SCUOLA NORMALE SUPERIORE

Cohomology and Combinatorics of Toric Arrangements

Ph.D. thesis in Mathematics

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A tutti i morti nel Mediterraneo.

To all the dead in the Mediterranean.

Contents

Introduction

This Ph.D. thesis presents my results obtained in the last three years. These results have appeared in the following preprints and articles: [Pag19b, Pag19d, CDD$^+$18, Pag18a, Pag18c, Pag19c, Pag18b, Pag19a, PP19b]. Initially, I investigated toric arrangements, a type of arrangements inspired by the hyperplane ones. Toric arrangements have been studied intensively in the last fifteen years both from topological and from combinatorial point of view. My results describe the cohomology ring of toric arrangements and their dependency from the combinatorial data. Later I have worked on another type of arrangements, i.e. elliptic arrangements, which are tougher than the toric case. I focused on the most regular case, i.e. the braid arrangements, that coincides with the configuration spaces of points in an elliptic curve. I have obtained some results on the unordered configuration space of points in an elliptic curve, and I have generalized some of them to configurations on closed orientable surfaces. Only very recently I have made some conjectures about the cohomology of braid elliptic arrangements.

Main results

An arithmetic matroid is a generalization of a matroid encoding a multiplicity function. It is used to encode the combinatorics of toric arrangements, but the theory is different from the classic setting. One main difference is the lack of a criptomorphism between arithmetic matroids and posets of layers, as shown by Theorem C. Another difference involves the representability problem: in the classical case is very hard to determine if a matroid is representable. While, in the arithmetic setting we have the following theorem.

Theorem A. *A surjective, torsion-free arithmetic matroid is representable if and only if it is strong GCD and orientable. In this case there exists a unique representation.*

Moreover, a torsion-free arithmetic matroid has a finite number of representations, explicitly classified.

Theorem A is proven by developing a theory of "orientable arithmetic matroids" that combines arithmetic matroids with orientable matroids. The proof involves both combinatorics and linear algebra, such as the Smith normal form.

In a joint work with Filippo Callegaro, Michele D'Adderio, Emanuele Delucchi and Luca Migliorini, we provide a description of the cohomology algebra of the complement of a toric arrangement. Our presentation is given by generators and relations, following the analogous Orlik-Solomon presentation for hyperplane arrangements.

Theorem B. *The rational cohomology algebra of a toric arrangement $\mathcal{A}$ is the algebra generated by $e_{W,A;B}$ for $W \in \mathcal{S}(\mathcal{A})$ with relations:*

$$e_{W,A;B}e_{W',A';B'} = \pm \sum_{L \in \pi_0(W \cap W')} e_{L,A\cup A';B\cup B'},$$

$$\sum_{i \in E} n_i e_{T,\emptyset;\{i\}} = 0,$$

$$\sum_{\substack{A \sqcup B \sqcup \{j\} = C \\ |B| \text{ even.} \\ W \supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \cup B)} e_{W,A;B} = 0.$$

An analogous but more complicate presentation holds for the cohomology ring with integer coefficients.

As a consequence of Theorem A, the rational cohomology of the complement algebra depends only on the poset of connected components of the tori in the arrangement (poset of layers).

We prove Theorem B by refining a previous theorem of De Concini and Procesi, based on the computation of the algebraic De Rham complex. We manipulate some polynomial identities and we study special coverings of toric arrangement; this two facts lead to the proof of Theorem B. Weaker results have been obtained studying the Leray spectral sequence for the inclusion of the complement in the ambient torus.

Let $\mathcal{A}$ and $\mathcal{A}'$ be the two toric arrangements described by the following integer matrices:

$$N = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}, \quad N' = \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}.$$

Theorem C shows that the rational cohomology ring is not determined by the arithmetic matroid and by the matroid over $\mathbb{Z}$, two structures encoding the combinatorics of toric arrangements. Nevertheless, the poset of layers cannot describe the cohomology ring with integer coefficients.

Theorem C. *The poset of layers $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A}')$ are non-isomorphic and the algebras $H(M(\mathcal{A}); \mathbb{Q})$ and $H(M(\mathcal{A}'); \mathbb{Q})$ are non-isomorphic. However, N and N' describe the same arithmetic matroid and the same matroid over $\mathbb{Z}$.*

Moreover, there exist two toric arrangements with isomorphic posets of layers and non-isomorphic cohomology algebras with integer coefficients.

This example was found by making the following observations. Consider an arithmetic matroid whose underlying matroid is modular, then there exist at most one representation of the arithmetic matroid. An analogous result holds for arithmetic matroids whose associated group is cyclic.

Let $\mathcal{C}_n(\Sigma_g)$ be the unordered configuration spaces of n points on the closed orientable surface Σ_g of genus g. For $g = 1$, we present the rational cohomology ring of $\mathcal{C}_n(E)$ by using the representation theory of $\mathrm{SL}_2(\mathbb{Q})$ as follows.

Theorem D. *The rational cohomology ring of $\mathcal{C}_n(E)$ is isomorphic to*

$$\wedge^\bullet \mathbb{V}_1 \otimes {}^{\mathrm{S}^\bullet \mathbb{V}_1[b]}\!\Big/_{(a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b)_{\mathrm{SL}_2(\mathbb{Q})}},$$

where a is a non-zero degree-one element of $\mathrm{S}^1 \mathbb{V}_1$ and b is an $\mathrm{SL}_2(\mathbb{Q})$-invariant variable of degree 3.

This theorem is obtained computing the $\mathfrak{S}_n$-invariant elements of the Križ model for the configuration space on the torus. The multiplicative structure is determine by finding a small set of generators and then we explicitly compute some non-vanishing of products in the Križ model.

In higher genus, we cannot determine the ring structure of the cohomology $H(\mathcal{C}_n(\Sigma_g))$. However, we used the richer representation theory of $\mathrm{Sp}(2g)$ to obtain the following results on the cohomology module. Consider the weight filtration W on $H(\mathcal{C}_n(\Sigma_g); \mathbb{Q})$ given by the mixed Hodge structure, the graded cohomology is $\mathrm{gr}^W H(\mathcal{C}_n(\Sigma_g); \mathbb{Q})$. We describe $\mathrm{gr}^W H(\mathcal{C}_n(\Sigma_g); \mathbb{Q})$ as a bi-graded symplectic representation of the mapping class group of Σ_g. Let V_ω be the irreducible representation of the symplectic Lie algebra of highest weight ω. Since we are working on finite dimensional representations of the group $\mathrm{Sp}(2g)$, it is useful to consider the Grothendieck ring R_g, i.e. the free $\mathbb{Z}$-module having as a basis the set of irreducible representations of $\mathrm{Sp}(2g)$ up to isomorphisms. Let $[V_\omega] \in R_g$ be the element corresponding to the representation V_ω for any dominant weight ω. The cohomological stability was proven for unordered configurations spaces, therefore we describe all the spaces $\mathcal{C}_n(\Sigma_g)$, for $n \in \mathbb{N}$, simultaneously. We do it by considering the formal power series in 3 variable over the ring R_g.

Theorem E. *For $g > 0$, we have the following equality in $R_g[[t, s, u]]$:*

$$\sum_{i,j,n} [\mathrm{gr}^W_{i+2j} H^{i+j}(\mathcal{C}_n(\Sigma_g))] t^i s^j u^n =$$
$$\frac{1}{1-u}\Big((1 + t^2 su^3)(1 + t^2 u) + (1 + t^2 su^2) t^{2g} su^{2(g+1)} +$$
$$+ (1 + t^2 su^2)(1 + t^2 su^3) \sum_{\substack{1 \le j \le g \\ i \ge 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i u^{j+2i} (1 + t^{2(g-j)} su^{2(g-j+1)}) \Big).$$

In the previous formula, only the representations associated to weights of the form $i\omega_1 + \omega_j$ appear. Notice that the coefficient of u^n is a polynomial in the ring $R_g[t,s]$ and it coincides with the mixed Hodge polynomial of $\mathcal{C}_n(\Sigma_g)$. By setting $t = s$, we obtain the Poincaré polynomial of $\mathcal{C}_n(\Sigma_g)$ with coefficients in R_g.

In order to obtain the analogous results with the ring $\mathbb{Z}$ instead of R_g, we need to compute their image under the map $\dim\colon R_g \to \mathbb{Z}$. By using the Weyl dimension formula we calculate the dimension of the representations of the symplectic group $\mathrm{Sp}(2g)$ appearing in our formula.

Lemma F. *We have*

$$\dim V_{i\omega_1+\omega_j} = \binom{2g+i+1}{i,j} \frac{2g+2-2j}{2g+2+i-j} \frac{j}{i+j}.$$

Therefore we obtain a closed formula for the mixed Hodge numbers and the Betti numbers of $\mathcal{C}_n(\Sigma_g)$; those formulas does not have any cancellations, indeed these numbers are sum of dimensions of the previous representations.

Overview

The chapters are meant to be self-contained. To achieve this, each chapter begins with the recalling of the needed results and notations from the previous ones.

In Chapter 1, we define and study orientable arithmetic matroids and we apply this new theory to realizable arithmetic matroids. We recover for orientable arithmetic matroids the basic constructions of matroids: deletion, contraction and duality. By studying GP-functions, we obtain the uniqueness of the orientation. Strengthening the condition “greatest common divisor”, we obtain a necessary condition for the realizability of torsion-free surjective arithmetic matroids. Orientability and the “strong GCD” property are equivalent to the representability of torsion-free surjective arithmetic matroids. Finally, by using a new operation between quasi-arithmetic matroids – called “reduction” – we classify all the representations of torsion-free arithmetic matroids.

In Chapter 2, the cohomology of the complement of toric arrangements is investigated. We start from the graded cohomology ring both with integer and rational coefficients. The main technique used is the Leray spectral sequence. We provide a presentation for the cohomology ring by covering in a non-trivial way a toric arrangement with unimodular toric arrangements. This result is presented firstly for the rational cohomology and then for the integer one. The last result concerns the generation in degree one of the cohomology ring.

In Chapter 3, we discuss the relation between the combinatorics of a toric arrangement (arithmetic matroid, poset of layers) and the cohomology ring of its complement. We also study discriminantal toric arrangements and the special case of toric arrangements whose associated matroid is modular. The main results of this chapter are the examples in the last two sections. The first one shows that the integral cohomology of toric arrangements is not determined by the poset of layers. The second example provides two toric arrangements describing isomorphic arithmetic matroids, but having different posets of layers and different rational cohomology algebras of the complements.

In Chapter 4, we focus on configuration spaces of closed orientable surfaces. We expose the representation theory of the Križ model for the cohomology. In genus one the Križ model is huge and complicate, and its cohomology is unknown, however we conjecture a closed formula for the mixed Hodge numbers in the ordered case. In the unordered case we provide a presentation of the cohomology ring keeping track of the mixed Hodge structure and of the action of the mapping class group. Finally, we obtain analogous statements for unordered configuration spaces in arbitrary genus, but only as a module.

Chapter 1

Arithmetic Matroids

The aim of this chapter is to relate two different generalizations of matroids: the oriented matroids and the arithmetic matroids. We want to give a definition of "oriented arithmetic matroid" and prove properties like the "uniqueness of orientation". This leads to a complete classification of all representation of a torsion-free arithmetic matroid.

Oriented matroids have a large use in mathematics and science (for general reference see [BLVS$^+$99, Oxl11]); they are related to the simplex method for linear programming, to the chirality of molecules in theoretical chemistry, and to knot theory. For instance, the Jones polynomial of a link is a specialization of the signed Tutte polynomial (see [Kau89]) of an oriented graphic matroid [Thi87, Jae88]. Another interesting fact is the correspondence between oriented matroids and arrangements of pseudospheres [FL78] that generalizes the correspondence between representable matroids and central hyperplane arrangements.

Arithmetic matroids – introduced in [DM13, BM14] – appear as a combinatorial object related to the cohomology of the complement of a toric arrangement [DP05, Moc12a, CDD$^+$18]. The study of toric arrangements is related to zonotopes, partition functions, box splines, and Dahmen-Micchelli spaces (see [DPV10, DP11, Moc12a]). All standard operations with matroids, e.g. deletion, contraction, duality, are generalized to arithmetic matroids in a consistently way. Recently, a new operation between two arithmetic matroids over the same matroid is discovered by Delucchi e Moci [DM18].

The correspondence between representable arithmetic matroids and central toric arrangements has not been generalized to the non-representable cases, so far. With the aim of filling this gap, we define a class of arithmetic matroids which we call *orientable arithmetic matroids* (see Definition 1.1.10) hoping that these correspond to "arrangements of pseudo-tori".

An $r \times n$ matrix with integer coefficients describes at the same time a central toric arrangement, an oriented matroid, and an arithmetic matroid. It comes natural to say that two matrices are equivalent if they describe two toric arrangements that differ by an automorphism of the ambient torus. Geometrically, the group $\mathrm{GL}_r(\mathbb{Z}) \times (\mathbb{Z}_2)^n$ acts on the space $\mathrm{M}(r, n; \mathbb{Z})$ by left multiplication and sign reverse of the columns. Two representation (i.e. matrices) of the arithmetic matroid are equivalent if and only if they belong to the same $\mathrm{GL}_r(\mathbb{Z}) \times (\mathbb{Z}_2)^n$-orbit.

The space $\mathrm{M}(r, n; \mathbb{Z})$ is included in $\mathrm{M}(r, n; \mathbb{Q})$ and the action of $\mathrm{GL}_r(\mathbb{Z}) \times \mathbb{Z}_2^n$ extends naturally to the one of $\mathrm{GL}_r(\mathbb{Q}) \times \mathbb{Z}_2^n$. Lemma 1.7.6 shows that all representations of an arithmetic matroid belong to the same $\mathrm{GL}_r(\mathbb{Q}) \times (\mathbb{Z}_2)^n$-orbit. By this fact, it can be easily deduced that representable arithmetic matroids have a unique orientation. We extend this result to the non-representable case, showing (Theorem 1.4.1) that orientable arithmetic matroids have a unique orientation (up to re-orientation).

Sections 1.1 to 1.4 and 1.6 are published in [Pag18a] and Sections 1.5 and 1.8 will appear in the preprint [PP19b] written with Giovanni Paolini. Sections 1.7 and 1.9 are part of the paper [Pag19b], the version below is quite different from the published version. [Pag19b, Theorem 5.6] contains a mistake, a corrected version of it is stated and proved below, see Theorem 1.9.9.

Plan

In Section 1.1 we start by recalling some standard definitions and by describing the basic construction in Section 1.2. We introduce a compatibility condition, eq. (GP), between the orientation and the multiplicity function of an oriented arithmetic matroid. The condition (GP) coincides with the Plücker relation for the Grassmannian and we prove that oriented arithmetic matroids are closed under deletion, contraction, and duality. Next, in Section 1.3, we show that the condition (GP) implies a generalization of the Leibniz rule for the determinant. We state and prove a result about the uniqueness of orientation (Section 1.4) so that it makes sense to talk of orientable arithmetic matroids instead of oriented arithmetic matroids. We state, in Section 1.5, the condition "strong GCD" that implies the representability of orientable arithmetic matroids, as proven in Section 1.6. Moreover, we show that orientable arithmetic matroids, upon forgetting the multiplicity function, are representable matroids (see Proposition 1.6.1). A torsion-free surjective arithmetic matroid has a unique representation up to equivalence; this fact is proven in Section 1.7. Later, in Section 1.8, we introduce a new operation on quasi-arithmetic matroids, called "reduction", in order to classify all representations of an arithmetic matroid (Section 1.9).

The entire discussion can be generalized to quasi-arithmetic matroids and so to matroids over $\mathbb{Z}$ (see [FM16]). It is not clear to the author how arithmetic matroids and orientable arithmetic matroids are related to matroids over hyperfields (see [BB16]).

1.1 Definitions

Let E be a *ground set*, i.e. a finite totally ordered set. We will frequently make use of r-tuples of elements of E, so with an abuse of notation for any set $A = \{a_1, \dots, a_r\} \subset E$ we will write A for the increasing tuple $(a_1, \dots, a_r)$.

1.1.1 Matroids

Let v_e for $e \in E$ be some elements in a finite generated abelian group H. As elements of the vector space $\mathbb{Q} \otimes H$, the elements v_e determine linear dependency relations. The family

$$\mathcal{C} := \min_{\subseteq} \{ C \subseteq E \mid \{v_e\}_{i \in C} \text{ is a linearly dependent set} \}$$

of index sets of minimal linear dependencies among this elements in H is the set of *circuits* of a *matroid* $\mathcal{M}$ on the set E We point to [Oxl11] for an introduction to this theory.

We give the definition of a matroid in terms of its basis, since [Oxl11, Theorem 1.2.3] shows that it is equivalent to the one given in terms of circuits.

Definition 1.1.1. A *matroid* over a the ground set E is a non-empty set $\mathcal{B} \subset \mathcal{P}(E)$ satisfying the following exchange property:

$$\forall B_1, B_2 \in \mathcal{B} \, \forall x \in B_1 \setminus B_2 \, \exists y \in B_2 \setminus B_1 \text{ such that } B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}. \quad (1.1)$$

Since this definition of matroids is cryptomorphic to the one involving the *rank function* (see [Oxl11, Theorem 1.3.2]), we denote a matroid with the pair (E, rk).

Throughout this paper we will denote the r-tuples $(y_i, x_2, \dots, x_r)$ by $\underline{x}_i$ and $(y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_r)$ by $\underline{y}^i$, for $i = 0, \dots, r$, where $\underline{x} = (x_2, \dots, x_r)$ and $\underline{y} = (y_0, \dots, y_r)$.

Definition 1.1.2 ([BLVS$^+$99, Definition 3.5.3])**.** A *chirotope* χ is a function $\chi \colon E^r \to \{-1, 0, 1\}$ such that:

(B0) it is not identically zero, i.e. $\chi \not\equiv 0$,

(B1) it is alternating, i.e. $\chi(\sigma \underline{x}) = \mathrm{sgn}(\sigma)\chi(\underline{x})$ for all $\sigma \in \mathfrak{S}_r$,

(B2) for all $x_2, \dots, x_r$ and all $y_0, \dots, y_r \in E$ such that

$$\chi(\underline{x}_i)\chi(\underline{y}^i) \geq 0,$$

for all $i > 0$, then we have

$$\chi(\underline{x}_0)\chi(\underline{y}^0) \geq 0.$$

We say that two chirotopes χ and χ' are the same chirotope if $\chi = \chi'$ or if $\chi = -\chi'$. This choice is not standard but useful for the notation.

Definition 1.1.3 ([BLVS+99, p. 134])**.** The *re-orientation* with respect to $A \subseteq E$ of a chirotope χ is the chirotope χ' defined by

$$\chi'(\underline{x}) = (-1)^{|A\cap\{x_1,\dots,x_r\}|}\chi(\underline{x}).$$

Two chirotopes are equivalent if one is a re-orientation of the other one.

The set $\{\{b_1,\dots,b_r\} \subset E \mid \chi(b_1,\dots,b_r) \neq 0\}$ is the matroid over E associated with the chirotope χ. A *signed circuit* in E is a function $c\colon C \to \{\pm 1\}$ where $C \subset E$. An *oriented matroid* is a collection of signed circuit that satisfies some properties listed in [BLVS+99, Definition 3.2.1]. A well-known cryptomorphism of Lawrence [Law82] between *oriented matroids* and chirotopes is stated in [BLVS+99, Theorem 3.5.5].

We can define the set of signed circuit associated with the chirotope χ as follow. Each circuit $C \subseteq E$ of the matroid associated with χ can be considered as an ordered set with the total order induced by E. Choose an ordered set $\underline{a} = (a_1, a_2, \dots, a_{r-s})$ such that $\mathrm{rk}(C \cup \underline{a}) = r$. Let c be the function defined by

$$c(i) = c_i \overset{\text{def}}{=} (-1)^i\chi(x_0,\dots,\hat{x_i},\dots,x_s,a_1,\dots,a_{r-s}),$$

it does not depend on the choice of the total order and, up to a global negation, the function c does not depend on the choice of $\underline{a}$.

Definition 1.1.4. The *set of signed circuit* associated with χ is the set of function $\{\, c, -c\colon C \to \{\, \pm 1 \,\} \mid C \text{ circuit} \,\}$ defined above.

For every matroid $\mathcal{M} = (E, \mathrm{rk})$ and every subset $A \subsetneq E$ we denote by $\mathcal{M}/A$ the *contraction* of A and with $\mathcal{M} \setminus A$ the *deletion* of A.

Let us recall the definition of arithmetic matroid introduced in [DM13, BM14].

Definition 1.1.5. A *molecule* (A, F, T) of the matroid (E, rk) is a triple of sets $A \sqcup F \sqcup T \subseteq E$ such that $\mathrm{rk}(F \sqcup A) = |F| + \mathrm{rk}(A)$ and $\mathrm{rk}(A \sqcup T) = \mathrm{rk}(A)$.

Definition 1.1.6. An *arithmetic matroid* is (E, rk, m) such that (E, rk) is a matroid and $m : \mathcal{P}(E) \to \mathbb{N}_+ = \{1, 2, \dots\}$ a function satisfying:

1. if $A \subseteq E$ and $x \in E$ is dependent on A, then $m(A \cup \{\, x \,\})|m(A)$;

2. if $A \subseteq E$ and $x \in E$ is independent from A, then $m(A)|m(A \cup \{\, v \,\})$;

3. if (A, F, T) is a molecule then

$$m(A)m(A \sqcup F \sqcup T) = m(A \sqcup F)m(A \sqcup T);$$

4. if (A, F, T) is a molecule then

$$\rho(A, A \sqcup F \sqcup T) \stackrel{\text{def}}{=} \sum_{A \subseteq S \subseteq A \sqcup F \sqcup T} (-1)^{|A|+|T|-|S|} m(S) \geq 0.$$

We call m the *multiplicity function*.

A *pseudo-arithmetic matroid* is a matroid with a multiplicity function that satisfies only 4. A *quasi-arithmetic matroid* is a matroid with a multiplicity function that satisfies $1-3$.

Definition 1.1.7. An arithmetic matroid (E, rk, m) is *representable* if there exists a finite generated abelian group H and a list of elements $(h_e)_{e\in E}$ of H such that

$$\text{rk}(A) = \text{rank}\big(\langle h_e \rangle_{e\in A}\big) \quad \text{and} \quad m(A) = \left|\text{Tor}\Big(H\big/\langle h_e \rangle_{e \in A}\Big)\right|$$

for all $A \subseteq E$. We call a such collection $(h_e)_{e\in E}$ in H a *representation* of the arithmetic matroid. The representation is *essential* if $\text{rk}\, H = \text{rk}(E)$.

For sake of notation we define for each representation $\Gamma_A = \langle h_e \rangle_{e\in A} < H$ and for each subgroup $K < H$ we set $[H : K] = |\text{Tor}(H/K)|$.

Definition 1.1.8. An arithmetic matroid (E, rk, m) is said to be *torsion-free* if $m(\emptyset) = 1$ and *surjective* if $m(E) = 1$.

A polynomial invariant for arithmetic matroids was introduced by Moci [Moc12a]:

Definition 1.1.9. The *arithmetic Tutte polynomial* of an arithmetic matroid (E, rk, m) is the following polynomial

$$T(x, y) = \sum_{A \subseteq E} m(A)(x-1)^{\text{rk}(E)-\text{rk}(A)}(y-1)^{|A|-\text{rk}(A)}.$$

The arithmetic Tutte polynomial has good properties: it has positive coefficients, satisfies the deletion-restriction property and can be defined in terms of *internal and external activities* of the arithmetic matroid. It specializes to the Poincaré polynomial of a toric arrangement, to the characteristic polynomial of the poset of layers, to the Hilbert series of the associated Dahmen-Micchelli space. Moreover, it counts the number of integer points of the associated zenotope and the connected components of arrangements in the compact torus $(S^1)^r$ ([Law11]). This polynomial is further generalized to the *G-Tutte polynomial*, see [LTY17, Tra18, TY19] and to the *universal Tutte character*, see [DFM17]. The arithmetic Tutte polynomial associated with a root system is computed in [ACH15] and for type A_n in [Ber19].

Definition 1.1.10. An *oriented arithmetic matroid* $(E, \mathrm{rk}, m, \chi)$ is a matroid (E, rk) of rank r together with two structures: a chirotope $\chi\colon E^r \to \{-1, 0, 1\}$ and a multiplicity function $m\colon \mathcal{P}(E) \to \mathbb{N}_+$ such that:

1. The unoriented matroid associated with the chirotope χ is the matroid (E, rk).

2. The triple (E, rk, m) is an arithmetic matroid.

3. For all $x_2, \ldots, x_r$ and all $y_0, \ldots, y_r \in E$ the following equality holds

$$\sum_{i=0}^{r} (-1)^i \chi(\underline{x}_i) m(\underline{x}_i) \chi(\underline{y}^i) m(\underline{y}^i) = 0, \tag{GP}$$

   where $\underline{x}_i = (y_i, x_2 \ldots, x_r)$ and $\underline{y}^i = (y_0, \ldots y_{i-1}, y_{i+1} \ldots y_r)$.

An arithmetic matroid is *orientable* if there exists a chirotope that makes the arithmetic matroid oriented.

Remark 1.1.11. Our property (GP), related to the Grassmannian-Plücker relations, implies the properties (GP$_r$), for all r, defined in [Len17b, Definition 10.3].

Notice that the compatibility condition (GP) involves only the values of the multiplicity function on the basis of (E, rk).

Remark 1.1.12. The condition (GP) implies (B2) of Definition 1.1.2.

Let H be a finite generated abelian group, and $\mathcal{B}$ be a basis of $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 1.1.13. A *representation* $(h_e)_{e \in E}$ in H of an oriented arithmetic matroid $(E, \mathrm{rk}, m, \chi)$ is a collection of elements in the finite generated abelian group H such that:

1. they are a representation of the arithmetic matroid (E, rk, m),

2. for each $A \in E^r$ we have

$$\chi(A) = \mathrm{sgn}(\det M_{\mathcal{B}}(\{\, h_a \otimes 1 \,\}_{a \in A})),$$

   where $M_{\mathcal{B}}(\{\, h_a \otimes 1 \,\}_{a \in A})$ is the matrix that represent the vectors $h_a \otimes 1$ in the basis $\mathcal{B}$.

Remark 1.1.14. The above definition does not depend on the choice of the basis since we consider χ and $-\chi$ the same chirotope.

1.2 Basic constructions

Deletion

The deletion of $A \subseteq E$ is an operation defined for matroids [Oxl11, p. 22], for oriented matroids [BLVS$^+$99, p. 133], and for arithmetic matroids [DM13, section 4.3] [BM14, section 3]. We now define a deletion operation for oriented arithmetic matroids.

Define $s = \operatorname{rk}(E \setminus A)$ and choose $\underline{a} = (a_1, a_2, \ldots, a_{r-s})$ be such that $\operatorname{rk}((E \setminus A) \cup \underline{a}) = r$. Let $\chi \setminus A \colon (E \setminus A)^s \to \{-1, 0, 1\}$ be the function defined by $\chi \setminus A(\underline{z}) = \chi(\underline{z} \cup \underline{a})$. The collection $(E \setminus A, \operatorname{rk} \setminus A, m \setminus A, \chi \setminus A)$ satisfies the first two conditions of Definition 1.1.10.

Proposition 1.2.1. *The collection* $(E \setminus A, \operatorname{rk} \setminus A, m \setminus A, \chi \setminus A)$ *is an oriented arithmetic matroid.*

Proof. Consider the elements $x_2, \ldots, x_s$ and $y_0, \ldots, y_s$ in $E \setminus A$. For all $0 \leq i \leq s$ such that $\chi(\underline{x}_i \cup \underline{a}) \neq 0$ and $\chi(\underline{y}_i \cup \underline{a}) \neq 0$, the triples $(\underline{x}_i, \underline{a}, \underline{y}^i)$ and $(\underline{y}^i, \underline{a}, \underline{x}_i)$ are molecules. The equality

$$m(\underline{x}_i \cup \underline{y}^i)^2 m(\underline{x}_i \cup \underline{a}) m(\underline{y}^i \cup \underline{a}) = m(\underline{x}_i \cup \underline{y}^i \cup \underline{a})^2 m(\underline{x}_i) m(\underline{y}^i)$$

follows from condition (3) applies to the two molecules. Notice that $\underline{x}_i \cup \underline{y}^i$ does not depend on i so we can denote it by $\underline{x} \cup \underline{y}$. We have

$$\begin{aligned} m(\underline{x} \cup \underline{y} \cup \underline{a})^2 \sum_{i=0}^{s} (-1)^i \chi(\underline{x}_i \cup \underline{a}) m(\underline{x}_i) \chi(\underline{y}^i \cup \underline{a}) m(\underline{y}^i) = \\ = m(\underline{x} \cup \underline{y})^2 \sum_{i=0}^{s} (-1)^i \chi(\underline{x}_i \cup \underline{a}) m(\underline{x}_i \cup \underline{a}) \chi(\underline{y}^i \cup \underline{a}) m(\underline{y}^i \cup \underline{a}). \end{aligned}$$

The right side is, up to a non-zero scalar, the equation (GP) applied to $x_2, \ldots, x_s, a_1, \ldots, a_{r-s}$ and $y_0, \ldots, y_s, a_1, \ldots, a_{r-s}$ for the oriented arithmetic matroid (E, χ, m). Therefore, we have proven the claimed equality

$$\sum_{i=0}^{s} \chi(\underline{x}_i \cup \underline{a}) m(\underline{x}_i) \chi(\underline{y}^i \cup \underline{a}) m(\underline{y}^i) = 0. \qquad \square$$

Contraction

The contraction of $A \subseteq E$ is an operation defined for matroids [Oxl11, p. 22], for oriented matroids [BLVS$^+$99, p. 134], and for arithmetic matroids [DM13, section 4.3] [BM14, section 3]. We now define a contraction operation for oriented arithmetic matroids.

Let A be a subset of E and call $r - s$ its rank. We choose an independent list $\underline{a} = (a_1, \ldots, a_{r-s})$ of elements in A. Define $\chi/A : (E \setminus A)^s \to \{-1, 0, 1\}$ as $\chi/A(\underline{z}) = \chi(\underline{z} \cup \underline{a})$, $\operatorname{rk}/A(S) = \operatorname{rk}(A \cup S) - \operatorname{rk}(A)$ and $m/A(S) = m(A \cup S)$.

Proposition 1.2.2. *The collection* $(E/A, \operatorname{rk}/A, m/A, \chi/A)$ *is an oriented arithmetic matroid.*

Proof. We call $T = A \setminus \underline{a}$ and fix the elements $x_2, \dots, x_s$ and $y_0, \dots, y_s$ of $E \setminus A$. For all i such that $\chi(\underline{x}_i \cup \underline{a}) \neq 0$ and $\chi(\underline{y}_i \cup \underline{a}) \neq 0$, the triples $(\underline{a}, \underline{x}_i, T)$ and $(\underline{a}, \underline{y}^i, T)$ are molecules of (E, rk). Thus

$$m(A)^2 m(\underline{x}_i \cup \underline{a}) m(\underline{y}^i \cup \underline{a}) = m(\underline{a})^2 m(\underline{x}_i \cup A) m(\underline{y}^i \cup A).$$

Since $m(A)$ and $m(\underline{a})$ are nonzero, then condition (GP) for $\underline{x}$ and $\underline{y}$ in the contracted matroid is equivalent to condition (GP) for $\underline{x} \cup \underline{a}$ and $\underline{y} \cup \underline{a}$ in the original matroid. □

Duality

The duality is an operation defined for matroids [Oxl11, chapter 2], for oriented matroids [BLVS+99, p. 135], and for arithmetic matroids [DM13, p. 339] [BM14, p. 5526]. We now define duality for oriented arithmetic matroids.

Recall that the set E is ordered. For every $\underline{z} = (z_1, \dots, z_k) \subseteq E$ we call $\underline{z}'$ the complement of $\underline{z}$ in E with some arbitrary order and let $\sigma(\underline{z}, \underline{z}')$ be the sign of the permutation that reorders the list $(\underline{z}, \underline{z}')$ as they appear in E. We define $\chi^* : E^{n-r} \to \{-1, 0, 1\}$ as

$$\chi^*(\underline{z}) = \chi(\underline{z}')\sigma(\underline{z}, \underline{z}')$$

and the multiplicity function $m^* : \mathcal{P}(E) \to \mathbb{N}_+$ as $m^*(\underline{z}) = m(\underline{z}')$.

Proposition 1.2.3. *The triple* (E, χ^*, m^*) *is an oriented arithmetic matroid.*

Proof. Let $\underline{x} = (x_2, \dots, x_{n-r})$ and $\underline{y} = (y_0, \dots, y_{n-r})$ be two sublists of E. Coherently with the notation above, let $\underline{x}' = (x'_0, \dots, x'_r)$ and $\underline{y}' = (y'_2, \dots, y'_r)$ be their complements. For every $0 \leq i \leq n-r$ the element y_i is equal to x_k or x'_j. In the first case $\chi^*(\underline{x}_i) = 0$ and in the second case

$$\chi^*(\underline{x}_i) = \chi(\underline{x}'^j)\sigma(\underline{x}_i, \underline{x}'^j) = (-1)^{n-r+1+j}\chi(\underline{x}'^j)\sigma(\underline{x}, \underline{x}').$$

Analogously, if $y_i = x'_j$ then

$$\chi^*(\underline{y}^i) = \chi(\underline{y}'_j)\sigma(\underline{y}^i, \underline{y}'_j) = (-1)^{n-r+i}\chi(\underline{y}'_j)\sigma(\underline{y}, \underline{y}')$$

where $\underline{y}'_j = (x'_j, y'_2, \dots, y'_r)$. If $y_i = x'_j$, then $m^*(\underline{x}_i) = m(\underline{x}'^j)$ and $m^*(\underline{y}^i) = m(\underline{y}'_j)$. Thus, up to a sign, the condition (GP) for $\underline{y}'$ and $\underline{x}'$ in the original matroid implies condition (GP) for $\underline{x}$ and $\underline{y}$ in the dual matroid. □

1.3 GP-functions

We now study functions satisfying a relation that looks like the Plücker relation for the Grassmannian. A posteriori all these functions are nothing else that the determinant $\det\colon V^r \to \mathbb{Q}$ restricted to a finite (multi-)set $E \subset V$.

Definition 1.3.1. A map $f\colon E^r \to \mathbb{Q}$ is a *GP-function* if it is alternating and for all $\underline{x} \in E^{r-1}$ and all $\underline{y} \in E^{r+1}$ the following equality holds

$$\sum_{i=0}^{r} (-1)^i f(y_i, x_2 \dots, x_r) f(y_0, \dots y_{i-1}, y_{i+1} \dots y_r) = 0$$

The main examples of GP-function are the function χm for every oriented arithmetic matroid. Another example is constructed as follow: given a map $i\colon E \to \mathbb{Q}^r$, the function $\underline{x} \mapsto \det(i(x_1), \dots, i(x_r))$ is a GP-function. The following theorem is a generalization of the Leibniz formula.

Theorem 1.3.2. *Let $f\colon E^r \to \mathbb{Q}$ be a GP-function. Then for all $(a_1, \dots, a_r)$ in E^r and $(b_1, \dots, b_r) \in E^r$ the following formula holds:*

$$\sum_{\sigma \in \mathfrak{S}_r} (-1)^{\operatorname{sgn} \sigma} \prod_{i=1}^{r} f(a_1, \dots, b_{\sigma(i)}, \dots, a_r) = f(a_1, \dots, a_r)^{r-1} f(b_1, \dots, b_r), \tag{1.2}$$

where $b_{\sigma(i)}$ substitutes a_i.

Proof. We prove lemma by induction, the base case $r = 2$ is trivial. We fix $(a_1, \dots, a_r) \in E^r$ and $(b_1, \dots, b_r) \in E^r$. Let $g\colon E^{r-1} \to \mathbb{Q}$ be the GP-function defined by

$$g(x_2, \dots, x_r) = f(a_1, x_2, \dots, x_r).$$

By inductive step we have

$$\sum_{\sigma \in \mathfrak{S}_{r-1}} (-1)^{\operatorname{sgn} \sigma} \prod_{i=2}^{r} g(a_2, \dots, c_{\sigma(i)}, \dots, a_r) = g(a_2, \dots, a_r)^{r-2} g(c_2, \dots, c_r). \tag{1.3}$$

The left hand side of the eq. (1.2) can be rewritten as:

$$\sum_{j=1}^{r} f(b_j, a_2, \dots, a_r) \sum_{\sigma \in \mathfrak{S}_{r-1}} (-1)^{\operatorname{sgn} \sigma + \operatorname{sgn} \tau_j} \prod_{i=2}^{r} f(a_1, \dots, b_{\sigma(\tau_j(i))}, \dots, a_r), \tag{1.4}$$

where $\tau_j = (1, j)$ and $\mathfrak{S}_{r-1}$ is the subgroup of $\mathfrak{S}_r$ of permutations that fix the element 1. Now, for every j, we use eq. (1.3) with $c_i = b_{\tau_j(i)}$ to manipulate expression (1.4):

$$f(a_1, \dots, a_r)^{n-2} \Big[f(b_1, a_2, \dots, a_r) f(a_1, b_2, \dots, b_r) - \sum_{j=1}^{r} f(b_j, a_2, \dots, a_r) \cdot \\ f(a_1, b_2, \dots, b_1, \dots, b_r) \Big]$$

that it is equal to left hand side of (1.2) since f is a GP-function. □

Lemma 1.3.3. *Let f and g be two GP-functions and $B \in E^r$. Suppose that $f(B) = g(B) \neq 0$ and $f(C) = g(C)$ for all $C \in E^r$ such that $|\{i \mid c_i \neq b_i\}| = 1$, then $f = g$.*

Proof. We use Theorem 1.3.2 for the function f and g. We set $\{a_1, \dots, a_r\} = B$ in eq. (1.2), the left hand side for f and g are equal, so

$$f(a_1, \dots, a_r)^{r-1} f(b_1, \dots, b_r) = g(a_1, \dots, a_r)^{r-1} g(b_1, \dots, b_r).$$

By hypothesis $f(a_1, \dots, a_r) = g(a_1, \dots, a_r) \neq 0$, thus we have $f(b_1, \dots, b_r) = g(b_1, \dots, b_r)$ for all b_i, $i = 1, \dots, r$. □

1.4 Uniqueness of the orientation

Theorem 1.4.1. *Let $(E, \mathrm{rk}, \chi, m)$ and $(E, \mathrm{rk}, \chi', m)$ be two oriented arithmetic matroids. Then χ' is a re-orientation of χ.*

We fix a total order on $E \simeq [n]$ such that $[r]$, the first r elements, are a basis of the matroid.

The basis graph of a matroid is first studied in [Mau73a] and [Mau73b].

Definition 1.4.2. The *basis graph* $\mathcal{BG}$ of a matroid $(E, \mathcal{B})$ is the graph on the set $\mathcal{B}$ of vertices with an edge between two vertices B_1 and B_2 if $|B_1 \setminus B_2| = 1$.

Once chosen a basis B_0 of a matroid, we define $\mathcal{BG}_1$ to be the induced subgraph of $\mathcal{BG}$ whose vertices are all vertices adjacent to B_0. Define $\mathcal{BG}_{\leq 1}$ the induced subgraph whose vertices are the ones adjacent to B_0 and B_0 itself.

Suppose that two GP-functions $\chi([r]) = \chi'([r])$, Lemma 1.4.6 proves that, up to reorientation, χ and χ' coincides on all vertices of distance one from $[r]$. Lemma 1.4.7 proves that $\chi(B) = \chi'(B)$ using Theorem 1.3.2.

Definition 1.4.3. Let $\mathcal{G}$ be the bipartite graph on vertices E and an edge between $i \in B_0$ and $j \in E \setminus B_0$ if $B_0 \setminus \{ i \} \cup \{ j \}$ is a basis. We call this graph the B_0-*fundamental circuit graph.*

Definition 1.4.4. The *Line graph* $L(\mathcal{G})$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the graph whose set of vertices is the set $\mathcal{E}$ of edges in $\mathcal{G}$. The graph $L(\mathcal{G})$ has an edge between e_1 and $e_2 \in \mathcal{E}$ if and only if the edges e_1 and e_2 are incident in $\mathcal{G}$.

The Line graph of $\mathcal{G}$ is the graph $\mathcal{BG}_1$. A *coordinatizing path* in $\mathcal{G}$ is a spanning forest of the graph $\mathcal{G}$. We choose a coordinatizing path P of the graph $\mathcal{G}$ and its Line graph $L(P)$ is an induced subgraph of $\mathcal{BG}_1$.

The following lemma is essentially proven in [Len19, Lemma 6].

Lemma 1.4.5. *Let (E, χ, m) be an oriented arithmetic matroid with basis graph $\mathcal{BG}$, B_0 be a vertex of $\mathcal{BG}$ and P be a coordinatizing path in $\mathcal{G}$, such that $L(\mathcal{G}) = \mathcal{BG}_1$. Then there exists a re-orientation χ' of χ such that $\chi'(B) = \chi'(B_0)$ for all vertices $B \in L(P)$.* □

We denote the point-wise product of two function χ and m with

$$\chi m(\underline{b}) \stackrel{\text{def}}{=} \chi(\underline{b}) \cdot m(\underline{b}).$$

We prove in our setting the equivalent of [Len19, Lemma 9].

Lemma 1.4.6. *Let (E, rk, m) be an arithmetic matroid with basis graph $\mathcal{BG}$, B_0 be a vertex of $\mathcal{BG}$ and P be a coordinatizing path in the graph $\mathcal{G}$, such that $L(\mathcal{G}) = \mathcal{BG}_1$. Let χ and χ' be two orientations of the arithmetic matroid (E, rk, m) such that $\chi(B) = \chi(B_0)$ and $\chi'(B) = \chi'(B_0)$ for all vertices $B \in L(P)$. If $\chi(B_0) = \chi'(B_0)$, then $\chi(B') = \chi'(B')$ for all $B' \in \mathcal{BG}_{\leq 1}$.*

Proof. Consider the subgraph $\mathcal{H}$ of $\mathcal{G}$ with the same set of vertices and with an edge between $i \in B_0$ and $j \in E \setminus B_0$ if and only if $\chi(B_0 \setminus \{i\} \cup \{j\}) = \chi'(B_0 \setminus \{i\} \cup \{j\}) \neq 0$. The graph $\mathcal{H}$ contains the chosen coordinatizing path P by hypothesis. Suppose that $\mathcal{H} \neq \mathcal{G}$ and let T ($T \neq \emptyset$) be the set of edges of $\mathcal{G}$ not contained in $\mathcal{H}$. For each $(i, j) \in T$ we can consider $l(i, j)$ the length of the minimal path in $\mathcal{H}$ connecting the vertices i and j. Obviously, $l(i, j)$ is a odd number greater than 2. Let us fix $(h, k) \in T$ with $l(h, k)$ minimal among all $l(i, j)$ for $(i, j) \in T$ and a minimal path $Q = (h = i_0, j_0, i_1, \ldots, i_t, j_t = k)$ in $\mathcal{H}$ between (h, k) (the equality $2t + 1 = l(h, k)$ holds). By minimality of (h, k), two vertices i_a and j_b are connected in $\mathcal{G}$ if and only if $a = b$, $a = b + 1$ or $b = t$ and $a = 0$.

Without loss of generality, we suppose $i_v = v + 1$ for $0 \leq v \leq t$, $B_0 = [r]$, and $j_v = r + v + 1$ for $0 \leq v \leq t$. Apply Theorem 1.3.2 with $a_i = i$ and $b_j = t + j + 2$ to the GP-functions χm and $\chi' m$. The product

$$\prod_{i=1}^{r} \chi m(a_1, \ldots, b_{\sigma(i)}, \ldots, a_r)$$

is non zero if and only if $(a_i, b_{\sigma(i)}) \in Q \cup \{(h, k)\}$ for all $i \leq t+1$ and $b_{\sigma(i)} = a_i$ for all $t + 1 < i \leq r$. The same implication holds for the function $\chi' m$. This happens only for two different permutations τ and η, say that $\tau(h) = k$ and $\tau(h) = j_0$. We define

$$\begin{aligned}
x &\stackrel{\text{def}}{=} \chi m(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r), \\
a &\stackrel{\text{def}}{=} \prod_{i \neq h} \chi m(a_1, \ldots, b_{\tau(i)}, \ldots, a_r), \\
b &\stackrel{\text{def}}{=} \prod_{i=1}^{r} \chi m(a_1, \ldots, b_{\mu(i)}, \ldots, a_r), \\
c &\stackrel{\text{def}}{=} \chi m(a_1, \ldots, a_r)^{r-1} \chi m(b_1, \ldots, b_r).
\end{aligned}$$

Thus, eq. (1.2) can be reduced to $ax + b = c$. The equivalent relation for χ' is $ax' + b = c'$ with $x' = \pm x$ and $c' = \pm c$. Since a, b, c and x are non-zero, then $x = x'$ and so

$$\chi(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r) = \chi'(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r).$$

This equality contradicts the supposition $\mathcal{H} \neq \mathcal{G}$. □

Lemma 1.4.7. *Let* (E, rk, m) *be an arithmetic matroid and* χ *and* χ' *two orientations of the arithmetic matroid that coincide on the elements of* $\mathcal{BG}_{\leq 1}$. *Then* $\chi = \chi'$.

Proof. By hypothesis both χm and $\chi' m$ are GP-functions, so by Lemma 1.3.3 they are equal. □

Theorem 1.4.1 follows from Lemmas 1.4.5 to 1.4.7.

1.5 The strong GCD property

Definition 1.5.1. An arithmetic matroid $M = (E, \mathrm{rk}, m)$ satisfies the *strong GCD property* if, for every subset $A \subseteq E$,

$$m(A) = \gcd\{\, m(B) \mid B \text{ basis and } |B \cap A| = \mathrm{rk}\, A \,\}.$$

Strong GCD arithmetic matroids are uniquely determined by the rank function and the multiplicity function restricted to the bases of the underlying matroid. The strong GCD property is equivalent to both (E, rk, m) and (E, rk^*, m^*) are GCD arithmetic matroids.

Lemma 1.5.2. *Let* M *be an arithmetic matroid. If* M *satisfies the strong GCD property, then it also satisfies the GCD property.*

Proof. For every independent set $I \subseteq E$, we have that

$$m(I) = \gcd\{\, m(B) \mid B \text{ basis and } I \subseteq B \,\}.$$

Then, for a generic subset $A \subseteq E$,

$$\begin{aligned} m(A) &= \gcd\{\, m(B) \mid B \text{ basis and } |B \cap A| = \mathrm{rk}(A) \,\} \\ &= \gcd\big\{ \gcd\{\, m(B) \mid B \text{ basis and } B \cap A = I \,\} \mid I \subseteq A \text{ and} \\ &\qquad |I| = \mathrm{rk}(I) = \mathrm{rk}(A) \big\} \\ &\overset{(*)}{=} \gcd\big\{ \gcd\{\, m(B) \mid B \text{ basis and } I \subseteq B \,\} \mid I \subseteq A \text{ and} \\ &\qquad |I| = \mathrm{rk}(I) = \mathrm{rk}(A) \big\} \\ &= \gcd\{\, m(I) \mid I \subseteq A \text{ and } |I| = \mathrm{rk}(I) = \mathrm{rk}(A) \,\}. \end{aligned}$$

The equality $(*)$ follows by $|I| = \mathrm{rk}(I) = \mathrm{rk}(A) \geq \mathrm{rk}(B \cap A) = |B \cap A|$. □

Lemma 1.5.3. *Let M be an arithmetic matroid. If M satisfies the strong GCD property, then its dual M^* also satisfies the strong GCD property.*

Proof. For every subset $A \subseteq E$, we have

$$\begin{aligned} m^*(A^c) = m(A) &= \gcd\{\, m(B) \mid B \text{ basis of } M \text{ and } |B \cap A| = \operatorname{rk}(A) \,\} \\ &\overset{(*)}{=} \gcd\{\, m^*(B^c) \mid B^c \text{ is a basis of } M^* \text{ and } |B^c \cap A^c| = \operatorname{rk}^*(A^c) \,\}. \end{aligned}$$

The equality $(*)$ follows by $|B^c \cap A^c| = |(B \cup A)^c| = |E| - (|B| + |A| - |B \cap A|) = |A^c| - |B| + |B \cap A| = |A^c| - \operatorname{rk}(E) + \operatorname{rk}(A) = \operatorname{rk}^*(A^c)$. □

Theorem 1.5.4. *Let M be an arithmetic matroid. Then M satisfies the strong GCD property if and only if both M and M^* satisfy the GCD property.*

Proof. If M satisfies the strong GCD property, then the same is true for M^* by Lemma 1.5.3, and therefore both M and M^* satisfy the GCD property by Lemma 1.5.2.

Conversely, suppose that M and M^* both satisfy the GCD property. By the GCD property of M, for every $A \subseteq E$, we have

$$m(A) = \gcd\{\, m(I) \mid I \subseteq A \text{ and } |I| = \operatorname{rk}(I) = \operatorname{rk}(A) \,\}. \tag{1.5}$$

By the GCD property for M^*, for every independent set $I \subseteq E$ we have

$$\begin{aligned} m(I) = m^*(I^c) &= \gcd\{\, m^*(B^c) \mid B^c \subseteq I^c \text{ and } |B^c| = \operatorname{rk}^*(B^c) = \operatorname{rk}^*(I^c) \,\} \\ &= \gcd\{\, m(B) \mid I \subseteq B \text{ and } |B^c| = \operatorname{rk}^*(B^c) = \operatorname{rk}^*(I^c) \,\}. \end{aligned}$$

The condition $|B^c| = \operatorname{rk}^*(B^c) = \operatorname{rk}^*(I^c)$ can be rewritten as $|B^c| = |B^c| - \operatorname{rk}(E) + \operatorname{rk}(B) = |I^c| - \operatorname{rk}(E) + \operatorname{rk}(I)$. The first equality implies that $\operatorname{rk}(B) = \operatorname{rk}(E)$. By the second equality, we obtain $|B^c| = |I^c| - \operatorname{rk}(E) + |I| = |E| - \operatorname{rk}(E)$, thus $|B| = \operatorname{rk}(E)$. Altogether, B is a basis. Then

$$m(I) = \gcd\{\, m(B) \mid I \subseteq B \text{ and } B \text{ is a basis} \,\}. \tag{1.6}$$

In particular, if $I \subseteq A \subseteq E$ and $|I| = \operatorname{rk}(I) = \operatorname{rk}(A)$, then $\operatorname{rk}(I) \le \operatorname{rk}(B \cap A) \le \operatorname{rk}(A)$ and therefore $|B \cap A| = \operatorname{rk}(B \cap A) = \operatorname{rk}(A)$. Putting together Equations (1.5) and (1.6), we finally obtain

$$m(A) = \gcd\{\, m(B) \mid B \text{ basis and } |B \cap A| = \operatorname{rk}(A) \,\}.$$

This proves the strong GCD property for M. □

Corollary 1.5.5. *Let M be a surjective, torsion-free, and representable arithmetic matroid. Then M satisfies the strong GCD property.*

Proof. By [DM13, Remark 3.1], a torsion-free representable arithmetic matroid has the GCD property. In particular, this applies to M. Since M is surjective and representable, its dual $M^* = (E, \operatorname{rk}^*, m^*)$ is torsion-free and representable, and thus it also satisfies the GCD property. By Theorem 1.5.4, we deduce that M satisfies the strong GCD property. □

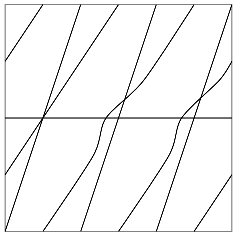


Figure 1.1: An arrangement of hypersurfaces in the compact torus.

As a final remark, notice that the strong GCD property is not preserved under deletion or contraction.

We show an example of orientable arithmetic matroid that is not representable.

Example 1.5.6. Let $([3], \mathrm{rk}, m)$ be the orientable arithmetic matroid associated with the matrix $\left(\begin{smallmatrix} 1 & 1 & 2 \\ 0 & n & n \end{smallmatrix}\right)$. Let m' be the multiplicity function defined by $m'([3]) = 1$ and $m'(A) = m(A)$ for all $A \subsetneq [3]$. The triple $([3], \mathrm{rk}, m')$ is a non-representable arithmetic matroid, since the multiplicity function does not have the GCD property. This matroid is orientable, indeed any orientation χ of $([3], \mathrm{rk}, m)$ is an orientation of $([3], \mathrm{rk}, m')$. Figure 1.1 represents an arrangement of hypersurfaces of T^2, the compact two dimensional torus, whose pattern of intersections coincides with the arithmetic matroid $([3], \mathrm{rk}, m')$ for $n = 3$.

1.6 Existence of a representation

Proposition 1.6.1. *Let (E, rk, m) be an orientable arithmetic matroid. Then the underlying matroid (E, rk) is representable over $\mathbb{Q}$.*

Proof. We choose an orientation χ of the arithmetic matroid (E, rk, m) and a basis $B_0 = (b_1, \ldots, b_r)$ of the matroid. For each $e \in E$, consider in $\mathbb{Q}^r$ the vector

$$v_e \stackrel{\text{def}}{=} (\chi m(b_1, \ldots, b_{i-1}, e, b_{i+1}, \ldots, b_r))_{1 \le i \le r}.$$

We choose a total order on $E = [n]$ such that $B_0 = [r]$. Let N be the matrix that represent the vectors v_i, for $i = 1, \ldots, n$, in the canonical basis of $\mathbb{Q}^r$. We claim that, for each $A \subseteq [n]$ of cardinality r, the functions $\det N[A]$ and $\chi m(B_0)^{r-1} \chi m(A)$ coincide. The claimed equality holds if $A = B_0$. If $A = \{1, \ldots, i-1, i+1, \ldots, r, j\}$, then

$$\begin{aligned} \det N[A] &= (-1)^{r-i} \frac{\chi m(1, \ldots, i-1, j, i+1, \ldots, r)}{\chi m([r])} \det N[[r]] \\ &= \frac{\chi m(A)}{\chi m(B_0)} \chi m(B_0)^r = \chi m(B_0)^{r-1} \chi m(A) \end{aligned} \tag{1.7}$$

The GP-function $\chi m(B_0)^{r-1}\chi m(\cdot)$ and $\det N[\cdot]$ coincide on $\mathcal{BG}_{\leq 1}$, thus by Lemma 1.3.3 $\chi m(B_0)^{r-1}\chi m(B) = \det N[B]$ for all $B \subset E$, $|B| = r$. The matroid defined by N is (E, rk) since they have the same set of basis. □

Theorem 1.6.2. *Let (E, rk, m) be an orientable arithmetic matroid with the strong GCD property. Then (E, rk, m) is representable.*

Proof. Consider a orientation of (E, rk, m), the vectors $v_e \in \mathbb{Q}^r$ for $e \in E$ defined in the proof of Proposition 1.6.1, and let Λ the lattice generated by $\{v_e\}_{e\in E}$. Let G be a finite abelian group of cardinality $m(\emptyset) = m(E)$. We claim that the elements $(v_e, 0)$ in $\Lambda \times G$ are a representation of the arithmetic matroid (E, rk, m).

Let (E, rk', m') be the arithmetic matroid described by the elements $(v_e, 0)$ in $\Lambda \times G$. Let Γ_B be the lattice generated by v_e for $e \in B$. By eq. (1.7) we have $\mathrm{rk}' = \mathrm{rk}$ and $|\det N[B]| = m(B_0)^{r-1}m(B)$ for all basis B. Therefore,

$$[\mathbb{Z}^r : \Lambda] = \gcd\{\, |\det N[B]| \mid B \text{ basis of } E\,\} = m(B_0)^{r-1}m(E)$$

and

$$m'(B) = |G|\frac{[\mathbb{Z}^r : \Gamma_B]}{[\mathbb{Z}^r : \Lambda]} = m(E)\frac{m(B_0)^{r-1}m(B)}{m(B_0)^{r-1}m(E)} = m(B).$$

The multiplicity functions m and m' coincides on all basis of the matroid (E, rk), hence by the strong GCD property $m = m'$. □

1.7 Uniqueness of representations

We consider the representation up to an equivalence relation.

Definition 1.7.1. Two representations $(h_e)_{e\in E} \subset H$ and $(h'_e)_{e\in E} \subset H'$ of the same arithmetic matroid are *equivalent* if there exist a isomorphisms of groups $f : H \to H'$ such that $f(h_e) = \pm h'_e$ for all $e \in E$.

In general an arithmetic matroid has a lot of non-equivalent representations.

Example 1.7.2. Fix $m \in \mathbb{N}_+$ and consider the arithmetic matroid on the empty set $E = \emptyset$, $\mathrm{rk}(\emptyset) = 0$ and $m(\emptyset) = m$. All the representations are of the form $\mathbb{Z}^k \times H$ where H is finite abelian groups with $|H| = m$. Moreover, two representations H and H' are equivalent if and only if $k = k'$ and $H \cong H'$.

Example 1.7.3. Fix $m \in \mathbb{N}_+$ and, for all $a \in \mathbb{Z}$ relative prime with m, consider the pair of vectors $\left(\binom{1}{0}, \binom{a}{m}\right)$ of $\mathbb{Z}^2$. These are representations of an arithmetic matroid over $E = \{1, 2\}$ with $m(E) = m$, two of them are equivalent if and only if $a \equiv \pm a' \mod m$.

Consider an arithmetic matroid (E, rk, m) and choose a orientation χ of the arithmetic matroid. Recall from Definition 1.1.4 the definition of a signed circuit. For each circuit C of (E, rk) define the vector

$$v_C = \sum_{x \in C} c(x) m(C \setminus \{ x \}) e_x \in \mathbb{Q}^E$$

depending on χ, where e_x is the canonical basis of $\mathbb{Q}^E$. Let V_χ be the subspace of $\mathbb{Q}^E$ generated by the v_C for all circuit C.

Let H be an essential representation of (E, rk, m), we choose a basis of $\mathcal{B}$ of $H_\mathbb{Q} = H \otimes_\mathbb{Z} \mathbb{Q}$ and define $\chi(x_1, \dots, x_r) = \mathrm{sgn}(\det M_\mathcal{B}(h_{x_1}, \dots, h_{x_r}))$ where $M_\mathcal{B}(h_{x_1}, \dots, h_{x_r})$ is the matrix that represents the vectors $h_{x_i} \otimes 1$ in the basis $\mathcal{B}$. By an abuse of notation we denote $h_{x_i} \otimes 1$ with $h_{x_i} \in H_\mathbb{Q}$.

Lemma 1.7.4. *Let H be an essential representation and V_χ as above. We have the following exact sequence*

$$0 \to V_\chi \to \mathbb{Q}^E \to H_\mathbb{Q} \to 0$$

where the surjective map is defined by $e_x \mapsto h_x$.

Proof. Since H is essential, then $H_\mathbb{Q}$ has dimension $r = \mathrm{rk}(E)$ and h_i, for $i \in E$, generate $H_\mathbb{Q}$ as vector space. For each circuit $C = (x_0, \dots, x_s)$ the vector space $W_C \subset H_\mathbb{Q}$ generated by $\{ h_{x_i} \}_{i=0,\dots,s}$ has dimension s. The function $w \colon W_C^{s+1} \to W_C$ defined by

$$(w_0, \dots, w_s) \mapsto \sum_{i=0}^{s} (-1)^i \det(w_0, \dots, \hat{w}_i, \dots, w_s) w_i$$

is multilinear and alternating, thus is identically zero. For $(w_0, \dots, w_s) = (h_{x_0}, \dots, h_{x_s})$ we obtain $\sum_{i=0}^{s} c(x_i) m(C \setminus \{ x_i \}) h_{x_i} = 0$. Therefore, V_χ is contained in $\ker(\mathbb{Q}^E \to H_\mathbb{Q})$.

Let $\sum_{x \in E} a_x e_x$ an element of $\ker(\mathbb{Q}^n \to H_\mathbb{Q})$. We prove by induction on $|\{ x \in E \mid a_x \neq 0 \}|$ that $\sum_{i \in E} a_x e_x \in V_\chi$. Since $\sum_{x \in E} a_x h_x = 0$, we have that $|\{ x \mid a_x \neq 0 \}|$ contains a circuit $C = (x_0, \dots, x_s)$. The element

$$\sum_{x \in E} a_x e_x - \frac{a_{x_0}}{c(x_0) m(C \setminus \{ x_0 \})} \sum_{i=0}^{s} c(x_i) m(C \setminus \{ x_i \}) e_{x_i}$$

belongs to V_χ by inductive hypothesis and so $\sum_{i \in E} a_x e_x \in V_\chi$. We have proven that $\ker(\mathbb{Q}^n \to H_\mathbb{Q})$ is contained in V_χ. □

The following corollary is an immediate consequence of Lemma 1.7.4

Corollary 1.7.5. *Let H be a representation of an arithmetic matroid. Then all linear relations in $H_\mathbb{Q}$ between the vectors $\{ h_e \}_{e \in E}$ are combination of the following ones*

$$\sum_{i \in C} c(i) m(C \setminus \{ i \}) h_i = 0$$

Lemma 1.7.6. *Let H and H' two essential representation of the same arithmetic matroid, then there exist a linear isomorphism $\varphi_{\mathbb{Q}}\colon H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ such that $\varphi_{\mathbb{Q}}(h_e) = \pm h'_e$ for all $e \in E$.*

Proof. Let χ and χ' be the two orientation of (E, rk, m) given by H and H'. By Theorem 1.4.1 there exists $A \subseteq E$ such that χ is the reorientation of χ' by A. Define the representation H'', equivalent to H', given by $H'' = H'$, $h''_a = -h'_a$ for $a \in A$ and $h''_e = h'_e$ for $e \in E \setminus A$. We prove that the assignment $h_e \mapsto h''_e$ for all $e \in E$ defines an isomorphism $\varphi_{\mathbb{Q}}\colon H_{\mathbb{Q}} \to H''_{\mathbb{Q}} = H'_{\mathbb{Q}}$. This follows from the diagram

$$\begin{array}{ccccc} V_\chi & \longrightarrow & \mathbb{Q}^n & \longrightarrow & H_{\mathbb{Q}} \\ & & & \searrow & \downarrow{\scriptstyle \varphi_{\mathbb{Q}}} \\ & & & & H''_{\mathbb{Q}} \end{array}$$

since $\chi = \chi''$, both $H_{\mathbb{Q}}$ and $H''_{\mathbb{Q}}$ are the cokernel of $V_\chi \mapsto \mathbb{Q}^n$. From the above diagram also follows that $e_x \mapsto h_x$ and $e_x \mapsto h''_x$, thus by commutativity $h_x \mapsto h''_x$ for all $x \in E$. □

Recall that an arithmetic matroid (E, rk, m) is torsion-free if $m(\emptyset) = 1$ and surjective if $m(E) = 1$.

Theorem 1.7.7. *A representable, surjective, torsion-free arithmetic matroid has a unique representation up to equivalence.*

Proof. Let H and H' be two essential representation of (E, rk, m), we show that H and H' are equivalent. We apply Lemma 1.7.6 and obtain $\varphi_{\mathbb{Q}}\colon H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ such that $\varphi_{\mathbb{Q}}(h_e) = \pm h'_e$. Since $m(\emptyset) = 1$ then $H \hookrightarrow H_{\mathbb{Q}}$ and $H' \hookrightarrow H'_{\mathbb{Q}}$. Thus is well defined the restricted map $\varphi = \varphi_{\mathbb{Q}|\Gamma_E}\colon \Gamma_E \to \Gamma'_E$. Since $m(E) = 1$, we have $H = \Gamma_E$ and $H' = \Gamma'_E$. □

Corollary 1.7.8. *Let $(E, \mathrm{rk}, m, \chi)$ be a representable, torsion-free, oriented arithmetic matroid. Let $\{ h_e \}_{e \in E} \subset H$ be an essential representation. Then the linear relations among these vectors $\{ h_e \}_{e \in E}$ are uniquely determined by the oriented arithmetic matroid.* □

Example 1.7.9. Let $(\{1, 2, 3\}, \mathrm{rk})$ be the matroid of three distinct lines in the real plane. The function m defined by:

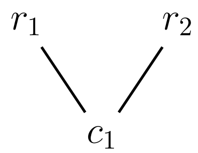
$$\begin{aligned} &m(\emptyset) = 1 \\ &m(e) = 1 \quad \text{for } e = 1, 2, 3 \\ &m(1, 2) = 10 \\ &m(1, 3) = 15 \\ &m(2, 3) = 25 \\ &m(1, 2, 3) = 5 \end{aligned}$$

defines an arithmetic matroid $([3], \mathrm{rk}, m)$. This arithmetic matroid is representable, indeed a possible representation is given by the matrix $\left(\begin{smallmatrix} -2 & -32 & -43 \\ 1 & 21 & 29 \end{smallmatrix}\right)$.

We choose a basis of the matroid, e.g. $\mathcal{B} = \{1, 2\}$ and we consider the matrix $A \in \mathrm{M}(2, 1; \mathbb{Q})$ representing the coordinates in the basis $\mathcal{B}$ of the third vector $\left(\begin{smallmatrix} -43 \\ 29 \end{smallmatrix}\right)$. The absolute value of the entries of A is easy to determine:

$$|a_{1,1}| = \frac{m(2,3)}{m(1,2)} = \frac{5}{2}$$
$$|a_{2,1}| = \frac{m(1,3)}{m(1,2)} = \frac{3}{2}$$

The associated matrix C is then $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and the associated bipartite graph is:



This graph has a unique maximal tree, that we call $\mathcal{A}$, hence the normal form of the matrix A (in normal form) is $\left(\begin{smallmatrix} \frac{5}{2} \\ \frac{3}{2} \end{smallmatrix}\right)$. A representation of the arithmetic matroid in normal form is given by $\left(\begin{smallmatrix} 2 & -32 & -43 \\ -1 & 21 & 29 \end{smallmatrix}\right)$ which is obtained from the one we had before by changing the sign of the first column.

1.8 Reduction of quasi-arithmetic matroids

In this section we introduce a new operation on quasi-arithmetic matroids, which we call *reduction*. We will use this construction in the algorithm that computes the representations of a torsion-free arithmetic matroid.

Definition 1.8.1 (Reduction)**.** Let $M = (E, \mathrm{rk}, m)$ be a quasi-arithmetic matroid. Its reduction is the quasi-arithmetic matroid $\overline{M} = (E, \mathrm{rk}, \overline{m})$ on the same groundset, with the same rank function, and with multiplicity function $\overline{m}$ is given by

$$\overline{m}(X) = \frac{\gcd\{\, m(B) \mid B \text{ is a basis, and } \mathrm{rk}(X) = |X \cap B| \,\}}{\gcd\{\, m(B) \mid B \text{ is a basis} \,\}}.$$

Given a matroid $\mathcal{M} = (E, \mathrm{rk})$ and two subsets $X, Y \subseteq E$, define

$$\mathcal{B}_{(X,Y)} = \{(B_1, B_2) \mid B_1 \text{ and } B_2 \text{ are bases of } \mathcal{M},\ \mathrm{rk}(X) = |X \cap B_1|, \\ \text{and } \mathrm{rk}(Y) = |Y \cap B_2|\}.$$

Lemma 1.8.2. *Let $\mathcal{M} = (E, \mathrm{rk})$ be a matroid, let (X, F, T) be a molecule and set $Y = X \sqcup T \sqcup F$ as in Definition 1.1.5. Then there is a bijection $\varphi\colon \mathcal{B}_{(X,Y)} \to \mathcal{B}_{(X\sqcup T,\, X\sqcup F)}$ given by*

$$\varphi(B_1, B_2) = \big((B_1 \setminus X) \cup (B_2 \cap (X \cup T)),\, (B_2 \setminus (X \cup T)) \cup (B_1 \cap X)\big).$$

Proof. Notice that $F \subseteq B_2$, because $\mathrm{rk}(Y) = \mathrm{rk}(Y \cap B_2) = \mathrm{rk}(X) + |B_2 \cap F|$ (the first equality is by definition of $\mathcal{B}_{(X,Y)}$, and the second equality is by definition of molecule).

We want to prove that $B_3 = (B_1 \setminus X) \cup (B_2 \cap (X \cup T))$ is a basis. The set $B_1 \setminus X$ is independent, and its rank (or cardinality) is equal to $|B_1| - |X \cap B_1| = \mathrm{rk}(E) - \mathrm{rk}(X)$ by definition of $\mathcal{B}_{(X,Y)}$. The set $B_2 \cap (X \cup T)$ is also independent, and (since $F \subseteq B_2$) its rank (or cardinality) is equal to $|B_2 \cap Y| - |F| = \mathrm{rk}(X) + |F| - |F| = \mathrm{rk}(X)$. Therefore $|B_3| \leq \mathrm{rk}(E)$. Applying property (2) of the rank function to the pair $(B_3, X \cup T)$, we obtain

$$\mathrm{rk}(B_3) + \mathrm{rk}(X \cup T) \geq \mathrm{rk}(B_3 \cup X \cup T) + \mathrm{rk}(B_3 \cap (X \cup T)).$$

Notice that $\mathrm{rk}(X \cup T) = \mathrm{rk}(X)$ (by definition of molecule), $B_1 \subseteq B_3 \cup X \cup T$, and $B_2 \cap (X \cup T) \subseteq B_3 \cap (X \cup T)$. Then

$$\mathrm{rk}(B_3) + \mathrm{rk}(X) \geq \mathrm{rk}(B_1) + \mathrm{rk}(B_2 \cap (X \cup T)) = \mathrm{rk}(E) + \mathrm{rk}(X).$$

Therefore $\mathrm{rk}(B_3) \geq \mathrm{rk}(E)$, and B_3 is a basis.

We want now to check that $|B_3 \cap (X \cup T)| = \mathrm{rk}(X \cup T)$. We have $B_1 \cap T = \emptyset$, because

$$\begin{aligned} \mathrm{rk}(X) + |T \cap B_1| &= |X \cap B_1| + |T \cap B_1| = |(X \cap B_1) \sqcup (T \cap B_1)| \\ &= |(X \cup T) \cap B_1| = \mathrm{rk}((X \cup T) \cap B_1) \\ &\leq \mathrm{rk}(X \cup T) = \mathrm{rk}(X). \end{aligned}$$

Thus $B_3 \cap (X \cup T) = B_2 \cap (X \cup T)$, and this set has cardinality $\mathrm{rk}(X) = \mathrm{rk}(X \cup T)$.

Similarly, $B_4 = (B_2 \setminus (X \cup T)) \cup (B_1 \cap X)$ is a basis, and $B_4 \cap (X \cup F)| = \mathrm{rk}(X \cup F)$. Therefore the map φ is well-defined.

The map $\psi\colon \mathcal{B}_{(X\sqcup T,\, X\sqcup F)} \to \mathcal{B}_{(X,Y)}$ defined by

$$\psi(B_3, B_4) = \big((B_3 \setminus (X \cup T)) \cup (B_4 \cap X),\, (B_4 \setminus X) \cup (B_3 \cap (X \cup T))\big)$$

can be verified to be the inverse of φ. Therefore φ is a bijection. □

Lemma 1.8.3. *Let $M = (E, \mathrm{rk}, m)$ be a quasi-arithmetic matroid, consider a molecule (X, F, T) and set $Y = X \sqcup T \sqcup F$ as in Definition 1.1.5. If $\varphi\colon \mathcal{B}_{(X,Y)} \to \mathcal{B}_{(X\sqcup T,\, X\sqcup F)}$ is the bijection of Lemma 1.8.2, and $(B_3, B_4) = \varphi(B_1, B_2)$, then*

$$m(B_1)\, m(B_2) = m(B_3)\, m(B_4).$$

Proof. Consider the following four molecules:

$$(B_1 \cap X,\ (B_2 \cap (X \cup T)) \cup B_1);$$
$$(B_2 \cap (X \cup T),\ (B_1 \cap X) \cup B_2);$$
$$(B_2 \cap (X \cup T),\ (B_2 \cap (X \cup T)) \cup B_1);$$
$$(B_1 \cap X,\ (B_1 \cap X) \cup B_2).$$

Applying axiom (A2) to these molecules, we get the following relations (we use the fact that $B_1 \cap T = \emptyset$, shown in the proof of Lemma 1.8.2):

$$m(B_1 \cap X)\, m((B_2 \cap (X \cup T)) \cup B_1) = m((B_1 \cup B_2) \cap (X \cup T))\, m(B_1); \tag{1.8}$$

$$m(B_2 \cap (X \cup T))\, m((B_1 \cap X) \cup B_2) = m((B_1 \cup B_2) \cap (X \cup T))\, m(B_2); \tag{1.9}$$

$$m(B_2 \cap (X \cup T))\, m((B_2 \cap (X \cup T)) \cup B_1) = m((B_1 \cup B_2) \cap (X \cup T))\, m(B_3); \tag{1.10}$$

$$m(B_1 \cap X)\, m((B_1 \cap X) \cup B_2) = m((B_1 \cup B_2) \cap (X \cup T))\, m(B_4). \tag{1.11}$$

Let $k = m((B_1 \cup B_2) \cap (X \cup T))$. Multiplying the previous equations in pairs, we obtain $k^2\, m(B_1)\, m(B_2) = k^2\, m(B_3)\, m(B_4)$, hence $m(B_1)\, m(B_2) = m(B_3)\, m(B_4)$. □

Theorem 1.8.4. *The reduction $\overline{M}$ of a quasi-arithmetic matroid (E, rk, m) is a torsion-free surjective quasi-arithmetic matroid, and satisfies the strong GCD property.*

Proof. Let $d = \gcd\{\, m(B) \mid B \text{ is a basis}\,\}$. We start by checking axiom (A1) of Definition 1.1.6. Consider a subset $X \subseteq E$ and an element $e \in E$.

- If $\mathrm{rk}(X \cup \{e\}) = \mathrm{rk}(X)$, then a basis B such that $\mathrm{rk}(X) = \mathrm{rk}(X \cap B)$ also satisfies $\mathrm{rk}(X \cup \{e\}) = \mathrm{rk}((X \cup \{e\}) \cap B)$. Therefore $d \cdot \overline{m}(X \cup \{e\}) \mid d \cdot \overline{m}(X)$.

- Similarly, if $\mathrm{rk}(X \cup \{e\}) = \mathrm{rk}(X) + 1$, then a basis B such that $\mathrm{rk}(X \cup \{e\}) = \mathrm{rk}((X \cup \{e\}) \cap B)$ also satisfies $\mathrm{rk}(X) = \mathrm{rk}(X \cap B)$. Therefore $d \cdot \overline{m}(X) \mid d \cdot \overline{m}(X \cup \{e\})$.

We now check axiom (A2). Let (X, Y) be a molecule, with $Y = X \sqcup T \sqcup F$ as in Definition 1.1.5. By definition of $\overline{m}$, we have that

$$d^2\, \overline{m}(X)\, \overline{m}(Y) = \gcd\{\, m(B_1)\, m(B_2) \mid (B_1, B_2) \in \mathcal{B}_{(X,Y)}\,\}.$$

Similarly,

$$d^2\, \overline{m}(X \cup T)\, \overline{m}(X \cup F) = \gcd\{\, m(B_3)\, m(B_4) \mid (B_3, B_4) \in \mathcal{B}_{(X \cup T,\, X \cup F)}\,\}.$$

By Lemmas 1.8.2 and 1.8.3, we obtain $d^2\,\overline{m}(X)\,\overline{m}(Y) = d^2\,\overline{m}(X\cup T)\,\overline{m}(X\cup F)$, hence $\overline{m}(X)\,\overline{m}(Y) = \overline{m}(X\cup T)\,\overline{m}(X\cup F)$.

Therefore $\overline{M}$ is a quasi-arithmetic matroid. By definition of $\overline{m}$, we also have that $\overline{m}(\emptyset) = \overline{m}(E) = 1$, i.e. M is torsion-free and surjective. It is also immediate to check that $\overline{M}$ satisfies the strong GCD property. □

It is not true in general that the reduction of an arithmetic matroid is an arithmetic matroid. We see this in the following example.

Example 1.8.5. Let $\mathcal{M} = (E, \mathrm{rk})$ be the uniform matroid of rank 2 on the groundset $E = \{1, 2, \ldots, 6\}$. Consider the multiplicity function $m\colon \mathcal{P}(E) \to \mathbb{N}_+$ defined as

$$\begin{aligned}
&m(\emptyset) = 1, && \\
&m(\{1\}) = m(\{2\}) = 2, && \\
&m(\{j\}) = 1 && \text{if } j > 2, \\
&m(\{X\}) = 1 && \text{if } |X \cap \{3,\ldots,6\}| \geq 2, \\
&m(\{i,j\}) = 2 && \text{if } i = 1,2 \text{ and } j > 2, \\
&m(\{1,2\}) = 4, && \\
&m(\{1,2,3\}) = 1, && \\
&m(\{1,2,j\}) = 2 && \text{if } j > 3.
\end{aligned}$$

Then $M = (E, \mathrm{rk}, m)$ is an arithmetic matroid (this can be checked using the software library [PP19a]). We have that $\overline{m}(X) = m(X)$ for every $X \subseteq E$, except that $\overline{m}(1,2,3) = 2$. The quasi-matroid $\overline{M} = (E, \mathrm{rk}, \overline{m})$ does not satisfy axiom (P) for the molecule $(\{1,2\}, E)$.

However, the reduction of a representable arithmetic matroid turns out to be a representable arithmetic matroid.

Theorem 1.8.6. *If $M = (E, \mathrm{rk}, m)$ is a representable arithmetic matroid, then its reduction $\overline{M}$ is also a representable arithmetic matroid.*

Proof. Let $(v_e)_{e\in E} \subseteq G$ be a representation of M. Denote by K the quotient of G by its torsion subgroup T. Let $\overline{G}$ be the sublattice of K generated by $\{\, \bar{v}_e \mid e \in E \,\}$, where $\bar{v}_e$ is the class of v_e in K. We are going to show that $(\bar{v}_e)_{e\in E} \subseteq \overline{G}$ is a representation of $\overline{M}$.

Let $M' = (E, \mathrm{rk}, m')$ be the arithmetic matroid associated with the representation $(\bar{v}_e)_{e\in E} \subseteq \overline{G}$. By construction, M' is representable, torsion-free (because $\overline{G}$ is torsion-free), and surjective (because the vectors $\bar{v}_e$ generate $\overline{G}$). Therefore, by Corollary 1.5.5, it satisfies the strong GCD property. As a consequence,

$$\gcd\{\, m'(B) \mid B \text{ basis} \,\} = m(E) = 1.$$

Let B be a basis of M. Since B is independent, we have that $T \cap \langle v_b \rangle_{b\in B} = \{0\}$. Then,

$$\begin{aligned} m(B) &= \left| {}^{G}\!/\!{}_{\langle v_b\rangle_{b\in B}} \right| = |T| \cdot \left| {}^{K}\!/\!{}_{\langle \bar{v}_b\rangle_{b\in B}} \right| = |T| \cdot \left| {}^{K}\!/\!{}_{\overline{G}} \right| \cdot \left| {}^{\overline{G}}\!/\!{}_{\langle \bar{v}_b\rangle_{b\in B}} \right| \\ &= |T| \cdot \left| {}^{K}\!/\!{}_{\overline{G}} \right| \cdot m'(B). \end{aligned}$$

If B varies among all bases of M, taking the gcd of both sides we get

$$\gcd \{ m(B) \mid B \text{ basis} \} = |T| \cdot \left| {}^{K}\!/\!{}_{\overline{G}} \right|.$$

Therefore

$$m'(B) = \frac{m(B)}{\gcd \{ m(B) \mid B \text{ basis} \}} = \overline{m}(B).$$

Since both M' and $\overline{M}$ satisfy the strong GCD property, $m'(X) = \overline{m}(X)$ for every subset $X \subseteq E$. This means that $\overline{M} = M'$ is representable. □

Finally, notice that the reduction does not commute with deletion and contraction. However, it commutes with taking the dual.

1.9 Classification of representations

The aim of this section is to classify all representations up to equivalence of a representable torsion-free arithmetic matroid (typically non-surjective).

Let (E, rk, m) be a realizable torsion-free arithmetic matroid and $(E, \mathrm{rk}, \overline{m})$ be its reduction. The triple $(E, \mathrm{rk}, \overline{m})$ is a representable arithmetic matroid by Theorem 1.8.6 and let $\{ h_e \}_{e\in E}$ in Γ be a representation of $(E, \mathrm{rk}, \overline{m})$. By Theorem 1.7.7, this representation is unique.

Each representation $\{ h'_e \}_{e\in E}$ in Λ of (E, rk, m) induces the representation $\{ h'_e \}_{e\in E}$ in $\Gamma_E \subset \Lambda$ of $(E, \mathrm{rk}, \overline{m})$. By uniqueness we identify Γ_E with Γ and h'_e with $s_e h_e$ for some $s_e \in \{ 1, -1 \}$. The representation $\{ s_e h'_e \}_{e\in E}$ in Λ is equivalent to $\{ h'_e \}_{e\in E}$ in Λ and it identifies the elements $s_e h'_e$ with the given one h_e. Therefore, each representation of (E, rk, m) is determined, up to equivalence, by a suitable extension of lattices $\Gamma \supset \Lambda$.

Let $f\colon \Lambda \to \Lambda'$ be an equivalence of representation, it restricts to a map $f\colon \Gamma \to \Gamma$. Is not true that $f(h_e) = h_e$, as shown in the following example.

Example 1.9.1. For a, m coprime integers consider, as in Example 1.7.3, the arithmetic matroid $([2], \mathrm{rk}, m)$, where $\mathrm{rk}(A) = |A|$, $m([2]) = m$ and $m(A) = 1$ for all $A \subsetneq E$. The representation described by the matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)$ is equivalent to the one described by the matrix $\left(\begin{smallmatrix} 1 & -a \\ 0 & m \end{smallmatrix}\right)$. However, since the associated matroid $([2], \mathrm{rk}, m)$ has no circuits, the two orientations are the same. The morphism $f\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ sends $h_1 \mapsto -h'_1$ and $h_2 \mapsto h'_2$.

Lemma 1.9.2. *Let $f\colon \Lambda \to \Lambda'$ be an equivalence of representation as above. If $i, j \in E$ are contained in a circuit, then there exists $s \in \{1, -1\}$ such that $f(h_i) = sh'_i$ and $f(h_j) = sh'_j$.*

Proof. There exists $I \subset E$ such that $I \sqcup \{i\}$ and $I \sqcup \{j\}$ are basis of the matroid. Let $s_i, s_j \in \{1, -1\}$ such that $f(h_i) = s_i h'_i$ and $f(h_j) = s_j h'_j$ and $\mathcal{B}$ be a basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that $f(\mathcal{B})$ is a basis of $\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}$ and therefore $M_{\mathcal{B}}(\{h_a \otimes 1\}_{a \in A}) = M_{f(\mathcal{B})}(\{f(h_a) \otimes 1\}_{a \in A})$ for any $A \in E^r$. We have

$$\chi(I \cup \{i\})\chi(I \cup \{j\}) = s_i s_j \chi(I \cup \{i\})\chi(I \cup \{j\}),$$

hence $s_i = s_j$ as claimed. □

Definition 1.9.3. Let (E, rk) be a matroid, we define $S = S(E, \mathrm{rk})$ as the subgroup of $\mathbb{Z}_2^E$ of elements s such that $s(i) = s(j)$ if there exists a circuit containing both i and j.

We regard S as the subgroup of $\mathrm{GL}(n; \mathbb{Z})$ of diagonal matrices with entries equal to ± 1. The representation $\{h_e\}_{e \in E}$ of $(E, \mathrm{rk}, \overline{m})$ induces a surjective morphism $p\colon \mathbb{Z}^E \to \Gamma$. By Lemma 1.7.4, the kernel of p is contained in V_χ where χ is the orientation induced by $\{h_e\}_{e \in E}$. Since, by definition, the elements $s \in S$ preserves $\ker p$, we regard S as a subgroup of $\mathrm{Aut}(\Gamma) \simeq \mathrm{GL}(r; \mathbb{Z})$.

Remark 1.9.4. The cardinality of S is equal to the number of irreducible component of the matroid (E, rk).

Remark 1.9.5. The matroid (E, rk, m) determines the isomorphism class of the group $G = \Lambda/\Gamma$. In fact G is the cokernel of the matrix $N(\mathcal{A})$ whose columns are the coordinate of χ_e in some basis. Then by the Smith normal form, its isomorphism class depends only on the greatest common divisor of the determinants of the minors of $N(\mathcal{A})$. The group G can be presented as the cokernel of D, where D is the $r \times r$ diagonal matrix with entries $d_i = \frac{e_i}{e_{i-1}}$, where $e_i = \gcd\{m(E) \mid |E| = i\}$.

A standard fact from commutative algebra is the correspondence between the group $\mathrm{Ext}^1(G, F)$ and the extension of the two $\mathbb{Z}$-modules F and G:

$$0 \to F \to X \to G \to 0 \tag{1.12}$$

up to equivalence, i.e. two extensions X and X' are equivalent if there exists the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & X' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Moreover every subgroup of $\mathrm{Aut}(G)$ or of $\mathrm{Aut}(F)^{\mathrm{op}}$ acts on $\mathrm{Ext}^1(G;F)$ by functoriality.

We are interested mainly in the case of the extension

$$0 \to \Gamma \to \Lambda \to G \to 0$$

that corresponds to an element $x \in \mathrm{Ext}^1(G,\Gamma)$. We will show that an equivalent representation Γ' will give another element x' that differs form x by the action of $S \times \mathrm{Aut}(G)$. Moreover, we will characterize all elements in $\mathrm{Ext}^1(G,\Gamma)/(S \times \mathrm{Aut}(G))$ that arise from representations of (E, rk, m). Now we want to characterize the cardinality of the torsion subgroup $|\mathrm{Tor}\, X|$ for X as in Equation (1.12). Suppose that F is free, then the torsion subgroup $\mathrm{Tor}\, X$ is isomorphic to its image in G.

Lemma 1.9.6. *Let F be a free $\mathbb{Z}$-module. Then the contravariant functor $\mathrm{Ext}^1(\cdot, F)$ from finite abelian groups to $\mathbb{Z}$-modules is an exact functor.*

Proof. A short exact sequence of finite abelian groups $0 \to H \xrightarrow{i} G \xrightarrow{p} K \to 0$ produces a short exact sequence

$$0 \to \mathrm{Ext}^1(K,F) \xrightarrow{p^*} \mathrm{Ext}^1(G,F) \xrightarrow{i^*} \mathrm{Ext}^1(H,F) \to 0$$

since $\mathrm{Hom}(H,F) = 0$ and $\mathrm{Ext}^2(K,F) = 0$ for all free $\mathbb{Z}$-module F and all finite group H. □

Lemma 1.9.7. *Let F be a free $\mathbb{Z}$-module, G a finite abelian group and x an element of $\mathrm{Ext}^1(G,F)$. Then:*

1. *There exists a unique subgroup $H \hookrightarrow G$, maximal among all subgroups H' such that $i^*_{H'}(x) = 0$.*

2. *There exists a unique quotient $G \twoheadrightarrow K$, minimal among all quotients K' such that $x \in \mathrm{Im}\, p^*_{K'}$.*

Moreover, such groups form an exact sequence $0 \to H \to G \to K \to 0$.

Proof. Suppose that, for two subgroups H and H' of G, $i^*_H(x) = 0$ and $i^*_{H'}(x) = 0$. There is a surjection $H \times H' \twoheadrightarrow HH' < G$ that gives an inclusion $\mathrm{Ext}^1(HH',F) \hookrightarrow \mathrm{Ext}^1(H,F) \times \mathrm{Ext}^1(H',F)$. The element $i^*_{HH'}(x)$ maps to $(0,0)$ so it must be zero ($i^*_{HH'}(x) = 0$ in $\mathrm{Ext}^1(HH',F)$). The arbitrariness of H and H' gives the first result.

The second point follows from the first making use of the following fact: for every exact sequence $0 \to H' \to G \to K' \to 0$ the element $i^*_{H'}(x)$ is zero if and only if $x \in \mathrm{Im}\, p^*_{K'}$. □

We make the following construction: for $A \subseteq E$, define Γ_A to be the subgroup of Γ generated by $\{ h_e \}_{e \in A}$ and $\mathrm{Rad}_\Gamma(\Gamma_A)$ its radical in the lattice Γ. Let F_A be the quotient of Γ by $\mathrm{Rad}_\Gamma(\Gamma_A)$ and notice that F_A is a free $\mathbb{Z}$-module. The exact sequence $0 \to \mathrm{Rad}_\Gamma(\Gamma_A) \to \Gamma \to F_A \to 0$ of free modules gives, for any finite abelian group G the exact sequence:

$$0 \to \mathrm{Ext}^1(G, \mathrm{Rad}_\Gamma(\Gamma_A)) \to \mathrm{Ext}^1(G, \Gamma) \xrightarrow{\pi_A} \mathrm{Ext}^1(G, F_A) \to 0. \tag{1.13}$$

Definition 1.9.8. Consider $x \in \mathrm{Ext}^1(G, \Gamma)$ and call $H_A(x)$ the maximal subgroup H of G given by Lemma 1.9.7 for the elements $\pi_A(x) \in \mathrm{Ext}^1(G, F_A)$. An element $x \in \mathrm{Ext}^1(G, \Gamma)$ is said *coherent with* (E, rk, m) if for all $A \subseteq E$ we have $\overline{m}(A)|H_A(x)| = m(A)$. Call $\mathcal{C}$ the subset of $\mathrm{Ext}^1(G, \Gamma)$ made by all coherent elements quotient by the action of $S \times \mathrm{Aut}(G)$.

Theorem 1.9.9. *Let (E, rk, m) be a representable torsion-free arithmetic matroid. The set $\mathcal{C}$ parametrizes all the representation up to equivalence.*

Proof. Let $[x] \in \mathcal{C}$ be a element such that x is coherent with (E, rk, m); x gives an extension $0 \to \Gamma \to \Lambda_x \to G \to 0$. If $S \subseteq E$ then there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & \Lambda_x & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F_A & \longrightarrow & \Lambda_x / \mathrm{Rad}_\Gamma \Gamma_A & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Call Λ_A the quotient $\Lambda_x / \mathrm{Rad}_\Gamma \Gamma_A$; we will show that the group $H_A(x)$ is the torsion subgroup of Λ_A. The exact sequence $0 \to F_A \to \Lambda_A \to G \to 0$ is represented by the element $\pi_A(x)$ (see eq. (1.13)). Therefore for all G' subgroup of G, $i_{G'}(\pi_A(x))$ is zero if and only if the upper short exact sequence of the following diagram splits.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_A & \longrightarrow & X & \longrightarrow & G' & \longrightarrow & 0 \\ & & \| & & \hookdownarrow & & \hookdownarrow & & \\ 0 & \longrightarrow & F_A & \longrightarrow & \Lambda_A & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

The upper short exact sequence splits if and only if X is included in the subgroup $F_A \times \mathrm{Tor}\,\Lambda_A$ of Λ_A. Indeed, if the upper sequence splits then $F_A \times G' \simeq X \subseteq \Lambda_A$ and G' is a torsion group, hence included in $\mathrm{Tor}\,\Lambda_A$. Viceversa, if $X \subseteq F_A \times \mathrm{Tor}\,\Lambda_A$ then the projection onto the first factor gives a retraction of $F_A \hookrightarrow X$ and so the sequence splits.

Hence, the maximal subgroup $H_A(x)$ is isomorphic to the torsion subgroup of Λ_A, that is, $\mathrm{Rad}_{\Lambda_x} \Gamma_A / \mathrm{Rad}_\Gamma \Gamma_A$. The obvious equality:

$$\left| \mathrm{Rad}_{\Lambda_x} \Gamma_A / \mathrm{Rad}_\Gamma \Gamma_A \right| \cdot \left| \mathrm{Rad}_\Gamma \Gamma_A / \Gamma_A \right| = \left| \mathrm{Rad}_{\Lambda_x} \Gamma_A / \Gamma_A \right| \tag{1.14}$$

implies the equality $|H_A(x)|\overline{m}(A) = m_x(A)$, where m_x is the multiplicity induced by the inclusion $\Gamma \hookrightarrow \Lambda_x$. Since x is a coherent element, we obtain the equality $m_x = m$. A different choice of x, say fxs, gives an equivalent representation since the five lemma applied to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & \Lambda_x & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \Lambda_{fxs} & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

proves that $\psi\colon \Lambda_x \to \Lambda_{fxs}$ is an isomorphism. Moreover, for each $e \in E$ we have that $\psi(h_e) = s(e)h_e$, thus the two representations are equivalent.

The surjectivity of the correspondence follows by taking a representation Λ of (E, rk, m). Indeed, the inclusion $\Gamma = \Gamma_E \to \Lambda$ has cokernel isomorphic to G by Remark 1.9.5. Thus Λ is represented by a element $x(\Lambda)$ of $\mathrm{Ext}^1(G, \Gamma)$. This element is coherent by eq. (1.14).

Let $x, y \in \mathrm{Ext}^1(G, F)$ be two coherent element and suppose that Λ_x and Λ_y are equivalent. Then by Lemma 1.9.2 we can find $s' \in S$ such that the equivalence $\psi'\colon \Lambda_{xs'} \to \Lambda_y$ induces the identity map on Γ. So, ψ' induces an automorphism f' of G and therefore the extensions $\Lambda_{f'xs'}$ and Λ_y are equivalent and then $y = f'xs'$. This completes the proof. □

Corollary 1.9.10. *For any centred toric arrangement, the data of the arithmetic matroid (E, rk, m) together with $[x] \in \mathcal{C}$ are a complete invariant system for the arrangement up to automorphisms of the torus.* □

Example 1.9.11. We continue Example 1.7.9 of the arithmetic matroid $([3], \mathrm{rk}, m)$. The group G is isomorphic to $\mathbb{Z}_5$. The multiplicity function $\overline{m}$ is defined by:

$$\begin{aligned} \overline{m}(\emptyset) &= 1 \\ \overline{m}(e) &= 1 \qquad \text{for } e = 1,2,3 \\ \overline{m}(1,2) &= 2 \\ \overline{m}(1,3) &= 3 \\ \overline{m}(2,3) &= 5 \\ \overline{m}(1,2,3) &= 1 \end{aligned}$$

The unique representation of $([3], \mathrm{rk}, \overline{m})$ is described by $\Gamma = \mathbb{Z}^2$ and by the elements:

$$\begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 3 \end{pmatrix}.$$

The relation $5h_1 + 3h_2 - 2h_3 = 0$ holds and the group $\mathrm{Ext}^1(G, \Gamma)$ is isomorphic to $\mathbb{Z}_5^2$. We look for the elements $x \in \mathrm{Ext}^1(G, \Gamma)$ which are coherent with the arithmetic matroid. Notice that the subgroups of G are only 0 and G (this is not true in general) and imposing the coherence conditions for x yields:

$$|H_\emptyset(x)| = \frac{m(\emptyset)}{m_U(\emptyset)} = 1 \Leftrightarrow H_\emptyset(x) = 0 \Leftrightarrow x \neq 0 \in \mathrm{Ext}^1(G, \Gamma)$$
$$|H_i(x)| = \frac{m(i)}{m_U(i)} = 1 \Leftrightarrow H_i(x) = 0 \Leftrightarrow \pi_i(x) \neq 0$$

The last implication holds for $i = 1, 2, 3$. By choosing the basis v_1, v_2 of Γ, we identify $x \in \mathrm{Ext}^1(G, \Gamma) \simeq \mathbb{Z}_5^2$ with pairs (a, b) such that $a, b \in \mathbb{Z}_5$. By standard commutative algebra $\Lambda_x = \mathrm{coker}((a, b, 5)\colon \mathbb{Z} \to \Gamma \oplus \mathbb{Z})$.

The conditions become, respectively:

$$\pi_1(x) = b \neq 0 \qquad \pi_2(x) = 2a - b \neq 0 \qquad \pi_3(x) = 3a - 4b \neq 0$$

For the remaining subsets $A \subset [3]$, which are all of rank two, we have that $\mathrm{Rad}_\Gamma \Gamma_A$ coincides with the whole lattice Γ. In particular $\pi_A(x) = 0$ and $H_A(x) = G$, therefore these coherence conditions are always satisfied.

The group $S \simeq \mathbb{Z}_2$ acts by changing sign of a and b and the group $\mathrm{Aut}(G) \simeq \mathbb{Z}_5^*$ acts by multiplication. Summing all up, the coherent elements are

$$\mathcal{C} = \{\, (a, b) \mid b, 2a - b \neq 0 \,\} \big/ S \times \mathbb{Z}_5^*,$$

notice that the conditions $2a - b \neq 0$ and $3a - 4b \neq 0$ are equivalent and the set $\mathcal{C}$ coincides with $\{\, a \in \mathbb{Z}_5 \mid a \neq 3 \,\}$ (take a representative with $b = 1$).

We are going to built a representation of the arithmetic matroid for each element of $\mathcal{C}$, such that any two of them are non-isomorphic. The group Λ_x is identified with the lattice in $\mathbb{Q}^2$ generated by $e_1, e_2, w = \frac{1}{5}(ae_1 + e_2)$ (recall that $x = (a, 1)$). A basis for Λ_x is given by $\{\, e_1, w \,\}$ and the three elements h_1, h_2, h_3 have the following coordinates:

$$C_a = \begin{pmatrix} 1 & 1 - 2a & 4 - 3a \\ 0 & 10 & 15 \end{pmatrix}$$

These are all the representations of the initial arithmetic matroid, up to equivalence.

Indeed, the initial representation is equivalent to C_1:

$$\begin{pmatrix} -2 & -32 & -43 \\ 1 & 21 & 29 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 10 & 15 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Example 1.9.12. We continue with Example 1.7.3 of the arithmetic matroid $([2], \mathrm{rk}, m)$. Let $e_1, e_2 \in \mathbb{Z}^2 = \Gamma$ be the unique representation of the arithmetic matroid $([2], \mathrm{rk}, \overline{m})$. All representations of (E, rk, m) are parametrize by the coherent elements in $\mathrm{Ext}^1(\mathbb{Z}_m, \Gamma) \simeq \mathbb{Z}_m^2$. We denote its elements by pairs

(a,b) and they are coherent if and only if $\gcd(a,m) = \gcd(b,m) = 1$. The group S is isomorphic to $\mathbb{Z}_2^2$ and acts on $\operatorname{Ext}^1(\mathbb{Z}_m, \Gamma)$ by changing the sign of the corresponding coordinates.

Therefore, the set $\mathcal{C}$ is $\{\, (a,b) \mid \gcd(a,m) = \gcd(b,m) = 1 \,\} / \mathbb{Z}_2^2 \times \mathbb{Z}_m^*$ and it can be identify with

$$\mathcal{C} = \{\, a \in [1, \frac{m}{2}] \mid \gcd(a,m) = 1 \,\},$$

by taking the representative with $b = 1$. The representations are given by the classes e_1, e_2 in $\Lambda_x = \operatorname{coker}((a,1,m) \colon \mathbb{Z} \to \Gamma \times \mathbb{Z})$, i.e. by the matrices

$$C_a = \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}.$$

Chapter 2

Cohomology

Sections 2.4, 2.5 and 2.11 are revised version of the corresponding section of [Pag19b], the rest of this Chapter is the article [CDD$^+$18], a joint work with Callegaro, D'Adderio, Delucchi and Migliorini.

2.1 Introduction

The topology of the complement of an arrangement of hyperplanes in a complex vector space is a classical subject, whose study received considerable momentum form early work of Arnold and Brieskorn (e.g., [Arn69, Bri73]) motivated by applications to the theory of braid groups and of configuration spaces. A distinguishing trait of this research field is the deep interplay between the topological and geometric data and the arrangement's combinatorial data, here usually understood to be the arrangement's *matroid*, a combinatorial abstraction of the linear dependencies among the hyperplanes' defining forms. A milestone in this direction is the presentation of the complement's integral cohomology algebra given by Orlik and Solomon [OS80], building on work of Arnold and Brieskorn. As we will explain below, this presentation is fully determined by the combinatorial (matroid) data and thus such an algebra can be associated with any matroid. Over the years, *Orlik-Solomon algebras* of general matroids have attracted interest in their own right [Yuz01].

In the wake of De Concini, Procesi and Vergne's work on the connection between partition functions and splines [DPV10] came a renewed interest in the study of complements of arrangements of subtori in the complex torus – a class of spaces which had already been considered by Looijenga in the context of moduli spaces [Loo93]. Following [DP05] we call such objects *toric arrangements*. Below we will briefly outline the state of the art on the topology of toric arrangements. This research direction was spurred particularly by the seminal work of De Concini and Procesi [DP05] which foreshadowed as rich an interplay between topology and combinatorics as is the case for hyperplane arrangements.

A crucial aspect that emerged in [DP05] and was confirmed by subsequent research in the topology of toric arrangements is that the matroid data naturally associated with every toric arrangement is not fine enough to encode meaningful geometric and topological invariants of the arrangement's complement. The quest for a suitable enrichment of matroid theory has been pursued from different points of view, i.e., by modeling the algebraic-arithmetic structure of the set of characters defining the arrangement [DM13, BM14, FM16] or by studying the properties of the pattern of intersections [DR18].

In this chapter we provide an Orlik-Solomon type presentation for the cohomology algebra of an arbitrary toric arrangement, generalizing De Concini and Procesi's work on the unimodular case. Our presentation with rational coefficients is fully determined by the intersection pattern. This presentation holds also for the integral cohomology algebra, but, in this case, it is not determined by the intersection pattern. In order to be able to state our results we provide some background.

Arrangements of hyperplanes and Orlik-Solomon Algebras

A *(central) hyperplane arrangement* is a finite set $\mathcal{A} = \{H_\lambda\}_{\lambda\in E}$ of codimension one linear subspaces in a complex vector space $V \simeq \mathbb{C}^n$. The space $M(\mathcal{A}) := V \setminus \cup\mathcal{A}$ is in a natural way an affine complex variety, hence its cohomology (over $\mathbb{C}$) is computed by the *algebraic* de Rham complex, as the quotient of the group of closed algebraic forms modulo that of exact algebraic ones (by Grothendieck's algebraic de Rham theorem [Gro66]).

We choose vectors $\{a_\lambda\}_{\lambda\in E} \subset V^*$ such that $H_\lambda = \ker a_\lambda$ and consider the free exterior algebra $\Lambda_\mathcal{A}$ over $\mathbb{Z}$ generated by the symbols $\{e_\lambda\}_{\lambda\in E}$. In $\Lambda_\mathcal{A}$ we define an ideal as follows: for every subset $A := \{ a_{\lambda_1}, \cdots, a_{\lambda_r} \} \subset \{ a_\lambda \}_{\lambda\in E}$ of linearly *dependent* vectors, we set

$$\partial e_A := \sum_{i=1}^{r} (-1)^{i-1} e_{\lambda_1} \cdots \widehat{e_{\lambda_i}} \cdots e_{\lambda_r} \tag{2.1}$$

and let $J_\mathcal{A}$ be the ideal generated by the ∂e_A's, where A runs over all linearly dependent subsets of E.

The quotient algebra $\Lambda_\mathcal{A}/J_\mathcal{A}$ is called the *Orlik-Solomon algebra* of the arrangement. The theorem of Orlik and Solomon states that the map $\Lambda_\mathcal{A} \to H^*(M(\mathcal{A}), \mathbb{Z})$ sending e_λ to the differential form $\frac{1}{2\pi \mathbf{i}} \operatorname{dlog} a_\lambda$ factors to an algebra isomorphism

$$\Lambda_\mathcal{A}/J_\mathcal{A} \xrightarrow{\simeq} H^*\big(M(\mathcal{A}), \mathbb{Z}\big).$$

Two consequences of this fact are:

1. $H^*(M(\mathcal{A}), \mathbb{Z})$ is generated in degree one;
2. the integral ring structure depends only on the structure of the family of linearly dependent subsets of $\{ a_\lambda \}_{\lambda\in E}$.

Definition 2.1.1. Recall the fixed total ordering of E. A *broken circuit* of E is any subset of the form $C \setminus \{ \min C \}$ where C is a circuit, i.e. a minimal dependent set.

A *no-broken-circuit set* (or *nbc*-set) is any subset of E that does not contain any broken circuit. The collection of all nbc sets is denoted $\mathrm{nbc}(\mathcal{A})$ (or $\mathrm{nbc}(\mathcal{M})$ if we want to stress the dependency from the matroid).

Remark 2.1.2. Every nbc-set is necessarily independent.

Theorem 2.1.3 (Orlik-Solomon [OS80])**.** *Let $\mathcal{A}$ be a central hyperplane arrangement, there is a an isomorphism:*

$$\Lambda_E \big/ J_E \xrightarrow{\simeq} H^\bullet(M(\mathcal{A}); \mathbb{Z})$$

Moreover a basis of Λ_E/J_E as $\mathbb{Z}$-module is given by the elements $e_S := \prod_{\lambda \in S} e_\lambda$ where S is a no broken circuit.

As we have explained more precisely in Section 1.1.1, the combinatorial data of the family of linearly dependent subsets of E is encoded in the arrangement's *matroid*. Thus, item (2) above can be rephrased by saying that the integral ring structure depends only on the matroid or equivalently, using a basic fact in matroid theory, that it depends only on the partially ordered set

$$\mathcal{S}(\mathcal{A}) := \{ \cap \mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A} \} \tag{2.2}$$

of all intersections of hyperplanes, ordered by reverse inclusion [OT92, 2.1].

The construction of Λ_E/J_E can be formally carried out for every abstract matroid, hence with every matroid is associated an *Orlik-Solomon algebra*, and this class of algebras enjoys a rich structure theory (see [Yuz01] for a survey). For instance, the matroid's Whitney numbers of the first kind count the dimensions of the algebra's graded pieces (hence, in the case of arrangements, the Betti numbers of the complement), and generating functions for these numbers can be obtained from classical polynomial invariants of matroids (e.g. the Tutte polynomial).

Toric arrangements

A *toric arrangement* is a finite set $\mathcal{A}$ of codimension one subtori in a complex torus $T \simeq (\mathbb{C}^*)^n$. The topological object of interest is, again, the complement $M(\mathcal{A}) := T \setminus \cup\mathcal{A}$. Each such subtorus can be defined as a coset of the kernel of some character of T. The arrangement is called *central* if every subtorus is the kernel of a certain character. If we fix one such defining character for every subtorus in $\mathcal{A}$ we can consider the matroid of linear dependencies among the resulting set of characters (e.g., viewed as a family of elements of the vector space obtained by tensoring the lattice of characters by $\mathbb{Q}$). This matroid does not depend on the choice of the characters.

Even to encode basic topological data such as the Betti numbers of the arrangement's complement, this "algebraic" matroid data must be refined, for instance by some "arithmetic" data given by the multiplicity function which keeps track of the index of sublattices spanned by subsets of the characters. This approach goes back to Lawrence [Law11]. An axiomatization of some crucial properties of this function is the foundation of the theory of *arithmetic matroids* [DM13, BM14]. By [DP05] and via Moci's arithmetic Tutte polynomial [Moc12a], the Betti numbers of the complement of a central toric arrangement can be computed from the associated arithmetic matroid.

Since intersections of subtori can be topologically disconnected, the "geometric" intersection data of a toric arrangement is customarily taken to be the poset of *layers*, i.e., connected components of intersections (see Definition 2.2.5). The significance of this poset was already pointed out by Zaslavsky [Zas77]. The paper [DR18] introduces *group actions on semimatroids* as an attempt for a unified axiomatization of posets of layers and multiplicity functions.

The line of research leading to the present work starts with [DP05] where a general result about the Betti numbers of the complement was obtained (see Theorem 2.2.16). Combinatorial models for the homotopy type of complements of toric arrangements were studied in [MS11, dD12], and minimality of such spaces was proved in [dD15]. Inspired by the seminal paper [DP95, MP98], De Concini and Gaiffi recently computed the cohomology of certain compactifications of $M(\mathcal{A})$ [DG18b, DG18a], see also [Moc12b] for related earlier work.

Associated graded of the rational cohomology of $M(\mathcal{A})$ were developed by Bibby [Bib16a] and Dupont [Dup16a], and the minimality result of [dD15] implies torsion-freeness of the integral cohomology. Dupont also proved rational formality of $M(\mathcal{A})$ in [Dup16b]. Further related work includes results about representation stability [Bib16b] and local system cohomology [DSY17].

Presentations of the graded rational algebra were discussed in [Bib16a].

The integral cohomology algebra was considered in [CD17] using purely combinatorial methods, but we point out that the formulas for the multiplication given there contain a mistake (see [CD19]). Here we take a different point of view. In particular, we obtain a presentation for the cohomology ring $H^*(M(\mathcal{A}), \mathbb{C})$ that can be seen as generalizing the one obtained for hyperplanes by Orlik-Solomon. In the *unimodular* case, i.e. when all the intersections of hypertori are connected, we recover the presentation that had been obtained in [DP05].

Results

In this chapter we provide Orlik-Solomon type presentations for the integral cohomology algebra of a general toric arrangement and we study its properties.

More precisely,

- We present the graded integral cohomology as the second page of the Leray spectral sequence for the inclusion of the complement in the ambient torus, see Theorem 2.4.3. This result generalize [Bib16a] and recovers the one of [CD17] by using different methods.

- We give a more combinatorial presentation of the cohomology algebra with rational coefficients (Corollary 2.5.4). This algebra depends only on the poset of layers, see Remark 2.5.2.

- We generalize De Concini and Procesi's presentation beyond the unimodular case, to all toric arrangements (Theorem 2.9.13). In the general case this algebra is not necessarily generated in degree one, and every minimal linear dependency among characters induces a number of relations equal to the number of connected components of the intersection of the involved characters (the case where every such dependency induces one relation is precisely the unimodular one studied by De Concini and Procesi).

  The data needed for the presentation of the rational cohomology is fully encoded in the poset of layers (Remark 2.9.15). Moreover, Theorem 3.5.2 shows that the cohomology ring structure cannot be recovered from the associated arithmetic matroid.

- We prove that the forms we choose as generators of the cohomology are integral. The relations involved in our presentation hold as relation of forms, not only of cohomology classes. Thereby we extend Dupont's result of rational formality to integral formality, and we obtain an Orlik-Solomon type presentation for the integral cohomology algebra as well (Theorem 2.10.4). Moreover, Theorem 3.4.2 shows that the integral cohomology algebra of the complement of a toric arrangement is not determined by the poset of layers.

- We give combinatorial criteria that determines whether the cohomology algebra is generated in degree one: see Theorem 2.11.5 for the case of rational coefficients and Theorem 2.11.6 for the case of integer coefficients. These criteria depend only on the poset of layers, see Remark 2.11.3.

Plan

The plan of the chapter is as follows: first, in Section 2.2 we recall a few definitions related to the topology and combinatorics of toric arrangements, and we reduce the study of all toric arrangements to the one of primitive arrangements in a connected torus. We study in Section 2.3 the Leray spectral sequence of the inclusion of the complement in the ambient torus. The spectral sequence for the constant sheaf with integer values collapses at the second page and converges to a gradation of the cohomology ring of the toric arrangement. In Section 2.4 we construct a bigraded algebra $\mathrm{A}(\mathcal{A})$ isomorphic to the graded cohomology with integer coefficients. In Section 2.5 we give a completely combinatorial presentation of an algebra $\mathrm{B}(\mathcal{A})$ isomorphic to the graded cohomology ring with rational coefficients. In Section 2.6 we introduce our choice of logarithmic forms associated with the arrangement's elements. Starting from De Concini and Procesi's work, we deduce some formal identities associated with minimal dependencies among the arrangement's defining characters. The technical tool towards treating the non-unimodular case are certain coverings of toric arrangements introduced in Section 2.7. Then, in Section 2.8 we put this tool to work and single out a special class of coverings (which we call "separating covers"). These coverings allow us to define some fundamental forms accounting for the single contributions in cohomology associated with different components of the same intersection. In Section 2.9 we prove that these forms generate the cohomology algebra and the relations generate the whole relation ideal. In Section 2.10 we extend our results to integral homology. Finally in Section 2.11, we give a purely combinatorial criteria to determine whether the cohomology ring (with rational or integer coefficients) is generated in degree one.

2.2 Basic definitions and notations

Generalities

Throughout, E will denote a finite set. For indexing purposes, we will fix an arbitrary total ordering $<$ of E (e.g., by identifying it with a subset of $\mathbb{N}$). We will also follow these conventions: we will consider every subset of E to be ordered with the induced ordering. For $A, B \subseteq E$, we will write (A, B) for the concatenation of the two totally ordered sets, i.e. if $A = \{ a_i < \cdots < a_l \}$ and $B = \{ b_i < \cdots < b_h \}$, then $(A, B) = (a_1, a_2, \ldots, a_l, b_1, \ldots, b_h)$, which is typically different from $A \cup B$.

Definition 2.2.1. Given $A, B \subseteq E$ such that $A \cap B = \emptyset$, let $\ell(A, B)$ denote the length of the permutation that takes (A, B) into $A \cup B$.

Definition 2.2.2. An element χ in an abelian group Λ is *primitive* if $\chi \notin n\Lambda$ for all integer $n > 1$.

Notice that the neutral element is never primitive.

Toric arrangements

Let $T = (\mathbb{C}^*)^d \times K$ be a complex torus (where K is a finite abelian group), and let $\Lambda = \mathrm{Hom}(T, \mathbb{C}^*)$ be the group of characters of T. Consider a list $\underline{\chi} \in \Lambda^E$

of elements of $\Lambda \simeq H^1(T, \mathbb{Z})$ and a tuple $\underline{b} \in (\mathbb{C}^*)^E$. The *toric arrangement* defined by $\underline{\chi}$ and $\underline{b}$ is

$$\mathcal{A} = \{H_i \mid i \in E\},$$

where $H_i = \chi_i^{-1}(b_i)$ is the level set of χ_i at level b_i, for all $i \in E$.

The toric arrangement is called *central* if $\underline{b} = (1, \ldots, 1)$, i.e., if H_i is the kernel of χ_i for all $i \in E$.

Definition 2.2.3. We define $M(\mathcal{A}) \subset T$ to be the *complement* of the toric arrangement $\mathcal{A}$, i.e.

$$M(\mathcal{A}) := T \setminus \bigcup_{H \in \mathcal{A}} H.$$

Definition 2.2.4. The toric arrangement $\mathcal{A}$ is called *unimodular* if $\cap_{i \in A} H_i$ is either connected or empty for all $A \subseteq E$.

Definition 2.2.5. For a given arrangement $\mathcal{A}$ in a torus T we define the *poset of layers* $\mathcal{S}(\mathcal{A})$ as the set of all connected components of nonempty intersections of elements of $\mathcal{A}$ ordered by reverse inclusion. The elements of $\mathcal{S}(\mathcal{A})$ are called *layers* of the arrangement $\mathcal{A}$.

Notice that the torus T is an element of $\mathcal{S}(\mathcal{A})$ since it is the intersection of the empty family of hypertori.

Definition 2.2.6. The toric arrangement $\mathcal{A}$ is called *essential* if the maximal elements in $\mathcal{S}(\mathcal{A})$ are points.

Reduction to connected tori

Since the study of a nice topological space can be reduced to the study of each connected component, we can assume the torus T to be connected in the following way.

Notice that each connected component is of the type $(\mathbb{C}^*)^d \times \{\, k \,\}$ for some $k \in K$ and that the decomposition $T = (\mathbb{C}^*)^d \times K$ is canonical. Therefore, each character χ can be written as $(\chi_1, \chi_2) \in \operatorname{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*) \times \operatorname{Hom}(K, \mathbb{C}^*)$. Thus the hypertorus $H = \chi^{-1}(b)$, with $b \in \mathbb{C}^*$, intersected with the connected component $(\mathbb{C}^*)^r \times \{\, k \,\}$ is the set $H(k) = \{\, t_1 \in (\mathbb{C}^*)^d \mid \chi_1(t_1) = b\chi_2(k)^{-1} \,\}$. It is an hypertorus in $(\mathbb{C}^*)^d \times \{\, k \,\}$ if $\chi_1 \neq 0$, is empty if $\chi_1 = 0$ and $\chi_2(k) \neq b$, and $H(k) = (\mathbb{C}^*)^d \times \{\, k \,\}$ otherwise. Given a toric arrangement $\mathcal{A} = (\underline{\chi}, \underline{b})$ in $(\mathbb{C}^*)^d \times K$, we consider only the connected component $(\mathbb{C}^*)^d \times \{\, k \,\}$ such that all $H_i(K)$ are different from the torus $(\mathbb{C}^*)^d \times \{\, k \,\}$ and, for such k, we define the arrangement $\mathcal{A}_k = (\underline{\chi}_1, \underline{b}\underline{\chi}_2(k)^{-1})$. The complement $M(\mathcal{A}) = \bigsqcup_k M(\mathcal{A}_k)$ is a disjoint union of complement of toric arrangements in connected tori and we have $H^\bullet(M(\mathcal{A}); \mathbb{Z}) = \prod_k H^\bullet(M(\mathcal{A}_k); \mathbb{Z})$.

Example 2.2.7. Consider the torus $\mathbb{C}^* \times (\mathbb{Z}_2)^2$, a generic element is (t, k, h) for $t \in \mathbb{C}^*$ and $k, h \in \{ 0, 1 \}$. Let $\mathcal{A}$ be the arrangement described by the following two equations

$$t^{-2}(-1)^k = \zeta_5, \qquad (-1)^k(-1)^h = -1,$$

where ζ_5 is a primitive 5th root of unity. The first describing character is primitive. The sets $H_2(0, 0)$ and $H_2(1, 1)$ are empty, $H_2(0, 1) = \mathbb{C}^* \times \{ 0 \} \times \{ 1 \}$, and $H_2(1, 0) = \mathbb{C}^* \times \{ 1 \} \times \{ 0 \}$. The arrangement $\mathcal{A}_{0,1}$ consists of one non-connected hypertorus $\{ t \mid t^{-2} = \zeta_5 \}$ and $\mathcal{A}_{1,0}$ has one non-connected hypertorus described by $\chi(t) = t^{-2}$ and $b = -\zeta_5$.

Now on we suppose all tori to be connected, hence the character group Λ will be a lattice isomorphic to $\mathbb{Z}^d$.

Remark 2.2.8. Once an isomorphism of Λ with $\mathbb{Z}^d$ is fixed, for every subset $A \subseteq E$ we can associate the integer $d \times |A|$ - matrix $N[A]$ whose columns are the characters in A, say in the fixed ordering of E.

Reduction to primitive and essential arrangement

If $\mathcal{A}$ is not essential, all maximal layers in $\mathcal{S}(\mathcal{A})$ are translates of the same torus subgroup W of T. This follows from the classical theory of hyperplane arrangements by applying [OT92, Lemma 5.30] to the lifting of $\mathcal{A}$ in the universal covering of T. By choosing any direct summand T' of W in T we can decompose the ambient torus as $T = W \times T'$. Hence, if we call $\mathcal{A}' = \{H \cap T' \mid H \in \mathcal{A}\}$ the arrangement induced by $\mathcal{A}$ in T', we have that $\mathcal{A}'$ is essential and $M(\mathcal{A}') = M(\mathcal{A})/W$. Moreover $M(\mathcal{A}) = W \times M(\mathcal{A}')$.

If an hypertorus is described by a non-primitive character $\chi = n\zeta$ for ζ primitive, then the n connected components of $\chi^{-1}(b)$ are equal to $\zeta^{-1}(c)$ for all $c \in \mathbb{C}^*$ such that $c^n = b$. Thus, each toric arrangement $\mathcal{A}$ can be described using only connected hypertori of codimension one.

Now on we assume that all toric arrangements are essential and described by primitive characters, i.e. all hypertori are connected.

Let $\mathcal{A} = (\chi_e, b_e)_{e \in E}$ be a toric arrangement, we call Γ_E (or Γ if there is no ambiguity) the subgroup of Λ generated by all characters χ_e, for $e \in E$. The hypothesis that $\mathcal{A}$ is essential implies the equality $\operatorname{rk} \Gamma = \operatorname{rk} \Lambda = r$.

Tangent space

Definition 2.2.9. Given a toric arrangement $\mathcal{A}$ in T and a point $p \in T$ we define the linear arrangement $\mathcal{A}[p]$ in the tangent space $\mathrm{T}_p(T)$ as the arrangement given by the hyperplanes $\mathrm{T}_p(H)$ for all $H \in \mathcal{A}$ such that $p \in H$ (see [OT92] for background on hyperplane arrangements).

For a given layer W of $\mathcal{A}$, a point $p \in W$ is *generic* if for any $H \in \mathcal{A}$ such that $W \not\subseteq H$ we have that $p \notin H$. We define the linear arrangement $\mathcal{A}[W]$ as the hyperplane arrangement $\mathcal{A}[p]$ for a generic point $p \in W$.

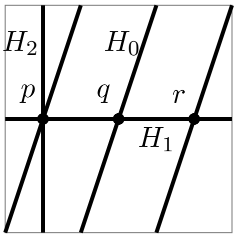


Figure 2.1: A picture of the arrangement $\mathcal{B}$.

Remark 2.2.10. Notice that the arrangement $\mathcal{A}[W]$ does not depend on the choice of the generic point p.

Example 2.2.11. Let x, y be the coordinates on the 2-dimensional torus T. We consider the arrangement $\mathcal{B}$ in $T = (\mathbb{C}^*)^2$ given by the following hypertori:

$$\begin{aligned} H_0 &:= \{ x^3 y = 1 \} ; \\ H_1 &:= \{ y = 1 \} ; \\ H_2 &:= \{ x = 1 \} . \end{aligned}$$

Notice that H_1 and H_2 as well as H_2 and H_0 intersect in a single point $p = (1, 1)$, while H_1 and H_0 intersect in three points: $p, q = (e^{\frac{2\pi i}{3}}, 1), r = (e^{\frac{4\pi i}{3}}, 1)$.

We can identify the group of characters Λ with $\mathbb{Z}^2$ generated by $\chi_1 = (0, 1)$, $\chi_2 = (1, 0)$. Hence $y = e^{\chi_1}, x = e^{\chi_2}$ and the hypertorus H_0 is associated with the character $\chi_0 = \chi_1 + 3\chi_2$.

The intersection of $\mathcal{B}$ with the compact torus is represented in Example 2.2.11. Along this chapter we will use this arrangement as a running example for the definitions and results that we introduce.

We identify the tangent space $\mathrm{T}_p(T)$ with $\mathbb{C}^2$, with coordinates $\bar{x}, \bar{y}$. The local arrangement $\mathcal{B}[p]$ is given by the hyperplanes with equations $3\bar{x} + \bar{y} = 0$, $\bar{y} = 0$, $\bar{x} = 0$, while the local arrangement $\mathcal{B}[q]$ has equations $3\bar{x} + \bar{y} = 0$, $\bar{y} = 0$.

Arithmetic matroids

There is additional enumerative data to be garnered from the set of characters $\{ \chi_e \}_{e \in E}$, when this is viewed as a subset of the lattice Λ. In particular, to every subset $A \subseteq E$ we can associate its *span* $\Gamma_A := \langle A \rangle \subseteq \Lambda$ and a lattice $\mathrm{Rad}_\Lambda \Gamma_A = (\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A) \cap \Lambda$.

The function

$$\mathrm{rk} \colon \mathcal{P}(E) \to \mathbb{N}, \quad A \mapsto \mathrm{rk}_{\mathbb{Z}} \Gamma_A$$

associates to every subset $A \subseteq E$ the rank of Γ_A as $\mathbb{Z}$-module. The function

$$m \colon \mathcal{P}(E) \to \mathbb{N}, \quad A \mapsto [\mathrm{Rad}_\Lambda \Gamma_A : \Gamma_A]$$

that associates to every subset A of E the cardinality of the torsion subgroup of the quotient Λ/Γ_A is the multiplicity function associated with the representation $\{\chi_e\}_{e\in E} \subset \Lambda$.

Remark 2.2.12.

(a) If $\mathcal{A}$ is a toric arrangement, then for all $A \subseteq E$ the integer $m(A)$ is the number of connected components of the intersection $\bigcap_{e\in A} H_e$ when this intersection is non-empty (cf. [Moc12a, Lemma 5.4]).

(b) Unimodularity of the list E is equivalent to m being constant equal to 1, and is equivalent to unimodularity of the arrangement $\mathcal{A}$.

(c) Given a matrix representation as in Remark 2.2.8, the number $m(A)$ equals the product of the elementary divisors of $N[A]$, i.e., the greatest common divisor of all its minors with size equal to the rank of $N[A]$ (cf. [Sta91, Theorem 2.2]). If $N[A]$ is a non-singular square matrix, then $m(A) = |\det N[A]|$.

It follows that the characters $\chi_e \in \Lambda$ for $e \in E$ are a representation of the arithmetic matroid $(E, \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$. We recall an important result of the previous chapter:

Corollary 2.2.13 (Corollary 1.7.5)**.** *If C is a circuit, then the following relation holds:*

$$\sum_{i\in C} c_i m(C \setminus \{i\})\chi_i = 0 \tag{2.3}$$

where $c_i \in \{1, -1\}$ are introduced in Definition 1.1.4 and depends only on the oriented arithmetic matroid.

Remark 2.2.14. If the arrangement is unimodular, from Corollary 2.2.13 we garner that every circuit can be realized by a minimal linear dependency all whose coefficients are integer units.

Theorem 2.2.15 ([DP05, Theorem 4.2])**.** *For each integer $k \geq 0$ we have a (noncanonical) decomposition, as W runs over $\mathcal{S}(\mathcal{A})$*

$$H^k(M(\mathcal{A})) = \bigoplus_{W\in\mathcal{S}(A)} H^{k-\mathrm{rk}\, W}(W) \otimes H^{\mathrm{rk}\, W}(M(\mathcal{A}[W])).$$

The following Theorem is essentially proved in [Loo93, subsection 2.4.3] and in [DP05, Remark 4.3]. The combinatorial version is stated in [Moc12a, Corollary 5.12]

Theorem 2.2.16. *If $\mathcal{A}$ is any toric arrangement in a torus T of dimension r, the Poincaré polynomial of the complement $M(\mathcal{A})$ is given in terms of the nbc-sets and the multiplicity function as*

$$\mathrm{Poin}(M(\mathcal{A}), t) = \sum_{j=0}^{r} N_j (t+1)^{r-j} t^j,$$

where, for $j = 0, \dots, r$,

$$N_j := \sum_{L \in \mathcal{S}_j} |\operatorname{nbc}_j(\mathcal{A}[L])|,$$

and $\operatorname{nbc}_j(\mathcal{A}[L])$ *is the set of no-broken-circuits of cardinality j in the arrangement $\mathcal{A}[L]$. In particular, the j-th Betti number of $M(\mathcal{A})$ is*

$$\beta_j(M(\mathcal{A})) = \sum_{i=0}^{j} N_i \binom{r-i}{j-i}.$$

Remark 2.2.17.

(a) The data given by the matroid $\mathcal{M}$ together with the function m determines an *arithmetic matroid.* We refer to [DM13] for a general abstract definition of an arithmetic matroid, and some of its properties.

(b) The poset $\mathcal{S}(\mathcal{A})$ determines the arithmetic matroid data. In fact, for any given set $A \subseteq E$ we can consider the set X of minimal upper-bounds in $\mathcal{S}(\mathcal{A})$: A is independent if and only if the poset-rank of the elements of X equals $|A|$, and the multiplicity of A equals $|X|$ (via Remark 2.2.12). On the other hand in Section 3.5 we will explicitly construct two toric arrangements with isomorphic arithmetic matroid data but non-isomorphic posets of layers.

Example 2.2.18. In the arrangement $\mathcal{B}$ introduced in Example 2.2.11 the only minimal dependent set of characters is $C = \{\chi_0, \chi_1, \chi_2\}$, hence this is the only circuit in the associated matroid. The relation $-\chi_0 + \chi_1 + 3\chi_2 = 0$ holds. The arithmetic matroid associated with $\mathcal{B}$ has set $E = \{\chi_0, \chi_1, \chi_2\} \subset \Lambda = \mathbb{Z}^2$ and the multiplicity function is given by

$$m(\{\chi_0, \chi_1\}) = 3,$$

while $m(A) = 1$ for all other subsets of E. In particular notice that $\mathcal{B}$ is a central, not unimodular arrangement. The Poincaré polynomial of $M(\mathcal{A})$ is

$$\mathrm{Poin}(M(\mathcal{B}), t) = 1 + 5t + 8t^2. \tag{2.4}$$

2.3 The Leray spectral sequence

In this section we state some general results on the Leray spectral sequence, see [Bre97] for a reference. The case of cohomology with rational coefficients has been studied by Bibby in [Bib16a]. We make use a result appeared for the first time in [Loo93] to compute the cohomology with integer coefficients of a toric arrangement. Using the Leray spectral sequence we obtain a nice presentation of a canonical bigradation of cohomology algebra of toric arrangements.

Let $j\colon M \hookrightarrow T$ be the natural inclusion, which is a continuous map between topological spaces. Let $\mathbb{Z}_M$ be the sheaf on M of locally constant functions with values in $\mathbb{Z}$.

We recall the definition of *higher direct image sheaves* for the map j and the sheaf $\mathbb{Z}_M$. Let us consider the presheaf defined by $U \mapsto \check{H}^q(j^{-1}(U);\mathbb{Z}_M)$. The associated sheaf is the q-direct image sheaf $R^q j_* \mathbb{Z}_M$.

Since $\mathbb{Z}$ is a ring, the cup product $\check{H}^q(j^{-1}(U);\mathbb{Z}_M) \otimes \check{H}^{q'}(j^{-1}(U);\mathbb{Z}_M) \to \check{H}^{q+q'}(j^{-1}(U);\mathbb{Z}_M)$ is defined in Čech cohomology, for details see [Bre97, Section II.7]. The cup product induces the map of sheaves $f_{q,q'}\colon R^q j_*\mathbb{Z}_M \otimes R^{q'} j_*\mathbb{Z}_M \to R^{q+q'} j_*\mathbb{Z}_M$. In the same way we can define $R^q j_* \mathbb{Q}_M$. We define the maps

$$\smile\colon \check{H}^p(T; R^q j_*\mathbb{Z}_M) \otimes \check{H}^{p'}(T; R^{q'} j_*\mathbb{Z}_M) \to \check{H}^{p+p'}(T; R^{q+q'} j_*\mathbb{Z}_M) \qquad (2.5)$$

as $(-1)^{p'q}$ times the composition of the cup product in the Čech cohomology and $f_{q,q'}$.

The inclusion j defines a natural map in cohomology $H^\bullet(T) \to H^\bullet(M)$ which is injective, so we identify $H^\bullet(T)$ with its image. We define a increasing filtration $\mathrm{F}_\bullet = \{\mathrm{F}_i\}_{i\in\mathbb{Z}}$ for the cohomology ring $H^\bullet(M)$ by

$$\mathrm{F}_i = \mathrm{Im}(H^{\leq i}(M;\mathbb{Z}) \otimes H^\bullet(T;\mathbb{Z}) \xrightarrow{\cup} H^\bullet(M;\mathbb{Z}))$$

for $i \geq 0$ and by $F_{-1} = 0$. The graded ring $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(M;\mathbb{Z})$ associated with the filtration $\mathrm{F}_\bullet$ is the ring $\bigoplus_{i\geq 0} \mathrm{F}_i \,/\, \mathrm{F}_{i-1}$.

Lemma 2.3.1 ([Bre97])**.** *There exists a spectral sequence of $\mathbb{Z}$-algebras which converges, as a bigraded algebra, to $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(M;\mathbb{Z})$. The second page of the spectral sequence is*

$$E_2^{p,q}(M) = \check{H}^p(T; R^q j_*\mathbb{Z}_M)$$

and the product coincides with the map defined in (2.5).

Proof. The existence and the convergence of the spectral sequence are proven in [Bre97, IV, Theorem 6.1]. The cup product in Leray spectral sequence is described in [Bre97, IV, section 6.8].

The limit of the spectral sequence is a graded ring associated with a filtration of $H^\bullet(M;\mathbb{Z})$ that can be determine as follows. The Leray spectral sequence can be identified to the first (or horizontal) spectral sequence of an appropriate double complex. The filtration in the double complex is described in [Bre97, A, section 2] and coincides with $\mathrm{F}_\bullet$. □

The *Brieskorn inclusion* is a natural map on the cohomology of hyperplane arrangements defined as follows. Fix a layer L of rank k in a hyperplane arrangement $\mathcal{A}$ with poset of intersection $\mathcal{S}$ and let $\mathcal{A}_L$ be the arrangement given by hyperplanes containing L. The Brieskorn inclusion is the composition

$$b_L\colon H^k(M(\mathcal{A}_L);\mathbb{Z}) \hookrightarrow \bigoplus_{W\in\mathcal{S}_k} H^k(M(\mathcal{A}_W);\mathbb{Z}) \xrightarrow{\sim} H^k(M(\mathcal{A});\mathbb{Z}),$$

where the second map is the *Brieskorn isomorphism* (see [OT92, Theorem 3.26, p. 65] or [Bri73, Lemma 3, p.27]).

From now on, let $j\colon M \to T$ be the open inclusion of complement of a toric arrangement in the corresponding torus, so the equality $j_*\mathbb{Z}_M = \mathbb{Z}_T$ holds. The higher direct image sheaves $R^q j_*\mathbb{Z}_M$ and $R^q j_*\mathbb{Q}_M$ has been partially described in [Loo93] and in [Bib16a], respectively. The analogous of the following lemma for the sheaf $R^q j_*\mathbb{Q}_M$ has been proven in [Bib16a, Lemma 3.1]. We adapt the proof of [Bib16a] in order to study the cup product structure in the case of integer coefficients.

Lemma 2.3.2. *Let i_W be the inclusion $W \hookrightarrow T$ for $W \in \mathcal{S}$. For all natural numbers q there exists an isomorphism of sheaves:*

$$\varphi_q\colon \bigoplus_{\operatorname{rk} W=q} (i_W)_*\mathbb{Z}_W \otimes_{\mathbb{Z}} H^q(M(\mathcal{A}[W]);\mathbb{Z}) \xrightarrow{\sim} R^q j_*\mathbb{Z}_M$$

Proof. Recall that the sheaf $R^q j_*\mathbb{Z}_M$ is the sheafification of the presheaf P, defined by:

$$P(U) \stackrel{\text{def}}{=} \check{H}^q(j^{-1}(U);\mathbb{Z}_M) = H^q(U\cap M;\mathbb{Z}).$$

We define the sheaves $\epsilon_W = (i_W)_*\mathbb{Z}_W \otimes_{\mathbb{Z}} H^{\operatorname{rk} W}(M(\mathcal{A}[W]);\mathbb{Z})$ for all $W \in \mathcal{S}$. Let $U \subset T$ be an open set, we have

$$\epsilon_W(U) = H^0(U\cap W;\mathbb{Z}) \otimes_{\mathbb{Z}} H^{\operatorname{rk} W}(M(\mathcal{A}[W]);\mathbb{Z}).$$

For all $x \in T$ we choose a neighbourhood basis of open sets U such that $U\cap M$ is isomorphic to a small neighbourhood of the origin in $M(\mathcal{A}[x])$. By definition $\mathcal{A}[x] = \mathcal{A}[W_x]$ where W_x is the minimal layer containing x. We call this basis of the topology $\mathcal{U}$. Define the morphism of sheaves $\varphi_W\colon \epsilon_W \to R^q j_*\mathbb{Z}_M$ such that for each open $U \in \mathcal{U}$ is

$$\varphi_W(U)\colon H^0(U\cap W;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(M(\mathcal{A}[W_x]);\mathbb{Z}) \to H^q(U\cap M;\mathbb{Z})$$

the pullback of the inclusion $U\cap M \hookrightarrow M(\mathcal{A}[W_x])$ composed with the cup product. Let φ_q be the direct sum map from $\epsilon_q := \oplus_{\operatorname{rk} W=q}\epsilon_W$ into $R^q j_*\mathbb{Z}_M$.

We show that φ_q is the desired isomorphism by checking on the stalks. The stalk of $R^q j_* \mathbb{Z}_W$ at x is

$$(R^q j_* \mathbb{Z}_M)_x = H^q(M(\mathcal{A}[W_x]); \mathbb{Z}),$$

and for $x \in T$ the map

$$(\varphi_q)_x \colon (\epsilon_q)_x = \bigoplus_{\substack{\operatorname{rk} W = q \\ W \ni x}} H^q(M(\mathcal{A}[W]); \mathbb{Z}) \to H^q(M(\mathcal{A}[W_x]); \mathbb{Z})$$

is the Brieskorn isomorphism, therefore φ_q is an isomorphism. □

In order to study the product map $f_{q,q'}$, we introduce the map

$$b_{W,W',L} \colon \epsilon_W \otimes \epsilon_{W'} \to \epsilon_L$$

which is defined as follows: if L is a connected component of $W \cap W'$ of rank $\operatorname{rk} L = \operatorname{rk} W + \operatorname{rk} W'$ we set:

$$b_{W,W',L}((\alpha \otimes a) \otimes (\gamma \otimes c)) = (\alpha_{|L} \gamma_{|L}) \otimes (b_{W,L}(a) \cup b_{W',L}(c))$$

where $b_{W,L}$ is the Brieskorn inclusion:

$$H^{\operatorname{rk} W}(M(\mathcal{A}[W]); \mathbb{Z}) \simeq H^{\operatorname{rk} W}(M(\mathcal{A}[L]_W); \mathbb{Z}) \hookrightarrow H^{\operatorname{rk} W}(M(\mathcal{A}[L]); \mathbb{Z}).$$

Otherwise we define $b_{W,W',L}$ to be zero. Now we can consider the direct sum map

$$b_{q,q'} = \bigoplus_{\substack{\operatorname{rk} W = q \\ \operatorname{rk} W' = q' \\ \operatorname{rk} L = q+q'}} b_{W,W',L} \colon \epsilon_q \otimes \epsilon_{q'} \to \epsilon_{q+q'}.$$

Lemma 2.3.3. *The isomorphism φ of Lemma 2.3.2 is compatible with $f_{q,q'}$ and $b_{q,q'}$, i.e. the diagram below commutes.*

$$\begin{array}{ccc}
\epsilon_q \otimes \epsilon_{q'} & \xrightarrow{\quad b_{q,q'} \quad} & \epsilon_{q+q'} \\
\downarrow{\scriptstyle \varphi_q \otimes \varphi'_q} & & \downarrow{\scriptstyle \varphi_{q+q'}} \\
R^q j_* \mathbb{Z}_M \otimes_{\mathbb{Z}} R^{q'} j_* \mathbb{Z}_M & \xrightarrow{\quad f_{q,q'} \quad} & R^{q+q'} j_* \mathbb{Z}_M
\end{array}$$

Proof. It is sufficient to show that $f_{q,q'} \circ \varphi_q \otimes \varphi_{q'}$ and $\varphi_{q+q'} \circ b_{q,q'}$ agree on all stalks. Let x be a point in T and $\alpha \otimes a$ and $\gamma \otimes c$ be elements of $(\epsilon_W)_x$ and of $(\epsilon_{W'})_x$, respectively. Let L be the connected component of $W \cap W'$ containing x and W_x be the minimal layer containing x. From the fact that

$$b_{L,W_x} \circ (b_{W,L}(a) \cup b_{W',L}(c)) = b_{W,W_x}(a) \cup b_{W',W_x}(c)$$

we have that both stalks are $\alpha\gamma b_{W,W_x}(a) \cup b_{W',W_x}(c)$. □

The next Theorem appeared first in [CD17]. An analogue on the rationals was proven in [Bib16a, Lemma 3.2] in a more general setting using some Hodge theory.

Theorem 2.3.4 ([CD17, Theorem 5.1.3])**.** *The Leray spectral sequence associated with the inclusion $M \hookrightarrow T$ degenerates at the second page. Hence the two algebras $E_2^{\bullet,\bullet}(M)$ and $\operatorname{gr}_{\mathrm{F}_\bullet} H^\bullet(M;\mathbb{Z})$ are isomorphic.*

Up to changing the coefficients, the filtration $\mathrm{F}_\bullet$ coincides with the one defined in [DP05, Remark 4.3]. From now on, we denote by $\operatorname{gr} H^\bullet(M;\mathbb{Z})$ the bigraded, graded commutative, $\mathbb{Z}$-algebra associated with $H^\bullet(M;\mathbb{Z})$ with respect to the filtration $\{ \mathrm{F}_n^\bullet \}_{n\in\mathbb{Z}}$.

2.4 Graded cohomology with integer coefficients

Let $\mathcal{A} = (\chi_e, b_e)_{e\in E}$ be a toric arrangement in T. We define the following algebra.

Definition 2.4.1. Let $\mathrm{A}^{p,q}(\mathcal{A})$ be the vector space

$$\mathrm{A}^{p,q}(\mathcal{A}) \stackrel{\text{def}}{=} \bigoplus_{W\in\mathcal{S}_q(\mathcal{A})} H^p(W;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(M(\mathcal{A}[W];\mathbb{Z}).$$

The graded vector space $\mathrm{A}^{\bullet,\bullet}(\mathcal{A}) = \bigoplus_{p,q} \mathrm{A}^{p,q}(\mathcal{A})$ is endowed of the following product. Let $\alpha\otimes a$ and $\gamma\otimes c$ two element in $H^p(W;\mathbb{Z})\otimes_{\mathbb{Z}} H^q(M(\mathcal{A}[W];\mathbb{Z})$ and in $H^{p'}(W';\mathbb{Z})\otimes_{\mathbb{Z}} H^{q'}(M(\mathcal{A}[W'];\mathbb{Z})$, respectively. If $\operatorname{rk} W + \operatorname{rk} W' \neq \operatorname{rk} W\cap W'$, then $(\alpha\otimes a)\cdot(\gamma\otimes c) = 0$. Otherwise, set

$$(\alpha\otimes a)\cdot(\gamma\otimes c) = (-1)^{p'q} \sum_{L\in\pi_0(W\cap W')} (i^*_{W,L}\alpha \cup i^*_{W',L}\gamma)\otimes(b_{W,L}(a)\cup b_{W',L}(b)),$$

where $i_{W,L}$ is the inclusion $L \hookrightarrow W$ and $b_{W,L}$ is the Brieskorn inclusion $H^q(M(\mathcal{A}[L]_W) \hookrightarrow H^q(M[L])$.

Theorem 2.4.2. *The second page of the Leray spectral sequence defined in Lemma 2.3.1 is isomorphic as a bigraded algebra to* $\mathrm{A}(\mathcal{A})$*.*

Proof. The isomorphism $\varphi_q\colon \epsilon_q \to R^q j_*\mathbb{Z}_M$ of Lemma 2.3.2 induces an isomorphism in cohomology:

$$\tilde{\varphi}_q : \bigoplus_{\operatorname{rk} W=q} H^\bullet(W;\mathbb{Z})\otimes_{\mathbb{Z}} H^q(M(\mathcal{A}[W]);\mathbb{Z}) \to E_2^{\bullet,q}(M)$$

Hence we have an isomorphism $\tilde{\varphi}\colon \mathrm{A}^{\bullet,\bullet}(\mathcal{A}) \to E_2^{\bullet,\bullet}(M)$; Lemma 2.3.3 ensures then that $\tilde{\varphi}$ is an isomorphism of algebras. □

As a consequence of the previous statements we obtain the following result.

Theorem 2.4.3. *For any toric arrangement $\mathcal{A}$, there exists an isomorphism of bigraded $\mathbb{Z}$-algebras:*

$$f\colon \mathrm{A}^{\bullet,\bullet}(\mathcal{A}) \to \mathrm{gr}_{\mathrm{F}}\, H^\bullet(M(\mathcal{A});\mathbb{Z})$$

Proof. The result follows since the map f is the composition the isomorphism given in Theorem 2.4.2 between $\mathrm{A}^{\bullet,\bullet}(\mathcal{A})$ and $E_2^{\bullet,\bullet}(M)$ and the isomorphism of Theorem 2.3.4. □

2.5 Graded cohomology with rational coefficients

In this section we give a purely combinatorial presentation of the bigraded algebra $\mathrm{gr}_{\mathrm{F}}\, H^\bullet(M(\mathcal{A});\mathbb{Q})$. We begin by defining the ring $\mathrm{B}(\mathcal{A})$, then we exhibit a basis of this $\mathbb{Q}$-vector space and finally we show an isomorphism between the objects $\mathrm{B}(\mathcal{A})$ and $\mathrm{gr}_{\mathrm{F}}\, H^\bullet(M(\mathcal{A});\mathbb{Q})$.

Definition 2.5.1. Let $\wedge[f_{W,A;B}]$ be the exterior algebra on generators $f_{W,A;B}$ where $A \sqcup B$ is an independent set and W a connected component of $\bigcap_{e\in A} H_e$; the bi-degree of the generator $f_{W,A}$ is $(|B|,|A|)$. The algebra $\mathrm{B}(\mathcal{A})$ is the quotient of quotient of $\wedge[f_{W,A;B}]$ by the following types of relations:

- For any two generators $f_{W,A;B}$, $f_{W',A';B'}$,

  $$f_{W,A;B} f_{W',A';B'} = 0$$

  if A, A', B, B' are not pairwise disjoint or if $A \sqcup B \sqcup A' \sqcup B'$ is a dependent set, and otherwise

  $$f_{W,A;B} f_{W',A';B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} f_{L,A\cup A';B\cup B'}. \tag{2.6}$$

- For every circuit $C \subseteq E$ a relation

  $$\sum_{i\in E} c_i m(C\setminus\{\,i\,\}) f_{T,\emptyset;\{\,i\,\}} = 0, \tag{2.7}$$

  where c_i are defined in Definition 1.1.4 (see also Corollary 2.2.13).

- For every circuit $C \subseteq E$ a relation

  $$\sum_{i\in C} (-1)^i f_{W,C\setminus\{\,i\,\};\emptyset} = 0. \tag{2.8}$$

Remark 2.5.2. The presentation of the algebra $\mathrm{B}(\mathcal{A})$ depends on the choice of an orientation χ of the arithmetic matroid $(E, \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$. However, the algebra $\mathrm{B}(\mathcal{A})$ depends only on the poset of layers $\mathcal{S}(\mathcal{A})$.

Let ω be a generator of $H^1(\mathbb{C}^*;\mathbb{Z})$, $\psi_i := \chi_i^*(\omega)$, and ψ_B be the product $\prod_{i\in B}\psi_i$.

Theorem 2.5.3. *The assignment*

$$f_{W,A;B} \mapsto (-1)^{\ell(B,A)}\psi_B \otimes e_A \in H^{|B|}(W) \otimes H^{|A|}(M(\mathcal{A}[W]))$$

induces an isomorphism $g\colon B(\mathcal{A}) \to A(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

As consequence we obtain:

Corollary 2.5.4. *The map*

$$f \circ g\colon \mathrm{B}(\mathcal{A}) \to \mathrm{gr_F}\, H^\bullet(M(\mathcal{A});\mathbb{Q})$$

is an isomorphism. □

We recall the definitions from [DP05].

Definition 2.5.5. A set $A \subseteq E$ is a *associated with* $W \in \mathcal{S}(\mathcal{A})$ if W is a connected component of $\bigcap_{i\in C} H_i$.

Let $W \in \mathcal{S}(\mathcal{A})$ be a layer and consider the subgroup $\Gamma_W < \Lambda$ given by all characters that vanish on W. Since $\mathcal{A}$ is essential, we choose $\dim W$ characters in our arrangement such that they form a basis of $\Lambda/\Gamma_W \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathcal{C}_W \subset E$ be the indexing set of these chosen characters. Notice that $|\mathcal{C}_W| = \dim W$.

Lemma 2.5.6. *Let $\mathcal{A}$ be an essential toric arrangement. Then a set of generators of* $\mathrm{B}(\mathcal{A})$ *as $\mathbb{Q}$-vector space is given by the elements $f_{W,S;T}$ with S no broken circuit associated with W and T a subset of $\mathcal{C}_W$.*

Proof. It is sufficient to show that any generic element $f_{W,A;B}$ can be written as linear combination of the ones $f_{W,S;R}$ with S no broken circuit associated with W and R a subset of $\mathcal{C}_W$. By relation eq.(2.8) we can write each $f_{W,A;B}$ as sum of certain $f_{W,S;B}$ with S a no broken circuit associated with W. Since $f_{W,S;B} = (-1)^{\ell(S,B)} f_{W,S;\emptyset} f_{T,\emptyset;B}$, by relations eq: (2.7) $f_{T,\emptyset;B}$ can be written as linear combination of some $f_{T,\emptyset;R}$ with $R \subseteq \mathcal{C}_W$ or $R \cap S \neq \emptyset$. In the first case $(-1)^{\ell(S,B)} f_{W,S;\emptyset} f_{T,\emptyset;R} = f_{W,S;R}$ that is an element in our set of generators. In the case $R \cap S \neq \emptyset$, the relations eq. (2.6) show that $f_{W,S;\emptyset} f_{T,\emptyset;R} = 0$ in $B(\mathcal{A})$. □

Lemma 2.5.7. *The map* $g\colon \mathrm{B}(\mathcal{A}) \to \mathrm{A}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *of Theorem 2.5.3 is well defined.*

Proof. We need to show that relations 2.6-2.8 hold in $\mathrm{A}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

- for each pair of generators $f_{W,A;B}, f_{W',A';B'}$ we have

$$g(f_{W,A;B}f_{W',A';B'}) = (-1)^{\ell(B,A)+\ell(B',A')}(\psi_B \otimes e_A)(\psi_{B'} \otimes e_{A'}).$$

If $A \sqcup A'$ is dependent than both side are zero, otherwise,

$$g(f_{W,A;B}f_{W',A';B'}) = (-1)^l \sum_{L \in \pi_0(W \cap W')} \psi_B\psi_{B'} \otimes e_A e_{A'},$$

where $l = \ell(B,A) + \ell(B',A') + |A||B'|$. Now $\psi_B\psi_{B'} \neq 0$ in $H(L;\mathbb{Z})$ if and only if $A \sqcup B \sqcup A' \sqcup B'$ is a dependent set, otherwise

$$g(f_{W,A;B}f_{W',A';B'}) = (-1)^{l+k} \sum_{L \in \pi_0(W \cap W')} g(f_{L,A\cup A';B\cup B'}),$$

where $k = \ell(B \cup B', A \cup A') + \ell(B,B') + \ell(A,A')$. Since both $\ell(B,A) + \ell(B',A') + \ell(B \cup A, B' \cup A')$ and $|A||B'| + \ell(B,B') + \ell(A,A') + \ell(B \cup B', A \cup A')$ are the sign of the permutation that reorders (B,A,B',A'), we conclude that $k + l \equiv l(A \cup B, A' \cup B') \mod 2$.

- The equation $\sum_{i\in E} c_i m(C \setminus \{ i \})\psi_i = 0$ holds by Corollary 2.2.13 and by the isomorphism $H^1(T;\mathbb{Z}) \simeq \Lambda$.

- The last relation holds because for each circuit C associated with W the equation $\sum_{i \in C}(-1)^i e_{C\setminus\{ i \}}$ holds in $H(M(\mathcal{A}[W];\mathbb{Z})$ (see eq. (2.1)). □

Proof of Theorem 2.5.3. The map g is surjective because $H^\bullet(W;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated in degree one by the elements ψ_i with $i \in \mathcal{C}_W$. Notice that this fact is false without tensoring by $\mathbb{Q}$ in the case $W = T$ and $\mathcal{A}$ a non-surjective arrangement.

Using Theorem 2.4.3 and lemma 2.5.6, the dimension of $\mathrm{B}(\mathcal{A})$ is at most

$$\sum_{W \in \mathcal{S}} 2^{\dim W} |\mathrm{nbc}_{\mathrm{rk}\, W}(\mathcal{A}[W])|.$$

This number is equal to $\dim_{\mathbb{Q}} \mathrm{A}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$, since by Theorem 2.2.16 we have

$$\dim_{\mathbb{Q}} \mathrm{A}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Poin}(M(\mathcal{A}),1) = \sum_{W \in \mathcal{S}} 2^{\dim W} |\mathrm{nbc}_{\mathrm{rk}\, W}(\mathcal{A}[W])|.$$

This completes the proof. □

Example 2.5.8. Consider the toric arrangement $\mathcal{B}$ of Example 2.2.11. The layers of rank two are 3 points:

$$p = (1,1) = H_0 \cap H_1 \cap H_2$$
$$q = (1,\zeta_3) \subset H_0 \cap H_1$$
$$r = (1,\zeta_3^2) \subset H_0 \cap H_1$$

An additive basis for the cohomology algebra of the complement is formed by

$$e_1,\, e_2,\, y_0,\, y_1,\, y_2$$

in degree one and by

$$e_{12},\, e_1y_0,\, e_1y_1,\, e_2y_2,\, y_{p,01},\, y_{p,02},\, y_{q,01},\, y_{r,01}$$

in degree two. Notice that this agree with eq. (2.4).

2.6 Some formal identities

In this section we derive some identities among the forms associated with a circuit $C \subseteq E$. For ease of notation we identify E as a subset of $\mathbb{N}$ with the natural order, and we suppose that $C = \{ 0, 1, \ldots, k \}$. Then the characters $\chi_0, \ldots, \chi_k$ exhibit a linear dependency, and we examine different cases according to the signs of the coefficients of this linear dependency.

The results of this section will be enough in order to treat the unimodular case, where (see Remark 2.2.14) such coefficients must be units.

Logarithmic forms

We will study presentations of the cohomology algebra that use, as generators, a distinguished set of logarithmic forms.

We call $\sqrt{-1}$ by $\mathbf{i}$.

Definition 2.6.1. For all $i \in E$ we set

$$\omega_i := \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - e^{\chi_i}), \qquad \text{and} \qquad \psi_i := \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(e^{\chi_i}). \tag{2.9}$$

For symmetry reasons, we also define the forms

$$\overline{\omega}_i := \frac{1}{2\pi\mathbf{i}} \, d\log(1 - e^{\chi_i}) + \frac{1}{2\pi\mathbf{i}} \, d\log(1 - e^{-\chi_i}) = 2\omega_i - \psi_i. \tag{2.10}$$

Given any $A = \{a_1 < \ldots < a_l\} \subseteq E$ we write

$$\psi_A := \psi_{a_1} \wedge \ldots \wedge \psi_{a_l}.$$

and

$$\omega_A := \omega_{a_1} \wedge \ldots \wedge \omega_{a_l}, \qquad \text{resp.} \qquad \overline{\omega}_A := \overline{\omega}_{a_1} \wedge \ldots \wedge \overline{\omega}_{a_l}.$$

Now, if $C = \{ \chi_0, \ldots, \chi_k \} \subseteq E$ is a circuit of a unimodular arrangement, and we assume that

$$\chi_0 = \sum_{i=1}^{k} \chi_1,$$

De Concini-Procesi in [DP05, p. 410, eq. (20)] (see also Remark 2.6.2 below) prove the formal relation

$$\partial\omega_C = \sum_{\substack{\min C \in A \subseteq C, \\ B(A)\neq\emptyset}} (-1)^{\epsilon(A)} \omega_A \psi_{B(A)} \tag{2.11}$$

where the fixed total ordering on E is understood,

$$\begin{aligned} i(A) &:= \max(C \setminus A), \\ B(A) &:= (C \setminus A) \setminus i(A), \\ \epsilon(A) &:= |A| + \ell(A, C \setminus A), \end{aligned}$$

and $\ell(A, C \setminus A)$ is the length of the permutation reordering $A, C \setminus A$ (see Definition 2.2.1).

Remark 2.6.2. Notice that in [DP11, eq. (15.3)] (and also in [DP05, eq. (20)]) there is a misprint concerning the sign: writing $[k]$ for $\{1, \dots, k\}$, the correct equation is

$$\omega_{[k]} = \sum_{I \subsetneq [k]} (-1)^{|I|+k+1+\ell(I,[k]\setminus I)} \omega_I \psi_{B(I\cup\{0\})} \omega_0. \tag{2.12}$$

To go from [DP11, eq. (15.3)] to our (2.11) it is enough to use the boundary relation

$$\partial\omega_C = \omega_{[k]} + \sum_{\substack{0 \in A \subseteq C, \\ |A|=k}} (-1)^{\epsilon(A)} \omega_A.$$

Example 2.6.3. Consider the unimodular arrangement $\mathcal{B}'$ in $T = (\mathbb{C}^*)^2$ given by the hypertori H_1, H_2, H_0, where $H_0 = \{xy = 1\}$. The relation $\chi_0 = \chi_1 + \chi_2$ holds and the forms associated with $\mathcal{B}'$ are

$$\begin{aligned} \omega_0 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - xy), & \omega_1 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - y), & \omega_2 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - x), \\ \psi_0 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(xy), & \psi_1 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(y), & \psi_2 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(x). \end{aligned}$$

The set $C = \{\chi_0, \chi_1, \chi_2\}$ is the only circuit and relation (2.11) gives

$$\omega_0\omega_1 - \omega_0\omega_2 + \omega_1\omega_2 = \omega_0\psi_1$$

as can be checked directly. In the following we will use the arrangement $\mathcal{B}'$ as a running example of a unimodular arrangement.

Lemma 2.6.4. *If $\chi_0 = \sum_{i=1}^k \chi_i$, we have the following identity.*

$$\omega_1 \cdots \omega_k = \omega_0 \prod_{i=2}^k (\omega_i - \omega_{i-1} + \psi_{i-1}) \tag{2.13}$$

Proof. We fix $j \in \{1, \dots, k\}$. Consider the non-zero products in the expansion of (2.13) that do not contain either the factor ω_j nor the factor ψ_j. In each one of these terms, all the factors ω_i for $i > j$ have to appear. Instead, due to the fact that $\omega_i \wedge \psi_i = 0$, exactly one of the two terms ω_i and ψ_i has to appear for $i < j$. So the sum of the products not containing ω_j or ψ_j will be

$$\omega_0 \prod_{1 \le i < j} (-\omega_i + \psi_i) \prod_{i > j} \omega_i.$$

Hence we have,

$$\begin{aligned} \omega_0 \prod_{i=2}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1}) &= \sum_{j=1}^{k} \omega_0 \prod_{1 \le i < j} (-\omega_i + \psi_i) \prod_{i > j} \omega_i = \\ &= \sum_{j=1}^{k} \sum_{\substack{0 \in A \subsetneq C, \\ i(A) = j}} (-1)^{|A_{\le j}| - 1} \eta_A \end{aligned}$$

where

$$\eta_A = \eta_0 \cdots \widehat{\eta_{i(A)}} \cdots \eta_k$$

and

$$\eta_i := \begin{cases} \omega_i & \text{if } i \in A \\ \psi_i & \text{otherwise.} \end{cases}$$

We conclude the purely formal identity

$$\omega_0 \prod_{i=2}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1}) = \sum_{0 \in A \subsetneq C} (-1)^{|A_{\le i(A)}| - 1} \eta_A. \tag{2.14}$$

Now we use our assumption $\sum_{i=1}^{k} \chi_i = \chi_0$. It entails that the form $\vartheta^{(0)}$ defined before Proposition 15.6 in [DP11] equals ω_0. In particular, again in the notation of [DP11], for $I \subset [k]$ we have

$$\Phi_I^{(0)} \overset{\text{def}}{=} (-1)^{\ell(I, [k] \setminus I)} \prod_{i \in I} \omega_i \prod_{j \in B(I \cup \{0\})} \psi_j \vartheta^{(0)} = (-1)^{i(I) - 1} \eta_{I \cup \{0\}}$$

noticing that the products $\eta_{I \cup \{0\}}$ already follow the standard ordering. We can now use [DP11, eq. (15.3)], i.e.,

$$\sum_{I \subsetneq [k]} (-1)^{|I| + k + 1} \Phi_I^{(0)} = \omega_1 \cdots \omega_k$$

in order to rewrite Equation (2.14). If we take $A = I \cup \{0\}$, since $k - i(A) = |A| - |A_{\le i(A)}|$, we obtain the claimed equality. □

Lemma 2.6.5. *If* $\sum_{i=0}^{k} \chi_i = 0$*, then we have*

$$\prod_{i=1}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1}) = 0 \tag{2.15}$$

or, using the forms $\overline{\omega}_i$ *defined in eq.* (2.10)*,*

$$\prod_{i=1}^{k} (\overline{\omega}_i + \psi_i - \overline{\omega}_{i-1} + \psi_{i-1}) = 0. \tag{2.16}$$

Proof. We start by a formal identity which can be readily verified, e.g., by induction on k.

$$\omega_1 \cdots \omega_k = \omega_1 \prod_{i=2}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1})$$

We can now expand the left-hand side using Lemma 2.6.4 applied to the identity $(-\chi_0) = \sum_{i>0} \chi_i$. Collecting terms we obtain

$$0 = \left(\omega_1 - \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(1 - e^{-\chi_0}) \right) \prod_{i=2}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1}).$$

Noticing that $\frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(1 - e^{-\chi_0}) = \omega_0 - \psi_0$ we conclude:

$$(\omega_1 - \omega_0 + \psi_0) \prod_{i=2}^{k} (\omega_i - \omega_{i-1} + \psi_{i-1}) = 0.$$

For the second equation we can immediately compute

$$2(\omega_i - \omega_{i-1} + \psi_{i-1}) = \overline{\omega}_i + \psi_i - \overline{\omega}_{i-1} + \psi_{i-1},$$

so multiplying formula (2.15) by 2^k we get the claimed identity. □

Example 2.6.6. We continue with the arrangement introduced in Example 2.6.3. Since the relation

$$\chi_0 = \chi_1 + \chi_2$$

holds, from Lemma 2.6.4 we have $\omega_1\omega_2 = \omega_0(\omega_2 - \omega_1 + \psi_1)$. In order to apply Lemma 2.6.5 we set $\chi_0' := -\chi_0$ (and hence $\omega_0' = \omega_0 - \psi_0$ and $\psi_0' = -\psi_0$) and if we consider the characters χ_0', χ_1, χ_2 and the ordering $(0, 1, 2)$ for the elements of the circuit we obtain

$$(\omega_1 - \omega_0' + \psi_0')(\omega_2 - \omega_1 + \psi_1) = 0,$$

while if we consider the ordering $(0, 2, 1)$ we obtain the relation

$$(\omega_2 - \omega_0' + \psi_0')(\omega_1 - \omega_2 + \psi_2) = 0$$

as one can easily check by direct computation. The relations that we can obtain with different orderings of the elements in the circuit are consequences of the two above.

Lemma 2.6.7. *If* $\sum_{i=0}^{k} c_i\chi_i = 0$ *where* $c_i = \pm 1$ *for all* i,

$$\prod_{i=1}^{k}(\overline{\omega}_i + c_i\psi_i - \overline{\omega}_{i-1} + c_{i-1}\psi_{i-1}) = 0, \tag{2.17}$$

Proof. We apply Lemma 2.6.5 to the identity $\sum_{i=0}^{k} \chi_i' = 0$ where we set $\chi_i' := c_i\chi_i$ for all i. A glance at Equations (2.9) and (2.10) shows that the forms $\overline{\omega}_i'$ and ψ_i' associated with the χ_i' satisfy $\overline{\omega}_i' = \overline{\omega}_i$ and $\psi_i' = c_i\psi_i$ for all i, proving the claimed equality. □

Definition 2.6.8. Given a subset $A \subseteq E$, for every $i \in E$ let

$$\overline{\eta}_i^A := \begin{cases} \overline{\omega}_i & \text{if } i \in A \\ \psi_i & \text{otherwise} \end{cases}.$$

Thus, if $B \subseteq E$ is disjoint from A we can define

$$\overline{\eta}_{A,B} := \prod_{i \in A\cup B} \overline{\eta}_i^A,$$

where the factors are in increasing order with respect to the total order on E.

Proposition 2.6.9. *Let* C *be a circuit of the matroid such that the corresponding minimal linear dependency has the form* $\sum_{i\in C} c_i\chi_i = 0$ *where* $c_i \in \{\pm 1\}$ *for all* i. *Then,*

$$\sum_{j\in C} \sum_{\substack{A,B\subset C \\ C=A\sqcup B\sqcup\{j\}}} (-1)^{|A_{\leq j}|} c_B \overline{\eta}_{A,B} = 0 \tag{2.18}$$

where, for every $B \subseteq E$, *we write* $c_B := \prod_{i\in B} c_i$. *Moreover, as a consequence of the equation above we have*

$$\sum_{j\in C} \sum_{\substack{A,B\subset C \\ C=A\sqcup B\sqcup\{j\} \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \overline{\eta}_{A,B} = 0. \tag{2.19}$$

In particular $\partial\overline{\omega}_C$ *corresponds to the sum of the terms with* $B = \emptyset$.

Proof. Equation (2.17) can be rewritten as follows:

$$\sum_{j=0}^{k} \prod_{i<j}(-\overline{\omega}_i + c_i\psi_i) \prod_{i>j}(\overline{\omega}_i + c_i\psi_i) = 0 \tag{2.20}$$

Expanding all the products and using Definition 2.6.8 we obtain the claimed formula (2.18).

Moreover, using the negated equation $\sum_{i=0}^{k} -c_i\chi_i = 0$, Lemma 2.6.7 gives

$$\prod_{i=1}^{k}(\overline{\omega}_i - c_i\psi_i - \overline{\omega}_{i-1} - c_{i-1}\psi_{i-1}) = 0. \tag{2.21}$$

Adding this relation to the one in (2.17), and decomposing the expansion of the product in two parts, one containing all the terms $\omega_A\psi_B$ with $|B|$ even and the other one containing all those terms with $|B|$ odd, it can be shown that each of the two parts must equal 0. □

In [DP05, Thm. 5.2] De Concini and Procesi prove that the complement of a unimodular toric arrangement is formal. They do this by showing that the rational cohomology ring is isomorphic to the sub-algebra of closed forms generated by $\omega_i = \mathrm{dlog}(e^{b_i} - e^{\chi_i})$ for $i \in E$ and $\psi_\chi = \mathrm{dlog}(e^\chi)$ for $\chi \in \Lambda$. The formal relations among these generators are implicit in [DP05, eq. (20)].

Notice that if the arrangement $\mathcal{A}$ is essential the forms $\psi_i = \mathrm{dlog}(e^{\chi_i})$ for $i \in E$ generate $H^1(T;\mathbb{Q})$. It follows that the relations stated in Proposition 2.6.9 above lead to a presentation of the cohomology ring with respect to the generators $\overline{\omega}_i$'s and ψ_i's. Hence we have the following reformulation of the result of [DP05].

Theorem 2.6.10. *Let $\mathcal{A}$ be an essential unimodular toric arrangement. The rational cohomology algebra $H^*(M(\mathcal{A}),\mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with*

- *Set of generators $e_{A;B}$, where A and B are disjoint and such that $A \sqcup B$ is an independent set; the degree of the generator $e_{A;B}$ is $|A \sqcup B|$.*
- *The following types of relations*
  - *For any two generators $e_{A;B}$, $e_{A';B'}$,*

$$e_{A;B}e_{A';B'} = 0 \tag{2.22}$$

if $A \sqcup B \sqcup A' \sqcup B'$ is a dependent set, and otherwise

$$e_{A;B}e_{A';B'} = (-1)^{\ell(A\cup B, A'\cup B')}e_{A\cup A';B\cup B'}. \tag{2.23}$$

  - *For every linear dependency $\sum_{i\in E} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, a relation*

$$\sum_{i\in E} n_i e_{\emptyset;\{i\}} = 0. \tag{2.24}$$

  - *For every circuit $C \subseteq E$, with linear dependency $\sum_{i\in C} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, a relation*

$$\sum_{j\in C} \sum_{\substack{A,B\subset C \\ C=A\sqcup B\sqcup\{j\} \\ |B| \text{ even.}}} (-1)^{|A_{\leq j}|} c_B e_{A;B} = 0 \tag{2.25}$$

where, for all $i \in C$, $c_i := \mathrm{sgn}\, n_i$ and $c_B = \prod_{i\in B} c_i$.

Remark 2.6.11. In order to check that the presentation above gives the same algebra described in [DP05], we can first notice that relation (2.23) implies that our algebra is generated in degree 1, by elements of the form $e_{\{i\};\emptyset}$ and $e_{\emptyset;\{i\}}$, that correspond respectively to the generators λ_{a_i,χ_i} and ω_i in [DP05, p. 410]. Then our relation (2.22) corresponds to relation (2) of [DP05, p. 410]; our relation (2.24) corresponds to relation (1) of [DP05, p. 410] and our relation (2.25) corresponds to relation (20') that is implicit in [DP05].

Example 2.6.12. Going on with the arrangement of Example 2.6.3 and using the relation $-\chi_0 + \chi_1 + \chi_2 = 0$ we obtain that the rational cohomology of the complement of arrangement $\mathcal{B}'$ has a presentation with generators

$$\overline{\omega}_0, \overline{\omega}_1, \overline{\omega}_2, \psi_0, \psi_1, \psi_2$$

where $-\psi_0 + \psi_1 + \psi_2 = 0$ and relation (2.19) (or equivalently relation (2.25)) gives

$$\overline{\omega}_0\overline{\omega}_1 - \overline{\omega}_0\overline{\omega}_2 + \overline{\omega}_1\overline{\omega}_2 - \psi_0\psi_1 - \psi_0\psi_2 + \psi_1\psi_2 = 0.$$

Note that $\psi_0\psi_1 + \psi_0\psi_2 = \psi_0\psi_0 = 0$ and hence the relation above can be simplified.

2.7 Coverings of arrangements

Recall that we consider a primitive arrangement $\mathcal{A}$ in a torus T.

Given a lattice Λ', $\Lambda \subseteq \Lambda' \subseteq \Lambda \otimes \mathbb{Q}$ we consider the Galois covering $U \to T$ associated with the subgroup $\Lambda'^* \subseteq \Lambda^* \simeq \pi_1(T)$ whose group of deck automorphisms is $(\Lambda'/\Lambda)^* \simeq \mathrm{Gal}(U/T)$.

Definition 2.7.1. Let $f\colon U \to T$ be a finite covering, and call $\mathcal{A}_U$ the lift of $\mathcal{A}$ through f to the torus U. More precisely, let

$$\mathcal{A}_U := \bigcup_{H \in \mathcal{A}} \pi_0(f^{-1}(H)),$$

the set of connected components of preimages of hypertori in $\mathcal{A}$. Moreover, given $i \in E$ let

$$a_i := |\pi_0(f^{-1}(H_i))|$$

denote the number of connected components of $f^{-1}(H_i)$. Given $q \in f^{-1}(H_i)$, let

$$H_i^U(q)$$

denote the connected component of $f^{-1}(H_i)$ containing q.

Remark 2.7.2. The previous definition ensures that $\mathcal{A}_U$ is again a primitive arrangement. It is, however, not necessarily central.

In fact, if we call $\widehat{\chi} := f \circ \chi$ the character of U induced by χ, we see that the connected components $f^{-1}(H_i)$ are associated with the (primitive) character $\frac{\widehat{\chi}_i}{a_i}$. More precisely, every $L \in \pi_0(f^{-1}(H_i))$ has equation

$$\frac{\widehat{\chi}_i}{a_i} = \frac{\widehat{\chi}_i}{a_i}(q)$$

where q is any point of L.

Logarithmic forms on coverings

Our next task is to describe the logarithmic forms on $M(\mathcal{A}_U)$ associated with $\mathcal{A}_U$.

Let $f\colon U \to T$ be a finite covering and let $M(\mathcal{A}_U)$ be as above. The algebraic de Rham complex $\Omega^\bullet_{M(\mathcal{A}_U)}$ splits as direct sum of subcomplexes

$$\Omega^\bullet_{M(\mathcal{A}_U)} \simeq \bigoplus_{\lambda \in \Lambda'/\Lambda} \Omega^\bullet_\lambda \tag{2.26}$$

where $\Omega^\bullet_\lambda$ consists of forms α such that for any $\tau \in \operatorname{Gal}(U/T)$ we have that $\tau^*(\alpha) = \lambda(\tau)\alpha$. In particular the subcomplex of invariant forms $\Omega^\bullet_{\mathbb{1}}$ is canonically identified with $\Omega^\bullet_{M(\mathcal{A})}$.

For any $i \in E$ and any point $q \in f^{-1}(H_i)$ we set

$$\omega_i^U(q) := \frac{1}{2\pi\mathbf{i}} \operatorname{d}\log\left(1 - e^{\frac{\widehat{\chi}_i}{a_i} - \frac{\widehat{\chi}_i}{a_i}(q)}\right) \tag{2.27}$$

for the logarithmic form in $\Omega^1_{M(\mathcal{A}_U)}$ associated with $H_i(q)$. Notice that this form does not depend on the choice of q in the same connected component.

Moreover, let

$$\psi_i^U := \frac{f^*(\psi_i)}{a_i} = \operatorname{d}\log e^{\frac{\widehat{\chi}_i}{a_i}}, \tag{2.28}$$

where the upper symbol * denotes as usual the pull-back.

More generally, given any $A \subseteq E$, choose $q \in f^{-1}(\cap_{i \in A} H_i)$ and let

$$\overline{\omega}_A^U(q) := \prod_{i \in A} \overline{\omega}_i^U(q) \quad \text{ and } \quad \omega_A^U(q) := \prod_{i \in A} \omega_i^U(q), \tag{2.29}$$

where the factors are taken in increasing order with respect to the index i and $\overline{\omega}_i^U(q) := 2\omega_i^U(q) - \psi_i^U$. Moreover, under the same set-up, for disjoint A and B let

$$\overline{\eta}_{A,B}^U(q) := \prod_{i \in A \sqcup B} \overline{\eta}_i^U(q)$$

where $\overline{\eta}_i^U(q) = \overline{\omega}_i^U(q)$ if $i \in A$ and $\overline{\eta}_i^U(q) = \psi_i^U$ if $i \in B$.

Proposition 2.7.3. *Let A be any set of indices and let W be a connected component of $\bigcap_{i\in A} H_i$. Let p be any point in W. The class*

$$\sum_{q\in f^{-1}(p)} \overline{\omega}_A^U(q)$$

is invariant with respect to the group G of deck automorphisms of f and it does not depend on the choice of the point p in W.

Proof. The only nontrivial case is when the characters associated with the indices in A are linearly independent, otherwise $\overline{\omega}_A^U(q) = 0$.

Let $\tau \in G$. Using the definitions we have the equalities

$$\begin{aligned}
\tau^*(\omega_i^U(q)) &= \tau^*\left(\frac{1}{2\pi\mathbf{i}} \operatorname{d}\log\left(1 - e^{\frac{\chi_i}{a_i} - \frac{\chi_i}{a_i}(q)}\right)\right)\\
&= \frac{1}{2\pi\mathbf{i}} \operatorname{d}\log\left(1 - e^{\frac{\chi_i}{a_i} + \frac{\chi_i}{a_i}(\tau) - \frac{\chi_i}{a_i}(q)}\right)\\
&= \frac{1}{2\pi\mathbf{i}} \operatorname{d}\log\left(1 - e^{\frac{\chi_i}{a_i} - \frac{\chi_i}{a_i}(\tau^{-1}q)}\right)\\
&= \omega_i^U(\tau^{-1}(q)).
\end{aligned}$$

Since the forms ψ_i are translation-invariant, we obtain immediately also

$$\tau^*(\overline{\omega}_i^U(q)) = \overline{\omega}_i^U(\tau^{-1}(q)).$$

If we write $A = \{a_1, \ldots, a_k\}$, we see that every form

$$\overline{\omega}_A^U(q) = \overline{\omega}_{a_1}^U(q)\overline{\omega}_{a_1}^U(q)\cdots\overline{\omega}_{a_k}^U(q)$$

satisfies

$$\tau^*\left(\overline{\omega}_A^U(q)\right) = \overline{\omega}_A^U(\tau^{-1}(q)).$$

The claim follows. □

The previous result allows us to give the following definition.

Definition 2.7.4. Let $\mathcal{A} = \{H_i\}_{i\in E}$ be a toric arrangement in the torus T and consider a finite covering $f : U \to T$. Consider an independent set $A \subseteq E$, let W be a connected component of $\cap_{i\in A} H_i$ and choose $p \in W$. Since the pullback map f^* is injective, we can define forms $\overline{\omega}_{W,A}^f$ and $\omega_{W,A}^f$ as the unique forms on $M(\mathcal{A})$ such that

$$f^*(\overline{\omega}_{W,A}^f) = \frac{1}{|\cap_{i\in A} H_i^U(q_0) \cap f^{-1}(p)|} \sum_{q\in f^{-1}(p)} \overline{\omega}_A^U(q)$$

and

$$f^*(\omega_{W,A}^f) = \frac{1}{|\cap_{i\in A} H_i^U(q_0) \cap f^{-1}(p)|} \sum_{q\in f^{-1}(p)} \omega_A^U(q)$$

where q_0 is any point in $f^{-1}(p)$.

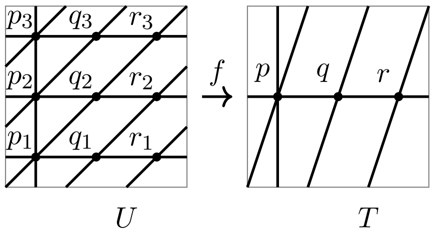


Figure 2.2: A picture on the compact torus of the covering described in Example 2.7.6.

Remark 2.7.5. If the arrangement $\mathcal{A}_U$ is unimodular the formula in the definition above becomes

$$f^*(\overline{\omega}^f_{W,A}) = \frac{1}{|L \cap f^{-1}(p)|} \sum_{q \in f^{-1}(p)} \overline{\omega}^U_A(q)$$

where L is any connected component of $f^{-1}(W)$.

Example 2.7.6. We can now consider the arrangement $\mathcal{B}$ of Example 2.2.11 and the covering $f : U = (\mathbb{C}^*)^2 \to T = (\mathbb{C}^*)^2$ given by $(u, v) \mapsto (u, v^3)$. The arrangement $\mathcal{B}_U$ is unimodular and is given by the 7 hypertori with equations $u = 1$, $v = e^{\frac{2\pi i a}{3}}$ and $uv = e^{\frac{2\pi i b}{3}}$ for $a, b = 0, 1, 2$.

The three subarrangements of $\mathcal{B}_U$ containing respectively the hypertori passing through $p_1 = (1, 1), p_2 = (1, e^{\frac{2\pi i}{3}}), p_3 = (1, e^{\frac{4\pi i}{3}})$ are, up to translation, isomorphic to the unimodular arrangement $\mathcal{B}'$, while the subarrangements passing through the other six points are all isomorphic to the boolean arrangement in $(\mathbb{C}^*)^2$ given by the hypertori with equations $x = 1$ and $y = 1$. We have the form

$$\begin{aligned} f^*(\overline{\omega}^f_{p,\{0,1\}}) =& \frac{-1}{4\pi^2} \left(\frac{1+uv}{1-uv} \frac{\mathrm{d}(uv)}{uv} \frac{1+v}{1-v} \frac{\mathrm{d}\, v}{v} + \frac{1+\zeta_3 uv}{1-\zeta_3 uv} \frac{\mathrm{d}(uv)}{uv} \frac{1+\zeta_3 v}{1-\zeta_3 v} \frac{\mathrm{d}\, v}{v} + \right. \\ & \left. + \frac{1+\zeta_3^2 uv}{1-\zeta_3^2 uv} \frac{\mathrm{d}(uv)}{uv} \frac{1+\zeta_3^2 v}{1-\zeta_3^2 v} \frac{\mathrm{d}\, v}{v} \right) = \\ =& \frac{-3}{4\pi^2} \frac{u^3v^6 + u^3v^3 + 4u^2v^3 + 4uv^3 + v^3 + 1}{uv\,(v^3-1)\,(u^3v^3-1)} \,\mathrm{d}\, u \,\mathrm{d}\, v \end{aligned}$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$ and hence, taking the pushforward and dividing by the degree, we get

$$\overline{\omega}^f_{p,\{0,1\}} = \frac{-1}{4\pi^2} \frac{x^3y^2 + x^3y + 4x^2y + 4xy + y + 1}{xy\,(y-1)\,(x^3y-1)} \,\mathrm{d}\, x \,\mathrm{d}\, y.$$

2.8 Separation

To deal with the non-unimodular case, the following definition turns out to be useful.

Definition 2.8.1. Let A be an independent subset of E. We say that a covering $f : U \to T$ *separates* A if, for any connected component W of $\cap_{i\in A} H_i$ and for all $i \in A$ there exist $q_i \in f^{-1}(H_i)$ such that $f(\cap_{i\in A} H_i^U(q_i)) = W$.

Remark 2.8.2. If $f : U \to T$ is a covering such that the arrangement $\mathcal{A}_U$ is unimodular, then f separates A for all independent set $A \subset E$.

Proposition 2.8.3. *Let $A \subseteq E$ be an independent set. There exists a covering $f : U \to T$ that separates A.*

Proof. Let Γ be a direct summand of $\operatorname{Rad}_\Lambda \Gamma_A$ in Λ, hence $\Lambda = \operatorname{Rad}_\Lambda \Gamma_A \oplus \Gamma$. Consider the lattice

$$\Lambda(A) := \left\langle \frac{\chi_i}{m(A)} \right\rangle_{i\in A} \oplus \Gamma \subseteq \Lambda \otimes \mathbb{Q}.$$

We have the tower of subgroups

$$\Gamma_A \subseteq \operatorname{Rad}_\Lambda \Gamma_A \subseteq \left\langle \frac{\chi_i}{m(A)} \right\rangle_{i\in A}$$

where by definition $[\operatorname{Rad}_\Lambda \Gamma_A : \Gamma_A] = m(A)$. Moreover since A is independent we have $[\langle \frac{\chi_i}{m(A)} \rangle_{i\in A} : \Gamma_A] = m(A)^{|A|}$. Hence we have $[\langle \frac{\chi_i}{m(A)} \rangle_{i\in A} : \operatorname{Rad}_\Lambda \Gamma_A] = m(A)^{|A|-1}$. The inclusion $\Lambda \subseteq \Lambda(A)$ induces a covering $f : U \to T$ of degree $[\Lambda(A) : \Lambda] = [\langle \frac{\chi_i}{m(A)} \rangle_{i\in A} : \operatorname{Rad}_\Lambda \Gamma_A] = m(A)^{|A|-1}$. The first equality follows since Γ is a direct summand of both terms in the left hand side.

This covering separates A. In fact, we claim that for every connected component W of $\bigcap_{i\in A} H_i$ and any choice of a point $q \in f^{-1}(W)$, the intersection $\bigcap_{i\in A} H_i^U(q)$ is connected. To prove this claim, let k denote the number of connected components of $\bigcap_{i\in A} H_i^U(q)$. We count in two different ways the number of connected components of $f^{-1}(\bigcap_{i\in A} H_i)$. On the one hand, for every i we have $m(A)$ connected components of $f^{-1}(H_i)$ and, once we have chosen for every i a connected hypertorus in $f^{-1}(H_i)$, their intersection has k connected components. In this way we have $km(A)^{|A|}$ such components. On the other hand, the number of connected components of the preimage of each of the $m(A)$ connected components of $\bigcap_{i\in A} H_i$ is at most $m(A)^{\operatorname{rk}(A)-1}$, the degree of the covering – hence we obtain a count of at most $m(A)^{\operatorname{rk}(A)} = m(A)^{|A|}$ components. We conclude $k = 1$. □

The following theorem motivates our definition of separating coverings.

Theorem 2.8.4. *Let $A \subset E$ be an independent set. If $f : U \to T$ and $g : V \to T$ are coverings that separate A, then $\overline{\omega}^f_{W,A} = \overline{\omega}^g_{W,A}$. Analogously we have $\omega^f_{W,A} = \omega^g_{W,A}$.*

In the proof we will make use of the following remark.

Remark 2.8.5. For every index i, let $H^U_{i,1}, \dots H^U_{i,m_i}$, denote the connected components of $f^{-1}(H_i)$ and $\overline{\omega}^U_{i,j} := \overline{\omega}_{H^U_{i,j}}$ be the associated forms.

If we assume that f separates A, then Definition 2.7.4 is equivalent to

$$f^*(\overline{\omega}^f_{W,A}) = \sum_{\substack{\mathbf{1} \leq \mathbf{j} \leq \mathbf{m} \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{i \in A} \overline{\omega}^U_{i,j_i}$$

where the sum is indexed using the componentwise ordering among integer A-tuples $\mathbf{1} := (1, \dots, 1)$, $\mathbf{j} := (j_i)_{i \in A}$, $\mathbf{m} := (m_i)_{i \in A}$.

Proof of Theorem 2.8.4. We give the proof for $\overline{\omega}^f_{W,A} = \overline{\omega}^g_{W,A}$, the other case being identical.

The theorem follows in its generality if we first assume that the statement holds when $g = f \circ h$, where $h : V \to U$ is a finite covering. In this case we have

$$\overset{g}{\overbrace{V \xrightarrow[h]{} U \xrightarrow[f]{} T}}$$

and

$$g^*(\overline{\omega}^f_{W,A}) = h^*(f^*(\overline{\omega}^f_{W,A})) = h^*\left(\sum_{\substack{\mathbf{1} \leq \mathbf{j} \leq \mathbf{m} \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{i \in A} \overline{\omega}^U_{i,j_i} \right)$$

where the multi-index $\mathbf{j}$ is as in the Remark 2.8.5. The last equality follows since f separates A.

Again Remark 2.8.5 applied to g gives

$$g^*(\overline{\omega}^g_{W,A}) = \sum_{\substack{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n} \\ \cap_i H^V_{i,k_i} \subseteq g^{-1}(W)}} \prod_{i \in A} \overline{\omega}^V_{i,k_i}.$$

where for all i, $H^V_{i,1}, \dots, H^V_{i,n_i}$ are the connected components of $g^{-1}(H_i)$, $\mathbf{k} = (k_1, \dots, k_{|A|})$, and $\mathbf{n} = (n_1, \dots, n_{|A|})$.

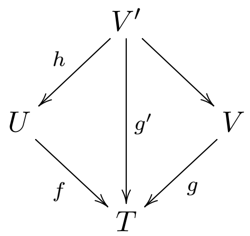
Now we have that

$$h^*(\overline{\omega}^U_{i,j_i}) = \sum_{h(H^V_{i,k_i}) = H^U_{i,j_i}} \overline{\omega}^V_{i,k_i}.$$

Hence from the previous equality we get

$$\begin{aligned} g^*(\overline{\omega}^f_{W,A}) &= \sum_{\substack{\mathbf{1}\leq\mathbf{j}\leq\mathbf{m} \\ \cap_i H^U_{i,j_i}\subseteq f^{-1}(W)}} \prod_{i\in A} h^*(\overline{\omega}^U_{i,j_i}) \\ &= \sum_{\substack{\mathbf{1}\leq\mathbf{j}\leq\mathbf{m} \\ \cap_i H^U_{i,j_i}\subseteq f^{-1}(W)}} \prod_{i\in A} \sum_{h(H^V_{i,k_i})=H^U_{i,j_i}} \overline{\omega}^V_{i,k_i} \\ &= \sum_{\substack{\mathbf{1}\leq\mathbf{j}\leq\mathbf{m} \\ \cap_i H^U_{i,j_i}\subseteq f^{-1}(W)}} \sum_{\substack{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n} \\ h(H^V_{i,k_i})=H^U_{i,j_i}}} \prod_{i\in A} \overline{\omega}^V_{i,k_i} \\ &= \sum_{\substack{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n} \\ \cap_i H^V_{i,k_i}\subseteq g^{-1}(W)}} \prod_{i\in A} \overline{\omega}^V_{i,k_i} \\ &= g^*(\overline{\omega}^g_{W,A}). \end{aligned}$$

Finally, in the general case of two coverings $f : U \to T$ and $g : V \to T$, we can consider the diagram



where $h : V' \to U$ is the pullback of g by f and $g' = f \circ h$. Since f separates A, then also g' separates A and we apply the first part of the proof to the maps f and g'. □

Remark 2.8.6. Since the covering $f : U \to V$ is finite, we have that $\overline{\omega}^f_{W,A} = f_*\overline{\omega}^U_A(q)$ for any $q \in f^{-1}(W)$ where f_* is the pushforward associated with the covering map f.

Using Theorem 2.8.4, we can state the following definition.

Definition 2.8.7. Given $A \subset E$ independent and given W a connected component of $\cap_{i\in A} H_i$, we define

$$\overline{\omega}_{W,A} := \overline{\omega}^f_{W,A}$$

and

$$\omega_{W,A} := \omega^f_{W,A}$$

where $f : U \to T$ is any covering that separates A.

Remark 2.8.8. We would like to convince the reader that the definition of the forms $\overline{\omega}_{W,A}$ and $\omega_{W,A}$ given above is the most natural choice in order to provide a set of form generating the cohomology of the toric complement.

As seen in (2.26), once we fix a covering $f : U \to T$ with Galois group G, the G-module $\Omega^1(M(\mathcal{A}_U))$ has a natural decomposition as a direct sum of semi-invariant modules associated with the characters of G. The forms defined above can be identified with certain G-invariant forms on $M(\mathcal{A}_U)$. We have that

$$\Omega^k_\lambda \Omega^{k'}_{\lambda'} \subseteq \Omega^{k+k'}_{\lambda+\lambda'}.$$

In particular, if the sum of the characters of the factors is the trivial character, we get invariant forms, which correspond to forms on $M(\mathcal{A})$.

The hypothesis that f separates A guarantees that we obtain enough semi-invariant 1-forms associated with the hypertori $f^{-1}(H_i)$, for $i \in A$, in order to obtain $m(A)$ independent invariant classes.

Lemma 2.8.9. *If $A, A' \subseteq E$ are such that $A \sqcup A'$ is an independent set and W, resp. W' are a choice of a connected component of $\bigcap_{i\in A} H_i$, resp. $\bigcap_{i\in A'} H_i$, we can compute*

$$\overline{\omega}_{W,A}\overline{\omega}_{W',A'} = (-1)^{\ell(A,A')} \sum_{L\in\pi_0(W\cap W')} \overline{\omega}_{L,A\sqcup A'}.$$

Proof. Consider a covering $f : U \to T$ that separates the independent set $A \sqcup A'$ (e.g. the one described in Proposition 2.8.3). Then, by definition, in order to evaluate the product $\overline{\omega}_{W,A}\overline{\omega}_{W',A'}$ we consider its pullback $f^*(\overline{\omega}_{W,A}\overline{\omega}_{W',A'})$ which, with Remark 2.8.5, equals

$$\Bigg(\sum_{\substack{\mathbf{1}\leq\mathbf{j}\leq\mathbf{m} \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{i\in A} \overline{\omega}^U_{i,j_i} \Bigg)\Bigg(\sum_{\substack{\mathbf{1}\leq\mathbf{j}'\leq\mathbf{m}' \\ \cap_{i'} H^U_{i',j'_{i'}} \subseteq f^{-1}(W')}} \prod_{i'\in A'} \overline{\omega}^U_{i',j'_{i'}} \Bigg)$$

$$= \sum_{\substack{\mathbf{1}\leq(\mathbf{j},\mathbf{j}')\leq(\mathbf{m},\mathbf{m}') \\ \cap_i H^U_{i,j_i} \cap_{i'} H^U_{i',j'_{i'}} \subseteq f^{-1}(W'\cap W)}} \prod_{i\in A} \overline{\omega}^U_{i,j_i} \prod_{i'\in A'} \overline{\omega}^U_{i,j'_{i'}}$$

$$= \sum_{\substack{\mathbf{1}\leq\bar{\mathbf{j}}\leq(\mathbf{m},\mathbf{m}') \\ \cap_{i\in A\sqcup A'} H^U_{i,\bar{j}_i} \subseteq f^{-1}(W'\cap W)}} (-1)^{\ell(A,A')} \prod_{i\in A\sqcup A'} \overline{\omega}^U_{i,\bar{j}_i},$$

where $\bar{\mathbf{j}} = (\mathbf{j}, \mathbf{j}')$. The latter equals, by definition

$$f^*\Bigg((-1)^{\ell(A,A')} \sum_{L\in\pi_0(W\cap W')} \overline{\omega}_{L,A\sqcup A'} \Bigg)$$

as was to be shown. □

Remark 2.8.10. Assume that $\cap_{i\in A} H_i$ is connected and call it W. Then, since the identity separates A, we have $\overline{\omega}_{W,A} = \overline{\omega}_A$.

Definition 2.8.11. Given $A \subset E$ independent and given W a connected component of $\cap_{i\in A} H_i$, we write $\overline{\eta}_{W,A,B}$ for the form

$$(-1)^{\ell(A,B)} \overline{\omega}_{W,A} \psi_B.$$

In the following if W is not a connected component of $\cap_{i\in A} H_i$ the expression $\overline{\eta}_{W,A,B}$ will be considered as meaningless and it will be treated as zero.

Remark 2.8.12. We have the following consequence of Lemma 2.8.9.
If $A, A', B, B' \subseteq E$ are such that $A \sqcup A' \sqcup B \sqcup B'$ is an independent set and W, resp. W' are a choice of a connected component of $\bigcap_{i\in A} H_i$, resp. $\bigcap_{i\in A'} H_i$, we can compute

$$\overline{\eta}_{W,A,B} \overline{\eta}_{W',A',B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} \overline{\eta}_{L, A\sqcup A', B\sqcup B'}.$$

Definition 2.8.13. We introduce the increasing filtration F of $H^*(M(\mathcal{A});\mathbb{Z})$ defined by

$$\mathrm{F}_i\, H^*(M(\mathcal{A});\mathbb{Z}) := \sum_{j\leq i} H^j(M(\mathcal{A});\mathbb{Z}) \cdot H^*(T;\mathbb{Z}).$$

Such a filtration is the Leray filtration of the inclusion $M(\mathcal{A}) \hookrightarrow T$. The same filtration, with rational coefficients, was introduced in [DP05, Remark 4.3.(2)]. The associated graded module is

$$\mathrm{gr}_k(H^*(M(\mathcal{A}))) = \bigoplus_{\substack{W\in\mathcal{L}(\mathcal{A})\\ \mathrm{codim}(W)=k}} H^*(W) \otimes H^k(M(\mathcal{A}[W])) \tag{2.30}$$

where $\mathcal{A}[W]$ is the hyperplane arrangement introduced in Definition 2.2.9.

Lemma 2.8.14. *Let $A, B \subseteq E$ such that $A \sqcup B$ is independent and let W be a connected component of $\cap_{i\in A} H_i$. Then, the image of $\overline{\eta}_{W,A,B}$ in the graded ring $\mathrm{gr}_{|A|}(H^*(M(\mathcal{A})))$ equals*

$$(-1)^{\ell(B,A)} 2^{|A|} \psi_B \otimes e_A \in H^{|B|}(W) \otimes H^{|A|}(M(\mathcal{A}[W])),$$

where e_A denotes the canonical generator in the top-degree of the OS-algebra of the hyperplane arrangement $\mathcal{A}[W]$ associated with the hyperplanes indexed by A (cf. Definition 2.2.9).

Proof. We consider the corresponding graduation gr^U for the lift to a unimodular covering $f : U \to T$ (e.g., the one separating A in Proposition 2.8.3).

By multiplicativity of gr_k, it suffices to prove the case $B = \emptyset$. We thus have to consider $\overline{\omega}_{W,A}$ which, by Remark 2.8.6, can be written as $\overline{\omega}_{W,A} = f_* \overline{\omega}_A^U(q)$, where q is a fixed point in $f^{-1}(W)$. Now,

$$\mathrm{gr}_k^U\left(\overline{\omega}_A^U(q)\right) = \mathrm{gr}_k^U\left(2^{|A|}\omega_A^U(q)\right),$$

hence

$$\mathrm{gr}_k(\overline{\omega}_{W,A}) = f_*(\mathrm{gr}_k\, \overline{\omega}_A^U(q)) = \mathrm{gr}_k(2^{|A|}\omega_{W,A}),$$

and, since $\exp_p^*(\omega_i) = e_i$, the class $[2^{|A|}\omega_{W,A}]$ maps to the element $2^{|A|} \otimes e_A$ in $H^0(W) \otimes H^{|A|}(M(\mathcal{A}[W]))$ as desired. □

2.9 Rational cohomology

Let $\mathcal{A} = \{H_0, \dots, H_k\}$ be a primitive, central and essential arrangement in the torus T. Suppose further that the associated matroid has exactly one circuit $C = E$, and hence $\mathrm{rk}\, E = k$. Let $\chi_0, \dots, \chi_k$ be the associated list of characters.

Recall that $\Gamma_C \subset \Lambda$ is the sublattice generated by the characters of C and $\mathrm{Rad}_\Lambda \Gamma_C$ is the intersection $(\mathbb{Q} \otimes_\mathbb{Z} \Gamma_C) \cap \Lambda$.

Definition 2.9.1. For every $i = 0, \dots, k$ set

$$a_i := \prod_{j \neq i} m(C \setminus \{j\}).$$

We call $\Lambda(C)$ the lattice in $\mathbb{Q} \otimes_\mathbb{Z} \Lambda$ generated by the elements $\frac{\chi_i}{a_i}$.

Remark 2.9.2. Since the matroid associated with $\mathcal{A}$ has exactly one circuit (i.e. $C = E$) and $\mathcal{A}$ is essential we have that $\Lambda = \mathrm{Rad}_\Lambda \Gamma_C$, hence $m(C)$ is precisely the index of Γ_C in Λ.

Lemma 2.9.3. *In $\Lambda(C)$ we have the relation*

$$\sum_{i=0}^{k} c_i \frac{\chi_i}{a_i} = 0 \tag{2.31}$$

where $c_i \in \{+1, -1\}$ for all i.

Proof. This follows from Corollary 2.2.13, since the product

$$a_i m(C \setminus \{i\}) = \prod_{j=0}^{k} m(C \setminus \{j\})$$

does not depend on the index i. □

Lemma 2.9.4. *The lattice* $\Lambda(C)$ *contains* Λ.

Proof. We split the claim into two inclusions:

$$\Lambda \overset{\text{(i)}}{\subseteq} \frac{1}{m(C)}\Gamma_C \overset{\text{(ii)}}{\subseteq} \Lambda(C).$$

Inclusion (i) follows from the fact that, by Remark 2.9.2 the quotient Λ/Γ_C is a group of cardinality $m(C)$, hence $m(C)\Lambda \subset \Gamma_C$.

For inclusion (ii), notice that every element of $\frac{1}{m(C)}\Gamma_C$ can be written as a combination

$$\frac{1}{m(C)}\sum_{i=0}^{k} n_i\chi_i = \sum_{i=0}^{k}\left(\frac{n_i a_i}{m(C)}\right)\frac{\chi_i}{a_i}$$

for some $n_i \in \mathbb{Z}$. Now, since C is a circuit, $m(C)$ divides every $m(C \setminus \{i\})$, $i = 0, \ldots, k$ (e.g., by Remark 2.2.12-(c)). Hence all parenthesized coefficients on the r.h.s. are integers, which means $\frac{1}{m(C)}\Gamma_C \subseteq \Lambda(C)$, as claimed. □

Lemma 2.9.5. *The inclusion of lattices* $\Lambda \subseteq \Lambda(C)$ *induces a covering of* T *of degree*

$$d = \prod_{j=0}^{k} m(C \setminus \{j\})^{k-1}.$$

Proof. It is enough to prove that d as defined above equals the index of Λ in $\Lambda(C)$.

Let us fix an index i and consider the inclusions

$$\Lambda_{C\setminus\{i\}} \subseteq \Lambda \subseteq \Lambda(C).$$

Since (by Lemma 2.9.3) the lattice $\Lambda(C)$ is generated by the basis $\{\frac{\chi_j}{a_j} \mid j \neq i\}$, the index of $\Lambda_{C\setminus\{i\}}$ in $\Lambda(C)$ is

$$[\Lambda(C) : \Gamma_{C\setminus\{i\}}] = \prod_{j\neq i} a_j.$$

On the other hand, $m(C \setminus \{i\})$ is by definition the index of $\Gamma_{C\setminus\{i\}}$ in $\Lambda = \mathrm{Rad}_\Lambda \Gamma_{C\setminus\{i\}}$. In conclusion, the desired index is

$$\begin{aligned}[\Lambda(C) : \Lambda] &= \frac{[\Lambda(C) : \Gamma_{C\setminus\{i\}}]}{[\Lambda : \Gamma_{C\setminus\{i\}}]} = \frac{\prod_{j\neq i} a_j}{m(C \setminus \{i\})} \\ &= \frac{\prod_{j\neq i}\prod_{l\neq j} m(C \setminus \{l\})}{m(C \setminus \{i\})} = \prod_{j=0}^{k} m(C \setminus \{j\})^{k-1}\end{aligned}$$

as claimed. □

Definition 2.9.6. Let

$$\pi_U : U \to T$$

denote the covering induced by the inclusion $\Lambda \subseteq \Lambda(C)$.

We denote by $\mathcal{A}_U$ the central arrangement in the torus U induced by the characters $\frac{\chi_i}{a_i}$ in $\Lambda(C)$. Notice that $\mathcal{A}_U$ is clearly primitive, since the $\frac{\chi_i}{a_i}$ form a basis of $\Lambda(C)$.

Lemma 2.9.7. *The arrangement $\mathcal{A}_U$ is unimodular.*

Proof. For every $j \in C$ the set $\{\frac{\chi_i}{a_i}\}_{i \neq j}$ is a basis of the lattice $\Lambda(C)$. In fact $C \setminus \{j\}$ is independent and by (2.31) we have that $\frac{\chi_j}{a_j}$ belongs to the lattice generated by the characters $\frac{\chi_i}{a_i}$.

Hence for every subset $A \subsetneq C$ we can choose $j \in C \setminus A$. Then the set $\{\frac{\chi_i}{a_i}\}_{i \in A}$ can be completed to a basis $\{\frac{\chi_i}{a_i}\}_{i \neq j}$ of $\Lambda(C)$. □

Notice that the number of connected component of $\pi_U^{-1}(H_i)$ is a_i. In fact for $j \neq i$ the character χ_i can be written in the basis $\{\frac{\chi_k}{a_k}\}_{k \neq j}$ as $\chi_i = a_i \frac{\chi_i}{a_i}$.

Lemma 2.9.8. *Let $A \subsetneq C$, let W be a connected component of $\bigcap_{i \in A} H_i$ and choose $p \in W$. Then, for every layer L of $\mathcal{A}_U$ such that $\pi_U(L) = W$, the number of preimages of p contained in L is*

$$|L \cap \pi_U^{-1}(p)| = m(A) \prod_{i=0}^{k} m(C \setminus \{i\})^{k-1-|A|} \prod_{i \in A} m(C \setminus \{i\}).$$

Proof. The cardinality of the preimage of p is equal to the degree of the covering, computed in Lemma 2.9.5. On the other hand, given W a connected component of $\cap_{i \in A} H_i$, the number of connected components of $\pi_U^{-1}(W)$ is equal to $\frac{\prod_{i \in A} a_i}{m(A)}$. Hence

$$\begin{aligned} |L \cap \pi_U^{-1}(p)| &= \frac{\prod_{i=0}^{k} m(C \setminus \{i\})^{k-1}}{\prod_{i \in A} a_i} m(A) \\ &= m(A) \prod_{i=0}^{k} m(C \setminus \{i\})^{k-1-|A|} \prod_{i \in A} m(C \setminus \{i\}). \end{aligned}$$ □

Example 2.9.9. In the case of the arrangement of Example 2.7.6 with matrix

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

we have that the lattice $\Lambda = \mathbb{Z}^2$ coincides with the lattice Λ^C. In this case we have $a_0 = 3, a_1 = 3, a_2 = 1$, hence the lattice $\Lambda(C)$ is generated by $\langle e_1 + \frac{e_2}{3}, \frac{e_2}{3}, e_1 \rangle$. In particular the inclusion $\Lambda \subset \Lambda(C)$ corresponds to the covering

$f\colon U \to T$ of Example 2.7.6 (see Figure 2.2). Notice that, with respect to the basis $\{ e_1, \frac{e_2}{3} \}$ of $\Lambda(C)$, the arrangement $\mathcal{B}_U$ is described by the matrix

$$\begin{pmatrix} 3 & 0 & 1 \\ 3 & 3 & 0 \end{pmatrix}.$$

Lemma 2.9.10. *For any $A, B \subseteq C$ such that $A \sqcup B$ is a maximal independent subset of C, for any connected component W of $\bigcap_{i\in A} H_i$ and $p \in W$ we have*

$$\pi_U^*(\overline{\eta}_{W,A,B}) = (-1)^{\ell(A,B)} \frac{m(A \cup B)}{m(A)} \sum_{q\in\pi_U^{-1}(p)} \overline{\omega}_A^U(q)\psi_B^U. \tag{2.32}$$

Proof. With Equation (2.28) we have

$$\pi_U^*(\psi_B) = \Big(\prod_{i\in B} a_i\Big)\psi_B^U.$$

and hence, applying this equality and Remark 2.7.5 to Definition 2.8.11, we get

$$\pi_U^*(\overline{\eta}_{W,A,B}) = (-1)^{\ell(A,B)} \frac{\prod_{i\in B} a_i}{|L \cap \pi_U^{-1}(p)|} \sum_{q\in\pi_U^{-1}(p)} \overline{\omega}_A^U(q)\psi_B^U. \tag{2.33}$$

The coefficient in formula (2.33) can be rewritten as

$$\begin{aligned}
\frac{\prod_{i\in B} a_i}{|L \cap \pi_U^{-1}(p)|} &= \frac{\prod_{i\in B} a_i}{m(A)\prod_{i=0}^k m(C\setminus\{i\})^{k-1-|A|}\prod_{i\in A} m(C\setminus\{i\})} \\
&= \frac{\prod_{i=0}^k m(C\setminus\{i\})^{|B|}}{m(A)\prod_{i=0}^k m(C\setminus\{i\})^{|B|-1}\prod_{i\in A\cup B} m(C\setminus\{i\})} \\
&= \frac{\prod_{i=0}^k m(C\setminus\{i\})}{m(A)\prod_{i\in A\cup B} m(C\setminus\{i\})} \\
&= \frac{m(A\cup B)}{m(A)},
\end{aligned}$$

and the claim follows. □

Definition 2.9.11. For any $A, B \subseteq C$ such that $A \sqcup B$ is an independent set and every $q \in \pi_U^{-1}(\bigcap_{i\in A} H_i)$, we set

$$\overline{\eta}_{A,B}^U(q) := (-1)^{\ell(A,B)}\overline{\omega}_A^U(q)\psi_B^U.$$

Recall from Proposition 2.6.9 that given a circuit C, for every $B \subset C$ we set $c_B = \prod_{i\in B} c_i$.

Theorem 2.9.12. *Let L be a connected component of $\cap_{i\in C} H_i$. We have*

$$\sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}\\ W\supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A\cup B)} \overline{\eta}_{W,A,B} = 0. \tag{2.34}$$

Proof. Now we fix a point $p \in \bigcap_{i\in C} H_i$ and we use relation (2.19) in $\mathcal{A}_U$. This gives us, for every $q \in \pi_U^{-1}(p)$,

$$\sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}}} (-1)^{|A_{\leq j}|} \overline{\eta}^U_{A,B}(q) c_B = 0.$$

Summing over all $q \in \pi_U^{-1}(p)$, we get

$$\begin{aligned}
0 &= \sum_{q\in\pi_U^{-1}(p)} \sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}}} (-1)^{|A_{\leq j}|} c_B \overline{\eta}^U_{A,B}(q)\\
&= \sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}}} (-1)^{|A_{\leq j}|} c_B \sum_{q\in\pi_U^{-1}(p)} \overline{\eta}^U_{A,B}(q)\\
&= \sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}\\ W\supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A\cup B)} \pi_U^*(\overline{\eta}_{W,A,B}).
\end{aligned}$$

Since π_U^* is an injective algebra homomorphism, we obtain the claimed equality. □

We now drop the assumption that the arithmetic matroid has a unique circuit and we go back to the general set-up of any arrangement $\mathcal{A}$ in a torus T.

Theorem 2.9.13. *Let $\mathcal{A}$ be an essential arrangement. The rational cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with*

- *Set of generators $e_{W,A;B}$, where W ranges over all layers of $\mathcal{A}$, A is a set generating W and B is disjoint from A and such that $A \sqcup B$ is an independent set; the degree of the generator $e_{W,A;B}$ is $|A \sqcup B|$.*
- *The following types of relations:*

– *For any two generators* $e_{W,A;B}$, $e_{W',A';B'}$,

$$e_{W,A;B}e_{W',A';B'} = 0$$

if $A \sqcup B \sqcup A' \sqcup B'$ *is a dependent set, and otherwise*

$$e_{W,A;B}e_{W',A';B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} e_{L,A\cup A';B\cup B'}. \tag{2.35}$$

– *For every linear dependency* $\sum_{i\in E} n_i\chi_i = 0$ *with* $n_i \in \mathbb{Z}$, *a relation*

$$\sum_{i\in E} n_i e_{T,\emptyset;\{i\}} = 0. \tag{2.36}$$

– *For every circuit* $C \subseteq E$, *with dependency* $\sum_{i\in C} n_i\chi_i = 0$ *with* $n_i \in \mathbb{Z}$, *and for every connected component* L *of* $\cap_{i\in C} H_i$ *a relation*

$$\sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even.}\\ W\supseteq L}} (-1)^{|A_{\le j}|} c_B \frac{m(A)}{m(A\cup B)} e_{W,A;B} = 0 \tag{2.37}$$

where, for all $i \in C$, $c_i := \operatorname{sgn} n_i$, $c_B = \prod_{i\in B} c_i$.

Proof. Consider the map

$$\Phi : \mathcal{E} \to H^*(M(\mathcal{A}), \mathbb{Q}), \quad e_{W,A;B} \mapsto [\overline{\eta}_{W,A,B}].$$

This map is well-defined – in fact, in the cohomology ring Equation (2.35) holds by Remark 2.8.12, Equation (2.36) already holds in the cohomology of the ambient torus, and Equation (2.37) holds by Theorem 2.9.12.

Now fix, for every independent $A \subseteq E$, a subset $D(A) \subseteq E$ such that $A \sqcup D(A)$ is a basis of the matroid. Then, notice that relations (2.37) and (2.36) allow us to express every generator in terms of generators $e_{W,A;B}$ where A is a no-broken-circuit set and B is a subset of $D(A)$. Then, with Lemma 2.8.14, the k-th graded part of the image of Φ equals $\operatorname{gr}_k H^*(M(\mathcal{A}), \mathbb{Q})$. We conclude that Φ is bijective, hence it defines the desired isomorphism. □

Remark 2.9.14. The relations in the presentation above hold for differential forms and not only for their cohomology classes. As a consequence the space $M(\mathcal{A})$ is rationally formal. This fact has been already observed by [DP05] for unimodular arrangements and proved by [Dup16b] in general.

Remark 2.9.15. Notice that all relations of type (2.36) are implied by those associated with minimal linear dependencies (i.e., circuits).

Moreover, the above presentation is completely encoded in the datum of the poset of layers of $\mathcal{A}$ (needed, e.g. for Relations (2.35), (2.37)) and in the (relative) sign pattern of the minimal linear dependencies. But by [Pag19b, Theorem 3.12] (see Corollary 2.2.13), the latter can also be recovered by the poset.

The complements of the two toric arrangements constructed in the already quoted paper [Pag19d] (see Remark 2.2.17) turn out to have non-isomorphic cohomology rings. Since the two arrangements have isomorphic matroids, this implies that the cohomology ring cannot be determined purely in terms of the arithmetic matroid.

Example 2.9.16. We can provide a presentation of the rational cohomology of the complement of arrangement $\mathcal{B}$. The cohomology ring is generated by:

$$\begin{aligned} \overline{\omega}_0 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}\left(\frac{(1-x^3y)^2}{x^3y}\right), & \psi_0 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(x^3y), \\ \overline{\omega}_1 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}\left(\frac{(1-y)^2}{y}\right), & \psi_1 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(y), \\ \overline{\omega}_2 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}\left(\frac{(1-x)^2}{x}\right), & \psi_2 =& \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(x) \end{aligned}$$

and

$$\begin{aligned} \overline{\omega}_{p,\{0,1\}} =& \frac{-1}{4\pi^2} \frac{x^3y^2+x^3y+4x^2y+4xy+y+1}{xy\,(y-1)\,(x^3y-1)} \,\mathrm{d}\,x\,\mathrm{d}\,y, \\ \overline{\omega}_{q,\{0,1\}} =& \frac{-1}{4\pi^2} \frac{x^3y^2+x^3y+4\zeta_3^2x^2y+4\zeta_3xy+y+1}{xy\,(y-1)\,(x^3y-1)} \,\mathrm{d}\,x\,\mathrm{d}\,y, \\ \overline{\omega}_{r,\{0,1\}} =& \frac{-1}{4\pi^2} \frac{x^3y^2+x^3y+4\zeta_3x^2y+4\zeta_3^2xy+y+1}{xy\,(y-1)\,(x^3y-1)} \,\mathrm{d}\,x\,\mathrm{d}\,y \end{aligned}$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$. The relations are

$$\begin{aligned} \overline{\omega}_i\psi_i =& 0 \qquad \forall\, i, \\ \overline{\omega}_0\overline{\omega}_1 =& \overline{\omega}_{p,\{0,1\}} + \overline{\omega}_{q,\{0,1\}} + \overline{\omega}_{r,\{0,1\}}, \\ -\psi_0+\psi_1+3\psi_2 =& 0, \\ \overline{\omega}_{p,\{0,1\}} - \overline{\omega}_0\overline{\omega}_2 + \overline{\omega}_1\overline{\omega}_2 =& -\psi_1\psi_2 + \psi_0\psi_2 + \frac{1}{3}\psi_0\psi_1 \end{aligned}$$

where the last relation can be verified checking the equalities

$$\begin{aligned} &\frac{x^3y^2+x^3y+4x^2y+4xy+y+1}{xy\,(y-1)\,(x^3y-1)} \,\mathrm{d}\,x\,\mathrm{d}\,y - \frac{x^3y+1}{y(x^3y-1)}\frac{x+1}{x(x-1)} \,\mathrm{d}\,y\,\mathrm{d}\,x + \\ &+ \frac{y+1}{y(y-1)}\frac{x+1}{x(x-1)} \,\mathrm{d}\,y\,\mathrm{d}\,x = \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{xy} = \\ &= -\operatorname{dlog}(y)\operatorname{dlog}(x) + \operatorname{dlog}(x^3y)\operatorname{dlog}(x) + \frac{1}{3}\operatorname{dlog}(x^3y)\operatorname{dlog}(y). \end{aligned}$$

2.10 Integral cohomology

Proposition 2.10.1. *The forms $\omega_{W,A}$ are integral forms.*

Proof. We first prove our statement in the case when W is a point, hence $|A| = n$ and $\omega_{W,A}$ is a n-form.

We will prove that for any integral cycle $S \in H_n(M(\mathcal{A}); \mathbb{Z})$ the integral

$$\int_S \omega_{W,A}$$

is an integer number. From the Universal Coefficients Theorem this implies that $\omega_{W,A}$ is an integral form.

Now, let $f : U \to T$ be any covering that separates A and such that the arrangement $\mathcal{A}_U$ is unimodular. For any cycle $S \in H_n(M(\mathcal{A}); \mathbb{Z})$ we have

$$\begin{aligned}\int_S \omega_{W,A} &= \frac{1}{\deg f} \int_{f^{-1}(S)} f^*(\omega_{W,A}) \\ &= \frac{1}{\deg f} \int_{f^{-1}(S)} \sum_{q \in f^{-1}(W)} \omega_A^U(q)\end{aligned}$$

where the last equality follows from Remark 2.7.5.

Now we can observe that the integral

$$\int_{f^{-1}(S)} \omega_A^U(q)$$

does not depend on the point $q \in f^{-1}(W)$. Moreover, since W is a point, $\deg f = |f^{-1}(W)|$. We thus have

$$\int_S \omega_{W,A} = \int_{f^{-1}(S)} \omega_A^U(q)$$

for any point $q \in f^{-1}(W)$. Since the arrangement $\mathcal{A}_U$ is unimodular we have (see (2.29)) $\omega_A^U(q) = \prod_{i \in A} \omega_i^U(q)$. By definition (2.27) each factor $\omega_i^U(q)$ is an integer form. Hence integrality of $\omega_A^U(q)$ implies integrality of $\omega_{W,A}$.

In the general case let W_0 be the translate of W containing the identity of T. We can consider the projection $\pi_{W_0} : T \to T'$, where $T' = T/W_0$. The W_0-invariant characters χ_i for $i \in A$ induce characters χ_i' of T', defining hypertori $H_i' = \pi_{W_0}(H_i) \subseteq T'$. Let $W' = \pi_{W_0}(W)$ be the component of $\bigcap_{i \in A} H_i'$ corresponding to W and consider the associated form $\omega'_{W',A}$ on T'. Then $\omega_{W,A} = \pi^*_{W_0}(\omega'_{W',A})$ and so integrality of $\omega_{W,A}$ follows from integrality of $\omega'_{W',A}$, which is granted because W' has dimension 0. ☐

Proposition 2.10.2. *For any independent set $A \sqcup B \subset E$ and for any layer W in $\cap_{i \in A} H_i$ the form $\frac{m(A)}{m(A \sqcup B)} \eta_{W,A,B}$ is integral.*

Proof. Let $\mathfrak{A} = \{b_1, \dots, b_{|A|}\}$ be a basis of Γ^A. We complete it to a basis $\mathfrak{A} \cup \mathfrak{B} = \{b_1, \dots, b_{|A|+|B|}\}$ of $\Gamma^{A \sqcup B}$. We can define the forms

$$v_j := \frac{1}{2\pi \mathbf{i}} \operatorname{d} \log(e^{b_j}).$$

Hence we can consider the square matrix $M = (m_{ij})$ such that for every $j \in A \sqcup B$ we have that

$$\psi_j = \sum_{i=1}^{|A|+|B|} m_{ij} v_j.$$

The matrix M is a block matrix of the form

$$M = \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

with M_1 a $|A| \times |A|$ matrix and M_3 a $|B| \times |B|$ matrix. For $j > |A|$ we have that $\omega_{W,A}\psi_j = \omega_{W,A} \sum_{i=|A|+1}^{|A|+|B|} m_{ij} v_j$, i.e., using only entries of M_3. Hence

$$\eta_{W,A,B} = \pm \omega_{W,A} \prod_{j=|A|+1}^{|A|+|B|} \psi_j = \pm \det(M_3) \omega_{W,A} \prod_{j=|A|+1}^{|A|+|B|} v_j.$$

Since $\det(M_3) = \frac{\det(M)}{\det(M_1)} = \frac{m(A \sqcup B)}{m(A)}$ we have that $\frac{m(A)}{m(A \sqcup B)} \eta_{W,A,B}$ is an integral form. □

Recall the filtration F of $H^*(M(\mathcal{A}))$ introduced in Definition 2.8.13:

$$\mathrm{F}_i \, H^*(M(\mathcal{A}); \mathbb{Z}) := \bigoplus_{j \leq i} H^j(M(\mathcal{A}); \mathbb{Z}) \cdot H^*(T; \mathbb{Z}).$$

From Definition 2.6.1 we have that

$$[\overline{\omega}_i] = 2[\omega_i] \text{ in } \mathrm{F}_1 / \mathrm{F}_0 \, H^*(M(\mathcal{A}); \mathbb{Z})$$

and

$$[\overline{\omega}_{W,A}] = 2^{|A|}[\omega_{W,A}] \text{ in } \mathrm{F}_{|A|} / \mathrm{F}_{|A|-1} \, H^*(M(\mathcal{A}); \mathbb{Z}) \tag{2.38}$$

Definition 2.10.3. The $\mathbb{Z}$-algebra $R \subset \Omega^*(M(\mathcal{A}))$ is the subalgebra generated by the closed forms $\omega_{W,A}\alpha$, where W runs among the layers of $\cap_{i \in A} H_i$ for A independent and $\alpha \in H^*(T; \mathbb{Z})$.

Theorem 2.10.4. *Let $\mathcal{A}$ be an essential toric arrangement. The integral cohomology ring of $M(\mathcal{A})$ is isomorphic to the algebra R:*

$$R \simeq H^*(M(\mathcal{A}); \mathbb{Z}).$$

In particular the space $M(\mathcal{A})$ is formal.

Proof. Since the relations given in the presentation of Theorem 2.9.13 are equalities between differential forms, the map $i : R \hookrightarrow H^*(M(\mathcal{A}); \mathbb{Z})$ sending each form to its cohomology class is an injective map of filtered modules. In particular it induces an homomorphism

$$\mathrm{gr}(i)\colon \mathrm{gr}(R) \to \mathrm{gr}(H^*(M(\mathcal{A}); \mathbb{Z}))$$

of graded modules. We claim that the map $\mathrm{gr}(i)$ is an isomorphism. Since the strictly filtered map i is injective, $\mathrm{gr}(i)$ is injective too.

We will prove that $\mathrm{gr}(i)$ is also surjective. As seen in Equation (2.30), the graded algebra decomposes as a direct sum

$$\mathrm{gr}_k(H^*(M(\mathcal{A}))) = \bigoplus_{\substack{W \in \mathcal{L}(\mathcal{A}) \\ \mathrm{codim}(W)=k}} H^*(W) \otimes H^k(M(\mathcal{A}[W]));$$

moreover the summand $H^*(W) \otimes H^k(M(\mathcal{A}[W]))$ is generated, as a $H^*(T; \mathbb{Z})$-module, by the elements $1 \otimes e_A$ for A independent set such that W is a connected component of $\bigcap_{i \in A} H_i$. From Equation (2.38) and Lemma 2.8.14 we have

$$2^{|A|}[\omega_{W,A}] = [\overline{\omega}_{W,A}] = 2^{|A|}(1 \otimes e_A)_W$$

where the inclusion of $H^*(W) \otimes H^k(M(\mathcal{A}[W]))$ in $\mathrm{gr}_k(H^*(M(\mathcal{A})))$ is understood.

Since the integral cohomology ring of the complement of an hyperplane arrangement is torsion free [OS80], it follows that the algebra $\mathrm{gr}_k(H^*(M(\mathcal{A})))$ is torsion free.

For every layer W and every set of indices A the element

$$(1 \otimes e_A)_W \in \mathrm{gr}_k(H^*(M(\mathcal{A})))$$

is the image of $\omega_{W,A}$ and hence $\mathrm{gr}(i)$ is surjective. Since $\mathrm{gr}(i)$ is an isomorphism, the claim follows. $\square$

Proposition 2.10.5. *The generators $\eta_{W,A,B}$ can be expressed in terms of the generators of the ring $R = H^*(M(\mathcal{A}); \mathbb{Z})$ as follows:*

$$\overline{\eta}_{W,A,B} = \sum_{C \subseteq A} (-1)^{|C|} 2^{|A \setminus C|} \frac{m(A \setminus C)}{m(A)} \eta_{L, A\setminus C, B \cup C}$$

where L is the unique connected component of $\cap_{i \in A \setminus C} H_i$ such that $W \subset L$.

Proof. Take any covering $f : U \to T$ that separates A, e. g. the one given in Proposition 2.8.3. From Definition 2.6.1, Definition 2.7.4, Lemma 2.9.10 and

Definition 2.8.7 it follows that

$$
\begin{aligned}
f^*(\overline{\eta}_{W,A,B}) &= (-1)^{\ell(A,B)} \frac{m(A \cup B)}{m(A)} \sum_{q \in f^{-1}(p)} \overline{\omega}_A^U(q) \psi_B^U = \\
&= (-1)^{\ell(A,B)} \frac{m(A \cup B)}{m(A)} \sum_{q \in f^{-1}(p)} \prod_{i \in A} (2\omega_i^U(q) - \psi_i^U) \psi_B^U = \\
&= \frac{m(A \cup B)}{m(A)} \sum_{\substack{q \in f^{-1}(p) \\ C \subseteq A}} (-1)^{\ell(A,B)+\ell(A \setminus C, C)+|C|} 2^{|A \setminus C|} \omega_{A \setminus C}^U(q) \psi_C^U \psi_B^U \\
&= f^* \left(\sum_{C \subseteq A} (-1)^{|C|} 2^{|A \setminus C|} \frac{m(A \setminus C)}{m(A)} \eta_{L, A \setminus C, B \cup C} \right)
\end{aligned}
$$

where L is the unique connected component of $\cap_{i \in A \setminus C} H_i$ containing W. The equality follows from the injectivity of the pull-back map. □

Example 2.10.6. For the arrangement $\mathcal{B}$ the previous relation gives the following presentations for the integral cohomology. We can take as generators the forms

$$
\begin{aligned}
\omega_0 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - x^3 y), & \psi_0 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(x^3 y), \\
\omega_1 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - y), & \psi_1 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(y), \\
\omega_2 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(1 - x), & \psi_2 &= \frac{1}{2\pi\mathbf{i}} \operatorname{dlog}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_{p,\{0,1\}} &= \frac{-1}{4\pi^2} \frac{x^2 y + x + 1}{(y-1)(x^3 y - 1)} \, \mathrm{d}\, x \, \mathrm{d}\, y, \\
\omega_{q,\{0,1\}} &= \frac{-\zeta_3}{4\pi^2} \frac{\zeta_3^2 x^2 y + \zeta_3 x + 1}{(y-1)(x^3 y - 1)} \, \mathrm{d}\, x \, \mathrm{d}\, y, \\
\omega_{r,\{0,1\}} &= \frac{-\zeta_3^2}{4\pi^2} \frac{\zeta_3 x^2 y + \zeta_3^2 x + 1}{(y-1)(x^3 y - 1)} \, \mathrm{d}\, x \, \mathrm{d}\, y
\end{aligned}
$$

and we have the equivalent for the relations obtained for the rational cohomology:

$$
\begin{aligned}
\omega_i \psi_i &= 0 \qquad \forall\, i, \\
\omega_0 \omega_1 &= \omega_{p,\{0,1\}} + \omega_{q,\{0,1\}} + \omega_{r,\{0,1\}}, \\
-\psi_0 + \psi_1 + 3\psi_2 &= 0, \\
\omega_{p,\{0,1\}} - \omega_0 \omega_2 + \omega_1 \omega_2 &= -\omega_0 \psi_2.
\end{aligned}
$$

As an example of application of Proposition 2.10.5 we can write the relation

$$\overline{\omega}_{p,\{0,1\}} = 4\omega_{p,\{0,1\}} - \frac{2}{3}(\psi_0\omega_1 + \omega_0\psi_1) + \frac{1}{3}\psi_0\psi_1,$$

that can be also checked directly using the formulas above and the formulas in Example 2.9.16.

2.11 Cohomology generated in degree one

In this section we will analyze the property of the cohomology ring of being generated in degree one. We will show that this property depends only on the poset of intersections and we will give a combinatorial criterion to determine when this property holds.

Lemma 2.11.1. *Let B be a graded algebra and $\{F_i\}_{i\in\mathbb{N}}$ an exhaustive filtration. Then B is generated in degree one if and only if $\operatorname{gr}_F B$ does.* □

Since $H^\bullet(T;\mathbb{Z}) \simeq H^\bullet(\mathbb{C}^*;\mathbb{Z})^{\otimes r}$ is generated in degree one, $H^\bullet(M;\mathbb{Z})$ is generated in degree one if and only if $\mathrm{A}^{0,\bullet}(\mathcal{A})$ is as well. A similar argument show that $H^\bullet(M;\mathbb{Q})$ is generated in degree one if and only if the same holds for $\mathrm{B}^{0,\bullet}(\mathcal{A})$.

Proposition 2.11.2. *Let $\mathcal{A}$ be a toric arrangement and $S \subseteq E$ be a independent set of cardinality k. The following formula holds in $\mathrm{A}(\mathcal{A})$:*

$$\prod_{s\in S} 1\otimes e_s = \sum_{\substack{W\in\mathcal{S}_k\\ W\in\bigcap S}} 1\otimes e_S,$$

where e_S belongs to $H^k(M(\mathcal{A}[W];\mathbb{Z})$.

Proof. The formula is an easy consequence of the definition of product in the algebra $\mathrm{A}(\mathcal{A})$. □

Remark 2.11.3. By Theorem 2.1.3, the elements

$$1\otimes e_S \in H^0(W;\mathbb{Z})\otimes H^k(M(\mathcal{A}[W];\mathbb{Z})$$

can be written uniquely as linear combinations of the $1\otimes e_{S'}$ with S' no broken circuit.

$$1\otimes e_S = \sum_{T \text{ nbc-set}} r^S_{W,T}(1\otimes e_T) \in H^0(W;\mathbb{Z})\otimes H^k(M(\mathcal{A}[W];\mathbb{Z})$$

The coefficients $r^S_{W,T}\in\mathbb{Z}$ are uniquely determined by the poset $\mathcal{S}_{\leq W}$.

Definition 2.11.4. Let R^k be the matrix whose columns are given by the vectors $(r^S_{W,T})_{W,T}$ for all S of cardinality k.

The matrix R^k are merely the coordinates of the element $1 \otimes e_S$ with respect to the basis $\{ 1 \otimes e_T \mid T \text{ nbc-set} \}$. Recall the numbers N_k of Theorem 2.2.16.

Theorem 2.11.5. *The algebra $H^\bullet(M(\mathcal{A}); \mathbb{Q})$ is generated in degree one if and only if all the matrices $\{R^k\}_{k \leq r}$ have rank equal to N_k.*

Proof. Fix the degree k. The cohomology algebra is generated in degree one if and only if $\mathrm{B}(\mathcal{A})$ is, this is equivalent to the fact that $\mathrm{B}^{0,\bullet}(\mathcal{A})$ is generated by $\mathrm{B}^{0,1}(\mathcal{A})$. This happens if and only if R^k has a right inverse; it has a right inverse if and only if R^k has rank equal to $\dim_{\mathbb{Q}} \mathrm{B}^{0,k}(\mathcal{A}) = N_k$. □

Theorem 2.11.6. *The algebra $H^\bullet(M(\mathcal{A}); \mathbb{Z})$ is generated in degree one if and only if all the matrices $\{R^k\}_{k \leq r}$ have N_k-th determinant divisor equal to one.*

Proof. As in the proof of Theorem 2.11.5, the algebra $\mathrm{A}(\mathcal{A})$ is generated in degree one if and only if R^k has a right inverse with integer coefficients. By the Smith normal form, this right inverse exists if and only if the N_k-th determinant divisor is equal to one. □

Chapter 3

Poset, Topology and Combinatorics

We show that the integral cohomology algebra of the complement of a toric arrangement is not determined by the poset of layers. Moreover, the rational cohomology algebra is not determined by the arithmetic matroid, however it is determined by the poset of layers.

Section 3.2 was appear in the first ArXiv version of [Pag19d], Sections 3.4 and 3.5 have appeared in the published version. Section 3.3 is part of [Pag19b].

3.1 Introduction

The study of the poset of layers of a toric arrangement is a new problem in this area of interest.

Definition 3.1.1 (Poset of layers)**.** The poset of layers of a toric arrangement $\mathcal{A}$ is the set of connected components of intersections of elements of $\mathcal{A}$, ordered by reverse inclusion.

Our interest was motivated by the attempt of finding an axiomatic definition of these posets, a cryptomorphism with arithmetic matroids, and the relation with the topology of toric arrangements. Very few is known about these posets: they intervals are geometric lattices. The particular case of graphical toric arrangements is studied in [AC17] and in [Bib16b] Bibby describes the poset of toric arrangements associated to root systems. In [DGP17] this description of the poset is used to prove the shellability of posets associated to root systems.

A natural question is the following:

Question 3.1.2. How the poset of layers varies among all representations of an arithmetic matroid?

A related poset is the *(arithmetic) independence poset* of a toric arrangement, defined in [Len17c, Definition 5], [Mar18, Section 2] (under the name of *poset of torsions*), and [DD18, Section 7] (under the name of *poset of independent sets*).

Definition 3.1.3 (Arithmetic independence poset)**.** The arithmetic independence poset of a toric arrangement $\mathcal{A}$ is the set of pairs (I, W) where $I \subseteq \mathcal{A}$ is an independent set and W is a connected component of $\bigcap I$. The order relation is defined as follows: $(I_1, W_1) \leq (I_2, W_2)$ if and only if $I_1 \subseteq I_2$ and $W_1 \supseteq W_2$.

D'Alì and Delucchi proved that both posets are homology Cohen-Macaulay over fields of all but a finite number of characteristics [DD18]. It was conjectured that the arithmetic independence poset is shellable, we confute the conjecture in Section 3.5. Notice that the non-arithmetic versions of these posets (the poset of flats and the independence poset of an ordinary matroid) are shellable, and therefore Cohen-Macaulay over fields of every characteristic.

Plan

In Section 3.2, we prove that, provided that the underline matroid is modular, all posets of layers are isomorphic. Representable modular matroids are direct sum of matroids of rank at most two (see [Oxl11, Proposition 6.9.1]). This decomposition holds as matroids, not as arithmetic matroids, thus we cannot apply this technique to our proof.

The study continues in Section 3.3 with the introduction of discriminantal toric arrangements in a, possibly disconnected, torus. We give the definition of poset isotopy (roughly speaking it consists in translating the hypertori without changing the poset of layers) and we prove that poset isotopy equivalent arrangements are diffeomorphic. Moreover we show that not all arrangements with the same characters and poset of layers are poset isotopy equivalent: this property depends on which connected component of the discriminantal arrangement they belong to.

In Section 3.4 we show that the integral cohomology algebra $H(M(\mathcal{A}), \mathbb{Z})$ of the complement of a (central) toric arrangement is not combinatorial, i.e. it does not depend only on the poset of layers (Theorem 3.4.2). This example gives a negative answer to Question 7.3.1 of [CD17].

In section 3.5, we show that arithmetic matroids and matroids over $\mathbb{Z}$ contain less information than the poset of layers. Indeed, we build two central toric arrangements with the same arithmetic matroid, the same matroid over $\mathbb{Z}$, but with non-isomorphic posets of layers (Theorem 3.5.2) and non-isomorphic cohomology algebra with rational coefficients. As consequence, there cannot exist a "cryptomorphism" between arithmetic matroids (respectively, matroids over $\mathbb{Z}$) and any class of posets such that – in the representable cases – the poset associated to the matroid coincides with the poset of layers of any representation.

3.2 Modular matroids and their posets

In this section we introduce a family of groups $\{K_\Lambda(S)\}_S$ related to a toric arrangements. We study their properties and we will use it to describe the poset of layers of a toric arrangement. This is the key point in order to show that the two arrangements of the Section 3.5 have different posets of layers.

The following class of modular matroids is quite small, though it contains free matroids and projective geometries. For a general reference on modular matroids, see [Oxl11, Section 6.9].

Definition 3.2.1. A pair of flats (S,T) is *modular* if the following equality holds

$$\mathrm{rk}(S) + \mathrm{rk}(T) = \mathrm{rk}(S \cap T) + \mathrm{rk}(S \cup T).$$

A flat S is *modular* if for all flats T the pair (S,T) is modular. A matroid is *modular* if all its flats are modular.

The groups $H_\Lambda(S)$

Let Λ be a lattice and $\{ v_e \}_{e \in E}$ be a finite set of elements of Λ. We define Γ_S, for $S \subseteq E$, to be the sub-lattice spanned by the vectors v_e, $e \in S$. The lattice Γ_E has a main role in the following discussion, therefore we address to it as Γ. Consider the functions m_Λ and m_Γ, from the subsets of E to the positive integer, defined by

$$m_\Lambda(S) \overset{\text{def}}{=} \left|\mathrm{Tor}\left({}^{\Lambda}\!/\!{}_{\Gamma_S}\right)\right| = \left|{}^{\mathrm{Rad}_\Lambda \Gamma_S}\!/\!{}_{\Gamma_S}\right| = \left|\mathrm{Ext}^1\left({}^{\Lambda}\!/\!{}_{\Gamma_S}, \mathbb{Z}\right)\right|$$

and by

$$m_\Gamma(S) \overset{\text{def}}{=} \left|\mathrm{Tor}\left({}^{\Gamma}\!/\!{}_{\Gamma_S}\right)\right| = \left|{}^{\mathrm{Rad}_\Gamma \Gamma_S}\!/\!{}_{\Gamma_S}\right| = \left|\mathrm{Ext}^1\left({}^{\Gamma}\!/\!{}_{\Gamma_S}, \mathbb{Z}\right)\right|.$$

This collection of vectors in Λ defines a matroid $([n], \mathrm{rk})$ that can be enriched by the multiplicity function m_Λ and the triple $([n], \mathrm{rk}, m_\Lambda)$ becomes a representable arithmetic matroid. Alternatively, we can enrich the matroid by the multiplicity function m_Γ and obtain the representable arithmetic matroid $([n], \mathrm{rk}, m_\Gamma)$.

Recall the definition of the radical of a sub-lattice $\Gamma' \subseteq \Lambda'$:

$$\mathrm{Rad}_{\Lambda'} \Gamma' \overset{\text{def}}{=} \{ v \in \Lambda' \mid \exists\, n \in \mathbb{N}_+ \text{ such that } nv \in \Gamma \}.$$

Consider for each subset S of $[n]$ the short exact sequence:

$$0 \to {}^{\mathrm{Rad}_\Gamma \Gamma_S}\!/\!{}_{\Gamma_S} \to {}^{\mathrm{Rad}_\Lambda \Gamma_S}\!/\!{}_{\Gamma_S} \to {}^{\mathrm{Rad}_\Lambda \Gamma_S}\!/\!{}_{\mathrm{Rad}_\Gamma \Gamma_S} \to 0.$$

We call the rightmost group $H_\Lambda(S) \overset{\text{def}}{=} \operatorname{Rad}_\Lambda \Gamma_S / \operatorname{Rad}_\Gamma \Gamma_S$. Fix two subsets $S \subset T$ of E and examine the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Rad}_\Gamma \Gamma_S / \Gamma_S & \longrightarrow & \operatorname{Rad}_\Lambda \Gamma_S / \Gamma_S & \longrightarrow & H_\Lambda(S) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow i_{S,T} & & \\ 0 & \longrightarrow & \operatorname{Rad}_\Gamma \Gamma_T / \Gamma_T & \longrightarrow & \operatorname{Rad}_\Lambda \Gamma_T / \Gamma_T & \longrightarrow & H_\Lambda(T) & \longrightarrow & 0 \end{array} \tag{3.1}$$

Since $\operatorname{Rad}_\Gamma \Gamma_T \cap \operatorname{Rad}_\Lambda \Gamma_S = \operatorname{Rad}_\Gamma \Gamma_S$, the map $i_{S,T}$ is injective. If $\operatorname{rk}(S) = \operatorname{rk}(T)$, then $i_{S,T}$ is the identity map because in this case we have the equalities $\operatorname{Rad}_\Gamma \Gamma_T = \operatorname{Rad}_\Gamma \Gamma_S$ and $\operatorname{Rad}_\Lambda \Gamma_T = \operatorname{Rad}_\Lambda \Gamma_S$. Let G be the torsion of the group Λ/Γ and notice that $H_\Lambda(E) = G$, thus the groups $H_\Lambda(S)$, $S \subseteq E$, are subgroups of G.

Definition 3.2.2. The *layer group* $\operatorname{LG}_\Lambda(S)$ of a representation Λ is the group

$$\operatorname{LG}_\Lambda(S) \overset{\text{def}}{=} \operatorname{Ext}^1\left(\operatorname{Rad}_\Lambda \Gamma_S / \Gamma_S, \mathbb{Z}\right) = \operatorname{Ext}^1\left(\Lambda / \Gamma_S, \mathbb{Z}\right).$$

We also define the *relative layer group*

$$\operatorname{LG}_\Gamma(S) \overset{\text{def}}{=} \operatorname{Ext}^1\left(\operatorname{Rad}_\Gamma \Gamma_S / \Gamma_S, \mathbb{Z}\right) = \operatorname{Ext}^1\left(\Gamma / \Gamma_S, \mathbb{Z}\right).$$

For any $S \subset T$ there is a natural map $\pi_{S,T} : \operatorname{LG}_\Lambda(T) \to \operatorname{LG}_\Lambda(S)$ which is injective if $\operatorname{rk}(S) = \operatorname{rk}(T)$ and surjective if $|T| - \operatorname{rk}(T) = |S| - \operatorname{rk}(S)$. We have a bijection between the connected components of $\bigcap_{s \in S} H_s$ and the elements of $\operatorname{LG}_\Lambda(S)$ since both has cardinality $m(S)$. Moreover, the group $H_\Lambda(S)$ has cardinality $\frac{m_\Lambda(S)}{m_\Gamma(S)}$.

The groups $\operatorname{LG}_\Lambda(S)$, together with the natural maps between them, determine the poset of layers of the central toric arrangement described by $v_i \in \Lambda$, as shown in [Len17a]. The following lemma holds for a pair of modular flats (see Definition 3.2.1).

Lemma 3.2.3. *Let S, T be a pair of modular flats. Then the following equality holds*

$$H_\Lambda(S) \cap H_\Lambda(T) = H_\Lambda(S \cap T).$$

Proof. The equality $\operatorname{Rad}_\Lambda \Gamma_S \cap \operatorname{Rad}_\Lambda \Gamma_T = \operatorname{Rad}_\Lambda \Gamma_S \cap \Gamma_T$ always holds. The modularity hypothesis implies that $\operatorname{Rad}_\Lambda \Gamma_S \cap \Gamma_T = \operatorname{Rad}_\Lambda \Gamma_{S \cap T}$, thus

$$\begin{aligned} H_\Lambda(S) \cap H_\Lambda(T) &= \operatorname{Rad}_\Lambda \Gamma_S \cap \operatorname{Rad}_\Lambda \Gamma_T / \operatorname{Rad}_\Lambda \Gamma_S \cap \operatorname{Rad}_\Lambda \Gamma_T \cap \Gamma = \\ &= \operatorname{Rad}_\Lambda \Gamma_{S \cap T} / \operatorname{Rad}_\Lambda \Gamma_{S \cap T} \cap \Gamma = H_\Lambda(S \cap T) \end{aligned}$$

□

The groups $K_\Lambda(S)$

For any subset $S \subseteq E$ we define the group $K_\Lambda(S) = \mathrm{Ext}^1(H_\Lambda(S), \mathbb{Z})$. For $S \subset T$ the dual of the diagram (3.1) is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\Lambda(T) & \longrightarrow & \mathrm{LG}_\Lambda(T) & \longrightarrow & \mathrm{LG}_\Gamma(T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_{S,T}} & & \downarrow{\scriptstyle \pi_{S,T}} & & \downarrow{\scriptstyle \gamma_{S,T}} & & \\ 0 & \longrightarrow & K_\Lambda(S) & \longrightarrow & \mathrm{LG}_\Lambda(S) & \longrightarrow & \mathrm{LG}_\Gamma(S) & \longrightarrow & 0 \end{array} \tag{3.2}$$

whose rows are exact. The map $p_{S,T}$ is always surjective and is an isomorphism if $\mathrm{rk}(S) = \mathrm{rk}(T)$.

Lemma 3.2.4. *Let T and S be two flats of a modular matroid (E, rk) and Λ be a representation of an arithmetic matroid (E, rk, m). The following diagram is a pushout diagram.*

$$\begin{array}{ccc} K_\Lambda(S \vee T) & \longrightarrow & K_\Lambda(S) \\ \downarrow & \ulcorner & \downarrow \\ K_\Lambda(T) & \longrightarrow & K_\Lambda(S \wedge T) \end{array}$$

Proof. By Lemma 3.2.3 the following diagram is a pullback diagram.

$$\begin{array}{ccc} H_\Lambda(S \wedge T) & \longrightarrow & H_\Lambda(S) \\ \downarrow & \lrcorner & \downarrow \\ H_\Lambda(T) & \longrightarrow & H_\Lambda(S \vee T) \end{array}$$

Applying the functor $\mathrm{Ext}^1(\bullet, \mathbb{Z})$ we obtain the claimed diagram. □

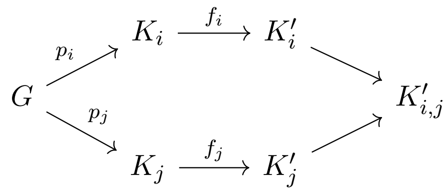
Let $p_i : G \twoheadrightarrow K_i$, for $i = 1, 2$, be two quotients of G by the subgroups L_i. We denote the pushout of p_i and p_j with

$$K_{i,j} \overset{\mathrm{def}}{=} K_i \sqcup K_j \big/ p_i(z) \sim p_j(z) = G \big/ L_i + L_j,$$

together with the two natural surjections $s_i : K_i \twoheadrightarrow K_{i,j}$ and $s_j : K_j \twoheadrightarrow K_{i,j}$. The pullback of s_i and s_j is

$$K_i \times_{K_{i,j}} K_j \overset{\mathrm{def}}{=} \{(x, y) \in K_i \times K_j \mid s_i(x) = s_j(y)\} = G \big/ L_i \cap L_j.$$

Lemma 3.2.5. *Let G and G' be two finite abelian groups of the same cardinality, and for $i \leq n$ let $p_i : G \twoheadrightarrow K_i$ and $p'_i : G' \twoheadrightarrow K'_i$ be quotients of G and G'. We denote with $K_{i,j}$ (and with $K'_{i,j}$) the pushout of p_i and p_j (respectively,* *of p'_i and p'_j). Suppose that there exist bijections $f_i\colon K_i \xrightarrow{1:1} K'_i$, for $i \leq n$, such that for every i, j the diagram*



(3.3)

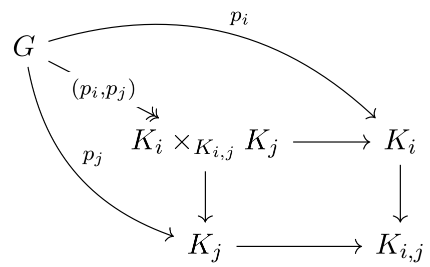
commutes and that the induced map $K_{i,j} \to K'_{i,j}$ is an isomorphism. Then there exist a bijection $f\colon G \to G'$ such that $p'_i \circ f = f_i \circ p_i$ for all i.

Proof. Suppose first $n = 1$, for all $x \in K_1$ the sets $p_1^{-1}(x)$ and $p_1'^{-1}(f_1(x))$ has the same cardinality. We can choose a bijection $f_x : p_1^{-1}(x) \to p_1'^{-1}(f_1(x))$ for all $x \in K_1$ and define $f(y) = f_{p_1(y)}(y)$. This function f is the sought function.

For $n > 1$, we want to reduce to the case $n - 1$. Let us fix $i \neq j$ and consider the pullback $K_i \times_{K_{i,j}} K_j$ of the two maps $K_i \to K_{i,j}$ and $K_j \to K_{i,j}$. We want to found a bijection

$$f_{i,j} : K_i \times_{K_{i,j}} K_j \xrightarrow{1:1} K'_i \times_{K'_{i,j}} K'_j.$$

Consider the pullback diagram



and observe that by diagram (3.3) the two map from $K_i \times_{K_{i,j}} K_j$ to $K'_{i,j}$ induced by f_i and f_j coincides. Notice that $|K_i \times_{K_{i,j}} K_j| = \frac{|K_i||K_j|}{|K_{i,j}|}$ and so $|K_i \times_{K_{i,j}} K_j| = |K'_i \times_{K'_{i,j}} K'_j|$. Since the pullbacks in the category of $\mathbb{Z}$-modules and in the category of sets coincide, we have a well defined bijection $f_{i,j} : K_i \times_{K_{i,j}} K_j \xrightarrow{1:1} K'_i \times_{K'_{i,j}} K'_j$. The bunch of $n-1$ maps $\{f_k\}_{k \neq i,j} \cup \{f_{i,j}\}$ satisfy the hypothesis, so by induction we construct the map $f\colon G \to G'$. □

Lemma 3.2.6. *Let Λ and Λ' be two representations of the same torsion-free arithmetic matroid whose underlying matroid is modular. Then the functor K_Λ and $K_{\Lambda'}$ are equivalent, i.e. there exist a bijection $f\colon K_\Lambda(E) \to K_{\Lambda'}(E)$ such that for every flat $S \subset E$ the map f induces a bijection $f_S\colon K_\Lambda(S) \to K_{\Lambda'}(S)$.*

Proof. We define f by induction on the poset of flats of the underlying matroid. Since $K_\Lambda(\emptyset) = 0 = K_{\Lambda'}(\emptyset)$, the base case is done. Suppose that we have defined $f_S : K_\Lambda(S) \to K_{\Lambda'}(S)$ for every flat S of rank less than k, compatibly with the restrictions. For each flat T of rank k consider the set $\{S_1, \dots, S_m\}$ of flats of rank $k-1$ contained in T. We apply Lemma 3.2.5 to $G = K_\Lambda(T)$, $G' = K_{\Lambda'}(T)$, $K_i = K_\Lambda(S_i)$ and $K'_i = K_{\Lambda'}(S_i)$. Lemma 3.2.4 implies $K_{i,j} = K_\Lambda(S_i \wedge S_j)$ and $K'_{i,j} = K_{\Lambda'}(S_i \wedge S_j)$, so the compatibility between f_{S_i}, f_{S_j} and $f_{S_i \wedge S_j}$ ensure the condition (3.3). Thus we have a bijection $f_T : K_\Lambda(T) \to K_{\Lambda'}(T)$ compatible with the restrictions. We repeat this procedure for every flat T of rank k and inductively for every flats. □

Theorem 3.2.7. *Let (E, rk, m) be a torsion-free arithmetic matroid such that the underlying matroid (E, rk) is modular. Then the posets of layers of all representations of the arithmetic matroid are isomorphic.*

Proof. An explicit description of the poset of layers of a central toric arrangement is given in [Len17a] in terms of the sets $\{\mathrm{LG}(S)\}_{S \subseteq E}$ and the maps $\pi_{S,T}$ between them. Any two representation Λ and Λ' contain the same representation Γ of $(E, \mathrm{rk}, \overline{m})$, as shown in Section 1.9. Observe that $\mathrm{LG}_\Lambda(S) = \mathrm{LG}_\Gamma(S) \times K_\Lambda(S)$ as a set and $\pi_{S,T} = (\gamma_{S,T}, p_{S,T})$ as map between sets. Analogously, $\mathrm{LG}_{\Lambda'}(S) = \mathrm{LG}_\Gamma(S) \times K_{\Lambda'}(S)$ and $\pi'_{S,T} = (\gamma_{S,T}, p'_{S,T})$. We observe that $K_\Lambda(S) = K_\Lambda(\overline{S})$, where $\overline{S}$ is the minimal flat containing S. Consider the bijections f_S of Lemma 3.2.6, the maps

$$\mathrm{Id} \times f_{\overline{S}} : \mathrm{LG}_\Gamma(S) \times K_\Lambda(\overline{S}) \to \mathrm{LG}_\Gamma(S) \times K_{\Lambda'}(\overline{S})$$

are compatible with $(\gamma_{S,T}, p_{S,T})$ since $f_S \circ p_{S,T} = p'_{S,T} f_T$ for all flats S and T. Therefore, the two poset of layers are isomorphic. □

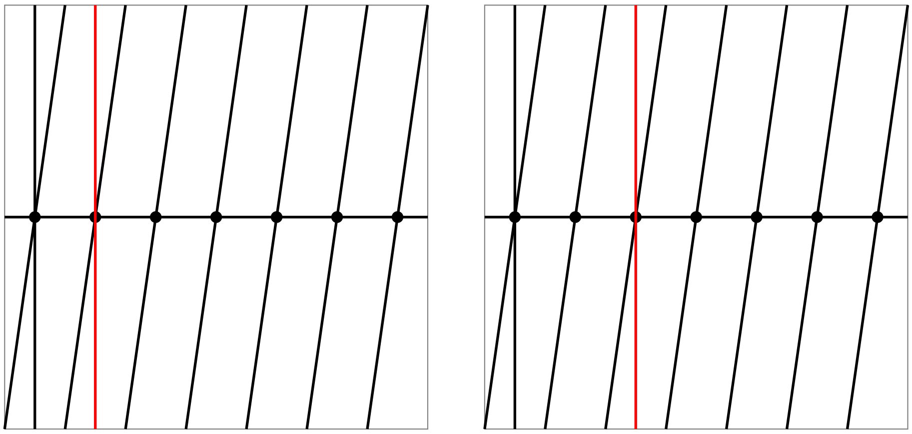
3.3 Discriminantal toric arrangements

We want to study the continuous deformations of a toric arrangement. Since the characters group Λ of a toric arrangement is a discrete set, no deformation can change the set of characters in $\mathcal{A}$, however it is possible that some hypertorus in the arrangement are translated.

A particular nice type of deformation is the *poset isotopy*, which is by definition a deformation that does not change the poset of layers. Two toric arrangements are said to be *poset isotopy equivalent* if there exists a poset isotopy that deforms one into the other. This notion has already been defined in the context of hyperplane arrangements, see [OT92, Definition 5.27].

There are two natural questions:

1. Are two poset isotopy equivalent toric arrangements diffeomorphic? Are they homeomorphic? Are they homotopy equivalent?



(a) $M(\mathcal{A})$ (b) $M(\mathcal{A}')$

Figure 3.1: Representations of the two arrangements on the compact subtorus. The red subtori are the hypertori H_4 and H'_4.

2. Are two toric arrangements with isomorphic poset of layers and same characters poset isotopy equivalent?

In this section we will give a negative answer to the second question and a positive one to the first question.

The following example was suggested by Filippo Callegaro and it is a counterexample to the second question.

Example 3.3.1. Let $\mathcal{A}$ and $\mathcal{A}'$ be the following two toric arrangements in $T^2 = \hom(\mathbb{Z}^2, \mathbb{C}^*)$:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{pmatrix}, (1, 1, 1, \zeta_7) \right\},$$
$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{pmatrix}, (1, 1, 1, \zeta_7^2) \right\},$$

where ζ is a primitive 7^{th}-root of unity.

These arrangements have the same poset of intersections $\mathcal{S}$ and the same characters χ_i for $i = 1, \dots, 4$, therefore by Theorem 2.10.4 their integral cohomology groups are isomorphic.

Definition 3.3.2. A layer $W \in \mathcal{S}$ is *generic* if there exist exactly $\operatorname{rk} W$ hypertori containing W.

A poset is *generic* if all layers in $\mathcal{S}$ are generic.

A poset $\mathcal{S}$ is *nearly generic* if there exists a layer $\overline{W} \in \mathcal{S}$ such that all layers W not containing $\overline{W}$ are generic.

Clearly, generic arrangements are nearly generic. We fix n characters χ_i in Λ and study the poset of layers of $\mathcal{A} = \{(\chi_i, b_i)\}_{i=1,\dots,n}$, where b_i ranges in $\mathbb{C}^*$.

Theorem 3.3.3. *The subset of* $(\mathbb{C}^*)^n$ *given by*

$$L(\mathcal{S}) = \{(b_i)_i \in (\mathbb{C}^*)^n \mid \mathcal{S}(\{\, \chi_i, b_i \,\}_{i=1,\dots,n}) \simeq \mathcal{S}\}$$

is a smooth locally closed subset of $(\mathbb{C}^*)^n$*. Moreover, if* $\mathcal{S}$ *is nearly generic then* $L(\mathcal{S})$ *is connected or empty.*

For each $\underline{b} \in (\mathbb{C}^*)^n$ define the hypertori $H_i = H_i(\underline{b}) = \mathcal{V}_T(1 - b_i\chi_i) \subset T$ for $i = 1, \dots, n$.

Lemma 3.3.4. *For all subset* $\underline{j}$ *of* $(1, \dots, n)$*, the set*

$$B_{\underline{j}} = \{\underline{b} \in (\mathbb{C}^*)^n \mid H_{j_1}(\underline{b}) \cap \cdots \cap H_{j_k}(\underline{b}) \neq \emptyset\}$$

is a connected torus in $(\mathbb{C}^*)^n$*. Moreover, the intersection of* $H_{j_1}, \dots, H_{j_k}$ *in* T *is independent, up to translation, of the point* $\underline{b} \in B_{\underline{j}}$*.*

Proof. Without loss of generality we suppose $j_i = i$, for $i = 1, \dots, k$ and study the subtorus Y of

$$(\mathbb{C}^*)^{n+r} = \operatorname{Spec} \mathbb{C}[b_1^{\pm 1}, \dots, b_n^{\pm 1}, z_1^{\pm 1}, \dots, z_r^{\pm 1}]$$

given by the equations $I = (1 - b_i\chi_i)_{i=1,\dots,k}$. The rings morphism

$$\mathbb{C}[b_1^{\pm 1}, \dots, b_n^{\pm 1}] \to \mathbb{C}[b_1^{\pm 1}, \dots, b_n^{\pm 1}, z_1^{\pm 1}, \dots, z_r^{\pm 1}] \big/ I$$

induces a projection between the associated tori:

$$\begin{array}{rccc} p\colon & Y & \longrightarrow & (\mathbb{C}^*)^n \\ & (\underline{b}, \underline{z}) & \longmapsto & \underline{b} \end{array}$$

The image of the map p is the closed subset B described by the contracted ideal

$$I^c = (1 - b_i\chi_i)_{i=1,\dots,k} \cap \mathbb{C}[b_1^{\pm 1}, \dots, b_n^{\pm 1}]$$

Given that the intersection of $H_1(\underline{b}), \dots, H_k(\underline{b})$ is non-empty if and only if $\underline{b}$ is in the image of p, B coincides with $B_{\underline{j}}$. The elimination ideal of a binomial ideal is still binomial: this is a standard fact about binomial ideals and Gröbner bases (for a proof see [ES96, Corollary 1.3]). In our case, since I is a binomial ideal, I^c is binomial and the closed subset B is a torus. Moreover, B is connected since it is the image of the connected torus Y under the map p.

The fibers of the map p are either empty or a torus. In the latter case the torus has codimension $\operatorname{rk}([k])$ and $m([k])$ connected components. The fibre of a point in B, seen as torus in T, is obtained from any other non-empty fibre by translations. □

Recall that χ_i, $i = 1, \dots, n$, are characters of a r-dimensional torus.

Remark 3.3.5. Let $\underline{j}$ be a sublist of $(1, \dots, n)$ of cardinality k. The torus Y is of dimension $n + r - k$, so the set $B_{\underline{j}}$ is of dimension $n + r - k - \operatorname{rk} \underline{j}$. In particular, $B_{\underline{j}}$ is a hypertorus in $(\mathbb{C}^*)^n$ if and only if $\underline{j}$ is a circuit (i.e. $\operatorname{rk} \underline{j} = r - k + 1$).

Definition 3.3.6. The centred toric arrangement $D(\chi_1, \chi_2, \dots, \chi_n)$ given by the sets $B_{\underline{j}}$,for all circuits $\underline{j}$, in $T' = (\mathbb{C}^*)^n$ is called *discriminantal toric arrangement* associated with the n characters χ_i, $i = 1, \dots, n$.

Proof of Theorem 3.3.3. If $\mathcal{S}$ is not $\mathcal{S}(\{(b_i, \chi_i)\}_{i=1,\dots,n})$ for some b_i, there is nothing to prove. Otherwise, for each layer W of $\mathcal{S}$, let $\underline{j}(W)$ be the ordered set:

$$\underline{j}(W) = \{i \mid A_i \leq W\}$$

where A_i is the atom of $\mathcal{S}$ associated with i.

The condition $\mathcal{S}(\{(a_i, \chi_i)\}_{i=1,\dots,n}) = \mathcal{S}$ is equivalent to

$$\forall\, \underline{a} \in (\mathbb{C}^*)^n\, \exists\, W \in \mathcal{S} \left(\underline{a} \in B_{\underline{j}} \Leftrightarrow \underline{j} \subseteq \underline{j}(W)\right)$$

By Lemma 3.3.4, the set

$$L(\mathcal{S}) = \bigcap_{W \in \mathcal{S}_r} B_{\underline{j}(W)} \setminus \bigcup_{\substack{\underline{j} \not\subseteq \underline{j}(W) \\ \forall W \in \mathcal{S}_r}} B_{\underline{j}}$$

is locally closed in $(\mathbb{C}^*)^n$ and open in the torus $\bigcap_{W \in \mathcal{S}_r} B_{\underline{j}(W)}$, hence it is also smooth.

If W is a generic layer, then we have that $B_{\underline{j}} = (\mathbb{C}^*)^n$ for all sets $\underline{j}$ included in $\underline{j}(W)$. Let $\mathcal{S}$ be a nearly generic poset and $\overline{W}$ non-generic maximal layer; then the equality $L(\mathcal{S}) = B_{\underline{j}(\overline{W})} \setminus \bigcup_{\text{some } \underline{j}} B_{\underline{j}}$ holds. If $L(\mathcal{S})$ is nonempty, it is an irreducible set; it is connected in the Zariski topology and thus also in the euclidean one. □

Example 3.3.7 (continuation of Example 3.3.1)**.** Let $\mathcal{S}$ be the poset associated with $\mathcal{A}$ or, equivalently, to $\mathcal{A}'$. The discriminantal toric arrangement associated with $\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{smallmatrix}\right)$ is the centred toric arrangement in $(\mathbb{C}^*)^4 = \operatorname{Spec} \mathbb{C}[\{a_i^{\pm 1}\}_{i \leq 4}]$ given by the subtori

$$\begin{aligned} B_{3,4} &= \mathcal{V}(1 - a_3 a_4^{-1}) \\ B_{1,2,3} &= \mathcal{V}(1 - a_1 a_2^{-1} a_3^7) \\ B_{1,2,4} &= \mathcal{V}(1 - a_1 a_2^{-1} a_4^7) \\ B_{1,2,3,4} &= \mathcal{V}(1 - a_1 a_2^{-1} a_3^7, 1 - a_3 a_4^{-1}). \end{aligned}$$

All the others $B_{\underline{j}}$ are equal to $(\mathbb{C}^*)^4$ and consequently the subset $L(\mathcal{S})$ is:

$$L(\mathcal{S}) = B_{1,2,3} \cap B_{1,2,4} \setminus B_{3,4} = \{\underline{a} \mid a_1^{-1} a_2 = a_3^7 = a_4^7,\, a_3 \neq a_4\}$$

Hence the set $L(\mathcal{S})$ is the disjoint union of six connected 2-dimensional tori. The two arrangements $\mathcal{A}$ and $\mathcal{A}'$ belong to different connected components so they cannot be deformed one into the other by means of translations. We have thus shown that $\mathcal{A}$ and $\mathcal{A}'$ have the same characters and the same poset of intersections but are not poset isotopy equivalent, see Definition 3.3.8.

Definition 3.3.8. A *deformation* of a toric arrangement is a collection of n hypersurfaces H_i in $(\mathbb{C}^*)^r \times B$ (where B is an algebraic variety over $\mathbb{C}$) such that for every point $b \in B$ the subset $H_i \cap \mathrm{pr}_2^{-1}(b)$ is a hypertorus in $(\mathbb{C}^*)^r \times \{b\}$. We call M_b the open set $(\mathbb{C}^*)^r \times \{b\} \setminus \bigcup_{i \leq n} H_i$.

A deformation is said to be a *poset-preserve deformations* if the poset of layers of M_b does not depend on the point $b \in B$.

We said that two toric arrangements $M_1, M_2 \subset T$ are *poset isotopic equivalent* if there exists a layers-preserve deformation in $T \times B$, B connected, such that the pair (T, M_1) (and (T, M_2)) is isomorphic to a fiber $(T \times \{b_1\}, M_{b_1})$ (and, respectively, to $(T \times \{b_2\}, M_{b_2})$).

The next result is an analogous to the one on hyperplane arrangements of Randell [Ran89].

Theorem 3.3.9. *If the toric arrangements $M, M' \subset T$ are poset isotopic equivalent then M and M' are diffeomorphic.*

Since the group character Λ of a torus T is a discrete set, two poset isotopy equivalent arrangements M and M' are described by the same characters $\chi_1, \dots, \chi_n$. Let $\mathcal{S}$ be the poset of layers of M or equivalently of M', the base B of the deformation can be chosen to be a connected component of $L(\mathcal{S})$. Call U the closure of B in $(\mathbb{C}^*)^n$, clearly U is a connected torus (possibly translated) of dimension m.

Consider the torus $T \times U$ with coordinates (z_i, a_j) as before, with hypertori defined by $1 - a_j\chi_j(z) = 0$ and call $\tilde{M}$ the toric arrangement $(T \times U) \setminus \bigcup_{i=1,\dots,n} \mathcal{V}(1 - a_j\chi_j(z))$. Choose a component-wise embedding of $T \times U$ in $(\mathbb{P}^1)^r \times (\mathbb{P}^1)^m$. The main result of [DG18b] is that there exists a smooth project variety $\mathcal{W}$ obtained from $(\mathbb{P}^1)^{r+m}$ by means of a suitable sequence of blow-ups that contains $\tilde{M}$. Call $\mathcal{W}_B$ the inverse image of $(\mathbb{P}^1)^r \times B$ through the natural map $\mathcal{W} \to (\mathbb{P}^1)^{r+m}$.

Let $w\colon \mathcal{W}_B \to B$ be the composition of the natural map $\mathcal{W}_B \to (\mathbb{P}^1)^r \times B$ with the projection onto the second component.

Lemma 3.3.10. *The map $w\colon \mathcal{W}_B \to B$ is a projective smooth map.*

Proof. The morphism $\mathcal{W}_B \to (\mathbb{P}^1)^r \times B$ is projective and smooth by base change of a blow-up map. The projection $(\mathbb{P}^1)^r \times B \to B$ is projective and smooth. So the composition w is smooth and projective. □

Lemma 3.3.11. *The variety $\mathcal{W}_B$ admits a Whitney stratification whose open stratum is naturally isomorphic to $\tilde{M}$.*

Proof. The complement $\mathcal{K}$ of $\tilde{M}$ in $\mathcal{W}_B$ is a union of some smooth divisors $\mathcal{K}_i$, $i = 1, \dots, k$. Moreover $\mathcal{K}$ is normal crossing (see [DG18b]). Consider the stratification given by the closed sets $\mathcal{K}_i$, $i = 1, \dots, k$. Each stratum has smooth closure in $\mathcal{W}_B$, therefore by [Ran89, Lemma] this is a Whitney stratification. □

To prove of Theorem 3.3.9 we follow the ideas of [Ran89].

Proof of Theorem 3.3.9. We apply the Thom's isotopy theorem ([GM88, Section I Theorem 1.5]) to the map w. Consider the Whitney stratification on $\mathcal{W}_B$ of Lemma 3.3.11. Since the poset of layers is the same for every point $b \in B$, the restriction of the map $w\colon \mathcal{W}_B \to B$ to every stratum is a submersion. Hence for all $b \in B$, there exists a smooth stratum-preserving map h such that the diagram below commutes:

$$\begin{array}{ccc} \mathcal{W}_B & \xrightarrow{\quad h \quad} & B \times w^{-1}(b) \\ & {\scriptstyle w}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & B & \end{array}$$

Thus for all $b' \in B$, M_b and $M_{b'}$ are diffeomorphic. □

3.4 A first example

The example that we will expose in this section is a generalization of [CD17, Example 7.3.2]. Using Theorem 2.6.10 and Theorem 2.10.4 we compute the cohomology algebra

Let $A = \bigoplus_{n\in\mathbb{N}} A^n$ be a graded-commutative R-algebra and consider for each $\alpha \in A^1$ the left multiplication $\delta^i_\alpha : A^i \to A^{i+1}$. The pair $(A; \delta_\alpha)$ is a complex for each $\alpha \in A^1$.

Definition 3.4.1. The k^{th} *resonance variety* of A is

$$\mathcal{R}^k(A) \stackrel{\text{def}}{=} \{\alpha \in A^1 \mid H^k(A, \delta_\alpha) \neq 0\}.$$

The k^{th} resonance varieties (with coefficients in the domain R) for a toric arrangement $\mathcal{A}$ is

$$\mathcal{R}^k(\mathcal{A}; R) \stackrel{\text{def}}{=} \mathcal{R}^k(H^\bullet(M(\mathcal{A}); R)).$$

We will use only the first resonance variety $\mathcal{R}^1(\mathcal{A}, R)$ of a toric arrangement $\mathcal{A}$, where R is the ring $\mathbb{Z}$ or $\mathbb{Q}$.

In this section we set $T = (\mathbb{C}^*)^2$. Consider the arrangements $\mathcal{A}$ and $\mathcal{A}^a_n$ in T defined respectively by the matrices

$$N = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } N^a_n = \begin{pmatrix} 1 & a & a+1 \\ 0 & n & n \end{pmatrix},$$

where n is a positive integer and $a, a+1$ are relatively prime to n.

We use Theorem 2.2.16 to calculate the Poincaré polynomials of these arrangements. The Poincaré polynomial of $M(\mathcal{A})$ is $1 + 5t + 6t^2$ and that of $M(\mathcal{A}_n^a)$ is $1 + 5t + (2n+4)t^2$. The arithmetic Tutte polynomial of the arithmetic matroid $([3], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ is $x^2 + x + y$, the one of $([3], \mathrm{rk}_{\mathcal{A}_n^a}, m_{\mathcal{A}_n^a})$ is $x^2 + x + ny + 2n - 2$. By the way, we have $\mathrm{rk}_{\mathcal{A}} = \mathrm{rk}_{\mathcal{A}_n^a}$.

Theorem 3.4.2. *Let $n > 5$ be a natural number relatively prime to 6, the arrangements $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ have isomorphic posets of layers but non isomorphic cohomology algebras with integer coefficients.*

From Theorem 2.9.13 the two arrangements $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ have isomorphic cohomology algebras with rational coefficients. We need a couple of lemmas to prove Theorem 3.4.2.

Lemma 3.4.3. *Let A be a graded-commutative algebra over $\mathbb{Q}$. The first resonance variety $\mathcal{R}^1(A)$ is a union (possibly infinite) of planes in A^1.*

Proof. If $\alpha \in \mathcal{R}^1(A)$, then there exists $\beta \in A^1 \setminus \alpha\mathbb{Q}$ such that $\alpha\beta = 0$. Thus, the plane generated by α and β is contained in $\mathcal{R}^1(A)$. We obtain the desired result from the arbitrariness of $\alpha \in \mathcal{R}^1(A)$. □

We use coordinates t_1, t_2 on T and we apply Theorem 2.6.10. The cohomology ring of $M(\mathcal{A})$ is generated by the closed forms

$$\begin{aligned} \omega_1 &= \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(1 - t_1), \\ \omega_2 &= \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(1 - t_2), \\ \omega_3 &= \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(1 - t_1 t_2), \end{aligned}$$

associated with the hypertori H_1, H_2, H_3 respectively, together with the forms $\psi_1 = \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(t_1)$ and $\psi_2 = \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log(t_2)$ (ψ_3 is equal to $\psi_1 + \psi_2$). The relations are:

$$\begin{aligned} \omega_1\omega_2 - \omega_1\omega_3 + \omega_2\omega_3 - \omega_3\psi_1 &= 0, \\ \omega_1\psi_1 &= 0, \\ \omega_2\psi_2 &= 0, \\ \omega_3\psi_1 + \omega_3\psi_2 &= 0. \end{aligned} \tag{3.4}$$

Lemma 3.4.4. *The first resonance variety $\mathcal{R}^1(\mathcal{A}; \mathbb{Q})$ of the complement of $\mathcal{A}$ is the union of the following five planes of $H^1(M(\mathcal{A}); \mathbb{Q})$;*

$$\begin{aligned} P_1 &= \langle \omega_1, \psi_1 \rangle, \\ P_2 &= \langle \omega_2, \psi_2 \rangle, \\ P_3 &= \langle \omega_3, \psi_1 + \psi_2 \rangle, \\ P_4 &= \langle \omega_1 - \omega_3, \omega_1 - \omega_2 - \psi_1 \rangle, \\ P_5 &= \langle \omega_2 - \omega_3, \omega_1 - \omega_2 + \psi_2 \rangle. \end{aligned}$$

Proof. The multiplication map $f : H^1(M(\mathcal{A})) \otimes H^1(M(\mathcal{A})) \to H^2(M(\mathcal{A}))$ is surjective and factors through $\wedge^2 H^1(M(\mathcal{A}))$. The kernel of

$$\tilde{f} \colon \wedge^2 H^1(M(\mathcal{A})) \longrightarrow H^2(M(\mathcal{A}))$$
$$\alpha \wedge \beta \longmapsto \alpha\beta$$

has dimension $4 = \binom{5}{2} - 6$, hence $L \stackrel{\text{def}}{=} \mathbb{P}(\ker \tilde{f}) \simeq \mathbb{P}^3$ is a linear subspace of $\mathbb{P}(\wedge^2 H^1(M(\mathcal{A}))) \simeq \mathbb{P}^9$.

An element $\alpha \in H^1(M(\mathcal{A}))$ belongs to the first resonance varieties if and only if there exists $\beta \in H^1(M(\mathcal{A}))$ such that $\alpha\beta = 0$ in $H^2(M(\mathcal{A}))$ and $\beta \notin \mathbb{C}\alpha$. This implies that $\alpha\wedge\beta$ is in $\ker \tilde{f}$ and so $[\alpha\wedge\beta]$ is in the linear subspace L. Viceversa if $[\gamma]$ belongs to L and is a decomposable tensor (i.e. belongs to $\operatorname{gr}(2, H^1(M(\mathcal{A}))))$ then $[\gamma] = [\alpha \wedge \beta]$ and the plane $\langle \alpha, \beta \rangle$ is contained in the first resonance variety.

Now we prove that the intersection $L \cap \operatorname{gr}(2, H^1(M(\mathcal{A})))$ is the disjoint union of five points. The relations in eq. (3.4) implies the following factorized equations

$$(\omega_1 - \omega_3)(\omega_1 - \omega_2 - \psi_1) = 0,$$
$$(\omega_2 - \omega_3)(\omega_1 - \omega_2 + \psi_2) = 0.$$

These equations ensure that the five different points $[P_i]$, $i = 1, \dots, 5$ lie in this intersection. The dimension of the Grassmannian $\operatorname{gr}(k, V)$ is $k(\dim V - k)$, which in our case is equal to 6. Moreover, when $k = 2$ its degree coincides with the Catalan number $C_{\dim V - 2}$. The formula for the degree of the Plücker embedding of the Grassmannian is due to Schubert in 1886, we refer to [GW11]. Hence $\operatorname{gr}(2, H^1(M(\mathcal{A})))$ has degree 5 and every $\mathbb{P}^3 \subset \mathbb{P}^9$ intersects $\operatorname{gr}(2, 5)$ scheme-theoretically in five points (this is the general case) or in a sub-variety of positive dimension.

We exclude the latter case by explicit computation. Fix the Plücker coordinates $[x_{ij}]_{1 \le i < j \le 5}$ of $\mathbb{P}^9$, where $\{\omega_1, \omega_2, \omega_3, \psi_1, \psi_2\}$ is the chosen basis of $H^1(M(\mathcal{A}))$. The coordinates of the five planes – in lexicographical order $[x_{1,2}, x_{1,3}, x_{1,4}, \dots, x_{4,5}]$ – are:

$$P_1 = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0],$$
$$P_2 = [0, 0, 0, 0, 0, 0, 1, 0, 0, 0],$$
$$P_3 = [0, 0, 0, 0, 0, 0, 0, 1, 1, 0],$$
$$P_4 = [1, -1, 1, 0, 1, 0, 0, -1, 0, 0],$$
$$P_5 = [1, -1, 0, 0, 1, 0, -1, 0, 1, 0].$$

Thus the linear subspace L has equation given by the ideal

$$I \stackrel{\text{def}}{=} (x_{15}, x_{24}, x_{45}, x_{12} + x_{13}, x_{13} + x_{23}, x_{13} - x_{34} + x_{35}).$$

The equation of the Grassmannian are given by the Pfaffian of principal minors of size four of a skew-symmetric matrix. Thus the defining ideal is

$$\begin{aligned}J \stackrel{\text{def}}{=} (&x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23},\\ &x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24}, x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34},\\ &x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34})\end{aligned}$$

and the sum of the two ideals is

$$\begin{aligned}I + J = (&x_{15}, x_{24}, x_{45}, x_{14}x_{25}, x_{14}x_{35}, x_{25}x_{34}, x_{12} + x_{13}, x_{13} + x_{23},\\ &x_{13} - x_{34} + x_{35}, x_{12}x_{34} + x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25}).\end{aligned}$$

This last ideal is zero dimensional; this computation was done in Sage [The18] and by hand. Therefore, the intersection of the subspace $\mathbb{P}(\ker \tilde{f})$ with the Grassmannian $\mathrm{gr}(2, H^1(M(\mathcal{A})))$ is (scheme theoretically) the union of five points. Since we have exhibit five distinct rational points, we obtain that the first resonance variety $\mathcal{R}^1(\mathcal{A}; \mathbb{Q})$ is the union of the five corresponding planes. □

The map $T \to T$ defined by $(t_1, t_2) \mapsto (t_1, t_1^a t_2^n)$ is a cyclic Galois covering. For every n and a the above map restricts to a Galois covering $\pi_a \colon M(\mathcal{A}_n^a) \to M(\mathcal{A})$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. The map π_a induces an inclusion

$$\pi_a^* : H^\bullet(M(\mathcal{A}); \mathbb{Z}) \hookrightarrow H^\bullet(M(\mathcal{A}_n^a); \mathbb{Z})$$

of cohomology rings with integer coefficients.

Since n is coprime with 2 and 3, $H^1(M(\mathcal{A}_n^a); \mathbb{Z})$ has rank five, equal to that of $H^1(M(\mathcal{A}); \mathbb{Z})$. Let $\alpha = \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log t_1$ and $\beta = \frac{1}{2\pi\mathbf{i}} \,\mathrm{d}\log t_2$ be the two canonical generators of $H^1(T; \mathbb{Z})$ as sub-lattice of $H^1(M(\mathcal{A}_n^a); \mathbb{Z})$: then the morphism π_a^* is:

$$\begin{aligned}\pi_a^*(\psi_1) &= \alpha,\\ \pi_a^*(\psi_2) &= n\beta + a\alpha,\\ \pi_a^*(\omega_i) &= \omega_i \qquad \text{for } i = 1, 2, 3.\end{aligned}$$

Lemma 3.4.5. *The first resonance variety $\mathcal{R}^1(\mathcal{A}_n^a; \mathbb{Z})$ is the union of the following five sub-lattices of $H^1(M(\mathcal{A}_n^a); \mathbb{Z})$:*

$$\begin{aligned}Q_1 &= \langle \omega_1, \alpha \rangle,\\ Q_2 &= \langle \omega_2, n\beta + a\alpha \rangle,\\ Q_3 &= \langle \omega_3, n\beta + (a+1)\alpha \rangle,\\ Q_4 &= \langle \omega_1 - \omega_3, \omega_2 - \omega_1 + \alpha \rangle,\\ Q_5 &= \langle \omega_2 - \omega_3, \omega_1 - \omega_2 + n\beta + a\alpha \rangle.\end{aligned}$$

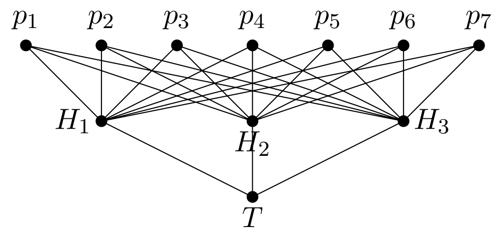


Figure 3.2: The Hasse diagram of the poset of layers of $\mathcal{A}_7^1$ which coincides with the one of $\mathcal{A}_7^2$

Proof. For $i = 1, 2$, the lattice $H^i(M(\mathcal{A}_n^a); \mathbb{Z})$ is embedded in $H^i(M(\mathcal{A}); \mathbb{Q})$ and the first resonance variety $R^1(\mathcal{A}_n^a; \mathbb{Z})$ is the intersection

$$\mathcal{R}^1(\mathcal{A}_n^a; \mathbb{Z}) = \mathcal{R}^1(\mathcal{A}; \mathbb{Q}) \bigcap H^1(M(\mathcal{A}_n^a); \mathbb{Z}). \qquad \square$$

Now we can complete the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. The posets of layers $\mathcal{S}(\mathcal{A}_n^1)$ and $\mathcal{S}(\mathcal{A}_n^2)$ are isomorphic because they have 3 connected hypertori that intersect in n points $(1, \zeta_n^i)$ for $i = 0, \ldots, n-1$ (where ζ_n is a n^{th} primitive root of unity). The Hasse diagram of the posets of layers in the case $n = 7$ is represented in Figure 3.2. Suppose that there exists an isomorphism $\varphi\colon H^\bullet(M(\mathcal{A}_n^1); \mathbb{Z}) \to H^\bullet(M(\mathcal{A}_n^2); \mathbb{Z})$; then φ must map $\mathcal{R}^1(\mathcal{A}_n^1; \mathbb{Z})$ isomorphically into $\mathcal{R}^1(\mathcal{A}_n^2; \mathbb{Z})$. Furthermore, φ sends each component Q_i^1 into another component $Q_{f(i)}^2$. For each (i, j), consider the cardinality $c^a(i, j)$ of the torsion subgroup of $H^1(M(\mathcal{A}_n^a); \mathbb{Z})/\langle Q_i^a, Q_j^a \rangle$ for $a = 1, 2$. The value of $c^a(i, j)$ is n when $(i, j) = (1, 2), (1, 3), (2, 3), (4, 5)$ and 1 otherwise, both for $a = 1$ and $a = 2$.Thus, φ maps Q_1^1, Q_2^1, Q_3^1 into Q_1^2, Q_2^2, Q_3^2 in some order. For $a = 1, 2$ the following equality holds

$$H^1((\mathbb{C}^*)^2; \mathbb{Z}) = \operatorname{Rad}\Big(\bigcap_{1 \le i < j \le 3} \langle Q_i^a, Q_j^a \rangle \Big),$$

hence φ preserves the sub-lattice $L \stackrel{\text{def}}{=} H^1((\mathbb{C}^*)^2; \mathbb{Z}) = \langle \alpha, \beta \rangle$. Now we claim that there is no linear map $\varphi_{|L} : L \to L$ that sends the three sub-lattices

$$\{\, Q_1^1 \cap L, Q_2^1 \cap L, Q_3^1 \cap L \,\}$$

into $\{\, Q_1^2 \cap L, Q_2^2 \cap L, Q_3^2 \cap L \,\}$ in some order. The three one-dimensional lattices are $Q_1^1 \cap L = \langle \alpha \rangle$, $Q_2^1 \cap L = \langle n\beta + \alpha \rangle$, $Q_3^1 \cap L = \langle n\beta + 2\alpha \rangle$ for the arrangement $\mathcal{A}_n^1$ and the lattices $Q_1^2 \cap L = \langle \alpha \rangle$, $Q_2^2 \cap L = \langle n\beta + 2\alpha \rangle$, $Q_3^2 \cap L = \langle n\beta + 3\alpha \rangle$ for the arrangement $\mathcal{A}_n^2$. In the case $a = 1$ we can find generators for two of those lattices (e.g. $-\alpha$ and $n\beta + \alpha$) such that their sum belongs to the sub-lattice nL. This property does not hold for the arrangement $\mathcal{A}_n^2$: indeed $\pm\alpha \pm (n\beta + 2\alpha), \pm\alpha \pm (n\beta + 3\alpha), \pm(n\beta + 2\alpha) \pm (n\beta + 3\alpha)$ are not in nL (here we use $n \neq 5$). Thus, we conclude that the map φ cannot exist. $\square$

3.5 A second example

The following example is constructed by looking for two toric arrangements with the following properties. The underline matroid is not a modular matroid. The two toric arrangements are coverings of the same toric arrangement with non cyclic Galois group. The smallest example of such arrangements must have rank at least three and four hypertori.

Consider the three matrices

$$N = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}, \quad N' = \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}, \quad N'' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

These integer matrices describe three central toric arrangements $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ in $T = (\mathbb{C}^*)^3$. Both $\mathcal{A}$ and $\mathcal{A}'$ are Galois coverings of $\mathcal{A}''$ with Galois group $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Let $([4], \mathrm{rk}, m)$ be the arithmetic matroid defined by $\mathrm{rk}(S) = \min(|S|, 3)$ and by

$$m(S) = \begin{cases} 1 & \text{if } |S| \leq 1 \\ 5 & \text{if } |S| = 2 \\ 25 & \text{if } |S| \geq 3 \end{cases}.$$

Remark 3.5.1. If we identify $\mathrm{Ext}^1(\mathbb{Z}_5^2, \mathbb{Z}^3) \simeq (\mathbb{Z}_5^2)^3$ using the canonical basis of $\mathbb{Z}^3$ and if we choose as representation of $(E, \mathrm{rk}, \overline{m})$ the columns of the matrix N'', then the representations N and N' are classified by the two different elements

$$\left[\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\right], \qquad \left[\begin{pmatrix} -1 & 0 & 1 \\ -4 & 1 & 0 \end{pmatrix}\right]$$

of $\mathcal{C}$, respectively.

Let $\mathcal{M}$ be the matroid over $\mathbb{Z}$ defined by

$$\mathcal{M}(S) = \begin{cases} \mathbb{Z}^3 & \text{if } |S| = 0 \\ \mathbb{Z}^2 & \text{if } |S| = 1 \\ \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} & \text{if } |S| = 2 \\ \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} & \text{if } |S| \geq 3 \end{cases}.$$

Theorem 3.5.2. *The matrices N and N' are representations of the arithmetic matroid $([4], \mathrm{rk}, m)$ and of the matroid $\mathcal{M}$ over $\mathbb{Z}$. Moreover, the posets $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A}')$ are not isomorphic.*

Proof. The first assertion follows from the Smith normal form of $N[S]$ and of $N'[S]$, the matrices obtained from N and N' by taking only the columns indexed by S. The second one follows from Lemma 3.5.3 below. □

The Poincaré polynomials of the complements $M(\mathcal{A})$ and $M(\mathcal{A}')$ coincide with

$$P(t) = P'(t) = 110t^3 + 41t^2 + 7t + 1.$$

The one of $M(\mathcal{A}'')$ is $P''(t) = 14t^3 + 17t^2 + 7t + 1$. The Tutte polynomial of the arithmetic matroid $([4], \mathrm{rk}, m)$ is $x^3 + x^2 + 25x + 25y + 48$ and the one associated with N'' is $x^3 + x^2 + x + y$.

Define $a \vee b$ as the set of all least upper bound of a, b in the poset of layers. Consider the following property

$$\exists \{i,j\} \cup \{k,l\} = [4] \, \forall a \in i \vee j, \forall b \in k \vee l \, (a \vee b \neq \emptyset). \tag{P}$$

In other words, the property (P) for $\mathcal{S}(\mathcal{A})$ (or for $\mathcal{S}(\mathcal{A}')$) says that there exists a choice of two hypertori H_i, H_j in $\mathcal{A}$ (resp. in $\mathcal{A}'$) such that every connected component of $H_i \cap H_j$ intersects every connected component of $H_k \cap H_l$.

Lemma 3.5.3. *The property* (P) *holds for* $\mathcal{S}(\mathcal{A})$ *but not for* $\mathcal{S}(\mathcal{A}')$*.*

Proof. We first discuss the poset $\mathcal{S}(\mathcal{A}')$. Consider $(i,j,k,l) = (1,2,3,4)$, there are five possible joins $1 \vee 2$ that correspond to the five layers

$$a_\mu : \begin{cases} x = 1 \\ y = \mu \end{cases},$$

where μ runs over all the fifth roots of unity. Analogously, the joins of 3 and 4 are the five layers

$$b_\zeta : \begin{cases} x = z^{-5} \\ y = \zeta z^5 \end{cases},$$

where ζ runs over all the fifth roots of unity. A join $a_\mu \vee b_\zeta$ exists if and only if the system

$$\begin{cases} x = 1 \\ y = \mu \\ z^5 = 1 \\ y = \zeta z^5 \end{cases}. \tag{3.5}$$

admits a solution. If $\zeta = \mu$, then the system has five solutions, otherwise there are no solutions. In particular, the property (P) does not hold for the poset $\mathcal{S}(\mathcal{A}')$.

The following case by case analysis shows that the three systems for the arrangement $\mathcal{A}$ analogous to (3.5) have always a unique solution:

$$\begin{cases} x = 1 \\ y = \mu \\ xz^5 = 1 \\ xyz^3 = \zeta \end{cases}, \quad \begin{cases} x = 1 \\ z = \mu \\ xy^5 = 1 \\ x^2y^3z = \zeta \end{cases}, \quad \begin{cases} x = 1 \\ yz = \mu \\ xy^5 = 1 \\ y = \zeta z \end{cases}. \qquad \square$$

Proposition 3.5.4. *The spaces $M(\mathcal{A})$ and $M(\mathcal{A}')$ have non-isomorphic cohomology algebras with rational coefficients, i.e.*

$$H^\bullet(M(\mathcal{A});\mathbb{Q}) \not\simeq H^\bullet(M(\mathcal{A}');\mathbb{Q}).$$

Proof. Suppose that an isomorphism $\varphi : H^\bullet(M(\mathcal{A});\mathbb{Q}) \to H^\bullet(M(\mathcal{A}');\mathbb{Q})$ exists. We claim that $\varphi(H^\bullet(T;\mathbb{Q})) = H^\bullet(T;\mathbb{Q})$ where T is the ambient torus. The proof of the claim is analogous to the one of Lemma 3.4.4. The first resonance variety of $M(\mathcal{A})$ and $M(\mathcal{A}')$ are the union of the four planes

$$\begin{aligned}
Q_1 &= \langle \omega_1, \alpha \rangle, & Q'_1 &= \langle \omega_1, \alpha \rangle,\\
Q_2 &= \langle \omega_2, 4\alpha + 5\beta \rangle, & Q'_2 &= \langle \omega_2, \alpha + 5\beta \rangle,\\
Q_3 &= \langle \omega_3, \alpha + 5\gamma \rangle, & Q'_3 &= \langle \omega_3, \alpha + 5\gamma \rangle,\\
Q_4 &= \langle \omega_4, 3\alpha + 5\beta + 5\gamma \rangle, & Q'_4 &= \langle \omega_4, 6\alpha + 5\beta + 5\gamma \rangle,
\end{aligned}$$

since the unique relations in cohomology of degree two are $\omega_i\psi_i = 0$ (see Theorem 2.9.13). Thus there exists a bijection $f : [4] \to [4]$ such that φ sends Q_i into $Q'_{f(i)}$, for $i = 1, \dots, 4$. Since $H^1(T;\mathbb{Q}) = \bigcap_{i=1}^4 \langle Q_j \rangle_{j\neq i}$ in $H^1(M(\mathcal{A});\mathbb{Q})$ and $H^1(T;\mathbb{Q}) = \bigcap_{i=1}^4 \langle Q'_j \rangle_{j\neq i}$ in $H^1(M(\mathcal{A}');\mathbb{Q})$, the map φ preserves the subspace $H^\bullet(T;\mathbb{Q})$. Consider now the quotients $S^\bullet = H^\bullet(M(\mathcal{A});\mathbb{Q})/(H^1(T;\mathbb{Q}))$ and $S'^\bullet = H^\bullet(M(\mathcal{A}');\mathbb{Q})/(H^1(T;\mathbb{Q}))$. The multiplication map $S^1 \times S^2 \to S^3$ has rank 51, instead the map $S'^1 \times S'^2 \to S'^3$ has rank 43. The rank of the two multiplication maps can be calculated with a computer. Therefore the map φ cannot be an isomorphism. □

The difference between the rank of $S^1 \times S^2 \to S^3$ and $S'^1 \times S'^2 \to S'^3$ can be explained intuitively.

For every $I \subseteq [4]$, the set of connected components of $\cap_{i\in I} H_i$ has a natural group structure, induced by the ambient torus. We call this group $\mathrm{LG}(I)$. Moreover given $I \subset J \subseteq [4]$, there exists a natural group homomorphism $\pi\colon \mathrm{LG}(J) \to \mathrm{LG}(I)$ that maps a connected component W to the unique connected component of $\bigcap_{i\in I} H_i$ containing W. In our case, since $\bigcap_{j\neq i} H_j = \bigcap_{j\in[4]} H_j$ for all $i \in [4]$, the map $\mathrm{LG}([4]) \to \mathrm{LG}([4] \setminus \{i\})$ is the identity. Call $\pi_{i,j}\colon \mathrm{LG}([4]) \to \mathrm{LG}(\{i,j\})$ the canonical projection.

Given I and J of cardinality two, there exists an isomorphism φ_I^J such that the following diagram commutes

$$\begin{array}{ccc}
\mathrm{LG}([4]) & \xrightarrow{\pi_I} & \mathrm{LG}(I) \\
\downarrow{\scriptstyle \pi_J} & \swarrow{\scriptstyle \varphi_I^J} & \\
\mathrm{LG}(J) & &
\end{array}$$

if and only if $\ker \pi_I = \ker \pi_J$. We compute all these kernels both for $\mathcal{A}$ and $\mathcal{A}'$ and we report it in Table 3.1, where e_1 and e_2 are two generators of $\mathrm{LG}([4]) \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Table 3.1: For every set $I \subset [4]$, $|I| = 2$, we describe the kernel of π_I and of π'_I.

I	$\ker \pi_I$	$\ker \pi'_I$
$\{1,2\}$	$\langle e_2 \rangle$	$\langle e_2 \rangle$
$\{1,3\}$	$\langle e_1 \rangle$	$\langle e_1 \rangle$
$\{1,4\}$	$\langle e_1 - e_2 \rangle$	$\langle e_1 - e_2 \rangle$
$\{2,3\}$	$\langle 4e_1 - e_2 \rangle$	$\langle e_1 - e_2 \rangle$
$\{2,4\}$	$\langle 2e_1 - e_2 \rangle$	$\langle 3e_1 - e_2 \rangle$
$\{3,4\}$	$\langle 3e_1 - e_2 \rangle$	$\langle e_2 \rangle$

From Theorem 2.9.13, we have that $S^\bullet$ is generated by the image of $\overline{\omega}_i := \overline{\omega}_{H_i,\{i\}}$ for $i \in [4]$, of $\overline{\omega}_{a,I}$ for $|I| = 2$ and $a \in \mathrm{LG}(I)$, and of $\overline{\omega}_{b,[4]\setminus\{i\}}$ for $i \in [4]$ and $b \in \mathrm{LG}([4])$. The linear relations are

$$\sum_{i=1}^{4} (-1)^i \overline{\omega}_{b,[4]\setminus\{i\}} = 0$$

for each $b \in \mathrm{LG}([4])$. The product $S^1 \times S^2 \to S^3$ is defined by

$$\overline{\omega}_i \overline{\omega}_{a,\{j,k\}} = (-1)^{l(\{i\},\{j,k\})} \sum_{b \in \pi_{j,k}^{-1}(a)} \overline{\omega}_{b,\{i,j,k\}}.$$

The analogous definitions and formulas hold for the arrangement $\mathcal{A}'$. In the algebra $S'^\bullet$ the following relations hold for $a \in \mathrm{LG}'(\{1,2\})$ and $c \in \mathrm{LG}'(\{1,4\})$:

$$(\overline{\omega}'_1 - \overline{\omega}'_2 + \overline{\omega}'_3 - \overline{\omega}'_4)(\overline{\omega}'_{a,\{1,2\}} + \overline{\omega}'_{\varphi^{3,4}_{1,2}(a),\{3,4\}}) = 0,$$
$$(\overline{\omega}'_1 + \overline{\omega}'_2 - \overline{\omega}'_3 - \overline{\omega}'_4)(\overline{\omega}'_{c,\{1,4\}} + \overline{\omega}'_{\varphi^{2,3}_{1,4}(c),\{2,3\}}) = 0,$$

since $\ker \pi'_{1,2} = \ker \pi'_{3,4}$ and $\ker \pi'_{1,4} = \ker \pi'_{2,3}$. These equations give ten independent relations, the corresponding relations in the algebra $S^\bullet$ are only two:

$$(\overline{\omega}_1 - \overline{\omega}_2 + \overline{\omega}_3 - \overline{\omega}_4)\left(\sum_{a \in \mathrm{LG}(\{1,2\})} \overline{\omega}_{a,\{1,2\}} + \sum_{b \in \mathrm{LG}(\{3,4\})} \overline{\omega}_{b,\{3,4\}} \right) = 0,$$
$$(\overline{\omega}_1 + \overline{\omega}_2 - \overline{\omega}_3 - \overline{\omega}_4)\left(\sum_{c \in \mathrm{LG}(\{1,4\})} \overline{\omega}_{c,\{1,4\}} + \sum_{d \in \mathrm{LG}(\{2,3\})} \overline{\omega}_{d,\{2,3\}} \right) = 0,$$

since $\ker \pi_{1,2} \neq \ker \pi_{3,4}$ and $\ker \pi_{1,4} \neq \ker \pi_{2,3}$.

By [DR18, Theorem E], the G-semimatroids described by N and N' are different.

D'Alì and Delucchi proved that both posets are homology Cohen-Macaulay over fields of all but a finite number of characteristics [DD18]. It was conjectured that the arithmetic independence poset is shellable. Notice that the non-arithmetic versions of these posets (the poset of flats and the independence poset of an ordinary matroid) are shellable, and therefore Cohen-Macaulay over fields of every characteristic.

Let M be the arithmetic matroid associated with the matrix N, by using the algorithm described [PP19b], we find that M has 13 non-equivalent essential representation. These 13 representations give rise to 3 non-isomorphic posets of layers. These 3 posets are realized by the matrices N, N' and

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & -5 \end{pmatrix}.$$

The homology of the order complex of the poset of layers (with the bottom element removed) is equal to $(0, \mathbb{Z}_5, \mathbb{Z}^{48})$ in all 3 cases. In particular, these posets of layers are not Cohen-Macaulay in characteristic 5, and therefore are not shellable. However, posets of layers of toric arrangements associated with root systems were proved to be shellable [DGP17, Pao18].

The arithmetic independence posets of the 13 representations of M are pairwise isomorphic. Their order complexes (with the bottom element removed) have homology $(0, \mathbb{Z}_5, \mathbb{Z}^{73})$. Therefore these posets are not Cohen-Macaulay in characteristic 5, and are not shellable.

Our observations settle a number of different conjectures about the posets associated with a toric arrangement, but also highlight the following problem.

Question 3.5.5. Let M be an arithmetic matroid. Are the arithmetic independence posets of the representations of M always pairwise isomorphic?

Chapter 4

Cohomology of Configurations Spaces

We study the rational cohomology of the configuration space of unordered points on closed orientable surfaces. In particular, we compute the mixed Hodge structure on the cohomology and the action of the mapping class group, by finding a series with coefficients in the Grothendieck ring of $\mathfrak{sp}(2g)$ that describes explicitly the decomposition of the cohomology into irreducible representations. From that we deduce the mixed Hodge numbers and the Betti numbers, obtaining a formula without cancellations. In the case of the 2-torus we compute the structure of the cohomology ring and we prove the formality over the rationals. We also conjecture the mixed Hodge numbers of the ordered configuration spaces on the 2-torus.

The results of Section 4.3 are essentially new and unpublished. Sections 4.2 and 4.4 appear in [Pag19c] and Sections 4.5 to 4.7 are taken from [Pag19a]. Some minor results also appear in [Pag18c].

4.1 Introduction

Configuration spaces of points are related to physics (state spaces of non-colliding particles on a manifold), robotics (motion planning), knot theory, and topology. Configuration spaces give invariants of the homeomorphism type of the base space. In the algebraic setting, configuration spaces are open in the moduli spaces of points.

The ordered configuration space of n points in a topological space X is

$$\mathcal{F}_n(X) = \{(p_1, \ldots, p_n) \in X^n \mid p_i \neq p_j\}.$$

We are interested in the unordered configuration space of X, that is

$$\mathcal{C}_n(X) = \{I \subset X \mid |I| = n\} = \mathcal{F}_n(X)/\mathfrak{S}_n.$$

Since the literature is very extensive, we compare our work only with the main results on the (co-)homology of configuration spaces. The first computation of the cohomology algebra of configuration spaces is due to Arnold [Arn69, Arn70] in the case of $\mathbb{R}^2$. This result has been generalized by Cohen, Lada, and May [CLM76] to the configuration space of $\mathbb{R}^n$ and later by Goresky and Macpherson [GM88]. Partially additive results have been obtained: by Bödigheimer and Cohen [BC88] for once-punctured oriented surfaces, by the same authors and Taylor [BCT89] for odd dimensional manifolds, and by Drummond-Cole and Knudsen [DCK17] for surfaces in general. However there is no description of the ring structure. The Betti numbers $\mathcal{C}_n(X)$ are described in the following cases: for $X = \mathbb{P}^2(\mathbb{R})$ by Wang [Wan02], for X a sphere by Salvatore [Sal04], for $X = \mathbb{P}^2(\mathbb{C})$ by Felix and Tanré [FT05] and for elliptic curves by Maguire and Schiessl [Mag16, Sch16].

The Euler characteristic of the configuration spaces of any even-dimensional orientable closed manifold M was computed by Felix and Thomas in [FT00] by the following formula:

$$\sum_{n=0}^{\infty} \chi(\mathcal{C}_n(M))u^n = (1+u)^{\chi(M)}.$$

In the case of surfaces, this formula can be obtained from eq. (4.21) by setting $t = s = -1$ and taking the dimension of the representations.

Results

We compute the rational cohomology of $\mathcal{C}_n(\Sigma_g)$ where Σ_g is the Riemann surface of genus g. We improve the previous results on configuration spaces on surfaces in three ways, see Theorem 4.7.11:

- we determine the mixed hodge numbers,
- we describe the action of the mapping class group on the gradation of the cohomology,
- we obtain a formula for Betti numbers without cancellations.

We also improve the previous results on configuration spaces in an elliptic curve in three ways.

- We describe the action of the mapping class group on the cohomology (no gradation for the torus), see Theorem 4.4.5.
- We give an equivariant description of the ring structure of the cohomology (Theorem 4.4.11).
- The formality result over the rationals is proven in Corollary 4.4.13.

Techniques

The main tools used are the Križ model and representation theory. The Križ model [Kri94, Tot96, FM94] is the differential graded commutative algebra over $\mathbb{Q}$ (for short DGCA)

$$\mathrm{A}^{\bullet,\bullet}(X,n) = {H^\bullet(X)^{\otimes n} \otimes \wedge^\bullet V}\big/_{I},$$

where V is the $\mathbb{Q}$-vector space with basis $\{G_{i,j}\}_{i<j}$ and

$$I = (G_{i,j}G_{j,k} + G_{j,k}G_{i,k} + G_{i,k}G_{i,j},\, G_{i,j}(x_i - x_j))_{\substack{i<j<k\in[n],\\ x\in H^\bullet(X)}},$$

where the element x_i is $1\otimes 1\otimes\cdots\otimes x\otimes\cdots\otimes 1 \in H^\bullet(X)^{\otimes n}$ with x in position i. The differential d on $\mathrm{A}(X,n)$ is defined by $\mathrm{d}(x_i) = 0$ and by $\mathrm{d}(G_{i,j}) = \Delta_{i,j}$, where $\Delta_{i,j}$ is the class of the diagonal in $H^\bullet(X)^{\otimes 2} \overset{i,j}{\hookrightarrow} H^\bullet(X)^{\otimes n}$. The bidegrees of the generators are $|x_i| = (|x|, 0)$ and $|G_{i,j}| = (0, 2d-1)$, where $d = \dim_{\mathbb{C}} X$. There is an isomorphism of bigraded algebras $H(\mathrm{A}(X,n),\mathrm{d}) \simeq H(\mathcal{F}_n(X))$, where the bigradation on the cohomology is given by the mixed Hodge structure. This model was generalized by Bibby [Bib16a] and Dupont [Dup16a] to general arrangements of hypersurfaces in smooth projective varieties.

Remark 4.1.1. The model $\mathrm{A}^{\bullet,\bullet}$ coincides with the Križ model $E^\bullet_\bullet$ introduced in [Kri94] up to shifting the degrees, i.e.

$$\mathrm{A}^{p,q} \cong E^{p+q}_q.$$

The dga $E^\bullet_\bullet$ is a rational model for X, as shown in [Kri94, Theorem 1.1].

The symmetric group $\mathfrak{S}_n$ acts on the algebra $\mathrm{A}(X,n)$ and the $\mathfrak{S}_n$-invariant subalgebra is a model for the space $\mathcal{C}_n(X)$, indeed we have the isomorphism $H(\mathrm{A}(X,n)^{\mathfrak{S}_n},\mathrm{d}) \simeq H^\bullet(\mathcal{C}_n(X))$. We use the results of [AAB14, Aza15] that describes the action of the symmetric group on the Križ model.

Let Σ_g be a Riemann surface, Γ_g be its mapping class group, and I_g be the Torelli subgroup. Recall the short exact sequence

$$0 \to I_g \to \Gamma_g \to \mathrm{Sp}(2g) \to 0.$$

The natural action of the subgroup I_g on $H^i(\mathcal{C}_n(\Sigma_g))$ is not trivial, but the induced action on $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(\mathcal{C}_n(\Sigma_g))$ is trivial. Indeed,we have $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(\mathcal{C}_n(\Sigma_g)) \simeq H^\bullet(\mathrm{A}(\Sigma_g,n)^{\mathfrak{S}_n},\mathrm{d})$ functorially, hence the isomorphism is Γ_g-equivariant. The action of Γ_g on the algebra $\mathrm{A}(\Sigma_g,n)^{\mathfrak{S}_n}$ is clearly symplectic thus I_g acts trivially on $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(\mathcal{C}_n(\Sigma_g))$.

The action of the Torelli group is studied in [Bia19] in the case of once punctured surfaces and it is non trivial on $H_2(\mathcal{C}_n(\Sigma_g \setminus \{*\}))$. From Theorem 4.7.11, we deduced that the filtration $\mathrm{F}_\bullet$ is trivial in cohomological degrees $0,1,2$ and also in degree 3 if $g=2$. Thus in this cases the action of the mapping class group is symplectic.

The case $g = 1$ is special and we denote Σ_1 by E. Indeed, I_1 is the trivial group and the action of the Torelli group is symplectic. Moreover, the Križ model splits as dga

$$\mathrm{A}^{\bullet,\bullet}(E, n) \cong \mathrm{B}^{\bullet,\bullet}(n) \otimes_{\mathbb{Q}} \mathrm{D}^{\bullet,0}$$

in equivariant way. Where $\mathrm{B}(n)$ is the model for the space

$$\mathcal{M}_n(E) \overset{\text{def}}{=} \{ (p_1, \dots, p_n) \in \mathcal{F}_n(E) \mid \sum p_i = 0 \} .$$

and $\mathrm{D} \cong H(E; \mathbb{Q})$. Notice that there exists a non canonical isomorphism $\mathcal{F}_n \cong E \times \mathcal{M}_n(E)$.

The mixed Hodge structure on the cohomology of algebraic varieties defines a bigrading compatible with the algebra structure (see [Del75, p.81] or [Voi07, Theorem 8.35]). In our case the bigrading given by the mixed Hodge structure coincides with the one given by the Leray spectral sequence as shown by Totaro [Tot96, Theorem 3] and by Gorinov [Gor17]. Explicitly, the subspace $\mathrm{A}^{p,q}(X, n)$ has weight $p + 2q$ and degree $p + q$.

Plan

In Section 4.2 we recall the Križ model for general surfaces. In Section 4.3 we specialize the model for genus one and we improve the result on the decomposition of the Križ model into irreducible representations, see Theorem 4.3.2. We also provide Conjectures 4.3.10 and 4.3.19 for the sums of mixed Hodge number in the ordered case. The unordered configuration spaces for genus one surfacesis described in Section 4.4, where the mixed Hodge numbers, the action of the mapping class group, the ring structure of the cohomology and formality result are discussed. In the last three sections we focus on the unordered configuration space for arbitrary genus. We present a simple model Section 4.5 and we study it as a representation of the symplectic group Section 4.6. Using these results we prove the main Theorem 4.7.11 in Section 4.7.

4.2 Representation theory on the Križ model

We study the action of the symmetric group $\mathfrak{S}_n$ and of $\mathrm{SL}_{2g}(\mathbb{Q})$ on the algebras A. Those actions are given by a geometric action on $\mathcal{F}_n(\Sigma_g)$. For a general reference on the representation theory of the Lie groups and of the Lie algebras we refer to [Hal15] and to [FH91], respectively. The cases of $\mathrm{SL}_2(\mathbb{C})$ and of $\mathfrak{sl}(2; \mathbb{C})$ can be found in [GW09].

Theorem 4.2.1 (Theorem 1.1 [Kri94]). *Let X be a smooth projective variety over the complex $\mathbb{C}$. The dga $\mathrm{A}(X, n)$ is a model for the configuration space $\mathcal{F}_n(X)$.*

Dimension formula and the braid hyperplane arrangement

The algebra $\mathrm{A}^{0,\bullet}(\Sigma_g, n)$ coincides with the cohomology of the braid hyperplane arrangement; we use this fact to compute the dimension of $\mathrm{A}^{p,q}(\Sigma_g, n)$.

The dimension of the cohomology of the braid hyperplane arrangement $H^k(M(A^H_{n-1}))$ coincides with $\left[{n \atop n-k}\right]$, the Stirling number of first kind. The Poincaré polynomial of the complement is

$$P_{n-1}(t) = \prod_{q=1}^{n-1}(1+qt) = \sum_{q=0}^{n-1}\left[{n \atop n-q}\right] t^q$$

by [Arn69, Corollary 2], while the (arithmetic) Tutte polynomial $T_n(x,y)$ was calculated in [GS96, Theorem 5.1] and has exponential generating function

$$\sum_{n=1}^{\infty}\frac{T_n(x,y)t^n}{n!} = \frac{\left(\sum_{n=0}^{\infty}(\frac{t}{y-1})^n \frac{y^{\binom{n}{2}}}{n!}\right)^{(y-1)(x-1)} - 1}{x-1}. \tag{4.1}$$

The (arithmetic) Tutte polynomial can be computed easily by using the recursion formula in [Pak93]:

$$T_{n+1}(x,y) = \sum_{k=1}^{n}\binom{n-1}{k-1}(x+y+y^2+\cdots+y^{k-1})T_k(1,y)T_{n+1-k}(x,y).$$

For any hyperplane arrangement $\mathcal{A}^H$ the Poincaré polynomial is a specialization of the Tutte polynomial, e.g. for the braid arrangement we have:

$$P_{n-1}(x) = x^{n-1}T_n\left(\frac{1}{x}, 0\right).$$

Proposition 4.2.2. *The dimension of* $\mathrm{A}^{p,q}(\Sigma_g, n)$ *is* $\left[{n \atop n-q}\right]\binom{2g(n-q)}{p}$. *Moreover, its Hilbert polynomial is*

$$P_{\mathrm{A}(\Sigma_g,n)}(t,s) = \prod_{q=0}^{n-1}((1+t)^{2g}+qs).$$

Proof. Notice that

$$\mathrm{A}^{p,q}(\Sigma_g, n) = \bigoplus_{W\in\mathcal{S}_n,\,\mathrm{rk}(W)=q} H^p(W)\otimes H^q(M(A^H[W]);\mathbb{Q}),$$

where $\mathcal{S}_n$ is the poset of layers that coincides with the poset of the partitions of $[n]$ [Bib16b, Section 3.2]. By considering only the dimensions, we have

$$\begin{aligned}\dim \mathrm{A}^{p,q}(\Sigma_g, n) &= \sum_{\mathrm{rk}(W)=q} \dim H^p(\Sigma_g^{n-q}) \dim H^q(M(A^H[W]); \mathbb{Q}) \\ &= \binom{2g(n-q)}{p} \dim H^q(M(A^H_{n-1}); \mathbb{Q}) \\ &= \begin{bmatrix} n \\ n-q \end{bmatrix} \binom{2g(n-q)}{p}.\end{aligned}$$

Finally, we have

$$\begin{aligned}P_{\mathrm{A}(\Sigma_g,n)}(t,s) &= \sum_{p,q} \dim \mathrm{A}^{p,q}(\Sigma_g, n) t^p s^q \\ &= \sum_{p,q} \begin{bmatrix} n \\ n-q \end{bmatrix} \binom{2g(n-q)}{p} t^p s^q \\ &= \sum_{q} \begin{bmatrix} n \\ n-q \end{bmatrix} (1+t)^{2g(n-q)} s^q \\ &= \prod_{q=0}^{n-1} ((1+t)^{2g} + qs).\end{aligned}$$

This completes the proof. □

Definition of the actions

Consider the action of $\mathfrak{S}_n$ on $\mathcal{F}_n$ defined by

$$\sigma^{-1} \cdot (p_1, \dots, p_n) = (p_{\sigma(1)}, \dots, p_{\sigma(n)})$$

for all $\sigma \in \mathfrak{S}_n$. This induces an action on A defined by

$$\begin{aligned}\sigma^{-1}((a_l)_i) &= (a_l)_{\sigma(i)}, \\ \sigma^{-1}((b_l)_i) &= (b_l)_{\sigma(i)}, \\ \sigma^{-1}(G_{i,j}) &= G_{\sigma(i),\sigma(j)}\end{aligned}$$

for all $1 \leq i < j \leq n$, $l = 1, \dots, g$ and all $\sigma \in \mathfrak{S}_n$. The mapping class group Γ_g of the surface Σ_g acts naturally on $\mathcal{F}_n(\Sigma_g)$ and on $\mathcal{C}_n(\Sigma_g)$.

Let f be an automorphism of Σ_g, the map induces the following vertical morphisms

$$\begin{array}{ccc} \mathcal{F}_n(\Sigma_g) & \hookrightarrow & \Sigma_g^n \\ {\scriptstyle f^n_{|\mathcal{F}_n}} \downarrow & & \downarrow {\scriptstyle f^n} \\ \mathcal{F}_n(\Sigma_g) & \hookrightarrow & \Sigma_g^n \end{array}$$

and by functoriality of the Leray spectral sequence it induces the action of Γ_g on $\mathrm{A}(\Sigma_g, n)$. We explicitly describe this action on the generators $G_{i,j}$, $(a_l)_i$, and $(b_l)_i$: since $f^n : E^n \to E^n$ fixes the divisor $\{p_i = p_j\}$, then $f \cdot G_{i,j} = G_{i,j}$. The other generators belongs to $\mathrm{A}^{1,0}(\Sigma_g, n) = H^1(\Sigma_g^n) \cong H^1(\Sigma_g)^{\oplus n}$. Therefore the action of Γ_g is symplectic and it factors through $\mathrm{Sp}_{2g}(\mathbb{Z})$. On $\mathrm{A}^{1,0}(\Sigma_g, n)$ it coincides with the diagonal action on $H^1(\Sigma_g)^{\oplus n}$.

It is clear that the action of $\mathfrak{S}_n$ and the one of $\mathrm{Sp}_{2g}(\mathbb{Z})$ commute.

Decomposition into $\mathfrak{S}_n$-representations

We recall a result of [AAB14, Theorem 3.15] on the decomposition of $\mathrm{A}(\Sigma_g, n)$ into $\mathfrak{S}_n$-modules.

Let $\lambda \vdash n$ be a partition of the number n, i.e. $\lambda = (\lambda_1, \dots, \lambda_l)$ such that $\lambda_i \geq \lambda_{i+1}$ and $\sum_{i=1}^{l} \lambda_i = n$. Let $\mathcal{B}$ be the ordered basis of $H^\bullet(\Sigma_g; \mathbb{Z})$ whose elements are $1, a_1, \dots, a_g, b_1, \dots, b_g, [p_g]$, where $[p_g]$ is the fundamental class of a point in Σ_g. We denote the total order on $\mathcal{B}$ using the symbol $\succeq$.

Definition 4.2.3. A *label* s of the partition λ is a function $s \colon \{1, \dots, l\} \to \mathcal{B}$, i.e. we label each block with an element of the chosen basis $\mathcal{B}$, such that if $\lambda_i = \lambda_{i+1}$ then $s(i) \succeq s(i+1)$.

Let C_k be the cyclic group of order k. For any partition $\lambda \vdash n$ define C_λ as the product of the cyclic groups C_{λ_i} for $i = 1, \dots, t$. It acts on $\{1, \dots, n\}$ in the natural way. Consider a labelled partition (λ, s) and define $N_{\lambda,s}$ as the group that permutes the blocks of λ with the same size and the same labels. The group $N_{\lambda,s}$ is a product of symmetric groups. Call $Z_{\lambda,s}$ the semidirect product $C_\lambda \rtimes N_{\lambda,s}$.

Example 4.2.4. We give an example for $g = 1$, let (λ, s) be the labelled partition $\lambda = (5, 5, 5, 5, 1, 1, 1) \vdash 23$ and $s = (z, z, z, 1, a, a, a)$, where $z := [p_1]$. The group $C_\lambda \cong (\mathbb{Z}_5)^4 < \mathfrak{S}_{23}$ is generated by the cycles $(1, 2, 3, 4, 5)$, $(6, 7, 8, 9, 10)$, $(11, 12, 13, 14, 15)$, and $(16, 17, 18, 19, 20)$. The subgroup $N_{\lambda,s} \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ is generated by the following permutations:

$$
\begin{aligned}
&(1,6)(2,7)(3,8)(4,9)(5,10),\\
&(1,11)(2,12)(3,13)(4,14)(5,15),\\
&(21,22),\\
&(21,23).
\end{aligned}
$$

Finally, $Z_{\lambda,s}$ is a group isomorphic to $(\mathbb{Z}_5 \wr \mathfrak{S}_3) \times \mathbb{Z}_5 \times \mathfrak{S}_3$.

Given two representations V, W of two groups G and H respectively, define the tensor representation $V \boxtimes W$ of $G \times H$ by the vector space $V \otimes W$ with the action $(g, h)(v \otimes w) = g(v) \otimes h(w)$.

We define the following one-dimensional representations. Let φ_n be a faithful character of the cyclic group on n elements and φ_λ the character of $C_\lambda \cong C_{\lambda_1} \times \cdots \times C_{\lambda_t}$ given by

$$\varphi_\lambda \stackrel{\text{def}}{=} \operatorname{sgn}_n |_{C_\lambda} \cdot (\varphi_{\lambda_1} \boxtimes \cdots \boxtimes \varphi_{\lambda_r}).$$

Define the degree deg of an element in $\mathcal{B}$ by its degree in the algebra $H^\bullet(\Sigma_g)$. Let $\alpha_{\lambda,s}$ be the one dimensional representation of $N_{\lambda,s} \cong \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l}$ defined on generators by

$$\alpha_{\lambda,s}(\sigma) \stackrel{\text{def}}{=} (-1)^{\lambda_i + \deg s(\lambda_i) + 1},$$

where σ is the permutation that exchange two blocks λ_i and λ_j with the same labels. In other words, the representation $\alpha_{\lambda,s}$ restricted to $\mathfrak{S}_{\mu_j}$ is the character $\operatorname{sgn}_{\mathfrak{S}_{\mu_j}}^{\otimes(\lambda_i + \deg s(\lambda_i)+1)}$, where λ_i is any block exchanged by $\mathfrak{S}_{\mu_j}$. Let $\xi_{\lambda,s}$ be the one dimensional representation of $Z_{\lambda,s}$ such that $\operatorname{Res}^{Z_{\lambda,s}}_{C_\lambda} \xi_{\lambda,s} = \varphi_\lambda$ and $\operatorname{Res}^{Z_{\lambda,s}}_{N_{\lambda,s}} \xi_{\lambda,s} = \alpha_{\lambda,s}$.

We define $|\lambda| = n - t$ for a partition $\lambda = (\lambda_1, \dots, \lambda_t)$ of n and for a label s the numbers $|s| = \sum_{i=1}^t \deg s(\lambda_i)$.

Compares the following results with [DPR14].

Theorem 4.2.5 ([AAB14, Theorem 3.15])**.** *For each labelled partition (λ, s) there exist a $\mathfrak{S}_n$-representations $\mathrm{A}(\lambda, s) \subset \mathrm{A}^{p,q}(\Sigma_g, n)$, with $p = |s|$ and $q = |\lambda|$, such that*

$$\mathrm{A}^{p,q}(\Sigma_g, n) = \bigoplus_{\substack{|\lambda|=q \\ |s|=p}} \mathrm{A}(\lambda, s)$$

as $\mathfrak{S}_n$-representation. Moreover:

$$\mathrm{A}(\lambda, s) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \operatorname{Ind}^{\mathfrak{S}_n}_{Z_{\lambda,s}} \xi_{\lambda,s}.$$

Example 4.2.6. Consider the marked partition (λ, s) of Example 4.2.4, the characters are shown in the following table.

	$(1,2,3,4,5)$	$(16,17,18,19,20)$	$(1,6)(2,7)(3,8)(4,9)(5,10)$	$(21,22)$
φ	ζ_5	ζ_5		
α			1	-1
ξ	ζ_5	ζ_5	1	-1

The action of $\mathfrak{S}_n \times \mathrm{Sp}_{2g}(\mathbb{Q})$

Now we describe the algebra $\mathrm{A}(\Sigma_g, n)$ as representation of $\mathfrak{S}_n \times \mathrm{Sp}_{2g}(\mathbb{Q})$.

A nice formula holds:

$$\wedge^p (V \boxtimes W) = \bigoplus_{\lambda \vdash p} \mathbb{S}_\lambda V \boxtimes \mathbb{S}_{\lambda'} W, \tag{4.2}$$

where λ' is the conjugate partition of λ and $\mathbb{S}_\lambda$ is the Schur functor, see [FH91] for a general reference. A proof of eq. (4.2) can be found in [Wey03, Corollary 2.3.3] or [FH91, Exercise 6.11(b)]. Moreover, the dimension of $\mathbb{S}_\lambda V$ is

$$\dim \mathbb{S}_\lambda V = s_\lambda(\underbrace{1, \cdots, 1}_{\dim V}) = \prod_{1 \le i < j \le \dim V} \frac{\lambda_i - \lambda_j + i - j}{i - j}, \tag{4.3}$$

where s_λ is the Schur polynomial as proven in [FH91, Theorem 6.3].

The root system associated with the Lie group $\mathrm{Sp}_{2g}(\mathbb{Q})$ is of type C_g. The roots are $\pm 2e_i$ for $1 \le i \le g$ and $\pm e_i \pm e_j$ for $1 \le i < j \le g$, where e_i are the canonical basis of $\mathbb{R}^g$. We choose as positive roots the vectors $2a_i$ for $1 \le i \le g$ and $e_i \pm e_j$ for $1 \le i < j \le g$ and so the simple roots are $\alpha_i := e_i - e_{i+1}$ for $1 \le i \le g-1$ and $\alpha_g := 2e_g$. The fundamental weights are $\omega_i = \sum_{j=1}^{i} e_j$ for $1 \le i \le g$ and the dominant weights are $\sum_{i=1}^{g} n_i \omega_i$ for $n_i \in \mathbb{N}$. The irreducible finite-dimensional representations of $\mathfrak{sp}(2g)$ are V_ω for all dominant weights ω.

Lemma 4.2.7. *The first row of the algebra* $\mathrm{A}(\Sigma_g, n)$ *decomposes as follows:*

$$\mathrm{A}^{p,0}(\Sigma_g, n) \cong \bigoplus_{\lambda \vdash p} \mathbb{S}_\lambda(\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathbb{1}_{n-1}) \boxtimes \mathbb{S}_{\lambda'}(V_{\omega_1}).$$

Proof. It is easy to see that

$$\mathrm{A}^{\bullet,0}(\Sigma_g, n) = \wedge^\bullet H^1(\Sigma_g^n) \cong \wedge^p \left(\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathbb{1}_{n-1} \boxtimes H^1(\Sigma_g)\right)$$

as representations of $\mathfrak{S}_n \times \mathrm{Sp}_{2g}(\mathbb{Q})$. Since $H^1(\Sigma_g) = V_{\omega_1}$, eq. (4.2) completes the proof. □

Let $T \simeq (\mathbb{Q}^*)^g \subset \mathrm{Sp}_{2g}(\mathbb{Q})$ be the maximal splitting torus of diagonal matrices. For a weight $\omega \in \mathrm{Hom}(T, \mathbb{Q}^*) = \mathbb{Z}^g$, we define the one dimensional representation $V(\omega)$ of T by $t \cdot v = \omega(t)v$.

Definition 4.2.8. The *weight* of a label s of the partition λ is the vector $\omega(s) \in \mathbb{Z}^g$ given by $(\omega(s))_i = |\{ j \mid s(\lambda_j) = a_i \}| - |\{ j \mid s(\lambda_j) = b_i \}|$.

Theorem 4.2.9. *The algebra* $\mathrm{A}(\Sigma_g, n)$ *decomposes as representation of* $\mathfrak{S}_n \times T$ *in the following way:*

$$\mathrm{A}^{p,q}(\Sigma_g, n) \cong \bigoplus_{\substack{|\lambda|=q \\ |s|=p}} \mathrm{A}(\lambda, s) \boxtimes V(\omega(s)).$$

Proof. It follows from Theorem 4.2.5 and the observation that $\mathrm{A}(\lambda, s)$ is preserved by the action of T. □

Let U be the subgroup of $\mathrm{Sp}_{2g}(\mathbb{Q})$ whose elements are the matrices upper triangular matrices with diagonal entries equal to one. We call an vector v in a representation V of $\mathrm{Sp}_{2g}(\mathbb{Q})$ an *highest weight vector* if $U \cdot v = v$ and $t \cdot v = \omega(t)v$ for all $t \in T$. In this case we said that the vector v has weight ω. For any weight $\omega \in \mathbb{Z}^g$ and any partition λ, define $U^p_{\lambda,\omega}$ as the set of highest weight vectors of weight ω of $\bigoplus_{|s|=p} \mathrm{A}(\lambda, s)$. It is not easy to compute $U^p_{\lambda,\omega}$, however it is useful for decomposing $\mathrm{A}(\Sigma_g, n)$.

Theorem 4.2.10. *The algebra* $\mathrm{A}(\Sigma_g, n)$ *decomposes as representation of the group* $\mathfrak{S}_n \times \mathrm{Sp}_{2g}(\mathbb{Q})$ *as follows:*

$$\mathrm{A}^{p,q}(\Sigma_g, n) = \bigoplus_{\substack{|\lambda|=q \\ \omega\in\mathbb{N}^g}} U^p_{\lambda,\omega} \boxtimes V_\omega,$$

where $\omega = \sum_{i=1}^g n_i\omega_i$ *runs over all dominants weights.*

Proof. Since $\mathrm{A}^{p,q}(\Sigma_g, n)$ is finite dimensional, it decomposes as direct sums of highest weight representations indexes by the highest weight vectors. □

4.3 Genus one: ordered configurations

The case of genus $g = 1$ has two advantages: we know a lot about the representation theory of $\mathrm{SL}_2(\mathbb{Q})$ and the group structure on E provides a decomposition of $\mathrm{A}(E, n)$ as tensor product of two algebras.

Decomposition into $\mathfrak{S}_n \times \mathrm{SL}_2(\mathbb{Q})$-representations

Recall also that in the case $g = 1$ the action of the mapping class group Γ_1 is symplectic. It follows trivially by the following classical result.

Theorem 4.3.1 (Theorem 2.5 [FM12])**.** *The mapping class group* Γ_1 *of the torus is isomorphic to* $\mathrm{SL}_2(\mathbb{Z})$ *and the isomorphism is given by the natural action of* Γ_1 *on* $H^1(E;\mathbb{Z})$*.*

For $g = 1$ we have $\mathrm{Sp}_2(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{Q})$. Since all dominant weights are of the form $k\omega_1$ for $k \in \mathbb{N}$, in this section we call the irreducible representations of $\mathrm{SL}_2(\mathbb{Q})$ by $\mathbb{V}_k := V_{k\omega_1}$. Let $T = \{ H_t \} \cong \mathbb{Q}^*$ be the maximal torus in $\mathrm{SL}_2(\mathbb{Q})$ generated by the diagonal matrices $H_t = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Let $\mathbb{V}_1$ be the irreducible representation $\mathbb{Q}^2$ with the standard action of matrix-vector multiplication and let $\mathbb{V}_k = \mathrm{S}^k\,\mathbb{V}_1$ be the irreducible representations given by the symmetric powers of $\mathbb{V}_1$. The representation $\mathbb{V}_k$ has dimension $k + 1$ and can be view as $\mathbb{Q}[x, y]_k$, i.e. the vector space of homogeneous polynomials in two variables. The action of T on the monomials is given by $H_t \cdot x^a y^{k-a} = t^{2a-k} x^a y^{k-a}$, thus

$\mathbb{V}_k$ decomposes, as representations of T

$$\mathbb{V}_k = \bigoplus_{a=0}^{k} V(2a-k), \tag{4.4}$$

where $V(2a-k)$ is the subspace where H_t acts with character t^{2a-k}, i.e. the subspace generated by $x^a y^{k-a}$. Since the group $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{C})$, each irreducible regular representation of $\mathrm{SL}_2(\mathbb{Q})$ is isomorphic to $\mathbb{V}_k$ for some $k \in \mathbb{N}$. For a proof see [GW09, Proposition 2.3.5] and use a density reasoning.

As a consequence we can decompose a representation V of $\mathrm{SL}_2(\mathbb{Q})$ using its decomposition $V = \oplus_{a\in\mathbb{Z}} V(a)^{\oplus n_a}$ as representation of T: indeed $V \cong \oplus_{k\in\mathbb{N}} \mathbb{V}_k^{\oplus m_k}$, where $m_k = n_k - n_{k+2}$. By setting $V = \mathbb{V}_m \otimes \mathbb{V}_n$, we obtain the following formula for $n \leq m$:

$$\mathbb{V}_m \otimes \mathbb{V}_n \cong \mathbb{V}_{m+n} \oplus \mathbb{V}_{m+n-2} \oplus \cdots \oplus \mathbb{V}_{m-n}. \tag{4.5}$$

As observed in Section 4.2, the group $\mathrm{SL}_2(\mathbb{Q})$ acts trivially on $G_{i,j}$ for all $1 \leq i < j \leq n$ and, for each $i \leq n$, the two dimensional subspace generated by a_i and b_i is isomorphic to $\mathbb{V}_1$ as representation of $\mathrm{SL}_2(\mathbb{Q})$.

We will use the decomposition of Theorem 4.2.10 to split $\mathrm{A}(E,n)$ into $\mathfrak{S}_n \times \mathrm{SL}_2(\mathbb{Q})$-modules.

Theorem 4.3.2. *The algebra $\mathrm{A}(E,n)$ decomposes as $\mathfrak{S}_n \times \mathrm{SL}_2(\mathbb{Q})$-representation in the following way:*

$$\mathrm{A}^{p,q}(E,n) \cong \bigoplus_{k=0}^{p} \Big(\bigoplus_{|\lambda|=q} U^p_{\lambda,k} \Big) \boxtimes \mathbb{V}_k. \tag{4.6}$$

Moreover, $\dim U^p_{\lambda,k}$ is zero if $k \not\equiv p$ modulo 2 and, otherwise, equals to

$$\dim U^p_{\lambda,k} = s_{1^k 2^a}(1^{n-q}) = \frac{k+1}{a+k+1}\binom{n-q+1}{a}\binom{n-q}{a+k},$$

where a is such that $k+2a=p$ and $q=|\lambda|$.

Proof. The first part follows from Theorem 4.2.10. Notice that, for each $\lambda \vdash n$ with $|\lambda| = q$, the vector space $\bigoplus_{\omega(s)=p} \mathrm{A}(\lambda,s)$ is isomorphic to $\mathrm{A}^{p,0}(E,n-q)$ as representation of $\mathrm{SL}_2(\mathbb{Q})$. The decomposition of Lemma 4.2.7 in the case $g=1$ simplifies as follows. The representation $\mathbb{S}_\lambda \mathbb{V}_1$ is non-zero if and only if λ has at most $\dim \mathbb{V}_1$ blocks, i.e. $\lambda = (b,a)$. In this case we have the equalities:

$$\mathbb{S}_\lambda \mathbb{V}_1 = \mathbb{S}_{b-a} \mathbb{V}_1 = \mathrm{S}^{b-a} \mathbb{V}_1 = \mathbb{V}_{b-a}.$$

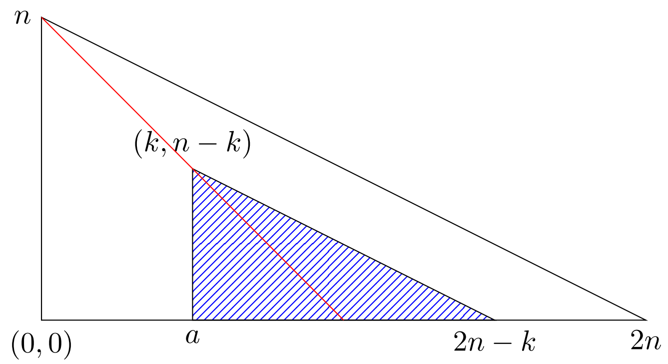


Figure 4.1: The representation $\mathbb{V}_k$ appears only in the darken triangle.

Thus $U^p_{\lambda,k} = \mathbb{S}_{1^k2^a}(\mathrm{Ind}^{\mathfrak{S}_{n-q}}_{\mathfrak{S}_{n-q-1}} \mathbb{1})$ as vector spaces, where $k = b - a$. Therefore, using eq. (4.3) we have:

$$\begin{aligned}\dim U^p_{\lambda,k} &= s_{1^k2^a}(1^{n-q}) \\ &= \prod_{\substack{i=1,\dots,a\\ j=a+1,\dots,a+k}} \frac{1+j-i}{j-i} \prod_{\substack{i=1,\dots,a\\ j=a+k+1,\dots,n-q}} \frac{2+j-i}{j-i} \prod_{\substack{i=a+1,\dots,a+k\\ j=a+k+1,\dots,n-q}} \frac{1+j-i}{j-i} \\ &= \prod_{i=1,\dots,a} \frac{n-q+2-i}{a-i+1}\frac{n-q+1-i}{a+k+2-i} \prod_{i=a+1,\dots,a+k} \frac{n-q+1-i}{a+k-i+1} \\ &= \frac{k+1}{a+k+1}\binom{n-q+1}{a}\binom{n-q}{a+k}.\end{aligned}$$

The last equalities are obtained by a cumbersome computation; alternatively, it can be obtained using the formula (iv) of [FH91, Exercise A.30]. □

The following corollary is immediate.

Corollary 4.3.3. *The multiplicity of* $\mathbb{V}_k$ *in* $\mathrm{A}^{p,q}(E,n)$ *is non zero if and only if* $k \equiv p \mod 2$ *and* $k \leq \min\{p, 2n-2q-p\}$. *Moreover, in this case the multiplicity is equal to* $\left[{n \atop n-q}\right] s_{1^k2^a}(1^{n-q})$, *where* $p = k + 2a$. □

Therefore, the representation $\mathbb{V}_a$ occurs only in the small triangle of Figure 4.1.

Splitting in cohomology

We fix the basis $1, a, b, [p]$ of $H(E;\mathbb{Q})$ and we define the following elements of $\mathrm{A}^{1,0}(E,n)$: $u_{i,j} := a_i - a_j$, $v_{i,j} := b_i - b_j$, for $i \neq j$, and $\gamma := \sum_{i=1}^n a_i$, $\overline{\gamma} := \sum_{i=1}^n b_i$.

We define, for $n > 0$, the dga $\mathrm{B}(n)$ as the subalgebra of $\mathrm{A}(E,n)$ generated by $u_{i,j}, v_{i,j}$ and $G_{i,j}$ for $1 \leq i < j \leq n$. It is a sub-dga since $\mathrm{d}(G_{i,j}) = u_{i,j}v_{i,j}$.

Let $\mathrm{D}(n)$ be the subalgebra of $\mathrm{A}^{\bullet,0}(E,n)$ generated by γ and $\overline{\gamma}$ endowed with the zero differential map. Notice that

$$\mathrm{A}(E,n) \cong \mathrm{B}(n) \otimes_{\mathbb{Q}} \mathrm{D}(n) \tag{4.7}$$

as differential algebras. Notice also that $\mathrm{D}(n)$ are all isomorphic to the cohomology ring of the elliptic curve E.

The following result is a particular case of [Bib16a, Theorem 3.3] and of [Dup16a, Theorem 1.2].

Theorem 4.3.4. *Let $n > 0$, the cohomology algebra of $\mathcal{F}_n(E)$ (or of $\mathcal{M}_n(E)$) with rational coefficients is isomorphic to the cohomology of the dga $\mathrm{A}(E,n)$ (respectively of $\mathrm{B}(n)$). Moreover, the n^2-sheeted covering*

$$\begin{aligned} E \times \mathcal{M}_n(E) &\to \mathcal{F}_n \\ (q, \underline{p}) &\mapsto (p_i + q)_{i=1,\dots,n} \end{aligned}$$

induces the isomorphism of eq. (4.7).

We have the analogous results of the previous subsection for the algebra $\mathrm{B}(n)$.

Corollary 4.3.5. *The multiplicity of $\mathbb{V}_k$ in $\mathrm{B}^{p,q}(n)$ is non zero if and only if $k \equiv p \mod 2$ and $k \leq \min\{p, 2n-2q-2-p\}$. Moreover, in this case the multiplicity is equal to $\left[{n \atop n-q}\right] s_{1^k 2^a}(1^{n-q-1})$, where $p = k + 2a$.*

Proof. It follows from the fact that $\mathrm{B}^{\bullet,0}(n) \cong \mathrm{A}^{\bullet,0}(E, n-1)$ as representation of $\mathrm{SL}_2(\mathbb{Q})$. □

Filtration and representation stability

Now we need to stress the dependence of $\mathrm{A}(\lambda, s)$ from n, thus we will write $\mathrm{A}_n(\lambda, s)$.

Definition 4.3.6. The *dimension of the support* of a labelled partition (λ, s) is the natural number $\tau(\lambda, s) = n - |\{ i \mid \lambda_i = 1,\ s(\lambda_i) = 1 \}|$.

Lemma 4.3.7. *Let (λ, s) be a labelled partition of n, then*

$$\mathrm{A}_n(\lambda, s) = \mathrm{Ind}_{\mathfrak{S}_{\tau(\lambda,s)} \times \mathfrak{S}_{n-\tau(\lambda,s)}}^{\mathfrak{S}_n} \mathrm{A}_{\tau(\lambda,s)}(\lambda, s).$$

Proof. It follows from Theorem 4.2.5 since $Z_{\lambda,s} = Z \times \mathfrak{S}_{n-\tau(\lambda,s)}$ for some $Z < \mathfrak{S}_{\tau(\lambda,s)}$ and $\xi_{\lambda,s|_{\mathfrak{S}_{n-\tau(\lambda,s)}}} = \mathbb{1}_{\mathfrak{S}_{n-\tau(\lambda,s)}}$. □

Let R be the $\mathbb{Z}$-algebra whose underline module is the free $\mathbb{Z}$-module generated by the elements $[\mathbb{V}_k]$ for $k \in \mathbb{N}$ and multiplication defined by the tensor product, see eq. (4.5). The ring R is called the *Grothendieck ring* of $\mathrm{SL}_2(\mathbb{Q})$, however we will use only as $\mathbb{Z}$-module. We denote for each finite dimensional representation $V = \oplus_k \mathbb{V}_k^{a_k}$ of $\mathrm{SL}_2(\mathbb{Q})$ by $[V] \in R$ the corresponding element $\sum_k a_k [\mathbb{V}_k]$.

Definition 4.3.8. The *Poincaré series* of a family $\mathrm{A}(n)$ of bigraded algebras with an action of $\mathrm{SL}_2(\mathbb{Q})$ is the formal series $Q_{\mathrm{A}}(t,s,r) \in R[[t,s,r]]$ defined by

$$Q_{\mathrm{A}}(t,s,r) \overset{\mathrm{def}}{=} \sum_{p,q,n} [\mathrm{A}^{p,q}(n)] t^p s^q r^n.$$

Definition 4.3.9. Let $\mathrm{F}_\bullet \mathrm{A}(E,n)$ be the increasing filtration defined by

$$\mathrm{F}_i \mathrm{A}(E,n) \overset{\mathrm{def}}{=} \bigoplus_{(\lambda,s) \text{ s.t. } \tau(\lambda,s)\leq i} \mathrm{A}(\lambda,s).$$

We define $\mathrm{F}_\bullet \mathrm{B}(n)$ as the induced filtration, i.e. $\mathrm{F}_i \mathrm{B}(n) := \mathrm{F}_i \mathrm{A}(E,n) \cap \mathrm{B}(n)$. Notice that the filtration is compatible with the decomposition of eq. (4.7).

We say that a filtration $\mathrm{F}_\bullet$ of a dga (A, d) is *strictly compatible* with the differential if $\operatorname{Im} \mathrm{d} \cap \mathrm{F}_i A = \mathrm{d}(\mathrm{F}_i A)$ for all i.

Conjecture 4.3.10. The filtration $\mathrm{F}_\bullet \mathrm{A}(E,n)$ is strictly compatible with the differential.

If the conjecture were true, then we would have the following:

$$\begin{aligned}
\mathrm{gr}_{\mathrm{F}} H^{p,q}(\mathcal{F}_n(E)) &\cong \mathrm{gr}_{\mathrm{F}} H^{p,q}(\mathrm{A}(E,n)) \\
&\cong H^{p,q}(\mathrm{gr}_{\mathrm{F}}(\mathrm{A}(E,n))) \\
&= \bigoplus_{i\leq n} H^{p,q}\left(\mathrm{F}_i \mathrm{A}(E,n) \big/ \mathrm{F}_{i-1} \mathrm{A}(E,n)\right) \\
&= \bigoplus_{i\leq n} H^{p,q}\Big(\bigoplus_{\tau(\lambda,s)=i} \mathrm{A}(\lambda,s)\Big) \\
&= \bigoplus_{i\leq n} \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_i\times\mathfrak{S}_{n-i}} H^{p,q}\left(\mathrm{A}(E,i) \big/ \mathrm{F}_{i-1} \mathrm{A}(E,i)\right).
\end{aligned} \tag{4.8}$$

We used, in order, that $\mathrm{A}(E,n)$ is a model for $\mathcal{F}_n(E)$, Conjecture 4.3.10, and Lemma 4.3.7.

Corollary 4.3.11. *Suppose that Conjecture 4.3.10 holds, then*

$$Q_{H(\mathcal{F}_\bullet(E))}(t,s,r) = \sum_{p,q,i} \left[H^{p,q}\left(\mathrm{A}(E,i) \big/ \mathrm{F}_{i-1} \mathrm{A}(E,i)\right)\right] t^p s^q \frac{r^i}{(1-r)^{i+1}}. \tag{4.9}$$

Proof. For any representation V of the group $\mathfrak{S}_i \times \mathfrak{S}_{n-i} \times \mathrm{SL}_2(\mathbb{Q})$ we have $[\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_i\times\mathfrak{S}_{n-i}} V] = \binom{n}{i}[V] \in R$ and for every $i \in \mathbb{N}$ we have $\sum_{n\geq i} \binom{n}{i} r^n = \frac{r^i}{(1-r)^{i+1}}$. The thesis follows from eq. (4.8). □

Corollary 4.3.12. *We have*

$$Q_{\mathrm{A}(E)}(t,s,r) = \sum_{k,a,q,n} \begin{bmatrix} n \\ n-q \end{bmatrix} s_{1^k 2^a}(1^{n-q})[\mathbb{V}_k] t^{k+2a} s^q r^n.$$

There exist natural numbers $m_{k,a,q,i} \in \mathbb{N}$ such that

$$Q_{\mathrm{A}(E)}(t,s,r) = \sum_{k,a,q,i} m_{k,a,q,i} [\mathbb{V}_k] t^{k+2a} s^q \frac{r^i}{(1-r)^{i+1}}.$$

Proof. We apply Corollary 4.3.3 for the first part. The second one follows by taking $m_{k,a,q,i}$ equals to the multiplicity of $\mathbb{V}_k$ in the representation

$$\mathrm{A}^{k+2a,q}(E,i) \big/ \mathrm{F}_{i-1}\, \mathrm{A}^{k+2a,q}(E,i). \qquad \square$$

This numbers $m_{k,a,q,i}$, for fixed value of k,a,q, are easy to calculate by solving the upper triangular system $A\underline{m} = \underline{y}$ where $A = (\binom{i}{j})_{i,j}$, $\underline{m} = (m_{k,a,q,i})_i$ and $\underline{y} = (\left[\begin{smallmatrix} i \\ i-q \end{smallmatrix}\right] s_{1^k 2^a}(1^{i-q}))_i$.

The particular case $m_{0,0,q,i}$ are listed in the sequence [OEI19, A259456].

The first row

Now we analyse the first rows of $\mathrm{A}^{\bullet,0}(E,n)$ and of $\mathrm{B}^{\bullet,0}(n)$ as representation of $\mathfrak{S}_n \times \mathrm{SL}_2(\mathbb{Q})$; from this we deduce the mixed hodge numbers $h^{p,0}(\mathcal{F}_n(E))$ and $h^{p,0}(\mathcal{M}_n(E))$.

Theorem 4.3.13. *For $n > 0$, the cohomology of $\mathcal{M}_n(E)$ in bidegree $(p,0)$ is given by*

$$H^{p,0}(\mathcal{M}_n(E)) = H^{p,0}(\mathrm{B}(n)) = \wedge^p V_n(1) \boxtimes \mathbb{V}_p = V_n(1^p) \boxtimes \mathbb{V}_p$$

as representations of $\mathfrak{S}_n \times \mathrm{SL}_2(\mathbb{Q})$ and it is of dimension $\binom{n-1}{p}(p+1)$.

Therefore we can describe $H^{\bullet,0}(\mathcal{F}_n(E))$ explicitly:

$$H^{\bullet,0}(\mathcal{F}_n(E)) = H^{\bullet,0}(\mathrm{A}(E,n)) = H^{\bullet}(E) \otimes H^{\bullet,0}(\mathcal{M}_n(E))$$

Proof of Theorem 4.3.13. First, notice that the representation $\mathbb{S}_{1^p} V_n(1) \boxtimes \mathbb{V}_p$ cannot lay in $\operatorname{Im} \mathrm{d}$ since $\mathbb{V}_p$ does not appear in $\mathrm{B}^{p-2,1}$ by Corollary 4.3.5. Therefore $H^{\bullet,0}(\mathcal{F}_n(E)) \supseteq V_n(1^p) \boxtimes \mathbb{V}_p$.

The range of $\mathrm{d}^{\bullet,1}$ is an ideal of $\mathrm{B}(n)$ generated in degree 2. Let $u_i = a_i - a_{i+1}$ and $v_i = b_i - b_{i+1}$ for $i = 1, \dots, n-1$ be a basis of $\mathrm{B}^{1,0}(n)$. Consider the monomial order on $\mathrm{B}(n) = \wedge V_n(1)$ induced by the deglex with $u_1 > u_2 > \cdots > u_{n-1} > v_{n-1} > v_{n-2} > \cdots > v_1$. For $i < j$ we have:

$$\mathrm{d}\, G_{i,j} = \left(\sum_{k=i}^{j-1} u_k \right) \left(\sum_{k=i}^{j-1} v_k \right),$$

and therefore in(Im d) contains all the elements of the form $u_i v_{j-1} = \mathrm{in}(\mathrm{d}\, G_{i,j})$ for $1 \leq i \leq j \leq n$. Therefore in(Im d) contains all monomials with two factors u_i and v_j with $i \leq j$. The remaining monomials are of the forms

$$u_{i_1} \cdots u_{i_k} v_{i_{k+1}} \cdots v_{i_p}$$

for $n-1 \geq i_1 > \cdots > i_p \geq 1$ and $0 \leq k \leq p$. Their number is $\binom{n-1}{p}(p+1)$ and it is an upper bound for the dimension of $\mathrm{B}^{p,0}(n)/\operatorname{Im} \mathrm{d}$. Finally, $H^{\bullet,0}(\mathcal{F}_n(E)) = V_n(1^p) \boxtimes \mathbb{V}_p$ since $\dim \wedge^p V_n(1) \boxtimes \mathbb{V}_p = \binom{n-1}{p}(p+1)$. □

Corollary 4.3.14. *We have*

$$Q_{H(\mathcal{M}_\bullet(E))}(t,0,r) = \sum_p [\mathbb{V}_p] t^p \left(\frac{r}{1-r} \right)^{p+1}$$

and

$$Q_{H(\mathcal{F}_\bullet(E))}(t,0,r) = 1 + \sum_p [\mathbb{V}_p](1 + [\mathbb{V}_1]t + t^2) t^p \left(\frac{r}{1-r} \right)^{p+1}$$

Proof. It follows from Theorem 4.3.13 using the following identity:

$$\sum_{n,p} \binom{n-1}{p} [\mathbb{V}_p] t^p r^n = \sum_p [\mathbb{V}_p] t^p \left(\frac{r}{1-r} \right)^{p+1}.$$

The second assertion follows from eq. (4.7) for $n > 0$, the case $n = 0$ is trivial. □

Lower bound for sums of mixed hodge numbers

Since Theorem 4.3.4 holds only for $n > 0$, we cannot prove the analogous statement of Corollary 4.3.11 for Q_{B}. Therefore we slightly modify the algebra B.

Definition 4.3.15. Let $\tilde{\mathrm{B}}(n)$ be the subalgebra of $\mathrm{B}(n)$ defined as $\mathrm{B}^{\bullet,>0}(n) + \operatorname{Im} \mathrm{d}$. By convention we set $\tilde{\mathrm{B}}(0) = 0$.

By definition we have $H^{p,q}(\tilde{\mathrm{B}}(n)) = H^{p,q}(\mathrm{B}(n))$ for all $p \geq 0$ and $q > 0$.

Corollary 4.3.16. *Suppose that Conjecture 4.3.10 holds, then*

$$Q_{H(\tilde{\mathrm{B}})}(t,s,r) = \sum_{p,q,i} \left[H^{p,q} \left(\tilde{\mathrm{B}}(i) \big/ \mathrm{F}_{i-1} \tilde{\mathrm{B}}(i) \right) \right] t^p s^q \frac{r^i}{(1-r)^{i+1}} \tag{4.10}$$

Proof. The monomials with $q = 0$ do not appear on both sides of eq. (4.10). We have $H^{p,q}(\tilde{\mathrm{B}}(i)/\mathrm{F}_{i-1}\tilde{\mathrm{B}}(i)) = H^{p,q}(\mathrm{B}(i)/\mathrm{F}_{i-1}\mathrm{B}(i))$ and $H^{p,q}(\tilde{\mathrm{B}}) = H^{p,q}(\mathrm{B})$ for $q > 0$. The thesis follows from eq. (4.7) dividing both sides of eq. (4.9) by $1 + [\mathbb{V}_1]t + t^2$. □

Corollary 4.3.17. *We have*

$$Q_{\tilde{\mathrm{B}}}(t,s,r) = \sum_{k,a,q,n} \begin{bmatrix} n \\ n-q \end{bmatrix} s_{1^k 2^a}(1^{n-q-1})[\mathbb{V}_k] t^{k+2a} s^q r^n.$$

There exist natural numbers $n_{k,a,q,i} \in \mathbb{N}$ such that

$$Q_{\tilde{\mathrm{B}}}(t,s,r) = \sum_{k,a,q,i} n_{k,a,q,i} [\mathbb{V}_k] t^{k+2a} s^q \frac{r^i}{(1-r)^{i+1}}.$$

Proof. We apply Corollary 4.3.5 for the first part. The second one follows from Corollary 4.3.16 with $n_{k,a,q,i}$ equals to the multiplicity of $\mathbb{V}_k$ in the representation ${\tilde{\mathrm{B}}^{k+2a,q}(i)}\big/{\mathrm{F}_{i-1}\,\tilde{\mathrm{B}}^{k+2a,q}(i)}$. □

We give a lower bound for the sum of hodge numbers $h^{p,q}(\mathcal{F}_n(E))$ on the "diagonal" $p+2q$ constant. We partially order R by saying $r \geq r'$ if $r-r'$ is an effective representation, moreover we partially order the series coefficient-wise.

Theorem 4.3.18. *Suppose that Conjecture 4.3.10 holds, then*

$$Q_{H(\mathcal{M}(E))}(t,t^2,r) \geq \sum_{k,b,i} |d_{k,b,i}|\, [\mathbb{V}_k] t^{k+2b} \frac{r^i}{(1-r)^{i+1}} + \sum_k [\mathbb{V}_k] t^k \left(\frac{r}{1-r}\right)^{k+1}, \tag{4.11}$$

where $d_{k,b,i} = \sum_{q=0}^{b} (-1)^q n_{k,b-q,q,i} \in \mathbb{N}$. Moreover,

$$Q_{H(\mathcal{F}(E))}(t,t^2,r) = 1 + (1 + [\mathbb{V}_1]t + t^2) Q_{H(\mathcal{M}(E))}(t,t^2,r).$$

Proof. We have proven that $H(\mathcal{M}(E)) = H(B)$ and, since $H^{p,0}(B)$ is contained in $\ker \mathrm{d}$, we have:

$$Q_{H(\mathcal{M}(E))}(t,t^2,r) = Q_{H(\tilde{\mathrm{B}})}(t,t^2,r) + Q_{H(\mathcal{M}(E))}(t,0,r).$$

Let $h_{k,a,q,i}$ be the multiplicity of $\mathbb{V}_k$ in $H^{p,q}\left({\tilde{\mathrm{B}}(i)}\big/{\mathrm{F}_{i-1}\,\tilde{\mathrm{B}}(i)}\right)$.

$$\begin{aligned}
Q_{H(\tilde{\mathrm{B}})}(t,t^2,r) &= \sum_{k,a,q,i} h_{k,a,q,i} [\mathbb{V}_k] t^{k+2a+2q} \frac{r^i}{(1-r)^{i+1}} \\
&\geq \sum_{k,b,i} \left| \sum_{q=0}^{b} (-1)^q h_{k,b-q,q,i} \right| [\mathbb{V}_k] t^{k+2b} \frac{r^i}{(1-r)^{i+1}} \\
&= \sum_{k,b,i} \left| \sum_{q=0}^{b} (-1)^q n_{k,b-q,q,i} \right| [\mathbb{V}_k] t^{k+2b} \frac{r^i}{(1-r)^{i+1}},
\end{aligned}$$

where the last equality is obtained by equating the "Euler characteristic" of subcomplexes and their cohomology. □

Conjecture 4.3.19. The inequality of eq. (4.11) is an equality.

We have verified the conjecture above until degree 7 in the variable r. Our method can lead to a lower bound of $\sum_q h^{c-2q,q}(n)$ for all c and n by taking the dimension of both sides of eq. (4.11).

Moreover, we conjecture the following:

Conjecture 4.3.20. The cohomology of $\mathrm{B}(i)/\mathrm{F}_{i-1}\mathrm{B}(i)$ is concentrate in degrees (p,q) with $p+q=i-2$ or $p+q=i-1$.

This conjecture is verified until total degree $p+q<7$.

Suppose that Conjecture 4.3.19 and Conjecture 4.3.20 hold, then the multiplicity of $\mathbb{V}_k$ in $H^{p,q}(\mathrm{B}(i)/\mathrm{F}_{i-1}\mathrm{B}(i))$ can be determine by the number $d_{k,b,i}$ (where $2b=p+2q-k$). More precisely, it is zero if $\operatorname{sgn} d_{k,b,i}\neq(-1)^q$ and equal to $|d_{k,b,i}|$ otherwise. This leads to an explicit combinatorial formula for the mixed hodge numbers for $\mathcal{F}_n(E)$.

Examples

Assume Conjecture 4.3.10, the numbers $d_{k,b,i}$ are calculated from the $n_{k,a,q,i}$. The value of $n_{k,a,q,i}$ are computed recursively in i from Corollary 4.3.17:

$$\begin{aligned}\begin{bmatrix} n \\ n-q \end{bmatrix} s_{1^k2^a}(1^{n-q-1}) &= \begin{bmatrix} n \\ n-q \end{bmatrix} \frac{k+1}{a+k+1}\binom{n-q}{a}\binom{n-q-1}{a+k} \\ &= \sum_{i=0}^{n}\binom{n}{i} n_{k,a,q,i}.\end{aligned}$$

We report $d_{k,b,i}$ for $i\leq 7$ in Table 4.1. Furthermore, by assuming Conjecture 4.3.19 we obtain, for all r and n, the following equality

$$\sum_{p+2q=r,q>0} \dim H^{p,q}(\mathcal{M}_n(E)) = \sum_{i=3}^{n}\sum_{k=0}^{\lfloor \frac{r}{2}\rfloor} |d_{r-2k,k,i}|\binom{n}{i}(r-2k+1).$$

The dimension of $H^{p,q}(\mathcal{M}_n(E))$ are computed for $n\leq 7$ from the algebra $\mathrm{B}(n)$ using [The18], see Tables 4.2 to 4.6.

4.4 Genus one: unordered configurations

We compute the cohomology with rational coefficients of the unordered configuration spaces of n points on the torus, taking care of the mixed Hodge structure and of the action of $\mathrm{SL}_2(\mathbb{Q})$. The integral cohomology groups are known only for small n in [Nap03, Table 2], where a cellular decomposition of ordered configuration spaces is given. In this section, we use the previous calculation of the Betti numbers of $\mathcal{C}_n(E)$ to determine the Hodge polynomial in the Grothendieck ring of $\mathrm{SL}_2(\mathbb{Q})$.

(k,b,i)	$d_{k,b,i}$	(k,b,i)	$d_{k,b,i}$
$(1,1,3)$	-1	$(3,2,6)$	35
$(0,2,4)$	-1	$(4,1,6)$	-10
$(1,2,4)$	2	$(0,4,7)$	-49
$(2,1,4)$	-3	$(0,5,7)$	154
$(0,3,5)$	5	$(1,3,7)$	14
$(1,3,5)$	-6	$(1,4,7)$	-70
$(2,2,5)$	11	$(1,5,7)$	-120
$(3,1,5)$	-6	$(2,3,7)$	-70
$(0,3,6)$	5	$(2,4,7)$	274
$(0,4,6)$	-26	$(3,2,7)$	35
$(1,3,6)$	6	$(3,3,7)$	-225
$(1,4,6)$	24	$(4,2,7)$	85
$(2,2,6)$	9	$(5,1,7)$	-15
$(2,3,6)$	-50		

Table 4.1: The non-zero value of $d_{k,b,i}$ for $i \leq 7$.

0	
1	2

Table 4.2: The dimension of the graded cohomology of $H(\mathcal{M}_2(E);\mathbb{Q})$.

0		
0	2	
1	4	3

Table 4.3: The dimension of the graded cohomology of $H(\mathcal{M}_3(E);\mathbb{Q})$.

0			
0	4		
0	8	10	
1	6	9	4

Table 4.4: The dimension of the graded cohomology of $H(\mathcal{M}_4(E);\mathbb{Q})$.

0				
0	12			
0	20	38		
0	20	50	24	
1	8	18	16	5

Table 4.5: The dimension of the graded cohomology of $H(\mathcal{M}_5(E);\mathbb{Q})$.

0					
0	48				
0	72	176			
0	60	260	152		
0	40	150	144	50	
1	10	30	40	25	6

Table 4.6: The dimension of the graded cohomology of $H(\mathcal{M}_6(E);\mathbb{Q})$.

0						
0	240					
0	336	976				
0	252	1491	1040			
0	140	1022	1232	425		
0	70	350	504	350	90	
1	12	45	80	75	36	7

Table 4.7: The dimension of the graded cohomology of $H(\mathcal{M}_7(E);\mathbb{Q})$.

We have $\mathrm{A}(E,n)^{\mathfrak{S}_n} = \mathrm{B}(n)^{\mathfrak{S}_n} \otimes_{\mathbb{Q}} \mathrm{D}$. and we use the results of Section 4.2 to compute $\mathrm{A}(\Sigma_g,n)^{\mathfrak{S}_n}$.

Theorem 4.4.1. *For $q > p+1$ we have $\mathrm{A}^{p,q}(\Sigma_g,n)^{\mathfrak{S}_n} = 0$.*

Proof. Let $\mathbb{1}_n$ be the trivial representation of $\mathfrak{S}_n$. We use Theorem 4.2.5 to show that

$$\langle \mathbb{1}_n, \mathrm{A}^{p,q}(\Sigma_g,n)\rangle_{\mathfrak{S}_n} = 0$$

for $q > p+1$. Indeed, it is enough to prove that

$$\langle \mathbb{1}_n, \mathrm{A}(\lambda,s)\rangle_{\mathfrak{S}_n} = 0$$

for all (λ,s) with $|\lambda| = q$ and $|s| = p$. By Frobenius reciprocity we have

$$\begin{aligned}\langle \mathbb{1}_n, \mathrm{Ind}_{Z_{\lambda,s}}^{\mathfrak{S}_n} \xi_{\lambda,s}\rangle_{\mathfrak{S}_n} &= \langle \mathbb{1}_{Z_{\lambda,s}}, \xi_{\lambda,s}\rangle_{Z_{\lambda,s}} \\ &= \langle \mathbb{1}_{C_\lambda}, \varphi_\lambda\rangle_{C_\lambda} \langle \mathbb{1}_{N_{\lambda,s}}, \alpha_{\lambda,s}\rangle_{N_{\lambda,s}} \\ &= \prod_{i=1}^{t} \langle \mathrm{sgn}_{|C_{\lambda_i}}, \varphi_{\lambda_i}\rangle_{C_{\lambda_i}} \prod_{j=1}^{l} \langle \mathbb{1}_{\mathfrak{S}_{\mu_j}}, \mathrm{sgn}_{\mathfrak{S}_{\mu_j}}^{\otimes(\lambda_i + \deg s(\lambda_i)+1)}\rangle_{\mathfrak{S}_{\mu_j}},\end{aligned}$$

where λ_i in any block exchanged by $\mathfrak{S}_{\mu_j}$. The scalar product $\langle \mathrm{sgn}_{|C_{\lambda_i}}, \varphi_{\lambda_i}\rangle_{C_{\lambda_i}}$ is non zero if and only if $\lambda_i = 1,2$ and $\langle \mathbb{1}_{\mathfrak{S}_{\mu_j}}, \mathrm{sgn}_{\mathfrak{S}_{\mu_j}}^{\otimes(\lambda_i + \deg s(\lambda_i)+1)}\rangle_{\mathfrak{S}_{\mu_j}}$ is non zero if and only if $\mu_j = 1$ or $\lambda_i + \deg s(\lambda_i) + 1$ is even. If $\mathrm{A}(\lambda,s)^{\mathfrak{S}_n} \neq 0$, then there exists at most one block λ_i such that $\lambda_i = 2$ and $s(\lambda_i) = 1$ and all other blocks with $\lambda_j = 2$ must be labelled with a_i, b_i for $i = 1, \dots, g$ or with $[p_g]$. Since q is the number of blocks of λ of size 2, we have

$$q = \sum_{i \text{ s.t. } \lambda_i = 2} 1 \leq 1 + \sum_{i \text{ s.t. } \lambda_i = 2} \deg(s(\lambda_i)) \leq 1 + \sum_i \deg(s(\lambda_i)) = 1 + p.$$

Consequently, $\mathrm{A}(\lambda, s)^{\mathfrak{S}_n} = 0$ for $q > p + 1$. □

In particular, for $g = 1$ we obtain the following corollary.

Corollary 4.4.2. *For $q > p + 1$ we have $\mathrm{B}^{p,q}(n)^{\mathfrak{S}_n} = 0$.* □

Observe that $H^\bullet(\mathcal{C}_n(E)) = H^\bullet(\mathcal{F}_n(E))^{\mathfrak{S}_n}$ by the Transfer Theorem. Define the series

$$T(t, s) = \frac{1 + t^3 s^4}{(1 - t^2 s^3)^2} = 1 + 2t^2 s^3 + t^3 s^4 + 3t^4 s^6 + 2t^5 s^7 + \dots$$

and let $T_n(t, s)$ be its truncation at degree n in the variable t.

The computation of the Betti numbers of unordered configuration space of n points in an elliptic curve was done simultaneously by [DCK17], [Mag16], and [Sch16] in different generality. We point to the last reference because [Sch16, Theorem] fits exactly our generality.

Theorem 4.4.3. *The Poincaré polynomial of $\mathcal{C}_n(E)$ is $(1 + t)^2 T_{n-1}(t, 1)$.*

Definition 4.4.4. The Hodge polynomial of $\mathcal{C}_n(E)$ in the Grothendieck ring R is

$$\sum_{i=0}^{2n} \sum_{k=i}^{2i} \left[W_k H^i(\mathcal{C}_n(E); \mathbb{Q}) \big/ W_{k-1} H^i(\mathcal{C}_n(E); \mathbb{Q}) \right] t^i s^k \quad \in R[t, s],$$

where $W_k H^i(\mathcal{C}_n(E); \mathbb{Q})$ is the weight filtration on $H^i(\mathcal{C}_n(E); \mathbb{Q})$. The ordinary Hodge polynomial is

$$\sum_{i=0}^{2n} \sum_{k=i}^{2i} \dim_{\mathbb{Q}} \left(W_k H^i(\mathcal{C}_n(E); \mathbb{Q}) \big/ W_{k-1} H^i(\mathcal{C}_n(E); \mathbb{Q}) \right) t^i s^k.$$

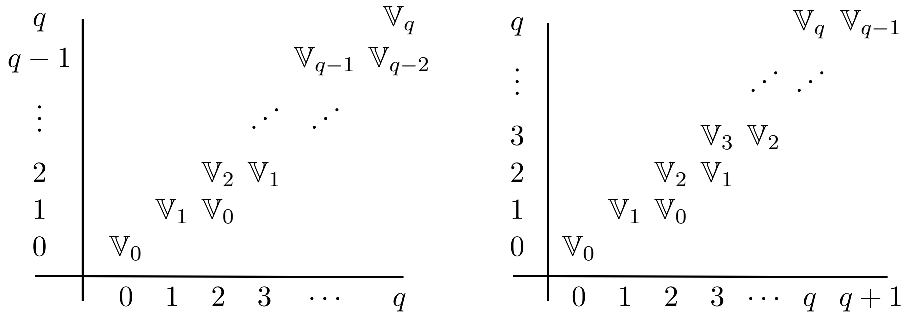
We prove a stronger version of Theorem 4.4.3.

Theorem 4.4.5. *The Hodge polynomial of $\mathcal{C}_n(E)$ with coefficients in the Grothendieck ring R is*

$$([\mathbb{V}_0] + [\mathbb{V}_1] ts + [\mathbb{V}_0] t^2 s^2) \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} [\mathbb{V}_i] t^{2i} s^{3i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} [\mathbb{V}_{i-1}] t^{2i+1} s^{3i+1} \right) \tag{4.12}$$

and the ordinary Hodge polynomial is $(1 + uv)^2 T_{n-1}(u, v)$.

Figure 4.2 represents the bigraded module $H(\mathrm{B}(n)^{\mathfrak{S}_n})$ that corresponds to the right factor of eq. (4.12).



(a) Case $n = 2q + 1$ odd. (b) Case $n = 2q + 2$ even.

Figure 4.2: The algebra $H(\mathrm{B}(n)^{\mathfrak{S}_n})$ as representation of $\mathrm{SL}_2(\mathbb{Q})$.

Some elements in cohomology

For the sake of notation, when we work with $g = 1$ the elements $(a_1)_i$ and $(b_1)_i$ for $i = 1, \dots, n$ will be denoted by a_i and b_i respectively.

Definition 4.4.6. Let $\alpha, \overline{\alpha} \in \mathrm{A}^{1,1}(E, n), \beta \in \mathrm{A}^{1,2}(E, n)$ be the elements

$$\alpha \stackrel{\text{def}}{=} \sum_{i,k<h} (a_i - a_k) G_{k,h}$$
$$\overline{\alpha} \stackrel{\text{def}}{=} \sum_{i,k<h} (b_i - b_k) G_{k,h}$$
$$\beta \stackrel{\text{def}}{=} \sum_{i,j,k<h} (3a_i - a_j - 2a_k)(b_j - b_k) G_{k,h}$$

where the sum is taken over pairwise distinct indices i, j, k, h with $k < h$.

Notice that the elements α and $\overline{\alpha}$ are defined only for $n > 2$ and β for $n > 3$. Remember that $\gamma, \overline{\gamma} \in \mathrm{D}^1$ were already defined as $\sum_i a_i$ and $\sum_i b_i$.

Lemma 4.4.7. *For $n > 2$, the element α belongs to $\mathrm{B}^{1,1}(n)^{\mathfrak{S}_n}$, is non-zero, and $\mathrm{d}\,\alpha = 0$.*

Proof. First observe that $\alpha = \sum_{i,k<h} u_{i,k} G_{k,h} \in \mathrm{B}^{1,1}(n)$. For all $\sigma \in \mathfrak{S}_n$ we have

$$\sigma\alpha = \sum_{i,k<h} u_{\sigma(i),\sigma(k)} G_{\sigma(k),\sigma(h)} = \alpha,$$

since $u_{\sigma(i),\sigma(k)} G_{\sigma(k),\sigma(h)} = u_{\sigma(i),\sigma(h)} G_{\sigma(k),\sigma(h)}$ in $\mathrm{A}(E, n)$. The elements $a_i G_{k,h}$ and $a_k G_{k,h}$ are linearly independent, so it is enough to observe that the coefficient of $a_3 G_{1,2}$ is 1. This proves that $\alpha \neq 0$. Finally, we compute $\mathrm{d}\,\alpha$:

$$\begin{aligned}\mathrm{d}\,\alpha &= \sum_{i,k<h} a_i \,\mathrm{d}\, G_{k,h} - a_k \,\mathrm{d}\, G_{k,h} \\ &= \sum_{i,k<h} a_i(a_k - a_h)(b_k - b_h) + a_k a_h (b_k - b_h) \\ &= \sum_{i,k,h} a_i a_k b_k - a_i a_h b_k + a_k a_h b_k \\ &= -\sum_{i,k,h} a_i a_h b_k = 0,\end{aligned}$$

where all sums are taken over pairwise distinct indices and the first two with the additional condition $k < h$. □

Lemma 4.4.8. *For $n > 3$, the element β belongs to $\mathrm{B}^{1,2}(n)^{\mathfrak{S}_n}$, is non-zero, and $\mathrm{d}\,\beta = 0$.*

Proof. Observe that

$$\beta = \sum_{i,j,k<h} (u_{i,j} + 2u_{i,k}) v_{j,k} G_{k,h} \in \mathrm{B}^{1,2}(n)$$

and that $\sigma\beta = \beta$ for all $\sigma \in \mathfrak{S}_n$ by the relations $u_{j,k} G_{k,h} = u_{j,h} G_{k,h}$ and $v_{j,k} G_{k,h} = v_{j,h} G_{k,h}$. Consider the map $\varphi \colon A \to \mathbb{Q}$ defined on generators by $\varphi(G_{1,2}) = 1$, $\varphi(a_3) = 1$ and $\varphi(b_4) = 1$ and zero on the other generators. The map φ is well defined and $\varphi(\beta) = 3$, thus $\beta \neq 0$. Using the computation in the proof of Lemma 4.4.7, we can observe that $\mathrm{d}\left(\sum_{i,j,h<k} 3a_i(b_j - b_k) G_{k,h}\right) = 0$. The claim $\mathrm{d}\,\beta = 0$ follows from:

$$\begin{aligned}\mathrm{d}(\beta) &= \mathrm{d}\Big(\sum_{j,k<h} (a_j + 2a_k)(b_j - b_k) G_{k,h}\Big) \\ &= \sum_{j,k<h} a_j b_j \,\mathrm{d}\, G_{k,h} + a_j b_k (a_k - a_h) b_h - 2a_k b_j a_h (b_k - b_h) - 2a_k b_k a_h b_h \\ &= \sum_{j,k,h} a_j b_j a_k b_k - a_i b_j a_h b_k + a_j b_k a_k b_h - 2a_k b_j a_h b_k - a_k b_k a_h b_h \\ &= 0,\end{aligned}$$

where the indexes of the sums are pairwise distinct. □

Lemma 4.4.9. *For $n > 2q$ the element α^q is non-zero.*

Proof. Let us rewrite α as $\alpha = \sum_{i,k<h} a_i G_{k,h} + (2 - n) \sum_{k<h} a_k G_{k,h}$. We show that the coefficient of the monomial $m = a_1 G_{1,2} a_3 G_{3,4} \dots a_{2q-1} G_{2q-1,2q}$ (defined for $n \geq 2q$) in α^q is non-zero for $n > 2q$.

This coefficient is

$$a_q = q! \sum_{\sigma \in \mathfrak{S}_q} \operatorname{sgn}(\sigma)(2-n)^{|\operatorname{Fix}\sigma|} 2^{q-|\operatorname{Fix}\sigma|}.$$

Where $q!$ are the ways to choose each $G_{2k-1,2k}$ one from each factors, and since $a_{2i-1}G_{2k-1,2k}$ has even degree we can suppose (up to the factor $q!$) that $G_{2k-1,2k}$ is taken from the k-th factor. Now x_{2i-1} arises from either x_{2i-1} or x_{2i} of the $\sigma(i)$-th factor for some permutation $\sigma \in \mathfrak{S}_q$. Since $m = \operatorname{sgn}(\sigma) \prod_{i=1}^{q} a_{2\sigma^{-1}(k)-1} G_{2k-1,2k}$ the contribution has the sign of the permutation σ. Finally, in α the monomial $a_{2i-1}G_{2k-1,2k}$ has coefficient $(2-n)$ if $i = k$ and 1 otherwise and the monomial $a_{2i}G_{2k-1,2k}$ has coefficient 0 if $i = k$ and 1 otherwise.

We claim that

$$\sum_{\sigma \in \mathfrak{S}_q} \operatorname{sgn}(\sigma) x^{|\operatorname{Fix}\sigma|} = (x-1)^{q-1}(x+q-1), \tag{4.13}$$

since both sides are the determinant of the matrix

$$\begin{pmatrix} x & 1 & \cdots & 1 \\ 1 & x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x \end{pmatrix}.$$

The left hand side of eq (4.13) is obtained by using the Laplace formula for the determinant and the right hand side by relating the determinant to the characteristic polynomial $(-t)^{q-1}(q-t)$ of the matrix $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. We use eq (4.13) with $x = \frac{2-n}{2}$ to obtain:

$$a_q = q! 2^q \sum_{\sigma \in \mathfrak{S}_q} \operatorname{sgn}(\sigma) \Big(\frac{2-n}{2}\Big)^{|\operatorname{Fix}\sigma|} = q! 2^q \Big(\frac{-n}{2}\Big)^{q-1} \Big(\frac{2q-n}{2}\Big)$$

Thus $a_q = (-1)^q q! n^{q-1}(n-2q)$ that is non-zero for $n > 2q$. ☐

Lemma 4.4.10. *For $n > 2q+1$ the element $\alpha^{q-1}\beta$ is non-zero.*

Proof. Let us rewrite β as

$$\beta = \sum_{i,j,k<h} a_i b_j G_{k,h} - 2(n-3) \sum_{i,k<h} (a_i b_k + a_k b_i) G_{k,h} - (n-3) \sum_{i,k<h} a_i b_i G_{k,h} + \\ + 2(n-2)(n-3) \sum_{k<h} a_k b_k G_{k,h}.$$

Let b_q be the coefficient in $\alpha^{q-1}\beta$ of the monomial

$$a_1 G_{1,2} a_3 G_{3,4} \dots a_{2q-1} G_{2q-1,2q} y_{2q+1}.$$

This monomial is defined for $n \geq 2q+1$ and we will show that $b_q \neq 0$ for $n > 2q+1$. The number b_q coincides with the coefficient of the same monomial in the product

$$\alpha^{q-1}\Big(\sum_{i,j,k<h} a_i b_j G_{k,h} - 2(n-3)\sum_{i,k<h} a_k b_i G_{k,h}\Big).$$

With further manipulation, we obtain that b_q is the coefficient of the above monomial in the expression

$$3\alpha^q b_{2q+1} + n\alpha^{q-1}\sum_{k<h} a_k G_{k,h} b_{2q+1}.$$

Using the computation in the proof of Lemma 4.4.9 we obtain

$$\begin{aligned} b_q &= 3(-1)^q q! n^{q-1}(n-2q) + nq(-1)^{q-1}(q-1)! n^{q-2}(n-2q+2) \\ &= 2(-1)^q q! n^{q-1}(n-2q-1). \end{aligned}$$

The number b_q is non zero for $n > 2q+1$. □

Proof of Theorem 4.4.5. It is enough to prove that the Hodge polynomial of $\mathrm{B}(n)^{SG_n}$ in the Grothendieck ring of $\mathrm{SL}_2(\mathbb{Q})$ is

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} [\mathbb{V}_i] u^{2i} v^{3i} + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} [\mathbb{V}_{i-1}] u^{2i+1} v^{3i+1}$$

Observe that $\operatorname{Im} \mathrm{d}^{q,p} = 0$ for $q > p+1$ by Corollary 4.4.2. From Lemma 4.4.7 and Lemma 4.4.9 we have that the elements α^k for $2k < n$ generate as $\mathrm{SL}_2(\mathbb{Q})$-module a subspace of dimension at least $k+1$ in $H^{k,k}(\mathrm{B}(n)^{\mathfrak{S}_n}, \mathrm{d})$. Analogously, from Lemma 4.4.8 and Lemma 4.4.10 the elements $\alpha^{k-1}\beta$ for $2k+1 < n$ generate as $\mathrm{SL}(\mathbb{Q})$-module a subspace of dimension at least k in $H^{k,k+1}(\mathrm{B}(n)^{\mathfrak{S}_n}, \mathrm{d})$. Since the Betti numbers of $\mathcal{C}_n(E)$ (Theorem 4.4.3) coincides with the above dimensions, we have that $H^{2k}(\mathrm{B}(n)^{\mathfrak{S}_n}) \cong \mathbb{V}_k u^{2k} v^{3k}$ and $H^{2k+1}(\mathrm{B}(n)^{\mathfrak{S}_n}) \cong \mathbb{V}_{k-1} u^{2k+1} v^{3k+1}$. □

The cohomology ring and formality

In this subsection we determine the cup product structure in the cohomology of $\mathcal{C}_n(E)$ and we prove the aforementioned formality result.

In the following we consider graded algebras with an action of $\mathrm{SL}_2(\mathbb{Q})$. We will write $(x_i \mid i \in I)_{\mathrm{SL}_2(\mathbb{Q})}$ for the ideal generated by the elements Mx_i for all $M \in \mathrm{SL}_2(\mathbb{Q})$ and all $i \in I$.

Theorem 4.4.11. *The cohomology ring of $\mathcal{C}_n(E)$ is isomorphic to*

$$\wedge^\bullet \mathbb{V}_1 \otimes {}^{\mathrm{S}^\bullet \mathbb{V}_1[b]}\Big/_{(a^{\lfloor\frac{n+1}{2}\rfloor}, a^{\lfloor\frac{n}{2}\rfloor} b, b^2)_{\mathrm{SL}(\mathbb{Q})}},$$

where a is a non-zero degree-one element in $V(1) \subset \mathbb{V}_1$ and b is an $\mathrm{SL}_2(\mathbb{Q})$-invariant variable of degree 3.

Proof. It is enough to prove that

$$H^\bullet(\mathrm{B}(n)^{\mathfrak{S}_n}) \cong \mathrm{S}^\bullet \mathbb{V}_1[b]/(a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b, b^2)_{\mathrm{SL}(\mathbb{Q})}.$$

Define the morphism $\varphi\colon \mathrm{S}^\bullet \mathbb{V}_1[b]/(a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b, b^2)_{\mathrm{SL}(\mathbb{Q})} \to H(\mathrm{B}(n)^{\mathfrak{S}_n})$ that sends a, b to α, β respectively. It is well defined because $H^k(\mathrm{B}(n)^{\mathfrak{S}_n}) = 0$ for $k \geq n$ and $\beta^2 = 0$ since it has odd degree. The map φ is surjective since $H(\mathrm{B}(n)^{\mathfrak{S}_n})$ is generated by α^i and $\alpha^i\beta$ as SL($\mathbb{Q}$)-module by Theorem 4.4.5. A dimensional reasoning shows the injectivity of the map φ. □

Corollary 4.4.12. *The cohomology $H^\bullet(\mathcal{C}_n(E))$ is generated as an algebra in degrees one, two and three.*

Proof. A minimal set of generators is given by $\alpha, \overline{\alpha}, \beta, \gamma, \overline{\gamma}$. □

Corollary 4.4.13. *The space $\mathcal{C}_n(E)$ is formal over the rationals.*

Proof. We prove that $\mathrm{B}(n)^{\mathfrak{S}_n}$ is formal. Consider the subalgebra K of $\mathrm{B}(n)^{\mathfrak{S}_n}$ generated by $\alpha, \overline{\alpha}, \beta$ endowed with the zero differential. It is concentrated in degrees (i, i) and $(i, i+1)$ because $\beta^2 = 0$. Since $K \cap \mathrm{Im}\,\mathrm{d} = 0$ (Corollary 4.4.2), $K \hookrightarrow \mathrm{B}(n)^{\mathfrak{S}_n}$ is a quasi-isomorphism. The fact that $K \cong H(\mathrm{B}(n)^{\mathfrak{S}_n})$ implies that the algebra $\mathrm{B}(n)^{\mathfrak{S}_n}$ is formal. As a consequence $\mathrm{A}(E, n)^{\mathfrak{S}_n}$ is formal. The space $\mathcal{C}_n(E)$ is formal since our model $\mathrm{A}(E, n)^{\mathfrak{S}_n}$ is equivalent to the Sullivan model. □

4.5 Models for $H(\mathcal{C}_n(X))$

We introduce a new and simple model for the cohomology of unordered configuration spaces. Furthermore, in the case of Riemann surfaces we describe a model uniform in the number of points.

Our first step consist in the simplification of the invariant part of the Križ model by taking a quotient $\mathrm{A}(\Sigma_g, n)^{\mathfrak{S}_n}/I_n$ where I_n is the acyclic ideal introduced in Definition 4.5.6. In the case of surfaces Σ_g, the problem is further simplified thanks to the introduction of a new dga $(\mathrm{U}_g, \mathrm{d})$, independent from n, that maps onto $(\mathrm{A}(\Sigma_g, n)^{\mathfrak{S}_n}/I_n, \mathrm{d})$. The dga $(\mathrm{U}_g, \mathrm{d})$ is filtered (c.f. Definition 4.3.9) and this filtration $\mathrm{F}_n\,\mathrm{U}_g$ is multiplicative and strictly compatible with the differential. This filtration computes the cohomology of our configuration spaces, indeed $H^\bullet(\mathrm{F}_n\,\mathrm{U}_g) \simeq H^\bullet(\mathcal{C}_n(\Sigma_g))$ as module.

The Bezrukavnikov basis for $\mathrm{A}(\Sigma_g, n)^{\mathfrak{S}_n}$

The results exposed in this section are similar to the ones obtained in [FT05].

We fix an ordered basis $\{ b_i \}_{i=1,\dots r}$ for the cohomology $H^\bullet(X; \mathbb{Q})$. We denote by $|b_i|$ the degree of $b_i \in H^{|b_i|}(X)$. In [Bez94] (see also [AAB14, Aza15]) a canonical Bezrukavnikov basis for $\mathrm{A}(X, n)$ was given. We need a such canonical basis for $\mathrm{A}(X, n)^{\mathfrak{S}_n}$.

Definition 4.5.1. An *invariant monomial* is an element $m \in \mathrm{A}(X,n)$ of the form

$$m = G_{1,2}(b_{i_1})_1 G_{3,4}(b_{i_2})_3 \dots G_{2l-1,2l}(b_{i_l})_{2l-1}(b_{j_1})_{2l+1} \dots (b_{j_k})_{2l+k}$$

where $I = (i_1, \dots i_l)$, $J = (j_1, \dots j_k)$ are non-decreasing sequences such that every integer r appears at most once in I if $|b_r|$ is even and at most once in J if $|b_r|$ is odd. Obviously, we also require $2l + k \leq n$.

We define the length of m as the natural number $2l + k$. By definition we have that $m_1 = \sigma(m_2)$ implies $m_1 = m_2$. We define a binary operation between invariant monomials.

Definition 4.5.2. Let m_1 and m_2 be two invariant monomials with indexing I_1, J_1 and I_2, J_2 respectively. Let I (and J) be the list $I_1 \cup I_2$ increasingly ordered (respectively $J_1 \cup J_2$). If the lists I and J define an invariant monomial m, then we define $m_1 \circ m_2 := sm \in E(X,n)$, where $s \in \{ \pm 1 \}$ is the following sign. Let $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ be two permutations such that $\sigma_1(m_1)\sigma_2(m_2) = \pm m$, the sign does not depend on the choice of the permutations thus we call it s. Otherwise, we define $m_1 \circ m_2 := 0$.

Lemma 4.5.3. *An additive basis for the invariants* $\mathrm{A}(X,n)^{\mathfrak{S}_n}$ *is given by the elements* $\frac{1}{(n-l(m))!}\sum_{\sigma\in\mathfrak{S}_n}\sigma(m)$ *where* m *runs over all invariant monomials in* $\mathrm{A}(X,n)$.

Proof. It is enough to prove that for each element x of the canonical Bezrukavnikov basis, the sum $\sum_{\sigma\in\mathfrak{S}_n}\sigma(x)$ is either zero or there exists a unique invariant monomial m such that

$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(x) = \sum_{\sigma\in\mathfrak{S}_n}\sigma(m).$$

Let (λ, s) be the unique partition such that $x \in \mathrm{A}(\lambda, s)$. If $\lambda_1 > 2$ then we have proven in Theorem 4.4.1 that $\mathrm{A}(\lambda,s)^{\mathfrak{S}_n} = 0$, so $\sum_{\sigma\in\mathfrak{S}_n}\sigma(x) = 0$.

If x is of the form $(b)_i(b)_j y$ with $|b|$ odd and y without indexes i, j, then

$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(x) = \binom{n}{2}\big((b)_i(b)_j + (b)_j(b)_i\big)\sum_{\mu\in\mathfrak{S}_{n-2}}\mu(y) = 0.$$

If x is of the form $G_{i,j}(b)_i G_{k,l}(b)_k y$ with $|b|$ even and y without indexes i, j, k, l, then

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_n}\sigma(x) &= \binom{n}{4}\sum_{\tau\in\mathfrak{S}_4}\tau(G_{i,j}(b)_i G_{k,l}(b)_k)\sum_{\mu\in\mathfrak{S}_{n-4}}\mu(y)\\
&= \binom{n}{4}\sum_{\tau\in\mathfrak{S}_4/\langle(i,k)(j,l)\rangle}\tau(G_{i,j}(b)_i G_{k,l}(b)_k + G_{k,l}(b)_k G_{i,j}(b)_i)\sum_{\mu\in\mathfrak{S}_{n-4}}\mu(y)\\
&= 0.
\end{aligned}$$

Otherwise, x is equal to $\sigma(m)$ for some $\sigma \in \mathfrak{S}_n$ and some invariant monomial m. Since the indexing of an invariant monomial is non-decreasing, the monomial m is unique. □

Recall from definition 4.3.9 the increasing filtration of $\mathrm{A}(X,n)^{\mathfrak{S}_n}$ given by

$$\mathrm{F}_i\,\mathrm{A}(X,n)^{\mathfrak{S}_n} = \Big\langle \sum_{\sigma\in\mathfrak{S}_n} \sigma(m) \;\Big|\; m \text{ invariant monomial with } 2l+k \leq i \Big\rangle.$$

Lemma 4.5.4. *For all monomials m_1 and m_2 of length l_1 and l_2 we have*

$$\frac{1}{(n-l_1)!}\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1)\frac{1}{(n-l_2)!}\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_2) = \frac{1}{(n-l_1+l_2)!}\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1\circ m_2)+x$$

for some $x \in \mathrm{F}_{l_1+l_2-1}\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$. In particular, the filtration $\mathrm{F}_i\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$ is multiplicative.

Proof. We compute the product of $\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1)$ and $\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_2)$ for two invariant monomials of length l_1 and l_2, respectively. If $l_1+l_2>n$ then the product belongs to $\mathrm{F}_{l_1+l_2}\,\mathrm{A}(X,n)^{\mathfrak{S}_n} = \mathrm{A}(X,n)^{\mathfrak{S}_n}$. We expand the product and obtain

$$\Big(\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1)\Big)\Big(\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_2)\Big) = \sum_{\sigma,\tau\in\mathfrak{S}_n}\sigma\big(m_1\tau(m_2)\big).$$

There are two cases:

- if $\tau(\{1,\dots,l_2\})\cap\{1,\dots,l_1\}\neq\emptyset$ then
$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1\tau(m_2))\in\mathrm{F}_{l_1+l_2-1}\,\mathrm{A}(X,n)^{\mathfrak{S}_n}.$$
- Otherwise, $\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1\tau(m_2)) = \sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1\circ m_2)$.

The number of permutations τ such that $\tau(\{1,\dots,l_2\})\cap\{1,\dots,l_1\}=\emptyset$ is equal to $\frac{(n-l_1)!(n-l_2)!}{(n-l_1-l_2)!}$. Thus we have:

$$\sum_{\sigma,\tau\in\mathfrak{S}_n}\sigma\big(m_1\tau(m_2)\big) = \frac{(n-l_1)!(n-l_2)!}{(n-l_1-l_2)!}\sum_{\sigma\in\mathfrak{S}_n}\sigma(m_1\circ m_2)+x,$$

for some $x\in\mathrm{F}_{l_1+l_2-1}\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$. □

Corollary 4.5.5. *The algebra $\mathrm{gr}_{\mathrm{F}}\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$ is generated in degrees one and two.*

Proof. Using Lemma 4.5.4, we deduce that $\mathrm{gr}_{\mathrm{F}}\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$ is generated by the classes of $\sum_{i=1}^n(b)_i\in\mathrm{F}_1\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$ and of $\sum_{i\neq j}G_{i,j}(b)_i\in\mathrm{F}_2\,\mathrm{A}(X,n)^{\mathfrak{S}_n}$ for b in the chosen basis of $H^\bullet(X)$. □

The $K(X,n)$ model

We present a simpler model for the cohomology of unordered configuration spaces in a compact orientable even dimensional manifold. Let I_n be the ideal of $\mathrm{A}(X,n)^{\mathfrak{S}_n}$ generated by the elements $y := \sum_{\sigma\in\mathfrak{S}_n}\sigma(G_{1,2}[X]_1)$ and x^2, where $x := \sum_{\sigma\in\mathfrak{S}_n}\sigma([X]_1)$.

Definition 4.5.6. Let $K(X,n)$ be the quotient $\mathrm{A}(X,n)^{\mathfrak{S}_n}/I_n$.

Lemma 4.5.7. *The quotient $K(X,n)$ is a dga and the induced map*

$$H^\bullet(\mathrm{A}(X,n)^{\mathfrak{S}_n}) \to H^\bullet(K(X,n))$$

is an isomorphism.

Proof. If $n=1$ then $I_1=0$ and there is nothing to prove; assume $n>1$. Let us check the containment $\mathrm{d}(I_n)\subseteq I_n$ on the generators of the ideal I_n: we have $\mathrm{d}\big((\sum_{\sigma\in\mathfrak{S}_n}\sigma([X]_1))^2\big)=0$ since $\mathrm{d}([X]_1)=0$. Recall that $\mathrm{d}(G_{1,2}) = \Delta_{1,2} = \sum(-1)^{|b|}b_1b_2^*$, where b runs over the chosen basis of $H^\bullet(X)$ and b^* is the Poincaré dual of b. Thus we have

$$\mathrm{d}(y) = \sum_{\sigma\in\mathfrak{S}_n}\sigma(\mathrm{d}(G_{1,2})[X]_1) = \sum_{\sigma\in\mathfrak{S}_n}\sigma([X]_1[X]_2),$$

and since

$$x^2 = \sum_{\sigma,\tau\in\mathfrak{S}_n}\sigma\big([X]_1\tau([X]_1)\big) = (n-1)!(n-1)\sum_{\sigma\in\mathfrak{S}_n}\sigma([X]_1[X]_2),$$

we have proven that $K(X,n)$ is a dga.

We claim that $H^\bullet(I,\mathrm{d})=0$, the sought isomorphism will follow from the long exact sequence associated to $I\to\mathrm{A}(X,n)\to K(X,n)$. Let $z=x^2a+yb$ be an element in I such that $\mathrm{d}(z)=0$. By Corollary 4.5.5 we can suppose $a,b\in\mathrm{F}_{n-2}\mathrm{A}(X,n)$. The relation

$$x^2\,\mathrm{d}(a) + \frac{x^2b}{(n-1)!(n-1)} - y\,\mathrm{d}(b) = 0$$

implies that $x^2\big(\mathrm{d}(a)+\frac{b}{(n-1)!(n-1)}\big)$ belongs to (y). The ideal (y) is generated as vector space by all the invariant monomials such that $i_l=[X]$. Because $\mathrm{d}(a)\in\mathrm{F}_{n-2}\mathrm{A}(X,n)$, it follows that $b=-(n-1)!(n-1)\,\mathrm{d}(a)+cy$. Finally, since $y^2=0$, we have $z=(n-1)!(n-1)\,\mathrm{d}(ya)$ and hence $H^\bullet(I,\mathrm{d})=0$. □

The stable algebra of surfaces

We want to compute all the cohomology groups of all configuration spaces $\mathcal{C}_n(\Sigma_g)$ simultaneously.

Fix a basis $\{b_i\}_i$ of the cohomology algebra $H^\bullet(X)$ and let $\{b_i^*\}_i$ be its dual basis. The cohomology class of the diagonal is $\Delta=\sum_i(-1)^{|b_i|}b_i\otimes b_i^*$, where $|b_i|$ is the degree of b_i.

Definition 4.5.8. The *stable algebra* U_g is the dga defined by

$$\mathrm{U}_g \overset{\text{def}}{=} \wedge^\bullet(\tilde{H}^\bullet(\Sigma_g) \oplus H^{\leq 1}(\Sigma_g)[1]) \big/ ([p_g]^2).$$

The generators in $H^p(X)[1]$ are of bidegree $(p, p+1)$ and will be denoted by $\tilde{b}$ for $b \in H^p(\Sigma_g)$. The ones in $\tilde{H}^p(\Sigma_g)$ are of bidegree $(p, 0)$.

The differential d of bidegree $(2, -1)$ is defined by:

$$\begin{aligned} \mathrm{d}(\tilde{1}) &= [p_g] - \sum_{i=1}^{2g} b_i b_i^*, & \\ \mathrm{d}(\tilde{b}) &= [p_g]b & \text{for } b \in H^1(\Sigma_g), \\ \mathrm{d}(b) &= 0 & \text{for } b \in H^\bullet(\Sigma_g). \end{aligned}$$

Define the increasing filtration $\mathrm{F}_\bullet \mathrm{U}_g$ of U_g by

$$\mathrm{F}_i \mathrm{U}_g = \langle x_1 x_2 \dots x_k \tilde{y}_1 \tilde{y}_2 \dots \tilde{y}_l \mid k + 2l \leq i \rangle.$$

We will omit the index g when the genus is unambiguous. Let $\lambda(n) = 2n - \chi(X)$ where χ is the Euler characteristic, from now on we will suppose $\lambda(n) \neq 0$. This assumption excludes only the case $\mathcal{C}_1(\mathbb{CP}^1)$.

We define the morphism $\varphi_n \colon \mathrm{U}_g \to K(\Sigma_g, n)$ given by:

$$\begin{aligned} \varphi_n(\tilde{1}) &= \sum_{i \neq j} G_{i,j} & \\ \varphi_n(\tilde{x}) &= \frac{\lambda(n)}{2} \sum_{i \neq j} G_{i,j} x_i & \text{for } x \in H^1(\Sigma_g) \\ \varphi_n([p_g]) &= \lambda(n) \sum_{i=1}^n [p_g]_i & \\ \varphi_n(x) &= \sum_{i=1}^n x_i & \text{for } x \in H^{\leq 1}(\Sigma_g). \end{aligned}$$

Lemma 4.5.9. *If $\lambda(n) \neq 0$, the map φ_n induces an isomorphism of chain complexes $\mathrm{F}_n \mathrm{U}_g \xrightarrow{\sim} K(\Sigma_g, n)$.*

Proof. Let us fix a basis $1, a^1, \dots a^g, b^1, \dots, b^g, [p_g]$ of $H^\bullet(\Sigma_g)$ and its dual basis $[p_g], b^1, \dots, b^g, -a^1, \dots, -a^g, 1$. The morphism φ_n is well defined since $\varphi_n([p_g]^2) = \sum_{i,j} [p_g]_i [p_g]_j \in I_n$. It is a morphism of differential algebras because

$$\begin{aligned}
d(\varphi_n(\tilde{1})) &= d\Big(\sum_{i\neq j} G_{i,j}\Big) \\
&= \sum_{i\neq j}\Big([p_g]_i + [p_g]_j - \sum_{k=1}^{g} a_i^k b_j^k - b_i^k a_j^k\Big) \\
&= 2(n-1+g)\sum_{i=1}^{n}[p_g]_i - \sum_{k=1}^{g}\Big(\sum_{i=1}^{n} a_i^k\Big)\Big(\sum_{i=1}^{n} b_i^k\Big) - \Big(\sum_{i=1}^{n} b_i^k\Big)\Big(\sum_{i=1}^{n} a_i^k\Big) \\
&= \varphi_n(d(\tilde{1})).
\end{aligned}$$

An analogous computation shows that $d(\varphi_n(\tilde{x})) = \varphi_n(d(\tilde{x}))$ for $x \in H^1(\Sigma_g)$.

By definition we have that $\varphi_n(F_i \operatorname{U}_g) \subseteq F_i K(\Sigma_g, n)$. It is enough to prove that the induced map $\operatorname{gr}_F F_n \operatorname{U}_g \to \operatorname{gr}_F K(\Sigma_g, n)$ is an isomorphism.

The basis for $\operatorname{gr}_F K(\Sigma_g, n)$ is given by the invariant monomials with $b_{i_l} \neq [p_g]$ and $b_{j_{k-1}} \neq [p_g]$, if $k > 1$. The surjectivity of φ_n follows from the multiplication law in $\operatorname{gr}_F K(\Sigma_g, n)$. Since $K(\Sigma_g, n)$ and $F_n \operatorname{U}_g$ have the same dimension the map $\varphi_{n|F_n A}$ is an isomorphism. □

4.6 Some facts of representation theory

We consider $\operatorname{gr}_{F_\bullet} H^\bullet(\Sigma_g;\mathbb{Q})$ as a representation of the Lie algebra $\mathfrak{sp}(2g)$ associated to the symplectic group. Call the fundamental weights of $\mathfrak{sp}(2g)$ $\omega_1, \dots, \omega_g$. The cohomology of Σ_g in degree one is given by the standard representation, i.e. $H^1(\Sigma_g) = V_{\omega_1}$.

Let $a = \tilde{1}$, $b = [p_g]$ and $V = H^1(\Sigma_g)$. The algebra U_g is isomorphic to $\wedge^\bullet V \otimes \operatorname{S}^\bullet V[a, b]/(b^2)$ with gradation given by: $\deg 1 \otimes v = (1, 1)$, $\deg v \otimes 1 = (1, 0)$, $\deg a = (0, 1)$ and $\deg b = (2, 0)$. The differential is given by $d(1 \otimes v) = b(v \otimes 1)$ and $d(a) = b + \omega$ where

$$\omega \overset{\text{def}}{=} -2\Big(\sum_{j=1}^{g} a_j \wedge b_j\Big) \otimes 1,$$

while on the other generators the differential is zero.

Before computing the cohomology of (U_g, d) we need to know the cohomology of $(\wedge^\bullet V \otimes \operatorname{S}^\bullet V, \tilde{d})$, where the differential $\tilde{d}$ is defined by $\tilde{d}(1 \otimes v) = v \otimes 1$ and $\tilde{d}(v \otimes 1) = 0$. The standard action of $\mathfrak{sl}(2g)$ on V induces an action on $(\wedge^\bullet V \otimes \operatorname{S}^\bullet V, \tilde{d})$, since the differential $\tilde{d}$ is $\mathfrak{sl}(2g)$-equivariant.

By an abuse of notation, we will call $\omega_1, \dots, \omega_{2g-1}$ the fundamental weights of $\mathfrak{sl}(2g)$ and W_λ its irreducible representations associated to a dominant weight λ. Set $\omega_{2g} = 0$.

Lemma 4.6.1. *The $\mathfrak{sl}(2g)$-representation $\wedge^j V \otimes \operatorname{S}^i V$ decomposes, for $j \leq 2g$, as*

$$\wedge^j V \otimes \operatorname{S}^i V = W_{i\omega_1 + \omega_j} \oplus W_{(i-1)\omega_1 + \omega_{j+1}}.$$

Proof. It is known that $\mathrm{S}^i V_{\omega_1} = W_{i\omega_1}$ and $\wedge^j V_{\omega_1} = W_{\omega_j}$. Let $\sigma = (1, j+1) \in \mathfrak{S}_{2g}$ be an element of the Weyl group of $\mathfrak{sl}(2g)$. The element $i\omega_1 + \sigma(\omega_j) = (i-1)\omega_1 + \omega_{j+1}$ is a dominant weight for $i > 0$. By the Parthasarathy–Ranga-Rao–Varadarajan conjecture (see [Lit94, Kum88]) $W_{i\omega_1+\omega_j}$ and $W_{(i-1)\omega_1+\omega_{j+1}}$ are contained in the tensor product $W_{i\omega_1} \otimes W_{\omega_j}$. Use the Weyl character formula to find

$$\dim W_{i\omega_1+\omega_j} = \binom{i+j-1}{i}\binom{i+2g}{i+j}.$$

The equality $\binom{i+j-1}{i}\binom{i+2g}{i+j} + \binom{i+j-1}{i-1}\binom{i+2g-1}{i+j} = \binom{2g}{j}\binom{i+2g-1}{i}$ completes the proof. □

Lemma 4.6.2. *The differential complex* $(\wedge^\bullet V \otimes \mathrm{S}^\bullet V, \tilde{\mathrm{d}})$ *is exact in positive degree.*

Proof. The differential

$$\tilde{\mathrm{d}} \colon W_{i\omega_1+\omega_j} \oplus W_{(i-1)\omega_1+\omega_{j+1}} \to W_{(i-1)\omega_1+\omega_{j+1}} \oplus W_{(i-2)\omega_1+\omega_j}$$

is a non-zero morphism of representations. Thus, we have $W_{i\omega_1+\omega_j} = (\ker \tilde{\mathrm{d}})^{j,i}$ and $W_{(i-1)\omega_1+\omega_{j+1}} = (\mathrm{Im}\,\tilde{\mathrm{d}})^{j+1,i-1}$ for $j \geq 0$. Obviously $(\mathrm{Im}\,\tilde{\mathrm{d}})^{0,0} = 0$, so

$$H^\bullet(\wedge^\bullet V \otimes \mathrm{S}^\bullet V, \tilde{\mathrm{d}}) = \mathbb{Q}.$$ □

Since the Lie algebra $\mathfrak{sl}(2g)$ does not act on U_g, we need to present a branching rule for $\mathfrak{sp}(2g) \subset \mathfrak{sl}(2g)$. For the sake of an uniform notation, we define $V_\lambda = 0$ if λ is not a dominant weight.

Lemma 4.6.3 (Branching rule)**.** *The* $\mathfrak{sl}(2g)$*-module* $W_{i\omega_1+\omega_j}$ *decomposes as* $\mathfrak{sp}(2g)$*-module in the following ways:*

$$W_{i\omega_1+\omega_j} = \bigoplus_{k=0}^{\lfloor\frac{j-1}{2}\rfloor} V_{i\omega_1+\omega_{j-2k}} \oplus \bigoplus_{k=0}^{\lfloor\frac{j-2}{2}\rfloor} V_{(i-1)\omega_1+\omega_{j-2k-1}} \quad \text{when } 2 \leq j \leq g,$$

$$W_{i\omega_1+\omega_j} = \bigoplus_{k=0}^{\lfloor\frac{2g-j-1}{2}\rfloor} V_{i\omega_1+\omega_{2g-j-2k}} \oplus \bigoplus_{k=0}^{\lfloor\frac{2g-j-2}{2}\rfloor} V_{(i-1)\omega_1+\omega_{2g-j-2k-1}} \quad \text{when } j \geq g.$$

Proof. We apply the result of [ST16, Theorem 1]. The diagram associated to $i\omega_1 + \omega_j$ has a hook shape with row of length $i+1$ and column of length j. Fill each box with labels in the ordered set $\{1 < ... < g < \bar{g} < ... < \bar{1}\}$, such that it becomes a semi-standard Young tableaux (SSYT) i.e. the rows are non-decreasing and columns are increasing. The word $w(T)$ – associated to a SSYT T – is the word obtained by reading the tableaux from right to left and from top to bottom. By convention, $e_{\bar{a}} = -e_a$. A word $w(T) = a_1a_2\dots a_k$

is admissible if for each $r \leq k$ the element $\sum_{s=1}^{r} e_{a_s}$ is a dominant weight for $\mathfrak{sp}(2g)$. The decomposition of $W_{i\omega_1+\omega_j}$ into $\mathfrak{sp}(2g)$-representations is given by

$$W_{i\omega_1+\omega_j} = \bigoplus_{w(T) \text{ adimissible}} V_{\lambda(T)},$$

where $\lambda(T) = \sum_{s=1}^{|w(T)|} e_{a_s}$.

Suppose $w(T)$ is admissible, then the first row of T is labelled only by ones. For $j \leq g$, all possible labels of the first column of T, from top to bottom, are the followings:

- $1, 2, \ldots, j-k, \overline{j-k}, \overline{j-k-1}, \ldots, \overline{j-2k+1}$, where k is an integer such that $0 \leq 2k \leq j-1$
- $1, 2, \ldots, j-k-1, \overline{j-k-1}, \ldots, \overline{j-2k}, \bar{1}$, where k is an integer such that $0 \leq 2k \leq j-2$ and $i > 0$.

Our decomposition follows, the case $j > g$ being analogous. □

The differential d involves the multiplication by ω, thus we need to study the operator $\omega\colon \wedge^i V \otimes \mathrm{S}^j V \to \wedge^i V \otimes \mathrm{S}^{j+2} V$.

Lemma 4.6.4 ([FH91, Theorem 17.5])**.** *The $\mathfrak{sp}(2g)$-representation $\wedge^j V$ is isomorphic to $\wedge^{2g-j} V$ and decomposes, for $j \leq g$, as*

$$\wedge^j V = W_{\omega_j} = \bigoplus_{k=0}^{\lfloor \frac{j}{2} \rfloor} V_{\omega_{j-2k}}.$$

Moreover, $(\ker \omega)^{2g-j} = V_{\omega_j} \subset \wedge^{2g-j} V$ *and* $(\operatorname{coker} \omega)^j = V_{\omega_j} \subset \wedge^j V$.

Lemma 4.6.5. *For $i > 0$ and $1 \leq j \leq g$, we have*

$$V_{\omega_j} \otimes \mathrm{S}^i V = V_{i\omega_1+\omega_j} \oplus V_{(i-1)\omega_1+\omega_{j+1}} \oplus V_{(i-1)\omega_1+\omega_{j-1}} \oplus V_{(i-2)\omega_1+\omega_j}.$$

Proof. We use Lemmas 4.6.1, 4.6.3 and 4.6.4:

$$\begin{aligned} V_{\omega_j} \otimes \mathrm{S}^i V &= \wedge^j V \otimes \mathrm{S}^i V \ominus \wedge^{j-2} V \otimes \mathrm{S}^i V \\ &= W_{i\omega_1+\omega_j} \oplus W_{(i-1)\omega_1+\omega_{j+1}} \ominus W_{i\omega_1+\omega_{j-2}} \ominus W_{(i-1)\omega_1+\omega_{j-1}} \\ &= V_{i\omega_1+\omega_j} \oplus V_{(i-1)\omega_1+\omega_{j+1}} \oplus V_{(i-1)\omega_1+\omega_{j-1}} \oplus V_{(i-2)\omega_1+\omega_j}. \end{aligned}$$ □

We denote by R_g the Grothendieck ring of $\mathfrak{sp}(2g)$.

Definition 4.6.6. Let W be a bigraded representation of the group $\mathfrak{sp}(2g)$. The Hilbert–Poincaré series of W is the formal power series

$$P_W(t, s) = \sum_{i,j} [W^{i,j}] t^i s^j \quad \in R_g[[t, s]].$$

Corollary 4.6.7. *The Hilbert–Poincaré series of the representation* $\wedge^\bullet V \otimes \mathrm{S}^\bullet V$ *is*

$$\begin{aligned} P_{\wedge V \otimes \mathrm{S} V}(t,s) = & \frac{t^{2(g+1)}-1}{t^2-1} + t^2 s \frac{t^{2g}-1}{t^2-1} + \\ & + (1+s)(1+t^2 s) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1+\omega_j}] \frac{t^{2(g-j+1)}-1}{t^2-1} t^{i+j} s^i. \end{aligned}$$

The Corollary follows from Lemmas 4.6.4 and 4.6.5.

Lemma 4.6.8. *We have*

$$\dim V_{i\omega_1+\omega_j} = \binom{2g+i+1}{i,j} \frac{2g+2-2j}{2g+2+i-j} \frac{j}{i+j}, \tag{4.14}$$

Proof. Recall that the positive roots of the Lie algebra $\mathfrak{sp}(2g)$ are $e_k \pm e_h$ for $1 \leq k < h \leq g$ and $2e_k$ for $1 \leq k \leq g$. Moreover $\rho = \sum_{k=1}^{g} (g+1-k) e_k$. Apply the Weyl formula for the dimension of a representation:

$$\begin{aligned} \prod_{k<h} \frac{\langle i\omega_1 + \omega_j + \rho, e_k - e_h \rangle}{\langle \rho, e_k - e_h \rangle} &= \frac{(g+i)!}{i!(g-j)!(j-1)!(i+j)} \\ \prod_{k<h} \frac{\langle i\omega_1 + \omega_j + \rho, e_k + e_h \rangle}{\langle \rho, e_k + e_h \rangle} &= \frac{(2g+i+1)!(g+1-j)!(2g+2-2j)}{(2g+i+1)!(2g+1-j)!(2g+i+2-j)} \\ \prod_{k=1}^{g} \frac{\langle i\omega_1 + \omega_j + \rho, 2e_k \rangle}{\langle \rho, 2e_k \rangle} &= \frac{g+1+i}{g+1-j}. \end{aligned}$$

We obtain eq. (4.14) by multiplying the right hand sides of the previous equations. ☐

4.7 The cohomology of configuration spaces

The formula for the Betti numbers of $\mathcal{C}_n(\Sigma_g)$ given in [DCK17] is different from ours (eq. (4.21)), which has no cancellations and a more geometric meaning.

The case of the sphere ($g = 0$) is essentially different from the case $g > 0$ and our approach is useless since $\mathfrak{sp}(2g)$ is trivial for $g = 0$. We refer to [Sal04, Sch18] for the proof of the following theorem.

Theorem 4.7.1. *The rational homology of* $\mathcal{C}_n(S^2)$ *is:*

$$\begin{aligned} & H^0(\mathcal{C}_n(S^2);\mathbb{Q}) = \mathbb{Q} & \\ & H^2(\mathcal{C}_2(S^2);\mathbb{Q}) = \mathbb{Q} & \\ & H^3(\mathcal{C}_n(S^2);\mathbb{Q}) = \mathbb{Q} & \text{for } n \geq 3 \\ & H^3(\mathcal{C}_n(S^2);\mathbb{Q}) = 0 & \text{otherwise.} \end{aligned}$$

The case of genus one surfaces is studied in [Sch16, Mag16].

From now on we assume $g > 0$. The following lemma is the proof of Conjecture 4.3.10 only for the invariant sub-algebra.

Lemma 4.7.2. *For $g > 0$ the filtration $\mathrm{F}_n \mathrm{U}_g$ is strictly compatible with the differential. Therefore,* $\mathrm{gr}_{\mathrm{F}_\bullet} H^\bullet(\mathrm{U}, \mathrm{d}) \simeq H^\bullet(\mathrm{gr}_{\mathrm{F}_\bullet} \mathrm{U}, \mathrm{gr}_{\mathrm{F}_\bullet} \mathrm{d})$.

The fact that the filtration $\mathrm{F}_n \mathrm{U}_g$ is strictly compatible with the differential d is related to the rational homological stability of $\mathcal{C}_n(\Sigma_g)$ proven in [Chu12, RW13] in more generality.

Proof of Lemma 4.7.2. We need to prove that $\mathrm{Im}\, \mathrm{d} \cap \mathrm{F}_n \mathrm{U} \subseteq \mathrm{d}(\mathrm{F}_n \mathrm{U})$ for all $n \geq 0$. Consider a generic element $ax + aby + z + bw$ in $\mathrm{F}_n \mathrm{U}$ with $x \in \mathrm{F}_{n-2} \mathrm{U}$, $y \in \mathrm{F}_{n-3} \mathrm{U}$, $z \in \mathrm{F}_n \mathrm{U}$, and $w \in \mathrm{F}_{n-1} \mathrm{U}$. Suppose that $\mathrm{d}(ax + aby + z + bw) \in \mathrm{F}_{n-1} \mathrm{U}$, then we have

$$\mathrm{d}(ax + aby + z + bw) = bx + \omega x + ab\tilde{\mathrm{d}}(x) + \omega by + b\tilde{\mathrm{d}}(z).$$

It follows that $\tilde{\mathrm{d}}(x) \in \mathrm{F}_{n-4} \mathrm{U}$, $\tilde{\mathrm{d}}(z) + \omega y \in \mathrm{F}_{n-2} \mathrm{U}$ and $\omega x \in \mathrm{F}_{n-1} \mathrm{U}$. Since the filtration, restricted to $\wedge^\bullet V \otimes \mathrm{S}^\bullet V$, is induced by the total degree, we can suppose x, y, z being homogeneous of total degree $n-2$, $n-3$, and n respectively. So we have $\tilde{\mathrm{d}}(x) = 0$, $\tilde{\mathrm{d}}(z) + \omega y = 0$, and $\omega x = 0$. We deduce from $\omega x = 0$ that $\deg(x) > 0$ and hence, from $\tilde{\mathrm{d}}(x) = 0$, that $x = \tilde{\mathrm{d}}(x')$ for some x' of total degree $n-1$. It follows that $\mathrm{d}(ax + aby + z + bw) = \mathrm{d}(x')$ for $x' \in \mathrm{F}_{n-1} \mathrm{U}$ and $\mathrm{Im}\, \mathrm{d} \cap \mathrm{F}_{n-1} \mathrm{U} \subseteq \mathrm{d}(\mathrm{F}_{n-1} \mathrm{U})$. ☐

From now on we will work in $\mathrm{gr}_\mathrm{F} \mathrm{U}$ with the differential $\mathrm{gr}_F \mathrm{d}$. The only difference between d and $\mathrm{gr}_\mathrm{F} \mathrm{d}$ is that $(\mathrm{gr}_\mathrm{F} \mathrm{d})(a) = \omega$. By abuse of notation we denote by d the differential of $\mathrm{gr}_\mathrm{F} \mathrm{U}$.

Lemma 4.7.3. *The kernel of the differential* d *is the direct sum of the following vector spaces:*

1. $(\ker \tilde{\mathrm{d}} \cap \ker \omega)[0, 1]$,
2. $(\mathrm{Im}\, \tilde{\mathrm{d}} \cap \ker \omega)[2, 2] \oplus \ker \tilde{\mathrm{d}}[2, 1]$,
3. $\ker \tilde{\mathrm{d}}$,
4. $\wedge^\bullet V \otimes \mathrm{S}^\bullet V[2, 0]$.

Proof. Consider a generic element $abx + ay + bz + w$ with $x, y, z, w \in \wedge^\bullet V \otimes \mathrm{S}^\bullet V$: its differential is

$$\mathrm{d}(ax + aby + z + bw) = \omega x + ab\tilde{\mathrm{d}}(x) + \omega by + b\tilde{\mathrm{d}}(z). \qquad (4.15)$$

Therefore $\mathrm{d}(ax + aby + z + bw) = 0$ if and only if $\omega x = 0$, $\tilde{\mathrm{d}}(x) = 0$ and $\omega y + \tilde{\mathrm{d}}(z) = 0$. The equations $\omega x = 0$ and $\tilde{\mathrm{d}}(x) = 0$, together with Lemma 4.6.2, imply that there exists x' such that $\tilde{\mathrm{d}}(x') = x$. The equation $\omega y + \tilde{\mathrm{d}}(z) = 0$ is equivalent to $\tilde{\mathrm{d}}(\omega y) = \omega \tilde{\mathrm{d}}(y) = 0$, thus $y \in \ker \tilde{\mathrm{d}} \oplus (\operatorname{Im} \tilde{\mathrm{d}} \cap \ker \omega)[0,1]$. Let z' be an element such that $\omega y = \tilde{\mathrm{d}}(z')$: then z is of the form $z' + z''$ for some $z'' \in \ker \tilde{\mathrm{d}}$ and w can be any element in $\wedge^\bullet V \otimes \mathrm{S}^\bullet V$. □

Lemma 4.7.4. *The image of the differential* d *is the direct sum of the following vector spaces:*

1. 0,
2. $\operatorname{Im} \tilde{\mathrm{d}}[2,1]$,
3. $\omega \ker \tilde{\mathrm{d}}$
4. $\operatorname{Im} \omega[2,0] + \operatorname{Im} \tilde{\mathrm{d}}[2,0]$.

Proof. Eq. (4.15) implies that the image of d has trivial intersection with the submodule $a \wedge^\bullet V \otimes \mathrm{S}^\bullet V$. Consider x such that $\tilde{\mathrm{d}}(x) \neq 0$, then the element $\omega x + ab\tilde{\mathrm{d}}(x)$ gives the addendum $\operatorname{Im} \tilde{\mathrm{d}}[2,1]$. Now suppose $\tilde{\mathrm{d}}(x) = 0$ and $\omega x \neq 0$, then ωx is in the image and generates a submodule isomorphic to $\omega \ker \tilde{\mathrm{d}}$.

Finally, $\operatorname{Im} \tilde{\mathrm{d}} \cap b \wedge^\bullet V \otimes \mathrm{S}^\bullet V$ coincides with $\operatorname{Im} \omega[2,0] + \operatorname{Im} \tilde{\mathrm{d}}[2,0]$ (in general this is not a direct sum). □

The following lemma is a consequence of Lemmas 4.7.3 and 4.7.4.

Lemma 4.7.5. *The cohomology* $H^\bullet(\mathrm{U}, \mathrm{d})$ *is generated by:*

1. $ay - x$ *if* $y = \tilde{\mathrm{d}}(x) \in \ker \tilde{\mathrm{d}} \cap \ker \omega$,

(2.1) $ab + \overline{\omega}$,

(2.2) $aby - x$ *if* $\tilde{\mathrm{d}}y \in \ker \tilde{\mathrm{d}} \cap \ker \omega$ *and* $\omega y = \tilde{\mathrm{d}}x$,

(3) y *if* $y \in \ker \tilde{\mathrm{d}} / \omega \ker \tilde{\mathrm{d}}$,

(4) by *if* $y \in \wedge^\bullet V \otimes \mathrm{S}^\bullet V / (\operatorname{Im} \tilde{\mathrm{d}} + \operatorname{Im} \omega)$.

Lemma 4.7.6. *The cohomologies of* $\ker \omega$, $\operatorname{Im} \omega$, *and* $\operatorname{coker} \omega$ *with respect to the differential* $\tilde{\mathrm{d}}$ *are given by:*

$$H^{0,0}(\operatorname{coker} \omega) = \langle 1 \rangle \tag{4.16}$$
$$H^{1,1}(\operatorname{coker} \omega) = \langle \overline{\omega} \rangle \tag{4.17}$$
$$H^{2,0}(\operatorname{Im} \omega) = \langle \omega \rangle \tag{4.18}$$
$$H^{g,i}(\operatorname{coker} \omega) = H^{g+1,i-1}(\operatorname{Im} \omega) = H^{g,i-2}(\ker \omega) \simeq V_{(i-2)\omega_1 + \omega_g} \tag{4.19}$$
$$H^{j,i}(\operatorname{coker} \omega) = H^{j+1,i-1}(\operatorname{Im} \omega) = H^{j,i-2}(\ker \omega) = 0. \tag{4.20}$$

Proof. Consider the two short exact sequences

$$0 \to \ker\omega \to \wedge^\bullet V \otimes \mathrm{S}^\bullet V \to \operatorname{Im}\omega[2,0] \to 0$$
$$0 \to \operatorname{Im}\omega \to \wedge^\bullet V \otimes \mathrm{S}^\bullet V \to \operatorname{coker}\omega \to 0.$$

By Lemma 4.6.2 we have $H^{j,i}(\operatorname{coker}\omega) \simeq H^{j+1,i-1}(\operatorname{Im}\omega)$ for $(j,i) \neq (0,0)$ and $H^{j,i}(\operatorname{Im}\omega) \simeq H^{j-1,i-1}(\ker\omega)$ for $(j,i) \neq (2,0)$. Eq. (4.16), (4.17) and (4.18) follow immediately from the long exact sequence in cohomology. Since $(\ker\omega)^{j,i} = 0$ for $j < g$ and $(\operatorname{coker}\omega)^{j,i} = 0$ for $j > g$, we deduce eq. (4.20). The only representation that can appear in $H^{g,i}(\operatorname{coker}\omega) \simeq H^{g,i-2}(\ker\omega)$ is $V_{(i-2)\omega_1+\omega_g}$. It is easy to see that the subspace $V_{i\omega_1+\omega_g} \subset V_{\omega_g} \otimes S^iV$ is contained in $\ker\omega \cap \ker \mathrm{d}$, but cannot lay in $\mathrm{d}(\ker\omega)$ since $(\ker\omega)^{g-1,i+1} = 0$. Finally, we have proven eq. (4.19). □

Lemma 4.7.7. *The Hilbert–Poincaré series of* $\ker\tilde{\mathrm{d}} \cap \ker\omega$ *is*

$$P_{\ker\tilde{\mathrm{d}}\cap\ker\omega}(t,s) = t^{2g} + (1+t^2s)\sum_{\substack{1\leq j\leq g\\ i\geq 0}} [V_{i\omega_1+\omega_j}]t^{2g-j+i}s^i.$$

Proof. Notice that $\ker\tilde{\mathrm{d}} \cap \ker\omega = \ker(\tilde{\mathrm{d}}_{|\ker\omega})$ and that

$$\begin{aligned} P_{\ker\omega}(t,s) = & \sum_{\substack{1\leq j<g\\ i\geq 0}} [V_{i\omega_1+\omega_j}]t^{2g-j+i}s^i(1+s)(1+t^2s)+ \\ & + t^{2g} + t^{2g}s + \sum_{i\geq 0}[V_{i\omega_1+\omega_g}]t^{g+i}s^i(1+t^2s+t^2s^2) \\ P_{H(\ker\omega)}(t,s) = & \sum_{i\geq 0}[V_{i\omega_1+\omega_g}]t^{g+i}s^i. \end{aligned}$$

Using the formula $(s+1)P_{\ker(\tilde{\mathrm{d}}_{|\ker\omega})}(t,s) = P_{\ker\omega}(t,s) + sP_{H(\ker\omega)}(t,s)$ we obtain the claimed equality. □

Lemma 4.7.8. *The Hilbert–Poincaré series of* $\ker\tilde{\mathrm{d}}/\omega\ker\tilde{\mathrm{d}}$ *is*

$$P_{\ker\tilde{\mathrm{d}}/\omega\ker\tilde{\mathrm{d}}}(t,s) = 1 + (1+t^2s)\sum_{\substack{1\leq j\leq g\\ i\geq 0}} [V_{i\omega_1+\omega_j}]t^{j+i}s^i.$$

Proof. Consider the exact sequence

$$0 \to \ker\tilde{\mathrm{d}} \cap \ker\omega \to \ker\tilde{\mathrm{d}} \xrightarrow{\omega} \ker\tilde{\mathrm{d}} \to {}^{\ker\tilde{\mathrm{d}}}\!\big/_{\omega\ker\tilde{\mathrm{d}}} \to 0.$$

We have

$$P_{\ker\tilde{\mathrm{d}}/\omega\ker\tilde{\mathrm{d}}}(t,s) = (1-t^2)P_{\ker\tilde{\mathrm{d}}}(t,s) + t^2P_{\ker\tilde{\mathrm{d}}\cap\ker\omega}(t,s),$$

and by Lemma 4.7.7 we obtain the claimed equality. □

Lemma 4.7.9. *The Hilbert–Poincaré series of* $\wedge^\bullet V \otimes \mathrm{S}^\bullet V/\operatorname{Im}\omega + \operatorname{Im}\tilde{\mathrm{d}}$ *is*

$$P_{\wedge V\otimes \mathrm{S}\, V/\operatorname{Im}\omega+\operatorname{Im}\tilde{\mathrm{d}}}(t,s) = (1+t^2 s)\big(1+s\sum_{\substack{1\le j\le g\\ i\ge 0}}[V_{i\omega_1+\omega_j}]t^{j+i}s^i\big).$$

Proof. Let K be the quotient $\wedge^\bullet V \otimes \mathrm{S}^\bullet V/\operatorname{Im}\omega + \operatorname{Im}\tilde{\mathrm{d}}$. Consider the exact sequence

$$0 \to \operatorname{Im}\tilde{\mathrm{d}} \cap \operatorname{Im}\omega \to \operatorname{Im}\tilde{\mathrm{d}} \oplus \operatorname{Im}\omega \to \wedge^\bullet V \otimes \mathrm{S}^\bullet V \to K \to 0$$

and notice that $\operatorname{Im}\tilde{\mathrm{d}} \cap \operatorname{Im}\omega = \ker\tilde{\mathrm{d}} \cap \operatorname{Im}\omega = \ker(\tilde{\mathrm{d}}_{|\operatorname{Im}\omega})$. We compute $P_{\ker(\tilde{\mathrm{d}}_{|\operatorname{Im}\omega})}$ using the formula

$$(1+s)P_{\ker(\tilde{\mathrm{d}}_{|\operatorname{Im}\omega})}(t,s) = P_{\operatorname{Im}\omega}(t,s) + sP_{H(\operatorname{Im}\omega)}(t,s)$$

relative to the bi-graded complex $(\operatorname{Im}\omega, \tilde{\mathrm{d}}_{|\operatorname{Im}\omega})$. Notice that $P_{\wedge V\otimes \mathrm{S}\, V}(t,s) = (1+s)P_{\operatorname{Im}\tilde{\mathrm{d}}} + 1$ by Lemma 4.6.2, so we obtain

$$\begin{aligned}(1+s)P_K &= (1+s)\big(P_{\wedge V\otimes \mathrm{S}\, V} - P_{\operatorname{Im}\tilde{\mathrm{d}}} - P_{\operatorname{Im}\omega} + P_{\ker(\tilde{\mathrm{d}}_{|\operatorname{Im}\omega})}\big)\\ &= sP_{\wedge V\otimes \mathrm{S}\, V} + 1 - sP_{\operatorname{Im}\omega} + sP_{H(\operatorname{Im}\omega)}\\ &= sP_{\operatorname{coker}\omega} + 1 + sP_{H(\operatorname{Im}\omega)}.\end{aligned}$$

The equalities

$$\begin{aligned}P_{H(\operatorname{Im}\omega)}(t,s) &= t^2 + t^2 s\sum_{i\ge 0}[V_{i\omega_1+\omega_g}]t^{g+i}s^i\\ P_{\operatorname{coker}\omega} &= 1 + t^2 s + (1+s)(1+t^2 s)\sum_{\substack{1\le j<g\\ i\ge 0}}[V_{i\omega_1+\omega_j}]t^{j+i}s^i +\\ &\quad + (1+s+t^2s^2)\sum_{i\ge 0}[V_{i\omega_1+\omega_g}]t^{g+i}\end{aligned}$$

complete the proof. ∎

Theorem 4.7.10. *The Hilbert-Poincaré series* $P_{H(\mathrm{U})}(t,s) \in R_g[[t,s]]$ *of the cohomology* $H(\mathrm{U}_g, \mathrm{d})$ *is*

$$(1+t^2 s)(1+t^2+t^{2g}s) + (1+t^2 s)^2\sum_{\substack{1\le j\le g\\ i\ge 0}}[V_{i\omega_1+\omega_j}]t^{j+i}s^i(1+t^{2(g-j)}s).$$

Proof. We use Lemma 4.7.5 and the computations of Lemmas 4.7.7 to 4.7.9:

$$P_{H(\mathrm{U})} = sP_{\ker\tilde{\mathrm{d}}\cap\ker\omega} + t^2 s + t^2 s^2 P_{\ker\tilde{\mathrm{d}}\cap\ker\omega} + P_{\ker\tilde{\mathrm{d}}/\omega\ker\tilde{\mathrm{d}}} + t^2 P_{\wedge V\otimes \mathrm{S}\, V/\operatorname{Im}\omega+\operatorname{Im}\tilde{\mathrm{d}}}.$$

∎

Let us define the series $Q_g(t,s,u)$ in the Grothendieck ring of $\mathfrak{sp}(2g)$ as:

$$Q_g(t,s,u) \stackrel{\text{def}}{=} \sum_{i,j,n} [\mathrm{gr}^W_{i+2j} H^{i+j}(\mathcal{C}_n(\Sigma_g))] t^i s^j u^n. \tag{4.21}$$

Theorem 4.7.11. *If $g > 0$, the series $Q_g(t,s,u) \in R_g[[t,s,u]]$ is equal to*

$$Q_g(t,s,u) = \frac{1}{1-u}\Big((1+t^2su^3)(1+t^2u) + (1+t^2su^2)t^{2g}su^{2(g+1)} + \\ + (1+t^2su^2)(1+t^2su^3) \sum_{\substack{1\leq j\leq g \\ i\geq 0}} [V_{i\omega_1+\omega_j}] t^{j+i} s^i u^{j+2i} (1+t^{2(g-j)}su^{2(g-j+1)})\Big).$$

Proof. Use Lemma 4.7.2 and notice that $Q_K(t,s,u) = P_K(tu, su)$ for any sub-quotient K of $\wedge^\bullet V \otimes \mathrm{S}^\bullet V$, thus:

$$(1-u)Q_g = su^2 P_{\ker \tilde{\mathrm{d}} \cap \ker \omega} + t^2su^3 + t^2s^2u^4 P_{\ker \tilde{\mathrm{d}} \cap \ker \omega} + \\ + P_{\ker \tilde{\mathrm{d}}/\omega \ker \tilde{\mathrm{d}}} + t^2uP_{\wedge V \otimes \mathrm{S} V/\operatorname{Im}\omega + \operatorname{Im}\tilde{\mathrm{d}}}.$$

□

From Theorem 4.7.11 and Lemma 4.6.8 we obtain a formula for the mixed Hodge numbers and for the Betti numbers of $\mathcal{C}_n(\Sigma_g)$.

Using this techniques, we can perform analogous computation for non-orientable closed surfaces and for once-punctured orientable surfaces: this two cases are easier than the our one. The same techniques can be applied to compute the invariants of configuration spaces of algebraic surfaces with irregularity zero.

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