La formula di Stirling

$$\begin{aligned}\ln(n!) &= \sum_{k=1}^{n} \int_1^k \frac{dt}{t} = \int_1^n \frac{n - \lfloor t \rfloor}{t}\, dt = \int_1^n \frac{(n + \frac{1}{2} - t) + (\{t\} - \frac{1}{2})}{t}\, dt \\ &= \left(n + \frac{1}{2}\right) \ln n - n + 1 + \int_1^n \frac{\{t\} - \frac{1}{2}}{t}\, dt.\end{aligned}$$

Dunque $n! \sim C n^{n+\frac{1}{2}} e^{-n}$, con $C := 1 + \int_1^\infty \frac{\{t\}-\frac{1}{2}}{t}$, [1] ma

$$\frac{\pi}{2} \leftarrow \frac{(2n)!!^2}{(2n+1)!!(2n-1)!!} = \frac{(2n)!!^4}{(2n+1)!(2n)!} \sim \frac{2^{4n} n!^4}{(2n)!^2} \cdot \frac{1}{2n} \sim \frac{2^{4n} C^4 n^{4n+2} e^{-4n}}{C^2 (2n)^{4n+1} e^{-4n}} \cdot \frac{1}{2n} = \frac{C^2}{4}.$$

Il limite $\frac{(2n)!!^2}{(2n+1)!!(2n-1)!!} \to \frac{\pi}{2}$ si ottiene integrando ripetutamente per parti $I(n) := \int_0^\pi \sin^n(t)\, dt$, ottenendo $I(n) = \frac{n-1}{n} I(n-2)$, che dà proprio (essendo $I(0) = \pi$ e $I(1) = 2$)

$$1 \leftarrow \frac{I(2n+1)}{I(2n)} = \left(\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots\right) \cdot 2 \cdot \left(\frac{2n-1}{2n} \cdots\right)^{-1} \cdot \pi^{-1}.$$

[1] L'integrale converge essendo $\int_k^{k+1} \left(\{t\} - \frac{1}{2}\right) = 0$, da cui $\left|\int_k^{k+1} \frac{\{t\}-\frac{1}{2}}{t}\, dt\right| = \left|\int_k^{k+1} \left(\{t\} - \frac{1}{2}\right)\left(\frac{1}{t} - \frac{1}{k}\right) dt\right| \le \frac{1}{2k^2}$.