BOUNDED COHOMOLOGY, QUOTIENT EXTENSIONS, AND HIERARCHICAL HYPERBOLICITY

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Abstract. We call a central extension bounded if its Euler class is represented by a bounded cocycle. We prove that a bounded central extension of a hierarchically hyperbolic group (HHG) is still a HHG; conversely if a central extension is a HHG, then the extension is bounded, and the quotient is commensurable to a HHG. Motivated by questions on hierarchical hyperbolicity of quotients of mapping class groups, we therefore consider the general problem of determining when a quotient of a bounded central extension is still bounded, which we prove to be equivalent to an extendability problem for quasihomomorphisms. Finally, we show that quotients of the 4-strands braid group by suitable powers of a pseudo-Anosov are HHG, and in fact bounded central extensions of some HHG. We also speculate on how to extend the previous result to all mapping class groups.

> Life is short and if you're looking
> for extension, you had best do well.
> 'Cause there's good deeds and then
> there's good intentions. They are
> as far apart as Heaven and Hell.
>
> Ben Harper

1. Introduction

Central extensions $1 \to K \to E \to G \to 1$ (which we often abbreviate with just E) are classified by their associated Euler class $\alpha_E \in H^2(G; K)$. We call a central extension *bounded* if its Euler class is bounded, i.e. it is represented by a bounded cocycle. One reason of interest is the fact that bounded central extensions are quasi-isometrically trivial [Ger92]. Another reason of interest comes from the study of hierarchically hyperbolic groups, see below.

In this paper we always consider central extensions with finitely generated kernel. We are interested in the following natural problem:

Problem 1.1. Given a bounded central extension E, which of its quotient central extensions $\bar{E}$ are bounded?

By a quotient central extension here we mean that there exists a diagram as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1. \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K & \longrightarrow & \bar{E} & \longrightarrow & \bar{G} & \longrightarrow & 1. \end{array}$$

Equivalently, $\bar{E} = E/N$, where N is a normal subgroup of E intersecting K trivially.

We note in the other direction that if a central extension has a bounded quotient central extension, then it is bounded (Lemma 2.12). However, not all quotient extensions of a bounded central extension are themselves bounded. For instance, the Heisenberg group is a quotient central extension of $F_2 \times \mathbb{Z}$ which is not bounded (see Example 2.13); in fact, any central extension can be realised as a quotient of a trivial extension of a free group (see Example 2.14).

By Proposition 2.9 (which we believe to be known to experts), boundedness of a central extension E is equivalent to the existence of a *quasihomomorphism* $E \to K$ that is the identity on K. Quasihomomorphisms, in the sense of e.g. [FK16], are also relevant for quotient central extensions, and indeed we show that boundedness of a quotient central extension is equivalent to an extendability problem for a quasihomomorphism. An informal statement is as follows:

Proposition 1.2 (see Proposition 2.16)**.** *Consider a bounded central extension*

$$1 \longrightarrow K \longrightarrow E \xrightarrow{\ \pi\ } G \longrightarrow 1.$$

with finitely generated kernel, and a quotient central extension $\bar{E} = E/N$. Then $\bar{E}$ is bounded if and only if a certain quasihomomorphism on $\pi(N)$ extends to G.

This has further connection to bounded cohomology in degree 3, which we discuss in Section 2.

The second and third authors encountered Problem 1.1 while studying quotients of mapping class groups, in particular relating to their conjecture on hierarchical hyperbolicity of quotients of mapping class groups [MS25, Question 3]. Hierarchically hyperbolic groups (HHG), as first defined by Behrstock, Hagen, and Sisto in [BHS17b], provide a common framework for, among others, mapping class groups of surfaces, most cubulated groups, fundamental groups of three-manifolds, and extra-large Artin groups [HS20, HRSS22, HMS24], and are therefore amenable to tools from both low-dimensional topology and the world of CAT(0) cube complexes. Showing that a given group is hierarchically hyperbolic yields a lot of information about it ([BHS21, HHP23, ANS+24, HHL23, DMS23, DMS25] is a highly non-exhaustive list). Hence it is natural to explore which group-theoretic procedures the class of HHG is closed under. These include taking graph products [BR22], relative hyperbolicity [BHS19], many graphs of groups [BR20], and several quotients [BHMS24, MS24, ABM+25]. In this direction, we characterise which central extensions preserve hierarchical hyperbolicity:

Theorem 1.3 (see Theorem 3.10)**.** *Let $1 \to K \to E \to G \to 1$ be a central extension with finitely generated kernel, and suppose that G is a hierarchically hyperbolic group. Then E is a hierarchically hyperbolic group if and only if the extension is bounded.*

We postpone to Section 4 the details of how Theorem 1.3 relates to hierarchical hyperbolicity of quotients of mapping class groups. In the other direction, we prove that a central quotient of a HHG is commensurable to a HHG:

Theorem 1.4 (see Theorem 3.13)**.** *Let $1 \to K \to E \to G \to 1$ be a central extension with finitely generated kernel. If E is a hierarchically hyperbolic group, then the extension is bounded.*

Suppose moreover that E has a HHG structure with cobounded product regions. Then there exists a finite-index subgroup $E' \leqslant E$ containing K such that E'/K is a hierarchically hyperbolic group.

The hypothesis that E has cobounded product regions is mild and natural, as explained in Definition 3.9. Since G is a finite-index overgroup of a HHG, it inherits the structure of a hierarchically hyperbolic *space*, given by the quasi-isometric inclusion $E'/K \hookrightarrow G$. However, even assuming cobounded product regions for E, one cannot hope that G itself is a hierarchically hyperbolic *group*: in Remark 3.14 we argue that the trivial central extension $E = \mathbb{Z} \times G$ of the $(3,3,3)$-triangle group G is HHG, while G is not by [PS23].

Finally, inspired by the setup explained in Section 4, we show that many quotients of the braid group B_4 on 4 strands are bounded central extensions. Recall here that B_4 is a central extension of a finite-index subgroup of the mapping class group of the five-holed sphere. The quotients we consider here are quotients by powers of pseudo-Anosovs, as studied for finite-type surfaces for instance in [DGO17, BHS17a].

Theorem 1.5 (see Theorem 4.3)**.** *Let h be a pseudo-Anosov element of B_4. Then there exists $M_0 \in \mathbb{N}_{>0}$ such that, for all multiples M of M_0, we have that $B_4/\langle\langle h^M \rangle\rangle$ is a bounded central extension of a HHG, and in particular it is a HHG.*

Outline of sections and arguments. We summarise here the techniques that are involved in our arguments. In what follows, we often restrict to central extensions with infinite cyclic kernel for the ease of exposition; in this setting, a quasihomomorphism with image in $\mathbb{Z}$ is simply called a *quasimorphism*.

In Section 2 we discuss generalities on central extensions and their quotients, boundedness, and quasihomomorphisms. It is worth mentioning Lemma 2.12, which states that, if a quotient extension is bounded, then so is the original extension. Next, we prove the characterisation of bounded quotient extensions, Proposition 2.16. The key observation is that a central extension $1 \to \mathbb{Z} \to E \xrightarrow{\pi} G \to 1$ is bounded if and only if it admits a quasimorphism $\phi\colon E \to \mathbb{Z}$ which is the identity on the kernel (see Proposition 2.9). The reader should think of ϕ as a "coarse homomorphic" retraction inducing a "coarse" splitting of the extension, as explained below.

Section 3 starts with background material on hierarchically hyperbolic groups (HHG). Roughly, a group Γ is a HHG if there exists a collection of hyperbolic spaces $\{\mathcal{C}U\}_{U\in\mathfrak{S}}$, together with coarsely Lipschitz "coordinate projections" $\pi_U\colon \Gamma \to \mathcal{C}U$ satisfying various conditions.

We uncover a connection between boundedness and hierarchical hyperbolicity of a central extension. In one direction, Theorem 3.10 shows that a bounded central extension of a HHG is itself a HHG. With an inductive argument, one can reduce to the case where the kernel is $\mathbb{Z}$, which was already settled by [AHPZ23, Theorem 5.14]; in turn, the proof of the latter is a refinement of [HRSS22, Corollary 4.3], which solved the case where the base is hyperbolic. We sketch here the core idea of both arguments, as it enlightens several ideas that appear repeatedly throughout our paper. From a bounded extension one gets an unbounded quasimorphism $\phi\colon E \to \mathbb{Z}$ as above, and the function $E \to \mathbb{Z} \times G$ mapping $e \in E$ to $(\phi(e), \pi(e))$ is a quasi-isometry. In other words, a bounded extension is also *quasi-isometrically trivial* in the sense of [Ger92], though the converse implication is not true in general (see [FS23, AM24] and Remark 3.12). Then the HHG structure for E will have one domain for every domain of G, with projection factoring through the quotient map π, and one quasiline to "detect" the $\mathbb{Z}$-factor. Such quasiline is built out of the quasimorphism ϕ, using an observation of Abbott, Balasubramanya, and Osin [ABO19, Lemma 4.15].

In the opposite direction, Theorem 3.13 proves that, if E is a HHG, then the central extension is bounded, and the quotient is a finite-index overgroup of a HHG. To get boundedness we again look for a quasimorphism on E which is the identity on $\mathbb{Z}$. Towards this, one first finds a domain $\mathcal{C}U$ on which the centre acts loxodromically, which must be a quasiline; then the required quasimorphism is the *Busemann quasimorphism* [Man08, Section 4.1] associated to the action of (a finite-index subgroup of) E on $\mathcal{C}U$, which roughly maps each $e \in E$ to its asymptotic translation length on the quasiline.

We now sketch how to prove hierarchical hyperbolicity of a finite-index subgroup of G. Let $\mathcal{C}U_1, \dots, \mathcal{C}U_k$ be the domains on which $\mathbb{Z}$ acts loxodromically. By taking linear combinations of the associated Busemann quasimorphisms, and then replacing each $\mathcal{C}U_i$ with a new quasiline (again using [ABO19, Lemma 4.15]), we can modify the HHG structure for E in such a way that $\mathbb{Z}$ acts loxodromically on a single $\mathcal{C}U_i$, call it $\mathcal{C}U$, and with uniformly bounded orbits on every other domain. Then the HHG structure of the quotient roughly coincides with what is left of the structure for E after we delete $\mathcal{C}U$.

In the above procedure, one has to restrict to a finite-index subgroup of E. This is unavoidable, as in Remark 3.14 we show that, while the $(3, 3, 3)$ triangle group G is not a HHG by [PS23], the direct product $\mathbb{Z} \times G$ is. In this sense, the statement of Theorem 1.4 is optimal.

In Section 4 we clarify how the study of hierarchical hyperbolicity of quotients of mapping class groups leads to questions on the boundedness of certain central extensions (Questions 4.1 and 4.2), and discuss possible strategies to tackle them.

Focusing on quotients by the normal closure of a high power of a pseudo-Anosov element, we solve the problem for the 4-strand braid group B_4 in Theorem 4.3. The key idea is that, if a further quotient extension has hyperbolic base, then it is bounded by [NR97], and hence so is the original quotient extension by Lemma 2.12. Thus the problem becomes to construct a suitable hyperbolic quotient of a five-punctured sphere, which we do using the machinery of *rotating families* from [Dah18] and of *short HHG* from [Man24].

For surfaces of higher genus, one can maybe adapt the above argument, conditionally to the residual finiteness of certain hyperbolic groups (see Remark 4.5), or try to exploit that the kernel is an increasing union of *convex-cocompact* subgroups (see Subsection 4.3).

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2. Bounded extensions and quasi(homo)morphisms

In this section we discuss generalities on central extensions and bounded cohomology. In particular, we state a characterisation of boundedness in terms of quasimorphisms (Proposition 2.9). The main result in this section is Proposition

2.16, which gives equivalent characterisations for a quotient central extension to be bounded.

Notation 2.1. Throughout this section, we use $\sim$ to denote equality *up to a uniformly bounded error.*

Definition 2.2. In this paper, *cocycle* will always refer to *inhomogeneous* 2*-cocycle*, namely a map $\omega\colon G^2 \to K$ such that

$$\delta\omega(g_1, g_2, g_3) = \omega(g_2, g_3) - \omega(g_1g_2, g_3) + \omega(g_1, g_2g_3) - \omega(g_1, g_2) = 0$$

for all $g_1, g_2, g_3 \in G$.

Let us briefly recall the dictionary between second cohomology and central extensions, referring the reader to [Bro94, Chapter IV.3] for details. Given a central extension $1 \to K \to E \to G \to 1$, we pick a section $\sigma\colon G \to E$ that is *normalised*, meaning that $\sigma(1) = 1$. We identify $\omega(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$ with an element of K, and then $\omega\colon G^2 \to K$ is a cocycle that is *normalised*, meaning that $\omega(1, g) = \omega(g, 1) = 0$ for all $g \in G$. This defines a class $[\omega] \in H^2(G; K)$, independent of the choice of σ, which is called the *Euler class* of the extension and we also denote by $[E]$. Conversely, given a normalised cocycle $\omega\colon G^2 \to K$ we define a group E with underlying set $K \times G$ and product

$$(k_1, g_1) \cdot (k_2, g_2) = (k_1 + k_2 + \omega(g_1, g_2), g_1g_2).$$

Then E is a central extension as above, and the map $g \to (0, g)$ is a normalised section. Equality in cohomology corresponds to equivalence of central extensions.

Definition 2.3. Let $(K, \|\cdot\|)$ be an Abelian group endowed with a norm. For another group G, a class $\alpha \in H^2(G; K)$ is *bounded* if it belongs to the image of the comparison map $H^2_b(G; K) \to H^2(G; K)$. More explicitly, α is bounded if there exists a cocycle $\omega : G^2 \to K$ such that $\alpha = [\omega]$, and such that $\|\omega(\cdot, \cdot)\|\colon G^2 \to \mathbb{Z}$ is uniformly bounded. A central extension $1 \to K \to E \to G \to 1$ is *bounded* if the corresponding Euler class $[E] \in H^2(G; K)$ is bounded.

Definition 2.4. Let $(K, \|\cdot\|)$ be as above, and let E be another group. A map $\chi\colon E \to K$ is a *quasihomomorphism* if there exists $D(\chi) \geqslant 0$, called the *defect* of χ, such that, for every $e_1, e_2' \in E$,

$$\|\chi(e_1) + \chi(e_2) - \chi(e_1e_2)\| \leqslant D(\chi).$$

A quasihomomorphism is *homogeneous* if it restricts to a homomorphism on every cyclic subgroup. When $K = \mathbb{Z}$ or $\mathbb{R}$ with the Euclidean norm, we say *quasimorphism* instead of quasihomomorphism.

Remark 2.5 (Homogeneisation)**.** Given a quasimorphism $\phi\colon E \to \mathbb{R}$, for every $g \in E$ the limit $\phi_h(g) = \lim_{n\to\infty} \frac{\phi(g^n)}{n}$ always exists. The map $\phi_h\colon E \to \mathbb{R}$ is a homogeneous quasimorphism; moreover, both $\|\phi - \phi_h\|$ and the defect of ϕ_h are bounded in terms of the defect of ϕ [Cal09, Lemma 2.21].

Real-valued quasimorphisms will only appear in the course of proofs about quasimorphisms with values in $\mathbb{Z}$. All of our statements will only involve the case in which K is a finitely generated Abelian group endowed with a word norm. In this case, a subset of K is bounded if and only if it is finite, so the notions above are independent of the choice of a norm. This allows to generalise the notion of quasihomomorphism to maps taking values in any discrete group, following Fujiwara–Kapovich [FK16].

Definition 2.6. A map $\chi\colon G \to H$ between groups is a *quasihomomorphism* if there exist a finite set $F \subset H$ such that, for every $g_1, g_2 \in G$, the difference between $\chi(g_1 g_2)$ and $\chi(g_1)\chi(g_2)$ lies in F (the way in which one takes the difference does not matter by [Heu20, Proposition 2.3]).

We record here an example of a quasimorphism that we shall use repeatedly throughout the paper.

Example 2.7 (Busemann quasimorphism, see e.g. [Man08, Section 4.1])**.** Let E be a group acting on a δ-hyperbolic metric space X, with Gromov boundary ∂X, and suppose the action fixes an ideal point $p \in \partial X$. Given a sequence $\{x_n\} \in X$ converging to p, the map $\phi_{\{x\}}\colon E \to \mathbb{R}$ defined by

$$\phi_{\{x\}}(g) = \limsup_{n\to\infty} \mathrm{d}(gx_0, x_n) - \mathrm{d}(x_0, x_n),$$

is a quasimorphism of defect at most 16δ. Then the Busemann quasimorphism ϕ is the homogeneisation of $\phi_{\{x\}}$, as in Remark 2.5. One can check that ϕ does not depend on the choice of the sequence; moreover, an element $g \in E$ acts loxodromically on X if and only if $\phi(g) \neq 0$. Note that, by construction, for every $g \in E$ one has that $|\phi(g)| \leqslant \mathrm{d}(x_0, gx_0)$; conversely, if X is a quasiline, then there exists a constant L such that $\mathrm{d}(x_0, gx_0) \leqslant |\mathrm{d}(gx_0, x_n) - \mathrm{d}(x_0, x_n)| + L$, for all sufficiently large n (if $\{x_0, gx_0, x_n\} = \{a, b, c\}$ and b is "between" a and c, one can see this by considering the distance between b and a geodesic $[a, c]$). Taking the limsup, one gets that $\mathrm{d}(x_0, gx_0)$ and $|\phi(g)|$ are within uniform distance.

We now turn to the characterisation of when a central extension is bounded, in terms of the existence of certain quasihomomorphisms.

Lemma 2.8. *Let $1 \to K \to E \to G \to 1$ be a central extension.*

- *Every bounded cocycle $\omega\colon G^2 \to K$ is cohomologous to a normalised bounded cocycle.*
- *If there exists a quasihomomorphic section $\sigma\colon G \to E$, then there exists a normalised quasihomomorphic section.*

Proof. For the first bullet, we have

$$0 = \delta\omega(g^{-1}, g, 1) = \omega(g, 1) - \omega(1, 1) + \omega(g^{-1}, g) - \omega(g^{-1}, g).$$

Therefore $\omega(g, 1) = \omega(1, 1)$, and similarly $\omega(1, g) = \omega(1, 1)$ for all $g \in G$. Let $b\colon G \to K$ be the constant function at $\omega(1, 1)$. Then $\delta b(g, h) = \omega(1, 1)$ for all $g, h \in G$. Setting $\omega' = \omega - \delta b$, we see that ω' is a normalised bounded cocycle cohomologous to ω.

For the second bullet, let $\sigma'(g) = \sigma(g)$ for all $g \neq 1$, and $\sigma'(1) = 1$. We need to check that $\sigma'(g_1)\sigma'(g_2)\sigma(g_1 g_2)^{-1}$ takes finitely many values over $g_1, g_2 \in G$. If $g_1, g_2, g_1 g_2 \neq 1$, then this follows from σ being a quasihomomorphism. If one of g_1, g_2 is equal to 1, then the equation above is just equal to 1. Finally, if $g_1 = g \neq 1$ and $g_2 = g^{-1}$, then the equation above is equal to

$$\sigma(g)\sigma(g^{-1}) = \big(\sigma(g)\sigma(g^{-1})\sigma(1)^{-1}\big)\sigma(1),$$

which again takes finitely many values because σ is a quasihomomorphism. □

Proposition 2.9. *Let $1 \to K \to E \xrightarrow{\pi} G \to 1$ be a central extension with finitely generated kernel. The following are equivalent:*

(1) $[E]$ is bounded, i.e. $[E]$ is represented by a (normalised) bounded cocycle.

(2) There exists a quasihomomorphism $\chi\colon E \to K$ such that $\chi|_K$ is the identity.
(3) There exists a (normalised) section $s\colon G \to E$ which is a quasihomomorphism.

Proof. By Lemma 2.8, we may assume that all cocycles and sections involved are normalised. Up to isomorphism of central extensions, E has underlying set $K \times G$, and product

$$(k_1, g_1) \cdot (k_2, g_2) = (k_1 + k_2 + \omega(g_1, g_2), g_1 g_2);$$

where $\omega\colon G^2 \to K$ is a normalised cocycle representing the central extension.

(1)⇒(2): Suppose that the extension is bounded, and choose ω to be a bounded cocycle. We set $\chi\colon E \to K$ to be the projection onto the first factor; notice that $\chi|_K$ is the identity. Then

$$\chi((k_1, g_1) \cdot (k_2, g_2)) = k_1 + k_2 + \omega(g_1, g_2) = \chi(k_1, g_1) + \chi(k_2, g_2) + \omega(g_1, g_2).$$

Thus the defect of χ is bounded by the norm of ω, and so it is bounded, i.e. χ is a quasihomomorphism.

(2)⇒(1): Suppose that there exists a quasihomomorphism $\chi\colon E \to K$ such that $\chi|_K$ is the identity. Let $b\colon G \to K$ be defined as $b(g) = \chi(0, g)$. Then

$$\begin{aligned}
(\omega - \delta b)(g_1, g_2) &= \omega(g_1, g_2) + b(g_1 g_2) - b(g_1) - b(g_2) \\
&= \chi(\omega(g_1, g_2), 1_G) + \chi(0, g_1 g_2) - \chi(0, g_1) - \chi(0, g_2) \\
&\sim \chi(\omega(g_1, g_2), g_1 g_2) - \chi(0, g_1) - \chi(0, g_2) \\
&= \chi((0, g_1) \cdot (0, g_2)) - \chi(0, g_1) - \chi(0, g_2) \sim 0.
\end{aligned}$$

Therefore $\omega - \delta b$ is bounded, and it is a cocycle cohomologous to ω. This shows that the extension is bounded.

(1) $\iff$ (3): This equivalence was already pointed out in [FS23] (see the discussion after Proposition 2.3 there). If $\sigma\colon G \to E$ is a quasihomomorphic normalised section, then the cocycle $\omega(g, h) = \sigma(gh)^{-1}\sigma(g)\sigma(h)$, which represents the Euler class, is bounded. The converse implication is [Heu20, Theorem C]. □

Corollary 2.10. *Let $1 \to K \to E \xrightarrow{\pi} G \to 1$ be a bounded central extension with finitely generated kernel. Let $L < K$ be a subgroup. Then the central extensions*

$$1 \to L \to E \to E/L \to 1$$

and

$$1 \to K/L \to E/L \to G \to 1$$

are also bounded.

Proof. We use the characterisation in the second item of Proposition 2.9. Let $\chi\colon E \to K$ be a quasimorphism such that $\chi|_K$ is the identity. Then $\chi|_L$ is the identity, which proves that the L-central extension is bounded. Composing χ with the projection $K \to K/L$, we get a quasimorphism $E/L \to K/L$ that is the identity on K/L, whcih proves that the K/L-central extension is bounded. □

Given a central extension $1 \to K \to E \xrightarrow{\pi} G \to 1$, let $N \leqslant E$ be a normal subgroup that intersects K trivially, so that $\pi\colon N \to \pi(N)$ is an isomorphism.

Then there is a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{p} & & \downarrow{p} & & \\ 1 & \longrightarrow & K & \longrightarrow & \overline{E} & \longrightarrow & \overline{G} & \longrightarrow & 1 \end{array} \tag{1}$$

where $\overline{E} = E/N$ and $\overline{G} = G/\pi(N)$, and both rows are central extension. We use the letter p to denote both quotients $E \to \overline{E}$ and $G \to \overline{G}$ by an abuse of notation. We say that the bottom row is a *quotient central extension* of the top row. The main focus of the paper is the following question (see Problem 1.1):

Question 2.11. Under which conditions is a quotient central extension bounded?

A first easy necessary condition is given by the following observation:

Lemma 2.12. *With the notation of Diagram* (1)*, the pullback* $p^*\colon H^2(\overline{G}; K) \to H^2(G; K)$ *sends* $[\overline{E}]$ *to* $[E]$*. In particular, if* $[\overline{E}]$ *is bounded, then* $[E]$ *is bounded.*

Proof. We choose a normalised section $\overline{\sigma}\colon \overline{G} \to \overline{E}$. We also choose an injective map $\tau\colon \overline{G} \to E$ such that $p\tau = \overline{\sigma}$ and $\tau(1_{\overline{G}}) = 1_E$. This way Diagram (1) is enriched as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{p} & \overset{\tau}{\nwarrow} & \downarrow{p} & & \\ 1 & \longrightarrow & K & \longrightarrow & \overline{E} & \underset{\overline{\sigma}}{\overset{\overline{\pi}}{\rightleftarrows}} & \overline{G} & \longrightarrow & 1 \end{array}$$

Now $\pi\tau$ is a section for $p\colon G \to \overline{G}$, so every element of G can be written uniquely as $g = \pi(n) \cdot \pi\tau(p(g))$ for some $n \in N$. We define $\sigma(g) = n \cdot \tau(p(g))$. First, note that σ is indeed a normalised section: $\sigma(1_G) = 1_E$, and if $g = \pi(n)\pi\tau(p(g))$, then

$$\pi\sigma(g) = \pi(n \cdot \tau(p(g))) = \pi(n) \cdot \pi\tau(p(g)) = g.$$

Secondly, we claim that $p\sigma = \overline{\sigma}p$, indeed

$$p\sigma(g) = p(n \cdot \tau(p(g))) = p\tau(p(g)) = \overline{\sigma}(p(g)),$$

which implies that the projection $p\colon E \to \overline{E}$ sends

$$\sigma(g)\sigma(h)\sigma(gh)^{-1} \mapsto \overline{\sigma}(p(g))\overline{\sigma}(p(h))\overline{\sigma}(p(gh))^{-1}.$$

This shows that the cocycle defined by σ is indeed the pullback of the cocycle defined by $\overline{\sigma}$, and concludes the proof. □

This condition is however not sufficient, as the following examples show:

Example 2.13. Consider the group $Z \times F_2$, where $Z = \langle z \rangle$ is infinite cyclic, and $F_2 = \langle x, y \rangle$ is free of rank 2. Let N be the normal closure of $z^{-1}[x, y]$, which intersects Z trivially. The quotient $(Z \times F_2)/N$ is the Heisenberg group H_3, and the quotient $F_2/\pi(N)$ is the free Abelian group $\mathbb{Z}^2$. So we have a map of central extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & Z \times F_2 & \longrightarrow & F_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z & \longrightarrow & H_3 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1 \end{array}$$

The top one splits, so it is trivial and in particularly bounded. The bottom one is not trivial, and in fact it maps to a generator of $H^2(\mathbb{Z}^2; \mathbb{R}) \cong \mathbb{R}$ under the change of coefficients map. Because $\mathbb{Z}^2$ is amenable, $H^2_b(\mathbb{Z}^2;\mathbb{R})$ vanishes [Fri17, Chapter 3], and so the bottom central extension cannot be bounded.

Example 2.14. More generally, we claim that every central extension can be expressed as a quotient of a trivial (thus bounded) central extension of a free group. Let $1 \to K \to E \to G \to 1$ be a central extension. Let F be a free group and $p\colon F \to G$ a presentation, which we lift to a homomorphism $\tilde{p}\colon F \to E$. Define $P\colon K \times F \to E$ to be the product of $\tilde{p}$ and the inclusion $K \to E$. An element $(k, w) \in K \times F$ belongs to the kernel of P if and only if $\tilde{p}(w) = k$; in particular K intersects this kernel trivially. So we indeed have a quotient central extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & K \times F & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

We can restate Question 2.11 in terms of extendability of a certain quasihomomorphism. We start with an easy lemma, which is a variation of a well-known fact regarding real-valued homogeneous quasimorphisms.

Lemma 2.15. *Let $N \leqslant G$ be a normal subgroup, and denote by $p\colon G \to G/N$ the quotient. Let $\chi\colon G \to K$ be a quasihomomorphism. If $\chi|_N \equiv 0$, then there exists a quasihomomorphism $\phi\colon G/N \to K$ such that χ is at a bounded distance from ϕp. Moreover, if $H \leqslant G$ is a subgroup such that $\chi|_H$ is a homomorphism, then ϕ can be chosen so that $\phi p|_H = \chi|_H$.*

Proof. Let $K = \mathbb{Z}^k \times T$, where $k \geqslant 0$ and T is a finite abelian group. Accordingly, write $\chi = (\chi^1, \dots, \chi^k, \chi^T)$, and note that $\chi^1, \dots, \chi^k$ are quasimorphisms (while χ^T can be any function $G \to T$). Recall that every quasimorphism $\chi^i\colon G \to \mathbb{R}$ is at a bounded distance from a unique homogeneous quasimorphism χ^i_h (Remark 2.5). The hypothesis, and uniqueness of homogeneous representatives, implies that $\chi^i_h|_N \equiv 0$ for all $1 \leqslant i \leqslant k$. Therefore, by [Cal09, Remark 2.90] there exist homogeneous quasimorphisms $\phi^i_h\colon G/N \to \mathbb{R}$ such that $\chi^i_h = \phi^i_h p$ for all $1 \leqslant i \leqslant k$. Setting ϕ^i to be the integral part of ϕ^i_h, we obtain a quasimorphism $\phi^i\colon G/N \to \mathbb{Z}$ such that $\phi^i p \sim \phi^i_h p = \chi^i_h \sim \chi^i$. Setting $\phi = (\phi^1, \dots, \phi^k, \phi^T)$, where $\phi^T\colon G/N \to T$ is any function, we obtain that χ is at a bounded distance from ϕp. Notice that ϕ is indeed a quasihomomorphism, as its entries are quasimorphisms.

Suppose that $\chi|_H$ is a homomorphism. Then by uniqueness of homogeneous representatives, $\chi^i|_H = \chi^i_h|_H = \phi^i_h p|_H$ for all i. In particular, ϕ^i_h takes integer values on $p(H)$, and so $\phi^i|_{p(H)} = \phi^i_h|_{p(H)}$. Moreover, because $\chi|_N \equiv 0$, there exists a map $\phi^T\colon G/N \to T$ such that $\phi^T p|_H = \chi^T|_H$. With this choice of ϕ^T, we have $\phi p|_H = \chi|_H$, as promised. □

Back to the setting of Diagram (1), we now provide the characterisation of boundedness of central extensions in terms of extendable quasihomomorphisms. Recall from Lemma 2.12 that, in order for the quotient central extension to be bounded, the original central extension needs to be bounded, so we assume this throughout.

Proposition 2.16. *With the notation of Diagram* (1)*, suppose that* $[E]$ *is bounded, and let $\chi\colon E \to K$ be a quasihomomorphism such that $\chi|_K$ is the identity, provided by Proposition 2.9. Then:*

- *The map $\chi\pi^{-1}\colon \pi(N) \to K$ is a well-defined quasihomomorphism, and it is almost invariant under conjugacy by G.*

- *If ω is a normalised bounded cocycle representing $[E]$, then χ can be chosen so that $\delta(\chi\pi^{-1}) = \omega|_{\pi(N)}$.*
- *$[\overline{E}]$ is bounded if and only if $\chi\pi^{-1}$ can be extended to a quasihomomorphism $G \to K$.*

Given a normal subgroup $\Lambda \leqslant \Gamma$, we say that a quasihomomorphism $\phi\colon \Lambda \to K$ is *almost invariant under conjugacy by* Γ if there exists a constant $C > 0$ such that

$$\|\phi(\gamma\lambda\gamma^{-1}) - \phi(\lambda)\| < C$$

for all $\gamma \in \Gamma, \lambda \in \Lambda$. Note that quasihomomorphisms on Γ are almost invariant under conjugacy by Γ, so this is a necessary condition for a quasihomomorphism on Λ to be extendable, which will come up in the next result.

Proof of Proposition 2.16. $\chi\pi^{-1}$ is well-defined because $\pi|_N$ is an isomorphism. Because χ is defined on E, it is almost invariant under conjugacy by E, and then it follows that $\chi\pi^{-1}$ is almost invariant under conjugacy by G.

Moving to the second bullet, from the proof of Proposition 2.9 we see that if ω is a normalised bounded cocycle representing E, seen as the Cartesian product $K \times G$ with the product twisted by ω, then χ can be chosen to be the projection onto the first factor, while π is the projection onto the second factor. Now every element $g \in N$ can be written as $(b(g), \pi(g))$, and it follows that $\chi\pi^{-1}\colon \pi(N) \to K$ is just the map $\pi(g) \mapsto b(g)$. Because N is a group and π is injective, for all $g_1, g_2 \in N$ it holds

$$\begin{aligned}(b(g_1g_2), \pi(g_1g_2)) &= (b(g_1), \pi(g_1)) \cdot (b(g_2), \pi(g_2)) \\ &= (b(g_1) + b(g_2) + \omega(\pi(g_1), \pi(g_2)), \pi(g_1g_2));\end{aligned}$$

which shows that $\delta(\chi\pi^{-1}) = \omega$ as cocycles $\pi(N)^2 \to K$.

We finally prove the equivalence in the third bullet. Suppose that $\chi\pi^{-1}$ extends to a quasihomomorphism $\phi\colon G \to K$. Consider the quasihomomorphism $\chi - \phi\pi\colon E \to K$. Notice that $\chi - \phi\pi$ restricts to the zero map on N and to the identity on K. Therefore, Lemma 2.15 yields a quasihomomorphism $\psi\colon \overline{E} \to K$ such that ψp is at a bounded distance from $\chi - \phi\pi$, and restricts to the identity on K. Thus $[\overline{E}]$ is bounded, by the second item of Proposition 2.9.

Conversely, suppose that $\psi\colon \overline{E} \to K$ is a quasihomomorphism that restricts to the identity on K. Then $\psi p\colon E \to K$ vanishes on N and is still the identity on K. Therefore $\chi - \psi p\colon E \to K$ vanishes on K, and coincides with χ on N. By Lemma 2.15, there exists a quasihomomorphism $\phi\colon G \to K$ such that $\phi\pi\colon E \to K$ is at a bounded distance from $\chi - \psi p$. So, up to changing ϕ by a bounded amount on $\pi(N)$, it is an extension of $\chi\pi^{-1}$. □

Remark 2.17. Shuhei Maruyama pointed out to us that both Propositions 2.9 and 2.16 can also be proved by diagram chasing, using the exact sequences from [KKM+21].

In view of Proposition 2.16, the problem of whether the quotient central extension $[\overline{E}]$ is bounded is equivalent to the problem of whether a specific quasihomomorphism $\pi(N) \to K$ extends to G. This problem is non-trivial. On the one hand, the easy obstruction for extendability, that is almost invariance under conjugacy by G, is excluded by the the first bullet of Proposition 2.16. On the other hand, the almost invariance under conjugacy is not sufficient for a quasimorphism to be extendable in general, as we have seen in Example 2.13. The problem of extendability has been studied at length in recent years, and produced various examples of non-extendable quasimorphisms, and criteria for extendability: see [KKM$^+$24] for a survey. In particular, we recall the following criteria:

Proposition 2.18 ([KKMM22, Proposition 1.6] and [KKM$^+$21, Theorem 1.9])**.** *Let $1 \to N \to G \to G/N \to 1$ be a short exact sequence of groups. Then any of the following conditions guarantees the extendability of every quasimorphism $N \to K$ that is almost invariant under conjugacy by G.*

(1) The extension virtually splits.
(2) $H_b^3(G/N;\mathbb{R}) = H_b^2(G/N;\mathbb{R}) = H^2(G/N;\mathbb{R}) = 0$.

In another direction, very far from the case of a normal subgroup, we have:

Proposition 2.19 ([HO13], see also [FPS15])**.** *Let $H \leqslant G$ be a hyperbolically embedded subgroup. Then every quasimorphism $H \to K$ extends to a quasimorphism $G \to K$.*

For both propositions, the original statement is about real-valued quasimorphisms, however these statements follow by splitting K as $\mathbb{Z}^k \times T$, choosing arbitrary extensions for the T part, and dealing with the $\mathbb{Z}$ factors by taking the integral part and modifying by a bounded amount.

Remark 2.20. By Proposition 2.16, the quasimorphism one has to extend is always such that its boundary is extendable. This shows a stark difference from the problem of extendability of bounded cohomology classes and extendability of quasimorphisms [KKM$^+$24, End of Section 1.3].

3. Hierarchical hyperbolicity of central extensions

In this Section we explore the connections between boundedness and hierarchical hyperbolicity for central extensions. We first recall the notion of hierarchically hyperbolic spaces and groups.

Definition 3.1 (Hierarchically hyperbolic space)**.** Let $\delta > 0$ and $\mathcal{X}$ be a (δ,δ)–quasigeodesic space. A *hierarchically hyperbolic space (HHS) structure with constant* δ for $\mathcal{X}$ is the data of an index set $\mathfrak{S}$ and a set $\{\mathcal{C}W : W \in \mathfrak{S}\}$ of δ-hyperbolic spaces $(\mathcal{C}W, \mathrm{d}_W)$ such that the following axioms are satisfied.

(1) **(Projections.)** For each $W \in \mathfrak{S}$, there exists a *projection* $\pi_W : \mathcal{X} \to 2^{\mathcal{C}W}$ that is a (δ,δ)–coarsely Lipschitz, δ–coarsely onto, δ–coarse map.
(2) **(Nesting.)** If $\mathfrak{S} \neq \varnothing$, then $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$ and contains a unique $\sqsubseteq$–maximal element, denoted by S. When $V \sqsubseteq W$, we say V is *nested* in W. For each $W \in \mathfrak{S}$, we denote by $\mathfrak{S}_W$ the set of all $V \in \mathfrak{S}$ with $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \sqsubsetneq W$ there is a specified non-empty subset $\rho_W^V \subseteq \mathcal{C}W$ with $\mathrm{diam}(\rho_W^V) \leqslant \delta$.
(3) **(Finite complexity.)** Any set of pairwise $\sqsubseteq$–comparable elements has cardinality at most δ.
(4) **(Orthogonality.)** The set $\mathfrak{S}$ has a symmetric relation called *orthogonality*. If V and W are orthogonal, we write $V \perp W$ and require that V and W are not $\sqsubseteq$–comparable. Further, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. We denote by $\mathfrak{S}_W^\perp$ the set of all $V \in \mathfrak{S}$ with $V \perp W$.

(5) **(Containers.)** For each $W \in \mathfrak{S}$ and $U \in \mathfrak{S}_W$ with $\mathfrak{S}_W \cap \mathfrak{S}_U^\perp \neq \varnothing$, there exists $Q \in \mathfrak{S}_W$ such that $V \sqsubseteq Q$ whenever $V \in \mathfrak{S}_W \cap \mathfrak{S}_U^\perp$. We call Q the *container of U in W*.
(6) **(Transversality.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V and W are *transverse*, denoted $V \pitchfork W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \pitchfork W$, there are non-empty sets $\rho_W^V \subseteq \mathcal{C}W$ and $\rho_V^W \subseteq \mathcal{C}V$, each of diameter at most δ.
(7) **(Consistency.)** For all $x \in \mathcal{X}$ and $U, V, W \in \mathfrak{S}$:
   - if $V \pitchfork W$, then $\min\left\{\mathrm{d}_W(\pi_W(x), \rho_W^V), \mathrm{d}_V(\pi_V(x), \rho_V^W)\right\} \leqslant \delta$,
   - if $U \sqsubseteq V$ and either $V \sqsubsetneq W$, or $V \pitchfork W$ and $W \not\perp U$, then $\mathrm{d}_W(\rho_W^U, \rho_W^V) \leqslant \delta$.
(8) **(Bounded geodesic image (BGI).)** For all $V, W \in \mathfrak{S}$ and for all $x, y \in \mathcal{X}$, if $V \sqsubsetneq W$ and $\mathrm{d}_V(\pi_V(x), \pi_V(y)) \geqslant \delta$, then every $\mathcal{C}W$–geodesic from $\pi_W(x)$ to $\pi_W(y)$ must intersect $\mathcal{N}_\delta(\rho_W^V)$.
(9) **(Large links.)** For all $W \in \mathfrak{S}$ and $x, y \in \mathcal{X}$, there exists a collection $\{V_1, \dots, V_m\} \subseteq \mathfrak{S}_W - \{W\}$ such that $m \leqslant \delta \mathrm{d}_W(\pi_W(x), \pi_W(y)) + \delta$, and for all $U \in \mathfrak{S}_W - \{W\}$, either $U \sqsubseteq V_i$ for some i, or $\mathrm{d}_U(\pi_U(x), \pi_U(y)) \leqslant \delta$.
(10) **(Partial realization.)** If $\{V_i\}$ is a finite collection of pairwise orthogonal elements of $\mathfrak{S}$ and $p_i \in \mathcal{C}V_i$ for each i, then there exists $x \in \mathcal{X}$ *realising* the tuple (p_i), meaning that, for every i and every $W \in \mathfrak{S}$:
   - $\mathrm{d}_{V_i}(\pi_{V_i}(x), p_i) \leqslant \delta$;
   - if $V_i \sqsubsetneq W$ or $W \pitchfork V_i$ then $\mathrm{d}_W(\pi_W(x), \rho_W^{V_i}) \leqslant \delta$.
(11) **(Uniqueness.)** There exists a function $\theta\colon [0, \infty) \to [0, \infty)$ so that for all $r \geqslant 0$, if $x, y \in \mathcal{X}$ and $\mathrm{d}_{\mathcal{X}}(x, y) \geqslant \theta(r)$, then there exists $W \in \mathfrak{S}$ such that $\mathrm{d}_W(\pi_W(x), \pi_W(y)) \geqslant r$.

We use $\mathfrak{S}$ to denote the HHS structure. We call an element $U \in \mathfrak{S}$ a *domain*, the associated space $\mathcal{C}U$ its *coordinate space*, and call the maps ρ_W^V the *relative projections* from V to W. The quantity δ is called a *hierarchy constant* for $\mathfrak{S}$. We often suppress reference to the projection maps, so for every $x, y \in \mathcal{X}$ and $U \in \mathfrak{S}$ we write $\mathrm{d}_U(x, y)$ to mean $\mathrm{d}_U(\pi_U(x), \pi_U(y))$.

Definition 3.2 (Hierarchically hyperbolic group)**.** A finitely generated group G is a *hierarchically hyperbolic group* (HHG) if the following hold.
(i) G acts metrically properly and coboundedly on a space $\mathcal{X}$ admitting a HHS structure $\mathfrak{S}$.
(ii) There is a $\sqsubseteq$–, $\perp$–, and $\pitchfork$–preserving action of G on $\mathfrak{S}$ by bijections such that $\mathfrak{S}$ contains finitely many G–orbits.
(iii) For each $W \in \mathfrak{S}$ and $g \in G$, there exists an isometry $g_W\colon \mathcal{C}W \to \mathcal{C}(gW)$ satisfying the following for all $V, W \in \mathfrak{S}$ and $g, h \in G$.
   - The maps $(gh)_W\colon \mathcal{C}W \to \mathcal{C}(ghW)$ and $g_{hW} \circ h_W\colon \mathcal{C}W \to \mathcal{C}(hW)$ coincide.
   - For each $x \in X$, $g_W(\pi_W(x)) = \pi_{gW}(g \cdot x)$ in $\mathcal{C}(gW)$.
   - If $V \pitchfork W$ or $V \sqsubsetneq W$, then $g_W(\rho_W^V) = \rho_{gW}^{gV}$ in $\mathcal{C}(gW)$.

We often drop the indices and denote each g_W simply by g. When the underlying HHS is not relevant, we denote a HHG by $(G, \mathfrak{S})$.

Remark 3.3 (Moral compass)**.** When reading the definitions above, the uninitiated reader should keep in mind the motivating example of a HHG, which is the mapping class group of an orientable, finite-type surface. In this context, $\mathcal{X}$ is the marking complex from [MM00, Section 2.5]; the elements of $\mathfrak{S}$ are isotopy classes of subsurfaces, with nesting given by inclusion and orthogonality corresponding to disjointness (both up to isotopy); finally, the coordinate space associated to a subsurface is the corresponding curve graph, onto which $\mathcal{X}$ maps via the subsurface projection. The various axioms from Definition 3.1 are abstractions of the properties of curve graphs and clean markings from [MM99, MM00].

3.1. **Variations on the axioms.**

Remark 3.4 (Normalisation)**.** For experts, Definition 3.1 is that of a *normalised* HHS, since the original definition [BHS19] only required the coordinate projections to have uniformly quasiconvex images. However it is always possible to normalise a HHS structure, by restricting each coordinate space $\mathcal{C}U$ to a neighbourhood of $\pi_U(\mathcal{X})$ (see e.g. [BHS19, Remark 1.3]).

Remark 3.5 (Bounded domain dichotomy)**.** A HHG acts cofinitely on the set of domains, and two domains in the same orbit have isometric coordinate spaces. Therefore, up to enlarging δ, we can and will assume that every $U \in \mathfrak{S}$ is either unbounded (meaning that $\mathcal{C}U$ is) or $\mathrm{diam}\mathcal{C}U \leqslant \delta$.

Remark 3.6 (Almost HHS structure)**.** Axiom (5) can be replaced by the following weaker requirement:

(5') **(Finite dimension.)** Every collection of pairwise orthogonal elements has cardinality at most δ.

The pair $(\mathcal{X}, \mathfrak{S})$ is an *almost HHS* if it satisfies all axioms from Definition 3.1, with (5) replaced by (5'). Every HHS has finite dimension by [BHS19, Lemma 2.1]; conversely, it was shown in [ABD21, Appendix A] that an almost HHS structure can be completed to a genuine HHS structure. Similarly, an almost HHG structure (defined as in Definition 3.2 with an action on an almost HHG) can be completed to a HHG structure, as argued in [ABD21, Remark A.7].

Remark 3.7 (Passing up)**.** The large links Axiom (9) can be replaced by the following weaker requirement:

(9') **(Passing up.)** For every $t > 0$, there exists an integer $P = P(t) > 0$ such that if $V \in \mathfrak{S}$ and $x, y \in \mathcal{X}$ satisfy $\mathrm{d}_{U_i}(x, y) > \delta$ for a collection of domains $\{U_i\}_{i=1}^{P}$ with $U_i \in \mathfrak{S}_V$, then there exists $W \in \mathfrak{S}_V$ containing some U_i, and such that $\mathrm{d}_W(x, y) > t$. We call $P\colon (0, \infty) \to (0, \infty)$ the *passing up function*.

It was shown in [BHS19, Lemma 2.5] that every HHS satisfies the Passing up axiom; conversely, in [Dur23, Section 4.8] it is argued that the passing up axiom, together with BGI and normalisation, imply the large links axiom. Remarkably, the container axiom is not involved in the argument, therefore an almost HHS structure (resp. HHG structure) satisfying the passing up axiom is a genuine almost HHS structure (resp. HHG structure).

The above discussion allows us to simplify any HHG structure, only keeping the unbounded domains:

Lemma 3.8. *Let G be a HHG, with structure $(\mathcal{X}, \mathfrak{S})$. Let*

$$\widetilde{\mathfrak{S}} = \{S\} \cup \{U \in \mathfrak{S} \mid \mathcal{C}U \text{ unbounded}\}.$$

Then $(\mathcal{X}, \widetilde{\mathfrak{S}})$*, with the same coordinate spaces, (relative) projections, and relations, is an almost HHG structure for* G*.*

Proof. Let δ be a HHG constant for the structure, which we can assume to be larger than the diameter of every bounded domain by Remark 3.5. We now check that $(\mathcal{X}, \mathfrak{S})$ satisfies the axioms of an almost HHG structure for G. Since we did not change the projections, the relations, and the relative projections, Axioms (1) to (4) and (6) to (8) still hold automatically, so we focus on the remaining ones.

- **Finite dimension (5').** Every collection of pairwise orthogonal elements has uniformly bounded cardinality, as this is true in $\mathfrak{S}$.
- **Large links (9)**. By Remark 3.7, it is enough to check the passing up axiom. In turn, the passing up axiom for $(\mathcal{X}, \mathfrak{S})$ produces some passing up function $P\colon (0,\infty) \to (0,\infty)$. Set
$$P'(t) = \begin{cases} P(\delta) & \text{if } t \leqslant \delta; \\ P(t) & \text{if } t \geqslant \delta. \end{cases}$$
Now let $t > 0$, $P' = P'(t)$, $V \in \widetilde{\mathfrak{S}}$ and $x, y \in \mathcal{X}$ satisfy $\mathrm{d}_{U_i}(x,y) > \delta$ for a collection of domains $\{U_i\}_{i=1}^{P'}$ with $U_i \in \widetilde{\mathfrak{S}}_V$. By the choice of P', there exists $W \in \mathfrak{S}$ which is nested in V, contains some U_i, and such that $\mathrm{d}_W(x,y) > \max\{t, \delta\}$. In particular $\mathrm{diam}\mathcal{C}W = \infty$ by Remark 3.5, so $W \in \widetilde{\mathfrak{S}}$.
- **Partial realisation (10)**. Let $\{V_i\} \subseteq \widetilde{\mathfrak{S}}$ a collection of pairwise orthogonal elements, and for every i let $p_i \in \mathcal{C}V_i$. Since the V_i are orthogonal in $\mathfrak{S}$, there exists some $x \in \mathcal{X}$ realising the tuple, so both bullets of the axiom are automatically satisfied in $\widetilde{\mathfrak{S}} \subseteq \mathfrak{S}$.
- **Uniqueness (11)**. Let $x, y \in \mathcal{X}$ and $r \geqslant 0$ be such that $\mathrm{d}_W(x,y) \leqslant r$ for every $W \in \widetilde{\mathfrak{S}}$. Notice that, if $W' \in \mathfrak{S} - \widetilde{\mathfrak{S}}$, then $\mathrm{d}_{W'}(x,y) \leqslant \mathrm{diam}\mathcal{C}W' \leqslant \delta$; thus x and y are uniformly close in every domain of $\mathfrak{S}$, and therefore they are uniformly close in $\mathcal{X}$ by the original uniqueness axiom.
- G still acts metrically properly and coboundedly on $\mathcal{X}$, and the G-action on $\mathfrak{S}$ restricts to $\widetilde{\mathfrak{S}}$ since it preserves the diameters of coordinate spaces. Hence $(\mathcal{X}, \widetilde{\mathfrak{S}})$ is an almost hierarchically hyperbolic *group* structure for G. □

Definition 3.9 ((Cobounded) product region)**.** Given $U \in \mathfrak{S}$, the *product region* associated to U is the subspace
$$P_U = \{x \in \mathcal{X} \mid \mathrm{d}_W(x, \rho^U_W) \leqslant \delta \text{ for all } U \pitchfork W \text{ or } U \subsetneq W\}.$$
If G is a HHG, then the stabiliser $\mathrm{Stab}_G(U)$ acts on P_U, and the action is proper as so is the G-action on $\mathcal{X}$. We say that G has *cobounded product regions* if the action is also cobounded. Under this assumption $\mathrm{Stab}_G(U)$ acts coboundedly on $\mathcal{C}U$ as well.

The assumption of having cobounded product regions is extremely natural, to the point that the authors of [BHS19] argue it should be part of the original definition. Intuitively, under this assumption, domain stabilisers are themselves hierarchically hyperbolic, and this allows for inductive arguments.

3.2. **Bounded central extensions and hierarchical hyperbolicity.** We now move to the cohomological characterisation of when a central extension of a HHG, with finitely generated kernel, is a HHG.

Theorem 3.10. *Let G be a HHG, and let $1 \to K \to E \to G \to 1$ be a central extension with finitely generated kernel. Then E is a HHG if and only if the extension is bounded.*

Remark 3.11. One direction of Theorem 3.10 extends [AHPZ23, Theorem 5.14], which states that a $\mathbb{Z}$-central extension of a HHG is a HHG. In turn, the argument there is a refinement of [HRSS22, Corollary 4.3], which dealt with the case with hyperbolic base and only relied on the fact that every $\mathbb{Z}$-central extension of a hyperbolic group is bounded [NR97]. As we have seen in Example 2.13, this is no longer automatically true for general HHG. Even assuming acylindrical hyperbolicity of the base is not enough: indeed $H^2(\mathbb{Z}^2 * \mathbb{Z}; \mathbb{Z}) \cong H^2(\mathbb{Z}^2; \mathbb{Z})$, which produces an unbounded class. The argument is the same as Example 2.13, using for instance that the comparison map $H^2_b(\mathbb{Z}^2 * \mathbb{Z}; \mathbb{R}) \to H^2(\mathbb{Z}^2 * \mathbb{Z}; \mathbb{R})$ is trivial [Li23, Example 4.7].

Remark 3.12. It is natural to ask whether one can also characterise the extension being just HHS rather than HHG. A sufficient, and potentially necessary condition for this is that the extension is quasi-isometrically trivial, which turns out to be equivalent to the Euler class of the extension being *weakly bounded*, in the sense of [NR97]. For many groups (for instance right-angled Artin groups), weakly bounded classes are bounded [FS23], and this is called property QITB (that is, quasi-isometrically trivial implies bounded). However, there are examples of quasi-isometrically trivial extensions which are not bounded [FS23, AM24]. To summarise, two intriguing questions arise from this:

(1) If a central extension of an HHG is an HHS, is the extension quasi-isometrically trivial?
(2) Do HHGs satisfy QITB? Equivalently, in view of Theorem 3.10, is a quasi-isometrically trivial extension of a HHG also a HHG?

Proof of Theorem 3.10. Let $K \cong \mathbb{Z}^n \times T$ for some $n \geqslant 0$ and T finite.

($\Leftarrow$) Assume that the extension $E \to G$ is bounded. A finite extension of a HHG is a HHG (via the geometric action on the same space). Moreover, a bounded $\mathbb{Z}$-central extension of a HHG is a HHG, as one sees by combining [ABO19, Lemma 4.15] and [AHPZ23, Theorem 5.14]. So by Corollary 2.10 and induction, we conclude that E is a HHG.

($\Rightarrow$) Now assume that E admits a HHG structure, coming from the action on some HHS $(\mathcal{X}, \mathfrak{S})$. Our goal is to produce a quasihomomorphism $\Psi \colon E \to K$ which is the identity on K, and then we will conclude by Proposition 2.9.

Fix a basepoint $x \in \mathcal{X}$, and let $z \in K$ be any infinite order element. As in [DHS17] let

$$\mathrm{Big}\,(z) = \{U \in \mathfrak{S} \mid \mathrm{diam} \pi_U(\langle z \rangle \cdot x) = \infty\},$$

that is, the set of coordinate spaces "witnessing" the infinite order of z. It is clear that $\mathrm{Big}\,(z)$ does not depend on the choice of the basepoint. Moreover, since z has infinite order, $\mathrm{Big}\,(z) \neq \varnothing$ (this is [DHS17, Proposition 6.4]), and it is a finite collection of pairwise orthogonal domains (this follows from combining [DHS17, Lemma 6.7] with [BHS19, Lemma 2.1]). Notice that E permutes $\mathrm{Big}\,(z)$, since for every $h \in E$ we have

$$\mathrm{diam} \pi_{hU}(\langle z \rangle \cdot x) = \mathrm{diam} \pi_U(h^{-1} \langle z \rangle \cdot x) = \mathrm{diam} \pi_U(\langle z \rangle h^{-1} \cdot x) = \infty.$$

Thus there exists a finite-index, normal subgroup $E' \leqslant E$ fixing every domain $\mathrm{Big}\,(z)$. We claim that, by arguing as in [ABO19, Lemma 4.20], one can show that $\mathcal{C}U$ is a quasiline for every $U \in \mathrm{Big}\,(z)$. Indeed, $Z' \coloneqq \langle z \rangle \cap E'$ has unbounded orbits on $\mathcal{C}U$, so E' does not act elliptically; moreover, the action is not parabolic as it is cobounded (this is because π_U is coarsely surjective). By the classification of group actions on hyperbolic spaces (see e.g. [ABO19, Theorem 4.2]), there must be a loxodromic element $h \in E'$. As Z' commutes with h, it must fix the endpoints $h^+, h^- \in \partial\mathcal{C}U$. In particular, the action of Z' is not parabolic, so some power of z is a loxodromic with endpoints $h^{\pm}$. Again, this means that every element in E', which commutes with Z', must fix both h^+ and h^-; hence, as the action is cobounded, $\mathcal{C}U$ must be a quasiline.

We now construct a quasimorphism $\phi_z \colon E \to \mathbb{R}$ which is unbounded on $\langle z \rangle$. As argued above, E' fixes the ideal endpoints of $\mathcal{C}U$, so let $\phi' \colon E' \to \mathbb{R}$ be the Busemann quasimorphism associated to the action, which is unbounded on Z'. Now [Man24, Lemma 3.4] takes ϕ' as input and produces an quasiline L on which E acts, and such that Z' (hence $\langle z \rangle$) still acts loxodromically. But z is central in E, which again implies that E acts on L without inversions. Then the required quasimorphism ϕ_z is the Busemann quasimorphism associated to the action on L.

Now let $\mathbb{Z}^n \leqslant K$ be a maximal torsion-free subgroup, and let $z_1 \in \mathbb{Z}^n$ be a primitive element. The quasimorphism $\phi_1 \coloneqq \phi_{z_1}$ restricts to a homomorphism on the abelian subgroup $\mathbb{Z}^n$ [Fri17, Corollary 2.12]; so let $z_2 \in \ker\phi_1$ be a primitive element. Proceeding analogously, one gets a basis $z_1, \dots, z_n$ for $\mathbb{Z}^n$ and a collection of quasimorphisms $\phi_1, \dots, \phi_n \colon E \to \mathbb{R}$ such that $\phi_i(z_j) = 0$ for every $i < j$, and $\phi_i(z_i) \neq 0$ for every i. In other words, the matrix $M_{ij} = \phi_i(z_j)$ is lower triangular without zeroes on the diagonal, and in particular invertible. Then let $\Psi \colon E \to \mathbb{R}^n$ defined by

$$\Psi(e) = M^{-1}(\phi_1(e), \dots, \phi_n(e))^T.$$

By construction Ψ is the identity on $\mathbb{Z}^n$. Up to taking the integer part, we can assume that Ψ takes values in $\mathbb{Z}^n$, and up to a further bounded modification we can assume that $\Psi|_K$ is the identity. As a bounded modification of a quasihomomorphism with abelian target is still a quasihomomorphism, Ψ satisfies the requirements of Proposition 2.9, and the proof is complete. □

Next, we prove a statement in the opposite direction:

Theorem 3.13. *Let $1 \to K \to E \to G \to 1$ be a central extension with finitely generated kernel. If E is a HHG, then the extension is bounded.*

Suppose moreover that E has a HHG structure with cobounded product regions. Then there exists a finite-index subgroup $E' \leqslant E$ containing K such that E'/K is a HHG.

Proof. The proof of $(\Rightarrow)$ of Theorem 3.10 did not use that G is a HHG, so it shows that if E is HHG, then the extension is bounded. We now focus on the rest of the statement, whose proof we split into several steps.

Step 1: the kernel fixes every unbounded domain. Consider a HHG structure $(\mathcal{X}, \mathfrak{S})$ for E. In this step we show that the K-action on the domain set $\mathfrak{S}$ fixes every unbounded domain (though K might act non-trivially on the corresponding coordinate spaces).

Fix $x_0 \in \mathcal{X}$, let $H_W = \mathrm{Stab}_E\,(W)$, and let $\mathcal{E}(H_W)$ be the collection of *eyries* of H_W, that is, the domains $V \in \mathfrak{S}$ such that $\mathrm{diam}\pi_V(H_W \cdot x_0) = \infty$ and which are

$\sqsubseteq$-maximal with this property (the definition is clearly independent of x_0). Notice that $W \in \mathcal{E}(H_W)$: indeed, by cobounded product regions H_W acts coboundedly on $\mathcal{C}W$, so $\operatorname{diam}\pi_W(H_W \cdot x_0) = \operatorname{diam}\mathcal{C}W = \infty$; furthermore, if $W \subsetneq V$ and we choose $x_0 \in P_W$, then $\pi_V(H_W \cdot x_0)$ coarsely coincides with ρ^W_V and is therefore bounded.

Now K acts on $\mathcal{E}(H_W)$, since for every $z \in K$ and $V \in \mathcal{E}(H_W)$ we have

$$\operatorname{diam}\pi_{zV}(H_W \cdot x_0) = \operatorname{diam}\pi_V(z^{-1}H_W \cdot x_0) = \operatorname{diam}\pi_V(H_W \cdot z^{-1}x_0) = \infty.$$

We are left to prove that $z^{-1}W = W$. By cobounded product regions, combined with the partial realisation Axiom (10), there is a constant $R > 0$ such that, for every tuple $(p_V) \in \prod_{V\in\mathcal{E}(H_W)} \mathcal{C}V$, there exists $g \in H_W$ such that $\mathrm{d}_V(gx_0, p_V) \leqslant R$ for every $V \in \mathcal{E}(H_W)$. Up to replacing R by a larger constant, we can also assume that $\mathrm{d}_W(x_0, zx_0) \leqslant R$. Hence, using that $\mathcal{C}W$ is unbounded, we can find $g \in H_W$ such that $\mathrm{d}_W(x_0, gx_0) > 4R$, while $\mathrm{d}_V(x_0, gx_0) \leqslant R$ for all $V \in \mathcal{E}(H_W) - \{W\}$. Now notice that

$$\begin{aligned}\mathrm{d}_{z^{-1}W}(x_0, gx_0) &= \mathrm{d}_W(zx_0, zgx_0) = \mathrm{d}_W(zx_0, gzx_0)\\ &\geqslant \mathrm{d}_W(x_0, gx_0) - \mathrm{d}_W(x_0, zx_0) - \mathrm{d}_W(gx_0, gzx_0)\\ &= \mathrm{d}_W(x_0, gx_0) - \mathrm{d}_W(x_0, zx_0) - \mathrm{d}_{g^{-1}W}(x_0, zx_0) \geqslant 2R.\end{aligned}$$

We used that z commutes with g in the first line, and that g fixes W in the last one. Since $z^{-1}W \in \mathcal{E}(H_W)$, we must have that $z^{-1}W = W$, as promised.

Step 2: choice of the finite-index subgroup. Now suppose $K \cong \mathbb{Z}^k \oplus T$ for some $k \geqslant 0$ and some finite abelian group T. Fix a generating set $z_1, \ldots, z_n$ of K such that the first k elements are a basis for $\mathbb{Z}^k$ and the remaining ones generate T. Let

$$\mathcal{U} = \bigcup_{i\leqslant k} \operatorname{Big}(z_i).$$

As in the proof of Theorem 3.10, E acts on each $\operatorname{Big}(z_i)$, so let E' be the finite-index subgroup of E that fixes each $U \in \mathcal{U}$. Notice that $K \leqslant E'$ by the previous paragraph; moreover, again by the proof of Theorem 3.10, for every $U \in \mathcal{U}$, the subgroup E' acts without inversions on the quasiline $\mathcal{C}U$.

Since $E' \leqslant E$ has finite index, $(\mathcal{X}, \mathfrak{S})$ is a HHG structure for E', and up to equivariant isometry we once and for all identify $\mathcal{X}$ with a Cayley graph for E' with respect to a finite generating set. By Lemma 3.8, E' has an almost HHG structure $(\mathcal{X}, \widetilde{\mathfrak{S}})$, where

$$\widetilde{\mathfrak{S}} = \{S\} \cup \{U \in \mathfrak{S} \mid \mathcal{C}U \text{ unbounded}\}.$$

We also notice that, by the arguments from [ANS⁺24, Section 3], every $W \in \widetilde{\mathfrak{S}} - \{S\}$ is orthogonal to every $U \in \mathcal{U}$ (in particular, any two $U, U' \in \mathcal{U}$ are orthogonal).

Step 3: reduction to a single big domain. We shall now quotient by one generator z_i of the kernel at a time. For short, let $z_1 = z$ and $Z = \langle z\rangle$. We first replace some coordinate spaces to get a new almost HHG structure where z has at most one big domain (this step is unnecessary if z has finite order). Let $\operatorname{Big}(z) = \{U_1, \ldots, U_j\}$, and for every $i = 1, \ldots, j$ let $\phi_i \coloneqq E' \to \mathbb{R}$ be the Busemann quasimorphism for the action on $\mathcal{C}U_i$. We first claim that these quasimorphisms are linearly independent. Indeed, if, say, ϕ_1 was a linear combination of the others, then there would be some function $f\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $\mathrm{d}_{U_1}(g, 1) \leqslant f(\mathrm{d}_{U_2}(g, 1), \ldots, \mathrm{d}_{U_j}(g, 1))$ for every $g \in E'$ (here we used that, as argued in Example 2.7, the absolute value of each quasimorphism coarsely coincides with the distance in the corresponding quasiline). However the elements of Big (z) are pairwise orthogonal, so the partial realisation axiom (10) implies that E' acts coboundedly on the product $\prod_{i=1}^{j} \mathcal{C}U_i$. In particular, one can find a sequence of $g_n \in E'$ for which $\mathrm{d}_{U_1}(g_n, 1) \to \infty$ while the other projections are uniformly bounded, contradicting the existence of f.

Next, set $\psi_1 = \phi_1$, and for every $2 \leqslant i \leqslant j$ let $\psi_i = \phi_i - \frac{\phi_i(z)}{\phi_1(z)}\phi_1$, which is non-trivial by linear independence. For every $1 \leqslant i \leqslant j$ let L_i be a Cayley graph for E' such that $\psi_i \colon L_i \to \mathbb{R}$ is a quasi-isometry, as constructed in [ABO19, Lemma 4.15]. Notice that $\psi_i(z) = 0$ whenever $i \geqslant 2$, so Z now acts elliptically on all quasilines excluding L_1.

Now replace each $\mathcal{C}U_i$ by the corresponding L_i, with projection given by the identity map $\mathcal{X} \to L_i$ as both are Cayley graphs for E'. We are left to check that this gives an almost HHG structure for E', as then in the new structure z will have a single big domain. All requirements only involving the relations on $\widetilde{\mathfrak{S}}$, or the projections between domains in $\widetilde{\mathfrak{S}} - \mathrm{Big}\,(z)$, still hold by construction, so we focus on the axioms which impose restrictions on the L_i.

- **Projections (1).** The projection $\mathcal{X} \to L_i$ is surjective and Lipschitz, as $\mathcal{X}$ is a Cayley graph for E' with respect to a finite generating set.
- **Consistency (7).** If two domains $U, V \in \widetilde{\mathfrak{S}}$ are transverse then none of them is in Big (z), so there is nothing more to check. Similarly, if $U \subsetneq V$ and both U and V have a well-defined projection to some W, then none of the three domains is in Big (z), and again we have nothing to prove.
- **Bounded geodesic image (8).** If some U_i is nested in some W then $W = S$, and $\mathcal{C}S$ is bounded by [ANS$^+$24, Lemma 3.1], so the axiom holds trivially. All other instances of the axiom already appeared in the original structure.
- **Large links (9).** As we only replaced finitely many coordinate spaces, the axiom still holds (possibly after replacing δ by a bigger constant)
- **Partial realisation (10).** Let $\{V_i\}_{i=1,\dots,m}$ a collection of pairwise orthogonal domains, each with a point $p_i \in \mathcal{C}V_i$. If $\{V_i\}$ only contains S then we have nothing to prove, because $\mathcal{C}S$ is bounded as pointed out before. Otherwise, up to enlarging the collection, we can assume that $\{V_i\}_{i=1,\dots,m}$ contains Big (z), in such a way that p_i belongs to L_i for all $i \leqslant j$. Let $g_1 = p_1$ (seen as an element of E'), and for all $2 \leqslant i \leqslant j$ choose $g_i \in E'$ such that $\phi_i(g_i) = \psi_i(p_i) + \psi_1(p_1)$. Let $p_i' = \pi_{\mathcal{C}U_i}(g_i)$. By the partial realisation axiom for the original structure, the tuple $(p_1' \dots, p_j', p_{j+1}, \dots, p_m)$ is realised by some $x \in \mathcal{X}$, and we are left to prove that x is uniformly close to p_i in each L_i. Since x is a realisation point, for every $i = 1, \dots, j$ we have that $\mathrm{d}_{\mathcal{C}U_i}(g_i, x) \leqslant \delta$, so that $\phi_i(x)$ and $\phi_i(g_i)$ are uniformly close. In turn, this means that $\psi_i(x)$ is uniformly close to $\psi_i(p_i)$ for all i, and therefore $\mathrm{d}_{L_i}(p_i, x)$ is uniformly bounded, as required.
- **Uniqueness (11)** Suppose $x, y \in \mathcal{X}$ have uniformly close projections to all domains in the new structure. If we show that their projections to each $\mathcal{C}U_i$ are also uniformly bounded then x and y are uniformly close, by the uniqueness axiom in the original structure. Up to the group action we can assume that $y = 1$, so that $\mathrm{d}_{L_i}(1, x)$ coarsely coincides with the absolute value of $\psi_i(x)$. Since all distances in the L_i are uniformly bounded, so are

the absolute values of ψ_i, hence of ϕ_i, and again this means that $\mathrm{d}_{\mathcal{C}U_i}(1, x)$ is uniformly bounded for all i.

- **Almost HHG structure.** All projections (old and new) are E'-equivariant, so we built an almost hierarchically hyperbolic *group* structure.

Step 4: HHG structure of the quotient. We shall now prove that $G' = E'/Z$ has the following almost HHG structure. Let $\mathcal{Y} = \mathcal{X}/Z$. Let $\overline{\mathfrak{S}} = \widetilde{\mathfrak{S}}$ if z has finite order, and $\overline{\mathfrak{S}} = \widetilde{\mathfrak{S}} - \{U_z\}$ otherwise, where U_z is the unique big domain for z in the new structure. The relations in $\overline{\mathfrak{S}}$ are the same as in $\widetilde{\mathfrak{S}}$.

Moving to the coordinate spaces, given a domain $W \in \widetilde{\mathfrak{S}}$, we denote by $[W]$ the corresponding domain in $\overline{\mathfrak{S}}$. Set $\mathcal{C}[W] = \mathcal{C}W/Z$ (this is well-defined, as $\mathcal{C}W$ is unbounded and therefore Z fixes W), and let $q_W : \mathcal{C}W \to \mathcal{C}[W]$ be the quotient projection, which is 1-Lipschitz. Given $[x] \in \mathcal{Y}$ (by which we mean the Z-orbit of a point $x \in \mathcal{X}$) and $[W] \in \overline{\mathfrak{S}}$, let $\pi_{[W]}([x]) = q_W(\pi_W(x))$ for any $x \in [x]$. Similarly, for every $[V], [W] \in \overline{\mathfrak{S}}$, set $\rho^{[V]}_{[W]} = q_W(\rho^V_W)$ whenever it is defined.

We now check that $(\mathcal{Y}, \widetilde{\mathfrak{S}}/Z)$ satisfies the axioms of an almost HHG structure. Most of the checks revolve around the fact that, for every $W \in \widetilde{\mathfrak{S}}$, the projection map $q_W : \mathcal{C}W \to \mathcal{C}[W]$ is a uniform quality quasi-isometry, because Z acts on $\mathcal{C}W$ with uniformly bounded orbits. To see this, notice that $\mathrm{Stab}_G(W)$ acts on $\mathcal{C}W$ coboundedly, and that for every $g \in \mathrm{Stab}_G(W)$

$$\mathrm{diam}\pi_W(Z \cdot gx_0) = \mathrm{diam}\pi_W(gZ \cdot x_0) = \mathrm{diam}\pi_W(Z \cdot x_0).$$

Let us now briefly go through all the axioms.

- **Projections (1).** For every $[W] \in \widetilde{\mathfrak{S}}/Z$, $\pi_{[W]}$ is a δ-coarsely onto, δ-coarse map, as the quotient projection $q_W : \mathcal{C}W \to \mathcal{C}[W]$ is 1-Lipschitz and surjective. Moreover, given $[x], [y] \in \mathcal{Y}$ let $x \in [x]$ and $y \in [y]$ realise the distance between $[x]$ and $[y]$, so that

$$\mathrm{d}_{[W]}([x], [y]) \leqslant \mathrm{d}_W(x, y) \leqslant \delta \mathrm{d}_{\mathcal{X}}(x, y) + \delta = \delta \mathrm{d}_{\mathcal{Y}}([x], [y]) + \delta.$$

- **Nesting (2).** By construction, the unique maximal element of $\widetilde{\mathfrak{S}}/Z$ is $[S]$; moreover, whenever $[V] \subsetneq [W]$ and $\mathcal{C}[W]$ is unbounded, we have that $\mathrm{diam}\rho^{[V]}_{[W]} \leqslant \mathrm{diam}\rho^V_W \leqslant \delta$.
- **Finite complexity, orthogonality, and finite dimension (3)-(5').** All requirements follow from the fact that $\overline{\mathfrak{S}} \subseteq \widetilde{\mathfrak{S}}$ with the same relations.
- **Transversality (6).** Uniform boundedness of projection points follows as for the nesting axioms.
- **Consistency, bounded geodesic image, and large links (7)-(9)** Since q_W is a uniform quasi-isometry for every $W \in \overline{\mathfrak{S}}$, all three axioms hold as they were true in $(\mathcal{X}, \widetilde{\mathfrak{S}})$, plus the fact that we defined (relative) projections via the original (relative) projections.
- **Partial realisation (10).** Let $[V_1], \ldots, [V_k] \in \overline{\mathfrak{S}}$ be pairwise orthogonal, and for every i let $p_i \in \mathcal{C}[V_i]$. For every i let $r_i \in q^{-1}_{V_i}(p_i)$, and let $x \in \mathcal{X}$ realise the collection $\{r_i\}_{i=1,\ldots,k}$. It is easy to see that $[x]$ realises $\{p_i\}_{i=1,\ldots,k}$ in $\mathcal{Y}$.
- **Uniqueness (11).** Let $[x], [y] \in \mathcal{Y}$, and let $r > 0$ be a constant such that $\mathrm{d}_{[W]}([x], [y]) \leqslant r$ for all $[W] \in \overline{\mathfrak{S}}$. Then any $x \in [x]$ and $y \in [y]$ have uniformly close projections to all $W \in \widetilde{\mathfrak{S}} - \{U_z\}$ (again, because every q_W is

a uniform quasi-isometry). Furthermore, since Z acts coboundedly on $\mathcal{C}U_z$, we can replace y by a Z-translate and uniformly bound $\mathrm{d}_{U_z}(x, y)$ as well. Then the uniqueness axiom for $(\mathcal{X}, \widetilde{\mathfrak{S}})$ yields that x and y are uniformly close in $\mathcal{X}$, and therefore $\mathrm{d}_{\mathcal{Y}}([x],[y]) \leqslant \mathrm{d}_{\mathcal{X}}(x,y)$ is uniformly bounded.
- **Almost HHG structure.** Since E' acted metrically properly and coboundedly on $\mathcal{X}$, then so does G' on $\mathcal{Y}$. Moreover, the cofinite E'-action on $\widetilde{\mathfrak{S}}$ already factored through G', so $(\mathcal{Y}, \overline{\mathfrak{S}})$ is an almost HHG structure for G'.

Step 5: conclusion. To conclude the proof of Theorem 3.13, we now iterate steps 3 and 4 above with z_2, then z_3 and so on, until we quotiented by the whole kernel K. □

Remark 3.14. In the statement of Theorem 3.13, one cannot hope that G is genuinely a HHG. Indeed, the $(3,3,3)$ triangle group G is not a HHG [PS23, Corollary 4.5]; however the direct product $G \times \mathbb{Z}$ is a HHG with cobounded product regions, as it acts geometrically on the standard cubulation of $\mathbb{R}^3$. We are thankful to Mark Hagen for explaining to us a proof of the latter fact, which we now present. Let $o = (0,0,0)$ and $p = (1,1,1)$. Let H_1 be the plane $\{x = y\}$, and similarly define $H_2 = \{y = z\}$ and $H_3 = \{x = z\}$. Let T_1, T_2, T_3 be the reflections across H_1, H_2, H_3, which preserve the standard cubulation of $\mathbb{R}^3$. Now let $F = \{x + y + z = 0\}$, which is the plane orthogonal to op passing through o, and let $L(x, y, z) = (x + 1, y + 1, z - 2)$, which is an integer translation along the line in which H_1 intersects F. Notice that T_1, T_2, T_3 preserve F as they all fix op, and moreover so does L. Then let $G = \langle \alpha, \beta, \gamma \rangle$, where

- $\alpha = T_1 \colon (x, y, z) \mapsto (y, x, z)$,
- $\beta = T_2 \colon (x, y, z) \mapsto (x, z, y)$,
- $\gamma = LT_3L^{-1} \colon (x, y, z) \mapsto (z - 3, y, x + 3)$.

G is a quotient of the $(3,3,3)$ triangle group H, since the above elements are all reflections and the product of any two of them has order 3 (geometrically, this is because the dihedral angle between any two of the planes is $\pi/3$). Also $\alpha \circ \beta \circ \gamma$ is given by $(x, y, z) \mapsto (x + 3, z - 3, y)$, which has infinite order (as one sees by looking at the first coordinate). However, H is *just-infinite*, meaning that its only infinite quotient is H itself, so $G \cong H$. Indeed, this follows from [McC68, Proposition 9], together with the fact that the point group is S_3, realised as an irreducible subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

Finally, let $S(x, y, z) = (x + 1, y + 1, z + 1)$, which commutes with L and with every T_i as $op \subset H_1 \cap H_2 \cap H_3$. Moreover $\langle S \rangle \cap G$ is trivial, since G fixes F. Hence $E = \langle S, G \rangle \cong \langle S \rangle \times G$ is a direct product. Notice that E preserves the standard cubulation of $\mathbb{R}^3$, as its generators do, and the action is proper and cocompact (as one can see by considering the plane F and its translates by powers of S); therefore, by e.g. [BHS17b, Remark 13.2], E has a HHG structure coming from the action on $(\mathbb{R}^3, \mathfrak{S})$, where $\mathfrak{S}$ consists of:

- the top element S;
- three elements for the coordinate planes, which are pairwise transverse;
- three elements for the coordinate lines, each of which is nested in the two planes it belongs and orthogonal to the third one.

It is easily seen that product regions in this structure coincide with the whole $\mathbb{R}^3$. Furthermore, since $\mathfrak{S}$ is finite, the finite-index subgroup of E that fixes $\mathfrak{S}$ pointwise acts coboundedly on $\mathbb{R}^3$, and therefore on every product region. This proves that E is a HHG with cobounded product regions, as required.

4. Quotients of mapping class groups

4.1. **The general problem.** We now discuss certain quotient extensions that it would be interesting to understand. The context is quotients of mapping class groups, and in particular the conjecture of the second and third author [MS25, Question 3], which we now state in a simplified form.

Let $\mathcal{MCG}(S)$ be the mapping class group of a finite-type surface S, and let $\mathcal{B} = \{\phi_1, \ldots, \phi_k\}$ be a collection of elements. The conjecture predicts that there exists $M \in \mathbb{N}_{>0}$ such that $\mathcal{MCG}(S)/\mathcal{N}$ is hierarchically hyperbolic, where

$$\mathcal{N} = \langle\!\langle \phi_1^M, \ldots, \phi_k^M \rangle\!\rangle.$$

There are various cases known in the literature. For instance, [BHMS24, Theorem 7.1] provides an affirmative answer if $\mathcal{B}$ consists of conjugacy representatives of all Dehn twists; [MS24] almost completely settles the conjecture for the five-punctured sphere; and [ABM$^+$25] will prove hierarchical hyperbolicity of quotients by random walks, which should be thought as the “typical” quotients (see also [BHS17a] for similar classes of quotients). The simplest unknown case is where $\mathcal{B}$ consists of a single Dehn twist, and even in that case one already encounters the problem of showing that a certain quotient central extension is bounded.

More precisely, if $p\colon \mathcal{MCG}(S) \to E$ is the quotient under consideration, let $H \leqslant E$ be the image under p of the stabiliser of a curve γ, with the property that no power of the Dehn twist around γ is conjugate into an element of $\mathcal{B}$. In any reasonable HHG structure, H should be itself hierarchically hyperbolic, as it would correspond to a standard product region. One of the simplest scenarios is where γ is non-separating, so that its stabiliser is the mapping class group of $S - \gamma$. Hence, a simplified, yet significant version of the problem is the following.

Question 4.1. Let U be a finite-type surface with one boundary component γ, let $\widehat{U}$ be the surface obtained by gluing a once-punctured disk to γ with puncture p, and let $\mathcal{MCG}(\widehat{U}, p)$ be the subgroup of the mapping class group of $\widehat{U}$ fixing p. There is a short exact sequence

$$1 \to \langle \tau_\gamma \rangle \to \mathcal{MCG}(U) \to \mathcal{MCG}(\widehat{U}, p) \to 1.$$

Let $\mathcal{N}_\gamma \leqslant \mathcal{MCG}(U)$ be the normal subgroup generated by all M-th powers of Dehn twists along curves which are supported on the interior of U. Is $\mathcal{MCG}(U)/\mathcal{N}_\gamma$ hierarchically hyperbolic, for a suitable choice of M?

It can be deduced from [Dah18] that $\mathcal{N}_\gamma$ intersects the kernel trivially, so the quotient is itself a central extension

$$1 \to \langle \tau_\gamma \rangle \to \mathcal{MCG}(U)/\mathcal{N}_\gamma \to \mathcal{MCG}(\widehat{U}, p)/\pi(\mathcal{N}_\gamma) \to 1.$$

Furthermore, the aforementioned [BHMS24, Theorem 7.1] yields that the base $\mathcal{MCG}(\widehat{U}, p)/\pi(\mathcal{N}_\gamma)$ is itself hierarchically hyperbolic, for a suitable choice of M; hence Question 4.1 is equivalent to the boundedness of the second central extension, by Theorem 3.10.

4.2. **Quotients by pseudo-Anosovs, low complexity.** Another interesting (and possibly easier) version of Question 4.1 is to ask about quotients by powers of pseudo-Anosovs rather than Dehn twists.

Question 4.2. Let U be a finite-type surface with one boundary component γ, and let $\widehat{U}$ be the surface obtained by gluing a once-punctured disk to γ with puncture p. Let h be a pseudo-Anosov mapping class, and let $\mathcal{N}_h$ be the normal closure of a sufficiently high power of h. Is the central extension

$$1 \to \langle \tau_\gamma \rangle \to \mathcal{MCG}(U)/\mathcal{N}_h \to \mathcal{MCG}(\widehat{U}, p)/\pi(\mathcal{N}_h) \to 1$$

hierarchically hyperbolic? Equivalently, since the base is hierarchically hyperbolic by [BHS17a, Theorem 6.2], is the extension bounded?

In this section we answer the question in the case where U is a 4-punctured disk, whose mapping class group is the braid group on 4 strands, and therefore $\widehat{U}$ is a 5-punctured sphere S_5. As we will see, it is crucial for the proof that modding out powers of Dehn twists in the mapping class group of the five-punctured sphere yields a hyperbolic group. Later, we also discuss a different strategy to approach the general case.

Theorem 4.3. *Let h be a pseudo-Anosov element of B_4, and let $\hat{h}$ be its image in $\mathcal{MCG}(S_5, p)$. Then there exists $M_0 \in \mathbb{N}_{>0}$ such that for all multiples M of M_0 the following holds. Let $\mathcal{N}_h \trianglelefteq B_4$ (resp. $\mathcal{N} \trianglelefteq \mathcal{MCG}(S_5, p)$) be the normal subgroup generated by h^M (resp. $\hat{h}^M$), with quotient map q (resp. $\widehat{q}$). Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle \tau_\gamma \rangle & \longrightarrow & B_4 & \longrightarrow & \mathcal{MCG}(S_5, p) & \longrightarrow & 1. \\ & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle\widehat{q}} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & B_4/\mathcal{N}_h & \longrightarrow & \mathcal{MCG}(S_5, p)/\mathcal{N} & \longrightarrow & 1. \end{array}$$

where the bottom row is a bounded central extension of a HHG. In particular, $B_4/\mathcal{N}_h$ is hierarchically hyperbolic.

One can understand subgroups generated by powers of pseudo-Anosovs using the technology of rotating families; this is done in [DGO17]. To prove the theorem, we will need an improvement of this technology from [Dah18]. We do not recall all the relevant definitions here, but we discuss the main points, giving the motivating example from [Dah18] along the way. A *composite projection system* is a set $\mathbb{Y}_*$ (for instance, the set of curves on a surface) together with additional data. One piece of data is, for each $X \in \mathbb{Y}_*$, a subset $Act(X)$ of $\mathbb{Y}_*$ (for instance, $Act(X)$ is the set of curves that intersect X). A *composite rotating family* on a composite projection system is given by groups G_X of automorphisms of $\mathbb{Y}_*$ fixing X, for all $X \in \mathbb{Y}_*$, satisfying certain conditions (for instance, G_X could be the subgroup generated by the Dehn twist around X).

We need the following technical proposition to make sure that the central extensions we will consider are still $\mathbb{Z}$-central extensions.

Proposition 4.4. *Let G be a group acting by automorphisms on a composite projection system $\mathbb{Y}_*$, and let Z be the kernel of the action. Suppose that we have subgroups $\{G_X\}_{X \in \mathbb{Y}_*}$ of G such that the following hold.*

- $Z \cap G_X = \{1\}$.

- *$G_{gX} = gG_Xg^{-1}$.*
- *G_X commutes with G_Y if $X \notin Act(Y)$.*
- *denoting $\hat{G}_X$ the image of G_X in G/Z, we have that $\{\hat{G}_X\}$ forms a composite rotating family on $\mathbb{Y}_*$.*

Then the quotient map $p\colon G \to G/Z$ restricts to an isomorphism

$$N = \langle\langle \{G_X\} \rangle\rangle_G \to \langle\langle \{\hat{G}_X\} \rangle\rangle_{G/Z} = \hat{N}.$$

Proof. Clearly we have $p(N) = \hat{N}$, so we can restrict p to N to obtain a surjective homomorphism $\bar{p}\colon N \to \hat{N}$. In order to show that this is an isomorphism, we construct an inverse $\iota : \hat{N} \to N$, using the first presentation of $\hat{N}$ provided by [Dah18, Theorem 2.2], which applies by the fourth bullet. The $\hat{G}_X$ are generators for this presentation, and thus have natural maps to the G_X by the first bullet. The relations for this presentation are conjugation relations, which are respected in view of the second bullet, and commutation relations, which are respected by the third bullet. Therefore, there is a well-defined homomorphism $N \to \hat{N}$, which when restricted to any $\hat{G}_X$ gives an inverse of $\bar{p}|_{G_X}$, and therefore ι is in fact the inverse of $\bar{p}$, as required. □

Proof of Theorem 4.3. Throughout this this proof we always assume that M is a multiple of some suitable $M_0 \in \mathbb{N}_{>0}$.

First of all, we need $\mathcal{N}_h$ to intersect $\langle \tau_\gamma \rangle$ trivially, justifying that there is indeed a diagram as in the statement, with the vertical arrow on the left being an isomorphism. This follows from Proposition 4.4, with $G = B_4$, $Z = \langle \tau_\gamma \rangle$, and G_X the conjugates of $\langle h^M \rangle$.

In order to show boundedness of the relevant central extension, we now argue that there is a further quotient central extension which is a central extension of a hyperbolic group. More precisely, we argue that there is a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & B_4/\mathcal{N}_h & \longrightarrow & \mathcal{MCG}(S_5, p)/\mathcal{N} & \longrightarrow & 1. \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & H & \longrightarrow & \hat{H} & \longrightarrow & 1, \end{array}$$

where $\hat{H}$ is hyperbolic. Since the comparison map for hyperbolic groups is surjective [NR97] (see also [Min01]), by Lemma 2.12 we then get that the central extension from the statement is bounded, as required.

The required quotient H can be taken to be the quotient of $G = B_4/\mathcal{N}_h$ by the normal subgroup $\mathcal{K}$ generated by suitable powers of all images of Dehn twists of B_4. We now explain why we can choose such a $\mathcal{K}$ that intersects trivially the image Z of $\langle \tau_\gamma \rangle$ in G, and such that $G/\mathcal{K}Z$ is hyperbolic. This uses results from [Man24, MS24] about *short HHG*. We do not need the whole definition here; what is important is that these are HHG with a specified collection of cyclic subgroups, and if all of those are finite then the group is hyperbolic, see [Man24, Lemma 2.14]. All relevant short HHG below also satisfy the assumption called colourability from [DMS23, Definition 2.8], which from now on we omit. Modding out suitable powers of generators of any subcollection of the specified cyclic subgroups yields another short HHG by [MS24, Theorem 4.1], with associated collection of specified subgroups simply the images of the specified subgroups of the original short HHG.

First of all, $\hat{G} = \mathcal{MCG}(S_5, p)/\mathcal{N}$ is a short HHG. This can be seen combining the following:

- $\mathcal{MCG}(S_5, p)$ is a short HHG, see [Man24, Subsection 2.3.1], with specified subgroups generated by Dehn twists.
- We can add any cyclic subgroup generated by a pseudo-Anosov to the collection of specified subgroups of the short HHG structure of $\mathcal{MCG}(S_5, p)$ by [Man24, Proposition 4.4].
- The aforementioned [MS24, Theorem 4.1] allows us to conclude that modding out suitable powers of h yields a short HHG (with the expected structure with specified subgroups coming from images of Dehn twists).

Now, for $\mathcal{K}$ as above we can consider its image $\hat{\mathcal{K}}$ in $\hat{G}$, and this is the kernel associated to a composite rotating family, by [MS24, Remark 3.8]. This allows us to apply Proposition 4.4 and deduce that $\mathcal{K}$ intersects Z trivially, as required. Moreover, $G/\mathcal{K}Z = \hat{G}/\hat{\mathcal{K}}$ is a short HHG by [MS24, Theorem 4.1], and all its specified subgroups are finite, which as mentioned above implies that the group is hyperbolic. This concludes the proof. □

Remark 4.5. Under the assumption that "enough" hyperbolic groups are residually finite, [BHMS24] constructs hyperbolic quotients of mapping class groups. One could in principle try to exploit these quotients, or variations, and a similar strategy as in the proof above to solve other cases of Question 4.2, or to solve it completely, but conditionally on the residual finiteness of the relevant hyperbolic groups. The main difficulty is ensuring that kernels to these hyperbolic quotients lift isomorphically to the corresponding central extension, as in Proposition 4.4.

4.3. **Quotients by pseudo-Anosov, high complexity?** In the setting of Question 4.2, results from [DGO17] allow us to write $\pi(\mathcal{N}_h)$ as the directed union of groups N_i, each of which is a free product of finitely many conjugates of the cyclic group generated by the relevant power of $\hat{h}$. Each of the N_i is *convex-cocompact*, and we expect that a variant of Theorem 2.19 allows one to extend any quasimorphism on N_i to $\mathcal{MCG}(U)/\pi(\mathcal{N}_h)$. If this extension could be realised in a way that the defect is uniformly controlled, then we would have a positive answer to the previous question, thanks to the following lemma:

Lemma 4.6. *Let G be a finitely generated group, $N \leqslant G$ a normal subgroup, and $\chi \colon N \to \mathbb{Z}$ a quasimorphism. Suppose that N can be expressed as a directed union of subgroups $\{N_i\}_{i \in I}$, and that for each $i \in I$ there is a quasimorphism $X_i \colon G \to \mathbb{Z}$ that extends $\chi|_{N_i}$, such that the defect of the X_i is uniformly bounded. Then χ extends to a quasimorphism $X \colon G \to \mathbb{Z}$.*

Proof. By using homogeneous representatives for one direction, and integer parts for the other, as we did before, we can reduce to the following statement. Let $\chi \colon N \to \mathbb{R}$ be a homogeneous quasimorphism. Suppose that N can be expressed as a directed union of subgroup $\{N_i\}_{i \in I}$, and that for each $i \in I$ there is a homogeneous quasimorphism $X_i \colon G \to \mathbb{R}$ that extends $\chi|_{N_i}$, such that the defect of the X_i is uniformly bounded. Then χ extends to a homogeneous quasimorphism $X \colon G \to \mathbb{R}$. Moreover, because G is finitely generated, thus countable, we can assume that the directed set I is just $\mathbb{N}$ with its well order.

Let us make a further reduction: it suffices to show that we can modify each X_i to obtain $X_i' \colon G \to \mathbb{R}$, which still extends $\chi|_{N_i}$ and has uniformly bounded defect, and moreover, for every $g \in G$, $\{|X_i'(g)|\}_{i \in \mathbb{N}}$ is uniformly bounded. Indeed, assuming this, fix a non-principal ultrafilter ω on $\mathbb{N}$. Then we can set $X(g) := \lim_\omega X_i(g)$, which is a well-defined real number. Moreover, it is a homogeneous quasimorphism, since ultralimits are norm non-increasing and linear.

Now consider the map $\alpha \colon G \to H_1(G;\mathbb{Q})$: this factors every homomorphism from G to $\mathbb{Q}$, and its kernel coincides with the set of elements with a power that belongs to $[G,G]$. In particular, if $g \in \ker(\alpha)$, then its stable commutator length $\mathrm{scl}(g)$ is finite, and by Bavard duality [Bav91] we have a bound $|X_i(g)| \leqslant 2\mathrm{scl}(g)D(X_i)$. This shows that for $g \in \ker(\alpha)$, we do not need to change X_i; note that this concludes the proof in the case in which G has finite abelianisation (e.g. for mapping class groups).

For the general case, let $x_1, \dots, x_n$ be elements of G that are mapped to a basis of $H_1(G;\mathbb{Q})$, chosen in such a way that $x_1, \dots, x_m \notin N$ while $x_{m+1}, \dots, x_n \in N$. Now, for each $j = 1, \dots, m$, let $\lambda_j \colon G \to \mathbb{Q}$ be the functional dual to x_j. For each $i \in \mathbb{N}$, we set

$$X_i' := X_i - \sum_{j=1}^{m} X_i(x_j) \cdot \lambda_j.$$

As a result, X_i' is a homogeneous quasimorphism with the same defect as X_i, the same restriction on N (in particular it is still an extension of $\chi|_{N_i}$) and moreover it vanishes on x_j for all $j = 1, \dots, m$. Now for an element $g \in G$, we can write $\alpha(g) = \sum a_j \alpha(x_j)$, for some $a_j \in \mathbb{Q}$. Thus, for a high enough power p, we can write

$$\alpha(g^p) = \alpha\left(\prod_{j=1}^{n} x_j^{p_j}\right)$$

for some integers $p_j \in \mathbb{Z}$. This way we have an expression $g^p = xyz$, where x is a product of powers of $x_1, \dots, x_m$; y belongs to N, and z belongs to $\ker(\alpha)$. By the first case treated before, $|X_i'(z)|$ is uniformly bounded. Moreover, $|X_i'(y)|$ is uniformly bounded because $y \in N_i$ for i large enough, at which point $X_i(y) = \chi(y)$. Finally, $|X_i'(x)|$ is uniformly bounded, since x is a product of m terms, each of which vanish under X_i'. All in all, this shows that $|X_i'(g^p)|$ is uniformly bounded, and thus $|X_i(g)| = \frac{1}{p}|X_i(g^p)|$ is uniformly bounded, which concludes the proof. □

As pointed out in Remark 2.20, in our setting there is a single bounded cocycle that extends $\delta\chi_i|_{N_i}$. It could be possible to exploit this to find the extensions X_i with uniformly bounded defect.

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