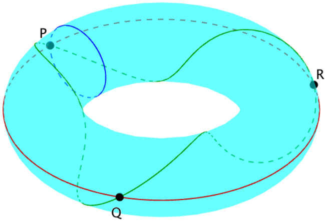
Roberto Pagaria

Scuola Normale Superiore

# Orlik-Solomon Relations for Toric Arrangements

Dobbiaco
February 21, 2018



Covered topics:

# Definitions

An *hyperplane arrangement* $\mathcal{A}^H$ in a vector space $V$ is a finite collection of (affine) hyperplanes $\{H_e\}_{e \in E}$.
The *complement* of the arrangement $\mathcal{A}^H$ is $M(\mathcal{A}^H) = V \setminus \bigcup_{e \in E} H_e$.

**Problem:** describe $H^\bullet(M(\mathcal{A}^H); \mathbb{Q})$.

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Each hyperplane $H_e$ can be defined as the zero locus of an element $\alpha_e \in V^*$ or equivalently of $-\alpha_e$. Consider the logarithmic form $\omega = \frac{\mathrm{d}z}{z}$ on $\mathbb{C}^*$ and $\omega_e = \frac{\mathrm{d}\alpha_e}{\alpha_e} = \alpha_e^*(\omega)$ on $V$.

## Remark

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**Remark**

The form $\omega_e$ does not depend on the sign of $\alpha_e$.

Define $\omega_I = \omega_{i_1}\omega_{i_2}\cdots\omega_{i_k}$ for every list $I = (i_1, \dots, i_k) \subseteq E$.
If $\alpha_{I_1}, \alpha_{I_2}, \dots, \alpha_{i_k}$ are linearly dependent, then $\omega_I = 0$.

# OS algebra

**OS-relations:** If $C \subset E$ is a minimal dependent set, then

$$\partial \omega_C := \sum_{i=0}^{k} (-1)^i \omega_{C \setminus c_i} = 0$$

or equivalently

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**Theorem (Orlik-Solomon 1980)**

*The algebra $H^\bullet(M(\mathcal{A}))$ is isomorphic to the external algebra on the set $\{\omega_e\}_{e \in E}$ with relations: $\omega_I = 0$ for any $I$ dependent and $\partial \omega_C = 0$ for any $C$ circuit.*

# Definitions

A *toric arrangement* $\mathcal{A}$ in the torus $T \simeq (\mathbb{C}^*)^n$ is a finite collection of (translates of) hypertori $\{D_e\}_{e \in E}$. Let $\Lambda \simeq \mathbb{Z}^n$ be the character group of $T$ and $\chi_e \in \Lambda$ a character defining $D_e$.

**In coordinates:** the characters are $\chi(t_1, \dots, t_n) = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ and the hypertori are

$$\{(t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} = b\}$$

The equations $\chi(\mathbf{t}) = b$ and $(-\chi)(\mathbf{t}) = b^{-1}$ define the same hypertorus.

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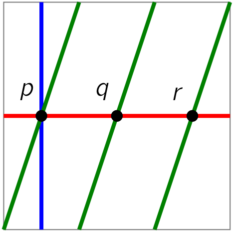
**Definition**

We say that $I \subset E$ is (in)dependent if the characters $\{\chi_e\}_{e\in I} \subset \Lambda \simeq \mathbb{Z}^n$ are linearly (in)dependent.

We want to study $M(\mathcal{A}) = T \setminus \bigcup_{e\in E} D_e$.

# Example




Analogously to the case of hyperplane arrangements, we define

$$\omega_e = (1-\chi_e)^*\omega = -\frac{\mathrm{d}(t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n})}{1 - t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}}.$$

Observe that

$$\omega_1 \cdot \omega_2 = \omega_{p,1,2} + \omega_{q,1,2} + \omega_{r,1,2};$$

these two-forms are linearly independent.

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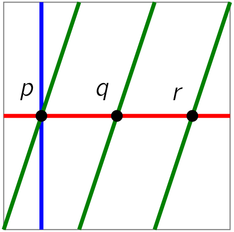


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Choose $f_p = x^2 + x + 1$ and define the form:

$$\omega_{p,1,2} := f_p \cdot \omega_1 \cdot \omega_2 = (x^2 + x + 1)\mathrm{d}\log(1 - x^3 y) \wedge \mathrm{d}\log(1-y)$$

The form $\omega_{p,1,2}$ depends on $f_p$, choosing $\tilde{f}_p = \frac{1}{y}(x^2 + x + 1)$ instead:

$$\tilde{\omega}_p := \tilde{f}_p \cdot \omega_1 \cdot \omega_2 = \omega_{p,1,2} + \omega_2 \cdot \mathrm{d}\log y$$

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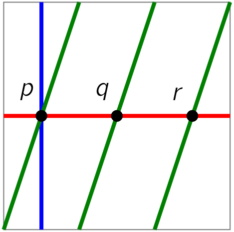


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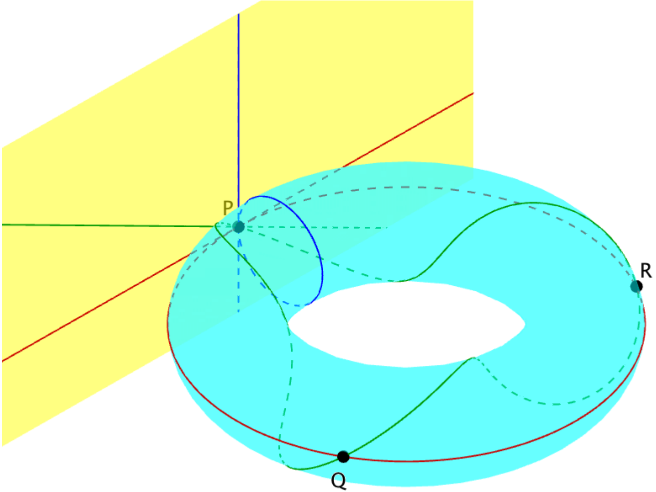
## Observation

*Because intersections of hypertori are, in general, not connected, the cohomology algebra is not always generated in degree one.*

We consider only forms $\omega_{W,A}$ where $W$ is a c.c. of $\cap_{a \in A} D_a$.




Consider the exponential map $\mathrm{T}_P\, T \to T$ and its pullback
$H^\bullet(M(\mathcal{A})) \twoheadrightarrow H^\bullet(M(\mathcal{A}[P]))$.
The forms $\omega_{Q,1,2}$, $\omega_{R,1,2}$ and those in $H^\bullet(T)$ belong to the kernel.

# The cohomology module

By the results about hyperplane we have:

$$\omega_{P,12} - \omega_{13} + \omega_{23} \equiv 0 \quad (H^1(T))$$

and more generally for $W$ c.c. of $\cap_{c \in C} D_c$

$$\partial \omega_C := \sum_{i=0}^{k} (-1)^i \omega_{W, C \setminus c_i} \equiv 0 \quad (H^1(T)) \tag{1}$$

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**Theorem (De Concini, Procesi 2005)**

*The graded ring* $\mathrm{gr}_{(H^1(T))} H^\bullet(M(\mathcal{A}))$ *is generated by* $\omega_{W,A}\psi$*, where* $\psi$ *is any element in* $H^\bullet(W)$ *with the relations of eq.* (1) *and multiplication given by*

$$\omega_{W,A}\omega_{W',A'} = \pm \sum_{L \text{ c.c. } W \cap W'} \omega_{L,AA'}$$

Analogous results are given by Bibby ('15) and Callegaro, Delucchi ('15).

# Unimodular case

A hypertori arrangement is *unimodular* if all intersections $\cap_{i \in A} D_a$ are empty or connected.
We choose $f_W = 1$, so $\omega_{W,A} = \omega_{a_1} \cdots \omega_{a_q}$ and define $\psi_e = \chi_e^*(\omega) \in H^1(T)$.

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**Theorem (De Concini, Procesi 2005)**

*If $\chi_0 = \chi_1 + \cdots + \chi_q$, then for the circuit $C = (0, 1, \ldots, q)$ the following relation in cohomology holds.*

$$\partial\omega_C = \sum_{\substack{0\in A \\ B\neq\emptyset}} (-1)^{\epsilon(A)} \omega_A \psi_B$$

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**Proof.**

Follows from the polynomial identity:

$$1 - \prod_{i=1}^{q} x_i = \sum_{I \subsetneq [q]} \prod_{i \in I} x_i \prod_{j \notin I} (1 - x_j)$$

□

When $\chi_0 + \chi_1 + \cdots + \chi_q = 0$ the following factorization holds:

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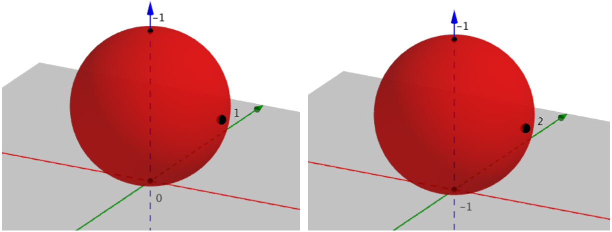
Choose now the following canonical form for every hyperplane $D_e$:

$$\overline{\omega}_e := \omega_e + \omega_e' = 2\omega_e - \psi_e = \frac{x_e + 1}{x_e(x_e - 1)} \mathrm{d}x_e$$




(e) Residues of $\omega_e$ (f) Residues of $\overline{\omega}_e$

## Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2017)

*If $\mathcal{A}$ is unimodular, the relations in cohomology are:*

$$\prod_{i=1}^{q}(\overline{\omega}_i + c_i\psi_i - \overline{\omega}_{i-1} + c_{i-1}\psi_{i-1}) = 0$$

*where* $\sum_i c_i\chi_i = 0$, $c_i = \pm 1$ *or, equivalently:*

$$\sum_{j=0}^{k}\sum_{A \not\ni j}(-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0$$

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Notice that $c_i\psi_i \in H^1(T)$ does not depend on the choice between $\chi_i$ and $-\chi_i$. A central arrangement is invariant for $z \mapsto z^{-1}$, hence:

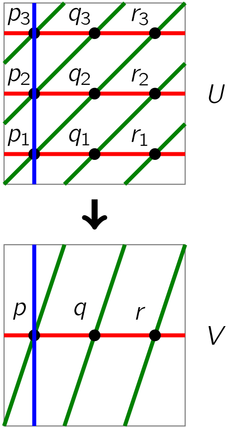
$$\sum_{j=0}^{k}\sum_{\substack{A \not\ni j \\ |B| \text{ even}}}(-1)^{|A_{\leq j}|}c_B\overline{\omega}_A\psi_B = 0 \tag{2}$$

# Coverings

Consider the covering $U \to T$ of the tori $u = x$, $v^3 = y$. The hypertori lift to:

$$1 - y \quad \mapsto \quad 1 - v^3$$

$$1 - x^3 y \quad \mapsto \quad 1 - u^3 v^3$$



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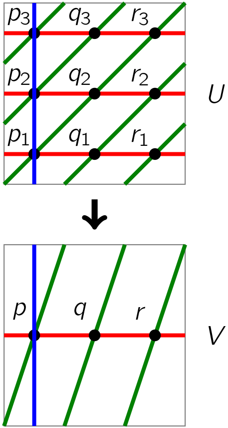
A form in $U$ is

$$\overline{\omega}^U_{p_1,1,2} = \overline{\omega}^U_1 \overline{\omega}^U_2 = \frac{v+1}{v(v-1)} \frac{uv+1}{uv(uv-1)} v \mathrm{d}u \mathrm{d}v,$$



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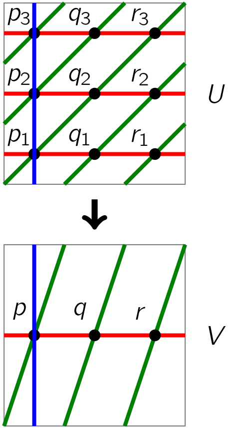
its pushforward is:

$$\overline{\omega}_{p,1,2} := \overline{\omega}^U_{p_1,1,2} + \overline{\omega}^U_{p_2,1,2} + \overline{\omega}^U_{p_3,1,2} =$$

$$= 3 \frac{u^3 v^6 + u^3 v^3 + 4u^2 v^3 + 4uv^3 + v^3 + 1}{uv\left(v^3 - 1\right)\left(u^3 v^3 - 1\right)} \mathrm{d}u \mathrm{d}v$$



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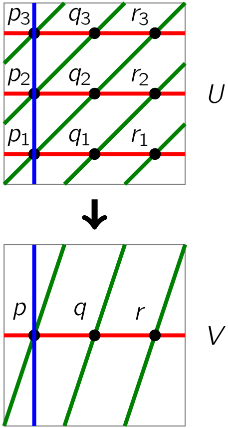
$$= \frac{x^3 y^2 + x^3 y + 4x^2 y + 4xy + y + 1}{xy\left(y - 1\right)\left(x^3 y - 1\right)} \mathrm{d}x \mathrm{d}y$$

$$= f_p \overline{\omega}_1 \overline{\omega}_2$$



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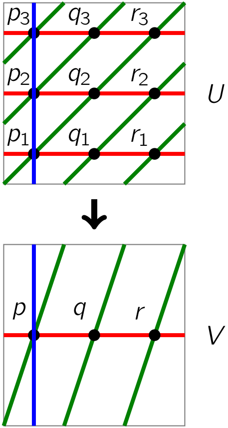
Remember eq. (2)

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0 \qquad (2)$$



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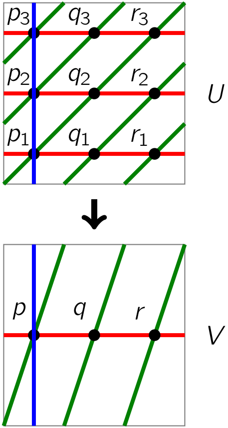
$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \frac{m(A)}{m(A \sqcup B)} c_B \overline{\omega}_{W,A} \psi_B = 0.$$



In our example the formula is:

$$\frac{x^3y^2+x^3y+4x^2y+4xy+y+1}{xy\left(y-1\right)\left(x^3y-1\right)}\mathrm{d}x\mathrm{d}y-\frac{x+1}{x(x-1)}\frac{y+1}{y(y-1)}\mathrm{d}x\mathrm{d}y+$$
$$+\frac{x+1}{x(x-1)}\frac{x^3y+1}{y(x^3y-1)}\mathrm{d}x\mathrm{d}y-\frac{1}{3}\mathrm{d}\log y\mathrm{d}\log x^3y+\mathrm{d}\log y\mathrm{d}\log x+$$
$$-\mathrm{d}\log x^3y\mathrm{d}\log x=0.$$

**Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2017)**

*The algebra $H^\bullet(M(\mathcal{A});\mathbb{Q})$ is generated by $\overline{\omega}_{W,A}$, $\psi_e$ with relations for every circuit $C$*

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**Thanks for listening!**