Dimostrazione elementare dei prodotti infiniti per $\sin(\pi z)$ e $\cos(\pi z)$

Poniamo per comodità $s = \sin t$ e $c = \cos t$. Per $n \geq 2$

$$
\begin{aligned}
I_n(z) \overset{def}{=} \int_0^{\pi/2} \cos(zt) c^n \, dt &= \frac{1}{z} \left[\sin(zt)c^n\right]_0^{\pi/2} + \frac{n}{z} \int_0^{\pi/2} \sin(zt) c^{n-1} s \, dt = \\
&= 0 - \frac{n}{z^2} \left[\cos(zt)c^{n-1}s\right]_0^{\pi/2} + \frac{n}{z^2} \int_0^{\pi/2} \cos(zt) \left[c^n - (n-1)c^{n-2}s^2\right] \, dt = \\
&= 0 + \frac{n}{z^2} \int_0^{\pi/2} \cos(zt) \left[nc^n - (n-1)c^{n-2}\right] \, dt
\end{aligned}
$$

$$\Longrightarrow z^2 I_n(z) = n^2 I_n(z) - n(n-1) I_{n-2}(z)$$

$$\Longrightarrow n(n-1) I_{n-2}(z) = (n^2 - z^2) I_n(z)$$

$$\boxed{\frac{I_{n-2}(z)}{I_{n-2}(0)} = \left(1 - \frac{z^2}{n^2}\right) \frac{I_n(z)}{I_n(0)}}$$

Calcoliamo esplicitamente $I_n(0)$ e $I_n(z)$ per $n = 0, 1$:

- $I_0(0) = \int_0^{\pi/2} dt = \frac{\pi}{2}$
- $I_0(z) = \int_0^{\pi/2} \cos(zt) \, dt = \frac{1}{z} \sin(\frac{\pi z}{2})$
- $I_1(0) = \int_0^{\pi/2} \cos t \, dt = 1$
- $I_1(z) = \int_0^{\pi/2} \cos(zt) \cos t \, dt =$ (integriamo per parti in due modi diversi)
  $= \left[\frac{\sin(zt)}{z} \cos t\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin(zt)}{z} \sin t \, dt$
  $\Longrightarrow zI_1(z) = \int_0^{\pi/2} \sin(zt) \sin t \, dt$
  $I_1(z) = \int_0^{\pi/2} \cos(zt) \cos t \, dt = [\cos(zt) \sin t]_0^{\pi/2} + \int_0^{\pi/2} z \sin(zt) \sin t \, dt$
  $\Longrightarrow \frac{1}{z} I_1(z) - \frac{1}{z} \cos(\frac{\pi z}{2}) = \int_0^{\pi/2} \sin(zt) \sin t \, dt$
  Confrontando, $zI_1(z) = \frac{1}{z} I_1(z) - \frac{1}{z} \cos(\frac{\pi z}{2}) \Longrightarrow (1 - z^2) I_1(z) = \cos(\frac{\pi z}{2})$

$$\boxed{\sin(\frac{\pi z}{2}) = \frac{\pi z}{2} \cdot \frac{I_0(z)}{I_0(0)}}\ , \quad \boxed{\cos(\frac{\pi z}{2}) = (1 - z^2) \frac{I_1(z)}{I_1(0)}}$$

La tesi seguirà facilmente se mostriamo $\lim_{n \to \infty} \frac{I_n(z)}{I_n(0)} = 1$:
essendo $\frac{1 - \cos(zt)}{t^2}$ limitata per $t \in (0, \frac{\pi}{2}]$, per qualche $A > 0$ vale

$$
\begin{aligned}
|I_n(0) - I_n(z)| &\leq \int_0^{\pi/2} |1 - \cos(zt)| \cos^n t \, dt \leq A \int_0^{\pi/2} t^2 \cos^n t \, dt \leq \\
&\overset{(t \leq \tan t)}{\leq} A \int_0^{\pi/2} t \cos^{n-1} t \sin t \, dt = A \left[-t \frac{\cos^n t}{n}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^n t}{n} \, dt = \frac{I_n(0)}{n}
\end{aligned}
$$

(abbiamo supposto $n \geq 1$). Dividendo per $I_n(0) > 0$ abbiamo la tesi. Quindi

$$\sin(\frac{\pi z}{2}) = \frac{\pi z}{2} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{(2k)^2}\right)$$

$$\cos(\frac{\pi z}{2}) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{(2k-1)^2}\right)$$

o anche

$$\boxed{\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}$$

$$\boxed{\cos(\pi z) = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right)}$$

Riferimenti bibliografici

[1] K. Venkatachaliengar. Elementary proofs of the infinite product for sin z and allied formulae. *The American Mathematical Monthly*, 69(6):pp. 541–545, 1962.