Distribution theory done right

or how I lost my mind dealing with locally convex topologies

Notes of course given in the year 2025/2026 by professor Pietro Majer on distribution theory from a functional analysis point of view, reorganized and written by Francesco Cerroni

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Introduction and notation

One of the most challenging problems a mathematician can face is giving a meaning to "physics math". To describe natural phenomena physicists use the so called *Dirac delta*: a "function" $\delta : \mathbb{R} \to \mathbb{R}$ defined as $\delta(0) = +\infty$, $\delta(x) = 0$ on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} \delta(x) f(x) dx = f(0) \text{ for every physically reasonable } f : \mathbb{R} \to \mathbb{R}.$$

To make sense of notions like this Laurent Schwartz introduced the concept of distributions: we can think the Dirac delta as an operator acting on an appropriate set of *test functions*. The most common choice for test functions is the space $C_0^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set. Of course we don't want to look at all linear maps $C_0^\infty(\Omega) \to \mathbb{R}$ but only to a subset of reasonable operators: the ones who are continuos with respect to a certain topology. It turns out the right topology is not easy to define: it is an inductive limit of metrizable topological vector spaces. This topology is not metrizable so it is necessary to invoke the general theory of locally convex topological vector spaces, in fact much of the mathematical research on this topic was driven by distributions. Not many people know that even Alexander Grothendieck, the famous algebraic geometer, did his PHD under the supervision of Schwartz on functional analysis. Nowadays in many fields of analysis turns out one can work with distributions without warring on the topological aspect: usually continuity for an operator $C_0^\infty(\Omega) \to \mathbb{R}$ is defined in terms of a "notion of convergence". Still, if someone wants to achieve a full understanding of the theory they should study the topological details. Also I personally believe that this is a really elegant and beautiful theory that is worth studying regarding the applications. Especially if you are like me and cannot stand when things are not clearly defined, I really recommend looking into it.

To give an insight of the power of distributions consider the problem of determining if a partial differential equation admits a solution. Take a linear partial differential operator $P \in \mathbb{R}[\partial_1, ..., \partial_N]$ (P is a polynomial in the "variables" $\partial_1, ..., \partial_N$ with **constants** real coefficients) who acts on C^∞ functions, given $g \in C^\infty(\mathbb{R}^N)$ we seek a solution to the partial differential equation

$$Pf = g \text{ for } f \in C^\infty(\mathbb{R}^N).$$

Now we will state three facts about distributions:

- $C^\infty(\mathbb{R}^n)$ functions can be embedded in the space of distributions,
- is it possible to define the notion of the derivative of a distribution, and all distributions are infinitely differentiable,
- we can extend the convolution operation to distributions.

In view of this facts the problem $Pf = g$ makes sense when f and g are distributions. Denote by δ_0 the Dirac delta centered in 0, suppose we find a distribution u that satisfies

$$Pu = \delta_0$$

Such u is called a *fundamental solution*. Then we can convolve both sides with g:

$$(Pu) * g = \delta_0 * g \Rightarrow P(u * g) = g$$

were we used the facts that convolution commutes with P (wich is a classical result for C^∞ functions) and convolving with δ_0 does not change a distribution. Therefore we can easily find solutions with arbitrary known data given a fundamental solution. In general $u * g$ is just a distribution, so the problem of finding a solution turned into the problem of determining the regularity of $u * g$. It is worth knowing that this method can be extended to the case of complex coefficients. Inspired by this and other techniques Lars Hörmander used distribution theory to give the first solid treatment of linear

partial differential equations [1]. Before his contributions there were result focusing on particular equations but mathematics was lacking of a general theory.

I decided not to include in the notes the general theory of topological vector spaces since it is already covered in many textbooks, in the appendix there are some facts used later on topological vector spaces and functional analysis (mostly without proofs).

For the sake of clarity we fix some notation:

- Ω will always denote an open set of $\mathbb{R}^N$
- $\mathcal{K}(\Omega)$ is the set of compact subsets of Ω
- $C^0(\Omega)$ is the space of continuos functions $\Omega \to \mathbb{R}$
- $C^k(\Omega)$ with $0 < k \leq \infty$ is the space of k-times differentiable functions $\Omega \to \mathbb{R}$
- $C_c^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions with compact support
- $C_0^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions f such that for every $\varepsilon > 0$ there exist a compact set $K \subset \Omega$ with $|f| < \varepsilon$ on $\Omega \setminus K$
- $C_b^k(\Omega)$, with $0 \leq k \leq \infty$, is the subspace of $C^k(\Omega)$ of functions f such that $\partial^\alpha f$ is bounded for every multindex $|\alpha| \leq k$
- $L^p(\Omega)$ is the set of measurable functions $f : \Omega \to \mathbb{R}$ such that $\int |f|^p < \infty$
- $L^1_{\text{loc}}(\Omega)$ is the set of measurable functions $f : \Omega \to \mathbb{R}$ such that for all $x \in \Omega$ there exist an open neighborhood $x \in U \subseteq \Omega$ with $f \in L^1(U)$.

1. Distributions

1.1. Topology on test spaces

Definition 1.1. For a fixed Ω we define the LCTVS

$$\mathcal{D}_K := \{f \in C_c^\infty(\Omega) \mid \operatorname{supp}(f) \subseteq K\}$$

with the topology induced by the family of seminorms

$$p_m(f) := \max_{|\alpha| \leq m} \|\partial^\alpha f\|_\infty, m \in \mathbb{Z}_{\geq 0}$$

Since $\mathcal{D}_K$ has a countable family of seminorms in indeed metrizable with a translation-invariant metric. Also note that the family of seminorms is filtered ($p_m(f) \leq p_n(f)$ for $m \leq n$).

Proposition 1.2. $\mathcal{D}_K$ is complete, therefore is a Frechét space.

Proof. Let (φ_n) be a Cauchy sequence in $\mathcal{D}_K$, (almost) by definition we have

$$p_K(\varphi_n - \varphi_m) \underset{n,m\to\infty}{\longrightarrow} 0 \text{ for all } K \in \mathcal{K}(\Omega).$$

In other words for every multindex α the sequence $(\partial^\alpha \varphi_n)$ is Cauchy with respect to the supremum norm. Since the supremum norm induces a complete topology all partial derivatives converge uniformly, say $\partial^\alpha \varphi_n \underset{\|\cdot\|_\infty}{\longrightarrow} \eta_\alpha$. It is then a standard fact that $\eta_0 \in C_c^\infty(\Omega)$, it is supported in K and for every multindex $\partial^\alpha \eta_0 = \eta_\alpha$. We conclude that $\varphi_n \underset{\mathcal{D}_K}{\longrightarrow} \eta_0$. □

Recall the following definition:

Definition 1.3. A partially ordered set I is said to be *directed* if for all $x, y \in I$ there exist $z \in I$ such that $x \leq z$ and $y \leq z$.

It is obvious that $\mathcal{K}(\Omega)$ is a directed set ordered by inclusion. As such, we can form directed limits with objects indexed by $\mathcal{K}(\Omega)$.

Definition 1.4. We define the space of test functions as the direct limit (in the category of topological vector spaces) of the spaces $\mathcal{D}_K$:

$$\mathcal{D}(\Omega) := \varinjlim_{\mathcal{K}(\Omega)} \mathcal{D}_K.$$

Note that if $K_1 \subset K_2$ are two compact sets the topology of $\mathcal{D}_{K_1}$ is the same as the subspace topology of $\mathcal{D}_{K_1} \subset \mathcal{D}_{K_2}$. We conclude that the limit in the definition is strict. Also $\mathcal{D}_{K_1}$ is closed in $\mathcal{D}_{K_2}$. In particular keep in mind that a subset $A \subset \mathcal{D}(\Omega)$ is limited if and only if A is contained and limited in some $\mathcal{D}_K$. Also if (K_n) is any exaustion of Ω then $\mathcal{D}(\Omega) \cong \lim \mathcal{D}_{K_n}$, and note that as vector spaces $\mathcal{D}(\Omega)$ is just $C_c^\infty(\Omega)$.

Proposition 1.5. The topology on $\mathcal{D}(\Omega)$ is not metrizable.

Proof. Fix an exhaustion (K_n) of Ω. Then $\mathcal{D}(\Omega)$ is the union of a countable family of closed sets

$$\mathcal{D}(\Omega) = \bigcup_n \mathcal{D}_{K_n}.$$

Since $\mathcal{D}_{K_n}$ is a proper subspace it has empty interior. If $\mathcal{D}(\Omega)$ were metrizable then it would be complete by the properties of strict limits, but then Baire's Theorem would imply that also $\mathcal{D}(\Omega)$ has empty interior, a contradiction. □

Definition 1.6. We define the LCTVS of smooth functions $\mathcal{E}(\Omega)$ as the space $C^\infty(\Omega)$ equipped with the seminorms

$$p_{K,m}(f) = \max_{|\alpha|\leq m} \|\partial^\alpha f\|_{K,\infty}, K \in \mathcal{K}(\Omega), m \in \mathbb{Z}_{\geq 0}$$

Observation 1.7. A sequence $(f_n) \subset \mathcal{E}(\Omega)$ converges if and only if every derivative is convergent on compact subsets.

Proposition 1.8. $\mathcal{E}(\Omega)$ is a complete metrizable LCTVS, and therefore Frechét.

Proof. Choose an exhaustion (K_n) of Ω, then is it easy to see that the countable family of seminorms $\left\{p_{K_n,m}\right\}_{n,m\geq 0}$ induce the topology on $\mathcal{E}(\Omega)$ and therefore is metrizable. For completeness the proof is similar to Proposition 1.2. $\square$

Proposition 1.9. We have a continuos inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$. Moreover the space $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ is dense with respect to the topology on $\mathcal{E}(\Omega)$.

Proof. By universal property, fix a compact $K \in \mathcal{K}(\Omega)$ and consider $\mathcal{D}_K \hookrightarrow \mathcal{E}(\Omega)$. This is continuos since for every $m \geq 0$ and $K' \in \mathcal{K}(\Omega)$ we have

$$p_{K',m}(f) \leq p_m(f) \text{ for all } f \in \mathcal{D}_K$$

Fix an exhaustion (K_n) and a family of smooth bump functions (η_n), with $\eta_n = 1$ on K_n. Then for every $f \in \mathcal{E}(\Omega)$ the sequence $(\eta_n f)$ is contained in $\mathcal{D}(\Omega)$ and $\eta_n f \underset{\mathcal{E}(\Omega)}{\longrightarrow} f$. $\square$

Notation. Given $\alpha = (\alpha_1, ..., \alpha_N), \beta = (\beta_1, ..., \beta_N)$ multindices we denote

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot ... \cdot \binom{\alpha_N}{\beta_N}.$$

We say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for every $i = 1, ..., N$.

Lemma 1.10. Let $\alpha = (\alpha_1, ..., \alpha_N)$ be a multindex and $f, g \in C^{|\alpha|}(\Omega)$, then

$$\partial^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f\ \partial^{\alpha-\beta} g.$$

Moreover we have

$$\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} = 2^{|\alpha|}.$$

Proof. A boring induction. $\square$

Lemma 1.11. (Leibniz estimate) Let $f, g \in C^m(\Omega)$, for every $|\alpha| \leq m$ we have the following estimate:

$$|\partial^\alpha(fg)(x)| \leq 2^{|\alpha|} \max_{\beta\leq\alpha} |\partial^\beta f(x)| \max_{\beta\leq\alpha} |\partial^\beta g(x)|$$

Proof. We use the previous lemma:

$$\begin{aligned} |\partial^\alpha(fg)(x)| &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |\partial^\beta f(x)|\ |\partial^{\alpha-\beta} g(x)| \leq \max_{\beta\leq\alpha} |\partial^\beta f(x)| \max_{\beta\leq\alpha} |\partial^\beta g(x)| \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \\ &= 2^{|\alpha|} \max_{\beta\leq\alpha} |\partial^\beta f(x)| \max_{\beta\leq\alpha} |\partial^\beta g(x)| \end{aligned}$$

$\square$

Definition 1.12. Let $\{\Omega_i\}_{i\in I}$ be an open covering of Ω, we say that a family $\{\eta_i\}_{i\in I}$ is a *smooth partition of unity subordinate to* $\{\Omega_i\}_{i\in I}$ if

1. each $\eta_i \in C^\infty(\Omega)$
2. supp $\eta_i \subset \Omega_i$ for all i
3. the family of supports is locally finite, i.e. every point $x \in \Omega$ has a neighborhood U such that the set
$$\{i \in I \mid \text{supp } \eta_i \cap U \neq \emptyset\} \subset I$$
is finite
4. $\sum_{i\in I} \eta_i(x) = 1$ for all $x \in \Omega$

Theorem 1.13. Smooth partitions of unity always exists.

Proof. Omitted since is a classical result. The main ingredient in the proof is the paracompacteness of Ω. Note that in our use cases usually one can construct by hand a partition of unity. $\square$

The following theorem gives a family of seminorms that generates a topology equal to the limit one.

Theorem 1.14. (Gårding-Lions seminorms) For every pair $\sigma, \mu : \Omega \to \mathbb{R}_{>0}$ of continuos functions we define the seminorm on $\mathcal{D}(\Omega)$
$$q_{\sigma,\mu}(f) = \sup_{\substack{x\in\Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x)\partial^\alpha f(x)|$$
by varying σ, μ we obtain a family of seminorms that generates the topology on $\mathcal{D}(\Omega)$.

Proof. We denote by τ_∞ the standard topology on $\mathcal{D}(\Omega)$ and with τ_q the topology generated by the seminorms. To show $\tau_q \subset \tau_\infty$ we use the universal property of inductive limits and show that for all $K \in \mathcal{K}(\Omega)$ the inclusion $\mathcal{D}_K \hookrightarrow (\mathcal{D}(\Omega), \tau_q)$ is continuos. Fix a seminorm $q_{\sigma,\mu}$ on $\mathcal{D}(\Omega)$, for every $f \in \mathcal{D}_K$ we have
$$q_{\sigma,\mu}(f) = \sup_{\substack{x\in\Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x)\partial^\alpha f(x)| \leq \left(\max_K \sigma\right) \max_{|\alpha| \leq \lceil \max_K \mu \rceil} \|\partial^\alpha f\|_\infty = \left(\max_K \sigma\right) p_{\lceil \max_K \mu \rceil}(f).$$

For the converse let $\varphi : \Omega \to [0, +\infty)$ be a continuos proper map (the preimage of compact sets is compact). We define
$$K_n := \varphi^{-1}([n-1, n+1]) \text{ for } n \geq 0$$
$$U_n := \varphi^{-1}((n-1, n+1)) \text{ for } n \geq 0$$
and note that $\bigcup_n U_n = \bigcup_n K_n = \Omega$, $U_n \subset K_n \subset \Omega$ for every n and $U_n \cap U_m \neq \emptyset$ if and only if $|n - m| \leq 1$. Fix an open, convex and balanced neighborhood of zero $V \in \tau_\infty$, for all n we have that $V \cap \mathcal{D}_{K_n}$ is an open set in $\mathcal{D}_{K_n}$ and therefore there exists m_n, δ_n such that
$$\left\{\varphi \in \mathcal{D}_{K_n} \mid p_{m_n}(\varphi) < \delta_n\right\} \subset V \cap \mathcal{D}_{K_n}.$$

We wanna patch the constants m_n, δ_n together, in order to achieve this take a smooth partition of unity (η_n) subordinate to (U_n). Now define:
$$\mu(x) := \sum_{n\geq 0} \eta_n(x) \cdot \max_{|j-n| \leq 1} (m_j)$$
$$\sigma(x) := \sum_{n\geq 0} \eta_n(x) \cdot \max_{|j-n| \leq 1} \left(\frac{2^{m_j+j+1} p_{m_j,K_j}(\eta_j)}{\delta_j}\right)$$

Note that if $x \in K_n$ then

$$\mu(x) \geq m_n$$
$$\sigma(x) \geq \frac{2^{m_n+n+1} p_{m_n,K_n}(\eta_n)}{\delta_n}$$

Now take any $f \in \mathcal{D}(\Omega)$ with $q_{\sigma,\mu}(f) < 1$, there exists $\overline{n}$ such that supp $f \subset U_0 \cup \ldots \cup U_{\overline{n}}$ and

$$f = \eta_0 f + \ldots + \eta_{\overline{n}} f$$

with $\eta_n f \in \mathcal{D}_{K_n}$ for every $n = 0, \ldots, \overline{n}$. Then by Leibniz estimate (Lemma 1.11)

$$p_{m_n}(\eta_n f) = \sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha (\eta_n f)(x) \leq 2^{|m_n|} p_{m_n,K_n}(\eta_n) p_{m_n,K_n}(f)$$

And also

$$1 > q_{\sigma,\mu}(f) = \sup_{\substack{x \in \Omega \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)| \geq \sup_{\substack{x \in K_n \\ |\alpha| \leq \mu(x)}} |\sigma(x) \partial^\alpha f(x)| \geq \frac{2^{m_n+n+1} p_{m_n,K_n}(\eta_n)}{\delta_n} \sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha f(x)|$$

where $\sup_{\substack{x \in K_n \\ |\alpha| \leq m_n}} |\partial^\alpha f(x)| = p_{m_n}(f)$. By the last two inequalities:

$$p_{m_n}(\eta_n f) \leq 2^{|m_n|} p_{m_n}(\eta_n) p_{m_n}(f) < \frac{2^{|m_n|} p_{m_n,K_n}(\eta_n) \delta_n}{2^{m_n+n+1} p_{m_n,K_n}(\eta_n)} = \frac{\delta_n}{2^{n+1}}$$

this implies that $\eta_n f \in 2^{-(n+1)} V \cap \mathcal{D}_{K_n}$. Finally note that:

$$f = \eta_0 f + \ldots + \eta_{\overline{n}} f \in \frac{1}{2} V + \ldots + \frac{1}{2^{\overline{n}+1}} V \subset \left(\frac{1}{2} + \ldots + \frac{1}{2^{\overline{n}+1}} \right) V = \left(1 - \frac{1}{2^{\overline{n}+1}} \right) V \subset V$$

Where we used the fact that, since V is convex, $aV + bV \subseteq (a+b)V$ for every $a, b \in \mathbb{R}_{\geq 0}$. This implies $\{ f \in \mathcal{D}(\Omega) | \; q_{\sigma,\mu}(f) < 1 \} \subset V$ and we conclude $\tau_\infty \subset \tau_q$. □

Definition 1.15. We define the space of distributions $\mathcal{D}'(\Omega)$ as the topological dual space of $\mathcal{D}(\Omega)$.

$$\mathcal{D}'(\Omega) := \{ u : \mathcal{D}(\Omega) \to R \text{ where } u \text{ is linear and continuos} \}$$

We also define the space of compactly supported distributions (the name will be clear afterword)

$$\mathcal{E}'(\Omega) := \{ u : \mathcal{E}(\Omega) \to R \text{ where } u \text{ is linear and continuos} \}$$

Notation. We adopt the duality notation: for $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ we write $\langle u, \varphi \rangle$ instead of $u(\varphi)$.

Proposition 1.16. For a linear map $u : \mathcal{D}(\Omega) \to \mathbb{R}$ the following are equivalent:
1. u is continuos, i.e. is a distribution
2. for all $K \in \mathcal{K}(\Omega)$, $u|_{\mathcal{D}_k}$ is continuos
3. for all $K \in \mathcal{K}(\Omega)$, there exists $m = m(K), C = C(K)$, such that $| \langle u, \varphi \rangle | \leq C p_{m(\varphi)}$ for all $\varphi \in \mathcal{D}_K$
4. there exists $\sigma, \mu \in C^0(\Omega)_+$ such that $\langle u, \varphi \rangle \leq q_{\sigma,\mu}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$
5. for all sequences (φ_n) converging to zero in $\mathcal{D}(\Omega)$ we have $\langle u, \varphi_n \rangle \longrightarrow 0$
6. for all compact sets $K \in \mathcal{K}(\Omega)$ and all sequences (φ_n) in $\mathcal{D}_K$ converging to zero we have $\langle u, \varphi_n \rangle \longrightarrow 0$

Proof. (1) is equivalent to (2) by the universal property of inductive limits. Statement (3) is just (2) stated in terms of seminorms. Statement (4) is continuity stated in terms of Gårding-Lions seminorms.

Statement (6) is equivalent to (2) because $\mathcal{D}_K$ is a metric space and is also equivalent to (5) since a convergin sequence (φ_n) in $\mathcal{D}(\Omega)$ is bounded, and so (φ_n) must lie $\mathcal{D}_K$ for some $K \in \mathcal{K}(\Omega)$. $\square$

It is natural to choose a topology for the space of distributions. Recall:

Definition 1.17. Let X be a vector space, let X^* be the dual of X and $F \subset X^*$ a subspace. We define the weak topology associated to F, denoted $\sigma(X, F)$, as the smallest topology on X making all the maps in F continuos.

Clearly with this topology all maps in F are continuos. It is remarkable that a simple excercise in linear algebra shows:

Exercise 1.18. The topological dual of $(X, \sigma(X, F))$ is indeed F.

We can then define a natural topology on $\mathcal{D}'(\Omega)$:

Definition 1.19. We define the topology on $\mathcal{D}'(\Omega)$ as the weak topology associated with the family of functionals $\mathcal{D}(\Omega)$, seen as embedded into the bidual space of $\mathcal{D}(\Omega)$. More explicitly it is the weak topology associated to the functionals

$$\begin{aligned} \tilde{\varphi} : \mathcal{D}'(\Omega) &\to \mathbb{R} \\ u &\mapsto \langle u, \varphi \rangle \end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Now we will make a simple, but powerful, observation.

Proposition 1.20. Let $T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ be a linear continuos map. Then the transpose $T^* : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is continuos.

Proof. We shall use the universal property of weak topology. Fix $\varphi \in \mathcal{D}(\Omega)$, we prove that the map

$$\begin{aligned} \tilde{\varphi} \circ T^* : \mathcal{D}'(\Omega) &\to \mathbb{R} \\ u &\mapsto \langle u, T(\varphi) \rangle \end{aligned}$$

is continuos. But this is just the functional on $\mathcal{D}(\Omega)$ represented by $T(\varphi)$ and is therefore continuos by the definition of the weak topology on $\mathcal{D}'(\Omega)$. $\square$

Proposition 1.21. By transposing the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ we obtain an injection

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$$

Proof. Suppose $u \in \mathcal{E}'(\Omega)$ has the property that $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, the we must show that $u = 0$. But this follows from the density of test functions in $\mathcal{E}(\Omega)$, Proposition 1.9. $\square$

We can embed all “reasonable” functions in $\mathcal{D}'(\Omega)$, this is why distributions are called generalized functions. Recall that the space $L^1_{\text{loc}}(\Omega)$ is topologized via the seminorms

$$\left\{ p_K(f) = \|f\|_{L^1(K)} \mid K \in \mathcal{K}(\Omega) \right\}.$$

Theorem 1.22. There is a continuos embedding

$$\begin{aligned} L^1_{\text{loc}}(\Omega) &\hookrightarrow \mathcal{D}'(\Omega) \\ f &\mapsto \int_\Omega f \cdot - \end{aligned}$$

Proof. To show continuity we use the universal property. Fix $\varphi \in \mathcal{D}(\Omega)$ and consider

$$
\begin{aligned}
L^1_{\text{loc}} &\to \mathbb{R} \\
f &\mapsto \int_\Omega f\varphi
\end{aligned}
$$

We have

$$\left|\int_\Omega f\varphi\right| \le \|f\|_{L^1(\text{supp}\varphi)}\|\varphi\|_\infty = p_{\text{supp}\varphi}(f)\|\varphi\|_\infty.$$

and this shows that the functional is continuos over L^1_{loc}. To show injectiveness we shall prove that

$$\int_\Omega f\varphi = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) \Rightarrow f = 0 \text{ a.e.}$$

but this is famous result known as "fundamental Lemma of calculus of variations". □

Exercise 1.23. Show that the following maps are continous:

$$
\begin{aligned}
\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) &\to \mathcal{D}(\Omega) \\
(f, g) &\mapsto fg
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(\Omega) \times \mathcal{E}(\Omega) &\to \mathcal{E}(\Omega) \\
(f, g) &\mapsto fg
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(\Omega) \times \mathcal{D}(\Omega) &\to \mathcal{D}(\Omega) \\
(f, g) &\mapsto fg
\end{aligned}
$$

Exercise 1.24. Recall that for a compact set $K \in \mathcal{K}(\Omega)$ the space $C^0(K)$ is a Banach space with the supremum norm, and we can define a topology on $C^0_c(\Omega)$ by

$$C^0_c(\Omega) := \varinjlim_{\mathcal{K}(\Omega)} C^0(K)$$

where the limits in inteded in the **category of topological spaces** (for the sake of the excercise). Show that the map

$$
\begin{aligned}
\mathbb{R} \times C^0_c(\Omega) &\to C^0_c(\Omega) \\
(\lambda, f) &\mapsto \lambda f
\end{aligned}
$$

is continuos, and determine if the summation map is continuos:

$$
\begin{aligned}
C^0_c(\Omega) \times C^0_c(\Omega) &\to C^0_c(\Omega) \\
(f, g) &\mapsto f + g
\end{aligned}
$$

1.2. Operations on distributions

Thanks to Proposition 1.20 we can extend an uncountable amount of operations on $\mathcal{D}(\Omega)$ to the distributions. It it left as an exercise to verify that such operations on $\mathcal{D}(\Omega)$ are continuos.

Definition 1.25. (multiplication by smooth functions) Fix $f \in \mathcal{E}(\Omega)$, we define multiplication by f as the transpose of

$$
\begin{aligned}
M_f : \mathcal{D}(\Omega) &\to \mathcal{D}(\Omega) \\
\varphi &\mapsto \varphi f
\end{aligned}
$$

Explicitly we have $\langle fu, \varphi \rangle = \langle u, f\varphi \rangle$.

Definition 1.26. (derivative) Fix a multindex α. The derivative operator is the transpose of

$$\begin{aligned} \partial^\alpha : \mathcal{D}(\Omega) &\to \mathcal{D}(\Omega) \\ \varphi &\mapsto \partial^\alpha \varphi \end{aligned}$$

multiplied by $(-1)^{|\alpha|}$. Explicitly we have $\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle$.

The reason behind the change of sign is to generalize the known formula of integration by parts, in fact for every $f \in C^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$:

$$\int_\Omega f \partial_i \varphi = - \int_\Omega \partial_i f \varphi$$

where the integral on $\partial\Omega$ is zero because φ has compact support.

Proposition 1.27. (Leibniz rule) For $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ we have

$$\partial_k(fu) = \partial_k(f)u + f\partial_k(u)$$

Proof. Just verify the relation for all $\varphi \in \mathcal{D}(\Omega)$. □

Note how all distributions have infinite derivatives. As such we can derive all L^1_{loc} functions. Also we can generalize known constructions:

Example 1.28. Sobolev functions $W^{1,2}(\Omega)$ are just L^2 functions with distributional derivative in L^2. A non trivial result shows that we can define the space of bounded variation functions $\mathrm{BV}(\Omega)$ as the space of L^1 functions with a Borel measure as distributional derivative.

1.3. Local nature of distributions

Sheaves are general objects used to track data on topological spaces that can be restricted and glued. Their use was popularized by Grothendieck who used them systematically in algebraic geometry. We shall prove that distributions form a sheaf.

Definition 1.29. (Presheaves) A *presheaf* $\mathcal{F}$ of vector spaces on a topological space X is the datum of:
- a vector space $\mathcal{F}(U)$ for every open set U
- a linear restriction map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for all open sets $V \subset U$

Such that for all triples $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \rho^V_W$, in other words we have a commutative triangle

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "\rho^U_W"] \arrow[dr, "\rho^U_V"'] & & \mathcal{F}(W) \\ & \mathcal{F}(V) \arrow[ur, "\rho^V_W"'] & \end{tikzcd}$$

Notation. If $\varphi \in \mathcal{F}(U)$ we write $\varphi|_V$ for $\rho^U_V(\varphi)$.

Definition 1.30. (Sheaves) A presheaf $\mathcal{F}$ is called *separated* if for every open set U and every open covering $U = \bigcup_{i \in I} U_i$ we have
- if $\varphi \in \mathcal{F}(U)$ and $\varphi|_{U_i} = 0$ for all i then $\varphi = 0$.

If $\mathcal{F}$ satisfies also

- for a datum $\{\varphi_i\}_{i\in I}$ with $\varphi_i \in \mathcal{F}(U_i)$, such that $\varphi_i|_{U_j} = \varphi_j|_{U_i}$ for all $i, j \in I$, there exist a (unique by the previous property) $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$;

then $\mathcal{F}$ is called a *sheaf.*

Example 1.31. For all $0 \leq m \leq \infty$, C^m is a sheaf on $\mathbb{R}^n$. On the other hand bounded continuous functions are a separated presheaf but not a sheaf. Similarly L^1 is not a sheaf but L^1_{loc} is. In a certain sense, we can define L^1_{loc} as the smallest sheaf containing L^1. This process is standard and is known as *sheafification.*

What distinguishes presheaves from sheaves is the local nature of the objects encoded.

Proposition 1.32. $\mathcal{D}'$ is a presheaf on $\mathbb{R}^n$.

Proof. Given $V \subset U$ and $u \in \mathcal{D}'(U)$, u is naturally an element of $\mathcal{D}'(V)$ since $\mathcal{D}(V) \subset \mathcal{D}(U)$. □

Lemma 1.33. Let $\Omega = \bigcup \Omega_i$ be an open covering. Then

$$\mathcal{D}(\Omega) = \bigoplus_i \mathcal{D}(\Omega_i),$$

where the direct sum is intended as sum of vector subspaces.

Proof. Let $\varphi \in \mathcal{D}(\Omega)$, by compacteness there exist $\Omega_1, ..., \Omega_n$ elements of the covering such that $\operatorname{supp}(\varphi) \subset \Omega_1 \cup ... \cup \Omega_n$. Now let $\eta_1, ..., \eta_n$ be a smooth partition of unity subordinate to $\Omega_1 \cup ... \cup \Omega_n$, clearly

$$\varphi = \eta_1\varphi + ... + \eta_n\varphi$$

and each $\eta_k\varphi$ is an element of $\mathcal{D}(\Omega_k)$. □

Corollary 1.34. $\mathcal{D}'$ is a separated presheaf.

Proposition 1.35. $\mathcal{D}'$ is a sheaf.

Proof. Let $\Omega = \bigcup_i \Omega_i$ be an open covering. Let $\{u_i\}$ be a collection of distributions with $u_i \in \mathcal{D}'(\Omega_i)$ that satisfies the compatibility condition of sheaf definition. Given $\varphi \in \mathcal{D}(\Omega)$ we can decopose φ as in the lemma as a sum $\varphi = \varphi_{i_1} + ... + \varphi_{i_n}$ with each $\varphi_{i_r} \in \mathcal{D}'\left(\Omega_{i_r}\right)$. Now we define $\langle u, \varphi\rangle := \langle u_{i_1}, \varphi_{i_1}\rangle + ... + \langle u_{i_n}, \varphi_{i_n}\rangle$, we just need to check that it is well defined. Let $\varphi = \varphi_{j_1} + ... + \varphi_{j_n}$ be another decomposition of φ, we should have

$$\sum_{r=1}^{n} \langle u_{i_r}, \varphi_{i_r}\rangle = \sum_{r=1}^{n} \langle u_{j_r}, \varphi_{j_r}\rangle .$$

The case $n = 2$ is easy because $\varphi_{i_1} = \varphi_{j_1} = 1$ on $\Omega_1 \setminus \Omega_2$, similarly $\varphi_{i_2} = \varphi_{j_2} = 1$ on $\Omega_2 \setminus \Omega_1$ and on $\Omega_1 \cap \Omega_2$ the distributions u_1, u_2 coincide. The general case follows by induction.

Let now $\Omega_{j_1}, ..., \Omega_{j_m}$ another covering of $\operatorname{supp} \varphi$ and $\varphi = \varphi_{k_1} + ... + \varphi_{k_m}$ a decomposition. We look for a common refinement. Let $\eta_{r,s}$ be a smooth partition of unity associated with the covering $\bigcup_{r,s} \Omega_{i_r} \cap \Omega_{k_s}$, then

$$\sum_{r=1}^{n} \langle u_{i_r}, \varphi_{i_r}\rangle = \sum_{r=1}^{n} \langle u_{i_r}, \sum_{s=1}^{m} \varphi_{i_r}\eta_{r,s}\rangle = \sum_{r,s=1}^{n,m} \langle u_{i_r}, \varphi_{i_r}\eta_{r,s}\rangle$$

$$\sum_{s=1}^{n} \langle u_{k_s}, \varphi_{k_s}\rangle = \sum_{s=1}^{n} \langle u_{k_s}, \sum_{r=1}^{m} \varphi_{k_s}\eta_{r,s}\rangle = \sum_{r,s=1}^{n,m} \langle u_{k_s}, \varphi_{k_s}\eta_{r,s}\rangle$$

Now note that $\varphi_{i_r}\eta_{r,s}, \varphi_{k_s}\eta_{r,s}$ are both supported in $\Omega_{i_r} \cap \Omega_{j_s}$ and by compatibility of distributions $u_{i_r} = u_{k_s}$ on $\Omega_{i_r} \cap \Omega_{j_s}$. With the help of the previous part we deduce the right expressions are equal by invariance of the chosen decomposition of φ (for a fixed open covering). □

Remark 1.36. The fact that $\mathcal{D}'$ is a sheaf let us define easily the notion of distributions on manifolds. Let M be a smooth real manifold, we define canonically $\mathcal{D}(U)$ for every affine chart U. Since affine charts form a base for the topology on M the sheaf axioms imply there exists a unique sheaf $\mathcal{D}'$ defined on the topological space M. The reader can check that elements of $\mathcal{D}'(V)$ acts naturally on elements of $C_0^\infty(V)$ for every open set $V \subset M$ (not necessarily affine). This process is analogous to the construction of the structure sheaf on an affine scheme in algebraic geometry.

Remark 1.37. The restriction map $\rho_V^U : \mathcal{D}'(U) \to \mathcal{D}'(V)$ in general is not surjective. In particular consider the case when $\partial V \cap U \neq \emptyset$, take $x \in \partial V \cap U$ and a sequence $(x_n) \subset V$ converging to x. Define the distribution $\langle u, \varphi \rangle := \sum_n \varphi(x_n)$, $u \in \mathcal{D}'(V)$. It is well defined since for every $\varphi \in \mathcal{D}(V)$ the sum is finite by compactness of $\operatorname{supp} \varphi$. Clearly u cannot be extended to a distribution on V. A particular case where ρ_V^U surjects is when V is a connected component of U.

1.4. Order of a distribution

Definition 1.38. We say that a distribution $u \in \mathcal{D}'(\Omega)$ has order equal or less than m if for all $K \in \mathcal{K}(\Omega)$ the exist $C = C(K)$ such that

$$|u(f)| \leq C p_m(f) \text{ for every } f \in \mathcal{D}_K.$$

The minimum of such integers is called the *order* of u. If there is not such m we say that u has infinite order.

Example 1.39. Let $x_0 \in \Omega$, the Dirac delta $\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0)$ is a distribution of order 0. In general $\varphi \mapsto \partial^\alpha \varphi(x_0)$ has order $|\alpha|$.

Example 1.40. Let (x_n) be a sequence in Ω that escapes from all compact sets and let α be a non-zero multindex. For $\varphi \in \mathcal{D}(\Omega)$ define

$$\langle u, \varphi \rangle := \sum_{n=1}^{\infty} \partial^{n \cdot \alpha} \varphi(x_n).$$

It is well defined beacuse for every φ the sum is actually finite. It easy to see that u is a distribution and has infinite order.

Theorem 1.41. Let $u \in \mathcal{D}'(\Omega)$ be a positive distribution, that is $\langle u, \varphi \rangle \geq 0$ for all $\varphi \geq 0$. If u has order zero then is a measure.

Proof. Fix $K \in \mathcal{K}(\Omega)$, by hypothesis there exists a $C > 0$ such that $|\langle u, \varphi \rangle| \leq C \|\varphi\|_\infty$ for all $\varphi \in \mathcal{D}_K$. Take now $\eta \in \mathcal{D}(\Omega)$ such that:

- $\eta|_K = 1$
- $0 \leq \eta \leq 1$ in Ω

and by the positivity hypothesis

$$\langle u, \varphi + \|\varphi\|_\infty \eta \rangle \geq 0 \text{ and } \langle u, \varphi - \|\varphi\|_\infty \eta \rangle \leq 0 \Rightarrow |\langle u, \varphi \rangle| \leq \langle u, \eta \rangle \|\varphi\|_\infty$$

— TODO

This implies that $u : \mathcal{D}_K \to \mathbb{R}$ is continuos with respect to the supremum norm by Hann-Banach we can extend u on $C^0(K)$. By the characterization of the dual space of $C^0(K)$ we conclude that there exists a measure μ on K such that

$$\langle u, \varphi \rangle = \int_K \varphi d\mu \text{ for all } \varphi \in \mathcal{D}_K$$

$\square$

1.5. Sequences of distributions

Proposition 1.42. Let (u_n) be a sequence in $\mathcal{D}'(\Omega)$ converging pointwise: for all $\varphi \in \mathcal{D}(\Omega)$ the sequence $\langle u_n, \varphi \rangle$ is convergent in $\mathbb{R}$. Then the map

$$u : \mathcal{D}(\Omega) \to \mathbb{R}$$
$$\varphi \mapsto \lim_{n \to \infty} \langle u_n, \varphi \rangle$$

is linear and continuos and so defines a distribution.

Proof. The linearity is clear. To prove continuity fix a compact set $K \in \mathcal{K}(\Omega)$ an consider $u|_{\mathcal{D}_K}$. Since the set $\{\langle u_n, \varphi \rangle | \ \varphi \in \mathcal{D}_K\}$ is bounded and $\mathcal{D}_K$ is a complete metric space we can apply the Banach-Steinhaus Theorem and conclude that the sequence (u_n) is uniformly continuos on $\mathcal{D}_K$. This implies that exists $C = C(K), m = m(K)$ such that:

$$|\langle u_n, \varphi \rangle| \leq C p_m(\varphi) \text{ for all } \varphi \in \mathcal{D}_K \text{ and all } n \in \mathbb{Z}_{>0} \tag{1}$$

and by passing to the limit on the left we find the required inequality. $\square$

Corollary 1.43. In the setting of the preceding proposition, if (φ_n) is a sequence converging to φ in $\mathcal{D}(\Omega)$ then $\langle u_n, \varphi_n \rangle \to \langle u, \varphi \rangle$.

Proof. Since (φ_n) is convergent the exist a $K \in \mathcal{K}(\Omega)$ with supp $\varphi_n \subseteq K$ for all n. We have

$$|\langle u_n, \varphi_n \rangle - \langle u, \varphi \rangle| \leq |\langle u_n, \varphi_n \rangle - \langle u_n, \varphi \rangle| + |\langle u_n, \varphi \rangle - \langle u, \varphi \rangle|.$$

The second term goes to zero by definition of u, while for the first we apply the inequality 1

$$|\langle u_n, \varphi - \varphi_n \rangle| \leq C p_m(\varphi - \varphi_n) \underset{n \to \infty}{\longrightarrow} 0$$

where we used the fact that $\varphi_n \underset{\mathcal{D}(\Omega)}{\longrightarrow} \varphi$. $\square$

1.6. Support

We can extend the notion of support to distribution. Recall that the support is define for L^1_{loc} functions:

Definition 1.44. Let $f \in L^1_{\text{loc}}(\Omega)$, we define the support as the complement of the maximal open set $\Omega_0 \subset \Omega$ with $f|_{\Omega_0} = 0$ a.e., that is Ω_0 is the union of all open sets $U \subset \Omega$ with $f|_U = 0$ a.e.

It is slightly non trivial that $f|_{\Omega_0} = 0$ a.e. because Ω_0 can be the union of a uncountable family of open sets. For a proof we refer to [2]. Clearly if f is continuos then this new definitions generalizes the classical one. Now we move onto distributions:

Definition 1.45. Let $u \in \mathcal{D}'(\Omega)$, we define the support as the complement of the maximal open set Ω_0 where $u|_{\Omega_0} = 0$. That is, let $\{U_i\}$ be the family of open subsets $U_i \subset \Omega$ with $u|_{U_i} = 0$, then $\Omega_0 := \bigcup_i U_i$ and supp $u := \Omega \setminus \Omega_0$.

The fact that $u|_{\Omega_0} = 0$ follows from the fact that distributions form separated presheaf by Corollary 1.34.

Example 1.46. The support of δ_{x_0} is just $\{x_0\}$.

Proposition 1.47. Let $u \in \mathcal{D}'(\Omega)$, $K \in \mathcal{K}(\Omega)$ and consider the following statements:

1. supp $u \subset \text{Int}(K)$

2. there exists $C > 0, m \in \mathbb{Z}_{\geq 0}$ such that $|\langle u, \varphi \rangle| \leq C p_{m,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$
3. supp $u \subset K$

then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Since supp u is closed and supp $u \subset K$ then it is compact. We can find an open set U with supp $u \subset U \subset K$ and a bump function ψ with
- $0 \leq \psi \leq 1$
- $\psi = 1$ on U
- supp $\psi \subset K$

This implies that for any $\varphi \in \mathcal{D}(\Omega)$

$$\{x \in \Omega \mid (1 - \psi(x)\varphi(x) \neq 0\} \subset \Omega \setminus U \subset \Omega \setminus \text{supp } u$$

and therefore

$$0 = \langle u, (1 - \psi)\varphi \rangle \Rightarrow \langle u, \varphi \rangle = \langle u, \psi\varphi \rangle .$$

Since $\psi\varphi \in \mathcal{D}_K$, by continuity of u there exists $C > 0, m \in \mathbb{Z}_{\geq 0}$ with

$$|\langle u, \varphi \rangle| = |\langle u, \varphi\psi \rangle| \leq C p_{m,K}(\psi\varphi)$$

and by Leibniz estimate (Lemma 1.11)

$$C p_{m,K}(\psi\varphi) \leq C 2^m p_{m,K}(\psi) p_{m,K}(\varphi)$$

and this is true for all $\varphi \in \mathcal{D}(\Omega)$.

(2) $\Rightarrow$ (3) Let $U \subset \Omega$ be an open set with $U \cap K = \emptyset$, we must show that $u|_U = 0$. Take any $\varphi \in \mathcal{D}(U) \subset \mathcal{D}(\Omega)$, we have $p_{m,K}(\varphi) = 0$ and $|\langle u, \varphi \rangle| \leq C p_{m,K}(\varphi) = 0$. □

Example 1.48. The implication (2) $\Rightarrow$ (1) is false, to see it just take any distribution with support a single point.

Example 1.49. The implication (3) $\Rightarrow$ (2) is a false, for a counterexample take $\Omega = \mathbb{R}$, $K = \{0\} \cup \left\{\frac{1}{n}\right\}_{n \in \mathbb{Z}_{>0}}$ and

$$\langle u, \varphi \rangle := \sum_{n \geq 1} \frac{1}{n} \left(\varphi\left(\frac{1}{n}\right) - \varphi(0) \right)$$

then supp $\varphi = K$, take now a sequence (φ_n) such that:
- $\varphi_n \in \mathcal{D}(\Omega)$
- $\varphi_n(x) = 0$ for $x \leq \frac{1}{n+1}$
- $\varphi_n(x) = 1$ for $x \in \left[\frac{1}{n}, 1\right]$
- $0 \leq \varphi_n \leq 1$

then we have $\varphi_n^{(k)}(x) = 0$ for all $k > 0$ and $x \in K$ so the only hope is to bound $|\langle u, \varphi_n \rangle|$ with $\|\varphi_n\|_{\infty,K}$. But:

$$\langle u, \varphi_n \rangle = \sum_{j \geq 1}^{n} \frac{1}{j} \underset{n \to +\infty}{\longrightarrow} +\infty$$

and $\|\varphi_n\|_{\infty,K} = 1$ for all n.

Exercise 1.50. Find some regularity hypothesis on K in order to make the implication (3) $\Rightarrow$ (2) true.

Remark 1.51. In the hypothesis of the first statement of the last proposition we can extend u on a continuos functional defined over $C^m(\Omega)$. Recall that $C^m(\Omega)$ is topologized via the seminorms $\{p_{m,K}\}_{K\in\mathcal{K}(\Omega)}$, we can define for every $\varphi \in C^\infty(\Omega)$:

$$\langle \tilde{u}, \varphi \rangle := \langle u, \psi\varphi \rangle$$

where ψ is the bump function used in the proof. The inequality remains valid:

$$| < \tilde{u}, \varphi > | \le C p_{m,K}(\varphi) \text{ for all } \varphi \in C^\infty(\Omega),$$

this implies $\tilde{u}$ is continuos on $C^\infty(\Omega) \subset C^m(\Omega)$. Then extend $\tilde{u}$ on all $C^m(\Omega)$ by density of $C^\infty(\Omega)$.

Now we can give a full characterization of distributions with compact support (and justify the name of $\mathcal{E}'(\Omega)$). Recall that we have an embedding $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ (Proposition 1.21).

Theorem 1.52. Let $u \in \mathcal{D}'(\Omega)$, then the following are equivalent:
1. supp u is compact
2. $u \in \mathcal{E}'(\Omega)$

Proof. (1) $\Rightarrow$ (2) Find a compact set $K' \in \mathcal{K}(\Omega)$ such that supp $u \subset \mathrm{Int}(K')$, then by the previous remark u extends on a continuos functional defined on $\mathcal{E}(\Omega)$.

(2) $\Rightarrow$ (1) By continuity of u with respect to the topology on $\mathcal{E}(\Omega)$ there must exists $C > 0, m \in \mathbb{Z}_{\ge 0}$ and $K \in \mathcal{K}(\Omega)$ such that

$$|\langle u, \varphi \rangle| \le C p_{m,K}(\varphi) \text{ for all } \varphi \in \mathcal{E}(\Omega).$$

By implication (2) $\Rightarrow$ (3) of the last proposition we get that u is supported in K. □

We can also characterize distributions with support equal to a point.

Proposition 1.53. Let $u \in \mathcal{D}'(\Omega)$ with supp $u = \{x_0\} \subset \Omega$, then there exists $m \in \mathbb{Z}_{\ge 0}$ and real constants $\{c_\alpha\}_{|\alpha| \le m}$ such that

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha \varphi(x_0) \text{ for all } \varphi \in \mathcal{D}(\Omega)\,.$$

Proof. Without loss of generality $x_0 = 0 \in \Omega$. Let η be a cutoff function:
- $\eta \in C^\infty(\mathbb{R}^N)$
- $\eta = 1$ on $\mathcal{B}(0,1)$
- $0 \le \eta \le 1$
- supp $\eta \subset \mathcal{B}(0,2)$

and define $\eta_\varepsilon := \eta\left(\frac{x}{\varepsilon}\right)$, note that for small ε (say ε_0) we have supp $\eta_{\varepsilon_0} \subset \Omega$. By hypothesis we have $\langle u, \varphi \rangle = \langle u, \eta_\varepsilon \varphi \rangle$ for all $\varepsilon < \varepsilon_0$. By continuity of u there must exists $C > 0$ and $m \in \mathbb{Z}_{\ge 0}$ such that

$$|\langle u, \eta_\varepsilon \varphi \rangle| \le C p_m(\eta_\varepsilon \varphi) \text{ for all } \varepsilon < \varepsilon_0, \varphi \in \mathcal{D}(\Omega)\,.$$

Now by generalized Leibniz rule:

$$|\partial^\alpha(\eta_\varepsilon \varphi)(x)| \le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \; |\partial^\beta \eta_\varepsilon(x) \partial^{\alpha-\beta}\varphi(x)| = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \; | \; \varepsilon^{-|\beta|} \partial^\beta \eta\left(\frac{x}{\varepsilon}\right) \partial^{\alpha-\beta}\varphi(x)|$$

Suppose now that $\varphi = o(\|x\|^m)$, and observe that if $x \in$ supp η_ε then $\|x\| < 2\varepsilon$. This implies that thinking of ε as a variable then $\varphi = o(\varepsilon^m)$, this is fundamental since the goal is taking the limit $\varepsilon \to 0^+$. Now, by Taylor's formula, we have that $\partial^{\alpha-\beta}\varphi = o\left(\varepsilon^{m-(|\alpha-\beta|)}\right)$ and furthermore there exists $M > 0$ such that $|\partial^\beta \eta\left(\frac{x}{\varepsilon}\right)| \le M$ for all $|\beta| \le m, x \in$ supp η. Finally we see that

$$|\partial^\alpha(\eta_\varepsilon\varphi)(x)| \le M \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \varepsilon^{-|\beta|}\ |\partial^{\alpha-\beta}\varphi(x)|$$

and $\varepsilon^{-|\beta|}\ |\partial^{\alpha-\beta}\varphi| = o\big(\varepsilon^{m-\ |\alpha-\beta|\ -\ |\beta|}\big) = o\big(\varepsilon^{m-|\alpha|}\big)$. Since we have a finite sum all the expression is $o\big(\varepsilon^{m-\ |\alpha|}\big)$, in particular we deduce that $p_m(\eta_\varepsilon\varphi) = o(1)$, and so

$$|\ \langle u,\varphi\rangle| = |\ \langle u,\eta_\varepsilon\varphi\rangle| \le Cp_m(\eta_\varepsilon\varphi) \underset{\varepsilon\to0^+}{\longrightarrow} 0$$

Now take any $\varphi \in \mathcal{D}(\Omega)$, by Taylor's formula:

$$\varphi = \sum_{|\alpha|\ \le m} \frac{\partial^\alpha\varphi(0)}{\alpha!} x^\alpha + R(x)$$

where $R = o(\|x\|^m)$. Then

$$\langle u,\varphi\rangle = \langle u, \sum_{|\alpha|\ \le m} \frac{\partial^\alpha\varphi(0)}{\alpha!} x^\alpha\rangle + \langle u, R\rangle = \sum_{|\alpha|\le m} \frac{\partial^\alpha\varphi(0)}{\alpha!}\ \langle u, x^\alpha\rangle$$

because $\langle u, R\rangle = 0$, and the thesis follows by setting $c_\alpha := \frac{\langle u,x^\alpha\rangle}{\alpha!}$. □

1.7. Equivalent seminorms on $\mathcal{D}_K$

Depending on the context it may be useful to have different families of seminorms that induce the same topology on $\mathcal{D}_K$.

Lemma 1.54. The family of norms $\left\{\|\partial^\alpha\varphi\|_\infty\right\}_{\alpha\in\mathbb{N}^N}$ generates the topology on $\mathcal{D}_K$

Proof. For any α we have $\|\partial^\alpha\varphi\|_\infty \le p_{|\alpha|}(\varphi)$. For the reverse inclusion let $m \ge 0$ and use the definition of p_m:

$$p_m(\varphi) = \max_{|\alpha|\ \le m} \|\partial^\alpha\varphi\|_\infty$$

□

Lemma 1.55. For any $\varphi \in \mathcal{D}(K)$ and multindex α we have

$$\|\varphi\|_\infty \le \frac{1}{\alpha!} R_\infty(K)^{|\alpha|} \|\partial^\alpha\varphi\|_\infty$$

where $R_\infty(K)$ is the maximum radius of a $\|\cdot\|_\infty$-ball inside K.

Proof. Fix $x \in K$, $y \notin K$. We apply Lagrange's reminder to the first entry of φ:

$$\varphi(x_1, ..., x_N) = \frac{1}{\alpha_1!} \partial_1^\alpha \varphi(\xi_1, x_2, ..., x_N)(x_1 - y_1)^{\alpha_1} \text{ for some } \xi_1 \in (y_1, x_1)$$

By iterating on every entry we get

$$\varphi(x_1, ..., x_N) = \frac{1}{\alpha!} \partial^\alpha\varphi(\xi_1, ..., \xi_N)(x-y)^\alpha$$

Since this is valid for every $x \in K$ and $y \notin K$:

$$\|\varphi\|_\infty \le \frac{1}{\alpha!} \|\partial^\alpha\varphi\|_\infty \sup_{x\in K} \inf_{y\notin K} \|x-y\|_\infty^{|\alpha|}$$

□

Proposition 1.56. let $A \subset \mathbb{N}^n$ be a set of multindices with the property

$$\sup_{\alpha \in A} \min_{1 \le i \le N} \alpha_i = +\infty,$$

equivalently for any $\beta \in \mathbb{N}^N$ there exist $\alpha \in A$ such that $\alpha \ge \beta$. Then the family of norms $\left\{\|\partial^\alpha \varphi\|_\infty\right\}_{\alpha \in A}$ generates the topology on $\mathcal{D}_k$.

Proof. Clearly this new topology is coarser than the standard one. To show the reverse inclusion take any $\beta \in \mathbb{N}^N$, then for every $\alpha \in \mathbb{N}^N$ we have by previous lemma:

$$\left\|\partial^\beta \varphi\right\|_\infty \le C\left\|\partial^{\beta+\alpha} \varphi\right\|_\infty$$

with $C = C(\alpha, K)$. Just take any α such that $\beta + \alpha \in A$. □

The following family of norms may be very useful for its simplicity.

Corollary 1.57. Denote $\varepsilon := (1, ..., 1)$. Then $\left\{\|\partial^{n\varepsilon} \varphi\|_\infty\right\}_{n \ge 0}$ generates the topology on $\mathcal{D}_K$.

Now we will find a family of norms based on the L^1-norm that generates the topology on $\mathcal{D}_K$.

Lemma 1.58. Denote $\varepsilon := (1, ..., 1)$. Then for any $\varphi \in \mathcal{D}(\Omega)$ and multindex α we have

$$\|\varphi\|_\infty \le \frac{1}{\alpha!} R_\infty(K)^{|\alpha|} \left\|\partial^{\alpha+\varepsilon} \varphi\right\|_{L^1(K)}$$

where $R_\infty(K)$ is the maximum radius of a $\|\cdot\|_\infty$-ball inside K.

Proof. Let $x \in K$, $y \notin K$, by the integral remainder in Taylor's formula

$$\varphi(x) = \int_{y_1}^{x_1} \frac{\partial_1^{\alpha_1+1}}{\alpha_1!} \varphi(t_1, x_2, ..., x_N)(x_1 - t_1)^{\alpha_1} dt_1 =$$

$$= \int_{y_1}^{x_1} ... \int_{y_N}^{x_N} \frac{\partial_1^{\alpha_1+1} ... \partial_N^{\alpha_N+1}}{\alpha!} \varphi(t_1, ..., t_N)(x_1 - t_1)^{\alpha_1} ... (x_N - t_N)^{\alpha_N} dt_1 ... dt_N$$

Let $Q(x, y) = \left\{z \in \mathbb{R}^N \mid \min\{x_i, y_i\} \le z_i \le \max(x_i, y_i)\right\}$, we have

$$\varphi(x) = (\Pi \operatorname{sgn}(x_i - y_i)) \int_{Q(x,y)} \frac{\partial^{\alpha+\varepsilon}}{\alpha!} \varphi(t)(x - t)^\alpha dt$$

Since this is valid for all $x \in K$, $y \notin K$ we get

$$\|\varphi\|_\infty \le \frac{1}{\alpha!} \left\|\partial^{\alpha+\varepsilon} \varphi\right\|_{L^1(K)} \sup_{x \in K} \inf_{y \notin K} \|x - y\|_\infty^{|\alpha|}$$

□

Proposition 1.59. The family of norms $\left\{\|\partial^\alpha \varphi\|_{L^1(K)}\right\}_{\alpha \in \mathbb{N}^N}$ generates the topology on $\mathcal{D}_k$.

Proof. Fix $\varphi \in \mathcal{D}_K$. We clearly have $\|\varphi\|_{L^1(K)} \le \mu(K) \|\varphi\|_\infty$, for the other inequality use the previous lemma to conclude that for any multindex α we have $\|\varphi\|_\infty \le C\|\varphi^{\alpha+\varepsilon}\|$ with $C = C(\alpha, K)$. □

Proposition 1.60. let $A \subset \mathbb{N}^n$ be a set of multindices with the property

$$\sup_{\alpha \in A} \min_{1 \le i \le N} \alpha_i = +\infty,$$

equivalently for any $\beta \in \mathbb{N}^N$ there exist $\alpha \in A$ such that $\alpha \ge \beta$. Then the family of norms $\left\{\|\partial^\alpha \varphi\|_{L^1(k)}\right\}_{\alpha \in A}$ generates the topology on $\mathcal{D}_k$.

Proof. Take any $\beta \in \mathbb{N}^N$, by the previous proof we know that

$$\frac{\|\partial^\beta \varphi\|_{L^1(K)}}{\mu(K)} \le \|\partial^\beta \varphi\|_\infty \le C \|\partial^{\beta+\alpha+\varepsilon} \varphi\|_{L^1(K)}$$

for all multindices α and C depends only on K. Just choose any α such that $\alpha + \varepsilon \in A$. $\square$

Corollary 1.61. Denote $\varepsilon := (1, ..., 1)$. Then $\left\{ \|\partial^{n\varepsilon} \varphi\|_{L^1(K)} \right\}_{n\ge 0}$ generates the topology on $\mathcal{D}_K$.

Now we move onto another estimate for $\|\varphi\|_\infty$ that involves the maximal $\|\cdot\|_1$-ball inside K.

Proposition 1.62. The family of seminorms

$$p_m^\circ(f) := \max_{|\alpha| = m} \|\partial^\alpha \varphi\|_\infty \text{ for } m \ge 0$$

induces the standard topology on $\mathcal{D}_K$.

Proof. Since $p_m^\circ \le p_m$ the topology defined by this seminorms is finer. To show the reverse inclusion just note that $p_m = \sum_{k=0}^m p_k^\circ$. $\square$

Lemma 1.63. For any $\varphi \in \mathcal{D}_K$ and $m \ge 1$ we have

$$\|\varphi\|_\infty \le \frac{1}{m!} R_1(K)^m p_m^\circ(\varphi)$$

where $R_1(K)$ is the maximal radius for a $\|\cdot\|_1$-ball inside K

Proof. Let $y \notin K$, $x \in K$ and $\varphi \in \mathcal{D}_K$. By one-dimensional Lagrange's remainder we get

$$\varphi(x) = \frac{1}{m!} \frac{d^m}{dt^m} (\varphi(x + t(y - x)))|_{t=\xi} = \frac{1}{m!} \sum_{|\alpha| = m} \binom{m}{\alpha} \partial^\alpha \varphi(x + \xi(y - x))(y - x)^\alpha$$

Then

$$|\varphi(x)| \le \frac{1}{m!} p_m^\circ(\varphi) \sum_{|\alpha| = m} \binom{m}{\alpha} |y - x|^\alpha = \frac{1}{m!} p_m^\circ(\varphi) \left(\sum_{i=1}^N |y_i - x_i| \right)^m = \frac{1}{m!} p_m^\circ(\varphi) \|y - x\|_1^m$$

Since the estimate is valid for every $x \in K$, $y \notin K$ we have

$$\|\varphi\|_\infty \le \frac{1}{m!} p_m^\circ(\varphi) \sup_{x\in K} \inf_{y \notin K} \|y - x\|_1^m$$

$\square$

1.8. Distributions as local sum of measures

Turn out distributions have a really "concrete" description. The main idea is to embed test functions in an easier space that has a well understood dual space. Fix a compact set $K \in \mathcal{K}(\Omega)$ and recall that the space $C^0(K)$ is Banach space with the supremum norm $\|\cdot\|_\infty$. First we will discuss the case of compactly supported distributions. Also recall that a compactly supported distribution has finite order.

Theorem 1.64. Let $u \in \mathcal{E}'(\Omega)$ have order m, then there exist $\{\mu_\alpha\}_{|\alpha| \le m}$ signed Radon measures on Ω such that

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha \varphi d\mu_\alpha \text{ for all } \varphi \in \mathcal{D}(\Omega)$$

Proof. Since $u \in \mathcal{E}'(\Omega)$ there exists $C > 0$ and a compact set $K \in \mathcal{K}(\Omega)$ such that

$$|\langle u, \varphi \rangle| \leq C p_{m,K}(\varphi) \text{ for all } \varphi \in \mathcal{D}(\Omega)$$

Without loss generality we assume that K is a neighborhood of $\operatorname{supp} u$, that is, there exist an open set U with $\operatorname{supp} u \subsetneq U \subsetneq K$. Now define the operator:

$$\begin{aligned} T : \mathcal{D}_K &\to C^0(K)^M \\ \varphi &\mapsto (\partial^\alpha \varphi \mid_K)_{|\alpha| \leq m} \end{aligned}$$

where $M = \#\{\alpha \mid |\alpha| \leq m\}$. This is clearly continuos and injective. Now define another operator:

$$\begin{aligned} \Lambda : \operatorname{Im}(T) &\to \mathbb{R} \\ \psi = (\partial^\alpha \varphi \mid_K)_{|\alpha| \leq m} &\mapsto \langle u, \varphi \rangle \end{aligned}$$

and

$$|\Lambda \psi| = |\langle u, \varphi \rangle| \leq C \max_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_\infty$$

but the last term is just $C\|\psi\|$, where $\|\cdot\|$ is the norm of $C^0(K)^M$. We conclude that Λ is continuos and by Hann-Banach Theorem it extends to a functional defined on all $C^0(K)$. By the characterization of the dual space of $C^0(K)$ (Theorem 1.26) we deduce the existence of $\{\mu_\alpha\}_{|\alpha| \leq m}$ Radon measures such that

$$\Lambda \psi = \langle u, \varphi \rangle = \sum_{|\alpha| \leq m} \int_K \partial^\alpha \varphi d\mu_\alpha = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha \varphi d\mu_\alpha \text{ for all } \varphi \in \mathcal{D}_K$$

To conclude note that the formula works for all $\varphi \in \mathcal{D}(\Omega)$ since $\operatorname{supp} u \subsetneq K$, in fact just take $\eta \in \mathcal{D}(\Omega)$ such that

- $\eta = 1$ on an open set U with $\operatorname{supp} u \subsetneq U \subsetneq K$
- $0 \leq \eta \leq 1$
- $\eta = 0$ outside K

then $\langle u, \varphi \rangle = \langle u, \eta\varphi \rangle + \langle u, (1-\eta)\varphi \rangle$, $\eta\varphi \in \mathcal{D}_K$ and $\langle u, (1-\eta)\varphi \rangle = 0$ since $\operatorname{supp}((1-\eta)\varphi) \cap \operatorname{supp} u = \emptyset$ □

Theorem 1.65. Let $u \in \mathcal{D}'(\Omega)$, then there exists a family of signed Radon measures $\{\mu_\alpha\}_{\alpha \in \mathbb{N}^N}$ such that for all $\varphi \in \mathcal{D}(\Omega)$

$$\int_\Omega \partial^\alpha \varphi d\mu_\alpha = 0 \text{ for all but finitely many } \alpha$$

and

$$\langle u, \varphi \rangle = \sum_\alpha \int_\Omega \partial^\alpha \varphi d\mu_\alpha \text{ for all } \varphi \in \mathcal{D}(\Omega)$$

Proof. Let $\Omega = \bigcup_{i \geq 1} \mathcal{B}_i$ be an open covering of Ω by open balls, furthermore suppose that the covering is locally finite and $\overline{\mathcal{B}_i} \subset \Omega$ for all i. Let $\{\eta_i\}$ be a partition of unity subordinate to the covering and notice that for all $\varphi \in \mathcal{D}(\Omega)$

$$\operatorname{supp} \varphi \cap \mathcal{B}_i = \emptyset \text{ for all but finitely many } i$$

so $\sum_i \eta_i u$ is a well-defined distribution and $\sum_i \eta_i u = u$, explicitly

$$\langle \sum_i \eta_i u, \varphi \rangle = \sum_i \langle \eta_i u, \varphi \rangle = \langle u, \sum_i \eta_i \varphi \rangle \, .$$

Now $\eta_i u$ is a compactly supported distribution ($\operatorname{supp} \eta_i u \subset \overline{\mathcal{B}_i}$), so there exists $\{\mu_{\alpha,i}\}_{|\alpha| \leq m_i}$ Radon measures representing $\eta_i u$ by the preceding Theorem. This implies

$$\langle u, \varphi \rangle = \langle \sum_i \eta_i u, \varphi \rangle = \sum_{i \geq 1} \sum_{|\alpha| \leq m_i} \int_\Omega \partial^\alpha \varphi d\mu_{\alpha,i}$$

to simplify the expression define the radon measure $\mu_\alpha = \sum_i \mu_{\alpha,i}$, note that for all $\varphi \in \mathcal{D}(\Omega)$

$$\int_\Omega \partial^\alpha \varphi d\mu_{\alpha,i} = 0 \text{ for all but finitely many } i$$

and therefore

$$\int_\Omega \partial^\alpha \varphi d\mu_\alpha = 0 = 0 \text{ for all but finitely many } \alpha$$

since $|\alpha| \leq m_i$ for all i. □

Corollary 1.66. In the setting of the last Theorem, if u has order m then

$$\langle u, \varphi \rangle = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha \varphi d\mu_\alpha$$

Proof. Just observe that $m_i \leq m$ for all i since m_i is the order of $\eta_i \mu$. This follows from the proof of Theorem 1.64. □

For the next Theorem we shall need the following lemma:

Notation. If $f : \mathbb{R}^N \to \mathbb{R}$ is any function we denote $f^\vee(x) := f(-x)$.

Lemma 1.67. If $f \in L^1(\mathbb{R}^N), g \in L^p(\mathbb{R}^N), h \in L^{p'}(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} (f * g) h = \int_{\mathbb{R}^N} g(f^\vee * h)$$

Proof. Easy verification, or look up [2](Prop. 4.16). □

Theorem 1.68. Let $u \in \mathcal{D}'(\Omega)$ and $K \in \mathcal{K}(\Omega)$ and denote $\varepsilon := (1, \ldots, 1)$. Then there exist $m \geq 0$ and $f \in L^\infty(K)$ such that

$$\langle u, \varphi \rangle = \int_K f \partial^{m\varepsilon} \varphi \text{ for all } \varphi \in \mathcal{D}_K$$

In addition, if $\Omega = \mathbb{R}^N$, f can be chosen to be continuos.

Proof. The proof is identical to Theorem 1.64, just use the equivalent norms $\left\{ \|\partial^{m\varepsilon} \varphi\|_{L^1(K)} \right\}_{n \geq 0}$ for the topology of $\mathcal{D}_K$ and embed $\mathcal{D}_K \hookrightarrow L^1(K)^M$. To conclude use the fact that the map

$$\begin{aligned} L^\infty(K) &\to L^1(K)' \\ g &\mapsto \int g \cdot - \end{aligned}$$

is an isometry.

For the second part denote by H the generalized Heaviside function: it is the characteristic function of the set $\{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, ..., N\}$. Notice that for all $\varphi \in \mathcal{D}(\Omega)$:

$$\varphi(x) = \int_{-\infty}^{x_1} \partial_1 \varphi(y_1, x_2, ..., x_N) dy_1 = ... = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_N} \partial_1 ... \partial_N \varphi(y) dy_N ... dy_1 =$$

$$= \int_{y \leq x} \partial^\varepsilon \varphi(y) dy = \int_{\mathbb{R}^N} \partial^\varepsilon \varphi(y) H(x - y) = \partial^\varepsilon \varphi * H$$

In the following we denote $H^\vee(x) = H(-x)$

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^N} f \partial^{m\varepsilon} \varphi = \int_{\mathbb{R}^N} f \big(\partial^{(m+1)\varepsilon} \varphi * H\big) = \int (f * H^\vee) \partial^{(m+1)\varepsilon} \varphi$$

and the function $f * H^\vee$ is continuos:

$$|f * H^\vee(x + z) - f * H^\vee(x)| \leq \int_{\mathbb{R}^N} |f(y) H(y - x - z) - f(y) H(y - x)| \; dy$$

$$\leq \|f\|_{L^\infty(K)} \int_K |H(y - x - z) - H(y - x)| \; dy$$

Note that $|H(y - x - z) - H(y - x)| = 0$ for all y such that $\|x - y\| > \|z\|$, therefore

$$\int_K |H(y - x - z) - H(y - x)| \; dy = \int_{\{y \mid \|x - y\| \leq \|z\|\} \cap K} |H(y - x - z) - H(y - x)| \; dy \leq$$

$$\leq \mu(\{y \mid \|x - y\| \leq \|z\|\} \cap K) \underset{z \to 0}{\longrightarrow} 0$$

□

1.9. Some exercises

Exercise 1.69. Consider the following operator

$$T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$$

$$\varphi \mapsto \lim_{\varepsilon \to 0^+} \int_{|x| \; \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

Show T is a well defined distribution of order 1.

Solution. Notice that the function $\frac{\varphi(0)}{x}$ is odd, therefore

$$\int_{|x| \; \geq \varepsilon} \frac{\varphi(x)}{x} = \int_{|x| \; \geq \varepsilon} \frac{\varphi(x) - \varphi(0)}{x}$$

Take $\varphi \in \mathcal{D}_{[-M, M]}$, we have by Lagrange's Theorem:

$$|\langle T, \varphi \rangle| \leq \int_{\varepsilon \leq \; |x| \; \leq M} \left| \frac{\varphi(x) - \varphi(0)}{x} \right| \leq 2M \|\varphi'\|_\infty$$

this shows that the integral is convergent by dominated convergence and T is continuos with order ≤ 1. To show that the order is 1 consider a function $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R})$:

- $\varphi_\varepsilon \geq 0$
- $\varphi_\varepsilon(0) = 1$
- $\varphi_\varepsilon = 1$ on $[\varepsilon, 1]$.

Then

$$T(\varphi_\varepsilon) \geq \int_\varepsilon^1 \frac{1}{x} \geq |\log(\varepsilon)|\ \|\varphi_\varepsilon\|_\infty \underset{\varepsilon \to 0^+}{\longrightarrow} +\infty$$

so we cannot bound $\langle T, \varphi \rangle$ with $\|\varphi\|_\infty$. $\square$

Exercise 1.70. Consider the following operator

$$\begin{aligned} T : \mathcal{D}(\mathbb{R}) &\to \mathbb{R} \\ \varphi &\mapsto \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x^2} dx - \frac{2}{\varepsilon}\varphi(0) \end{aligned}$$

Show T is a well defined distribution of order 2.

Solution. Notice that for $M > 0$

$$\frac{2}{\varepsilon} = \int_{\varepsilon \leq |x| \leq M} \frac{1}{x^2} dx + \frac{2}{M}$$

so it follows

$$\int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x)}{x^2} dx - \frac{2}{\varepsilon}\varphi(0) = \int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} dx - \frac{2}{M}\varphi(0)$$

Suppose $\varphi \in \mathcal{D}_{[-M,M]}$, then by Langrange's reminder

$$\left| \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} \right| \leq \frac{1}{2}x^2 \|\varphi''\|_\infty$$

so we get the bound

$$|\langle T, \varphi \rangle| = \left| \int_{\varepsilon \leq |x| \leq M} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} dx - \frac{2}{M}\varphi(0) \right| \leq \frac{1}{2}M^2 \|\varphi''\|_\infty + \frac{2}{M}\|\varphi\|_\infty$$

So by dominated convergence theorem the integral converges and T ha order less or equal than 2. $\square$

Exercise 1.71. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ with $f = O(\|x\|^{-m})$ as $x \to 0$. Then, as a distribution, f extends to an element $T \in \mathcal{D}'(\mathbb{R}^N)$ of order $\leq m$. Also characterize all the possible extension of f.

Solution. We define T using Taylor's expansion:

$$\langle T, \varphi \rangle := \int_{\|x\| \geq 1} f\varphi + \int_{\|x\| \leq 1} f \left(\varphi - \sum_{|\alpha| < m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha \right)$$

checking that is well defined and continuos is similar to the previous exercises with Lagrange's reminder:

$$\left| \varphi(x) - \sum_{|\alpha| < m} \frac{\partial^\alpha \varphi(0)}{\alpha!} x^\alpha \right| \leq C p_m(\varphi) \|x\|^m$$

for some constant $C > 0$. To characterize all the extensions take T_1, T_2 extensions of f. Notice that supp $(T_1 - T_2) = \{0\}$, so by Proposition 1.53 we have

$$\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle + \sum_{|\alpha| \leq r} c_\alpha \partial^\alpha \varphi(0) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^N)$$

So two extensions of f differ by a linear combination of derivatives of Dirac deltas. □

Exercise 1.72. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, $f > 0$, such that $\frac{1}{f} = o(\|x\|^m)$ as $x \to 0$, that is f grows more than any polynomial. Then f do not admit, as a distribution, an extension on $\mathcal{D}'(\mathbb{R}^N)$.

Solution. By contradiction suppose T is an extension of f. We seek a sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ such that $\varphi_n \underset{\mathcal{D}(\mathbb{R}^N)}{\longrightarrow} 0$ but not on $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$. Let $\eta \in \mathcal{D}(\mathbb{R}^N)$ be such that:

- $0 \leq \eta \leq 1$
- $\operatorname{supp} \eta \subset \mathcal{B}(0,4) \setminus \mathcal{B}(0,1)$
- $\eta = 1$ on $\mathcal{B}(0,3) \setminus \mathcal{B}(0,2)$

and define $\eta_k := \varepsilon_k \eta(kx)$, where (ε_k) is a sequence that will be chosen at then end. Notice that

$$\partial^\alpha \eta_k(x) = \varepsilon_k k^{|\alpha|} \partial^\alpha \eta(kx) \Rightarrow p_m(\eta_k) \leq \varepsilon_k k^m p_m(\eta) \text{ for all } m \geq 0$$

Since T is an extension of f:

$$\langle T, \eta_k \rangle = \int_{\mathbb{R}^N \setminus \{0\}} f \eta_k \geq \varepsilon_k \int_{\frac{2}{k} \leq \|x\| \leq \frac{3}{k}} f \geq \varepsilon_k \frac{C}{k^N} \inf_{0 < \|x\| < \frac{3}{k}} f$$

where we used the fact that the measure of $\mathcal{B}\left(0, \frac{3}{k}\right) \setminus \mathcal{B}\left(0, \frac{2}{k}\right)$ is given by a constant C multiplied by $\frac{1}{k^N}$. Now define

$$\varepsilon_k := \frac{k^N}{C \inf_{0 < \|x\| < \frac{3}{k}} f}$$

from this we get $\langle T, \eta_k \rangle = 1$ and $p_m(\eta_k) \leq \varepsilon_k k^m p_m(\eta) \underset{k \to +\infty}{\longrightarrow} 0$ since $\frac{1}{f} = o(\|x\|^m)$ for every $m \geq 0$. To conclude note that $\eta_k \underset{\mathcal{D}(\mathbb{R}^N)}{\longrightarrow} 0$ but does not converge in $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ since the supports of the η_k are not contained in any compact set $K \subset \mathbb{R}^N \setminus \{0\}$, but this would imply that $\langle T, \eta_k \rangle \underset{k \to +\infty}{\longrightarrow} 0$, a contradiction. □

Exercise 1.73. Let $T \in \mathcal{D}'(\mathbb{R}^N)$, $f \in \mathcal{E}(\mathbb{R}^N)$ and suppose $f = 0$ on $\operatorname{supp} T$. Then is is true that $fT = 0$?

Proof. The answer is negative since a distribution is capable of capturing the behavior of functions in an infinitesimal neighborhood of the support. In fact, take the distribution $T \in \mathcal{D}'(\mathbb{R})$ defined as

$$\langle T, \varphi \rangle := \varphi'(0)$$

and note $\operatorname{supp} T = \{0\}$. Take $\varphi \in \mathcal{D}(\mathbb{R})$, $f \in \mathcal{E}(\mathbb{R})$ that satisfies the following:

- $\varphi(0) = 1$ in a neighborhood of 0
- $f(0) = 0$
- $f'(0) \neq 0$

Then

$$< fT, \varphi > = < T, f\varphi > = (f\varphi)'(0) = f'(0)\varphi(0) + f(0)\varphi'(0) = f'(0) \neq 0.$$

□

2. Convolution

We can extend the convolution operation to distributions. Our first step will be to define the convolution of a distribution and a test function. We start by recalling some known facts about convolution taken from [2](Chapter 2).

Definition 2.1. Let $f, g \in L^1(\mathbb{R}^N)$, we define their convolution as

$$(f * g)(x) := \int_{\mathbb{R}^N} f(y)g(x-y)dy$$

Proposition 2.2. (Young inequality) For $f, g \in L^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} f * g = \int_{\mathbb{R}^N} f \int_{\mathbb{R}^N} g$$

and therefore $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$.

Proof. Easy application of Fubini-Tonelli Theorem:

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(y)g(x-y)dy\right)dx = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(y)g(x-y)dx\right)dy = \int_{\mathbb{R}^N} f(y)dy \int_{\mathbb{R}^N} g(x-y)dx$$

□

Remark 2.3. The inequality tells us that convolution $* : L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ is a bilinear continuos map.

We can also convolve L^1_{loc} functions with C^0_C functions:

Proposition 2.4. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $g \in C^0_C(\mathbb{R}^N)$. Then the function $f * g$ is well defined and continuos. Furthermore if $g \in C^1_C(\mathbb{R}^N)$ then $f * g$ is differentiable and

$$\partial_k(f * g) = f * (\partial_k g) \text{ for all } k = 1, ..., N$$

Also is worth knowing the following:

Proposition 2.5. Let $f, g \in L^1(\mathbb{R}^N)$, then $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$.

Now we move onto distributions. Note that given $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$ the expression

$$(f * g)(x) = \int_{\mathbb{R}^N} f(y)g(x-y)dy$$

can be wrote as $(f * g)(x) = \langle f, y \mapsto g(x-y)\rangle$ if we interpret f as a distribution. We take this as a definition:

Definition 2.6. Let $u \in \mathcal{D}'(\mathbb{R}^N)$ and $\varphi \in \mathcal{D}'(\mathbb{R}^N)$, we define a function called the convolution of u and φ as

$$(u * \varphi)(x) := \langle u, y \mapsto g(x-y)\rangle$$

From now on for simplicity we will write $g(x - \cdot)$ instead of $y \mapsto g(x - \cdot)$. The following lemma will be fundamental to establish the main properties of convolution:

Lemma 2.7. Let $U \subset \mathbb{R}^N$, $V \subset \mathbb{R}^M$ be two open set. Suppose $\varphi \in \mathcal{E}(U \times V)$ is a smooth function such that there exist a compact set $K \in \mathcal{K}(U)$ that satisfies $\operatorname{supp} \varphi \subset K \times V$. Let $u \in \mathcal{D}'(U)$, then

- $\Phi(y) := \langle u, \varphi(\cdot, y)\rangle$ is smooth
- for all $\beta \in \mathbb{N}^M$ we have $\partial^\beta \Phi = \langle u, \partial^\beta_y \varphi(\cdot, y)\rangle$

Proof. We do first the case of $|\beta| = 1$. Fix $y \in Y$ and $\varepsilon > 0$ such that $\overline{\mathcal{B}(y,\varepsilon)} \subset V$. Take $h \in \mathbb{R}^M$ with $\|h\| < \varepsilon$ and expand φ:

$$\varphi(x, y+h) = \varphi(x,y) + \sum_{k=1}^{M} h_k \partial_y^k \varphi(x,y) + \psi(x,y,h)$$

and $\psi(x,y,h) = o(\|h\|)$. Now apply u:

$$\Phi(y+h) = \varphi(y) + \sum_{k=1}^{M} h_k \left\langle u, \partial_y^k \varphi(\cdot, y)\right\rangle + \left\langle u, \psi(\cdot, y, h)\right\rangle$$

we need to show that $\langle u, \psi(\cdot,y,h)\rangle = o(\|h\|)$ in order to show that the above expression is the Taylor expansion of Φ. Since $\psi(\cdot,y,h) \in \mathcal{D}_K$ (recall that y, h are fixed) by continuity of u we have

$$|\left\langle u, \psi(\cdot,y,h)\right\rangle| \leq C \max_{\substack{|\alpha| \leq m \\ x \in K}} \|\partial_x^\alpha \psi(x,y,h)\|_\infty \text{ for some } C > 0, m \geq 0.$$

We need an estimate on the second term. Let $j = 1, ..., N$:

$$\partial_x^\alpha \varphi(x, y+h) = \partial_x^\alpha \varphi(x,y) + \sum_{k=1}^{M} h_k \partial_y^k \partial_x^\alpha \varphi(x,y) + \partial_x^\alpha \psi(x,y,h)$$

Since $\partial_x^\alpha \psi$ is the reminder of the Taylor expansion of $\partial_x^\alpha \varphi(x, y+h)$ (with respect to y) we can apply Langrage's reminder to get an estimate:

$$\|\partial_x^\alpha \psi(x,y,h)\| \leq \frac{1}{2}\|h\|^2 \max_{|\beta|=2} \left\|\partial_y^\beta \partial_x^\alpha \varphi\right\|_{\infty, K\times \overline{\mathcal{B}(y,\varepsilon)}}$$

and this completes the proof by showing $\psi(\cdot,y,h) = o(\|h\|)$. The case for a general β follows by iterating the previous proof $|\beta|$ times. □

Corollary 2.8. The convolution of a distribution and a test function $u * \eta$ is smooth.

Proof. Use the preceding lemma with $\varphi(x,y) = \eta(y-x)$ in $\mathcal{E}(\mathbb{R}^N \times \mathbb{R}^N)$. □

Remark 2.9. We have $\partial^\alpha(u * \varphi) = (\partial^\alpha u) * \varphi = u * (\partial^\alpha \varphi)$. In fact the equality $\partial^\alpha(u * \varphi) = u * (\partial^\alpha \varphi)$ is the second point of the lemma. For the other equality:

$$(\partial^\alpha u * \varphi)(x) = \langle \partial^\alpha u, \varphi(x - \cdot)\rangle = (-1)^{|\alpha|}\left\langle u, \partial^\alpha \varphi(x-\cdot)\right\rangle = (-1)^{2|\alpha|}\left\langle u, (\partial^\alpha\varphi)(x-\cdot)\right\rangle = u * (\partial^\alpha \varphi)$$

Proposition 2.10. For $u \in \mathcal{D}'(\mathbb{R}^N), \varphi \in \mathcal{D}(\mathbb{R}^N)$ we have

$$\operatorname{supp}(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp} \varphi$$

Proof. If $(u * \varphi)(x) = \langle u, \varphi(x - \cdot)\rangle \neq 0$ implies that $\operatorname{supp}(\varphi(x-\cdot)) \cap \operatorname{supp} u \neq \emptyset$, in other words $(x - \operatorname{supp}\varphi) \cap \operatorname{supp} u \neq \emptyset$. The last condition is equivalent to $x \in \operatorname{supp}\varphi + \operatorname{supp} u$. □

Proposition 2.11. For $u \in \mathcal{D}'(\mathbb{R}^N), \varphi \in \mathcal{D}(\mathbb{R}^N), \psi \in C_C^0(\mathbb{R}^N)$ we have

$$(u * \varphi) * \psi = u * (\varphi * \psi)$$

Proof. We want to approximate $\varphi * \psi$ by a Riemann sum. For $\delta > 0$ define

$$S_\delta(x) = \sum_{k \in \mathbb{Z}^N} \varphi(x - \delta k)\psi(\delta k)\delta^n = \int_{\mathbb{R}^N} \varphi\Big(x - \delta\Big\lfloor\frac{y}{\delta}\Big\rfloor\Big)\psi\Big(\delta\Big\lfloor\frac{y}{\delta}\Big\rfloor\Big)dy$$

Since $\varphi * \psi$ is compactly supported it is uniformly continuos, therefore there exists $\omega = \omega(\delta) > 0$ such that

$$\left|\varphi(x-y)\psi(y) - \varphi\left(x - \delta\left\lfloor\frac{y}{\delta}\right\rfloor\right)\psi\left(\delta\left\lfloor\frac{y}{\delta}\right\rfloor\right)\right| \leq w(\delta)$$

and $\omega(\delta) \underset{\delta\to 0^+}{\longrightarrow} 0$ uniformly in x, this implies $S_\delta \underset{\|\cdot\|_\infty}{\longrightarrow} \varphi * \psi$. By looking at derivatives it can be shown that $S_\delta \underset{\mathcal{D}(\mathbb{R}^N)}{\longrightarrow} \varphi * \psi$, thus by continuity of u we get $\langle u, S_\delta\rangle \to \langle u, \varphi * \psi\rangle$. Also

$$\lim_{\delta\to 0^+} \langle u, S_\delta\rangle = \lim_{\delta\to 0^+} \sum_{k\in\mathbb{Z}^N} (u * \varphi)(x - \delta k)\psi(\delta k) = (u * \varphi) * \psi$$

□

Now we will see how convolution lets us approximate distributions with smooth functions.

Remark 2.12. We always have $\langle u, \varphi\rangle = (u * \varphi^\vee)(0)$, in fact $(u * \varphi^\vee)(0) = \langle u, \varphi^\vee(0 - \cdot)\rangle = \langle u, \varphi\rangle$.

Proposition 2.13. Let $u \in \mathcal{D}'(\mathbb{R}^N)$ and let $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^N)$ be a sequence satisfying:
- $\varphi_j \geq 0$
- $\int_{\mathbb{R}^N} \varphi_j = 1$
- for every $\varepsilon > 0$ we have $\operatorname{supp} \varphi_j \subset \mathcal{B}(0, \varepsilon)$ definitely

Then

$$\mathcal{E}(\mathbb{R}^N) \ni u * \varphi_j \underset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} u$$

where it is understood that $u * \varphi$ is embedded as a function into $\mathcal{D}'(\mathbb{R}^N)$.

Proof. Set $u_{\varphi_j} := u * \varphi_j$ and take any $\psi \in \mathcal{D}(\mathbb{R}^N)$:

$$\langle u_{\varphi_j}, \psi_j\rangle = \left(\left(u * \varphi_j\right) * \psi_j^\vee\right)(0) = \left(u * \left(\varphi_j * \psi^\vee\right)\right)(0) = \left(u * \left(\varphi_j * \psi^\vee\right)\right)(0) = \langle u, \varphi_j * \psi^\vee\rangle$$

and since ψ is smooth it is a classical result that $\varphi_j * \psi^\vee \underset{\mathcal{D}(\mathbb{R}^N)}{\longrightarrow} \psi^\vee$. For a proof look up [2] (Chapter 2). From this we get $\langle u_{\varphi_j}, \psi\rangle \longrightarrow \langle u, \psi\rangle$ and the thesis follows by the definition of the weak topology on $\mathcal{D}'(\mathbb{R}^N)$. □

Corollary 2.14. $\mathcal{E}(\mathbb{R}^N) \subset \mathcal{D}'(\mathbb{R}^N)$ is dense.

We can achieve a better result:

Proposition 2.15. For any open set $\mathcal{D}(\Omega) \subset \mathcal{D}'(\Omega)$ is dense.

Proof. Let $(\eta_n) \subset \mathcal{D}(\Omega)$ be a sequence such that $0 \leq \eta_n \leq 1$ and $\{\eta_n = 1\}_{n\geq 1}$ is an increasing sequence of compact sets that invades Ω. Take a sequence $(\varphi)_n \subset \mathcal{D}(\mathbb{R}^N)$ as in the last proposition with the additional property that for all n: $\operatorname{supp} \eta_n + \operatorname{supp} \varphi_n \subset \Omega$. Now consider $(\eta_n u) * \varphi_n$, we have

$$\operatorname{supp}((\eta_n u) * \varphi_n) \subset \operatorname{supp} \eta_n + \operatorname{supp} \varphi_n \subset \Omega$$

since $\operatorname{supp} \eta_n + \operatorname{supp} \varphi_n$ we deduce that $(\eta_n u) * \varphi_n \in \mathcal{D}(\Omega)$. Now take any $\psi \in \mathcal{D}(\Omega)$, definitely $\operatorname{supp} \psi \subset \{\eta_n = 1\}$ and thus for $n \gg 0$ it is true that $\langle(\eta_n u) * \varphi_n, \psi\rangle = \langle u * \varphi_n, \psi\rangle$ and by previous proposition this converges to $\langle u, \psi\rangle$. □

Remark 2.16. The above proof works because $\eta_n u$ is compactly supported and thus can be extended to be an element of $\mathcal{D}'(\mathbb{R}^N)$. To achieve this just take a bump function ξ that is 1 on a neighborhood of $\operatorname{supp} \eta_n$ and is 0 outside Ω. Then define $\widetilde{\eta_n u}(\varphi) := (\eta_n u)(\xi\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

A Appendix

A.1 Topological vector spaces

Definition 1.1. A topological vector space (TVS) over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ is a vector space X equipped with a topology such that the maps

$$\begin{aligned} X \times X &\to X & \qquad \mathbb{K} \times X &\to X \\ (x, y) &\mapsto x + y & \qquad (\alpha, x) &\mapsto \alpha x \end{aligned}$$

are continuos.

Observation 1.2. Since translations are homeomorphism the topology of a TVS is determined by the open neighborhoods of 0.

Definition 1.3. A TVS is called locally convex (LCTVS) if it admits a basis for the open neighborhoods consisting of convex sets.

Proposition 1.4. Let X be a LCTVS. Then there exist a family of seminorms $\mathcal{P} = \{p_i\}_{i \in I}$ such that

$$V(p_i, n) = \left\{x \in X \mid p_{i(x)} < \frac{1}{n}\right\} \text{ with } i \in I, n \in \mathbb{Z}_{>0}$$

is a prebase that generates the topology of X.

Proposition 1.5. If X is a LCTVS topologized via a countable family of seminorms $\mathcal{P} = \{p_n\}$ such that for all $x \in X$ there exist n with $p_n(x) \neq 0$. Then X is metrizable with the translation-invariant distance

$$d(x, x') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x, x')}{1 + p_n(x, x')}$$

A complete metrizable LCTVS with a translation-invariant distance is often called a *Frechét* space.

Definition 1.6. A subset $A \subset X$ of a vector space is said to be:
- *balanced* if $\alpha A \subseteq A$ for every $|\alpha| \leq 1$
- *absorbing* if for every $x \in X$ there exist $t > 0$ such that $x \in tA$

Observation 1.7. In a TVS all open neighborhoods of 0 are absorbing.

Proposition 1.8. Every TVS X admits a basis $\mathcal{U}$ for the open neighborhoods of 0 such that every set $U \in \mathcal{U}$ is balanced. If X is a LCTVS then one can construct $\mathcal{U}$ such that all the neighborhoods are convex.

Definition 1.9. Given two subsets $A, B \subset X$ of a vector space we say that B *absorbs* A is there exist $t > 0$ such that $A \subset tB$. We say that a subset $A \subset X$ of a TVS is *bounded* if it is absorbed by every neighborhood of 0.

Definition 1.10. Let X be a TVS. A sequence $(x_n) \subset X$ is a Cauchy sequence if for all open neighborhoods U of zero we have $x_n - x_m \in U$ definitively. X is called *complete* if all Cauchy sequences converge.

It easy to check that Cauchy sequences are bounded.

Proposition 1.11. Let X be a TVS, then
- X is always T_3 (you can separate points and closed sets)
- X is T_0 if and only if is T_1
- If X is T_0 then X is T_2

- Let Y be a direct summand of $\overline{\{0\}}$, then X is isomorphic (as TVS) to $\overline{\{0\}} \oplus Y$ and Y is T_2.

Definition 1.12. (Inductive limits in TVS) Let (X_n) be a sequence of TVS with continuos injections $X_n \hookrightarrow X_{n+1}$. Let

$$X_\infty := \bigcup_{n \geq 1} X_n$$

we define the topology τ_∞ on X_∞ by constructing a family of balanced open neighborhoods of 0:

$$\sum_{n \geq 1} V_k := \bigcup_{n \geq 1} \sum_{k=1}^{n} V_k$$

where each V_n is an open balanced neighborhood of 0 in X_n. The TVS (X_∞, τ_∞) is called the *direct limit* of the sequence (X_n) and is denoted by

$$(X_\infty, \tau_\infty) = \varinjlim X_n$$

It easy to prove that an inductive limit of LCTVS is again a LCTVS.

Theorem 1.13. (Universal property of TVS limit) Let (X_n) be a sequence of TVS with continuos injections $X_n \hookrightarrow X_{n+1}$. Denote with $X_\infty = \lim X_n$ and let Y be a TVS, then a linear map $L : X_\infty \to Y$ is continuos if and only if $L|_{X_n} : X_n \to Y$ is continuos for all n.

Corollary 1.14. The limit topology τ_∞ on X_∞ is the finest TVS topology making all inclusions $X_n \hookrightarrow X_\infty$ continuos.

Proof. Let τ be another TVS topology making all inclusion continuos. Apply the universal property to show $\mathrm{id} : (X, \tau_\infty) \to (X, \tau)$ is continuos. □

Observation 1.15. The limit topology on X_∞ is in general different from the limit topology in the category of topological spaces. This means if $A \subset X_\infty$ is a subset such that $A \cap X_n$ is open for every n then A is **not** necessarily open in X_∞.

Definition 1.16. An inductive limit of TVS is said to be *strict* if the maps $X_n \hookrightarrow X_{n+1}$ are topological embeddings.

Strict limits have very nice properties:

Theorem 1.17. (Properties of strict limits) Let $X_\infty = \lim_n X_n$ be a strict limit of TVS. Then:
1. For all n the map $X_n \hookrightarrow X_\infty$ is a topological embedding
2. Fix $n_0 > 0$ and $C \subset X_{n_0}$, then C is closed in X_∞ if and only if is closed X_n for every $n \geq n_0$. In particular if every X_n is closed in X_{n+1} then every X_n is closed in X_∞.
3. If every X_n is T_0 then X_∞ is T_0
4. If X_n is closed in X_{n+1} for every n then a subset $A \subset X_\infty$ is bounded if and only if there exist n_0 with $A \subset X_{n_0}$ and A is bounded in X_{n_0}.
5. If X_n is closed in X_{n+1} and X_n is complete for every n then the limit is complete.

A.2 Functional analysis

Usually the Banach-Steinhaus Theorem is presented in the context of Banach spaces. To generalize it we need some definitions and results. The reader who is familiar with the standard notion of equicontinuity should not be bothered by the following definition:

Notation. If X, Y are TVS we write $L(X, Y)$ for the vector space of continuos linear maps $X \to Y$.

Proposition 1.18. A family $\{L_i\}_{i \in I} \subset L(X, Y)$ is called *equicontinuos* if for all open neighborhoods U of $0 \in Y$ there exist an open neighborhood V of $0 \in X$ such that

$$L_i(V) \subseteq U \text{ for all } i \in I.$$

Definition 1.19. A subspace $A \subseteq X$ of a topological space is called *nowhere dense* if $\operatorname{Int}\left(\overline{A}\right) = \emptyset$. A is said to be of *first category* if it is a countable union of nowhere dense subsets, otherwise is said to be of *second category*.

Theorem 1.20. (Baire's category theorem) Let $\{U_n\} \subseteq X$ be a countable collection of dense open sets, where X is

- a complete metric space, or
- a locally compact Hausdorff space;

then $\bigcap_n U_n$ is dense. In particular X is of second category.

Theorem 1.21. (Banach-Steinhaus) Let X, Y be TVS and $(L_i)_{i \in I} \subset L(X, Y)$ a pointwise bounded family on a second category subspace, that is to say the set

$$\{x \in X \mid \text{the set } \{L_i(x) \mid i \in I\} \subset Y \text{ is bounded}\}$$

is of second category. Then $(L_i)_{i \in I}$ is equicontinuos.

Now we discuss how we can characherize the dual space of continuos functions on a locally compact space.

Definition 1.22. Let (X, Σ) be a measurable space, a signed measure on μ is a function

$$\mu : \Sigma \to [-\infty, +\infty]$$

that is countably additive, $\mu(\emptyset) = 0$ and at most one of $+\infty, -\infty$ is in the image of μ.

Definition 1.23. Let X be a topological space endowed with a sigma algebra Σ, and let μ be a (signed) measure on X.

- Σ is *Borel* if all Borel sets are measurable
- μ is *inner regular* if for all measurable sets $A \subset X$ we have

$$\mu(A) = \sup_{K \subset A \text{ compact}} \mu(K)$$

- μ is *outer regular* if for all measurable sets $A \subset X$ we have

$$\mu(A) = \inf(A \subset U \text{ open})\mu(U)$$

- μ is *regular* if it is inner and outer regular
- μ is *locally finite* if $|\mu(K)| < +\infty$ for all compact sets $K \subset X$

If Σ, μ satisfies all the above properties then μ is called a (signed) *Radon* measure.

Theorem 1.24. (Hann decomposition) If X is a measurable space and μ a signed measure on X, then there exists two measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$.

Definition 1.25. In the context of the preceding theorem we define $|\mu| := \mu^+ + \mu^-$ and the *total variation* of μ as $|\mu|(X)$. If the total variation is finite we say that μ is *bounded*.

Theorem 1.26. (Riesz representation) Let X be a compact Hausdorff topological space and $\Phi : C(X, \mathbb{R}) \to \mathbb{R}$ a continuos linear functional. Then there exist a unique signed bounded Borel measure μ on X such that

$$\Phi(f) = \int_X f d\mu \text{ for all } f \in C(X, \mathbb{R})$$

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