Duality of matrix pencils, singular pencils and linearizations

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Outline

- An old tool and a new one
  - Wong chains
  - Duality
- Applications
  - Symplectic/Hamiltonian pencils
  - Fiedler linearizations
  - $\mathbb{L}_1$ linearizations

Why Wong chains

Multiple eigenvectors are ill-defined:

$$Aw = 0$$
$$A(v + \alpha w) = 0$$

Jordan chains are ill-defined:

$$Aw = 0$$
$$A(v + \alpha w) = w$$

Eigenvectors of singular pencils are ill-defined:

$$(L_1 x + L_0)w = 0 \quad \forall x$$
$$(L_1 \lambda + L_0)(v + \alpha w) = 0$$

Things are more complicated for longer singular chains + Jordan chains

So, what's well defined?

Subspaces!

Wong chains [Wong KT, '74]

Wong chain for $L_1 x + L + 0$ attached to $\lambda \in \mathbb{C} \cup \{\infty\}$

$$\mathcal{W}_0 = \{0\}$$
$$\mathcal{W}_{k+1} = (L_1\lambda + L_0)^{-1}(L_1\mu + L_0)\mathcal{W}_k$$

for any $\mu \neq \lambda$, $\mu \in \mathbb{C} \cup \{\infty\}$ (evaluation suitably defined at ∞)

Pencil generalization of the usual algorithm to compute Jordan chains.

$\mathcal{W}_1 = \text{span}\{\text{first vectors of each eigvl-}\lambda \text{ and singular Kronecker chain}\}$

$\mathcal{W}_2 = \text{span}\{\text{first two vectors of each eigvl-}\lambda \text{ and singular chain}\}$

. . . and so on

Duality

New tool [Noferini, P., submitted]

Definition

$R_1 x + R_0$ right dual of $L_1 x + L_0$ if

$$\ker \begin{bmatrix} L_1 & L_0 \end{bmatrix} = \operatorname{im} \begin{bmatrix} R_0 \\ -R_1 \end{bmatrix}$$

Symmetric version: $\begin{bmatrix} L_1 & L_0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = 0$, maximal ranks

Why useful? Like inverses or matrix functions, "transform" Kronecker invariants without computing them

Used implicitly in doubling algorithms, never a first-class citizen though

Duality and Kronecker form

Theorem

The Kronecker forms of $L_1 x + L_0$ and its right dual $R_1 x + R_0$ are related:

$L_1 x + L_0$	$R_1 x + R_0$
any regular block	same block
$(k-1) \times k$ singular	$k \times (k+1)$ singular
$(k+1) \times k$ singular	$k \times (k-1)$ singular
1×0 singular	nothing
nothing	0×1 singular

Example

$$
\begin{bmatrix}
\boxed{\begin{matrix} \lambda - s & 1 \\ 0 & \lambda - s \end{matrix}} & & \\
& \boxed{\begin{matrix} s & 1 \end{matrix}} & \\
& & \boxed{\begin{matrix} s & \\ 1 & s \\ & 1 \end{matrix}}
\end{bmatrix}
\qquad
\begin{bmatrix}
\boxed{\begin{matrix} \lambda - s & 1 \\ 0 & \lambda - s \end{matrix}} & & \\
& \boxed{\begin{matrix} s & 1 & \\ & s & 1 \end{matrix}} & \\
& & \boxed{\begin{matrix} s \\ 1 \end{matrix}}
\end{bmatrix}
$$

How things change under duality

Theorem

$M(x)$ minimal basis of $R(x) \implies \dfrac{R(\mu)}{x-\mu}M(x)$ minimal basis of $L(x)$

For all $\mu \in \mathbb{C} \cup \{\infty\}$, $L(x)$ and $R(x)$ dual

Theorem

$\mathcal{W}_k$ Wong chain at λ for $R(x) \implies \dfrac{R(\mu)}{\lambda-\mu}\mathcal{W}_k$ Wong chain at λ for $L(x)$

For all $\mu \neq \lambda$, $L(x)$ and $R(x)$ dual

Everything works well “projectively” for ∞

Results for Wong chains imply easily results for eigenvectors, Jordan chains

Structured pencils

Definition [Lin, Mehrmann, Xu '99]

$S(x) = S_1 x + S_0 \in \mathbb{C}[x]^{2n\times 2n}$ symplectic if $S_1 J S_1^* = S_0 J S_0^*$, $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

$J(S_0^* x + S_1^*)$ right dual of $S(x)$. Results above give immediately:

- Eigenvalue pairing $(\lambda, 1/\lambda^*)$ under general hypotheses
- Constraints on singular blocks: right min. indices $=$ left m.i. -1

Analogous results for Hamiltonian pencils $H_1 J H_0^* = -H_0 J H_1^*$

Recap definitions

Given grade-d matrix polynomial $A(x) = \sum_{i=0}^{d} A_i x^i$, 1^{st} companion form

$$C(x) := \begin{bmatrix} A_d x + A_{d-1} & A_{d-2} & A_{d-1} & \cdots & A_0 \\ I & -Ix & & & \\ & I & -Ix & & \\ & & \ddots & \ddots & \\ & & & I & -Ix \end{bmatrix}$$

It is a strong linearization, i.e., (recall Fernando's talk)

- same eigenvalues and partial multiplicities as $A(x)$
- same $\dim \ker_{\mathbb{C}(x)} A(x)$ and $\dim \ker_{\mathbb{C}(x)} A(x)^T$
  (fixes number, but not size, of singular Kronecker blocks)

Minimal basis: minimal-degree polynomial basis of $\ker_{\mathbb{C}(x)} A(x)$

Fiedler pencils

For a grade-d matrix polynomial, one can construct special matrices $F_0, F_1, \ldots, F_d \in \mathbb{C}^{nd \times nd}$ such that

$$sF_d - F_{\sigma(0)} F_{\sigma(1)} \cdots F_{\sigma(d-1)}$$

is a strong linearization for any permutation σ
(choosing $\sigma(i) = d - i$ gives 1^{st} companion form)

Proofs:

- [Antoniou, Vologiannidis '04] regular case
- [De Terán, Dopico, Mackey '09] singular case (more complicated)

We can make things simpler using duality

Same plan as above

- show that what is done in the regular case is duality
- duality works also for singular case, results come without effort

Proving that Fiedlers are linearizations

Regular case: [Antoniou, Vologiannidis '04]
Idea: reduce to 1st companion form

1. Start from the first companion form, a known linearization

$$F_{10}x + F_9F_8F_7F_6F_5F_4F_3F_2F_1F_0$$

2. Prove (using regularity) that it stays a linearization after swapping blocks of "descending" F_i

$$F_{10}x + F_6F_5F_4F_3F_2F_1F_0F_9F_8F_7$$

3. Repeat! Thanks to commutation properties, it's enough to work with "block descending" sequences

$$F_{10}x + F_1F_0F_5F_4F_3F_2F_6F_9F_8F_7$$

Singular case: point 2 is a duality, so everything works verbatim!

Minimal bases, Wong chains and eigenvectors

Theorem [De Terán, Dopico, Mackey '09] (reworded)

(m.i. in a Fiedler pencil) = (m.i. in companion) − (number of swaps)

Similarly, using formulas for minimal bases / Wong chains:
Let $c_1, \dots, c_t$ block ends for $F(x)$, set $F_{j:i} := F_{j-1}F_{j-2}\cdots F_i$

- $\mu = 0$: $T_0 = F_{c_1:0}F_{c_2:0}\cdots F_{c_t:0}$

$M(x)$ m.b. for companion $\implies$ $\frac{1}{x^t}T_0M(x)$ m.b. for $F(x)$
$\mathcal{W}_k$ Wong chain for companion, $\lambda \neq 0$ $\implies$ $T_0\mathcal{W}_k$ W.c. for $F(x)$

- $\mu = \infty$: $T_\infty = F_{d:c_t}^{-1}F_dF_{d:c_{t-1}}^{-1}F_d\cdots F_{d:c_1}^{-1}F_d$

$M(x)$ m.b. for companion $\implies$ $T_\infty M(x)$ m.b. for $F(x)$
$\mathcal{W}_k$ Wong chain for companion, $\lambda \neq \infty$ $\implies$ $T_\infty\mathcal{W}_k$ W.c. for $F(x)$

$\mathbb{L}_1$ spaces

Let B span $\ker \begin{bmatrix} A_d & A_{d-1} & \cdots A_0 \end{bmatrix}$; a right dual of the 1st companion is

$$D(x) = \left(\begin{bmatrix} 1 & -x & & & \\ & 1 & -x & & \\ & & \ddots & \ddots & \\ & & & 1 & -x \end{bmatrix} \otimes I_n \right) B$$

[Mackey[2], Mehl, Mehrmann '06] introduced a space $\mathbb{L}_1$ of pencils $L(x)$ that satisfy

$$\begin{bmatrix} L_1 & L_0 \end{bmatrix} \begin{bmatrix} 0 & I_{nd} \\ -I_{nd} & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_0 \end{bmatrix} = 0$$

Almost the definition of duality, missing rank condition on $\begin{bmatrix} L_1 & L_0 \end{bmatrix}$

Relaxing the full Z-rank condition

...then they introduce an additional condition (full Z-rank) which makes them strong linearizations

Theorem (part [M⁴], part new)

$A(x)$ regular matrix polynomial. For $L(x)$ in $\mathbb{L}_1$, the following are equivalent:

1. strong linearization of $A(x)$
2. regular pencil
3. full Z-rank
4. left dual of $D(x)$ new!
5. no common left kernel for L_1, L_0 new! (simplest to check)

If $A(x)$ singular, $3 \implies 4 \implies 1$
(but not the other way round, counterexamples)

Conclusions

Why is duality cool?

- natural concept, makes many relations explicit
- allows to play with minimal indices without computing them
- simplifies proofs: singular cases often come for free
- suggests new view and new linearizations

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Thanks for your attention

$$\begin{bmatrix} L_1 & L_0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ R_0 \end{bmatrix}$$