A COMBINATION THEOREM FOR THE TWIST CONJECTURE FOR ARTIN GROUPS

OLI JONES, GIORGIO MANGIONI, AND GIOVANNI SARTORI

Abstract. We reduce a strong version of the twist conjecture for Artin groups to Artin groups whose defining graphs have no separating vertices. This produces new examples of Artin groups satisfying the conjecture, and sheds more light on the isomorphism problem for Artin groups. Along the way we also prove a combination result for the ribbon property for vertices.

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This project is easy money.

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Introduction

Artin groups are a rich class of groups generalising braid groups and with strong connections to Coxeter groups. They are defined via the following presentation: given a finite simplicial graph Γ with vertices $\mathrm{V}(\Gamma)$, edges $\mathrm{E}(\Gamma)$, and for each edge $\{a,b\} \in \mathrm{E}(\Gamma)$ a label $m_{ab} \in \mathbb{N}_{\geqslant 2}$, the associated *Artin group* A_Γ is

$$\langle \mathrm{V}(\Gamma) \mid \mathrm{prod}(a,b,m_{ab}) = \mathrm{prod}(b,a,m_{ab})\,\forall\{a,b\} \in \mathrm{E}(\Gamma)\rangle,$$

where $\mathrm{prod}(u,v,n)$ denotes the prefix of length n of the infinite alternating word $uvuvuv\ldots$. When a and b do not span an edge, we abuse notation and set $m_{ab} = \infty$.

One of the most prominent open questions for Artin groups is the *isomorphism problem*, which specialises Dehn's isomorphism problem [Deh12], and asks for an algorithm that, given two labelled graphs, determines if they give rise to isomorphic Artin groups. The question is completely open in general, with some results when one or both graphs are restricted to some subclasses [Bau81, Dro87, Par04, Vas22]. In this paper we are concerned with the twist conjecture, which can be seen as a first step towards the isomorphism problem. Before giving more details, we need some terminology.

An *Artin system* is a pair (A,S) where A is a group isomorphic to some A_Γ, and the isomorphism maps $S \subseteq A$ bijectively onto the vertices of Γ (see Definition 2.1). The set S is called an *Artin generating set* for A, and the graph Γ, which we also denote by Γ_S, is its *defining graph*. A *standard parabolic subgroup* of (A,S) is the subgroup A_Y generated by some $Y \subseteq S$; by a result of van der Lek, (A_Y,Y) is itself an Artin system [VdL83].

Two Artin generating sets for the same Artin group are *twist equivalent* if they are related by a sequence of *elementary twists*, whose definition we briefly recall (we postpone all details to Definition 3.2). Given an Artin generating set S, let $Y \subseteq S$ be a subset which separates Γ_S. Suppose that the Artin system (A_Y,Y) is spherical (meaning the corresponding Coxeter group is finite) and does not decompose as a direct product. Performing an elementary twist of S along Y roughly amounts to conjugating one of the connected components of $\Gamma_S - \Gamma_Y$ by the *Garside element* of (A_Y, Y) (this is a distinguished element corresponding to the longest element in the associated Coxeter group, see [BS72]). Two graphs Γ and Γ' are then said be *twist equivalent* if $\Gamma \cong \Gamma_S$ and $\Gamma' \cong \Gamma_U$ for some twist equivalent Artin generating sets S and U of an Artin group A.

In all the solved cases of the isomorphism problem, two graphs give rise to isomorphic Artin groups if and only if they are twist equivalent, so it is natural to ask if this is always the case:

Question A ([Cha16, Problem 28])**.** If two labelled graphs Γ and Δ give rise to isomorphic Artin groups, are they necessarily twist equivalent?

As a first step in this direction, Brady, McCammond, Mühlherr, and Neumann conjectured the following:

Conjecture B (Weak twist conjecture, [BMMN02, Conjecture 8.2])**.** *Let (A, S) be an Artin system. If $U \subseteq A$ is an Artin generating set with $R_S = R_U$, then Γ_S and Γ_U are twist equivalent.*

Here R_S, called the *reflection set* of (A, S), is the union of the conjugacy classes of S in A, and similarly for R_U. The corresponding conjecture for Coxeter groups, which is [BMMN02, Conjecture 8.1], has been proven to be false in [RT08]; however, Artin groups are believed to be more rigid than their Coxeter counterparts.

In this paper we consider the following stronger version of the twist conjecture:

Conjecture C (Strong twist conjecture)**.** *Let (A, S) be an Artin system. If $U \subseteq A$ is an Artin generating set with $R_S = R_U$, then S and U are twist equivalent.*

The analogue of the strong twist conjecture for Coxeter groups (with the appropriate extra conditions to avoid the known counter examples) was already considered by Mühlherr in [Müh06, Conjecture 2].

A combination theorem for the strong twist conjecture. A *big chunk* of a simplicial graph is a connected induced subgraph without separating vertices which is maximal with these properties. Given an Artin system (A, S), a *big chunk parabolic* is a subgroup conjugated to some A_X, where $X \subseteq S$ spans a big chunk in Γ_S. In our previous work, we proved that every isomorphism between two Artin systems induces a bijection between the sets of big chunks, which preserves the isomorphism type [JMS25]. In light of this, one might hope to recover the strong twist conjecture for an Artin system from the twist conjecture for its big chunk parabolics. We prove just that in our first main result, under a further mild hypothesis:

Theorem D (see Theorem 4.11)**.** *Let (A, S) be an Artin system, with Γ_S connected. Suppose that the following holds:*

- *For every $X \subseteq S$ spanning a big chunk, (A_X, X) satisfies the strong twist conjecture;*
- *For every $Y \subseteq S$ spanning a clique, (A_Y, Y) satisfies the vertex ribbon property.*

Then (A, S) satisfies the strong twist conjecture.

First introduced by Paris [Par97], a *ribbon* can be intuitively thought of as a minimal element that conjugates two standard parabolic subgroups of an Artin system (see Definition 4.1, which for our purposes only describes ribbons between standard generators). The *vertex ribbon property* states that any element that conjugates two standard generators decomposes as a product of ribbons, and is conjectured to hold for all Artin systems. In the course of proving Theorem D, we also obtain a combination result for the vertex ribbon property, see Theorem G below.

As an application of Theorem D, we exhibit new families of Artin systems satisfying the strong twist conjecture. Our result covers Artin systems for which even the weak twist conjecture was not previously known (see Figure 1 for an example).

Corollary E (see Corollary 4.16)**.** *Let (A, S) be an Artin system, with Γ_S connected. Suppose that, for every $X \subseteq S$ spanning a big chunk, the corresponding parabolic (A_X, X) is of one of the following types:*

(1) (A_X, X) is right-angled (i.e. all edge labels are 2);
(2) (A_X, X) is of large-type (i.e. all edge labels are at least 3) and triangle-free (i.e. no induced subgraph of Γ_X is a triangle);
(3) (A_X, X) is of large-type and free-of-infinity (i.e. Γ_X is a complete graph);
(4) (A_X, X) is of type XXXL (i.e. all edge labels are at least 6);
(5) (A_X, X) is of type A_n for $n \geqslant 3$; B_n for $n \geqslant 3$; or D_n for $n \geqslant 4$ and $n \neq 5$.

Then (A, S) satisfies the strong twist conjecture.

In Lemmas 4.12-4.14-4.15, by combining existing results in the literature, we show that the classes of Artin systems appearing as big chunks above satisfy the strong twist conjecture. Furthermore, work of Godelle [God03, God07] proves that the Artin systems supported on cliques of Γ_S enjoy the vertex ribbon property.

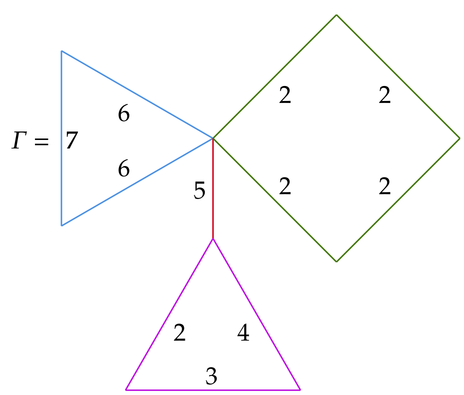

FIGURE 1. In the labelled graph Γ above, all big chunks (here, in different colours) are of the type described in Corollary E, so that A_Γ satisfies the strong twist conjecture.

Theorem D and Corollary E should be compared to [RT13], where the authors describe a JSJ decomposition of Coxeter groups over FA subgroups, and produce a similar combination theorem for the (strong) twist conjecture, allowing them to conclude that all Coxeter systems with chordal defining graphs satisfy the strong twist conjecture. Indeed, underlying our techniques is an explicit JSJ decomposition for any Artin group, which we constructed in our previous work [JMS25].

If we drop the assumption on the ribbon property in Theorem D, we still get a sufficient condition for the weak twist conjecture:

Theorem F (see Theorem 3.7)**.** *Let (A, S) be an Artin system, with Γ_S connected. Suppose that, for every $X \subseteq S$ spanning a big chunk, (A_X, X) satisfies the strong twist conjecture. Then (A, S) satisfies the weak twist conjecture.*

In the above setup, we actually show that every Artin generating set U with $R_S = R_U$ is related to S by a finite sequence of elementary twists and *Dehn twists* (see Definition 3.4). Indeed, Theorem D follows from Theorem F by using the vertex ribbon property to describe Dehn twists as compositions of elementary twists.

A combination theorem for the vertex ribbon property. As a collateral result of the proof of Theorem D, we can reduce the vertex ribbon property to the free-of-infinity Artin groups.

Theorem G (see Corollary 4.9)**.** *Let (A, S) be an Artin system. Suppose that, for any $Y \subseteq S$ spanning a clique in Γ_S, (A_Y, Y) satisfies the vertex ribbon property. Then (A, S) satisfies the vertex ribbon property.*

The above result is a consequence of a more general combination theorem, stating that if an Artin system (A, S) admits a visual splitting and the factors satisfy the vertex ribbon property then so does the whole (A, S) (see Proposition 4.7).

Organisation of the paper. Section 1 contains generalities on simplicial actions on trees and *deformation spaces*, in the sense of Forester [For01], which are the main technical tools of this paper. In Section 2 we recall some properties of Artin groups; then, building on results from our previous paper [JMS25], we associate a deformation space to every Artin system, whose trees roughly correspond to the maximal visual splittings over big chunk parabolics.

In Section 3, we prove a technical intermediate theorem, involving a generalised version of the strong twist conjecture that allows Dehn twists (as well as elementary twists and conjugations), which in particular implies Theorem F. In Section 4, we upgrade this theorem under the further mild assumption that the big chunks satisfy the ribbon property. In particular, we obtain Theorem D, which we then combine with the strong twist conjecture for several classes of Artin groups to get Corollary E (see Corollaries 4.11 and 4.16, respectively). In the process we also prove the combination result for the ribbon property, which then yields Theorem G (see Corollary 4.9).

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1. Background on deformation spaces

We first recall some properties of simplicial actions on trees, referring to [Ser03] for further generalities. We shall work in the following setting:

Notation 1.1. By *tree* we mean a simply connected simplicial graph, equipped with the metric where each edge has length one. Given a group G, a *G-tree* (T, Ω) is a tree T endowed with an action $\Omega \colon G \to \mathrm{Aut}(T)$ by simplicial isometries, without edge inversions, and *minimal* (i.e. no proper sub-tree is invariant under the action). We often suppress the reference to the action Ω when it is not relevant or it is clear from the context.

Throughout we will write $\mathrm{Stab}_\Omega(S)$ for the pointwise stabiliser of $S \subseteq T$, or $\mathrm{Stab}_G(S)$ if there is no danger of confusing the action.

Remark 1.2. Bass [Bas93, Proposition 7.9] proved that, if G is finitely generated, a minimal G-action on a tree T is also cocompact.

The *translation length* of an element $g \in G$ is defined as $|g| \coloneqq \inf_{x \in T} \mathrm{d}_T(x, gx)$. The *minset* of g is the subtree spanned by all points x which realise the translation length. If $|g| = 0$ the element is called *elliptic*, and its minset is the sub-tree of all fixed points of g. If otherwise $|g| > 0$ the element is *loxodromic*, and its minset is a geodesic line on which g acts by translations.

For the remainder of this section we follow [GL07], and recall some properties of *deformation spaces*, a notion originally due to Forester [For01]. A deformation space is a space parameterised by G-trees.

Definition 1.3. Two G-trees (T, Ω) and (T', Ω') are *G-equivariantly isometric* if there is a simplicial isometry $f \colon T \to T'$ such that, for every $g \in G$, we have that $f \circ \Omega(g) = \Omega'(g) \circ f$.

Definition 1.4. We say an edge $e = \{u, v\}$ of a G-tree (T, Ω) is *collapsible* if $\mathrm{Stab}_G(u) \leqslant \mathrm{Stab}_G(v)$, and u, v lie in different G-orbits. The associated *elementary collapse* produces a new G-tree (T', Ω') by removing e and identifying u and v, and propagating this move equivariantly across T. The inverse of an elementary collapse is called an *elementary expansion*. A finite sequence of elementary collapses and expansions is called an *elementary deformation*.

Elementary collapses preserve elliptic subgroups, in the following sense:

Lemma 1.5. *If (T', Ω') is obtained from (T, Ω) by an elementary collapse, every edge stabiliser (resp. vertex stabiliser) for Ω' is also an edge stabiliser (resp. vertex stabiliser) for Ω.*

Proof. Say the collapse $\varphi \colon T \to T'$ identifies the endpoints of $e = \{u, v\}$ where $\mathrm{Stab}_\Omega(u) \leqslant \mathrm{Stab}_\Omega(v)$. If x' is an open edge of T', or a vertex which is not in the G-orbit of $\varphi(e)$, then φ is injective on $\varphi^{-1}(x')$, so by G-equivariance of φ we have that $\mathrm{Stab}_{\Omega'}(x') = \mathrm{Stab}_\Omega(x)$ for any edge (resp. vertex) x in the preimage of x'.

To conclude the proof, we now show that

$$\mathrm{Stab}_{\Omega'}(\varphi(e)) = \mathrm{Stab}_\Omega(v). \tag{1}$$

Let $g \in \mathrm{Stab}_{\Omega'}(\varphi(e))$. Towards a contradiction, assume that $g \cdot v \neq v$. Because g fixes $\varphi(e)$, the vertices v and $g \cdot v$ have the same image under φ. Then there is a path of G-translates $e_1 = g_1 \cdot e, e_2 = g_2 \cdot e_1, \ldots, e_k = g_k \cdot e_{k-1}$ of e connecting v to $g \cdot v$. Notice that $k \geqslant 2$, as v and u are in different G-orbits; moreover $g_1 \cdot v = v$, as the path connects v to $\varphi(v)$, and for the same reason g_2 fixes $g_1 \cdot u$ but not v. However $g_2 \in \mathrm{Stab}_\Omega(g_1 \cdot u) = \mathrm{Stab}_\Omega(u)^{g_1} \leqslant \mathrm{Stab}_\Omega(v)$, which is absurd. □

Definition 1.6. Given a G-tree (T, Ω), the *deformation space* containing (T, Ω) is the simplicial realisation of the following partial order. Take the underlying set to be all G-trees related to (T, Ω) by an elementary deformation, up to equivariant isometry. Then say $(T', \Omega') \geqslant (T'', \Omega'')$ if (T', Ω') admits a sequence of elementary collapses to (T'', Ω'').

Remark 1.7. For experts, what we refer to here as a deformation space may more accurately be called the simplicial spine. This is a deformation retract of the genuine deformation space, which is obtained by considering a space of metric trees (instead of only simplicial trees). We restrict to the spine for the ease of exposition.

The following theorem of Forester is a very convenient characterisation of when two G-trees are in the same deformation space:

Theorem 1.8. [For01, Theorem 1.1] *Let (T, Ω) and (T', Ω') be cocompact G-trees. Then the following are equivalent:*

(1) (T, Ω) and (T', Ω') are related by an elementary deformation (i.e. they belong to the same deformation space).
(2) (T, Ω) and (T', Ω') have the same elliptic subgroups.

Definition 1.9. A G-tree is *reduced* if no elementary collapse is possible.

Remark 1.10. If a G-tree T is not reduced, one obtains a (possibly non-unique) reduced G-tree T' by collapsing collapsible edges until no collapse is possible. This procedure eventually ends, as T/G is finite.

Definition 1.11. Given a G-tree T, we say (the orbit of) an edge e is *surviving* if there exists a sequence of elementary collapses from T to a reduced tree, such that e is not collapsed. We say T is surviving if every orbit of edges is surviving.

A particularly well-behaved subclass of deformation spaces are the *non-ascending* ones. We first need a definition.

Definition 1.12 (see [GL17, Section 1.2])**.** A deformation space $\mathcal{D}$ is *irreducible* if there exists $(T, \Omega) \in \mathcal{D}$ and two loxodromic elements for the action Ω whose commutator is again loxodromic.

Definition 1.13. A deformation space $\mathcal{D}$ is *non-ascending* if it is irreducible and there is no G-tree $(T, \Omega) \in \mathcal{D}$ containing an edge e with the following properties:

(1) Both endpoints of e are in the same G-orbit,
(2) The subgroup $\mathrm{Stab}_\Omega(e)$ is equal to the stabiliser of one endpoint and properly contained in the other.

The following theorem explains our interest in non-ascending deformation spaces:

Theorem 1.14. *Let $\mathcal{D}$ be a non-ascending deformation space, and let $\mathcal{F}$ be the subcomplex of $\mathcal{D}$ spanned by surviving trees. Two trees $T_1, T_2 \in \mathcal{F}$ are related by an elementary deformation where every intermediate tree may be taken to be surviving.*

Proof. Such an elementary deformation is a path from T_1 to T_2 in $\mathcal{F}$, so we have to show that $\mathcal{F}$ is connected. This follows from the fact that $\mathcal{D}$ is connected by construction, and deformation retracts onto $\mathcal{F}$ by [GL07, Theorem 7.6]. □

We conclude the section with a sufficient condition for a deformation space to be non-ascending:

Lemma 1.15. *Suppose $\mathcal{D}$ is an irreducible deformation space containing a G-tree T such that T/G is a tree. Then $\mathcal{D}$ is non-ascending.*

Proof. Towards a contradiction, suppose that T' is a G-tree with an edge e as forbidden by Definition 1.13. Then the image of e in T'/G is a loop, and in particular the latter is not a tree. Furthermore, the *Betti number* of a G-tree (i.e. the rank of the fundamental group of the quotient) is a constant across all trees in a deformation space [GL07, Section 4]; hence no $T \in \mathcal{D}$ is such that T/G is a tree, against the hypothesis. □

We can rephrase the requirement on T/G in the above Lemma with the existence of a suitable fundamental domain for the G-action.

Definition 1.16 (combinatorial fundamental domain)**.** Let (T, Ω) be a G-tree; a *combinatorial fundamental domain* is a subtree $K \subseteq T$ such that:

(1) for every vertex $v \in \mathrm{V}(T)$, $|G \cdot v \cap \mathrm{V}(K)| = 1$;
(2) for every edge $e \in \mathrm{E}(T)$, $|G \cdot e \cap K| = 1$.

Lemma 1.17. *A G-tree (T, Ω) admits a combinatorial fundamental domain if and only if T/G is a tree.*

Proof. If T/G is a tree then any lift of the quotient to T is a combinatorial fundamental domain. Conversely, if T admits a combinatorial fundamental domain K then $T/G \cong K$ is a tree, since the orbits of T are in bijection with K. □

2. Background on Artin groups

In this section, we recall some well known facts on Artin groups, and results from [JMS25] which we will need. We also introduce a deformation space for Artin groups, depending only on the union of the conjugacy classes of the generating set.

Definition 2.1. Given a finite, labelled simplicial graph Γ, let A_Γ be the associated Artin group, as in the Introduction. Given a group A and a finite subset S, we say that (A, S) is an *Artin system* if $A \cong A_{\Gamma_S}$ for some graph Γ_S, called the *defining graph* of (A, S), and the isomorphism sends S bijectively to the vertices of Γ_S. We say that S is an *Artin generating set* of A. Two Artin systems (A, S) and (A', S') are isomorphic if there is an isomorphism $A \cong A'$ mapping S to S'.

Remark 2.2. The defining graph of an Artin system is well defined up to labelled graph isomorphism. As such, we will freely talk about subgraphs spanned by generators in S, implicitly identifying them with vertices in Γ_S.

Definition 2.3. If an Artin system (A, S) is such that Γ_S is an edge with label $m \geqslant 3$, then A is called a *dihedral Artin group.* If $S = \{a, b\}$, the centre of A is infinite cyclic, generated by

$$z_{ab} = \begin{cases} \Delta_{ab} & \text{if } m \text{ is even;} \\ \Delta_{ab}^2 & \text{if } m \text{ is odd,} \end{cases}$$

where $\Delta_{ab} = \mathrm{prod}(a, b, m_{ab})$ is the *Garside element* (see e.g. [BS72]).

Definition 2.4. Given an Artin system (A, S), a *standard parabolic subgroup* (with respect to S) is a subgroup generated by some $U \subseteq S$, which we denote by A_U. By a theorem of Van der Lek, $A_U \cong A_{\Gamma_U}$, where Γ_U is the induced subgraph of Γ_S spanned by the vertices in U [VdL83, Theorem 4.13]. We say a subgroup of an

A is a *parabolic subgroup* (still with respect to S) if it is conjugate to a standard parabolic subgroup.

Remark 2.5 (One-endedness)**.** An Artin group A is one-ended if and only if some (equivalently, any) Γ such that $A \cong A_\Gamma$ is connected and has at least two vertices (see e.g. [JMS25, Remark 2.6] for further details).

A vertex v of a simplicial graph Γ is *separating* if the subgraph spanned by $\mathrm{V}(\Gamma) - \{v\}$ is disconnected. The presence of a separating vertex (almost) fully characterises when an Artin group *splits over* $\mathbb{Z}$ (that is, it admits a graph of groups decomposition with infinite cyclic edge stabilisers):

Theorem 2.6 ([JMS25, Theorem A])**.** *Let A be a one-ended Artin group. Then A splits over $\mathbb{Z}$ if and only if, for some (equivalently every) Artin system (A, S),*

- $|S| = 2$*, or*
- Γ_S *has a separating vertex.*

Under the hypothesis of the second bullet of the theorem, there exist $U, V \subsetneq S$ such that $\Gamma_U \cap \Gamma_V$ is a vertex $s \in S$ while $U \cup V = S$. Hence $A = A_U *_{\langle s \rangle} A_V$, and we call such a decomposition a *visual splitting* over a separating vertex.

Theorem 2.6 motivates the following definition.

Definition 2.7. Given a graph Γ, a *big chunk* is a connected induced subgraph of Γ without separating vertices, which is maximal (with respect to inclusion) with these properties. If $\Gamma = \Gamma_S$ for some Artin system (A, S), a *big chunk parabolic* is a subgroup of A conjugated to some $\langle U \rangle$, where $U \subseteq S$ spans a big chunk in Γ_S.

The next Theorem allows us to speak of the number and isomorphism types of big chunks parabolics of an Artin group, without any reference to Artin generating sets:

Theorem 2.8 ([JMS25, Theorem 5.6])**.** *Let Γ and Γ' be finite, connected labelled simplicial graphs. Let $\mathcal{BC}(\Gamma)$ be the set of big chunks of Γ, and define $\mathcal{BC}(\Gamma')$ in a similar fashion. Given an isomorphism $\varphi\colon A_\Gamma \to A_{\Gamma'}$, there exists a bijection $\varphi_\#\colon \mathcal{BC}(\Gamma) \to \mathcal{BC}(\Gamma')$ such that:*

(1) For every $\Lambda \in \mathcal{BC}(\Gamma)$, $A_\Lambda \cong A_{\varphi_\#(\Lambda)}$.
(2) If Λ is not a leaf of label 2, then $A_{\varphi_\#(\Lambda)}$ is a conjugate of $\varphi(A_\Lambda)$.
(3) If Λ is an even leaf then so is $\varphi_\#(\Lambda)$.

Moreover, if φ maps standard generators of Γ to conjugates of standard generators of Γ', then we can arrange that $A_{\varphi_\#(\Lambda)}$ is a conjugate of $\varphi(A_\Lambda)$ for every $\Lambda \in \mathcal{BC}(\Gamma)$.

We will use this theorem only in the form of the following immediate corollary. For an Artin system (A, S), call $R_S \coloneqq \{s^a \mid s \in S, a \in A\}$ the *set of reflections.*

Corollary 2.9. *Let A be an Artin group and $S, U \subseteq A$ be Artin generating sets such that $R_S = R_U$. Then $H \leqslant A$ is a big chunk parabolic subgroup of (A, S) if and only if it is a big chunk parabolic subgroup of (A, U).*

Proof. This follows by applying the "moreover" part of Theorem 2.8 to the isomorphism $A_{\Gamma_S} \cong A \cong A_{\Gamma_U}$. □

2.1. **A deformation space for Artin groups.** We now describe a deformation space for an Artin system (A, S), only depending on the set of reflections R_S, whose trees roughly correspond to maximal visual splittings over separating vertices. Towards proving Theorem F, we restrict our attention to one-ended Artin groups.

Definition 2.10 (see [JMS25, Definition 5.2])**.** Let A be a one-ended Artin group. For every Artin system (A, S) let M_S be the graph of groups decomposition of A defined as follows.

- The underlying graph of M_S has one black vertex for every big chunk of Γ_S, and one white vertex for every separating vertex of Γ_S.
- The vertex group associated to a black vertex (henceforth, a black vertex group) is the corresponding standard big chunk parabolic, while the vertex group associated to a white vertex (henceforth, a white vertex group) is generated by the corresponding separating vertex, seen as an element of S.
- There is an edge between a white vertex and a black vertex if the corresponding separating vertex belongs to the corresponding big chunk. The edge group is the same as the white vertex group, which embeds in the black vertex group via the subgraph embedding.

See Figure 2 for an example. Let $\mathcal{D}^S$ be the deformation space of M_S.

We notice that M_S is well-defined up to equivariant isometry, as two identifications of S with Γ_S can only differ by a graph automorphism of Γ_S, which descends to an equivariant isometry of M_S.

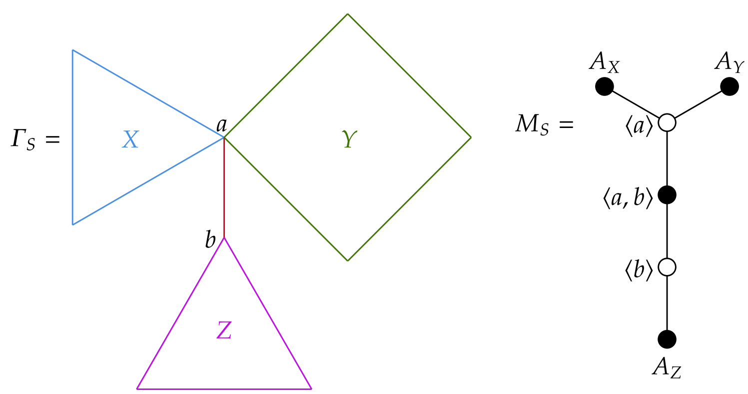

FIGURE 2. Let (A, S) be an Artin system with defining graph Γ_S as above (this is the graph from Figure 1). The separating vertices are a and b, and the subsets $X, Y, Z, \{a, b\} \subset S$ each span a big chunk in Γ_S. The decomposition M_S has one white vertex for every separating vertex, and one black vertex for every big chunk. Notice that all elements of S are elliptic in M_S.

Remark 2.11 (M_S is surviving)**.** Every edge of M_S corresponds to a pair $\{v, \Delta\}$, where Δ is a big chunk of Γ_S and $v \in \Delta$ separates Γ_S. If for every such v we collapse one of the adjacent edges of M_S, we get a reduced tree, as two big chunks cannot contain each other. Furthermore, every separating vertex belongs to at least two big chunks, so for every edge e of M_S one can find a collapse as above where e survives. See Figure 3 for an example.

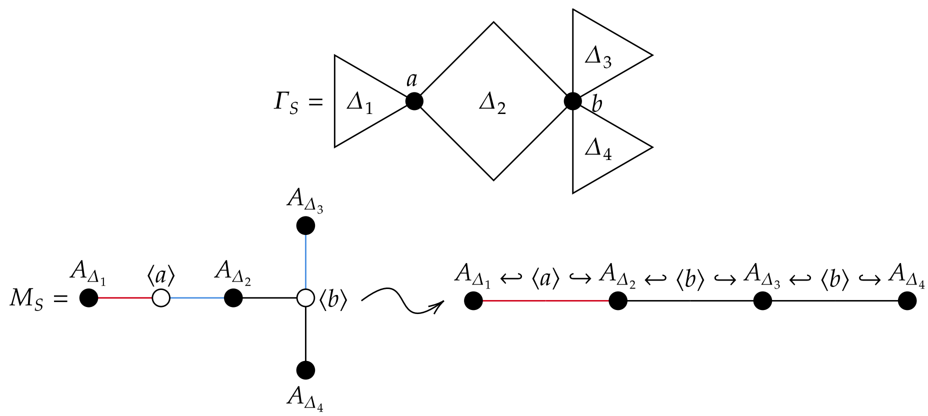

FIGURE 3. Each Δ_i is a big chunk of Γ_S. If we fix an edge (here, in red), we can always collapse one edge of M_S for every white vertex (here, the blue collection) to get a reduced tree where the fixed edge survives.

Lemma 2.12. *Let S and U be Artin generating sets of A such that $R_S = R_U$. Then $\mathcal{D}^S = \mathcal{D}^U$.*

Proof. We will use the characterisation from Theorem 1.8, hence it suffices to check that M_S and M_U have the same elliptic subgroups. In turn, by construction elliptic subgroups of M_S are precisely the subgroups of big chunk parabolics of S, and similarly for U; hence the lemma follows from Corollary 2.9. □

In view of the Lemma, from now on we shall refer to $\mathcal{D}^S$ as $\mathcal{D}^R$ where $R = R_S$, in all situations where only the dependence on the reflection set is relevant.

3. Reducing the twist conjecture

For the next definition, we say that an Artin system (A, S) is *indecomposable* if S cannot be partitioned into two non-empty, disjoint subsets Y, Z such that $m_{yz} = 2$ for every $y \in Y$ and $z \in Z$. We also say that an Artin system (A, S) is *spherical* if the associated *Coxeter group* $A/\langle\langle s^2 \mid s \in S\rangle\rangle$ is finite.

Definition 3.1 (Garside element, [Gar69])**.** If (A, S) is spherical and indecomposable, there is a distinguished element $\Delta_S \in A$, which we call the *Garside element.* For our purposes, it is enough to know that:

- If $S = \{a\}$ then $\Delta_S = a$.
- If $S = \{a, b\}$ then $\Delta_S = \Delta_{ab}$ is as in Definition 2.3.

Definition 3.2 (Twist equivalence)**.** Let (A, S) be an Artin system, and let $J \subseteq S$ be such that (A_J, J) is spherical and indecomposable. Let $J^\perp$ be the generators in $S - J$ which commute with J. Suppose that $S - (J \cup J^\perp)$ is a disjoint union $B \sqcup C$, where B, C are non-empty and any $b \in B$ is not adjacent to any $c \in C$ in

Γ_S (in other words, $\Gamma_{J\cup J^\perp}$ separates Γ_S). The *elementary twist* of B around J is the map $\tau\colon S \to A$ defined by

$$\tau(s) := \begin{cases} s & \text{if } s \in C \cup J \cup J^\perp; \\ \Delta_J s \Delta_J^{-1} & \text{if } s \in B. \end{cases}$$

The image $\tau(S)$ is again an Artin generating set for A [BMMN02, Theorem 4.5]. Two Artin generating sets S, S' for A are *twist equivalent*, and we write $S \sim S'$, if S' is obtained from S via a finite sequence of elementary twists and conjugations. Two simplicial graphs Γ, Γ' are twist equivalent, and we write $\Gamma \sim \Gamma'$, if there exist Artin systems (A, S) and (A, S') such that $\Gamma_S = \Gamma$, $\Gamma_{S'} = \Gamma'$, and $S \sim S'$.

We are interested in the following property:

Definition 3.3. An Artin system (A, S) satisfies the *strong twist conjecture* if for every Artin generating set $U \subseteq A$ with $R_S = R_U$ is twist equivalent to S. An Artin group A satisfies the strong twist conjecture if every Artin system (A, S) does.

Before exploring which Artin groups enjoy the above property, we first define another class of isomorphisms of Artin groups:

Definition 3.4 (Dehn Twist)**.** Let (A, S) be an Artin system, and $r \in S$ separate Γ_S, so $S - \{r\}$ is a disjoint union $B \sqcup C$ where B, C are non-empty and for all $b \in B$ and $c \in C$, b and c are not adjacent in Γ_S. A *Dehn twist* of B about r is a map $\delta\colon S \to A$ defined as follows:

$$\delta(s) := \begin{cases} s & \text{if } s \in C \cup \{r\}; \\ hsh^{-1} & \text{if } s \in B, \end{cases}$$

where h is any element centralising r. It is easily seen that $\delta(S)$ is an Artin generating set for A, and that $\Gamma_{\delta(S)}$ is isomorphic to Γ_S.

Definition 3.5. Two Artin generating sets S, U for A are *generalised twist equivalent*, and we write $S \sim_D U$, if they are related by a finite sequence of elementary twists, Dehn Twists, and conjugations. An Artin system (A, S) satisfies the *generalised strong twist conjecture* if for every Artin system (A, U) with $R_S = R_U$ we have that $S \sim_D U$. An Artin group A satisfies the generalised strong twist conjecture if every Artin system (A, S) does.

Remark 3.6. Notice that if $S \sim_D U$ then $R_S = R_U$, as both types of twists replace some generators by conjugates, and $\Gamma_S \sim \Gamma_U$, as Dehn twists do not change the isomorphism type of the defining graph. In particular, the generalised strong twist conjecture implies the weak twist conjecture from the introduction (see Conjecture B).

We are now ready to state the main result of this section, which is Theorem F:

Theorem 3.7. *Let (A, S) be an Artin system, with A one-ended. Suppose that, whenever $X \subseteq S$ spans a big chunk, the Artin system (A_X, X) satisfies the strong twist conjecture. Then (A, S) satisfies the generalised strong twist conjecture.*

3.1. **Proof of Theorem 3.7.** For the rest of the section, unless otherwise stated, we work under the following assumption:

Notation 3.8. Let (A, S) be an Artin system, with A one-ended. Let $R = R_S$ be the associated reflection set, and let $\mathcal{D}^R$ be the corresponding deformation space.

We start by observing the following:

Lemma 3.9. *Every A-tree in $\mathcal{D}^R$ admits a combinatorial fundamental domain, as in Definition 1.16.*

Proof. By construction M_S/A is a tree, and therefore so is T/A for every $T \in \mathcal{D}^R$ by invariance of the Betti number across the deformation space. Then the statement follows from Lemma 1.17. □

Corollary 3.10. *$\mathcal{D}^R$ is non-ascending, unless Γ_S consists of a single big chunk.*

Proof. By Lemma 1.15, it is enough to show that, if Γ_S has at least two big chunks, then $\mathcal{D}^R$ is irreducible, and in turn it suffices to exhibit two loxodromic elements for the action on M_S whose commutator is again loxodromic. Let $X, Y \subseteq S$ span different big chunks, let $a \in X - Y$ and $b \in Y - X$, and set $g = ab$ and $h = a^2b$, which are both loxodromic as they they are products of two elliptic elements with disjoint stable fixed point sets (this is a standard ping-pong type argument, see e.g. [Man17]). Their commutator $abab^{-1}a^{-2}$ is conjugate to $[b, a] = b(ab^{-1}a^{-1})$, which again is a product of elliptic elements with disjoint stable fixed point sets. □

Lemma 3.11. *Let $\mathcal{F} \leqslant \mathcal{D}^R$ be the subcomplex spanned by surviving trees. For every $(T, \Omega) \in \mathcal{F}$ and every $e \in \mathrm{E}(T)$, $\mathrm{Stab}_\Omega(e)$ is a cyclic S-parabolic subgroup.*

Proof. For the sake of light notation, we will identify A-trees with the underlying tree. The claim holds for M_S by how it was constructed in Definition 2.10. If Γ_S only has one big chunk then $\mathcal{D}^R$ consists of a single point, and there is nothing to prove. Otherwise we can assume that $\mathcal{D}^R$ is non-ascending, by Corollary 3.10.

Let M' be a reduced tree obtained from M_S by a finite sequence of elementary collapses, which exists as observed in Remark 1.10. Since each collapse preserves the set of stabilisers of those edges that are not collapsed, the claim holds for M'. If T is a reduced tree, then T and M' are related by a finite sequence of slide moves [GL07, Theorem 7.2], which preserve the set of edge stabilisers (notice that the theorem requires the deformation space to be non-ascending). In particular, the claim holds for each reduced tree. Finally, if T is a surviving tree, then, for every edge e of T, there exists a reduced tree T' and a finite sequence of elementary collapses $T \to T'$ that do not collapse e. Again, because elementary collapses preserve the set of stabilisers of edges that are not collapsed, the claim holds for T. □

Definition 3.12 (S-tree)**.** Let (A, S) be an Artin system; an A-tree (T, Ω) is an *S-tree* if there exists a combinatorial fundamental domain $K \subseteq T$ for the action Ω such that:

(1) for every $x \in \mathrm{V}(K) \cup \mathrm{E}(K)$, $\mathrm{Stab}_\Omega(x)$ is a standard S-parabolic;
(2) every element of S fixes a vertex of K.

The following two propositions, which are the core arguments underlying Theorem 3.7, show that performing an elementary deformation on an S-tree produces an S'-tree, where $S' \sim_D S$. In the proofs we will follow the strategy of [Jon24, Lemma 4.8 and Lemma 4.9].

Proposition 3.13. *Let $(T, \Omega), \in \mathcal{D}^R$ be an S-tree, and let $(T', \Omega') \in \mathcal{D}^R$ be an A-tree that is obtained from (T, Ω) by collapsing one edge. Then (T', Ω') is an S-tree.*

Proof. Let $\varphi\colon T \to T'$ be the collapse map, and let $K \subseteq T$ be a combinatorial fundamental domain for Ω that makes (T, Ω) into an S-tree. Since φ is A-equivariant and K contains exactly one edge for every orbit, there exists a unique edge $e \in K$ that is collapsed, say with endpoints u and v such that $\mathrm{Stab}_\Omega(u) \leqslant \mathrm{Stab}_\Omega(v)$.

We claim that $K' = \varphi(K)$ makes (T', Ω') into an S-tree, that is, that K' satisfies the conditions of Definition 3.12. Firstly, K' is a combinatorial fundamental domain, by equivariance of φ.

Let us now show that stabilisers of simplices in K' are standard S-parabolic subgroups. Because φ restricts to a bijection $K - e \to K - \{\varphi(e)\}$, we only need to check that $\mathrm{Stab}_{\Omega'}(\varphi(e))$ is a standard S-parabolic. In this case Equation (1) gives $\mathrm{Stab}_{\Omega'}(\varphi(e)) = \mathrm{Stab}_\Omega(v)$, which by hypothesis is a standard S-parabolic.

Finally, every $s \in S$ fixes some vertex $x \in V(K)$, so s must also fix $\varphi(x) \in K'$. □

Proposition 3.14. *Let $(T, \Omega), (T', \Omega') \in \mathcal{F}$ be surviving trees such that (T, Ω) is obtained from (T', Ω') by an elementary collapse. If (T, Ω) is an S-tree, then there is an Artin generating set $S' \subseteq A$ such that $S \sim_D S'$ and (T', Ω') is an S'-tree.*

Proof. We denote by $\varphi\colon T' \to T$ the collapsing map. Let $K \subseteq T$ be a combinatorial fundamental domain for Ω that makes (T, Ω) into an S-tree. Since K is a combinatorial fundamental domain and φ is A-equivariant, there exists an edge $e = \{u, v\}$ of T' that gets collapsed to a vertex $\varphi(e) \in K$, say with $\mathrm{Stab}_{\Omega'}(u) \leqslant \mathrm{Stab}_{\Omega'}(v)$. In particular, $\mathrm{Stab}_\Omega(\varphi(e))$ is a standard S-parabolic subgroup of A. Note that, by Lemma 3.11, every edge stabiliser of T' is a cyclic S-parabolic subgroup of A. Now consider the chain

$$\mathrm{Stab}_{\Omega'}(e) = \mathrm{Stab}_{\Omega'}(u) \leqslant \mathrm{Stab}_{\Omega'}(v) = \mathrm{Stab}_\Omega(\varphi(e)) \leqslant A, \tag{2}$$

where the equality is Equation (1). This is a chain of inclusions of S-parabolic subgroups of A, so by [BP23, Theorem 1.1] $\mathrm{Stab}_{\Omega'}(e)$ is a parabolic subgroup of $\mathrm{Stab}_{\Omega'}(v)$ (meaning that $\mathrm{Stab}_{\Omega'}(e)$ is conjugated *by an element of* $\mathrm{Stab}_{\Omega'}(v)$ to a standard S-parabolic *contained inside* $\mathrm{Stab}_{\Omega'}(v)$). Up to replacing e by a $\mathrm{Stab}_{\Omega'}(v)$-translate of it, we may therefore assume that $\mathrm{Stab}_{\Omega'}(e)$ is a standard S-parabolic subgroup of $\mathrm{Stab}_{\Omega'}(v)$, hence of A.

Let us now decompose the fundamental domain K as $\bigcup_{i=1}^n K_i$, where the K_i's are the maximal subtrees of K having $\varphi(e)$ as a vertex of valence one, so that $K_i \cap K_j = \{\varphi(e)\}$ whenever $i \neq j$. Such decomposition induces subsets $\{S_i\}_{i=0}^n$ of S, where $\langle S_0 \rangle_A = \mathrm{Stab}_\Omega(\varphi(e))$ and, for every $i \in \{1, \dots, n\}$ and every $s \in S_i$, the subtree of K fixed by s is contained in $K_i - \{\varphi(e)\}$.

Claim 3.15. *The union $\bigcup_{i=0}^n S_i$ is a partition of S.*

Proof of Claim 3.15. Let $s \in S$. By definition of S-tree, s fixes a vertex of K. If s fixes $\varphi(e)$, then $s \in S_0$; if not, then there exists $i \in \{1, \dots, n\}$ such that s fixes a vertex w_i of K_i, so $s \in S_i$. If there existed a different index j such that s fixes a vertex w_j of K_j, then s would fix the unique path between w_i and w_j, hence $\varphi(e)$, giving a contradiction. This shows that the S_i's are all disjoint. □

The tree structure of K gives a decomposition of A as an amalgamated product of the factors $\{\langle S_0 \cup S_i \rangle\}_{i=1}^n$ over the subgroup $\langle S_0 \rangle$.

Claim 3.16. *The decomposition $\langle S_1 \cup S_0 \rangle *_{\langle S_0 \rangle} \cdots *_{\langle S_0 \rangle} \langle S_n \cup S_0 \rangle$ is a visual splitting for (A, S).*

Proof of Claim 3.16. Because $\bigcup_{i=0}^n S_i = S$, it is sufficient to show that, for every distinct indices $i, j \in \{1, \dots, n\}$, for every $s \in S_i$ and for every $t \in S_j$, $\langle s, t\rangle_A \cong F_2$. To this end, let us observe that $\mathrm{Stab}_\Omega(\varphi(e))$ is a standard parabolic, so $\mathrm{Stab}_\Omega(\varphi(e)) \cap \langle s\rangle = \mathrm{Stab}_\Omega(\varphi(e)) \cap \langle t\rangle = \{1\}$ by [VdL83]. Hence s and t have disjoint stable fixed-point sets, in particular separated by $\varphi(e)$, which no non-trivial power of s or t fix. A standard ping-pong argument now shows that s and t generate a non-abelian free group, as required. □

We now define simultaneously an Artin generating set $S' \subseteq A$ that is generalised twist-equivalent to S, and a subtree $K' \subseteq T'$ that makes (T', Ω') into an S'-tree. To be precise, we set $S'_0 = S_0$, and we shall inductively replace each S_i with S'_i for every $i = 1, \dots, n$. Letting $U_i = \bigcup_{j \leqslant i} S'_j \cup \bigcup_{j>i} S_j$ for all i, two generators $a, b \in U_i$ may generate a dihedral only if either they both belong to some S_j (or S'_j), or if one of them is in S_0; hence every U_i will give an amalgamated product decomposition over $\langle S'_0\rangle$ for A. Furthermore, at each step we shall show that $U_i \sim_D U_{i+1}$. Then the required S' will be U_n, which will therefore be generalised twist equivalent to S.

Let us define $L'_i := \overline{\varphi^{-1}(K_i - \{\varphi(e)\})} \subseteq T'$. Because φ is injective outside of the orbit of e, L'_i is isomorphic to K_i as a graph and contains some $w_i \in \varphi^{-1}(e)$ as a vertex of valence one. Notice that, since $\mathrm{Stab}_{\Omega'}(v) = \mathrm{Stab}_\Omega(e) = \mathrm{Stab}_{\Omega'}\left(\varphi^{-1}(e)\right)$, the only vertices in $\varphi^{-1}(e)$ are v and all $\mathrm{Stab}_{\Omega'}(v)$-translates of u.

If w_i is either v or u, we simply set $K'_i := L'_i \cup e$ and $S'_i := S_i$. This way $U_{i-1} = U_i$, and there is nothing to prove. Let us now assume that $w_i = h \cdot u$ for some $h \in \mathrm{Stab}_{\Omega'}(v) - \{1\}$. In order that K' is connected, we set $K'_i := h^{-1} \cdot L'_i \cup e$. Let f be the edge of L'_i that has $h \cdot u$ as an endpoint. Because $\varphi\colon T' \to T$ restricts to an isometry $L'_i \to K_i$, f is not collapsed, so we have that $\mathrm{Stab}_{\Omega'}(f) = \mathrm{Stab}_\Omega(\varphi(f))$. In particular, the latter is a cyclic standard S-parabolic, say generated by some $s \in S$. In turn, the edge $h^{-1} \cdot f$ has u as an endpoint and $h^{-1}\,\mathrm{Stab}_{\Omega'}(f)\,h$ is an S-parabolic subgroup of A contained in $\mathrm{Stab}_{\Omega'}(u)$, hence an S-parabolic subgroup of $\mathrm{Stab}_{\Omega'}(u)$ by [BP23, Theorem 1.1]. Therefore, up to replacing h by a $\mathrm{Stab}_{\Omega'}(u)$-translate (which might change L'_i, but does not move $w_i = hu$), we may assume that $h^{-1}\,\mathrm{Stab}_{\Omega'}(f)\,h$ is a standard S-parabolic subgroup, say generated by $t \in S$. Since h^{-1} conjugates s to t, it follows that there is a sequence $\{s_i\}_{i=0}^k \subseteq S$ such that [Par97, Corollary 4.2]:

(1) $s_0 = t$, $s_k = s$, and for every $i \in \{0, \dots, k-1\}$, $\{s_i, s_{i+1}\}$ spans an odd dihedral S-parabolic;
(2) $h^{-1} \in \mathrm{C}_A(t)\Delta_{s_0 s_1} \cdots \Delta_{s_{k-1} s_k}$, where $\mathrm{C}_A(t)$ denotes the centraliser of t.

We also remark that $s, t \in S_0$ as they both lie in $\mathrm{Stab}_{\Omega'}(v) = \mathrm{Stab}_\Omega(\varphi(e))$.

Before defining S'_i, we observe the following fact:

Claim 3.17. *For every $x \in S_i$ and for every $r \in S_0$, if $\langle x, r\rangle$ is a spherical dihedral subgroup, then $r = s$.*

Proof of Claim 3.17. Let $x \in S_i$ and let $r \in S_0 - \{s\}$. Because $x \in S_i$, the stable fixed point set of x intersects L'_i and does not contain f. On the other hand, r fixes v but not f, so its stable fixed-point in T' lies on the opposite side of f with respect to L'_i. Again, a standard ping-pong argument shows that $\langle x, r\rangle \cong F_2$. □

As the above argument holds for any $i = 1, \dots, n$, it implies that, if two vertices of Γ_{S_0} are connected by a path, then we can choose such path to belong entirely to

Γ_{S_0}, as for every i there is a unique vertex in S_0 from which one can enter or exit S_i. In particular, we can assume that all s_i from Item (1) belong to S_0.

Set $S'_i \coloneqq h^{-1}S_ih$, and recall that we defined $U_{i-1} = \bigcup_{j<i} S'_j \cup \bigcup_{j\geqslant i} S_j$ and $U_i = \bigcup_{j\leqslant i} S'_j \cup \bigcup_{j>i} S_j$. Consider the map $\alpha_i\colon U_{i-1} \to U_i$ which maps $a \in U_{i-1}$ to

$$\alpha_i(a) = \begin{cases} h^{-1}ah & \text{if } a \in S_i; \\ a & \text{otherwise.} \end{cases}$$

To conclude that U_{i-1} and U_i are generalised twist equivalent, it is enough to prove the following:

Claim 3.18. *α_i is a sequence of elementary twists and a Dehn Twist.*

Proof of Claim 3.18. By inspection of Item (2), conjugating S_i by h amounts to first conjugating by $\Delta_{s_{k-1}s_k}$, where $s_k = s$. Notice that both s_{k-1} and s belong to S_0, thus to U_{i-1}, and the edge $\{s_{k-1}, s\}$ separates $\Gamma_{U_{i-1}}$ in view of Claim 3.17; hence conjugating S_i by $\Delta_{s_{k-1}s_k}$ is an elementary twist τ_1, with respect to U_{i-1}. Notice that, since $\{s_{k-1}, s\}$ is an odd edge, s_{k-1} now separates $\Gamma_{\tau_1(S_i)}$ from the rest of $\Gamma_{\tau_1(U_{i-1})}$, as the twist "slides" S_i along the odd edge. Proceeding inductively, one sees that conjugating S_i by $\Delta_{s_0s_1}\dots\Delta_{s_{k-1}s_k}$ is a sequence τ of elementary twists along edges, after which $s_0 = t$ separates $\Gamma_{\tau(S_i)}$ from the rest of $\Gamma_{\tau(U_{i-1})}$. Now S'_i is obtained from $\tau(S_i)$ by conjugating by an element in the centraliser of t, and this is a Dehn Twist in the visual splitting of $(A, \tau(U_{i-1}))$ along t. This proves that U_{i-1} and U_i are generalised twist equivalent, as required. ☐

Let $K' = \bigcup_i K'_i \subseteq T'$. We are now left to show that K' makes (T', Ω') into an S'-tree. K' has exactly one more edge than K (that is, the edge e), thus it is a finite subgraph of T', and it is connected by construction of the K'_i's, each of which is connected and contains e. It follows that K' is a finite subtree of T'. For the sake of clarity, we shall break down the remaining part of the proof into smaller claims.

Claim 3.19. *K' is a combinatorial fundamental domain for (T', Ω').*

Proof of Claim 3.19. Let $x \in \mathrm{V}(T')$. If $\varphi(x)$ lies in the Ω-orbit of $\varphi(e)$, then x lies in the Ω'-orbit of either u or v. The fact that $|A \cdot x \cap \mathrm{V}(K')| = 1$ follows by observing that K is a combinatorial fundamental domain and that u and v belong to distinct Ω'-orbits. If $\varphi(x)$ is not in the orbit of $\varphi(e)$, then there exist $k \in K - \{\varphi(e)\}$ and $g \in A$ such that $\varphi(x) = g \cdot k$. Because φ is injective away from the orbit of e, $\varphi^{-1}(k) = k'$ for some k' that either belongs to K' or such that a $\mathrm{Stab}_{\Omega'}(v)$-translate of it belongs to K'. In both cases, the orbit of k' intersects K'. Because φ is A-equivariant, $\varphi(x) = g \cdot \varphi(k')$ implies that $\varphi(x) = \varphi(g \cdot k')$; because φ is injective on these points, it follows that $x = g \cdot k'$.

The proof that $\mathrm{E}(K')$ contains exactly one element for each Ω'-orbit of edges runs analogously. ☐

Claim 3.20. *For every $x \in \mathrm{V}(K') \cup \mathrm{E}(K')$, $\mathrm{Stab}_{\Omega'}(x)$ is a standard S'-parabolic subgroup.*

Proof of Claim 3.20. We have that $\mathrm{Stab}_{\Omega'}(v) = \mathrm{Stab}_{\Omega}(\varphi(e)) = \langle S_0 \rangle$ by construction of K'; moreover, as argued in Equation (2), $\mathrm{Stab}_{\Omega'}(u) = \mathrm{Stab}_{\Omega}(e)$ is a standard S-parabolic inside $\langle S_0 \rangle$. Since $S_0 = S'_0$, both stabilisers are standard S'-parabolics as well.

Let now $x \in \mathrm{V}(K') - \{u, v\}$. From injectivity of φ away from the orbit of e and A-equivariance, it follows that $\mathrm{Stab}_{\Omega'}(x) = \mathrm{Stab}_{\Omega}(\varphi(x))$. By construction of K', there exists $h \in \mathrm{Stab}_{\Omega'}(v)$ and $i \in \{1, \ldots, n\}$ such that $\varphi(x) \in h^{-1} \cdot K_i$. Because K makes (T, Ω) into an S-tree, there exists $U \subseteq S_i$ such that $\mathrm{Stab}_{\Omega}(h \cdot \varphi(x)) = \langle U \rangle$, that is, $\mathrm{Stab}_{\Omega}(\varphi(x)) = \langle h^{-1}Uh \rangle$, which is a standard S'-parabolic subgroup of A, since $h^{-1}Uh \subseteq h^{-1}S_ih = S'_i$.

Again, the proof that edge-stabilisers are standard S'-parabolics runs analogously. □

Claim 3.21. *Every element $s' \in S'$ fixes a vertex of K'.*

Proof of Claim 3.21. Every $s' \in S'_0$ fixes e. If instead $s' \in S' - S'_0$, there exist $h \in \mathrm{Stab}_{\Omega'}(v)$, $i \in \{1, \ldots, n\}$ and $s \in S_i \subseteq S$ such that $s' = h^{-1}sh$. By construction of K_i, s acts elliptically on T, fixing a subtree of K_i and therefore s' acts elliptically, fixing a subtree of $h^{-1} \cdot L'_i \subseteq K_i$. □

The proof of Proposition 3.14 is now complete. □

Proposition 3.22. *Given a one-ended Artin group A with Artin generating sets S and U such that $R_S = R_U$, there exist equivariantly isometric A-trees T_S and $T_{U'}$ with infinite cyclic edge stabilisers, such that T_S is an S-tree and $T_{U'}$ is a U'-tree, for some Artin generating set $U' \sim_D U$. Moreover, if A has at least two big chunks, T_S contains an edge and is minimal as an A-tree.*

Proof. Take M_S and M_U as in Definition 2.10. We claim the tree M_S is an S-tree. Take the fundamental domain K to be the one described in Definition 2.10. Every standard big chunk S-parabolic occurs as a point stabiliser in K, and every $s \in S$ belongs to some big chunk parabolic, so every $s \in S$ fixes a point in K. Conversely, every simplex stabiliser in K is either a standard big chunk S-parabolic, or generated by some $s \in S$ separating Γ_S, so is in particular a standard S-parabolic. Likewise, M_U is a U-tree.

One easily checks from Definition 2.10 that if A has at least two big chunks, then M_S is minimal and is not just a point.

Now, since $R_S = R_U$, Lemma 2.12 gives that M_S and M_U are in the same deformation space $\mathcal{D}^R$, where $R = R_S$. If $\mathcal{D}^R$ consists of a single point then M_S and M_U are equivariantly isometric, and we are done. Otherwise $\mathcal{D}^R$ is non-ascending by Lemma 3.10. Moreover, both M_S and M_U are surviving by Remark 2.11, so Theorem 1.14 produces a sequence of elementary collapses and expansions taking M_U to a tree T' which is equivariantly isometric to M_S. By applying Propositions 3.13 and 3.14, we can realise T' as a U'-tree, where U' is as required by the statement. □

Proof of Theorem 3.7. Recall that we are given a one-ended Artin group A and an Artin system (A, S) such that, for every $X \subseteq S$ spanning a big chunk, (A_X, X) satisfies the strong twist conjecture. Our goal is to prove that every Artin generating set U for A such that $R_S = R_U = R$ is generalised twist equivalent to S. We proceed by induction on the number of big chunks of A. The base case is immediate.

Before moving to the inductive step, let us clarify the inductive hypothesis. Let $\mathcal{V}$ be the collection of proper subsets $V \subsetneq S$ such that A_V is one-ended and V is of the form $V = X_1 \cup \ldots \cup X_i$, where each X_j spans a big chunk in Γ_S. By induction we can assume that, for every $V \in \mathcal{V}$, the Artin system (A_V, V) satisfies the generalised strong twist conjecture.

Now, replacing U by some $U' \sim_D U$ does not change the set of reflections, so in light of Proposition 3.22 we may assume that there are equivariantly isometric A-trees (T_S, Ω_S) and (T_U, Ω_U) with infinite cyclic edge stabilisers, such that T_S is an S-tree and T_U is a U-tree (we henceforth suppress the actions in our notation). Write K_S and K_U to be the corresponding combinatorial fundamental domains.

We now use the S-tree and U-tree structure to produce visual splittings of A with respect to S and U such that both decompositions have the same factors, towards applying the inductive hypothesis to these factors. By Proposition 3.22, T_S has at least one edge, since A has at least two big chunks; so fix an arbitrary edge e of K_S, and let $\mathrm{Stab}_{\Omega_S}(e) = \langle r \rangle$ for some $r \in S$. Take $f : T_S \to T_U$ to be an equivariant isometry. Up to conjugating U, we may assume that $f(e) \in \mathrm{E}(K_U)$. By equivariance of f it follows that $\mathrm{Stab}_{\Omega_U}(f(e)) = \langle r \rangle$, and so either $r \in U$ or $r^{-1} \in U$.

In fact, it must be that $r \in U$. If this was not the case, then since $r \in R_S = R_U$, there would be $u \in U$ conjugate to r. However U would have one generator conjugate to the inverse of another, and this contradicts the existence of a homomorphism $A_U \to \mathbb{Z}$ mapping every generator to 1.

Write $\overline{T_S}$ (resp. $\overline{T_U}$) for the equivariant quotient A-tree obtained by collapsing every edge in T_S (resp. T_U) not in the orbit of e (resp. $f(e)$). By a routine diagram chase one sees that f induces an equivariant isometry $\overline{f} : \overline{T_S} \to \overline{T_U}$.

By equivariance of the quotient map, $\overline{T_S}$ has one orbit of edges. We claim it has two orbits of vertices. Suppose not, then, writing $e = \{x, y\}$, this means that there is a path in T_S from x to gy (for some $g \in A$), not passing through any edge in the orbit of e. However, this implies that T_S/A is not simply connected, contradicting the fact that T_S has a combinatorial fundamental domain by Lemma 1.17.

We write $S = S_1 \cup S_2$, where each S_i is the set of generators fixing a vertex of K_S on one side of e. This can be done since K_S is a fundamental domain making T_S an S-tree. Notice that $S_1 \cap S_2 = \{r\}$. Using the equivariance of the quotient map, it is not hard to show that $\overline{T_S}$ is the Bass-Serre tree for the splitting $A = A_{S_1} *_{\langle r \rangle} A_{S_2}$.

The arguments above apply verbatim with U in place of S, substituting $f(e)$ for e to obtain a splitting $A = A_{U_1} *_{\langle r \rangle} A_{U_1}$, where $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \{r\}$. Notice that, by the equivariance of $\overline{f}$, the stabilisers of the endpoints of the image of e in $\overline{T_S}$ and $\overline{T_U}$ are the same subgroups of A, so (up to exchanging U_1 and U_2), $A_{S_i} = A_{U_i}$ for $i \in \{1, 2\}$. We denote these subgroups simply by A_i.

The next two claims will allow us to use the inductive hypothesis:

Claim 3.23. $S_1, S_2 \in \mathcal{V}$.

Proof of Claim 3.23. We first notice that $S_1, S_2 \neq S$, since otherwise $\overline{T_S}$, and hence T_S, would not be minimal. Furthermore, each $X \subseteq S$ spanning a big chunk in S lies in either S_1 or S_2. Indeed, A_X fixes a point p in T_S, since T_S is in the same deformation space as M_S, and therefore has the same elliptic subgroups by Theorem 1.8. This means that each $x \in X - \{r\}$ must fix a point in K_S belonging to the same connected component of $T_S - e$ as p.

Finally, A_{S_i} is one-ended since Γ_{S_i} is connected. To see this, we shall prove that, for every $x \in S_i$ and any simple path $\gamma \subseteq \Gamma_S$ connecting x to r, we have that $\gamma \subseteq \Gamma_{S_i}$. Indeed, notice that x and the first vertex of γ after it, call it z, must belong to a common big chunk, say spanned by $X \subseteq S$. Moreover, by the above argument X must belong to one of S_1 and S_2, and it must be that $X \subseteq S_i$ since

$x \neq r$ belongs to S_i. Hence $z \in S_i$ as well. If $z = r$ we stop; otherwise we repeat this procedure, eventually showing that $\gamma \subseteq \Gamma_{S_i}$, as required. □

Claim 3.24. *For $i = 1, 2$ let $R^{A_i}_{S_i}$ and $R^{A_i}_{U_i}$ be the reflection sets for S_i and U_i in A_i. Then $R^{A_i}_{S_i} = R^{A_i}_{U_i}$.*

Proof of Claim 3.24. Let $s \in S_i$, and we have to find some $u' \in U_i$ which is conjugate to s in A_i. Since $R_S = R_U$, there is $u \in U$ and $g \in A$ such that $s = gug^{-1}$. Moreover, notice that Γ_{U_i} is a union of big chunks of Γ_U (this follows as in Claim 3.23, applied to U), so by e.g. [JMS25, Remark 3.6], there exists a retraction $\rho\colon A \to A_i$ mapping every generator $v \in U$ to

$$\rho(v) = \begin{cases} v & \text{if } v \in U_i; \\ r & \text{if } v \notin U_i. \end{cases}$$

Since ρ is the identity on A_i, we get that $s = \rho(s) = \rho(g)\rho(u)\rho(g)^{-1}$. Hence s is conjugate to $\rho(u) \in U_i$ by $\rho(g) \in A_i$, as required. □

By induction, there exists $\psi : S_1 \to A_1$ which is a generalised twist equivalence between S_1 and U_1.

Claim 3.25. *ψ extends to a generalised twist equivalence $\hat{\psi}\colon S \to A$; moreover $S' \coloneqq \hat{\psi}(S) = U_1 \cup hS_2h^{-1}$, where $h \in A_1$ is such that $\psi(r) = hrh^{-1}$.*

Proof of Claim 3.25. Write ψ as a sequence $\psi_l \circ \ldots \circ \psi_1$ of elementary twists, conjugations, and Dehn twists. By an inductive argument, it is enough to show that ψ_1 extends to an elementary twist, conjugation, or Dehn twist of A with respect to S. If ψ_1 is the conjugation by some $h_1 \in A_1$ then let $\hat{\psi}_1 : S \to A$ be the conjugation by h_1, and we have nothing to show.

Next, suppose ψ_1 is a Dehn twist around a separating vertex v of Γ_{S_1}. Then v is separating in Γ_S as well, since $S_1 \cap S_2 = \{r\}$; furthermore, either $v = r$ or S_2 is in the same connected component of $\Gamma_S - \{v\}$ as r. If $\psi_1(r) = r$ (that is, if either $v = r$ or r is not in one of the connected components of $\Gamma_{S_1} - \{v\}$ which get conjugated), we can define $\hat{\psi}_1\colon S \to A$ to be the identity on S_2, and notice that this is a Dehn twist around v in Γ_S. Otherwise $\psi_1(r) = h_1rh_1^{-1}$ for some $h_1 \in A_1$ commuting with v, and if we define $\hat{\psi}_1$ by mapping every $s \in S_2$ to $h_1sh_1^{-1}$ we still get a Dehn twist around v in Γ_S.

A similar argument, with the required adjustments, proves that if ψ_1 is an elementary twist then it can be extended to an elementary twist $\hat{\psi}_1\colon S \to A$, thus proving the claim. □

Now notice that $r, hrh^{-1} \in U_1$ are conjugate in A_1. It follows from [Par97, Corollary 4.2] that there is a path of odd edges $\{e_1, \ldots, e_n\}$ in Γ_{U_1} from hrh^{-1} to r. For $i = 1, \ldots, n$ write Δ_i for the Garside elements of e_i, so that $\Delta_n \ldots \Delta_1 h$ centralises r.

As in the proof of Claim 3.18, successively conjugating hS_2h^{-1} by Δ_1, Δ_2, and so on, is a sequence of twists along separating edges (meaning that every e_i is separating in the defining graph of $U_1 \cup \Delta_{i-1} \ldots \Delta_1 hS_2h^{-1}(\Delta_{i-1} \ldots \Delta_1)^{-1}$). Hence

$$S' = U_1 \cup hS_2h^{-1} \sim U_1 \cup (\Delta_n \ldots \Delta_1)hS_2h^{-1}(\Delta_n \ldots \Delta_1)^{-1} \eqqcolon S'',$$

and we note that $\Gamma_{S''}$ has r as a separating vertex. In particular, since $\Delta_n \ldots \Delta_1 h$ centralises r, we can Dehn twist S'' by this element to obtain $U_1 \cup S_2$. We have now seen that $S_1 \cup S_2 \sim_D U_1 \cup S_2$.

We finally notice that, again in view of Claims 3.23 and 3.24, there is a generalised twist equivalence between S_2 and U_2 in A_2. As above, this can be used to show that $U_1 \cup S_2 \sim_D U_1 \cup U_2$, thus completing the proof of Theorem 3.7. □

4. A genuine combination theorem using ribbons

In Section 3 we proved that an Artin system (A, S) satisfies the generalised strong twist conjecture, provided that its big chunk parabolics satisfy the strong twist conjecture. In order to to improve the conclusion of Theorem 3.7 to the genuine strong twist conjecture, we need to guarantee that, with respect to any Artin generating set that is twist-equivalent to S, Dehn twists around separating vertices can be written as a composition of elementary twists and conjugations. We prove that this condition is satisfied, whenever sufficiently many parabolic subgroups of (A, S) enjoys the *vertex ribbon property*. The latter describes the elements that conjugate standard generators via ribbons, which intuitively can be thought as minimal conjugating elements. The notion of a ribbon was introduced by Paris in [Par97] and then studied in detail by Godelle (see Theorem 4.4 below).

Definition 4.1 (ribbons)**.** Let (A, S) be an Artin system and let $x, y \in S$. An element $g \in A$ such that $x = gyg^{-1}$ is an *elementary* (x, y)*-ribbon* (with respect to S) if one of the following conditions hold:

(1) the elements x and y are distinct, $m_{xy} \in \mathbb{N}_{\geqslant 3}$ is odd and $g = \Delta_{xy}$ (or its inverse);
(2) the elements x and y coincide and one of the following holds:
  (a) $g = x$ (or its inverse);
  (b) there is $t \in S$ such that $m_{xt} \in \mathbb{N}_{\geqslant 4}$ is even and $g = \Delta_{xt}$ (or its inverse);
  (c) there is $t \in S$ such that $m_{xt} = 2$ and $g = t$ (or its inverse).

An element $g = g_1 \cdots g_n$ such that $x = gyg^{-1}$ is an (x, y)*-ribbon* (with respect to S) if there exist $x_0, \ldots, x_n \in S$ such that $x_0 = x$, $x_n = y$ and, for every $i \in \{1, \ldots, n\}$, g_i is an elementary (x_{i-1}, x_i)-ribbon. We denote by $\mathrm{Ribb}_S(x, y)$ the set of (x, y)-ribbons with respect to S.

We say that the pair (x, y) satisfies the *vertex ribbon property* (in (A, S)) if, for every $g \in A$, if $x = gyg^{-1}$, then $g \in \mathrm{Ribb}_S(x, y)$. We say that (A, S) satisfies the *vertex ribbon property* if every pair (x, y) with $x, y \in S$ satisfies the ribbon property with respect to S.

Remark 4.2. Let (A, S) be an Artin system and $a \in S$. From Definition 4.1, it follows that, for every (a, a)-ribbon $h \in A$, there exists a sequence $(a_i, b_i)_{i=0}^n$ of pairs of elements in S, such that the following hold.

- $a_0 = b_n = a$.
- For every i, either $\{a_i, b_i\}$ span a dihedral subgroup, and we set $m_i = m_{a_i b_i}$, or $a_i = b_i$, and with a little abuse of notation we set $m_i = 1$.
- $a_{i+1} = a_i$ if m_i is even, and $a_{i+1} = b_i$ if m_i is odd.
- Set $t_i = b_i$ if $m_i \leqslant 2$, while $t_i = \Delta_{a_i b_i}$ if $m_i \geqslant 3$.
- There exist $\varepsilon_i \in \{\pm 1\}$ such that $h = t_n^{\varepsilon_n} \ldots t_0^{\varepsilon_0}$.

In other words, the $\{a_i\}_{i=0}^n$ are the vertices of an odd loop $\gamma \subseteq \Gamma_S$, to which some even "spikes" are glued, as in Figure 4.

Remark 4.3. We will freely use that, if $g \in \mathrm{Ribb}_S(s, t)$ and $h \in \mathrm{Ribb}_S(t, r)$, then $gh \in \mathrm{Ribb}_S(s, r)$ and $g^{-1} \in \mathrm{Ribb}_S(t, s)$.

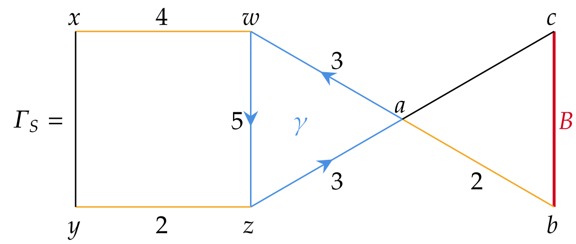

FIGURE 4. An example of a "loop with spikes" associated to an (a,a)-ribbon, as in Remark 4.2. In the picture the ribbon is $h = b\Delta_{az}y^{-1}\Delta_{wz}\Delta_{wx}\Delta_{aw}^{-1}$, so we set $(a_0,b_0) = (a,w)$, $(a_1,b_1) = (w,x)$, $(a_2,b_2) = (w,z)$, $(a_3,b_3) = (z,y)$, $(a_4,b_4) = (z,a)$, and $(a_5,b_5) = (a,b)$. The blue path γ corresponds to the collection of elementary ribbons along odd edges, while the orange "spikes" are given by the elementary ribbons along even edges. In the proof of Proposition 4.5, we also consider a subset $B \subset S$ which a separates from the rest of the vertices (here, in red).

In this paper, for the sake of explicitly describing Dehn twists, we only defined ribbons between vertices, which are are an instance of a more general definition of ribbons, encoding "minimal" group elements that conjugate standard parabolic subgroups (see e.g. [God07, Definition 1.3]). Godelle conjectured that the vertex ribbon property (and a more general "ribbon property") holds for all Artin groups, and proved it for several classes, some of which we list here specialised to the generality we shall need later:

Theorem 4.4 ([God03, God07])**.** *Let (A,S) be an Artin system of either spherical or large type. Then (A,S) satisfies the vertex ribbon property.*

The next lemma shows the relevance of the vertex ribbon property in this paper:

Proposition 4.5. *Let (A,S) be an Artin system. If δ is a Dehn Twist of (A,S) around a vertex $a \in S$, and (a,a) satisfies the vertex ribbon property in S, then $\delta(S) \sim S$.*

Proof. By definition of a Dehn twist, there exist a non-trivial decomposition $S - \{a\} = B \sqcup C$ such that $m_{bc} = \infty$ for every $b \in B$ and $c \in C$, and an element $h \in A$ centralising a, such that $\delta(S) = C \cup \{a\} \cup hBh^{-1}$. By the vertex ribbon property for (a,a), $h \in \mathrm{Ribb}_S(a,a)$, so for every $i = 0,\dots,n$ let $(a_i,b_i) \in S \times S$, m_i, t_i, and ε_i be as in Remark 4.2. Up to decomposing h into a product of smaller (a,a)-ribbons, we can assume that the path γ passes through a only at its endpoints, so either $\gamma \subseteq \Gamma_{C\cup\{a\}}$ or $\gamma \subseteq \Gamma_{B\cup\{a\}}$. We can assume that we are in the first case up to swapping B and C, because conjugating B by h is the same as conjugating C by h^{-1} and then conjugating the whole generating set by h.

Set $h_{-1} = 1$ and, for every $i = 0,\dots,n$, let $h_i = t_i^{\varepsilon_n} \dots t_0^{\varepsilon_0}$. Set $B_i = h_iBh_i^{-1}$ and $S_i = C \cup \{a\} \cup B_i$, so that $S_n = \delta(S)$. We shall now prove by induction on i that $S_i \sim S$, and furthermore that a_{i+1} separates Γ_{B_i} from the rest of Γ_{S_i}. The base case $i = -1$ holds vacuously. Now suppose the conclusion holds for $i-1$. There are four cases to consider, depending on m_i.

If $m_i = 1$ then $t_i = a_i$, so conjugating B_{i-1} by t_i (or its inverse) is an elementary twist around $\{a_i\}$. If $m_i = 2$ then $t_i = b_i$, so conjugating B_{i-1} by t_i (or its inverse) is an elementary twist around $\{b_i\}$ (notice that $a_i \in \{b_i\}^\perp$ by construction). If $m_i \geqslant 4$ is even, then $t_i = \Delta_{a_i b_i}$, so conjugating B_{i-1} by t_i (or its inverse) is an elementary twist along the edge $\{a_i, b_i\}$. In each of these three cases $\Gamma_{S_i} \cong \Gamma_{S_{i-1}}$ and $a_{i+1} = a_i$, which still separates Γ_{B_i} from the rest of Γ_{S_i}.

The last case to consider is when $m_i \geqslant 3$ is odd, so that $t_i = \Delta_{a_i b_i}$. The only difference with the even case is that Γ_{S_i} is obtained from $\Gamma_{S_{i-1}}$ by "sliding" $\Gamma_{B_{i-1} \cup \{a_i\}}$ along the edge $\{a_i, a_{i+1}\}$, as in Figure 5. More formally, if $x \in C \cup \{a\}$ and $y \in B_i$ are such that $m_{xy} \neq \infty$, then $x = a_{i+1}$. Therefore a_{i+1} now separates Γ_{B_i} from the rest of Γ_{S_i}. This concludes the induction, and in turn the proof that $\delta(S) \sim S$. □

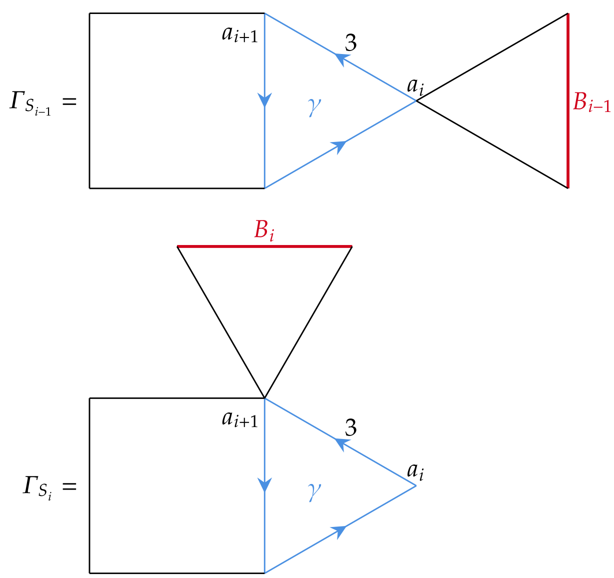

FIGURE 5. An elementary twist along the odd edge $\{a_i, a_{i+1}\}$ "slides" B_{i-1}, so that a vertex in B_i can only be connected to a_{i+1} in Γ_{S_i}.

We now study which procedures preserve the vertex ribbon property. We first observe that, to verify the vertex ribbon property, it is sufficient to understand centralisers of standard generators. We will freely use this fact in the sequel.

Lemma 4.6. *An Artin system (A, S) satisfies the vertex ribbon property if and only if for all $s \in S$, (s, s) satisfies the vertex ribbon property in S.*

Proof. The forward direction is obvious. For the reverse direction, suppose $s, t \in S$ and $t = gsg^{-1}$. Since s and t are conjugate, it follows from [Par97] that there is a path of odd edges between them, and so there is $h \in \mathrm{Ribb}_S(s, t)$ (which is the product of the Garside elements of these edges). We see that $s = hth^{-1} = hgs(hg)^{-1}$, so $hg \in \mathrm{Ribb}_S(s, s)$ by assumption. Hence $g \in h^{-1}\,\mathrm{Ribb}_S(s, s) = \mathrm{Ribb}_S(t, s)$ as required. □

Proposition 4.7. *Let (A, S) be an Artin system. Suppose that $A = A_B *_{A_D} A_C$ where $B, C \subseteq S$ and $D = B \cap C$. Suppose further that (A_B, B) and (A_C, C) satisfy the vertex ribbon property. Then (A, S) satisfies the vertex ribbon property.*

Proof. We consider the Bass-Serre tree T for the splitting in the statement. We write v_B and v_C for the vertices corresponding to A_B and A_C respectively, and e for the edge between those vertices, which is stabilised by A_D.

By Lemma 4.6, it is enough to prove that, for every $s \in S$ and every $w \in A$ which commutes with s, $w \in \mathrm{Ribb}_S(s, s)$. Without loss of generality assume that $s \in B$, so that $sv_B = v_B$. We write T^s for the subtree of T fixed by s, and notice that if $x \in T^s$ then $wx \in T^s$ as well. There are several cases to consider, according to the shape of T^s.

Case 1. We suppose first that s only fixes v_B. Then $wv_B = v_B$ so $w \in A_B$. Since (A_B, B) satisfies the vertex ribbon property by hypothesis, we have that $w \in \mathrm{Ribb}_B(s, s) \subseteq \mathrm{Ribb}_S(s, s)$.

Case 2a. Next, suppose that s fixes e, i.e. $s \in D$. We characterise edges in T^s:

Claim 4.8. *Every edge ge in T^s can be written as fe, where $f \in \mathrm{Ribb}_S(s, t)$ for some $t \in D$.*

Proof of Claim 4.8. We proceed by induction on the distance between the midpoints of ge and e. The base case is clear by taking $t = s$ and $f = 1$.

For the inductive step, fix $ge \in T^s$ such that $g \in \mathrm{Ribb}_S(s, t)$. Suppose without loss of generality that gv_C is the endpoint furthest from e, and consider an edge sharing this endpoint in T^s, which we may write as ghe for some $h \in A_C$. Notice that, since $sghe = ghe$, we have that $h^{-1}th = h^{-1}g^{-1}sgh \in A_D$. By [BP23, Theorem 1.1], $h^{-1}\langle t\rangle h$ is parabolic inside of A_D, so up to postmultiplying h by an element of A_D (which does not change the edge ghe), we may assume $h^{-1}th = r$ for some $r \in D$. By the assumption that (A_C, C) satisfies the vertex ribbon property, we see that $h \in \mathrm{Ribb}_C(t, r) \subseteq \mathrm{Ribb}_S(t, r)$, and so $gh \in \mathrm{Ribb}_S(s, r)$. This completes the proof of the claim. □

Now, take $w \in A_S$ such that $wsw^{-1} = s$, and therefore $we \in T^s$. By the Claim, we may write $we = fe$ where $f \in \mathrm{Ribb}_S(s, t)$ for some $t \in D$. In turn $w = fh$, where $h \in A_D \leqslant A_B$ is such that $hsh^{-1} = t$, and as such $h \in \mathrm{Ribb}_B(t, s) \subseteq \mathrm{Ribb}_S(t, s)$ by assumption that (A_B, B) satisfies the vertex ribbon property. It follows that $w \in \mathrm{Ribb}_S(s, s)$ as required.

Case 2b. Finally, suppose that s fixes an edge, and we claim that we can reduce to Case 2a. Indeed, since s fixes v_B, it must also fix some edge of the form be, where $b \in A_B$. Then $b\langle s\rangle b^{-1}$ is a parabolic subgroup of A_D, so by [BP23] there exists $d \in A_D$ and $s' \in D$ such that $b\langle s\rangle b^{-1} = d^{-1}\langle s'\rangle d$. By looking at the map $A_S \to \mathbb{Z}$ sending every generator to 1 we see that $bsb^{-1} = d^{-1}s'd$. Now, since $db \in A_B$ conjugates $s \in B$ to $s' \in D \subseteq B$, the vertex ribbon property for (A_B, B) yields that $db \in \mathrm{Ribb}_B(s', s) \subseteq \mathrm{Ribb}_S(s', s)$. Furthermore, since $wsw^{-1} = s$, the element $w' = dbw(db)^{-1}$ commutes with s', and the latter fixes e as it belongs to A_D. Then $w' \in \mathrm{Ribb}_S(s', s')$ by Case 2a, and therefore $w = (db)^{-1}w'db \in \mathrm{Ribb}_S(s, s)$, as required. □

Corollary 4.9. *Let (A, S) be an Artin system. Suppose that, for any $Y \subseteq S$ spanning a clique in Γ_S, (A_Y, Y) satisfies the vertex ribbon property. Then (A, S) satisfies the vertex ribbon property.*

Proof. We proceed by induction on the cardinality of S, the base case $|S| = 1$ being trivial. Now suppose the conclusion holds for every Artin system (A', S') such that $|S'| < |S|$. If Γ_S is a clique there is nothing to prove. Otherwise let $u, v \in S$ be non-adjacent in Γ_S, and set $B = S - \{v\}$, $C = S - \{w\}$, and $D = B \cap C$. Every clique in Γ_B or Γ_C is also a clique in Γ_S; so by induction (A_B, B) and (A_C, C) satisfy the vertex ribbon property, and therefore so does (A, S) by Proposition 4.7. □

The vertex ribbon property is also preserved under elementary twists, provided that it holds for cliques:

Lemma 4.10. *Let (A, S) be an Artin system, and let S' be obtained from S by an elementary twist. Suppose further that, for any $Y \subseteq S$ spanning a clique in Γ_S, (A_Y, Y) satisfies the vertex ribbon property. Then for any $Y' \subseteq S'$ spanning a clique in $\Gamma_{S'}$, $(A_{Y'}, Y')$ satisfies the vertex ribbon property. In particular (A, S') satisfies the vertex ribbon property.*

Proof. By definition of an elementary twist, there exist $J, T, R \subseteq S$ such that, if we set $D = J \cup J^{\perp}$, $B = R \cup D$, $C = T \cup D$, then $S = B \cup C$ while $S' = B' \cup C$, where $B' = \Delta_J R \Delta_J^{-1} \cup D$. Notice that there is an isomorphism $(A_B, B) \to (A_{B'}, B')$ (see e.g. [BMMN02, Definition 4.4]), so for every $Y' \subseteq S'$ spanning a clique in $\Gamma_{B'}$ (resp. Γ_C), $(A_{Y'}, Y')$ has the vertex ribbon property by assumption. Furthermore any clique of $\Gamma_{S'}$ is contained in either $\Gamma_{B'}$ or Γ_C, as no $b' \in B' - D$ is adjacent to any $c \in C - D$ in $\Gamma_{S'}$. Finally, (A, S') satisfies the vertex ribbon property by Corollary 4.9. □

We are now ready to prove that, if we assume the vertex ribbon property for cliques, then we can promote the generalised twist equivalence from Theorem 3.7 to a genuine twist equivalence.

Theorem 4.11. *Let (A, S) be an Artin system, with A one-ended. Suppose that the following hold:*

- *For any $X \subseteq S$ spanning a big chunk in Γ_S, (A_X, X) satisfies the strong twist conjecture;*
- *For any $Y \subseteq S$ spanning a clique in Γ_S, (A_Y, Y) satisfies the vertex ribbon property.*

Then (A, S) satisfies the strong twist conjecture.

Proof. Theorem 3.7 produces a sequence $S = S_0, \ldots, S_k = U$ of Artin generating sets such that, for every i, S_{i+1} is obtained from S_i by a conjugation, an elementary twist, or a Dehn Twist (with respect to S_i).

It is now enough to inductively prove that every S_i is twist equivalent to S; we shall also prove that, for every $Y \subseteq S_i$ spanning a clique in Γ_{S_i}, (A_Y, Y) satisfies the vertex ribbon property. There is nothing to prove for the base case $S_0 = S$. Now assume that S_i satisfies the inductive hypothesis. By Corollary 4.9, (A, S_i) has the vertex ribbon property, so Proposition 4.5 yields that any Dehn twist of A (with respect to S_i) is a compositions of elementary twists. Thus $S_{i+1} \sim S_i \sim S$; moreover, by Lemma 4.10, whenever $Y' \subseteq S_{i+1}$ spans a clique in $\Gamma_{S_{i+1}}$, $(A_{Y'}, Y')$ has the vertex ribbon property. This concludes the inductive step, and the proof of Theorem 4.11. □

4.1. **New examples of the strong twist conjecture.** We now collect several examples of Artin groups satisfying the strong twist conjecture, by gathering results from the literature.

Lemma 4.12. *Let (A, S) be an Artin system of large type (i.e. for every $a, b \in S$, $m_{ab} \geqslant 3$). Suppose that Γ_S is connected and has no separating vertex. Assume further that (A, S) is either:*

(1) a dihedral Artin group;
(2) of type XXXL (i.e. for every $a, b \in S$, $m_{ab} \geqslant 6$);
(3) triangle-free (i.e. for every $a, b, c \in S$, $\max\{m_{ab}, m_{bc}, m_{ac}\} = \infty$);
(4) free-of-infinity (i.e. for every $a, b \in S$, $m_{ab} < \infty$).

Then (A, S) satisfies the strong twist conjecture.

Proof. Let U be an Artin generating sets for A such that $R_S = R_U$. By [MV24, Corollary B], $\Gamma_S \sim \Gamma_U$; therefore, up to replacing U by a twist-equivalent generating set, we can assume that $\Gamma_S \cong \Gamma_U$, that is, $U = \varphi(S)$ for some $\varphi \in \mathrm{Aut}(A)$. We split the proof into the various cases from the list above.

Case (1): Let $S = \{a, b\}$ and let $m = m_{ab}$. If m is odd, then [GHMR00, Theorem C] gives that $\mathrm{Out}(A) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the global inversion, and therefore φ must be a conjugation since $R_S = R_{\varphi(S)}$.

Now assume that m is even. Up to composing φ with a conjugation and the graph automorphism swapping a and b, we can assume that $\varphi(a) = a$. Then, by e.g. inspecting the proof of [Jon24, Lemma 2.9], one gets that φ is either trivial or $\varphi(b) = a^{-1}b^{-1}a^{-1}$. The latter cannot happen since $a^{-1}b^{-1}a^{-1}$ and b have different images in the abelianisation, contradicting that $R_S = R_{\varphi(S)}$.

In view of the above case, we henceforth assume that $|S| \geqslant 3$, and denote Γ_S simply by Γ.

Case (2): Let $\mathcal{G}$ be the collection of simplicial graphs obtained from Γ via a sequence of elementary edge twists. For every $\Gamma' \in \mathcal{G}$ let $S' \sim S$ be an Artin generating set such that $\Gamma' \cong \Gamma_{S'}$, and fix an isomorphism $\varphi_{S'}\colon A \to A$ consisting of a sequence of edge twists from S to S' (if $\Gamma' = \Gamma$ we choose $S' = S$ and $\varphi_S = \mathrm{id}$). Let $\mathcal{S}$ be the collection of such Artin generating sets.

Then [BMV24, Theorem 9.6] states that $\mathrm{Aut}(A)$ is generated by the following elements:

- The element $\overline{\sigma} \coloneqq \varphi_{S''}^{-1} \circ \sigma \circ \varphi_{S'}$ whenever $S', S'' \in \mathcal{S}$ and $\sigma\colon (A, S') \to (A, S'')$ is an isomorphism of Artin systems. By construction $\overline{\sigma}$ fixes S setwise, since σ identifies S' with S''.
- The element $\bar{t} \coloneqq \varphi_{S''}^{-1} \circ t \circ \varphi_{S'}$ for every $S', S'' \in \mathcal{S}$ and every edge twist t taking S' to S''. By construction $\bar{t}$ is a composition of edge twists.
- Inner automorphisms.
- The global inversion i mapping every generator $s \in S$ to s^{-1}.

The subgroup $N \leqslant \mathrm{Aut}(A)$ generated by the first three types of elements is the kernel of the map $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ induced by mapping every generator $s \in S$ to 1. Moreover $\varphi \in N$, as $\varphi(s)$ is conjugated to s for every $s \in S$, thus $\varphi(S) \sim S$ by construction. This proves that (A, S) satisfies the strong twist conjecture.

Case (3): By combining [Cri05, Theorems 1 and 2], the automorphism group of A has exactly the same description as in the XXXL case, so the strong twist conjecture for (A, S) follows identically.

Case (4): By [Vas25, Theorem A], the automorphism group of A is generated by graph automorphisms, the global inversion, and conjugations; hence a simplified version of the proof above yields the strong twist conjecture for (A, S) in this case. □

Remark 4.13. In fact, if (A, S) is large-type triangle-free, the arguments of [Cri05] show that (A, S) satisfies the strong twist conjecture even if Γ_S has a separating vertex.

Lemma 4.14. *Let (A, S) be a right-angled Artin system (i.e. for all $a, b \in S$, $m_{ab} \in \{2, \infty\}$), with A one-ended. Then (A, S) satisfies the strong twist conjecture.*

Proof. Let U be an Artin generating set for A such that $R_S = R_U$. By combining [Dro87] and [Bau81], there is an isomorphism of Artin systems $\varphi\colon (A, S) \cong (A, U)$. Furthermore, by [Lau95] (see also [Ser89]), $\mathrm{Aut}(A)$ is generated by graph automorphisms, transvections, inversions, and partial conjugations (all with respect to S). Since the commutator subgroup is characteristic, the abelianisation map $\mathrm{Ab} : A \to \mathbb{Z}^n$ induces a map $\mathrm{Ab}_* : \mathrm{Aut}(A) \to \mathrm{GL}(n, \mathbb{Z})$. One easily checks that the subgroup $H \leqslant \mathrm{Aut}(A)$ preserving R_S is the preimage under Ab_* of

$$\mathrm{Ab}_*(\langle\{\sigma\colon (A, S) \to (A, S) \text{ is a graph automorphism}\}\rangle),$$

so since $\varphi \in H$ it must lie in a coset of the form $\ker(\mathrm{Ab}_*)\sigma$ for some graph automorphism σ. It is now enough to notice that σ fixes S setwise, and, by [Lau95, Theorem 2.2], $\ker(\mathrm{Ab}_*)$ is generated by the partial conjugations, which are exactly elementary twists in the sense of Definition 3.2. This proves that $U = \varphi(S) \sim S$. □

Now we turn to Artin systems of spherical type. Restricting to indecomposable systems (where there is no visual decomposition as a direct product) there are four infinite families. One family is the dihedral Artin groups, which we have already considered. Spherical systems in the other three families also satisfy the strong twist conjecture. The only exception is D_5, whose outer automorphism group is not known. In Figure 6, we list the Coxeter graphs of the Artin groups of type A_n, B_n and D_n.

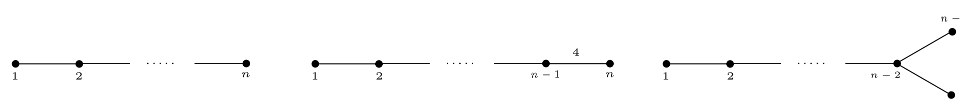

FIGURE 6. From left to right, the Coxeter graph of the spherical Artin group of type A_n, B_n and D_n. The notation is different from the one we have been using throughout: here non-adjacent vertices in the Coxeter graph correspond to commuting generators, while unlabelled edges correspond with braid relations of length 3.

Lemma 4.15. *Let (A, S) be a spherical Artin system of type A_n, with $n \geqslant 3$; B_n, with $n \geqslant 3$; or D_n, with $n \geqslant 4$ and $n \neq 5$. Then (A, S) satisfies the strong twist conjecture.*

Proof. Write W_S for the quotient Coxeter group, obtained as the quotient of A with kernel $K_S := \langle\langle s^2 \mid s \in S\rangle\rangle$. The group W_S is finite since (A, S) is spherical. Suppose U is an Artin generating set of A with $R_S = R_U$, and define W_U and K_U

as before. Since $R_S = R_U$, notice that $K_S = K_U$, so $W_S \cong W_U$ and in particular W_U is finite. Hence (A, U) is also spherical by definition.

By Paris' solution to the isomorphism problem within the class of spherical Artin groups, it follows that $\Gamma_S \cong \Gamma_U$ [Par04], so there is an automorphism $\psi : A \to A$ such that $\psi(S) = U$.

We will now show that $[\psi] \in \mathrm{Out}(A)$ has a representative that simply permutes S. Since such an automorphism fixes S setwise, and replacing S by a conjugate generating set is a twist equivalence, this will complete the proof.

We now divide into cases based on the isomorphism type of A. Given $\varphi_1, \varphi_2 \in \mathrm{Aut}(A)$, we will write $\varphi_1 \sim \varphi_2$ if $[\varphi_1] = [\varphi_2]$ in $\mathrm{Out}(A)$.

Case A_n: In this case $\psi \sim \iota^i$ for $i \in \{0, 1\}$, where ι inverts each element of S [DG81]. Since $R_U = R_S$, it cannot be that $i = 1$, since then a generator in S would be conjugate to the inverse of a generator in S, contradicting the existence of the map to $\mathbb{Z}$ sending each $s \in S$ to 1.

Case D_n, $n \neq 5$: In this case $\psi \sim \sigma\iota^i$, for $i \in \{0, 1\}$, where ι is as before and σ is a possibly trivial permutation of the generators [Sor21, CP24]. Since σ does not invert any generators, it must be that $i = 0$ for the same reason as in the previous case.

Case B_n: This case is more complicated. Write $r_1, \dots, r_n$ for the standard generating set, where $m_{r_{n-1}r_n} = 4$, and set $\Delta_B = (r_1 \dots r_n)^n$ and $\delta = r_{n-1} \dots r_1 r_1 \dots r_{n-1}$. By inspecting the proof of [CC05, Proposition 10] and the consequent remark, $\psi \sim T^k \iota^j \mu^i$, for $k \in \mathbb{Z}$ and $i, j \in \{0, 1\}$, where ι is as above while T and μ have the following forms (in each case $1 \leqslant \ell \leqslant n - 1$):

$$T\colon \begin{cases} r_\ell \mapsto r_\ell \Delta_B \\ r_n \mapsto r_n \Delta_B^{-(n-1)} \end{cases} \qquad \text{and } \mu\colon \begin{cases} r_\ell \mapsto r_\ell^{-1} \\ r_n \mapsto \delta r_n \end{cases}$$

(Via different methods, Paris and Soroko recovered the same result for the case $n \geqslant 5$ [PS25, Corollary 2.6]. The reader should notice that the generators extracted from the proof of [CC05, Proposition 10] and those presented in the statement of [PS25, Corollary 2.6] are the same, up to composition with inner automorphisms.)

We consider the induced action of $\mathrm{Out}(A)$ on the abelianisation, which is isomorphic to $\mathbb{Z}^2$ and generated by $\overline{r_1}$ (which is equal to $\overline{r_\ell}$ for $1 \leqslant \ell \leqslant n - 1$) and $\overline{r_n}$. Since $T^k \tau^j \mu^i$ preserves the generators setwise up to conjugacy, the action on the abelianisation can be at most a permutation of coordinates. Now, by looking at the action on the $\overline{r_1}$ coordinate, since $|\overline{\Delta_B}| > 2$, we see that $k = 0$ and $i - j = 0$. Now, by looking at the action on the second coordinate, we see that $i = 0$, so $j = 0$, completing the proof. □

Corollary 4.16. *Let (A, S) be an Artin system, with A one-ended. Suppose that, for every $X \subseteq S$ spanning a big chunk, (A_X, X) is as in one of Lemmas 4.12-4.14-4.15. Then (A, S) satisfies the strong twist conjecture.*

Proof. Lemmas 4.12-4.14-4.15 ensure that (A_X, X) satisfies the strong twist conjecture whenever $X \subseteq S$ spans a big chunk. Now let $Y \subseteq S$ span a clique, which must be contained in some big chunk, say spanned by $X \subseteq S$. Notice that if (A_X, X) is of large type (resp. spherical type, right-angled) then so is (A_Y, Y). Then (A_Y, Y) enjoys the vertex ribbon property, either by Theorem 4.4, or because a right-angled

Artin system with complete defining graph is a free abelian group with the standard generating set, for which the vertex ribbon property clearly holds. Hence the statement follows from Theorem 4.11. □

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