Vectorization

Goal represent images, and 'linear functions of their pixels', in a linear algebra framework.

Image $\iff$ rectangular array (matrix) of intensity values of pixels, e.g. in $[0, 1]$.

In this context, a $m \times n$ image = a vector of data in $\mathbb{R}^{mn}$.

Vectorization gives an explicit way to map it to a vector.

Vectorization: definition

$$
\operatorname{vec} X = \operatorname{vec} \left[\begin{array}{c|c|c|c} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{array} \right] := \left[\begin{array}{c} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ \hline x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \\ \hline \vdots \\ \hline \\ x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{array} \right].
$$

Vectorization: comments

Column-major order: leftmost index 'changes more often'. Matches Fortran, Matlab standard (C/C++ prefer row-major instead).

Converting indices in the matrix into indices in the vector:

$$(X)_{ij} = (\operatorname{vec} X)_{i+mj} \qquad \text{0-based,}$$

$$(X)_{ij} = (\operatorname{vec} X)_{i+m(j-1)} \qquad \text{1-based.}$$

vec(AXB)

First, we will work out the representation of a simple linear map, $X \mapsto AXB$ (for fixed matrices A, B of compatible dimensions).

If $X \in \mathbb{R}^{m \times n}$, $AXB \in \mathbb{R}^{p \times q}$, we need the $pq \times mn$ matrix that maps $\operatorname{vec} X$ to vec(AXB).

$$(AXB)_{hl} = \sum_j (AX)_{hj}(B)_{jl} = \sum_j \sum_i A_{hi} X_{ij} B_{jl}$$

$$= \begin{bmatrix} A_{h1}B_{1l} & A_{h2}B_{1l} & \dots & A_{hm}B_{1l} \,\big|\, A_{h1}B_{2l} & A_{h2}B_{2l} & \dots & A_{hm}B_{2l} \,\big|\, \dots \\ \big|\, A_{h1}B_{nl} & A_{h2}B_{nl} & A_{hm}B_{nl} \end{bmatrix} \operatorname{vec} X$$

Kronecker product: definition

$$\operatorname{vec}(AXB) = \begin{bmatrix} b_{11}A & b_{21}A & \dots & b_{n1}A \\ b_{12}A & b_{22}A & \dots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1q}A & b_{2q}A & \dots & b_{nq}A \end{bmatrix} \operatorname{vec} X$$

Each block is a multiple of A, with coefficient given by the corresponding entry of B^T.

Definition

$$X \otimes Y := \begin{bmatrix} x_{11}Y & x_{12}Y & \dots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \dots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{m1}Y & x_{m2}Y & \dots & x_{mn}Y \end{bmatrix}.$$

so the matrix above is $B^T \otimes A$.

Properties of Kronecker products

$$X \otimes Y = \begin{bmatrix} x_{11} Y & x_{12} Y & \dots & x_{1n} Y \\ x_{21} Y & x_{22} Y & \dots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{m1} Y & x_{m2} Y & \dots & x_{mn} Y \end{bmatrix}.$$

- $\operatorname{vec} AXB = (B^T \otimes A) \operatorname{vec} X$. (Warning: not B^*, if complex).
- $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, when dimensions are compatible. Proof: $B(DXC^T)A^T = (BD)X(AC)^T$.
- $(A \otimes B)^T = A^T \otimes B^T$.
- orthogonal $\otimes$ orthogonal $=$ orthogonal.
- upper triangular $\otimes$ upper triangular $=$ upper triangular.
- One can "factor out" several decompositions, e.g.,

$$A \otimes B = (U_1 S_1 V_1^T) \otimes (U_2 S_2 V_2^T) = (U_1 \otimes U_2)(S_1 \otimes S_2)(V_1 \otimes V_2)^T.$$

Examples