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Neostability Theory

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Contents

Readme

Disclaimer

This notes have been typeset in LaTeX "on the fly" during the course "Neostability Theory" held by D. García at the University of Leeds in the fall of 2016/2017, and they have not been reviewed yet. They are primarily intended for personal use, and in particular they are *not* the official notes of the course. As a consequence, they can be *very* inaccurate, messy, and they may contain serious errors. Emails pointing out errors, mistakes, etc. are very welcome.

Deliberate omissions are marked [like this], while MISSING denotes that I was unable to transcribe something (which can be a single word, an entire theorem, etc.)

Info

You can find this notes on `http://poisson.phc.unipi.it/~mennuni/Mennuni_Neostability_Theory_notes.pdf` (but they could be moved; in case, check my Leeds webpage[1]). You can contact me at `mmrm@leeds.ac.uk`. This version has been compiled on December 30, 2016. To get the source code click on the paper clip.

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[1] Which does not exist yet, otherwise I would have linked that.

Chapter 1

21/09

1.1 Overview

We will have 4 weeks with4 hours and 6 with 2 hours.

We will work in first order logic. The contents/goals of the course will be:

- Stable theories: definability of types, the counting types theorems, the order property, the binary tree property,...
- Simple theories: the tree property, independence relations,...
- NIP (dependent) theories: VC-dimension, Keisler measures,...

1.2 Languages, Formulas, Theories, Structures,...

Definition 1.1. A language L is a collection of constant, function and relation symbols. By abuse of notation, L will also denote the set of formulas in the language L, i.e. of the form

$$\psi(x_1, \dots, x_n; y_1, \dots, y_m) \equiv \forall x_1 \exists x_2 \forall x_3 \dots \exists x_n \varphi(\bar{x}; \bar{y})$$

where $\varphi(\bar{x}; \bar{y})$ is a Boolean combination of basic relations (atomic formulas) in L with variables $(\bar{x}; \bar{y})$, of the form $t_1 = t_2$ or $R(t_1, \dots, t_k)$.

Definition 1.2. An L-structure M is given by interpreting every constant c with an element $c^M \in M$, every function symbol f_i of arity n_{f_i} with a function $M^{n_{f_i}} \to M$, and each relation symbol R_i of arity n_{R_i} with a subset $R_i^M \subseteq M^{n_{R_i}}$.

Example 1.3. $L = \{\} = \{=\}$. Here the terms are variables, the basic relations are of the form $x = y$. If M is an infinite set, it is an L-structure.

We will use standard abuses of notation as $\bigwedge_{i,j}$, $\neq$, etc.

Definition 1.4. *Definable sets* are solutions of formulas in M.

Example 1.5. Consider the formula $\varphi(x, y) \equiv x = y$. In M^2 this formula defines the diagonal:

$$\varphi(M^2) = \{(a, b) \in M^2 \mid M \vDash \varphi(a, b)\} = \{(a, b) \in M^2 \mid a = b\}$$

Definition 1.6. Given $A \subseteq M$ we define $L(A) = L \cup \{c_a \mid a \in A\}$, where each c_a is a constant symbol.

This allows to use parameters and have formulas like[1] $\varphi(x, y) \equiv x = a$ or $\psi(x, y) \equiv y = b$ where $a, b \in M$. Notice that $x = a$ does not mention the variable y. Anyway we can pad it by rewriting as $x = a \wedge y = y$. In general we will write something like $\varphi(x, y)$ to denote that we are regarding the defined set as a subset of M^2, even if φ does not mention all of its free variables.

Definition 1.7. A *sentence* is a formula without free variables. A *theory* is a consistent set of sentences.

Example 1.8. Let $L = \{R\}$, where R is a binary relation symbol, and let M be an infinite graph, where R is interpreted as the edge relation. Examples of formulas here are

- xRa
- $\forall x\ (\neg xRx)$
- $\forall x, y\ (xRy \to yRx)$

(Models of the last two axioms will be called graphs/simple graphs/undirected graphs/no loops, etc.)

Example 1.9. Consider $\{P_{m,n} \mid m, n \geq 1\}$, where

$$P_{m,n} \equiv \forall x_1, \dots, x_m\ \forall y_1, \dots, y_n \exists z\ \bigwedge_{i,j} x_i \neq y_j \to \bigwedge_{i,j} zRx_i \wedge \neg zRy_j$$

These axioms, together with the graph axioms, give the theory of the *random graph*. We say "the" theory of the random graph because this theory is *complete*.

Definition 1.10. A theory T is *complete* if for every L sentence σ either $T \vdash \sigma$ or $T\sigma\neg\sigma$. Since our theories will implicitly be considered deductively closed we will also write $\sigma \in T$ or $\neg\sigma \in T$.

Example 1.11.

[1] We will write a instead of c_a.

- Let $L_{\text{groups}} = \{\cdot, ^{-1}, e\}$, symbols for respectively a binary function, a unary function and a constant. We could have only used $\{\cdot\}$ but the larger language has the following advantage: if G is a group and H is an L_{groups}-substructure of G, then $H \leq G$.[

- $L_{\text{rings}} = \{+, \cdot, 0, 1, -\}$

- $L_{\text{ordered rings}} = \{+, \cdot, 0, 1, -, <\}$

Notice that if M is infinite and $L = \{=\}$, the definable subsets of M^1 (i.e. in one variable) are either finite or cofinite, as can be seen by proving quantifier elimination[2] via induction on formulas.

Definition 1.12. If all the definable subsets of every model of a theory are finite or cofinite, we call the theory *strongly minimal.*

Fact 1.13. Let $M = (\mathbb{C}, +, \cdot, 0, 1, -)$. Then:

- M is strongly minimal

- $\mathrm{Th}(M)$ has quantifier elimination and is axiomatised by ACF_0, the theory of algebraically closed fields of characteristic 0.

- The definable sets are Boolean combinations of basic relations, which in this case are of the form $t_0 = t_1$, for t_0, t_1 terms, i.e. an equality of two polynomials, or WLOG $p(x_0, \dots, x_n) = 0$. These are called *constructible sets.*

- Notice that, even if in one variable the definable sets are the same as the ones definable in the trivial language $\{=\}$, in higher dimensions you have new ones.

Fact 1.14. Let $M = (\mathbb{R}, +, \cdot, 0, 1, -, <)$. Then:

- (Tarski): $\mathrm{Th}(M)$ has quantifier elimination.

- The definable sets are Boolean combinations of solutions of equalities and inequalities of polynomials. These are called *semialgebraic sets.*

- By quantifier elimination, definable subsets of M^1 are finite unions of intervals and points.

Definition 1.15. If T has an order and is such that, in every of its models, the 1-dimensional definable sets are finite unions of intervals and points, we call T *o-minimal.*

[2] I.e. that every formula is equivalent to a quantifier-free one.

Remark 1.16. Let M be an L-structure. If we enrich the language to $L' = L \cup \{R_\varphi(\bar{x}) \mid \varphi(\bar{x}) \in L\}$ and interpret the new relation symbols in the natural way, i.e. $\models R_\varphi(\bar{a}) \Leftrightarrow M \models \varphi(\bar{a})$, then the L'-theory of M trivially eliminates quantifiers. But this is basically useless, as the purpose of quantifier elimination is to understand definable sets, and this construction gives no new information. But finding an intermediate language in which we still have quantifier elimination *and* understand the definable sets can be useful[3].

1.3 Types

Definition 1.17. Let $M \models T$ and $A \subseteq M$. A *(partial) n-type* over A (in M)[4] is a finitely satisfiable (in M) set of formulas with n free variables in the language $L(A)$.

Example 1.18. Let $M = \mathbb{R}$. The set $\{x^2 = 1+1\}$ is a 1-type over $\emptyset$. The set $\{x = \pi, y^2 = x\}$ is a 2-type over $\{\pi\}$. A more interesting example is $\{n \cdot x < 1 \mid 0 \neq n < \omega\}$, which is the same as saying $\{x < 1/n \mid 0 \neq n < \omega\}$. This is a 1-type over $\emptyset$, and it is satisfied by every nonpositive element in $\mathbb{R}$. If we instead consider

$$\{0 < n \cdot x < 1 \mid 0 \neq n < \omega\}$$

this is finitely satisfiable in $\mathbb{R}$ (hence a type), but no element of $\mathbb{R}$ satisfies the whole type.

Definition 1.19. If $M \subseteq N$, we say that M is an *elementary substructure* of N and write $M \prec N$ if every $L(M)$ formula true in M (we denote this theory with $\operatorname{El Diag}(M)$)[5] is true in N.

[Tarski-Vaught test]

Fact 1.20. If $\pi(x)$ is a type in M over A, then there is $M' \succ M$ that realises $\pi(x)$.

Proof. Apply compactness to the $L \cup \{c\}$ theory $\operatorname{El Diag}(M) \cup \pi(c)$. □

1.4 Saturation, Monster Models

Definition 1.21. Let κ be a cardinal. We say that M is *κ-saturated* if for every $A \subseteq M$ with $|a| < \kappa$ and every n-type $\pi(\bar{x})$ over A, M realises $\pi(\bar{x})$.

[3] Finding a minimal language with quantifier elimination should be called something like *Morleyzation* of a theory.

[4] But we will omit M soon.

[5] But warning: in the official notes this is denoted with $\operatorname{Diag}(M)$. In the literature Diag has sometimes another meaning, so I prefer to use this other notation.

Example 1.22.

If $L = \{=\}$ and T is the theory of infinite set, if $|M| = \kappa$, then M is κ-saturated.

$\mathbb{R}$ with the ordered field structure is not $\aleph_0$-saturated since we already showed there is a non-realised $\emptyset$-type.

Theorem 1.23. Let M be a structure and κ a cardinal $\geq |L|$. Then there is a κ-saturated $N \succ M$ such that $|N| \leq |M|^\kappa$.

A monster model is, ideally, a model $\mathbb{M}$ that embeds all models $M \vDash T$ and realises all types over subsets of $\mathbb{M}$ (and more things). This is of course impossible, so we need to take a different approach. Some books say that $\mathbb{M}$ is κ-saturated with "big enough" κ, where "big enough" means "bigger than all models you are interested in". But what if you are interested in *all* models? So our approach would be to consider κ to be bigger than all models you use in a proof. For example we will say " if $M \vDash T$ and[6] $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ for some $A \subseteq M$, then there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(a) = b$". This is just to avoid to say "there is $N \succ M$ such that (same thing with $\mathbb{M}$ replaced by N", which could be messy if we have a lot of different models in the same proof. Instead, we just use a big $\mathbb{M}$ where to embed all we will use. This is analogous to what you do in calculus where you first assume there is a δ doing stuff and then estimate the size of δ. The same can be done with κ here.

Indeed, this is not the full story. The point is that we would like to have a $|\mathbb{M}|$-saturated $\mathbb{M}$. This is the best we could hope for because

Fact 1.24. M cannot be $|M|^+$-saturated: just consider the type $\{x \neq m \mid m \in M\}$.

Anyway $|M|$-saturated M do not always exist. Sometimes they do under additional hypotheses, such as GCH (try to plug it in Theorem 1.23. . .). Another approach is to assume the existence of a *regular* cardinal $\kappa \geq \aleph_1$ such that for all $\lambda < \kappa$ $2^\lambda < \kappa$ (a *strongly inaccessible*) cardinal).

Anyway, let's see the proof of Theorem 1.23:

Proof.

Claim. There is $M' \succ M$ such that M' realises all 1-types over subsets of M of cardinality $\leq \kappa$, and $|M'| \leq |M|^\kappa$.

Proof of Claim. First notice that realising all 1-types is sufficient: if you have an n-type $p(x_1, \ldots, x_n)$ you just need to realise first all formulas which only involve x_1, then add a realisation to the parameters and iterate.

Also, notice that $|\{A \subseteq M \mid |A| \leq \kappa\}| \leq |M|^\kappa$. Given $A \subseteq M$, $|A| \leq \kappa$, then the space of *complete* types $S_1^M(A)$ has cardinality $\leq 2^{|L|+|A|} = 2^\kappa$.

[6] $\mathrm{tp}(b/A)$ are all $L(A)$-formulas satisfied by b.

Hence there are at most $|M|^\kappa$ possible types over subsets of M of cardinality $\leq \kappa$. Let us thus enumerate this set as $\{p_\alpha(x) \mid \alpha < |M|^\kappa\}$.

We are now going to construct a chain of models this way: we set $M_0 = M$, for nonzero limit α we set $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, and $M_{\alpha+1}$ realises p_α and $M_{\alpha+1} \succ M_\alpha$, $|M_{\alpha+1}| \leq |M_\alpha|^\kappa$. We do this with compactness plus Löwenheim-Skolem. Then we set $M' = \bigcap_{\alpha<|M|^\kappa} M_\alpha$. Since a union of an elementary chain of models is an elementary extension of each element of the chain, $M' \succ M$, and every p_α is realised by construction. $\square$ CLAIM

To complete the proof, we do another inductive construction: we let $N_0 = M$. Set $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ for nonzero limit α, and set $N_{\alpha+1} = (N_\alpha)'$, where the latter is given by the claim. This is again an elementary extension of M because it is a union of an elementary chain. Let now $A \subseteq N$ be of cardinality $< \kappa$. The idea now would be: if $A \subseteq N_\alpha$ for some α then $N_{\alpha+1} \subseteq N$ realises every type with parameters in A. (there is some technical problem with ensuring this, which we will fix, but modulo this the proof is finished) The fix is that the cofinality of $|M|^\kappa$ is greater than κ, which can be proved via König's Lemma: if $f: \kappa \to |M|^\kappa$ is cofinal we have

$$|M|^\kappa = \kappa = \sum_{i<\kappa} f(i) < \prod_{i<\kappa} |M|^\kappa = (|M|^\kappa)^\kappa = |M|^\kappa$$

which is clearly absurd. $\square$

Chapter 2

23/09

Next week we will receives notes, exercises, bibliography, etc.

[finished the proof or Theorem 1.23 (reported on last lesson's notes), and Tarski-Vaught]

2.1 Remarks on The Monster Model

The monster $\mathbb{M}$ has this features:

- $\mathbb{M}$ is κ-saturated for "very big" κ.
- $\mathbb{M}$ is *strongly κ-homogeneous*, i.e. whenever $A, B \subseteq \mathbb{M}$, $|A|, |B| < \kappa$, and $f\colon A \to B$ is a partial elementary map, then f can be extended to an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$.
- $\mathbb{M}$ is κ-universal: if $M \vDash T$ and $|M| < \kappa$ then there is an elementary embedding $f\colon M \to \mathbb{M}$, i.e. $M \cong f(M) \prec \mathbb{M}$. This in particular means that we cannot build a monster if T is not complete.

E.g. small subfields of $\mathbb{C}$ can be swapped by an automorphism. For time reasons we will not see how to ensure this. Anyway models constructed as in the proof of Theorem 1.23[1] *do* satisfy this properties. An idea of how you prove homogeneity is the following.

Assume $M = \bigcup_{i<\alpha} M_i$ where each M_i is saturated with respect to the cardinality of the previous ones. Suppose $A, B \subseteq M_\alpha$ (again, by cofinality reasons). You extend the map $f\colon A \to B$ by a back-and-forth argument using saturation.

Example 2.1. An example of $\aleph_0$-saturated model which is not $\aleph_0$-strongly-homogeneous is, in the theory of discrete linear orders without endpoints, $(\mathbb{Z} \times (\mathbb{R} \sqcup \mathbb{Q}))$ with the antilexicographical order: it is $\aleph_0$-saturated because

[1] They are called *special* models, or something like that.

you can show that $\aleph_0$-saturated models of that theory are of the form $\mathbb{Z} \times I$, for I a dense linear order. But $(0, 0_{\mathbb{R}}) \mapsto (0, 0_{\mathbb{Q}})$ is a partial elementary map that does not extend to an automorphism.

[details of the example above]

Definition 2.2.

- Let $\kappa(\mathbb{M})$ be the saturation of the monster. A set is *small* if it has cardinality less than $\kappa(\mathbb{M})$.
- We will write $\vDash \varphi(\bar{a})$ for $\mathbb{M} \vDash \varphi(\bar{a})$.
- Let $\Phi(\bar{x})$ and $\Psi(\bar{x})$ be small sets of formulas. We say that $\Phi(\bar{x}) \vdash \Psi(\bar{x})$ to mean that for all $\bar{a} \in \mathbb{M}$ such that[2] $\vDash \Phi(\bar{a})$, then $\vDash \Psi(\bar{a})$

Fact 2.3. Let $\varphi(\bar{x})$ be an $L(\mathbb{M})$ formula and $\Phi(\bar{x})$ be a consistent small set of $L(\mathbb{M})$-formulas. If $\Phi(\bar{x}) \vdash \varphi(\bar{x})$, then there is a finite set $\Phi_0(\bar{x}) \subseteq \Phi(\bar{x})$ such that $\Phi_0 \vdash \Phi$.

Proof. Assume otherwise, and define the type

$$\Gamma(\bar{x}) \coloneqq \Phi(\bar{x}) \cup \{\neg\varphi(\bar{x})\}$$

Since $\Phi \vdash \varphi$, the monster $\mathbb{M}$ has no realisations of Γ. However, Γ is a type over a small set of parameters, which is finitely consistent, so by compactness and saturation it should have realisations; this is a contradiction. $\square$

Fact 2.4. Let A be a small set. Then $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ if and only if there is[3] $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(a) = b$

Proof. The map being the identity on A and sending a to b is a partial elementary map by hypothesis. By strong homogeneity it extends to an automorphism which will be the required σ. $\square$

Lemma 2.5. Let X be a definable set. Then X is A-definable if and only if $\sigma(X) = X$ for every $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.

From now one we will write x or b even if x or b are tuples, instead of, say, $\bar{x}$ and $\bar{b}$.

Proof. $\Rightarrow$ Let $X = \varphi(\mathbb{M}, \bar{a})$, for some $\varphi(\bar{x}, \bar{y}) \in L$ and $\bar{a} \in A$. We have $\sigma(X) = \varphi(\mathbb{M}, \sigma(\bar{a}))$. But then

$$\begin{aligned} b \in \sigma(X) &\Leftrightarrow b = \sigma(b') \text{ for some } b' \in X \\ \Leftrightarrow \vDash \varphi(b', a) \text{ and } \sigma(b') = b &\Leftrightarrow \vDash \varphi(\sigma(b'), \sigma(a)) \Leftrightarrow \vDash \varphi(b, \sigma(a)) \end{aligned}$$

[2] I.e. $\bar{a}$ realises all formulas in $\Phi(\bar{x})$.

[3] I.e. an automorphism of $\mathbb{M}$ that fixes A pointwise.

So if $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ we have $\sigma(X) = \varphi(\mathbb{M}, \sigma(a)) = \varphi(\mathbb{M}, a) = X$.

$\Leftarrow$ Suppose $X = \varphi(\mathbb{M}, b)$ for some b. Consider $p(y) = \mathrm{tp}(b/A)$. We want to prove that $p(y) \vdash \forall x\big(\varphi(x,y) \leftrightarrow \varphi(x,b)\big)$. If $b' \vDash p(y)$ then $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$. This implies that there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(b) = b'$ and $\varphi(\mathbb{M}, b) = \varphi(\mathbb{M}, b')$. So $b' \vDash \forall x(\varphi(x,y) \leftrightarrow \varphi(x,b))$. Therefore there is a finite $p_0(y) \subseteq \varphi(y)$ such that $p_0(y) \vdash \forall x\big(\varphi(x,y) \leftrightarrow \varphi(x,b)\big)$. Let $\psi(y)$ be the conjunction of p_0. Consider

$$\theta(x) \coloneqq \exists y(\psi(y) \wedge \varphi(x,y))$$

which has parameters in A. Then

$$c \in \theta(\mathbb{M}) \iff \exists b' \; \mathbb{M} \vDash \psi(b') \wedge \varphi(c, b') \iff c \in \varphi(\mathbb{M}, b') = \varphi(\mathbb{M}, b) \iff c \in X$$

$\square$

Remark 2.6. The hypothesis that X is already definable is very important. For example, in DLO (Dense Linear Orders), consider a monster $\mathbb{M}$ and embed $\mathbb{Q}$ in it. Consider the set

$$X = \{m \in \mathbb{M} \mid 0 < x < 1/n \mid n < \omega\}$$

This set is not definable, but it is *type-definable*, over $\mathbb{Q}$ i.e. it is an infinite intersection of definable sets: just write $X = \bigcap_{n<\omega} 0 < x < 1/n$. Therefore if $\sigma \in \mathrm{Aut}(\mathbb{M}/\mathbb{Q})$ we have that $\sigma(X) = X$, even if X is not definable.

Definition 2.7. If X is such that for every $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ we have that $\sigma(X) = X$, we say that X is *A-invariant*.

The whole point is that we want to say "small" in the same spirit as we say "finite".

Remark 2.8. If for you "small" means "countable", then $\mathbb{C}$ is a monster model.

2.2 Almost A-definable Sets

As with completions and eliminations of quantifiers, you do not know if your theory has elimination of quantifier, but there is also a construction that yields a "sibling" theory which has it.

Notation 2.9. Sets of parameters will be often implicitly assumed to be small.

Lemma 2.10. Let $X \subseteq \mathbb{M}^n$ be definable and A be small. The following are equivalent:

1. X is *almost A-definable*, i.e. there is an A-definable equivalence relation E with finitely many classes and X is the union of some of those.

2. The set $\{\sigma(X) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite.

3. The set $\{\sigma(X) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.

Proof. $1 \Rightarrow 2$ Assume $E(x, y; a)$ is the formula witnessing that X is almost A-definable. Then $X = \bigcup_{i=1}^{n} E(\mathbb{M}, b_i; a)$, and $\{E(\mathbb{M}, b_j; a) \mid 1 \leq j \leq m + n\}$ is the full set of classes in $\mathbb{M}$. Notice that if $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ then $\sigma(E(\mathbb{M}, b_i; a)) = E(\mathbb{M}, \sigma(b_i); a) = E(\mathbb{M}, b_j; a)$. Therefore $|\{\sigma(X) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}| \leq \binom{n+m}{n}$.

$2 \Rightarrow 3$:)

$3 \Rightarrow 1$ Let $X = \varphi(\mathbb{M}, b)$, so $\{\sigma(X) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\} = \{\sigma(\varphi(\mathbb{M}, \sigma(b))) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$. By hypothesis, there is a small number $\alpha < \kappa(\mathbb{M})$ of realisations $\langle b_i \mid i < \alpha\rangle$ of $\mathrm{tp}(b/A)$ such that for any $b' \models \mathrm{tp}(b/A)$ there is $i < \alpha$ such that $\varphi(\mathbb{M}, b') = \varphi(\mathbb{M}, b_i)$.

Claim. [4] In fact, there are finitely many realisations $b_1, \ldots, b_k$ of $\mathrm{tp}(b/A)$ such that

$$\forall b' \models \mathrm{tp}(b/A)\ \exists i \leq k\ \varphi(\mathbb{M}, b') = \varphi(\mathbb{M}, b_i)$$

Proof of Claim. Let

$$\Gamma(y) \coloneqq \mathrm{tp}(b/A) \cup \{\neg\forall x\big(\varphi(x, y) \leftrightarrow \varphi(x, b_i)\big) \mid i < \alpha\}$$

This is the same as

$$\Gamma(y) = \mathrm{tp}(b/A) \cup \{\neg\forall x\big(\varphi(\mathbb{M}, y) \neq \varphi(\mathbb{M}, b_i)\big) \mid i < \alpha\}$$

If the Claim is not true, Γ is finitely consistent, and since Γ is a type over a small set of parameters, it has a realisation in $\mathbb{M}$. This is a contradiction. $\square_{\text{CLAIM}}$

This means that $\mathrm{tp}(b/A) = p(y) \vdash \bigvee_{i=1}^{k} \forall x\big(\varphi(x, y) \leftrightarrow \varphi(x, b_i)\big)$. Therefore there is $\psi(y)$ such that $\psi(y) \vdash \bigvee_{i=1}^{k} \forall x\big(\varphi(x, y) \leftrightarrow \varphi(x, b_i)\big)$. Define

$$E(x_1, x_2) \coloneqq \forall y\ \big(\psi(y) \to (\varphi(x_1, y) \leftrightarrow \varphi(x_2, y))\big)$$

in other words, $a_1 E a_2$ iff they agree on $\varphi(x, b_i)$ for all $i = 1, \ldots, k$. So each class can be codified by a function $s\colon \{1, \ldots, k\} \to \{0, 1\}$. Therefore

$$\varphi(\mathbb{M}, b) = \bigcup\{E_s \mid s(i) = 1 \text{ whenever } \varphi(\mathbb{M}, b_i) \cap \varphi(\mathbb{M}, b) \neq \emptyset\}$$

$\square$

[4] Which is 2, by the way.

Definition 2.11. We say that a is *algebraic* over A if there is $\varphi(x) \in L(A)$ such that

- $\models \varphi(a)$, and
- $\varphi(\mathbb{M})$ is finite.

We say that a is *definable* over A if there is $\varphi(x) \in L(A)$ such that

- $\models \varphi(a)$, and
- $\varphi(\mathbb{M}) = \{a\}$.

Example 2.12. In algebraically closed fields (say of characteristic 0) being algebraic over A just means being algebraic over A in the algebraic sense, and being definable over A means being in the field generated by A.

Definition 2.13. The *algebraic closure* $\mathrm{acl}(A)$ of A is[5] $\{b \in \mathbb{M}^1 \mid b \text{ is algebraic over } A\}$. The *definable closure* $\mathrm{acl}(A)$ of A is $\{b \in \mathbb{M}^1 \mid b \text{ is definable over } A\}$.

[5]Remember that in our notational conventions b can be in general a tuple, so the $\mathbb{M}^1$ is to stress that it is a single element (in Definition 2.11 it could be a tuple.).

Chapter 3

14/10

Remark 3.1. In Lemma 2.10 there is another equivalent statement, namely "X is M^{eq}-definable. We will see the definition now.

3.1 Imaginaries and M^{eq}

Definition 3.2. Let M be and L-structure and $T = \mathrm{Th}(M)$. Let $\mathrm{ER}(T)$ be the set of $\emptyset$-definable equivalence relations. We define

$$L^{\mathrm{eq}} \coloneqq L \cup \{\underbrace{S_E \mid E \in \mathrm{ER}(T)}_{\text{new sorts}}\} \cup \{\underbrace{f_E \mid E \in \mathrm{ER}(T)}_{\text{functions}}\}$$

[recap on multi-sorted structure and difference between vector spaces with a single sort and functions for scalar multiplication and vector spaces with a separate sort for the field]

Note that equality is an $\emptyset$-definable equivalence relation, so we have a sort $S_=$ which will, when interpreted, be isomorphic to M.

Definition 3.3. We define M^{eq} this way: each sort S_E is interpreted as

$$S_E \coloneqq \{\bar{a}/E \mid E \in \mathrm{ER}(T) \text{ of arity } n, \bar{a} \in M^n\}$$

Moreover we have function symbols f_E interpreted as $f_E \colon M^n \to S_E$ sending $\bar{a}$ to $\bar{a}/E$.

We call $S_=$ the *home sort*. Its elements are called *real elements*. Elements in other sorts are called *imaginary*.

Notice that all a/E are in $\mathrm{acl}^{\mathrm{eq}}(a)$, as witnessed by the formula $y = f_E(a)$.

What we would like is not to have imaginary elements, i.e. to define them in terms of the real ones.

Definition 3.4. We define T^{eq} to be the L^{eq}-theory

$$T^{\mathrm{eq}} = T \cup \{\forall y \in S_E\ \exists \bar{x} \in S_=\ (f_E(\bar{x}) = y)\} \cup \{\forall \bar{x}_1, \bar{x}_2\ (f_E(\bar{x}_1) = f_E(\bar{x}_2) \leftrightarrow E(\bar{x}_1, \bar{x}_2))\}$$

Lemma 3.5.

1. If $M = S_=$, then for every $\varphi(\bar{x}) \in L$ and $\bar{a} \in M$, we have $M \vDash \varphi(\bar{a}) \iff M^{\mathrm{eq}} \vDash \varphi(\bar{a})$.
2. If $M \vDash T$, then $M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$.
3. Every $M^* \vDash T^{\mathrm{eq}}$ is of the form $M^* = M^{\mathrm{eq}}$ for some $M \vDash T$.
4. $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$.
5. Given equivalence relations $E_1, \dots, E_k \in \mathrm{ER}(T)$ and $\varphi(x_1, \dots, x_k) \in L^{\mathrm{eq}}$ (with $x_i \in S_{E_i}$) there is some $\psi(\bar{y}_1, \dots, \bar{y}_k) \in L$ such that

$$T^{\mathrm{eq}} \vdash \forall \bar{y}_1, \dots, \bar{y}_k \in S_= \Big(\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))\Big)$$

We will skip the proof for now (but the first 4 points are easy). Point 5 of the Lemma is the real reason we define T^{eq}: the point is "eliminating imaginaries" in the same way as the Morleyzation of a structure eliminates quantifiers. Even in this case, the real deal is eliminating imaginaries in a language which is understandable enough. Let us give the precise definition of "eliminating imaginaries".

Definition 3.6. A theory T *eliminates imaginaries* if for every $M \vDash T$ and $e \in M^{\mathrm{eq}}$ there is $\bar{d} \in M$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{d})$ and $\bar{d} \in \mathrm{dcl}^{\mathrm{eq}}(e)$.

Example 3.7.

- Algebraically closed fields eliminate imaginaries.
- Infinite sets do not eliminate imaginaries.
- Vector spaces in the two-sorted language *do not* eliminate imaginaries: consider

$$x \in V \sim y \in V \iff \exists \lambda \in \mathbb{K} \setminus \{0\}\ (x = \lambda \cdot y)$$

The idea here is "every definable $X \subseteq \mathbb{M}^n$ corresponds to an imaginary", in the following sense: Let $X = \varphi(\mathbb{M}, \bar{b})$ and consider $E(\bar{y}_1, \bar{y}_2) := \forall \bar{x}\big(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2)\big)$. We have $\bar{b}/E_\varphi \in M^{\mathrm{eq}}$ and the following properties:

1. For all[1] $\sigma \in \mathrm{Aut}(\mathbb{M})$, we have $\sigma(X) = X$ iff $\sigma(\bar{b}/E_\varphi) = \bar{b}/E_\varphi$. This is because we have $\sigma(X) = X$ iff $\mathbb{M} \vDash \forall \bar{x}\ (\varphi(\bar{x}, \sigma(\bar{b})) \leftrightarrow \varphi(x, \bar{b}))$ iff $\sigma(\bar{b}) E_\varphi \bar{b}$ iff $\sigma(\bar{b}/E_\varphi) = \bar{b}/E_\varphi$.

[1] Note that automorphisms of $\mathbb{M}$ can be extended to automorphisms of $\mathbb{M}^{\mathrm{eq}}$ in the natural way (it is not obvious that it is well-defined on equivalence classes, but it is).

2. X is L^{eq}-definable over $\bar{b}/E$: consider

$$\psi(\bar{x}; z) \coloneqq \exists \bar{y}\ (\varphi(\bar{x}, \bar{y}) \wedge f_{E_\varphi}(\bar{y}) = z)$$

and plug $\bar{b}/E$ in place of z.

3. $\bar{b}/E$ is the unique element e in the sort S_E such that $\psi(\bar{x}, e)$ defines X. This is because if $\psi(\bar{x}, e)$ defines X then $\varphi(\bar{x}, \bar{b}')$ defines X and $f_E(b') = e$, and this means $e = \bar{b}'/E = \bar{b}/E$.

Definition 3.8. $\bar{b}/E_\varphi$ is called the *code* of X.

So the point of elimination of imaginaries is that definable sets have codes: here a definable set is fixed setwise by an automorphism iff it fixes its code.

Example 3.9. A code of a plane is the tuple of coefficients of the equations defining it.

Chapter 4

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Proof of Lemma 3.5. As anticipated, we are only going to prove the last point. The proof is by induction on φ. Is φ is atomic, then either $\varphi \in L$ (and there is nothing to do), otherwise it is of the form $f_E(\bar{x}) = f_E(\bar{y})$. In this case just let $\psi_\varphi(\bar{x}, \bar{y}) \coloneqq E(\bar{x}, \bar{y})$.

The $\neg$ and $\wedge$ cases are clear, so we are left to deal with $\exists$. Assume $\varphi(\bar{x})$ has the form $\exists z\ \Phi(\bar{x}, z)$. Modulo T^{eq}, this is equivalent to

$$\exists \tilde{z}\ \big(\Phi(\bar{x}, f_E(\tilde{z}))\big)$$

where $\bar{x}$ has sort $S_R = (S_{E_1}, \dots, S_{E_k})$ and z has sort S_E. By inductive hypothesis there is $\psi_\Phi(\bar{y}, \bar{w})$ such that

$$T^{\mathrm{eq}} \vdash \forall \bar{y}, \bar{w}\ \big(\psi_\Phi(\bar{y}, \bar{w}) \leftrightarrow \Phi(f_R(\bar{x}), f_E(\bar{w})\big)$$

We claim that then it is sufficient to set

$$\psi_\varphi \coloneqq \exists \bar{w}\ (\psi_\Phi(\bar{y}, \bar{w}))$$

In fact take $\bar{a}$ in the home sort of $\mathbb{M}^{\mathrm{eq}}$, arbitrary. Then

$$\begin{aligned}
\mathbb{M}^{\mathrm{eq}} \vDash \varphi(f_R(\bar{a})) \iff \exists \bar{b} \in \mathbb{M}\ \mathbb{M}^{\mathrm{eq}} \vDash \Phi(f_R(\bar{a}), f_E(\bar{b}))\\
\iff \exists \bar{b} \in \mathbb{M}\ \mathbb{M}^{\mathrm{eq}} \vDash \psi_\Phi(\bar{a}, \bar{b})\\
\iff \mathbb{M}^{\mathrm{eq}} \vDash \psi_\varphi(\bar{a})
\end{aligned}$$

□

Corollary 4.1. $\mathbb{M}$ is *stably embedded* in $\mathbb{M}^{\mathrm{eq}}$ (if you know the definition; anyway we will say it later).

Remark 4.2. If T eliminates imaginaries and $e, \bar{d}$ are as in the definition, up to taking conjunctions we can assume that interdefinability is witnessed by a single formula $\varphi(\bar{x}, y)$.

Some comments:

- Moving from T to T^{eq} preserves definable sets. But, for particular theories, what is the right[1] language in which you have elimination of imaginaries?[2]

Example 4.3.

- Some very trivial examples of imaginaries are diagonal tuples (a, a) or $(a, a, a, \dots, a)$ (where the equivalence relation is "being equal as a tuple").

- Finite sets: you can write a formula defining the equivalence relation $E_2((x_1, x_2), (y_1, y_2)) \coloneqq \{x_1, x_2\} = \{y_1, y_2\}$.

- Quotients: let G be a definable group and $H \leq G$ be $\emptyset$-definable. Consider $E_H(g_1, g_2) \coloneqq g_1 H = g_2 H$, which can be defined with $\exists h\ (H(h) \wedge g_1 h = g_2)$.

- Consider in $\mathbb{C} \vDash \mathsf{ACF}_0$ the formula $\varphi(x, y, a, b) \coloneqq a \cdot x + b \cdot y = 0$. Then consider the equivalence relation $E((z_1, z_2), (w_1, w_2)) \coloneqq \exists \lambda \neq 0\ (\lambda z_1 = w_1 \wedge \lambda z_2 = w_2)$. Then $(a, b)/E$ works as a code for the set defined by φ.

Fact 4.4. The theory T_∞ of infinite sets does not eliminate imaginaries: in particular, it does not eliminate $\{a, b\}$ for $a \neq b$.

Proof. Assume $e = \{a, b\}$ is interdefinable with a real tuple $\bar{d}$. In fact, if this was to be the case, any $\sigma \in \mathrm{Aut}(\mathbb{M}/e)$ would fix $\bar{d}$ by Lemma 2.5. However, no c is fixed by all these σ: if $c \neq a, b$, then we can just swap it with some other c' fixing a and b; if $c = a$, say[3], we can exchange a and b and $\{a, b\}$ would be fixed anyway. □

The trick with theories like ACF_0 is that you can encode finite sets with polynomials, and then you get elimination of imaginaries via strong minimality. E.g. encode $\{a, b\}$ with $x^2 - (a + b)x + ab$.

Proposition 4.5. The following are equivalent:

1. T eliminates imaginaries and there are at least two $\emptyset$-definable elements.

2. For every $\emptyset$-definable equivalence relation E on $\mathbb{M}^n$ there are m and a $\emptyset$-definable $f \colon \mathbb{M}^n \to \mathbb{M}^m$ such that $\bar{a} E \bar{b} \iff f(\bar{a}) = f(\bar{b})$.

[1] I.e. "minimal", or "more natural", or the like.
[2] Same as with elimination of quantifiers.
[3] The case $c = b$ is analogous.

Proof.

2 ⇒ 1 Let $e = \bar{a}/E$, and consider $\bar{d} = f(\bar{a})$. Let us show that they are interdefinable. Consider

$$\varphi(x, y) = \exists \bar{z}\ (f_E(\bar{z}) = y \wedge f(\bar{z}) = x)$$

We want to see that $\{\bar{d}\} = \varphi(\mathbb{M}^n, e)$ and $\{e\} = \varphi(\bar{d}, \mathbb{M}^{\mathrm{eq}})$, but this is just true by definition. To construct the two $\emptyset$-definable elements define

$$(x_1, x_2)E(y_1, y_2) \iff (x_1 = x_2 \leftrightarrow y_1 = y_2)$$

This has exactly two equivalence classes. Codify E with some f, and note that the image of f has exactly two elements, each of them $\emptyset$-definable by considering $f(x, x)$ and $f(x, y)$ for x and y two distinct[4] elements of $\mathbb{M}$.

1 ⇒ 2 Fix an $\emptyset$-definable equivalence relation E. For every $e = \bar{a}/E$ there is a tuple $\bar{d}_e \in \mathbb{M}^{m_e}$ and a formula $\varphi_e(\bar{x}, y)$ such that φ_e witnesses the interdefinability[5] of $\bar{d}$ and e. Now consider the fact that $\{\varphi_e(\bar{x}, y) \mid e \in S_E\}$ covers $S(\mathbb{M})$. By compactness there are formulas $\varphi_1(\bar{x}_1, y_1), \ldots, \varphi_k(\bar{x}_k, y_k)$ such that [...]. We therefore find sets $D_1, \ldots, D_k$ who partition $\mathbb{M}^n$ (the preimages of the appropriate imaginaries). This defines a function, but it could be not into a single $\mathbb{M}^m$ because the m_e depend on e. That's where you use interdefinability: we have $\{D_i \to \mathbb{M}^{m_i} \mid i \leq k\}$ and we amalgamate them in $\mathbb{M}^{m_1+\ldots+m_k+k}$ padding tuples with some "noise". Anyway the noise should be carefully chosen to avoid clashes, but his can be done up to taking larger k. We'll see in more detail in next lessons/the official notes of the course.[6] □

[4] We are tacitly assuming that we have at least two elements in our structure.

[5] Up to conjunctions: if $\{b\} = \zeta(a, \mathbb{M})$ and $\{a\} = \theta(\mathbb{M}, b)$, look at $\zeta(x, y) \wedge \theta(x, y)$.

[6] It is one of those usual coding tricks: use the two constants as 0s and 1s and look in the last k coordinates (probably $\lceil \log_2 k \rceil$ suffice) to know in which coordinates to look.

Chapter 5

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5.1 Indiscernible Sequences

This is somehow a generalisation of the behaviour of converging sequences.

Definition 5.1. Let I be a linear order. A sequence $\langle a_i \mid i \in I \rangle$ is A-*indiscernible* (or *indiscernible over* A) if for every formula $\varphi(x_1, \dots, x_n) \in L(A)$, whenever $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ we have $\models \varphi(a_{i_1}, \dots, a_{i_n}) \iff \models \varphi(a_{j_1}, \dots, a_{j_n})$.

So the idea is that ordered tuples all have the same A-type.

Example 5.2. These are some examples:

1. Take $\langle a_i = a \mid i \in I \rangle$.

2. Let $T = T_\infty$. Then any sequence $\langle a_i \mid i \in \omega \rangle$ with different elements $a_i \neq a_j$, each not in A, is A-indiscernible.

3. If $T = \mathsf{DLO}$, any increasing sequence $\langle a_n \mid n \in \omega \rangle$ is $\emptyset$-indiscernible. Note that here, if we permute the a_i, the type changes (this was not the case in the previous example): if $a_i < a_j$ and we swap them... In other words, for all $m < n$ we have $\mathrm{tp}(a_1, a_2) = \mathrm{tp}(a_m, a_n)$, but $\mathrm{tp}(a_1, a_2) \neq \mathrm{tp}(a_2, a_1)$. Also, if we take $A = \{b\}$ with $a_2 < b < a_3$, say, the sequence will not be A-indiscernible. In fact,

   Fact 5.3. A-indiscernible sequences in DLO are monotone sequences in a specific A-cut[1].

[1] An A-cut S is a subset of A closed under initial segments (i.e. downwards), and saying that something lies in S means that it's bigger than (all the stuff in) S but smaller than (all the stuff in) $A \setminus S$

4. If $A \subseteq M \vDash \mathsf{ACF}_0$, how does a non-constant indiscernible sequence $\langle a_i \mid i < \omega\rangle$ over A look like? Take a single element, say a_1. Since all the other ones must have the same A-type, and they are infinitely many and different, we cannot have $a_1 \in \operatorname{acl}(A)$. So a_1 is transcendental over A, and so are all the other a_i. Take now a_2 and consider $\operatorname{tp}(a_2/Aa_1)$. By indiscernibility, this must be the same as $\operatorname{tp}(a_j/Aa_1)$ for $j > 1$, and for the same reason as above it cannot be algebraic. Continuing this way, we realise that $\langle a_i \mid i < \omega\rangle$ is a sequence of A-algebraically independent transcendentals. In this case, we can still permute the sequence and get another A-indiscernible one (being algebraically independent is something that does not depend on the order). [definition of "transcendental" in fields]

Indiscernible sequences are a very useful tool: for example they come handy in inductive arguments because if you spot a relation between, say a_1, a_{20}, a_{55}, you can move it to a_1, a_2, a_3.

Theorem 5.4 (Ramsey's Theorem)**.** For every coloring c of the n-elements subsets $[\mathbb{N}]^n$ of $\mathbb{N}$ into k colors there is an infinite $I \subseteq \mathbb{N}$ such that $c \restriction [I]^n$ is constant. We say that I is *homogeneous*.

In *arrow notation*,

$$\aleph_0 \to (\aleph_0)^n_k$$

Proof. Not here. □

Theorem 5.5 (Erdös-Rado)**.** In arrow notation,

$$(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$$

I.e. every time you color the $n+1$-elements subsets of $(\beth_n(\kappa))^+$ with κ colors there is an homogeneous subset of size κ^+.

Proof. Not here. □

Remark 5.6. There is the following estimate for *Ramsey numbers* (two colors and 2-subets) $R(k) > 2^{\frac{k}{2}}$ (e.g. $R(3) = 6$, $R(4) = 17$, i.e. $6 \to (3)^2_2$, $17 \to (4)^2_2$ and 6, 17 are minimal.).

5.2 Expanding and Shrinking Indiscernibles

We can use these theorems for *expanding and shrinking indiscernibles.*

Definition 5.7. Given a sequence[2] $I = \langle a_i \mid i < \omega\rangle$, the *Ehrenfeucht-Mostowski type* of I over A, denoted $\mathrm{EM}(I/A)$, is the set of formulas (in infinitely many variables[3]) $\varphi(x_1, \dots, x_n) \in L(A)$ such that for all $i_1 < \ldots < i_n$ we have $\vDash \varphi(a_{i_1}, \dots, a_{i_n})$.

[2] Actually, we can give the same definition with any infinite linear order in place of ω.
[3] ω of them, even if I is indexed on a bigger set.

So, if I is A-indiscernible, its Ehrenfeucht-Mostowski type is basically its type. But if you take, for instance, $1, 2, 1, 2, 1, 2, \dots$ in $\mathbb{Q}$, you will not get a complete type, but you still get formulas as $x > 0$.

Proposition 5.8. Let $\bar{a} = \langle a_i \mid i < \omega \rangle$ be an arbitrary sequence in $\mathbb{M}$ and A a small set of parameters. Then, for any small linear order I there is an A-indiscernible sequence $\langle b_i \mid i \in I \rangle$ such that whenever Δ is a finite set of $L(A)$-formulas, there are $j_1 < \ldots < j_n$ in ω such that for all $\varphi \in \Delta$

$$\vDash \varphi(b_1, \ldots, b_n) \iff \vDash \varphi(a_{j_1}, \ldots, a_{j_n})$$

Proof. Let $L' = L \cup \{c_i \mid i \in I\}$ and let T' be the L'-theory given by:

1. $\{\varphi(c_{j_1}, \ldots, c_{j_n}) \in L(A) \mid \forall i_1 < \ldots < i_n \vDash \varphi(a_{i_1}, \ldots, a_{i_n})\}_{j_1 < \ldots < j_n \in I}$
2. $\{\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, j_n) \mid \psi \in L(A), i_1 < \ldots < i_n \in I, j_1 < \ldots, j_n \in I\}$

Notice that our Proposition is equivalent to the fact that T' is consistent: since everything is small, a model of T' can be embedded in $\mathbb{M}$, and the realisations of the c_js will be our b_js. Let us do it more precisely:

Claim. T' consistent is enough.

Proof of Claim. Since everything is small, then there are elements $\langle b_j \mid j \in I \rangle$ such that $\langle b_j \mid j \in I \rangle = \langle c_j^{\mathbb{M}} \mid j \in I \rangle$. By 2 above, $\langle b_j \mid j \in I \rangle$ is A-indiscernible.

Now let $\Delta \in \mathscr{P}_{\text{fin}}(L(A))$, consider $\varphi_\Delta = \bigwedge_{\psi \in \Delta} \psi$ and assume WLOG[4] that $\vDash \varphi_\Delta(b_1, \ldots, b_n)$. If for all $j_1 < \ldots < j_n < \omega$ we had $\vDash \neg\varphi_\Delta(a_{j_1}, \ldots, a_{j_n})$, then we would have had $\neg\varphi_\Delta \in \text{EM}(\langle a_i \mid i < \omega \rangle / A)$, so $\neg\varphi_\Delta$ would have been inn 1 above and we get the absurd $\vDash \neg\varphi_\Delta(b_1, \ldots, b_n)$. $\square_{\text{CLAIM}}$

We are now going to use compactness to show that T' is consistent. Let $T_0 \in \mathscr{P}_{\text{fin}}(T')$ and let[5] $\{\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_m\} = \Phi_0$ be the collection of all formulas mentioned in T_0, except we replace the constants c_j with variables x_j. There are at most 2^{k+m} types in the formulas from Φ_0. Now consider the coloring obtained this way: given $j_1 < \ldots < j_n$ in ω, define

$$c(j_1, \ldots, j_n) = \text{tp}_{\Phi_0}(a_{j_1}, \ldots, a_{j_n})$$

In other words, color the set of indexes $\{j_1, \ldots, j_n\}$ with the Φ_0-type realised by the corresponding a_js. By the instance $\aleph_0 \to (\aleph_0)^n_{2^{k+m}}$ of Ramsey's Theorem we have an infinite subsequence of the a_js all[6] with the same Φ_0-type, and this completes the compactness argument. $\square$

[4] Up to negating the ψs which do not hold for $b_1, \ldots, b_n$.

[5] Clearly, we mean that the φs are coming from 1 and the ψ from 2

[6] More precisely, we should speak of the ordered n-tuples coming from them.

Corollary 5.9. Let $\langle a_i \mid i \in I\rangle$ be a small A-indiscernible sequence and $J \supseteq I$ another small linear order. Then, there is an A-indiscernible sequence $\{b_j \mid j \in J\}$ in $\mathbb{M}$ extending it, i.e. such that for all $j \in I$ we have $a_j = b_j$.

Proof. Use the previous Proposition together with saturation an homogeneity of $\mathbb{M}$: first find some suitable $(b_j \mid j \in J)$ with the Proposition and then, since $\operatorname{tp}((a_i \mid i \in I)/A) = \operatorname{tp}((b_j \mid j \in I)/A)$ and everything is small, we can swap them with some element of $\operatorname{Aut}(\mathbb{M}/A)$. $\square$

5.3 Stable Formulas

Definition 5.10. Let T be a complete theory.

1. We say that $\varphi(\bar{x}, \bar{y})$ has the *k-order property* (in T) iff there are elements $\bar{a}_1, \dots, \bar{a}_k$ and $\bar{b}_1, \dots, \bar{b}_k$ in $\mathbb{M}$ such that $\models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j$.
2. We say that $\varphi(\bar{x}, \bar{y})$ is *unstable* (has the *order property*) if, for all $k < \omega$, it has the k-order property.
3. *Stable* means "not unstable".

Philosophically speaking, "$\varphi(x, y)$ has the order property if it encodes and infinite order"[7].

Example 5.11.

- It is easy to see that in any theory, the formula $x = y$ is stable: if $a_1 = b_2$ and $a_1 = b_3$, then we also should have $a_2 \neq b_2$ but $a_2 = b_3$, contradicting transitivity.
- The formula $x < y$ in DLO has the order property. Just take an increasing sequence $a_n = b_n$ for all $n \in \mathbb{N}$.
- Let T be the theory of an equivalence relation with infinitely many infinite classes. For the same reason as in the first example, xEy is going to be stable.
- If T is the theory of the random graph, the formula $\varphi(x, y) := xRy$ has the order property. Just choose the appropriate a_i and b_j inductively using the random graph axiom. Anyway, we will see later that here there is no formula defining an order with infinite chains.

Note that even unstable theories have a "stable part": for example we just said that $x = y$ is stable in every theory.

[7] The "philosophically" is because the infinite linearly order set will not in general be definable. But there will be something like a type-definable partial order with an infinite chain or stuff like that. Whether the statement is true depends on how you interpret "encodes".

Chapter 6

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6.1 Some Remarks On Stability

Remark 6.1. By compactness, $\varphi(\bar{x}, \bar{y})$ is unstable (in T) iff for some $M \vDash T$ there are $\langle \bar{a}_i, \bar{b}_i \mid i < \omega \rangle$ such that $M \vDash \varphi(\bar{a}_i, \bar{b}_j) \iff i < j$.

Exercise 6.2. You can assume (a_i) and (b_i) to be indiscernible.

Proposition 6.3. Assume $\varphi(\bar{x}, \bar{y})$ is unstable. Then for every linear order I there is $M \vDash T$ and $\langle \bar{a}_i, \bar{b}_i \mid i \in I \rangle \subseteq M$ such that $M \vDash \varphi(\bar{a}_i, \bar{b}_j) \iff i < j$ in I.

Proof. Compactness. Just write the set of formulas

$$\{\varphi(\bar{x}_i, \bar{y}_j) \mid i < j\} \cup \{\neg\varphi(\bar{x}_i, \bar{y}_j) \mid i \geq j\}$$

It suffices to show that this is finitely consistent. But each finite subset of this can easily be shown to be finitely consistent under our hypotheses. Then any $|I|^+$-saturated[1] $M \vDash T$ will realise our type. □

6.2 Counting Types

Definition 6.4. Given a[2] theory T, for each infinite cardinal κ define the *stability function* of T to be

$$g_T(\kappa) \coloneqq \sup\{|S_1(M)| \mid M \vDash T, |M| = \kappa\}$$

(actually it all started with Shelah noticing that the behaviour of this function is related to combinatorial patterns as the ones we saw above).

Definition 6.5. We define, for κ an infinite cardinal,

$$\operatorname{ded} \kappa \coloneqq \sup\{|I| \mid (I, <) \text{ is a linear order with a dense } J \subseteq I \text{ with } |J| = \kappa\}$$

[1] Actually $|I|$-saturation should suffice.

[2] Complete, as usual.

Fact 6.6. $\kappa \le \operatorname{ded}\kappa \le 2^\kappa$.

Proof. By density, every element of I is then determined by its cut in J. The lower bound is trivial. □

Example 6.7. $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$

Proposition 6.8. For all infinite cardinals κ, we have $\kappa < \operatorname{ded}\kappa$.

Proof. Assume μ is minimal such that $2^\mu > \kappa$. Consider the lexicographic order on[3] $2^{\le\mu}$. Let now $I = 2^{\le\mu}$ and $J = 2^{<\mu}$. By choice of μ, we have

$$|J| = |2^{<\mu}| = \sup\{2^\lambda \mid \lambda < \mu\} \le \kappa$$

On the other hand $|I| \ge 2^\mu$. It only remains to show that J is dense in I, but this is obvious: follow two functions until they coincide. At some point they have to split, and this point is less than μ and the corresponding node belongs to J. Then $\operatorname{ded}\kappa \ge 2^\mu > \kappa$. □

Proposition 6.9. If T is unstable, then for all $\kappa \ge |T|$ we have $g_T(\kappa) \ge \operatorname{ded}\kappa$.

Proof. Let[4] $\varphi(x, y)$ be an unstable formula in T. Let I be a dense linear order of size[5] $\operatorname{ded}\kappa$ with a dense subset J of size κ. Pick $\langle a_i, b_i \mid i \in I\rangle$ witnessing the order property for $\varphi(x, y)$. Then

$$|S_\varphi(\underbrace{\{b_j \mid j \in J\}}_{B})| \ge \operatorname{ded}\kappa$$

because if $a_i \ne a_{i'}$ are in I there is some $j \in J$ such that $i < j < i'$ so $\varphi(x, b_j)$ is in $\operatorname{tp}(a_i/B) \setminus \operatorname{tp}(a_{i'}/B)$. By Löwenheim-Skolem there is some $M \models T$ such that $B \subseteq M$ and $|M| = \kappa$. This shows that $|S_x(M)| \ge |S_\varphi(B)| \ge \operatorname{ded}\kappa$. □

6.3 On $\operatorname{ded}\kappa$

Obviously, GCH $\Rightarrow \operatorname{ded}\kappa \ge 2^\kappa$.

Fact 6.10 (Mitchell(1972))**.** If $\operatorname{cof}\kappa \ge \aleph_1$, then there is a cardinal preserving Cohen extension forcing $\operatorname{ded}\kappa < 2^\kappa$.

Fact 6.11 (Chernikov-Kaplan-Shelah)**.** Starting with GCH you can force

$$\aleph_{\omega+\omega} = \operatorname{ded}\aleph_\omega < (\operatorname{ded}\aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1} = 2^{\aleph_\omega}$$

[3] Sequences of 0s and 1s of length $\le \mu$, i.e. the functions from some ordinal $\lambda \le \mu$ to 2. You can think of the order as projecting the relative tree on a line. For example $0 < 01$.

[4] From now on, I will deliberately write x instead of $\bar{x}$, and the like.

[5] Here we are cheating a bit because we did not show that the supremum is attained. But for the purposes of this proof this is just an abuse of notation: just show the same thing for every $\lambda < \operatorname{ded}\kappa$...

Open Problem 6.12. Can we have these two strict inequalities simultaneously?

$$\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0} < 2^\kappa$$

Theorem 6.13 (Keisler-Shelah[6])**.** Let T be countable. Then the function $g_T(\kappa)$ coincides with one of the following six, and they correspond to properties of the theory:

1. κ, corresponding to T being ω-stable (in this case it is the same as totally transcendental)
2. $\kappa + 2^{\aleph_0}$ corresponding to T being superstable (but not ω-stable)
3. $\kappa^{\aleph_0}$ corresponding to T being stable (but not superstable)
4. $\operatorname{ded}\kappa$ [multi-order]
5. $(\operatorname{ded}\kappa)^{\aleph_0}$ corresponding to T being NIP (but none of the previous)
6. 2^κ corresponding to T being none of the above (i.e. having IP)

Corollary 6.14. In an universe with GCH, you cannot detect NIP by counting types.

Proof. In that case, $(\operatorname{ded}\kappa)^{\aleph_0} = 2^\kappa$. □

Lemma 6.15. Let T be a complete theory, and let $\varphi(x, y)$, $\psi(x, z)$ be stable formulas. Then the following hold:

1. $\varphi^*(y, x) := \varphi(x, y)$ is stable.
2. $\neg\varphi(x, y)$ is stable.
3. $\varphi \wedge \psi$ and $\psi \vee \varphi$ are stable.
4. If $y = uv$ and $c \in M^{|v|}$ then $\varphi(x, uc)$ is stable.
5. If T is stable, then T^{eq} is stable.

Proof.

1. Reverse the order.
2. Reverse the order.

[6] Actually mostly done by Shelah, last two functions done by Keisler.

3. By the previous point, it suffices to prove one of the two. So assume that $\theta(x; yz) \coloneqq \varphi \vee \psi$ is unstable, as witnessed by sequences $\langle a_i, b_i c_i \mid i < \omega \rangle$. We have

$$\vDash \theta(a_i; b_j c_j) \iff \vDash \varphi(a_i, b_j) \vee \psi(a_i, c_j) \iff i < j$$

Define

$$P \coloneqq \{(i,j) \in \mathbb{N}^2 \mid i < j \text{ and } \vDash \varphi(a_i, b_j)\}$$
$$Q \coloneqq \{(i,j) \in \mathbb{N}^2 \mid i < j \text{ and } \vDash \psi(a_i, c_j)\}$$

This is a 2-coloring of the pairs in $\mathbb{N}$, and it suffices to use Ramsey's theorem: an infinite homogeneous set for P would say that φ is unstable, while an infinite homogeneous set for Q would say that ψ is unstable.

4. If $\varphi(x; uc)$ is unstable, as witnessed by $\langle a_i, b_i \rangle$, then $\langle a_i, b_i c \rangle$ witnesses that $\varphi(x, uv)$ is unstable.

5. Painful but easy.

□

There will be homework between today and tomorrow, to be handed in two weeks.

Chapter 7

14/11

7.1 This Week's Goals

The goal for this week is to prove these theorems (some of the definitions are still going to be given).

Theorem 7.1. Let T be a complete theory, and $\varphi(x, y)$ a formula. The following are equivalent:

1. $\varphi(x, y)$ is stable.
2. $R_\varphi(x = x) < \omega$.
3. All φ-types are definable.
4. For every $\kappa \geq |L|$ and $M \vDash T$ of size κ, we have $|S_\varphi(M)| \leq \kappa$.
5. There is some $M \vDash T$ of size κ such that $|S_\varphi(M)| < \operatorname{ded} \kappa$.

This is a *local* statement. It can be turned in the following *global* one.

Theorem 7.2. Let T be a complete theory. The following are equivalent:

1. T is stable.
2. There is no indiscernible sequence $\langle a_i \mid i < \omega \rangle$ and formula $\varphi(x, y)$ such that $\vDash \varphi(a_i, a_j) \iff i < j$.
3. For all $\kappa \geq \aleph_0$ we have $g_T(\kappa) \leq \kappa^{|T|}$.
4. There is some κ such that $g_T(\kappa) \leq \kappa$.
5. There is some κ such that $g_T(\kappa) < \operatorname{ded} \kappa$.
6. All formulas $\varphi(x, y)$ with $|x| = 1$ are stable.

These can be used for the homework.

7.2 Some Combinatorics

Lemma 7.3 (Erdös-Makkai)**.** Suppose H is infinite and $\mathcal{C} \subseteq \mathscr{P}(H)$ is such that $|\mathcal{C}| > |H|$. Then, for every $n \in \mathbb{N}$, there are $h_1, \dots, h_n \in H$ and $C_0, \dots, C_n \in \mathcal{C}$ such that

$$\{h_1, \dots, h_n\} \cap C_j = \{h_1, \dots, h_j\} \tag{$\star_n$}$$

In other words $h_i \in C_j \iff i \leq j$.

Proof. For a given n, $(\star_n)$ holds for $\mathcal{C}$ if and only if it holds for $H \backslash \mathcal{C} = \{H \backslash C \mid C \in \mathcal{C}\}$: just define $h'_1 = h_n, \dots, h'_n = h_1$ and $C'_0 = H \setminus C_n, \dots, C'_n = H \setminus C_0$. Then

$$h'_i \in C'_j \iff h_{n-i+1} \notin C_{n-j} \iff n-i+1 > n-j \iff i < j+1 \iff i \leq j$$

Assume the result holds for n (the case $n = 0$ is trivial). Pick $C \in \mathcal{C}$. Then either

$$|\mathcal{C} \cap C| = |\{X \cap C \mid X \in \mathcal{C}\}| > \kappa := |H| \tag{7.1}$$

or

$$|\mathcal{C} \cap C| \leq \kappa \tag{7.2}$$

If (7.1) holds, then there is $c \in C$ such that $|\{Y \in \mathcal{C} \cap C \mid c \notin Y\}| > \kappa$. This is because otherwise

$$|\{X \cap C \mid X \in \mathcal{C}\} \setminus \{C\}| \leq \left| \bigcup_{c \in C} \{Y \in \mathcal{C} \cap C \mid c \notin Y\} \right| \leq \kappa \cdot \kappa = \kappa$$

Take such $c \in C$ and let $\mathcal{D} = \{Y \in C \cap \mathcal{C} \mid c \notin Y\}$. Then $|\mathcal{D}| > |C|$ and $\mathcal{D} \subseteq \mathscr{P}(C)$. By inductive hypothesis, there are $Y_0, \dots, Y_n \in \mathcal{D}$ and $c_1, \dots, c_n \in C$ such that $c_i \in Y_j \iff i \leq j$. Now by definition $Y_j = C_j \cap C$ for some $C_j \in \mathcal{C}$. Set $h_{n+1} := c$ and $C_{n+1} := C$.

If (7.2) then we can do the same trick with $H \setminus C$ in place of C, because then[1] $|(H \setminus \mathcal{C}) \cap (H \setminus C)| > \kappa$ and we can apply what we said in the beginning. □

This has the following consequence:

Proposition 7.4. Assume B is an infinite set of parameters and $\varphi(x, y)$ is a formula such that $|S_\varphi(B)| > |B|$. Then $\varphi(x, y)$ is unstable.

[1] The map $\mathcal{C} \to (\mathcal{C} \cap C) \times ((H \setminus \mathcal{C}) \cap (H \setminus C))$ sending $C_0 \mapsto (C_0 \cap C, (H \setminus C_0) \cap (H \setminus C))$ is injective: if C_0, C_1 differ by some element c, it either lies in C or in $H \setminus C$, and this will be detected by one of the two components of the map.

Proof. Consider, for each $a \in \mathbb{M}$,

$$\mathrm{tp}_\varphi(a/B) = \{\varphi(x,b)^{\eta_b} \mid \mathbb{M} \models \varphi(a,b)^{\eta_b}\}$$

where η_b can be 0 or 1 and we mean that $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$. Then set

$$S_a = \{b \in B \mid \models \varphi(a,b)\}$$

and

$$\mathcal{C} = \{S_a \mid a \in \mathbb{M}\}$$

Since by hypothesis $|\mathcal{C}| > |B|$, we can apply the Erdös-Makkai Theorem, and we get that, for each n, there are $a_1, \ldots, a_n, a_0$ and $b_1, \ldots, b_n$ such that

$$i \leq j \iff b_i \in S_{a_j} \iff \models \varphi(a_j, b_i)$$

And this means that $\varphi^*(y,x) = \varphi(x,y)$ is unstable, and we can conclude by point 1 of Lemma 6.15. □

7.3 Shelah's Local Rank

Definition 7.5. We say that $\varphi(x,y)$ has the *binary tree property* iff there are tuples $\langle b_\sigma \mid \sigma \in 2^{<\omega}\rangle$ such that for every $\eta \in 2^\omega$ this set of formulas is consistent:

$$\{\varphi(x, b_{\eta\restriction k})^{\eta(k)} \mid k < \omega\}$$

In other words the branches of this tree are consistent:

$$\begin{array}{cccccccc}
 & & & \varphi(x,b_\Lambda) & & & & \\
 & & / & & \backslash & & & \\
 & \varphi(x,b_0) & & & & \neg\varphi(x,b_1) & & \\
 / & & \backslash & & / & & \backslash & \\
\varphi(x,b_{00}) & & \neg\varphi(x,b_{01}) & & \varphi(x,b_{10}) & & \neg\varphi(x,b_{11}) & \\
/ \quad \backslash & & / \quad \backslash & & / \quad \backslash & & / \quad \backslash & \\
\ldots \quad \ldots & & \ldots \quad \ldots & & \ldots \quad \ldots & & \ldots \quad \ldots &
\end{array}$$

Definition 7.6. Let $\varphi(x,y)$ be a formula. The R_φ*-rank* of X (a definable set) is defined as follows:

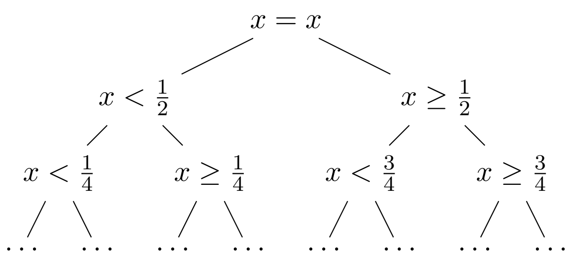
- $R_\varphi(X) \geq 0$ iff X is consistent (nonempty).
- $R_\varphi(X) \geq n+1$ iff there is a tuple $a \in \mathbb{M}^{|y|}$ such that $R_\varphi(X \wedge \varphi(x,a)) \geq n$ and $R_\varphi(X \wedge \neg\varphi(x,a)) \geq n$

This rank somehow measures "how stable" a formula is.

Remark 7.7. Note that in the tree generated here siblings have the same parameters.

Example 7.8. Let's try with $X \coloneqq x = x$ and see what happens. First, use $\varphi(x, y) \coloneqq x = y$. Since both $x = a$ and $x \neq a$ are consistent, the rank is at least 1. Anyway x_a has rank 0, since we cannot split it anymore. (The other one will of course continue to split on the "$\neq$" branch.

Example 7.9. What if $\varphi(x, y) \coloneqq x < y$? This has infinite rank:



Remark 7.10. Saying "rank $\geq n$" is definable! We will see later in detail.

Theorem 7.11. $\varphi(x, y)$ is stable iff $R_\varphi(x = x) < \omega$.

Proof.

$\Leftarrow$ Assume $\varphi(x, y)$ is unstable, as witnessed by $\langle a_i, b_i \mid i \in [0, 1]\rangle$. It then suffices to do as in Example 7.9.

$\Rightarrow$ Assume $\varphi(x, y)$ has the binary tree property (it is implied by having infinite rank). Note that if $B = \{b_\sigma \mid \sigma \in 2^{<\omega}\}$ then $|B| = \aleph_0$ and $|S_\varphi(B)| = 2^{\aleph_0}$. Apply Proposition 7.4. $\square$

Definition 7.12. Suppose $p \in S(A)$. We say that p *is definable over* B if for every $\varphi(x, y)$ there is $\psi_\varphi(y) \in L(B)$ such that $\varphi(x, a) \in p \iff \models \psi_\varphi(a)$. We say that $p \in S_x(A)$ is *definable* iff it is definable over A.

Example 7.13. Let T be the theory of equality. Here there are two kind of types: the type

$$P_b \coloneqq \{x = b\} \cup \{x \neq c \mid c \neq b\}$$

and the type

$$P = \{x \neq a \mid a \in M\}$$

These are both definable. For example, let us look at the formula $x = y$. For the first one, notice that $x = a \in P_b \iff a = b$. For the second one, notice that $x = a \in P \iff a \neq a$. Then take respectively $y = b$ and $y \neq y$.

Lemma 7.14. The following facts hold:

1. For each $n \in \omega$, the set $\{e \mid R_\varphi(\theta(x, e)) \geq n\}$ is definable.

2. If $R\varphi(\theta(x)) = n$ then for every $a \in \mathbb{M}^{|y|}$ either $R_\varphi(\theta(x) \wedge \varphi(x, a)) < n$ or $R_\varphi(\theta(x) \wedge \neg\varphi(x, a)) < n$.

Proof. The second statement is basically definition. The first one is because you can write a big, painful but easy formula: $R_\varphi(\theta(x,e)) \geq n$ iff

$$\models \exists (y_\sigma)_{\sigma \in 2^{\leq n}} \exists (x_\eta)_{\eta \in 2^{n+1}} \Big(\bigwedge_{\eta \in 2^n} \theta(x_\eta, e) \wedge \bigwedge_{k=1}^{n} (\varphi(x_\eta, y_{\eta \restriction k}))^{\eta(k)} \Big) \qquad \square$$

Theorem 7.15. If $\varphi(x,y)$ is stable, then all φ-types are definable.

Proof. Let $p \in S_\varphi(A)$. Since φ is stable, there is some $n_\varphi \in \omega$ such that $R_\varphi(x = x) = n_\varphi$. Then there is $p_0 \in \mathscr{P}_{\text{fin}}(p)$ such that $R_\varphi(p_0)$ is minimal[2] among the finite subtypes of p. How do you construct it? Start with any formula, say $\varphi(x, a_1) \in p$, and check if there is anything in p that brings the rank down when conjuncted. Take conjunctions and keep doing that until you can. Then let p_0 be the resulting finite conjunction. Then we can define p using the previous Lemma: set[3]

$$\psi_\varphi(y) \coloneqq R_\varphi(p_0) = R_\varphi(p_0 \wedge \varphi(x,y))$$

If $\varphi(x,y)$ is in p, then the two ranks will be equal by choice of p_0. Otherwise the rank goes down by the second part of the Lemma. To conclude, notice that this is an $L(A)$-formula. $\square$

[2] Of course with $R_\varphi(p_0)$ we mean $R_\varphi(\bigwedge p_0)$.

[3] More precisely, if $R_\varphi(p_0) = n$, let $\psi_\varphi(y)$ be

$$\Big(R_\varphi(p_0 \wedge \varphi(x,y)) \geq n\Big) \wedge \neg \Big(R_\varphi(p_0 \wedge \varphi(x,y)) \geq n+1\Big)$$

Chapter 8

17/11

8.1 More Characterisations of Stability

Proposition 8.1. The following are equivalent:

1. T is unstable.
2. There is a formula $\theta(x, y)$ and a sequence $\{c_i \mid i < \omega\}$ such that $\models \theta(c_i, c_j) \iff i < j$.

Proof. $1 \Rightarrow 2$ If T is unstable, there are φ and $\langle a_i, b_i \mid i < \omega\rangle$ such that $\models \varphi(a_i, b_i)$ iff $i < j$. Just let $c_i = a_i b_i$ and $\theta(x_1, x_2; y_1, y_2)$ be $\varphi(x_1, y_2)$.

$2 \Rightarrow 1$ Just let $\varphi = \theta$ and $a_i = b_i = c_i$. □

I will always be a linear order.

Definition 8.2. We say that a sequence $\langle a_i \mid i \in I\rangle$ is *totally indiscernible* if for any $i_1, \ldots, i_n$ (pairwise different) in I and any $j_1, \ldots, j_n$ (pairwise different) in I we have $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n})$.

Example 8.3.

- In DLO, $\langle a_n = n \mid n < \omega\rangle$ is indiscernible over $\emptyset$ but not totally indiscernible, since $a_1 < a_2$ but $a_2 \not< a_1$.
- In ACF_0, a sequence $\langle a_i = \pi_i \mid i < \omega\rangle$ of algebraically independent transcendentals is totally indiscernible.

Proposition 8.4. T is stable if and only if every indiscernible sequence is totally indiscernible.

Proof.

$\Leftarrow$ Assume that T is unstable, as witnessed[1] by some $\varphi(x, y)$ and $\langle a_i \mid i < \omega\rangle$ such that $\models \varphi(a_i, a_j) \iff i < j$. By Proposition 5.8 applied with

[1] Using the previous Proposition.

$\Delta = \{\varphi(x,y), \neg\varphi(x,y), \varphi(y,x), \neg\varphi(y,x)\}$ there is an indiscernible sequence $\langle a'_i \mid i < \omega\rangle$ and $i < j$ such that

$$\vDash \varphi(a'_1, a'_2) \wedge \neg\varphi(a'_2, a'_1) \iff \vDash \varphi(a_i, a_j) \wedge \neg\varphi(a_j, a_i) \iff i < j$$

So $\varphi(x,y) \wedge \neg\varphi(y,x) \in \mathrm{tp}(a'_1, a'_2) \setminus \mathrm{tp}(a'_2, a'_1)$, and our sequence is not totally indiscernible.

($\Rightarrow$) Suppose $\langle a_i \mid i \in I\rangle$ is an indiscernible sequence which is not totally indiscernible. Again by Proposition 5.8 we can then get another sequence $\langle a'_i \mid i \in \mathbb{Q}\rangle$ with the same property. This means that there is a formula $\varphi(x_1, \ldots, x_n)$, some $r_1 < \ldots < r_n \in \mathbb{Q}$, and some permutation σ of $\mathbb{Q}$ such that

$$\vDash \varphi(a_{r_1}, \ldots, a_{r_n}) \wedge \neg\varphi(a_{r_{\sigma(1)}}, \ldots, a_{r_{\sigma(n)}})$$

Since every finite permutation is a product of consecutive transpositions, there is some $j \in \{1, \ldots, n\}$ such that

$$\vDash \varphi(a_{r_1}, \ldots, a_{r_j}, a_{r_{j+1}}, \ldots, a_{r_n}) \wedge \neg\varphi(a_{r_1}, \ldots, a_{r_{j+1}}, a_{r_j}, \ldots, a_{r_n})$$

(not really, but almost: up to another permutation of $\{1, \ldots n\}$). Choose

$$\psi(x,y) \coloneqq \varphi(a_{r_1}, \ldots, a_{r_{j-1}}, x, y, a_{r_{j+2}}, \ldots, a_{r_n}) \wedge \neg\varphi(a_{r_1}, \ldots, a_{r_{j-1}}, y, x, a_{r_{j+2}}, \ldots, a_{r_n})$$

Then $\psi(x,y)$ has the order property: since we used $\mathbb{Q}$, we have infinitely many guys to choose from to witness it. In fact, if $\langle k_i \mid i < \omega\rangle$ is an increasing sequence in r_j, r_{j+1} then $\psi(a_{k_i}, a_{k_j}) \iff i < j$. $\square$

Proposition 8.5. Assume T is stable. Then for every $\varphi(x,y)$ there is a $k_\varphi < \omega$ such that for every indiscernible sequence $\langle a_i \mid i \in I\rangle$, for all b, either $|\{i \in I \mid \vDash \varphi(a_i, b)\}| < k_\varphi$ or $|\{i \in I \mid \vDash \neg\varphi(a_i, b)\}| < k_\varphi$.

Proof. If $\varphi(x,y)$ is stable, then $\varphi(x,y)$ does not have the k-order property for some $k < \omega$. Then just let $k_\varphi \coloneqq k$ (or $k+1$, depending on you definitions). Indeed, if this did not work, we could find an indiscernible sequence $\langle a_i\rangle$ such that $\varphi(x,b)$ holds for at least k-many elements and $\neg\varphi(x,b)$ does not hold for at least k-many elements. Using total indiscernibility we then get

$$\vDash \exists y \Big(\bigwedge_{0 \le j < k} \varphi(a_j, y) \wedge \bigwedge_{k \le j < 2k} \neg\varphi(a_j, y) \Big)$$

and in particular, for each $i \le k$, again by total indiscernibility,

$$\vDash \exists y \Big(\bigwedge_{0 \le j < i} \varphi(a_j, y) \wedge \bigwedge_{i \le j < k} \neg\varphi(a_j, y) \Big)$$

And if b_i is a witness to the above existential, then $\vDash \varphi(a_j, b_i) \Leftrightarrow j < i$. $\square$

8.2 Some Examples of Stable Theories

Do not expect complete proofs here.

Theorem 8.6. Every strongly minimal theory is stable.

Proof Sketch. Basically the only 1-types over a model are the ones saying "i am this guy" or "i am different from all these guys", and for a model M we have $|S_n(M)| = |S_1(M)| = |M|$. $\square$

Recall that these theories are strongly minimal:

1. ACF_0, ACF_p
2. Vector spaces
3. Graphs, finite balancing with no cycle

A bigger class between "strongly minimal" and "stable" is "ω-stable".

Definition 8.7 (Assume the language is countable)**.** A theory is ω-stable iff whenever $|A| = \aleph_0$ then $|S_n(A)| \leq \aleph_0$ for all $n < \omega$.

These can be understood either via Morley rank or via the (omitting of the) binary tree property: the same one we had for stable formulas, except you are allowed to change the formula while going down the tree.

Example 8.8. These are ω-stable but not strongly minimal:

- An equivalence relation with infinitely many infinite classes (will have Morley rank 2).
- The theory DCF_0 of differentially closed fields[2] of characteristic 0. (Robinson)

Example 8.9. Examples of stable, but not ω-stable theories are

- The theory DCF_p of differentially closed fields of characteristic p. (Card Wood)
- The theories $\mathsf{SCF}_{p,e}$ (separably closed fields[3]).

Example 8.10. Groups:

- All abelian groups are stable.
- All algebraic groups over an ACF are stable, since they are definable in a stable theory.

[2] See later.
[3] See later.

- The free group F_n on n generators is stable (Sela) (not ω-stable if $n \geq 2$).

Conjecture 8.11 (Cherlin)**.** Every simple[4] group of finite Morley rank is an algebraic group over an algebraically closed field.

This has been open since the $'70s$.
What about fields?

Theorem 8.12 (Macyntire)**.** Every ω-stable field is algebraically closed.

Conjecture 8.13. Every stable field is separably closed.

What are separably closed fields? Call an algebraic element *separable* if its minimal polynomial has only simple roots. Then separably closed means that all separable elements over the field are already in the field. The *degree of imperfection* is $e \coloneqq [K \mid K^p]$.

Fact 8.14. $\mathsf{SCF}_{p,e}$ is model complete, has quantifier elimination after you add a base of K as a vector space over K^p, it is stable[5] and not ω-stable.

The reason it is not ω-stable is that there is a descending chain $K > K^p > K^{p^2} > \ldots$ and this cannot happen in an ω-stable theory.[6]

Differentially closed fields are differential fields (fields with a derivation) such that for any differential polynomials $f, g \in K\{y\} = K[y, \partial y, \partial^2 y, \ldots]$ such that $\mathrm{ord}(f) > \mathrm{ord}(g)$ there is a such that $f(a) = 0$ and $g(a) \neq 0$.

Graphs?

Fact 8.15. Every planar graph is stable.

Remark 8.16. We did not see one of the implications (all φ-types are definable implies few φ-types). This is in the official notes. The proof is basically: count the definitions.

[4] In the algebraic sense.
[5] Delon?
[6] (Morley Rank, Morley Degree) goes down.

Chapter 9

21/11

9.1 Stable = NIP + NSOP

9.1.1 Motivation

It all started with this "classification map". Some dividing lines (Shelah):

- Independence property
- Tree property
- Order property

The idea was that structures satisfying one of these are "bad structures". On the other side, not satisfying any of them brings us in the "structured side", e.g. in the NOP (i.e. stable) case every type is definable. The independence property is somehow associated to "randomness". The prototypical example of NIP theories are the stable ones or DLO, the prototypical example of IP theory is the random graph.

The idea is: if T is unstable, then either T is "random" or T "has[1] a linear order". Lets start giving precise definitions.

9.1.2 Definitions

Definition 9.1. We say that $\varphi(x; y)$ has the *strict order property* if there is a sequence $\langle b_i \mid i < \omega \rangle$ such that for all $i < j < \omega$ we have $\varphi(\mathbb{M}, b_i) \subsetneq \varphi(\mathbb{M}, b_j)$. In other words

$$\models \forall x \, (\varphi(x, b_i) \to \varphi(x, b_j)) \land \exists x \, (\varphi(x, b_j) \land \neg\varphi(x, b_i))$$

Example 9.2.

[1] In the sense of "defines". Remember that an unstable formula defines and order on a set which may not be definable.

- The formula $x < y$ in DLO has the strict order property, as witnessed by any increasing sequence $b_1 < b_2 < \ldots$, because then $\{a \mid a < b_1\} \subsetneq \{a \mid a < b_2\} \subsetneq \ldots$.
- xRy in the random graph does *not* have SOP. This will be a particular case of a more general result which we will show later.

Definition 9.3. T has SOP if some formula in T does, and T is NSOP otherwise.

Theorem 9.4. Let T be a complete theory. The following are equivalent:

1. T has SOP.
2. There is a formula $\psi(x_1, x_2) \in L$ defining a preorder with infinite chains.
3. There is a formula $\psi(u_1, u_2) \in L^{\mathrm{eq}}$ defining a partial order with infinite chains.[2]

Definition 9.5. A *preorder* is a binary relation which is reflexive and transitive.

In other words, $x \preceq y \wedge y \preceq x$ does not imply $x = y$.

Proof of the Theorem.
$1 \Rightarrow 2$ Assume $\varphi(x; y)$ has the SOP, as witnessed by $\{b_i \mid i < \omega\}$. Define

$$\psi(y_1, y_2) \coloneqq \forall x\ (\varphi(x, y_1) \to \varphi(x, y_2))$$

This obviously defines a preorder on $\mathbb{M}^{|y|}$, and by hypothesis it has an infinite chain: namely, $\langle b_i \mid i < \omega \rangle$.

Note that this is a formula that defines a preorder *on the whole structure*, not just on the b_i.

$2 \Rightarrow 3$ Just define the equivalence relation

$$E(y_1, y_2) = \psi(y_1, y_2) \wedge \psi(y_2, y_1)$$

Since ψ is a preorder, this induces a partial order on $\mathbb{M}^{|y|}/E$. The formula will be

$$\hat{\psi}(u_1, u_2) = \exists x_1, x_2\ (\psi(x_1, x_2) \wedge f_E(x_1) = u_1 \wedge f_E(x_2) = u_2)$$

$3 \Rightarrow 1$ Given such a $\psi(u_1, u_2)$, by Lemma 3.5 we have a formula φ such that

$$T^{\mathrm{eq}} \vDash \forall x, y\ (\varphi(x, y) \leftrightarrow \psi(f_E(x), f_E(y)))$$

so it suffices to look at φ. $\square$

[2]Here $|u_1| = |u_2| = 1$.

Definition 9.6. We say that $\varphi(x; y)$ has the *independence property* (IP) if there are elements $\langle a_i \mid i < \omega \rangle$ and $\langle b_S \mid S \subseteq \omega \rangle$ such that

$$\vDash \varphi(a_i, b_S) \iff i \in S$$

We say that T has the independence property if some formula in T has. Otherwise we say that T is NIP, or *dependent*.

In other words, "φ has IP if it can encode the power set of $\mathbb{N}$".

Remark 9.7. This is basically the same idea as VC-dimension in statistics and yes, there are connections.

Example 9.8.

- $x = y$ and $x < y$ in DLO do not have IP (are NIP). For the second formula, the point is that we can get a configuration of the form

$$a_1 < b_{\{1\}} < a_2 < b_{\{2\}}$$

but $1 \notin \{2\}$.

- xRy in the random graph has IP: take any sequence of different elements $\langle a_i \mid i < \omega \rangle$, and take, for all $S \subseteq \omega$, the partial type

$$P_S(y) \coloneqq \{a_i R y \mid i \in S\} \cup \{\neg a_i R y \mid i \notin S\}$$

which is finitely consistent by the random graph axioms. Then let b_S be any realisation of P_S in the monster.

Remark 9.9. By compactness, for IP, or for SOP, it is enough to show that there are arbitrarily long finite sequences with the desired property, e.g. the same as in the definitions but replacing ω with arbitrarily large $n \in \omega$.

Example 9.10. Some examples of NIP theories:

- All stable theories.
- All o-minimal theories, e.g. DLO, RCF or $\mathrm{Th}(\mathbb{R}_{\exp})$.
- Algebraically closed valued fields.
- Trees.

As usual, sequences witnessing "bad properties" can be assumed to be indiscernible. For NIP, there is even a better thing:

Definition 9.11. Given a formula $\varphi(x;y)$ we define the *alternation number* of φ as

$$\operatorname{alt}(\varphi) = \max\{n \mid (*_n) \text{ happens}\}$$

$$\exists \text{ indisc. } \langle a_i \rangle_{i<\omega} \text{ and } b \in \mathbb{M} \text{ s.t. } \vDash \bigwedge_{i=0}^{n-1} \varphi(a_{2i}, b) \wedge \neg\varphi(a_{2i+1}, b) \qquad (*_n)$$

If there is no maximum we set $\operatorname{alt}(\varphi) = \infty$.

Example 9.12. In DLO, suppose that $\langle a_i \mid i < \omega \rangle$ is increasing. Then, if b is smaller than all the a_i, or bigger than all of them, there will be no alternation on $x < b$. If b is "in the middle", it will be 1. Since we are taking the maximum on the indiscernible sequences, this means that $\operatorname{alt}(x < y) \geq 1$.

Theorem 9.13. φ is NIP if and only if $\operatorname{alt}(\varphi) < \infty$.

Proof.

$\Leftarrow$ This is very easy if you assume the following:

Claim. If $\varphi(x, y)$ has IP then there are an indiscernible $\langle a_i \mid i < \omega \rangle$ and elements $\langle b_S \mid S \subseteq \omega \rangle$ such that $\vDash \varphi(a_i, b_S) \iff i \in S$.

Given the Claim, it is sufficient to take as S the even numbers. But the Claim follows from Proposition 5.8.

$\Rightarrow$ Assume $\operatorname{alt}(\varphi) = \infty$. By compactness there is an indiscernible $\langle a_i \mid i < \omega \rangle$ and some $b \in \mathbb{M}^{|y|}$ such that

$$\forall i < \omega \ \vDash \varphi(a_{2i}, b) \wedge \neg\varphi(a_{2i+1}, b)$$

From here we will extract the independence property for arbitrarily large sequences. In other words, we are now going to show that, given $n < \omega$, there are $\langle b_S \mid S \subseteq n \rangle$ such that for all $i < n$ we have $\vDash \varphi(a_i; b_S) \iff i \in S$. Given S, we can find indexes $(i_0, \ldots, i_{n-1})$ such that i_j is even iff $j \in S$: just take $i_j \coloneqq 2j + 1 - \chi_S(j)$. So for all $S \subseteq n$ we have

$$\vDash \exists y \Big(\bigwedge_{j \in S} \varphi(a_{i_j}, y) \wedge \bigwedge_{j \notin S} \neg\varphi(a_{i_j}, y) \Big)$$

this is in $\operatorname{tp}(a_{i_0}, \ldots, a_{i_{n-1}})$ which is, by indiscernibility, the same as $\operatorname{tp}(a_0, \ldots, a_{n-1})$. Just collect as the "true" b_S a witness for this existential with respect to them. □

Theorem 9.14. T is stable if and only if it is both NIP and NSOP.

Since IP and SOP imply unstability[3], this follows from the following stronger local statement.

[3] For IP, just consider $b_{\{0,\ldots,j-1\}}$.

Lemma 9.15. If $\varphi(x,y)$ is unstable, then either $\varphi(x,y)$ has IP or there is a formula $\theta(x,b)$ such that $\varphi(x,y) \wedge \theta(x,b)$ has SOP.

Proof. Let $\langle a_i \mid i \in \mathbb{Q}\rangle$ and $\langle b_i \mid i \in \mathbb{Q}\rangle$ witness the order property with $\leq$ instead of $<$ for φ, and assume that the second sequence is indiscernible. Assume that $\varphi(x,y)$ is NIP. Then there is $n < \omega$ such that the formula

$$\varphi(x,b_0) \wedge \neg\varphi(x,b_1) \wedge \varphi(x,b_2) \wedge \neg\varphi(x,b_3) \wedge \ldots \wedge \varphi(x,b_{2n-2}) \wedge \neg\varphi(x,b_{2n-1})$$

is inconsistent, by Theorem 9.13. On the other hand

$$\neg\varphi(x,b_0) \wedge \neg\varphi(x,b_1) \wedge \ldots \wedge \neg\varphi(x,b_{n-1}) \wedge \varphi(x,b_n) \wedge \ldots \wedge \varphi(x,b_{2n-1})$$

is consistent, as witnessed by a_n because of the order property. So we have n-many "no" and n-many "yes" . Up to a permutation given by a composition of swapping a "yes" with a "no", we can get to a 'yes-no-yes-no-…" sequence[4]. Therefore, there is a function $\eta\colon 2n \to 2$ and $i_0 < 2n$ such that

$$\bigwedge_{i<i_0} (\varphi(x,b_i))^{\eta(i)} \wedge \neg\varphi(x,b_{i_0}) \wedge \varphi(x,b_{i_0+1}) \wedge \bigwedge_{i>i_0+1} (\varphi(x,b_i))^{\eta(i)}$$

is consistent but

$$\underbrace{\bigwedge_{i<i_0} (\varphi(x,b_i))^{\eta(i)}}_{①} \wedge \varphi(x,b_{i_0}) \wedge \neg\varphi(x,b_{i_0+1}) \wedge \underbrace{\bigwedge_{i>i_0+1} (\varphi(x,b_i))^{\eta(i)}}_{②}$$

is inconsistent. This is because the swappings bring us from a consistent thing to an inconsistent one, so at a certain point we must stop being consistent. Let $\theta(x,\bar{b}) \coloneqq ① \wedge ②$, where $b = b_0, \ldots, b_{i_0-1}, b_{i_0+2}, \ldots, b_{2n-1}$.

Let us show that $\psi(x,y) \coloneqq \varphi(x,y) \wedge \theta(x,b)$ has the strict order property. Choose an increasing sequence $\langle r_n \mid n < \omega\rangle$ in (i_0, i_0+1) and let $c_n = b_{r_n}$. We have to check that

1. $\vDash \forall x\ (\psi(x,c_i) \to \psi(x,c_{i+1}))$
2. $\vDash \exists x\ (\psi(x,c_{i+1}) \wedge \neg\psi(x,c_i))$

For the second thing, just use as a witness something like $a_{\frac{r_n+r_{n+1}}{2}}$. For the first part: if it does not hold there is a such that

$$\vDash \theta(a,b) \wedge \varphi(a,b_{r_i}) \wedge \neg\varphi(a,b_{r_i+1})$$

but this a cannot exist by construction when $r_i = i_0$ and $r_{i+1} = i_0 + 1$. Apply indiscernibility. □

[4]Start swapping the last "no" with the first "yes" and bring the "no" to the last position. Bring the moved "yes" to the first position. Then iterate.

From the next time we will follow the presentation from *Simple Theories* by Frank Wagner (mostly Chapter 2), but with more details[5]. Other references are *Simple Theories and Elimination of Hyperimaginaries* by Casanovas and *A Course in Model Theory* by Tent and Ziegler.

[5]Actually filling the details in this book is probably the best way to learn simplicity, but of course it requires time.

Chapter 10

24/11

"Map of the universe": `www.forkinganddividing.com`, by G. Conant. [various comments]

Fact 10.1. Triangle-free generic graph: it is TP_2, but *rosy*.

10.1 Dividing

Definition 10.2. Let $k \in \mathbb{N}$. We say that a formula $\varphi(x; b)$ *k-divides over A* iff there is a sequence $\langle b_i \mid i < \omega \rangle$ such that

- for all $i < \omega$ we have $\mathrm{tp}(b_i/A) = \mathrm{tp}(b/A)$, and
- $\{\varphi(x; b_i) \mid i < \omega\}$ is k-inconsistent, meaning that every one of its subsets with k elements is inconsistent.

We say that a partial type $\pi(x)$ *k-divides over A* iff $\pi(x) \vdash \varphi(x; b)$ for some formula $\varphi(x; b)$ that k-divides.

We say that a formula/type *divides* iff it divides for some $k \in \mathbb{N}$.

Example 10.3. The formula $x = b$ divides over A if and only if $b \notin \mathrm{acl}(A)$.

Proof. $\Leftarrow$ Let $\{b_i \mid i < \omega\}$ be different realisations of $\mathrm{tp}(b/A)$. Clearly, $\{x = b_i \mid i < \omega\}$ is 2-inconsistent.

$\Rightarrow$ If $b \in \mathrm{acl}(A)$, then $\mathrm{tp}(b/A)$ has only finitely many realisations. Therefore, infinitely many b_i will be equal, so there is no way to have k-inconsistency: you can always find k equal guys.[1] $\square$

Example 10.4. Let T be the theory of an equivalence relation with infinitely many infinite classes. Then xEb divides over $\emptyset$.

[1] Actually that set will be finite so k-inconsistency... Another way of saying it is: there is no infinite sequence of b_i.

Proof. There is just one 1-type over $\emptyset$. Pick each b_i in a different equivalence class. Then $\{xEb_i \mid i < \omega\}$ is 2-inconsistent. $\square$

Example 10.5. In the random graph, xRb does not divide over $\emptyset$. This is because any attempt to have k-inconsistency will clash with the random graph axiom.

We will see later that in the random graph the only dividing formulas are the ones of the form $x = b$.

Example 10.6. In DLO, consider $x < b$. This does not divide over $\emptyset$ because there is no minimum. Anyway, $a < x < b$ does; it is easy to get 2-inconsistency: just take $a_1 < b_1 < a_2 < b_2 < \dots$. All the (a_i, b_i) have the same type over $\emptyset$ because $a_i < b_i$ is the only thing to check and it always holds.

Example 10.7 (With no proof)**.** In ACF $\operatorname{tp}(a/B)$ divides over $A \subseteq B$ if and only if

$$\operatorname{tr}_{\deg}(a/B^{\mathrm{alg}}) < \operatorname{tr}_{\deg}(a/A^{\mathrm{alg}})$$

10.2 Forking

Definition 10.8. A formula $\theta(x)$ *forks over A* iff $\theta(x) \vdash \bigvee_{i=1}^{n} \varphi_i(x, b_i)$ such that each $\varphi_i(x, b_i)$ divides over A.

The idea is that dividing should correspond to "small" formulas. However, it is not true that dividing is closed under finite unions/disjunctions, and that's why you need forking. In other words, forking is the ideal generated by the dividing formulas.

Clearly, dividing implies forking.

Example 10.9. Let M be the circle S^1 and define $R(x, y, z)$ to hold iff x is different from y and z and lies in the small arc of y and z (if y and z are opposite, then we agree that no x satisfies it). Then, if b, c are not opposite, $R(x, b, c)$ is consistent but divides over $\emptyset$ (there is quantifier elimination). But $x = x \vdash \bigvee_{i=1}^{4} R(x, b_i, c_i)$. So the ideal can be improper.

It will turn out that the above theory is NIP. Anyway, these pathologies will not arise in *simple* theories.

Lemma 10.10 (Standard Lemma)**.** For every infinite sequence I, every small set of parameters A and any small linear order J, there is an A-indiscernible sequence $\langle b_j \mid j \in J\rangle$ realising $\mathrm{EM}(I/A)$.

Proof. With Proposition 5.8. $\square$

Proposition 10.11. Some properties of forking and dividing:

1. Dividing implies forking.

2. If two formulas $\varphi_1(x), \varphi_2(x)$ fork over A, then $\varphi_1 \vee \varphi_2$ forks over A.

3. If p, q are partial types, $p \vdash q$ and q divides (forks) over A, so does p.

4. $\varphi(x;b)$ k-divides over A if and only if it k-divides over a for all finite $a \in A$.

5. $\varphi(x,b)$ divides over A if and only if there is an A-indiscernible sequence $\langle b_i \mid i < \omega \rangle$ such that $b_0 \equiv_A b$ and $\{\varphi(x;b_i) \mid i < \omega\}$ is inconsistent.

6. A partial type $\pi(x)$ k-divides (forks) over A if and only if there is a finite conjunction $\theta(x,c)$ of formulas in π that k-divides (forks) over A.

7. No $p \in S_n(A)$ divides over A.

8. Let $A \subseteq B \subseteq C$. If $\mathrm{tp}(a/C)$ does not divide (fork) over A, then $\mathrm{tp}(a/C)$ does not divide (fork) over B and $\mathrm{tp}(a/B)$ does not divide (fork) over A.

Proof.

1. Trivial.

2. Trivial.

3. Trivial.

4. Compactness: write down "there are infinitely many guys with the same type over A such that…"

5. $\Leftarrow$ By indiscernibility all the b_i have the same type of b over A. Let us check that there is k-inconsistency for some k. If not, for every $k < \omega$ there are $i_1 < \ldots < i_k$ such that $\{\varphi(x;b_{i_k}) \mid 1 \leq j \leq k\}$ is consistent. By indiscernibility, for all k we have $\vDash \exists x \ \varphi(x,b_1), \ldots, \varphi(x,b_k)$. By compactness, $\{\varphi(x,b_i) \mid i < \omega\}$ is consistent.

   $\Rightarrow$ Use the Standard Lemma to turn a sequence I witnessing dividing into an indiscernible one. Since I is k-inconsistent, this is written in $\mathrm{EM}(I/A)$ and will still be true for the indiscernible one.

6. For forking it is obvious by compactness. For dividing, proceed as follows. Assume $\pi(x) \vdash \varphi(x,b)$ and $\varphi(x,b)$ divides over A. Then there is a finite conjunction $\theta(x,c)$ of formulas in π such that $\theta(x,c) \vdash \varphi(x,b)$. Let $\langle b_i \mid i < \omega \rangle$ witness k-dividing over A for some k. Since $b_i \equiv_A b$ there is $\alpha_i \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\alpha_i(b) = b_i$. Consider $\{\alpha_i(c) \mid i < \omega\}$.

Clearly, they all have the same type as c over A. Suppose $\{\theta(x, \alpha_i(c)) \mid i < \omega\}$ is not k-inconsistent, say

$$a \vDash \bigwedge_{i=1}^{k} \theta(x, \alpha_i(c))$$

then

$$a \vDash \bigwedge_{i=1}^{k} \varphi(x, b_i)$$

a contradiction.

7. If we have $\varphi(x, a)$ with $a \in A$, then there is just one realisation of $\mathrm{tp}(a/A)$.

8. Assume $A \subseteq B \subseteq C$. We want to prove that if $\mathrm{tp}(a/B)$ divides over A or $\mathrm{tp}(a/C)$ divides over B, then $\mathrm{tp}(a/C)$ divides over A. If $q = \mathrm{tp}(a/B)$ divides over A, so does $p = \mathrm{tp}(a/C) \vdash q$ by a previous point. If $\mathrm{tp}(a/C)$ divides over B, there is a formula $\varphi(x, c) \in \mathrm{tp}(a/C)$ that divides over B. Then $\varphi(x, c)$ divides over A, because every B-indiscernible sequence is A-indiscernible[2].

$\square$

We finish this lesson with another example.

Example 10.12. If $a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A)$, then $\mathrm{tp}(a/Ab)$ forks over A. In other words, if you become algebraic you have to fork.

Proof. Pick an algebraic formula $\varphi(x; b) \in \mathrm{tp}(a/Ab)$. Then this type implies $\bigvee_{i \leq \ell} x = a_i$, where $\{a_i \mid i \leq \ell\}$ is the set of realisations of $\varphi(x; b)$, and each of this formulas divides. $\square$

Next lessons in MALL2.

[2]Or, directly, having the same type over B implies having the same type over A.

Chapter 11

01/12

11.1 Finitely Satisfiable Types

Definition 11.1. A (partial) type $p \in S(B)$ (π) is *finitely satisfiable in A* iff for every formula $\varphi(x, b) \in p$ (finite conjunction of formulas in π) there is $a \in A^{|x|}$ such that $\models \varphi(a, b)$.

Definition 11.2. Let $B \supseteq A$, $p \in S(A)$, $q \in S(B)$, and $q \supseteq p$. If q is finitely satisfiable in A, we say that q is a *coheir* of p.

Example 11.3. Let T_E be the theory an equivalence relation E with infinitely many infinite classes. Let A be a model M with $\aleph_0$ classes of size $\aleph_0$ and $B = N \succ M$ with $\aleph_1$ classes of size $\aleph_1$.

Let $p = \{\neg xEa \mid a \in M\} \in S(M)$. Let $N \ni b \models p$ and $q' = \mathrm{tp}(b/N)$. Clearly, $q' \supseteq p$. Let $q = \{\neg xEb \mid b \in N\}$. We also have $q \supseteq p$. Note that q is finitely satisfiable in M, hence a coheir of p, but q' is not: look at the formula xEb (or $x = b$).

Remark 11.4. Note that p need not be a coheir of itself, for example if $A = \emptyset$. Of course p is always a coheir of itself if A is a model.

Definition 11.5. Assume $p \in S(A)$ and $q \supseteq p$. We say that q is a *forking extension* of p if q forks over A. Otherwise, we say that q is a *non-forking extension* of p.

Note that the previous definition also works if q is a partial type.

In the example above, xEb forks over M, so q' is a forking extension of p. To see this, notice that all the stuff in $N \setminus M$ has the same type over M and choose the b_i in different classes. The idea is that the information in p is "I am not related to anybody". In this sense, q is the extension of p that resembles it the most. Note that T_E is stable, so every type is definable. q has the same definition as p.

Example 11.6. Let T be DLO. Here there are four kinds of types, over $\mathbb{Q}$, say:

- Realised types.
- Types at $\pm\infty$.
- Irrational cuts, e.g. $\{x \geq q \mid q < \sqrt{2}\} \cup \{x < q \mid q > \sqrt{2}\}$. I.e. a cut where there is no maximum of the left part and no minimum of the right part.
- Infinitesimal cuts a^+ and a^-. For example $P_{a^+} = \{x < a' \mid a' > a\} \cup \{x > a' \mid a' \leq a\}$. ($a^-$ will be the analogous thing on the left)

Let $p = P_{a^+} \in S_1(\mathbb{Q})$, and consider its extensions with parameters in a bigger model N. These can be of the form

- $x = a_\varepsilon$. This divides.
- An irrational cut $\{x > b \mid b \in S\} \cup \{x < b' \mid b' \in S'\}$. This is not a coheir. Is it a forking extension? Yes: take a formula $b < x < b'$ and argue as in Example 10.6.
- $q_{a^+} = \{x < b' \mid b' > a\} \cup \{x > b' \mid b' \leq a\}$. This is again not finitely satisfiable: look at $a < x < b'$.
- An irrational cut $q = \{x < a' \mid a' \in M, a' > a\} \cup \{x > b \mid b \in N, b < (a, +\infty) \cap M\}$ (here $M = \mathbb{Q}$). This is finitely satisfiable in M. Note that this is not exactly what we expected: the "idea" behind q is not the "same" as the idea of p (being "just on the right of a").

We will see that finitely satisfiable types cannot fork, and this applies to q. Anyway, q_{a^+} is non-forking as well, morally speaking because intervals cannot move away from a. So the other implication does not hold in general. It will be true in a stable theory, and that is why Example 11.3 was working so nice.

Remark 11.7. If a partial type π is finitely satisfiable over A, then it does not fork[1] over A.

Proof. Suppose first that π divides over A. Then there is a finite conjunction $\theta(x)$ of formulas in π such that $\theta(x) \vdash \varphi(x, b)$ and there is a sequence $\{b_i \mid i < \omega\}$ with $b_i \equiv_A b$ and $\{\varphi(x, b_i) \mid i < \omega\}$ is k-inconsistent for some k. Since π was finitely satisfiable in A, there is $a \in A$ such that $\models \theta(a)$, hence $\models \varphi(a, b)$. Since $a \in A$ and $b_i \equiv_A b$, we also have $\models \varphi(a, b_i)$, and this is a contradiction.

[1] In particular it does not divide.

Now let us show that π does not fork over A. Assume that $\theta(x)$ is a finite conjunction of formulas in π such that $\theta(x) \vdash \bigvee_{i=1}^{n} \varphi_i(x, b_i)$, each $\varphi(x, b_i)$ dividing over A. Let $a \in A$ realise θ. Then $a \vDash \bigvee_{i=1}^{n} \varphi_i(x, b_i)$, so there is $i \leq n$ such that $\vDash \varphi(a, b_i)$, and we can argue as in the previous case. □

Theorem 11.8. Let $A \subseteq B$ and π a partial type over B. The following facts hold:

1. If π is finitely satisfiable in A, then it does not fork over A.

2. If π is finitely satisfiable in A, then it has a completion $p \in S(B)$ which is finitely satisfiable in A.

3. If π does not fork over A, then it has a completion $p \in S(B)$ which does not fork over A.

Proof.

1. Done just above.

2. Assume π is finitely satisfiable in A. Let $\pi' = \pi \cup \{\neg\varphi(x, b) \mid \varphi(x, b) \in L(B)$ is not finitely satisfiable in $A\}$. If π' is consistent, then any completion p of π' will do. Take Γ_0 be a finite subset of π' of the form

$$\Gamma_0 \subseteq \pi \cup \{\neg\varphi_1(x, b_1), \ldots, \neg\varphi_m(x, b_m)\}$$

   If it is not consistent, then for some finite conjunction $\theta(x)$ of formulas in π we have $\theta(x) \vdash \bigvee_{i \leq m} \varphi_i(x, b_i)$. Since π is finitely satisfiable in A, there is $a \in A$ such that $\vDash \theta(a)$. Then for some $i \leq m$ we also have $\vDash \varphi_i(a, b_i)$, contradicting the fact that $\varphi_i(x, b_i)$ is not finitely satisfiable in A.

3. Do the same proof as in the previous point but with $\{\neg\varphi(x, b) \mid \varphi(x, b) \in L(B)$ forks over $A\}$.

□

Corollary 11.9. Every coheir is a non-forking extension.

11.2 Some Technical Things About Forking

Proposition 11.10. Let $\pi(x, b)$ be a partial type. Then $\pi(x, b)$ does not divide over A if and only if for every A-indiscernible sequence $\{b_i \mid i < \omega\}$ with $b_0 = b$ the set $\bigcup_{i<\omega} \pi(x, b_i)$ is consistent.

Proof.

$\Leftarrow$ Assume $\pi(x,b)$ divides over A, so for a finite conjunction θ of formulas in π we have $\theta(x,b) \vdash \varphi(x,c)$ which divides over A. So there is an A-indiscernible $\langle c_i \mid i<\omega\rangle$ such that $\{\varphi(x,c_i) \mid i<\omega\}$ is k-inconsistent and $c_i \equiv_A c$. Let $\alpha \in \operatorname{Aut}(\mathbb{M}/A)$ be such that $\alpha(c_0)=c$. Consider $\langle c_i' := \alpha(c_i) \mid i<\omega\rangle$. For each i, let $\sigma_i \in \operatorname{Aut}(\mathbb{M}/A)$ be such that $\sigma_i(c)=c_i'$. Consider $\langle b_i := \sigma_i(b) \mid i<\omega\rangle$. Clearly, $\sigma_i(b) \equiv_A b$. Moreover, $\{\theta(x,b_i) \mid i<\omega\}$ is k-inconsistent, since it implies $\langle\varphi(x,c_i) \mid i<\omega\rangle$ which is. The indiscernible sequence is provided by the Standard Lemma, plus an automorphism to move the first guy to b. By construction, that union is inconsistent.

$\Rightarrow$ Let $\langle b_i \mid i<\omega\rangle$ be an A-indiscernible sequence with $b_0=b$ and such that $\bigcup_{i<\omega}\pi(x,b_i)$ is inconsistent. By compactness, there are finitely many formulas $\theta_1(x,b_{i_1}),\dots,\theta_n(x,b_{i_n})$ such that $\theta_i(x,b)\in\pi(x,b)$ and their conjunction is inconsistent. Define

$$\theta(x,b)=\bigwedge_{i=1}^{n}\theta_i(x,b)$$

(note that we replaced all the b_{i_j} with b). We claim that $\theta(x,b)$ divides over A. In fact, using the sequence $\langle b_i \mid i<\omega\rangle$, we get inconsistency by construction. $\square$

Remark 11.11. The above even works when b is an infinite (but small) tuple.

Another thing that can be proven by moving things around with automorphisms is:

Proposition 11.12. The following are equivalent:

1. $\operatorname{tp}(a/Ab)$ does not divide over A.
2. For any A-indiscernible sequence I with $b\in I$ there is $a'\equiv_{Ab} a$ such that I is Aa'-indiscernible.
3. For any A-indiscernible sequence I with $b\in I$ there is an Aa-indiscernible sequence J such that $I\equiv_{Ab} J$.
4. For any A-indiscernible sequence I with $b\in I$ there are $a'\equiv_{Ab} a$ and $J'\equiv_{Ab} I$ such that J' is Aa'-indiscernible.

So in some sense (not) dividing detects whether you can have "slightly more indiscernibility".

Remark 11.13. $I\equiv_{Ab} J$ is stronger than $\mathrm{EM}(I/Ab)=\mathrm{EM}(J/Ab)$. The second thing can even hold if I and J have different lengths.

"Official hours" go on until next Thursday.

Chapter 12

05/12

12.1 More Technical Things About Dividing

Proof of Proposition 11.12.

$4 \Rightarrow 3$ Pick $\alpha \in \mathrm{Aut}(\mathbb{M}/Ab)$ such that $\alpha(a') = a$ and let $J = \alpha(J')$. Then $J \equiv_{Ab} J' \equiv_{Ab} I$. Moreover, since J' is Aa'-indiscernible, J is Aa-indiscernible.

$3 \Rightarrow 2$ Pick $\alpha \in \mathrm{Aut}(\mathbb{M}/Ab)$ moving J to I. Choose $a' = \alpha(a)$.

$2 \Rightarrow 1$ Let $p(x, b) := \mathrm{tp}(a/Ab)$. By Proposition 11.10 it is enough to show that if $I = \langle b_i \mid i < \omega \rangle$ is an indiscernible sequence with $b_0 = b$, then $\bigcup_{i \in I} p(x, b_i)$ is consistent. By hypothesis, there is $a' \vDash p(x, b)$ such that I is Aa'-indiscernible. In particular, $a' \vDash p(x, b_i)$ for all $i < \omega$.

$1 \Rightarrow 4$ Denote $p(x, b) = \mathrm{tp}(a/Ab)$ and let I be an A-indiscernible sequence containing b. By compactness, it is sufficient to consider the case $I = \langle b_i \mid i < \omega \rangle$. Say $b = b_{i_0}$ (we could also assume $i_0 = 0$). By hypothesis, $\bigcup_{i<\omega} p(x, b_i)$ is consistent. Let $a' \vDash \bigcup_{i<\omega} p(x, b_i)$. Since $a' \vDash p(x, b_{i_0})$, we have $a' \equiv_{Ab} a$. By the Standard Lemma, there is an Aa'-indiscernible sequence $J'' = \langle b''_i \mid i < \omega \rangle$ such that $\mathrm{EM}(J''/Aa') \supseteq \mathrm{EM}(I/Aa')$. In particular, by choice of a', we have $b''_{i_0} \vDash p(a', y)$. Hence $b''_{i_0} \equiv_{Aa'} b_{i_0} = b$. Let $\alpha \in \mathrm{Aut}(\mathbb{M}/Aa')$ be such that $\alpha(b''_{i_0}) = b$ and define $J' := \alpha(J'')$. Then J' is still Aa'-indiscernible, so we just need to check that $J' \equiv_{Ab} I$. Note that both I and J' contain b. Take any $L(Ab)$-formula $\varphi(x_{i_1}, \ldots, x_{i_n}, b)$, with $i_1 < \ldots < i_n$. Suppose $\vDash \varphi(b'_{i_1}, \ldots, b'_{i_n}, b = b'_{i_0})$. Then

$$\varphi(x_{i_1}, \ldots, x_{i_n}, x_{i_0}) \in \mathrm{EM}(J'/A) = \mathrm{EM}(J''/A) = \mathrm{EM}(I/A)$$

It follows that $\varphi(x_{i_1}, \ldots, x_{i_n}, x_{i_0}) \in \mathrm{EM}(I/A)$, and therefore $\vDash \varphi(b_{i_1}, \ldots, b_{i_n}, b)$. $\square$

Proposition 12.1. Suppose $A \subseteq B$ and $\mathrm{tp}(a/B)$ does not divide over A and $\mathrm{tp}(c/Ba)$ does not divide over Aa. Then $\mathrm{tp}(ac/B)$ does not divide over A.

Example 12.2. If a is transcendental over K and c is transcendental over $K(a)$, then ac is transcendental over K. (Say $A = \mathbb{Q}^{\mathrm{alg}}$). Note that you cannot just assume that $\mathrm{tp}(c/B)$ does not divide over A: think of what happens when $c = a^2$, say.

Proof of the Proposition. It is enough to show that $\mathrm{tp}(ac/Ab)$ does not divide over A for any finite $b \subseteq B$. Let I be an A-indiscernible sequence containing b. Since $\mathrm{tp}(a/Ab)$ does not divide over A, by Proposition 11.12 there is an Aa-indiscernible I' with $I' \equiv_{Ab} I$. In particular $b \in I'$. Since $\mathrm{tp}(c/(Aa)b)$ does not divide over A, there is a Aac-indiscernible sequence I'' containing b and such that $I'' \equiv_{Aab} I$. In particular, $I'' \equiv_{Ab} I$, and this implies that $\mathrm{tp}(ac/Ab)$ does not divide over A again by Proposition 11.12. $\square$

12.2 Simple Theories

This time too, we define the "bad" behaviour first.

Definition 12.3. A formula $\varphi(x, y)$ has the *k-tree property* iff there are elements $\langle b_\sigma \mid \sigma \in \omega^{<\omega} \rangle$ such that the following things hold.

- For every $\eta \in \omega^\omega$, the set $\{\varphi(x, b_{\eta \upharpoonright i}) \mid i \in \omega\}$ is consistent.
- For every $\sigma \in \omega^{<\omega}$, the set $\{\varphi(x, b_{\sigma^\frown i}) \mid i \in \omega\}$ is k-inconsistent.

We say that $\varphi(x, y)$ has the *tree-property* iff it has the k-tree property for some k.

The picture is a tree where every branch is consistent, but every time you pick the sons of a particular node, you have k-inconsistency.

Definition 12.4. A theory T is *simple* if no formula has the tree property.

Remark 12.5. Some comments:

1. The tree property implies the binary tree property. Hence stable theories are simple.[1]
2. DLO is not simple: let $\varphi(x, y)$ be $y_1 < x < y_2$. Then φ has the 2-tree property: start with an interval $(b_1^\Lambda, b_2^\Lambda)$. Find inside it infinitely many pairwise disjoint intervals. Iteratively, choose in each interval in infinitely many intervals. Then branches are intersections of encapsulated intervals, hence are consistent, since a finite intersection is just the smallest interval, but sons of the same node are pairwise disjoint, so 2-inconsistent.

Theorem 12.6. The following facts hold:

[1] Actually it is not this straightforward.

1. If T is simple, then T is NSOP.
2. T is stable if and only if T is simple and NIP.

Proof.

1. Exercise. Hint: extend the previous example. Get an infinite chain and then use compactness to get a dense infinite chain.
2. We already know that T is stable iff T is NSOP and NIP. Use the previous point for one inclusion and the previous Remark for the other one.

$\square$

Example 12.7. The triangle-free generic graph is NSOP but is not simple.

Definition 12.8 (Dividing Sequence)**.** Let $\Delta = \{\psi_1(x;y_1), \dots, \psi_\ell(x;y_\ell)\}$ be a finite set of partitioned formulas. A Δ-*k-dividing-sequence* over A is a sequence $\langle \varphi_i(x;b_i) \mid i < \delta \in \mathrm{On}\rangle$ such that

1. each $\varphi_i(x;b_i)$ k-divides over $A \cup \langle b_j \mid j < i\rangle$, and
2. $\varphi_i(x;y_i) \in \Delta$, and
3. $\{\varphi_i(x;b_i) \mid i < \delta\}$ is consistent.

In this case, we call δ the *length* of the Δ-k-dividing sequence.

Note the analogy with the DLO tree example: each node of the tree 2-divides over the previous parameters, and Δ is just $\{y_1 < x < y_2\}$.

Lemma 12.9. The following hold:

1. If $\varphi(x;y)$ has the k-tree property, then for every set of parameters A and ordinal μ there is a $\{\varphi\}$-k-dividing sequence over A of length μ.
2. If no formula in Δ has the k-tree property[2] there is no infinite Δ-k-dividing sequence.

Proof. The idea is simple, but it gets technical.

1. By compactness, we can assume the tree property for $\varphi(x;y)$ with a λ-branching tree of height μ for any $\lambda \geq \omega$. That is, there are parameters $\langle b_\sigma \mid \sigma \in \lambda^{<\mu}\rangle$ such that
   - for all $\sigma \in \lambda^{<\mu}$ the set $\{\varphi(x_i, b_{\sigma^\frown i}) \mid i < \lambda\}$ is k-inconsistent, and
   - for all $\eta \in \lambda^\mu$, the set $\{\varphi(x, b_{\eta\restriction i}) \mid i < \mu\}$ is consistent.

[2] In particular if we are in simple theory.

Let us take $\lambda = (2^{|T|+|A|+|\mu|})^+$. Then, we can construct a path $\eta \in \lambda^\mu$ and tuples $\langle c_\alpha \mid \alpha < \mu \rangle$ as follows. The idea is that we are branching so much that we get infinitely many guys with the same type. In other words, by pigeonhole and choice of λ, we can find an infinite sequence $\langle b_{\langle n_i \rangle} \mid i < \omega \rangle$ of tuples all with the same type over A. Let $c_0 = b_{\langle n_0 \rangle}$. Then, by construction, $\varphi(x, c_0)$ divides over A. Now iterate taking into account the previous parameters: if we have already constructed $\langle c_\beta \mid \beta < \alpha \rangle$, we have $c_\beta = b_{\eta \upharpoonright \beta}$ for some $\eta \in \lambda^\mu$. As the set $\{\varphi(x, b_{(\eta\upharpoonright\alpha)^\frown i}) \mid i < \lambda\}$ is k-inconsistent, we can find by pigeonhole and choice of λ an infinite sequence $\langle b_{(\eta\upharpoonright\alpha)^\frown i_n} \mid n < \omega \rangle$ of tuples all with the same type over $A \cup \{c_\beta \mid \beta < \alpha\}$. Let $c_\alpha = b_{(\eta\upharpoonright\alpha)^\frown i_0}$. This produces the required sequence.

2. Since Δ is finite, one of its the formulas $\varphi(x; y)$ will have been used infinitely often, and there are tuples $\langle b'_i \mid i < \omega \rangle$ such that $\varphi(x; b'_i)$ k-divides over $A \cup \{b'_j \mid j < i\}$. Construct $\langle b_\sigma \mid \sigma \in \omega^{<\omega} \rangle$ witnessing the k-tree property as follows. Find an A-indiscernible sequence $b'_0 = b_0^0, b_0^1, \ldots$ such that $\{\varphi(x; b_0^j) \mid j < \omega\}$ is k-inconsistent. This will be the first level of the tree: we set $b_{\langle j \rangle} = b_0^j$. Inductively, suppose we have $\langle b_\sigma \mid \sigma \in \omega^{\leq n} \rangle$ such that for all $\sigma, \sigma' \in \omega^n$ we have $b_\sigma \equiv_A b_{\sigma'}$ and such that $b_{\langle \underbrace{0, 0, \ldots, 0}_{n} \rangle} = b'_{n-1}$. Since $\varphi(x; b_n)$ k-divides over $Ab'_{<n}$ there is an $Ab_{\langle 0 \rangle} b_{\langle 0,0 \rangle}, \ldots, b_{\langle 0,0,\ldots,0 \rangle}$-indiscernible sequence $b'_n = b_n^0, b_n^1, \ldots$ such that $\{\varphi(x; b_n^j) \mid j < \omega\}$ is k-inconsistent. Set $b_{\langle 0,0,\ldots,0 j \rangle} := b_n^j$. For each $\sigma \in \omega^n$, let $\alpha \in \mathrm{Aut}(\mathbb{M}/A)$ be such that $\alpha_\sigma(b_{\langle 0,0,\ldots,0 \rangle}) = b_\sigma$. It is sufficient to set $b_{\sigma^\frown j} := \alpha_\sigma(b_{\langle 0,0,\ldots,0,j \rangle})$.

$\square$

This does not generalise to the case Δ infinite. Things can go wrong even in stable theories.

Chapter 13

08/12

13.1 Shelah's Local D-rank

This will not be used later in the course, but is common in the literature. It can be defined as "the foundational rank of k-dividing". What does it mean?

Definition 13.1. Let $\varphi(x, y)$ be a formula and $k \in \mathbb{N}$. We define $D(-, \varphi, k)$ on partial types inductively as follows:

- $D(\pi(x), \varphi, k) \geq 0$ iff $\pi(x)$ is consistent.
- $D(\pi(x), \varphi, k) \geq \alpha + 1$ iff there is b such that $D(\pi(x) \cup \{\varphi(x, b)\}, \varphi, k) \geq \alpha$ and $\varphi(x, b)$ k-divides over[1] $\operatorname{dom} \pi(x)$.
- $D(\pi(x), \varphi, k) \geq \alpha$, for α limit, iff for all $\beta < \alpha$ we have $D(\pi(x), \varphi, k) \geq \beta$.

Unwinding definitions we have

Proposition 13.2. $D(\pi(x), \varphi, k) \geq \alpha$ if and only if there is a φ-k-dividing sequence of length α on $\operatorname{dom} \pi(x)$ and consistent with $\pi(x)$.

From Lemma 12.9 it follows that

Corollary 13.3. T is simple if and only if for all $k < \omega$ and $\varphi(x, y) \in L$ we have $D(x = x, \varphi, k) < \omega$.

13.2 Independence and Morley Sequences

Definition 13.4. Let A, B, C be small subsets of $\mathbb{M}$. We say that A *is independent from* B *over* C, denoted $A \mathop{\smile\!\!\!\!\!\!|}_C B$ iff for every finite $a \in A$ the type $\operatorname{tp}(a/BC)$ does not fork over C.

[1] I.e. the parameters mentioned in π.

This is probably the most important definition in this course. It generalises a lot of notions of independence already used in mathematics.

Example 13.5. Let $C \subseteq B \models \mathsf{ACF}$. Let A be made of transcendentals over C. Then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $\mathrm{tr}_{\mathrm{deg}}(A/C) = \mathrm{tr}_{\mathrm{deg}}(A/B)$. Note that $\geq$ always holds. Compare with Example 10.7.

The general idea is that $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$ means that $B \cup C$ has more information about A than C does. Like, A is a criminal, C is a government, and B is another one.

Example 13.6. In vector spaces, $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $\dim(A/B \cup C) = \dim(A/C)$, where $\dim(A/C) = \dim(\langle A, C\rangle) - \dim(\langle C\rangle)$.

Definition 13.7. A *Morley sequence over A* is an A-indiscernible sequence $\langle b_i \mid i \in I\rangle$ such that $b_i \mathop{\smile\!\!\!\!\!\!\mid}_A \{b_j \mid j < i\}$. We denote $b_{<i} = \{b_j \mid j < i\}$.

Example 13.8. Let T be the theory of an equivalence relation with infinitely many infinite classes. An indiscernible sequence made of elements in different equivalence classes is a Morley sequence. An indiscernible sequence made of elements all in the same class is not a Morley sequence: since $xEc_1 \in \mathrm{tp}(c_2/Ac_1)$, we have $c_2 \not\mathop{\smile\!\!\!\!\!\!\mid}_A c_1$.

Example 13.9. In vector spaces, a sequence of linearly independent vectors.

Example 13.10. In strongly minimal theories, there is a unique non-dividing type in one variable. Morley sequences arise from it. In particular indiscernible sequences are either constant or Morley sequences.

Proposition 13.11. Let T be a complete theory. The following are equivalent:

1. T is simple.
2. (Local Character) For all $p \in S_n(B)$ there is some $A \subseteq B$ such that $|A| \leq |T|$ and p does not divide over A.
3. There is some cardinal κ such that if $p \in S_n(M)$ then there is $A \subseteq M$ of cardinality $\leq \kappa$ such that p does not divide over A.

We will see later that in simple theories forking equals dividing. So local character means that given a and B, there is a "small" A (in the cardinality sense) that already contains all the information that B had about a.

We need some preliminary lemmas.

Lemma 13.12 (Finite Character)**.** $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if for any finite $A_0 \subseteq A$ and finite $B_0 \subseteq B$ we have $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$.

Proof. Easy. □

Lemma 13.13. Suppose $\varphi(x, b)$ divides over A. The following hold:

1. For any small $B \supseteq A$ there is $B' \equiv_A B$ such that $\varphi(x, b)$ divides over B'.

2. For any small $B \supseteq A$ there is $b' \equiv_A b$ such that $\varphi(x, b')$ divides over B.

Proof. Of course, the proof involves automorphisms.

1. Use the other part and let $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ be such that $\sigma(b') = b$. Set $B' = \sigma(B)$.

2. By hypothesis there is $\langle b_i \mid i < \omega \rangle$ such that $b = b_0$, $b_i \equiv_A b$ and $\{\varphi(x, b_i) \mid i < \omega\}$ is k-inconsistent. In particular, $\mathrm{tp}(b_i/A)$ is non-algebraic, therefore $\{\sigma(b) \mid \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is not small by Lemma 2.10, and so there is σ_0 such that $\sigma_0(b) \notin B$. Call $b_0'' := \sigma_0(b_0)$ and $I_0 := \langle \sigma_0(b_i) \mid i < \omega \rangle$. Since $\mathrm{tp}(\sigma_0(b_1)/Ab_0'')$ is non-algebraic, there is $\sigma_1 \in \mathrm{Aut}(\mathbb{M}/Ab_0'')$ such that $b_1'' := \sigma_1(\sigma_0(b)) \notin B$. Let $I_1 := \langle \sigma_1(\sigma_0(b_i)) \mid i < \omega \rangle$, and notice it starts with b_0'', b_1''. Iterating this argument, we get a sequence $I'' = \{b_i'' \mid i < \omega\}$ which is

   - A-indiscernible,
   - such that $\{\varphi(x, b_i'') \mid i < \omega\}$ is k-inconsistent, and
   - such that $I'' \cap B = \emptyset$.

   By the Standard Lemma there is $J = \{b_i' \mid i < \omega\}$ which is B-indiscernible and such that $\mathrm{EM}(J/B) \supseteq \mathrm{EM}(I''/B)$. □

Proof of Proposition 13.11.

$1 \Rightarrow 2$ Suppose local character fails for $p \in S_n(B)$. Since no type over B divides over B, we can assume that $|B| \geq |T|^+$. By hypothesis, there is $\varphi_0(x, b_0) \in p$ that k_0-divides over $\emptyset$. Also, there is $\varphi_1(x, b_1) \in p$ that k_1-divides over b_0, and we can iterate this for all $\alpha < |T|^+$, getting a dividing sequence. By pigeonhole, there is $\langle i_j \mid j < \omega \rangle \subseteq |T|^+$ such that $\varphi_{i_j}(x, y) = \varphi(x, y)$ and $k_{i_j} = k$. Therefore there is an infinite φ-k-dividing sequence over $\emptyset$, call it $\langle \varphi(x, b_{i_j}) \mid j < \omega \rangle$. This shows that T is not simple.

$2 \Rightarrow 3$ Take $B = M$ and $\kappa = |T|$.

$3 \Rightarrow 1$ The idea is taking a sufficiently long φ-k-dividing sequence and then construct $M = \bigcup_{i<\kappa^+} M_i$ and use regularity of κ^+. Let us spell out the details.

If T is not simple, there is an infinite φ-k-dividing sequence $\langle \varphi(x, b_i) \mid i < \kappa^+ \rangle$. So $\varphi(x, b_i)$ k-divides over $\{b_j \mid j < i\}$, and $\{\varphi(x, b_i) \mid i < \kappa^+\}$ is consistent. If $\{b_i \mid i < \kappa^+\}$was a model M_i and $\{\varphi(x, b_i) \mid i < \kappa^+\} \subseteq p$ for some p we would be done. The second thing is of course equivalent to its consistency. The first one requires some work.

Claim. There are tuples $\langle b_i' \mid i < \kappa^+ \rangle$ and models $\langle M_i \mid i < \kappa^+ \rangle$ such that

- For all $i \leq j < \kappa^+$ the formula $\varphi(x, b'_j)$ divides over M_i.
- $|M_i| \leq \kappa$.
- $M_i \preceq M_j$ for $i < j < \kappa^+$.
- $\{\varphi(x, b_i) \mid i < \kappa^+\}$ is consistent.
- $b'_i \in M_{i+1}$.

Proof of Claim. We prove this by induction. Base case: since $\varphi(x, b_0)$ divides over $\emptyset$, by Lemma 13.13 there is a small model M_0 with $|M_0| \leq \kappa$ and $\varphi(x, b_0)$ divides over M_0. Take $b'_0 = b_0$. By Löwenheim-Skolem, there is a model $M_1 \supseteq M_0 \cup \{b_0\}$, with $M_0 \prec M_1$ and $|M_1| \leq |M_0 \cup \{b_0\}'| + |T| \leq \kappa$. Now M_1 is small and $\varphi(x, b_1)$ divides over $A = b'_0$, so there is $b'_1 \equiv_A b_1$ such that $\varphi(x, b'_1)$ divides over M_1, again by Lemma 13.13. Let $\sigma \in \mathrm{Aut}(\mathbb{M}/Ab'_0)$ move b_1 to b'_1. Let $I = \langle \sigma_1(b_i) \mid 2 \leq i < \kappa^+\rangle$. This has the same properties as the original sequence, but on top of b'_0, b'_1.

The successor case is basically the same as the case b_1: we have by induction $\langle M_i \mid i \leq \alpha\rangle$ and $\langle b'_i \mid i \leq \alpha\rangle$ and $I' = \langle b''_0, \ldots, b''_\alpha, b_{\alpha+1}, \ldots\rangle$. Call $I = \langle b_{\alpha+1}, \ldots, \rangle$ By Löwenheim-Skolem there is $M_{\alpha+1} \succ M_\alpha$ containing b_α and $|M_{\alpha+1}| \leq \kappa$. By the construction above, $\varphi(x, b_{\alpha+1})$ divides over $M_\alpha \cup b_{\leq\alpha}$, so there is $b'_{\alpha+1} \equiv_{M_\alpha \cup b_{\leq\alpha}} b_{\alpha+1}$ such that $\varphi(x, b'_{\alpha+1})$ divides over $M_{\alpha+1}$. If $\sigma \in \mathrm{Aut}(\mathbb{M}/M_\alpha \cup b_{\leq\alpha})$ sends $b_{\alpha+1}$ to $b'_{\alpha+1}$, rename I to be $\langle \sigma(b_i) \mid \alpha + 2 \leq i < \kappa^+\rangle$.

For the limit step: let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. This has still size $\leq \kappa$. We know that $\varphi(x, b_\alpha)$ divides over $A := \{b'_i \mid i < \alpha\}$. Apply the usual Lemma, find b'_α and rename the sequence.

Note that moving everything every time is what gets us consistency. We can let p be any completion of $\{\varphi(x, b_i) \mid i < \kappa^+\}$ to $S_x(M)$. $\square_{\text{CLAIM}}$

Now, by hypothesis, there is $A \subseteq M$ of size $|A| \leq \kappa$ such that p does not divide over A. Since κ^+ is regular we have $A \subseteq M_i$ for some $i < \kappa^+$, and $\varphi(x, b_{i+1}) \in p$ divides over $M_i \supseteq A$ by construction, so we get a contradiction. $\square$

Corollary 13.14. If T is simple, then any type $p \in S(A)$ does not fork over A.

Proof. Suppose $p \in S(A)$ forks over A, as witnessed by $p \vdash \bigvee_{\ell<m} \varphi_\ell(x, b_\ell)$. Let $\Delta := \{\varphi_\ell(x, y_\ell) \mid \ell < m\}$. We will show that there is an infinite Δ-k-dividing sequence. By compactness, it is enough to show that there are Δ-k dividing sequences of length n for arbitrarily large $n < \omega$, consistent with $p(x)$.

By induction, assume we have constructed $\langle \psi_i(x, a_i) \mid i < n\rangle$ which is Δ-k-dividing and is consistent with $p(x)$. By Lemma 13.13 we can assume that it is a Δ-k-dividing sequence over $Ab_0 \dots b_{m-1}$. Since p implies that disjunction, each extension of it must be consistent with one of the disjoints; so WLOG $\varphi_0(x, b_0)$ is consistent with $\{\psi_i(x, a_i) \mid i < n\}$. Now a Δ-k-dividing sequence over A is $\langle \varphi_0(x, b_0), \psi_0(x, a_0), \dots, \psi_{n-1}(x, a_{n-1}) \rangle$.

The base step is just the above with the empty sequence. $\square$

Fact 13.15. In DLO, if $b_1 < a < b_2$, then $a \not\downarrow_{\emptyset} b$ but $b \downarrow_{\emptyset} a$.

Chapter 14

12/12

The central thing today is this theorem. We will prove things around it. It is a characterisation of simplicity that does not mention forking (almost).

Theorem 14.1 (Kim-Pillay)**.** Let T be a complete theory and $\mathop{\smile\!\!\!\!\!\!|}^0$ a ternary relation between finite tuples and sets $a \mathop{\smile\!\!\!\!\!\!|}^0_A B$ satisfying:

0. (Invariance[1]) If $\alpha \in \mathrm{Aut}(M)$ then $a \mathop{\smile\!\!\!\!\!\!|}^0_A B$ if and only if $\alpha(a) \mathop{\smile\!\!\!\!\!\!|}^0_{\alpha(A)} \alpha(B)$.

1. (Existence) For every a, A, B, there is $a' \equiv_A a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^0_A B$.

2. (Finite Character) $a \mathop{\smile\!\!\!\!\!\!|}^0_A B$ if and only if for all finite $b \subseteq B$ we have $a \mathop{\smile\!\!\!\!\!\!|}^0_A b$.

3. (Monotonicity and Transitivity[2]) If $A \subseteq B \subseteq C$ then $a \mathop{\smile\!\!\!\!\!\!|}^0_A C$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}^0_A B$ and $a \mathop{\smile\!\!\!\!\!\!|}^0_B C$.

4. (Symmetry) $a \mathop{\smile\!\!\!\!\!\!|}^0_A b$ if and only if $b \mathop{\smile\!\!\!\!\!\!|}^0_A a$

5. (Local Character) There is a cardinal κ such that for all a and B there is $B_0 \subseteq B$ of size $|B_0| < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!|}^0_{B_0} B$.

6. (Independence Theorem) Let $M \models T$ such that

   - $a' \equiv_M b'$,
   - $a \mathop{\smile\!\!\!\!\!\!|}^0_M a'$,
   - $b \mathop{\smile\!\!\!\!\!\!|}^0_M b'$,
   - $a \mathop{\smile\!\!\!\!\!\!|}^0_M b$.

[1] This is satisfied for example by non-forking and by algebraic closure.
[2] Note that $\Rightarrow$ always holds for forking independence, in any theory.

Then there is c such that $c \equiv_{Ma} a'$, $c \equiv_{Mb} b'$ and $c \mathop{\smile\!\!\!\!\!\mid}^0_M ab$

Then T is simple and $\mathop{\smile\!\!\!\!\!\mid}^0$ is the forking independence $\mathop{\smile\!\!\!\!\!\mid}$.

This can be used to check that a theory is simple. For instance, in the random graph, if you put $a \mathop{\smile\!\!\!\!\!\mid}^0_A B$ iff $\operatorname{acl}(a) \cap \operatorname{acl}(B) = \operatorname{acl}(a) \cap \operatorname{acl}(A)$ (iff $a \cap B = a \cap B$, since the algebraic closure in the random graph is trivial), then all those conditions will be satisfied. This also tells us what forking independence is!

The converse holds: all those properties hold for $\mathop{\smile\!\!\!\!\!\mid}$ in a simple theory. We already saw that some of those do.

About the Independence Theorem: it implies that if we have $p(x,a) \cap q(x,b) \supseteq p_0 \in S(M)$ and $a \mathop{\smile\!\!\!\!\!\mid}_M b$ then there is $c \vDash p(x,a) \cup q(x,b)$ (so you can amalgamate the types) and moreover $c \mathop{\smile\!\!\!\!\!\mid}_M ab$. Here a', b' are realisations of p_0. Also, $a \mathop{\smile\!\!\!\!\!\mid}^0_M a'$ and $b \mathop{\smile\!\!\!\!\!\mid}^0_M b'$ say that $\operatorname{tp}(a'/Ma)$ does not fork over M, and similarly for b and b'.

Proving the Equivalence Theorem for simple theories is not easy. What we are going to do is check the other properties for simple theories and $\mathop{\smile\!\!\!\!\!\mid}$, then assume the Equivalence Theorem and see the Kim-Pillay one.

We start with a Corollary of Theorem 11.8 and Corollary 13.14.

Corollary 14.2 (Existence)**.** If T is simple, every type over A has a non-forking extension.

Proof. If T is simple and $p \in S(A)$, then by Corollary 13.14 p does not fork over A. But in any theory, by Theorem 11.8, if p does not fork over A, for any $B \supseteq A$ there is $q \supseteq p$ such that $q \in S(B)$ and q does not fork over A. □

Lemma 14.3. If $p \in S(B)$ does not fork over A, there is an infinite Morley sequence in[3] p over A which is B-indiscernible. If T is simple, every $p \in S(A)$ has an infinite Morley sequence.

Proof. Let $p_0 := p$ and take $a_0 \vDash p_0$. Since $a_0 \mathop{\smile\!\!\!\!\!\mid}_A A$ there is a non-forking extension p_1 of p over Ba_0. Take $a_1 \vDash p_1$. Iterate for a very large λ, getting $\langle a_i \mid i < \lambda \rangle$ such that $a_i \mathop{\smile\!\!\!\!\!\mid}_A Ba_{<i}$ for all $i < \lambda$. This is almost a Morley sequence, except it need not be indiscernible. By Erdös-Rado[4], there is a B-indiscernible sequence $\langle a'_i \mid i < \omega \rangle$ such that there are $i_1 < \ldots < i_n < \lambda$ such that $\operatorname{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \operatorname{tp}(a'_1, \ldots, a'_n/A)$. In particular, $\langle a'_i \mid i < \omega \rangle$ is still independent.

If T is simple, every p does not fork over A. □

The following says that you can characterise dividing in simple theories by just looking at Morley sequences.

[3] I.e. the elements of the sequence all realise p

[4] The Standard Lemma does not suffice because we also want to preserve independence.

Proposition 14.4 (Kim's Lemma)**.** Let T be simple. Let $\pi(x,b)$ be a partial type. Suppose $\langle b_i \mid i<\omega\rangle$ is a Morley sequence over A in $\mathrm{tp}(b/A)$ such that $\bigcup_{i<\omega}\pi(x,b_i)$ is consistent. Then $\pi(x,b)$ does not divide over A.

So Kim's Lemma says that you can check dividing just on one indiscernible sequence, provided it is Morley. In general you cannot just check $\bigcup_{i<\omega}\pi(x,b_i)$ for an arbitrary indiscernible sequence: for example, for equivalence relations, xEb divides over $\emptyset$, but there is some indiscernible sequence that does not witness dividing: take everything in the same equivalence class. Anyway, if you take an indiscernible sequence with everything in different classes, it will witness dividing.

Proof of the Lemma. By compactness and the previous Lemma (and possibly Erdös-Rado[5].), for any linear order I there is a Morley sequence $\langle b_i \mid i\in I\rangle$ in $\mathrm{tp}(b/A)$. Do this with $I=(|T|^+)^*$, where the $*$ denotes taking the reverse order. Also, we can construct the sequence in such a way that $\bigcup_{i\in I}\pi(x,b_i)$ is consistent. Let c realise it. By local character, there is $i_0\in I$ such that $\mathrm{tp}(c/A\cup\{b_i \mid i\in I\})$ does not divide over $A\cup\{b_i \mid i>i_0\}$ (by regularity of $|T|^+$; you can take an end segment because if $B\supseteq A$ and you don't fork over A then you don't fork over B). Write this as $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{Ab_{>i_0}} \langle b_i \mid i\in I\rangle$, where the "d" is for "dividing". In particular, $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{Ab_{>i_0}} b_{\geq i_0}$. Put $B=Ab_{i_0}$ and $a=\langle b_i \mid i>i_0\rangle$. Then $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{Aa} Ba$. On the other hand, by A-independence, $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{A} B$. Therefore, $ac \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{A} B$ by Proposition 12.1. Hence, $\mathrm{tp}(c\langle b_i \mid i>i_0\rangle/Ab_{i_0})$ does not divide over A. In particular, $\pi(x,b_{i_0})\subseteq\mathrm{tp}(c/b_{i_0})$ does not divide over A. By automorphisms, $\pi(x,b)$ does not divide over A. □

Proposition 14.5. Let T be simple. Then $\pi(x,b)$ divides over A if and only if $\pi(x,b)$ forks over A. In other words, in simple theories forking equals dividing.

Proof. Dividing always implies forking, so let's take care of the other implication. Suppose $\pi(x,b)$ does not divide over A, and suppose $\pi(x,b)\vdash\psi(x,b)=\bigvee_{\ell<m}\varphi_\ell(x,b)$. We will show that for some $\ell<m$ the formula $\varphi_\ell(x,b)$ does not divide over A. Let $\langle b_i \mid i<\omega\rangle$ be a Morley sequence over A in $\mathrm{tp}(b/A)$. Then $\{\psi(x,b_i) \mid i<\omega\}$ is consistent because $\pi(x,b)\vdash\psi(x,b)$ and ψ does not divide over A by hypothesis. In particular, there is $\ell<m$ and an infinite $I\subseteq\omega$ such that $\{\varphi_\ell(x,b_i) \mid i\in I\}$ is consistent. Since $\langle b_i \mid i\in I\rangle$ is still a Morley sequence over A in $\mathrm{tp}(b/A)$, by Kim's lemma $\varphi_\ell(x,b)$ does not divide over A. □

Corollary 14.6. Forking in simples theories satisfies Local Character.

Proof. We know it for dividing. □

[5] To say "I'm independent" you write down the negation of all forking formulas

Proposition 14.7. The following hold in a simple theory:

Symmetry $A \perp_C B \iff B \perp_C A$

Monotonicity and Transitivity Suppose $B \subseteq C \subseteq D$. Then $A \perp_B C$ if and only if $A \perp_C D$ and $A \perp_B C$.

Proof.

Symmetry By finite character, it suffices to show that $a \perp_C b \Rightarrow b \perp_C a$. Suppose $a \perp_C b$. By the previous results there is a Morley sequence $\langle a_i \mid i < \omega \rangle$ over C in $\mathrm{tp}(a/Cb)$ which is indiscernible over Cb. Let $p(x, y) = \mathrm{tp}((a, b)/C)$. We will show that $\bigcup_{i<\omega} p(a_i, y)$ is consistent, and by Kim's Lemma and forking equals dividing this suffices. But by construction $b \models \bigcup_{i<\omega} \varphi(a_i, y)$.

Monotonicity and Transitivity We already know one implication. We already know that $a \perp^{\mathrm{d}}_A B$ and $c \perp^{\mathrm{d}}_{Aa} Ba$ implies $ac \perp^{\mathrm{d}}_A B$ by Proposition 12.1. Notice that by definition $c \perp^{\mathrm{d}}_{Aa} Ba \iff c \perp^{\mathrm{d}}_{Aa} B$. Since now we know symmetry and forking equals dividing, we also know that

$$B \perp_A a \text{ and } B \perp_{Aa} C \Rightarrow B \perp_A ac$$

Replacing

- B by A
- A by B
- Aa by C
- Aac by D

we get (also moving things from the bottom to the right)

$$A \perp_B C \text{ and } A \perp_C D \Rightarrow A \perp_B D$$ □

The only thing left to prove for forking in simple theories is the Independence Theorem.

The results mentioned today can be used in the homework.

Chapter 15

15/12

I was not there. Proof of the Kim-Pillay theorem. See the official notes.