Introduction to the mathematical theory of incompressible fluids

Paula Luna, Vittorio Baroncini, Filippo Giovagnini

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1.1 Preliminary results and notation

Definition 1.1: *Let us define the following operators:*

$$\mathbb{Q} := -\nabla(-\Delta)^{-1}\operatorname{div},$$

and:

$$\mathbb{P} := Id - \mathbb{Q}.$$

Remark 1.2: We notice that, passing to the Fourier transform, we can write:

$$\mathcal{F}(\mathbb{P}f)(\xi) = \hat{f} - \frac{1}{|\xi|^2}\left(\xi \cdot \hat{f}\right)\xi$$

Remark 1.3: We also notice that, by Calderon-Zygmund theory we have:

$$\mathbb{P}, \mathbb{Q} : \mathbb{L}^p \to \mathbb{L}^p \quad \text{for any } 1 < p < \infty$$

1.2 Conserved quantities

Let us consider the following problem:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

If we take the scalar product with respect to u we obtain:

$$\int_{\mathbb{R}^d} u \cdot \partial_t u + \int_{\mathbb{R}^d} u \cdot (u \cdot \nabla)u + \int_{\mathbb{R}^d} u \cdot \nabla\pi = 0$$

but we notice that:

$$\int_{\mathbb{R}^d} u \cdot (u \cdot \nabla)u = \sum_{j,k} \int_{\mathbb{R}^d} u^j \partial_j u^k u^k = \frac{1}{2}\sum_{j,k} \int_{\mathbb{R}^d} u^j \partial_j |u^k|^2 = \frac{1}{2}\sum_{j} \int_{\mathbb{R}^d} u^j \partial_j |u|^2 = \int_{\mathbb{R}^d} u \cdot \nabla |u|^2 \underset{\operatorname{div} u = 0}{=} 0$$

Therefore we obtained that:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 = 0$$

1.3 Elements of Littlewood Paley theory

Remark 1.4: Let $\chi \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that:

$$\begin{cases} \chi = 1 \text{ on } B(0,1), \\ \chi = 0 \text{ outside } B(0,2). \end{cases} \tag{2}$$

then we define $\varphi(\xi) = \chi(\xi) - \chi(2\chi)$ and for every $j \geq 0$ we define $\varphi_j(\xi) = \varphi$