpha\}_{|\alpha| \leq m}$ such that

$$\langle u, \varphi \rangle = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \varphi(x_0) \text{ for all } \varphi \in \mathcal{D}(\Omega).$$

Proof. Without loss of generality $x_0 = 0 \in \Omega$. Let η be a cutoff function:

- $\eta \in C^\infty(\mathbb{R}^N)$
- $\eta = 1$ on $\mathcal{B}(0, 1)$
- $0 \leq \eta \leq 1$
- $\text{supp } \eta \subset \mathcal{B}(0, 2)$

and define $\eta_\varepsilon := \eta(\frac{x}{\varepsilon})$, note that for small ε (say ε_0) we have $\text{supp } \eta_{\varepsilon_0} \subset \Omega$. By hypothesis we have $\langle u, \varphi \rangle = \langle u, \eta_\varepsilon \varphi \rangle$ for all $\varepsilon < \varepsilon_0$. By continuity of u there must exist $C > 0$ and $m \in \mathbb{Z}_{\geq 0}$ such that

$$|\langle u, \eta_\varepsilon \varphi \rangle| \leq C p_m(\eta_\varepsilon \varphi) \text{ for all } \varepsilon < \varepsilon_0, \varphi \in \mathcal{D}(\Omega).$$

Now by generalized Leibniz rule:

$$|\partial^\alpha (\eta_\varepsilon \varphi)(x)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta \eta_\varepsilon(x) \partial^{\alpha-\beta} \varphi(x)| = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\varepsilon^{-|\beta|} \partial^\beta \eta\left(\frac{x}{\varepsilon}\right) \partial^{\alpha-\beta} \varphi(x)|$$

Suppose now that $\varphi = o(\|x\|^m)$, and observe that if $x \in \text{supp } \eta_\varepsilon$ then $\|x\| < 2\varepsilon$. This implies that thinking of ε as a variable then $\varphi = o(\varepsilon^m)$, this is fundamental since the goal is taking the limit $\varepsilon \rightarrow 0^+$. Now, by Taylor's formula, we have that $\partial^{\alpha-\beta} \varphi = o(\varepsilon^{m-(|\alpha-\beta|)})$ and furthermore there exists $M > 0$ such that $|\partial^\beta \eta(\frac{x}{\varepsilon})| \leq M$ for all $|\beta| \leq m, x \in \text{supp } \eta$. Finally we see that

$$|\partial^\alpha (\eta_\varepsilon \varphi)(x)| \leq M \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \varepsilon^{-|\beta|} |\partial^{\alpha-\beta} \varphi(x)|$$

and $\varepsilon^{-|\beta|} |\partial^{\alpha-\beta} \varphi| = o(\varepsilon^{m-|\alpha-\beta|-|\beta|}) = o(\varepsilon^{m-|\alpha|})$. Since we have a finite sum all the expression is $o(\varepsilon^{m-|\alpha|})$, in particular we deduce that $p_m(\eta_\varepsilon \varphi) = o(1)$, and so

$$|\langle u, \varphi \rangle| = |\langle u, \eta_\varepsilon \varphi \rangle| \leq C p_m(\eta_\varepsilon \varphi) \xrightarrow{\varepsilon \rightarrow 0^+} 0$$

Now take any $\varphi \in \mathcal{D}(\Omega)$, by Taylor's formula:

$$\varphi = \sum_{|\alpha| \leq m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha + R(x)$$

where $R = o(\|x\|^m)$. Then

$$\langle u, \varphi \rangle = \langle u, \sum_{|\alpha| \leq m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha \rangle + \langle u, R \rangle = \sum_{|\alpha| \leq m} \frac{\partial^\alpha \varphi(0)}{\alpha!} \langle u, x^\alpha \rangle$$

because $\langle u, R \rangle = 0$, and the thesis follows by setting $c_\alpha := \frac{\langle u, x^\alpha \rangle}{\alpha!}$. □

1.7. Some exercises

Exercise 1.52. Consider the following operator

$$T : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$\varphi \mapsto \lim_{\varepsilon \rightarrow 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

Show T is a well defined distribution of order 1.

Solution. Notice that the function $\frac{\varphi(0)}{x}$ is odd, therefore

$$\int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} = \int_{|x| \geq \varepsilon} \frac{\varphi(x) - \varphi(0)}{x}$$

Take $\varphi \in \mathcal{D}_{[-M, M]}$, we have by Lagrange's Theorem:

$$|\langle T, \varphi \rangle| \leq \int_{\varepsilon \leq |x| \leq M} \left| \frac{\varphi(x) - \varphi(0)}{x} \right| \leq 2M \|\varphi'\|_\infty$$

this shows that the integral is convergent by dominated convergence and T is continuous with order ≤ 1 . To show that the order is 1 consider a function φ_ε :

- $\varphi_\varepsilon \geq 0$
- $\varphi_\varepsilon(0) = 1$
- $\varphi_\varepsilon = 1$ on $[\varepsilon, 1]$.

Then

$$T(\varphi_\varepsilon) \geq \int_\varepsilon^1 \frac{1}{x} \geq |\log(\varepsilon)| \|\varphi_\varepsilon\|_\infty \xrightarrow{\varepsilon \rightarrow 0^+} +\infty$$

so we cannot bound $\langle T, \varphi \rangle$ with $\|\varphi\|_\infty$. □

Exercise 1.53. Consider the following operator

$$T : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$\varphi \mapsto \lim_{\varepsilon \rightarrow 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x^2} dx - \frac{2}{\varepsilon} \varphi(0)$$

Show T is a well defined distribution of order 2.

Solution. Notice that for $M > 0$

$$\frac{2}{\varepsilon} = \int_{\varepsilon \leq |x| \leq M} \frac{1}{x^2} dx + \frac{2}{M}$$

so it follows

$$\int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x)}{x^2} dx - \frac{2}{\varepsilon} \varphi(0) = \int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} dx - \frac{2}{M} \varphi(0)$$

Suppose $\varphi \in \mathcal{D}_{[-M,M]}$, then by Lagrange's reminder

$$\left| \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} \right| \leq \frac{1}{2}x^2\|\varphi''\|_\infty$$

so we get the bound

$$|\langle T, \varphi \rangle| = \left| \int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} dx - \frac{2}{M}\varphi(0) \right| \leq \frac{1}{2}M^2\|\varphi''\|_\infty + \frac{2}{M}\|\varphi\|_\infty$$

So by dominated convergence theorem the integral converges and T has order less or equal than 2. $\square$

Exercise 1.54. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ with $f = O(\|x\|^{-m})$ as $x \rightarrow 0$. Then, as a distribution, f extends to an element $T \in \mathcal{D}'(\mathbb{R}^N)$. Also characterize all the possible extension of f .

Solution. We define T using Taylor's expansion:

$$\langle T, \varphi \rangle := \int_{\|x\| \geq 1} f\varphi + \int_{\|x\| \leq 1} f \left(\varphi - \sum_{|\alpha| < m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha \right)$$

checking that is well defined and continuous is similar to the previous exercises with Lagrange's reminder:

$$\left| \varphi(x) - \sum_{|\alpha| < m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha \right| \leq Cp_m(\varphi)\|x\|^m$$

for some constant $C > 0$. To characterize all the extensions take T_1, T_2 extensions of f . Notice that $\text{supp}(T_1 - T_2) = \{0\}$, so by Proposition 1.51 we have

$$\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle + \sum_{|\alpha| \leq r} c_\alpha \partial^\alpha \varphi(0) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^N)$$

So two extensions of f differ by a linear combination of derivatives of Dirac deltas. $\square$

Exercise 1.55. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, $f > 0$, such that $\frac{1}{f} = o(\|x\|^m)$ as $x \rightarrow 0$, that is f grows more than any polynomial. Then f do not admit, as a distribution, an extension on $\mathcal{D}'(\mathbb{R}^N)$.

Solution. By contradiction suppose T is an extension of f . We seek a sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ such that $\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R}^N)} 0$ but not on $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$. Let $\eta \in \mathcal{D}(\mathbb{R}^N)$ be such that:

- $0 \leq \eta \leq 1$
- $\text{supp } \eta \subset \mathcal{B}(0, 4) \setminus \mathcal{B}(0, 1)$
- $\eta = 1$ on $\mathcal{B}(0, 3) \setminus \mathcal{B}(0, 2)$

and define $\eta_k := \frac{\eta(kx)}{k!}$. Notice that

$$\partial^\alpha \eta_k(x) = \frac{1}{k!} k^{|\alpha|} \partial^\alpha \eta(kx) \Rightarrow p_m(\eta_k) \leq \frac{1}{k!} k^m p_m(\eta) \xrightarrow{k \rightarrow \infty} 0 \text{ for all } m \geq 0$$

so $\eta_k \xrightarrow{\mathcal{D}(\mathbb{R}^N)} 0$ but does not converge in $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ since the supports of the η_k are not contained in any compact set $K \subset \mathbb{R}^N \setminus \{0\}$. Since T is an extension of f :

$$\langle T, \eta_k \rangle = \int_{\mathbb{R}^N \setminus \{0\}} f \eta_k \geq \frac{1}{k!} \int_{\frac{2}{k} \leq \|x\| \leq \frac{3}{k}} f \geq \frac{1}{k!} \frac{C}{k^n} \inf_{0 < \|x\| < \frac{3}{k}} f \xrightarrow{k \rightarrow \infty} +\infty$$



Exercise 1.56. Let $T \in \mathcal{D}'(\mathbb{R}^N)$, $f \in \mathcal{E}(\mathbb{R}^N)$ and suppose $f = 0$ on $\text{supp } T$. Then it is true that $fT = 0$?

A Appendix

A.1 Topological vector spaces

Definition 1.1. A topological vector space (TVS) over $\mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C})$ is a vector space X equipped with a topology such that the maps

$$\begin{array}{ll} X \times X \rightarrow X & \mathbb{K} \times X \rightarrow X \\ (x, y) \mapsto x + y & (\alpha, x) \mapsto \alpha x \end{array}$$

are continuous.

Observation 1.2. Since translations are homeomorphism the topology of a TVS is determined by the open neighborhoods of 0.

Definition 1.3. A TVS is called locally convex (LCTVS) if it admits a basis for the open neighborhoods consisting of convex sets.

Proposition 1.4. Let X be a LCTVS. Then there exist a family of seminorms $\mathcal{P} = \{p_i\}_{i \in I}$ such that

$$V(p_i, n) = \left\{ x \in X \mid p_i(x) < \frac{1}{n} \right\} \text{ with } i \in I, n \in \mathbb{Z}_{>0}$$

is a prebase that generates the topology of X .

Proposition 1.5. If X is a LCTVS topologized via a countable family of seminorms $\mathcal{P} = \{p_n\}$ such that for all $x \in X$ there exist n with $p_n(x) \neq 0$. Then X is metrizable with the translation-invariant distance

$$d(x, x') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x, x')}{1 + p_n(x, x')}$$

A complete metrizable LCTVS with a translation-invariant distance is often called a *Frechét* space.

Definition 1.6. A subset $A \subset X$ of a vector space is said to be:

- *balanced* if $\alpha A \subseteq A$ for every $|\alpha| \leq 1$
- *absorbing* if for every $x \in X$ there exist $t > 0$ such that $x \in tA$

Observation 1.7. In a TVS all open neighborhoods of 0 are absorbing.

Proposition 1.8. Every TVS X admits a basis $\mathcal{U}$ for the open neighborhoods of 0 such that every set $U \in \mathcal{U}$ is balanced. If X is a LCTVS then one can construct $\mathcal{U}$ such that all the neighborhoods are convex.

Definition 1.9. Given two subsets $A, B \subset X$ of a vector space we say that B *absorbs* A if there exist $t > 0$ such that $A \subset tB$. We say that a subset $A \subset X$ of a TVS is *bounded* if it is absorbed by every neighborhood of 0.

Definition 1.10. Let X be a TVS. A sequence $(x_n) \subset X$ is a Cauchy sequence if for all open neighborhoods U of zero we have $x_n - x_m \in U$ definitively. X is called *complete* if all Cauchy sequences converge.

It is easy to check that Cauchy sequences are bounded.

Proposition 1.11. Let X be a TVS, then

- X is always T_3 (you can separate points and closed sets)
- X is T_0 if and only if it is T_1
- If X is T_0 then X is T_2

- Let Y be a direct summand of $\overline{\{0\}}$, then X is isomorphic (as TVS) to $\overline{\{0\}} \oplus Y$ and Y is T_2 .

Definition 1.12. (Inductive limits in TVS) Let (X_n) be a sequence of TVS with continuous injections $X_n \hookrightarrow X_{n+1}$. Let

$$X_\infty := \bigcup_{n \geq 1} X_n$$

we define the topology τ_∞ on X_∞ by constructing a family of balanced open neighborhoods of 0:

$$\sum_{n \geq 1} V_k := \bigcup_{n \geq 1} \sum_{k=1}^n V_k$$

where each V_n is an open balanced neighborhood of 0 in X_n . The TVS (X_∞, τ_∞) is called the *direct limit* of the sequence (X_n) and is denoted by

$$(X_\infty, \tau_\infty) = \varinjlim X_n$$

It easy to prove that an inductive limit of LCTVS is again a LCTVS.

Theorem 1.13. (Universal property of TVS limit) Let (X_n) be a sequence of TVS with continuous injections $X_n \hookrightarrow X_{n+1}$. Denote with $X_\infty = \varinjlim X_n$ and let Y be a TVS, then a linear map $L : X_\infty \rightarrow Y$ is continuous if and only if $L|_{X_n} : X_n \rightarrow Y$ is continuous for all n .

Corollary 1.14. The limit topology τ_∞ on X_∞ is the finest TVS topology making all inclusions $X_n \hookrightarrow X_\infty$ continuous.

Proof. Let τ be another TVS topology making all inclusion continuous. Apply the universal property to show $\text{id} : (X, \tau_\infty) \rightarrow (X, \tau)$ is continuous. $\square$

Observation 1.15. The limit topology on X_∞ is in general different from the limit topology in the category of topological spaces. This means if $A \subset X_\infty$ is a subset such that $A \cap X_n$ is open for every n then A is **not** necessarily open in X_∞ .

Definition 1.16. An inductive limit of TVS is said to be *strict* if the maps $X_n \hookrightarrow X_{n+1}$ are topological embeddings.

Strict limits have very nice properties:

Theorem 1.17. (Properties of strict limits) Let $X_\infty = \varinjlim X_n$ be a strict limit of TVS. Then:

1. For all n the map $X_n \hookrightarrow X_\infty$ is a topological embedding
2. Fix $n_0 > 0$ and $C \subset X_{n_0}$, then C is closed in X_∞ if and only if is closed X_n for every $n \geq n_0$. In particular if every X_n is closed in X_{n+1} then every X_n is closed in X_∞ .
3. If every X_n is T_0 then X_∞ is T_0
4. If X_n is closed in X_{n+1} for every n then a subset $A \subset X_\infty$ is bounded if and only if there exist n_0 with $A \subset X_{n_0}$ and A is bounded in X_{n_0} .
5. If X_n is closed in X_{n+1} and X_n is complete for every n then the limit is complete.

A.2 Functional analysis

Usually the Banach-Steinhaus Theorem is presented in the context of Banach spaces. To generalize it we need some definitions and results. The reader who is familiar with the standard notion of equicontinuity should not be bothered by the following definition:

Notation. If X, Y are TVS we write $L(X, Y)$ for the vector space of continuous linear maps $X \rightarrow Y$.

Proposition 1.18. A family $\{L_i\}_{i \in I} \subset L(X, Y)$ is called *equicontinuous* if for all open neighborhoods U of $0 \in Y$ there exist an open neighborhood V of $0 \in X$ such that

$$L_i(V) \subseteq U \text{ for all } i \in I.$$

Definition 1.19. A subspace $A \subseteq X$ of a topological space is called *nowhere dense* if $\text{Int}(\overline{A}) = \emptyset$. A is said to be of *first category* if it is a countable union of nowhere dense subsets, otherwise is said to be of *second category*.

Theorem 1.20. (Baire's category theorem) Let $\{U_n\} \subseteq X$ be a countable collection of dense open sets, where X is

- a complete metric space, or
- a locally compact Hausdorff space;

then $\bigcap_n U_n$ is dense. In particular X is of second category.

Theorem 1.21. (Banach-Steinhaus) Let X, Y be TVS and $(L_i)_{i \in I} \subset L(X, Y)$ a pointwise bounded family on a second category subspace, that is to say the set

$$\{x \in X \mid \text{the set } \{L_i(x) \mid i \in I\} \subset Y \text{ is bounded}\}$$

is of second category. Then $(L_i)_{i \in I}$ is equicontinuous.

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