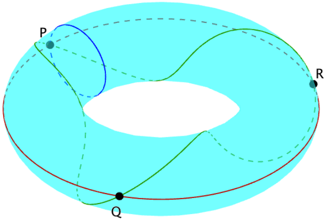
Roberto Pagaria

Scuola Normale Superiore

Arithmetic Matroids and their Representations

Combinatorics Seminar



at KTH Royal Institute of Technology

April 3, 2019

Hyperplane arrangements

We will see:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] \arrow[dr] & \text{OMat} \arrow[r] & \text{Mat} \arrow[dl, leftrightarrow] \\
M(\mathcal{A}_{\mathbb{C}}) & \mathcal{L}(\mathcal{A}) \arrow[l, dashed]
\end{tikzcd}$$

Where:

- $\mathcal{A}$ is a real hyperplane arrangement,
- $M(\mathcal{A}_{\mathbb{C}})$ is the complement,
- Mat and OMat are a matroid and an oriented matroid,
- $\mathcal{L}(\mathcal{A})$ is the lattice of intersections.

A real **hyperplane arrangement** $\mathcal{A}$ is a finite collection of hyperplanes H_i, $i \in E$ in $\mathbb{R}^k$.
The **complement** $M(\mathcal{A}_{\mathbb{C}})$ is the topological space $\mathbb{C}^k \setminus \cup_{H \in \mathcal{A}} H \otimes \mathbb{C}$.

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The **poset of intersections** is the partially ordered set whose elements are $\cap_{H \in I} H$, for $I \subseteq E$ ordered by reverse inclusion.

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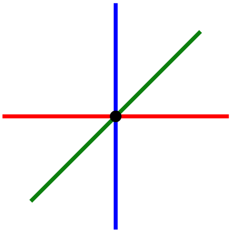

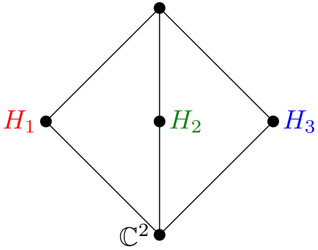

The poset of intersections is a **geometric lattice**, i.e. a graded atomistic semimodular lattice.

A **matroid** is a finite set E with a rank function $\mathrm{rk}\colon 2^E \to \mathbb{N}$ that satisfies:

1. $\mathrm{rk}(I) \leq |I|$,
2. if $I \subset J$ then $\mathrm{rk}(I) \leq \mathrm{rk}(J)$,
3. $\mathrm{rk}(I \cap J) + \mathrm{rk}(I \cup J) \leq \mathrm{rk}(I) + \mathrm{rk}(J)$.

A **matroid** is a finite set E with a rank function rk: $2^E \to \mathbb{N}$ that satisfies:

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A k-**flat** is a subset $I \subseteq E$ maximal among all subsets of rank k.
A set I is **(in-)dependent** if $\mathrm{rk}(I) < |I|$ (resp. $\mathrm{rk}(I) = |I|$).
A **basis** $B \subseteq E$ is an independent set such that $\mathrm{rk}(E) = \mathrm{rk}(B) (= |B|)$.
A **circuit** $C \subseteq E$ is a minimal dependent set.

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Facts:

1. the poset of flats coincides with the poset of the intersections.
2. the matroid is uniquely determined by its bases.

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Example

The previous arrangement defines the matroid $E = \{1, 2, 3\}$ and $\mathrm{rk}(I) = \mathrm{codim}(\cap_{i \in I} H_i) = \min\{|I|, 2\}$.
The flats are $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}$.
The bases are $\{1, 2\}, \{1, 3\}, \{2, 3\}$.

Explicit construction

Any hyperplane H_i in $\mathbb{R}^k$ is defined by $v_i \in (\mathbb{R}^k)^*$, unique up to scalars. An arrangement $\mathcal{A}$ can be defined by a matrix

$$V = (v_i) \in M(k, n; \mathbb{R}).$$

For each $I \in [n]$, let $V[I] = (v_i)_{i \in I}$.

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For each $I \in [n]$, let $V[I] = (v_i)_{i \in I}$.

The matroid associated with A is the set $[n]$ with rank function

$$\text{rk}(I) := \text{rank}(V[I])$$

and does not depend on the choice of V.

Oriented matroids

Definition

A **chirotope** of rank k over E is $\chi\colon E^k \to \{0, 1, -1\}$ such that

1. $\chi(\sigma\underline{x}) = \mathrm{sgn}(\sigma)\chi(\underline{x})$ for each $\sigma \in \mathfrak{S}_k$,
2. $(-1)^i\chi(y_i, x_2, \ldots, x_k)\chi(x_1, y_1, \ldots, \hat{y}_i, \ldots, y_k) \geq 0$ for all i, then $\chi(\underline{x})\chi(\underline{y}) \geq 0$.

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Example: The chirotope defined by an arrangement $\mathcal{A}$ in $\mathbb{R}^k$ is of rank k and is defined by:

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Definition

Two chirotopes χ and χ' are equivalent if there exists $A \subseteq E$ such that

$$\chi'(\underline{x}) = (-1)^{|A \cap \underline{x}|}\chi(\underline{x}).$$

Tutte polynomial

Definition

The **Tutte polynomial** of a matroid (E, rk) is

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The Poincaré polynomial of $M(\mathcal{A}_{\mathbb{C}})$ coincides with

$$P_{M(\mathcal{A}_{\mathbb{C}})}(q) = q^n T\left(\frac{q+1}{q}, 0\right),$$

where T is the Tutte polynomial of the matroid represented by $\mathcal{A}$.

Topology

The cohomology algebra $H^{\bullet}(M(\mathcal{A}_{\mathbb{C}}))$ is isomorphic to the Orlik-Solomon algebra of the associated matroid.

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However, the homotopy type is not determined by the matroid as shown by Rybnikov.

Definition

The **OS-algebra** of a matroid (E, rk) is the external algebra on generators ω_e for $e \in E$ and relations

$$\sum_{i=1}^{r} (-1)^i \omega_{c_1} \dots \hat{\omega}_{c_i} \dots \omega_{c_r} = 0$$

for each circuit $C = \{c_1, \dots, c_r\} \subset E$.

Hyperplane arrangements

We have seen:

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Toric arrangements

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$$\begin{tikzcd}
\mathcal{A}_T \arrow[r] \arrow[d] \arrow[dr] & \text{OAMat} \arrow[l, dashed, bend right] \arrow[r, hook] & \text{AMat} \\
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Where:

- $\mathcal{A}_T$ is a toric arrangement,
- $M(\mathcal{A}_T)$ is the complement,
- AMat and OAMat are arithmetic matroid and orientable arithmetic matroid,
- $\mathcal{S}(\mathcal{A}_T)$ is the lattice of layers.

An hypertorus T in an algebraic torus $(\mathbb{C}^*)^k$ is the set

$$T = \{(t_1, \dots, t_k) \in (\mathbb{C}^*)^k \mid t_1^{v_1} t_2^{v_2} \cdots t_k^{v_k} = 1\}$$

for a vector $(v_i) \in \mathbb{Z}^k$.

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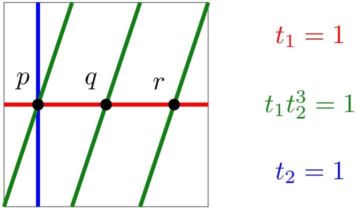
A **toric arrangement** $\mathcal{A}_T$ is a collection of hypertori H_i, for $i \in [n]$.

Example: the integer matrix

$$V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

defines the following toric arrangement:



Poset of layers

A **layer** is a connected component of the intersection of some hypertori of $\mathcal{A}_T$.
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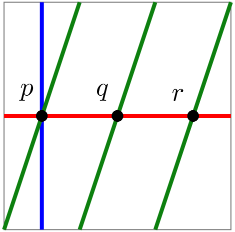

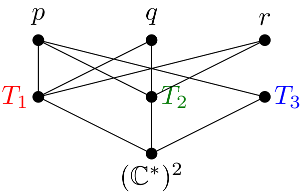

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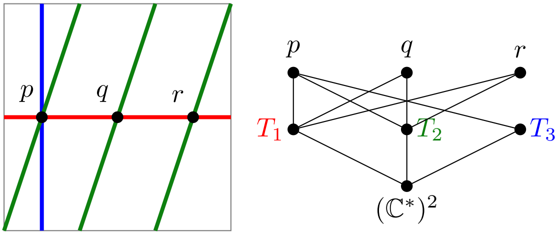

Fact: The poset of layers is a geometric semilattice.

Arithmetic matroids

A **molecule** is $S = R \sqcup T \sqcup F \subseteq E$ such that $\mathrm{rk}(R \sqcup T) = \mathrm{rk}(R)$ and $\mathrm{rk}(R \sqcup F) = \mathrm{rk}(R) + |F|$.

Definition (Brändén – Moci, D'Adderio – Moci 2014)

An **arithmetic matroid** is a matroid (E, rk) together with a multiplicity function $m\colon 2^E \to \mathbb{N}_+$ such that:

1. if $\mathrm{rk}(I \cup e) = \mathrm{rk}(I)$, then $m(I \cup e) | m(I)$; otherwise $m(I) | m(I \cup e)$,
2. for each molecule $m(R)m(R \cup T \cup F) = m(R \cup F)m(R \cup T)$,
3. for each molecule $\sum_{R \subseteq I \subseteq S} (-1)^{|R|+|F|-|I|} m(I) \geq 0$.

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Example

A toric arrangement A_T defines an arithmetic matroid by $\text{rk}(I) := \text{codim}(\cap_{i \in I} T_i)$ and $m(I) =$ # c.c. of $\cap_{i \in I} T_i$.

Explicit construction

An hypertorus T is the set

$$\{(t_1, \dots, t_k) \in (\mathbb{C}^*)^k \mid t_1^{v_1} t_2^{v_2} \cdots t_k^{v_k} = 1\}$$

for a vector $(v_i) \in \mathbb{Z}^k$. We collect these data in a matrix

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This matrix is defined up to left multiplication by GL$(k, \mathbb{Z})$ and reverse sign of the columns.

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The associated arithmetic matroid is defined by

$$\mathrm{rk}(I) := \mathrm{rank}(V[I])$$

and by

$$m(I) := \gcd_{|J|=|I|} |\det(V[I]_J)|.$$

Orientable arithmetic matroid

An **oriented arithmetic matroid** is (E, χ, m) such that (E, χ) is an oriented matroid and (E, rk, m) is an arithmetic matroid with the compatibility condition: for all $x_2, \dots, x_k$ and $y_0, \dots, y_k$

$$\sum_{i=0}^{k} (-1)^i \chi(\underline{x}_i) m(\underline{x}_i) \chi(\underline{y}^i) m(\underline{y}^i) = 0, \qquad \text{(GP)}$$

where $\underline{x}_i = (y_i, x_2 \dots, x_k)$ and $\underline{y}^i = (y_0, \dots y_{i-1}, y_{i+1} \dots y_k)$.

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Condition (GP) involves only the value of m on the bases of (E, rk).

Theorem (P. 2018)

If an arithmetic matroid is orientable then the orientation is unique up to re-orientation.

Representability problem

We ask whether an arithmetic matroid is representable and how many different representations exist.

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An arithmetic matroid is **strong GCD** if for all $I \subset E$

$$m(I) = \text{gcd}\{m(B) | B \text{ basis and } |B \cap I| = \text{rk}\, I\}.$$

Theorem (P. 2018)

Suppose that $m(\emptyset) = m(E) = 1$, then (E, rk, m) is representable if and only if

1. *it is orientable,*
2. *it is strong GCD.*

Moreover, the representation is unique.

Arithmetic Tutte polynomial

Definition (Moci 2011)

The **arithmetic Tutte polynomial** of an arithmetic matroid (E, rk) is

$$T'(x,y) := \sum_{A \subseteq E} m(A)(x-1)^{\mathsf{rk}(E)-\mathsf{rk}(A)}(y-1)^{|A|-\mathsf{rk}(A)}.$$

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The Poincaré polynomial of $M(\mathcal{A}_T)$ coincides with

$$P_{M(\mathcal{A}_T)}(q) = q^n T'\left(\frac{2q+1}{q}, 0\right),$$

where T' is the arithmetic Tutte polynomial of the arithmetic matroid represented by $\mathcal{A}_T$.

Topology

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2018)

The rational cohomology algebra of the complement $M(\mathcal{A}_T)$ is generated by ψ_i and by $\omega_{W,I}$, for I independent and W c. c. of $\cap_{i \in I} T_i$, with relations

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The rational cohomology algebra of the complement $M(\mathcal{A}_T)$ is generated by ψ_i and by $\omega_{W,I}$, for I independent and W c. c. of $\cap_{i\in I} T_i$, with relations

$$\omega_{W_1,I_1}\omega_{W_2,I_2} = \pm \sum_{L \text{ c. c. } W_1 \cap W_2} \omega_{L,I_1 \sqcup I_2} \tag{1}$$

$$\omega_{W,I}\psi_i = 0 \quad \text{if} \quad i \in I \tag{2}$$

$$\sum_{j\in C} \sum_{\substack{A\sqcup B \sqcup \{j\} = C \\ |B| \text{ even}}} (-1)^{|A_{\le j}|} c_B \frac{m(A)}{m(A\cup B)} \omega_{W,A}\psi_B = 0, \tag{3}$$

$$\sum_{j\in C} (-1)^j c_{C\setminus j} m(C\setminus j)\psi_{C\setminus j} = 0 \tag{4}$$

where C is a circuit and $c_B \in \{\pm 1\}$ depending on the chosen orientation.

Example: The two toric arrangements described by

$$N_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 7 & 7 \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 7 \end{pmatrix},$$

have the same poset of layers but different integral cohomology algebras.

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Example: The two toric arrangements described by

$$N_3 = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} \text{ and } N_4 = \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix},$$

have different posets of layers and different rational cohomology algebra. However, they have the same arithmetic matroid.

Toric arrangements

We have seen:

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