# Ultrafilters, congruences, and profinite groups

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Università di Pisa

based on joint work with
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- In this talk, we mainly adopt the latter.

# Definable types

## Review of the basics

$T$ complete, work in a monster, $a, b, \ldots, x, y, \ldots$ finite tuples

- $q(y) \in S(M)$ is *definable* iff each $d_q\varphi := \{d \in M : \varphi(y, d) \in q(y)\}$ is $M$-definable.

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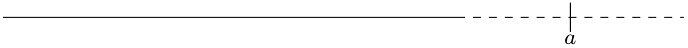
**Example** $M = (\mathbb{Q}, <)$ and $q(y) = \{y > d : d \in M\}$.



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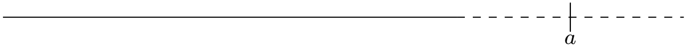

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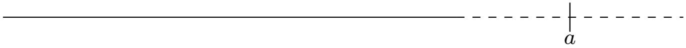

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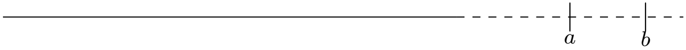

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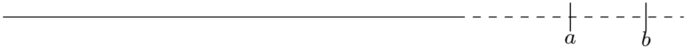

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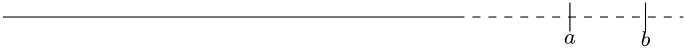


- $p \otimes q$ is itself definable and $\otimes$ is associative.

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Proof. Use Ellis theory to find $u \in \beta\mathbb{Z}$ with $0 < u = u \oplus u$. Pick the colour $A \in u$. Since $u = u \oplus u$, there is $z_0 \in A$ with $A \cap (A - z_0) \in u$.

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  FALSE in general, e.g. random graph (note: historically $\otimes$, $\oplus$ were first defined on ultrafilters)
- Spelling out $\oplus$, with $+$ the usual sum, $A \in p \oplus q \iff \{n : A - n \in q\} \in p$

A typical application in Ramsey theory/additive combinatorics:

### Theorem (Hindman)

$\forall$ finite colouring of $\mathbb{Z}$, there are $(z_i)_{i<\omega}$ s.t. all $z_{i_0} + \ldots + z_{i_n}$ have the same colour.

Proof. Use Ellis theory to find $u \in \beta\mathbb{Z}$ with $0 < u = u \oplus u$. Pick the colour $A \in u$. Since $u = u \oplus u$, there is $z_0 \in A$ with $A \cap (A - z_0) \in u$. Again since $u = u \oplus u$, there is $z_1 \in A \cap (A - z_0)$ with $A \cap (A - z_0) \cap (A - z_1) \cap (A - (z_0 + z_1)) \in u$. Repeat. $\square$

# Into the wild

## HIC SVNT TENSORES

- Let $M$ be $\mathbb{Z}$ with the *full* structure: every subset of every $\mathbb{Z}^n$ is $\emptyset$-definable.
- Over $M$, all types are $\emptyset$-definable. $S_1(M) = \beta\mathbb{Z} :=$ Space of ultrafilters over $\mathbb{Z}$.
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Want finite products instead? Work in $(\beta\mathbb{Z}, \odot)$.

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- Recap: $(\beta\mathbb{Z}, \oplus, \odot)$, applications in Ramsey theory/combinatorics.
- What's the next thing you do in arithmetic? (hint: look at the title of this slide)

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Write $u \equiv^{\mathrm{s}}_{w} v$ iff for some/all $(d, a, b) \vDash w \otimes u \otimes v$ we have $d \mid (a - b)$.

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Maybe there is a better notion of congruence?

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If $\mathrm{BD}(A) > 0$ and $A^{\complement}$ is thick, there are $u, v \in \beta\mathbb{Z} \setminus \mathbb{Z}$ such that $A \in u \oplus v$ and $A^{\complement} \in v \oplus u$.

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- Failure of transitivity: take any infinite $w$ containing all $n\mathbb{Z}+1$. Then $0 \equiv_w w \equiv_w 1$.

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The following are equivalent for $w \in \beta\mathbb{Z} \setminus \{0\}$:

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# Examples and non-examples

✗ Infinite primes: $w(x) \otimes w(y) \vdash |x| < |y|$ are primes!

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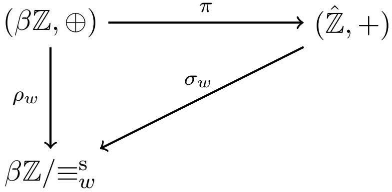

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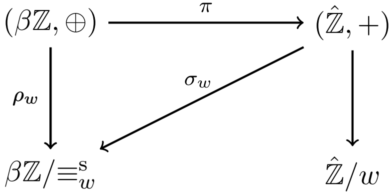

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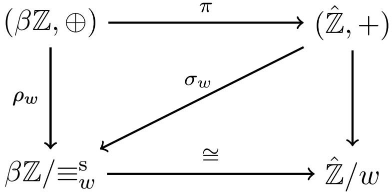

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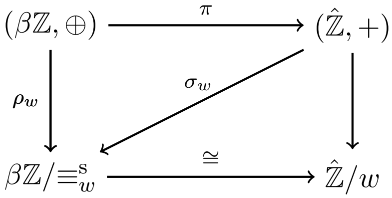


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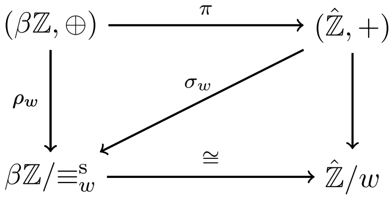


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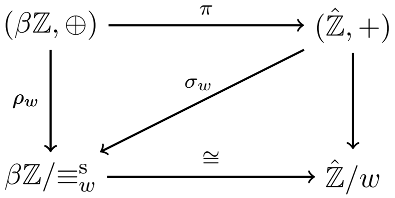


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$w$ is self-divisible iff $\ker \sigma_w$ is closed iff $\beta\mathbb{Z}/\equiv^{\mathrm{s}}_w$ is profinite iff $\beta\mathbb{Z}/\equiv^{\mathrm{s}}_w \cong \prod_{p\in\mathbb{P}} G_{p,w}$

$$G_{p,w} = \begin{cases} \mathbb{Z}/p^n\mathbb{Z}, & \text{if } n = \max\{k : p^k\mathbb{Z} \in w\} \\ \mathbb{Z}_p, & \text{otherwise.} \end{cases}$$

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$D(w) := \{n : n\mathbb{Z} \in w\}$

$w \,\tilde{\mid}\, u := u \equiv_w 0$

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$\varphi_w :=$ associated supernatural number

**Theorem 3.10.** *For every $w \in \beta\mathbb{Z} \setminus \{0\}$, the following are equivalent.*

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Thanks for listening!

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all the $\uparrow$
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details