Lecture: Image deblurring and denoising

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- Digital images are matrices (tensors) where each entry contains the intensity of the corresponding pixel (or a pointer to a RGB map of colors)
- Blur occurs in the process of recording

Image deblurring is the process of recover the original image using a mathematical model of the blurring process

Informations about lost details is hidden but can be recovered knowing the blurring function

Because of the noise no hope to fully recover the exact image

Blur in images

Blurring always arise: it is unavoidable that scene information contaminates neighboring pixels.

Blur within the camera

- Camera lens out of focus
- Lens not perfectly crafted (aberrations)

Blur outside the camera

- Motion of camera/object during exposure
- Outside events such as atmospheric turbolence for astronomical images

Errors in Digital Images

Digital images always are affected by errors (noise)

- false light, defects in the recoding process, analog-to-digital conversion
- truncation errors (quantization): integer approximation of a continuous quantity
- Noise generated during the conversion of the light to an electrical signal
- Short noise: a type of electronic noise due to the discrete nature of electric charges

Errors in Digital Images

Digital images always are affected by errors (noise)

- false light, defects in the recoding process, analog-to-digital conversion (salt-and-pepper)
- truncation errors (quantization): integer approximation of a continuous quantity (uniform noise)
- Noise generated during the conversion of the light to an electrical signal (Gaussian noise)
- Short noise: a type of electronic noise due to the discrete nature of electric charges (Poisson noise)

We have only statistical information about the noise

Modelling the blurring process

We need a mathematical model relating the blurred image to the true image

[P.C.Hansen-Nagy-O'Leary]
$X \in \mathbb{R}^{m \times n}$ original true image
$B \in \mathbb{R}^{m \times n}$ blurred image

A simple model

A simple case is when the blurring of the columns in the image is independent of the blurring of the rows.

$$A_c \in \mathbb{R}^{m \times m}, A_r \in \mathbb{R}^{n \times n}$$

$$A_c X A_r^T = B$$

- When left multiply by A_c we are applying the same vertical blurring to the columns of X
- When right multiply by A_r we are applying the same horizontal blurring to the rows of X
- No matter if we apply first the blurring of columns or rows

$$(A_c X) A_r^T = A_c (X A_r^T)$$

Deblurring

If

$$A_c X A_r^T = B$$

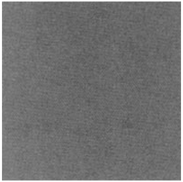
then

$$X_{\text{naive}} = A_c^{-1} B A_r^{-T}$$

Solving linear systems instead

$$\begin{cases} A_c Y = B \\ X A_r^T = Y \end{cases}$$

This is what we get



The noise completely destroies the image

Why does the naive approach fail?

- X exact image (unknown)
- $B_{\text{exact}} = A_c X A_r^T$ also unknown!

What we have is

$$B = B_{\text{exact}} + E$$

where E represent the noise in the recorded image.

Here we assumed that the noise is additive and statistically uncorrelated with the image.

E is unknown as well, only statistical properties are known.

Why does the naive approach fail?

We have

$$X_{\text{naive}} = A_c^{-1} B A_r^{-T} = A_c^{-1}(B_{\text{exact}} + E) A_r^{-T} = X + A_c^{-1} E A_r^{-T}$$

The quantity $A_c^{-1} E A_r^{-T}$ is called inverted noise, if it has larger elements than X it will dominate the solution.

That is indeed the typical situation

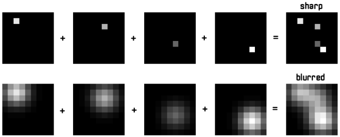
Since $B = A_c X A_r^T + E$ we have

$$\frac{\|X_{\text{naive}} - X\|_F}{\|X\|_F} \leq cond(A_c) cond(A_r) \frac{\|E\|_F}{\|A_c X A_r^T\|_F},$$

$$cond(A) = \sigma_1/\sigma_n$$

Linear model of blurring

We assume that the model of the blurring process is linear



Usually this is a good approximation of the reality, and makes our life easier!

Our model $B = A_c X A_r^T$ far too simple
If vectorize the image B, we have

$$\mathbf{b}_{\text{exact}} = Vec(B_{\text{exact}}) = Vec(AcXA_r^T) = \underbrace{(A_r \otimes A_c)}_{A}\mathbf{x}$$

$$\mathbf{b}_{\text{exact}} = A\mathbf{x}, \quad \mathbf{e} = Vec(E)$$

we have

$$\mathbf{b} = \mathbf{b}_{\text{exact}} + \mathbf{e}$$

Why does the naive approach fail?

From the linear model

$$Ax = b$$

we have

$$\mathbf{x}_{\text{naive}} = A^{-1}\mathbf{b}$$

we expect failure in reconstructing the true image because of the inverse noise

As we observed

$$\mathbf{x}_{\text{naive}} = A^{-1}(\mathbf{b}_{\text{exact}} + \mathbf{e}) = \mathbf{x} + A^{-1}\mathbf{e}$$

SVD analysis

Consider the SVD of $A \in \mathbb{N} \times \mathbb{N}$

$$A = U\Sigma V^T = \sum_{i=1}^{N} \sigma_i \mathbf{u_i} \mathbf{v_i^T}$$

then

$$A^{-1} = V\Sigma^{-1} U^T = \sum_{i=1}^{N} \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$$

Then

$$\mathbf{x}_{\text{naive}} = A^{-1}\mathbf{b} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

and the inverted noise contribution is

$$A^{-1}\mathbf{e} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^T \mathbf{e}}{\sigma_i} \mathbf{v}_i$$

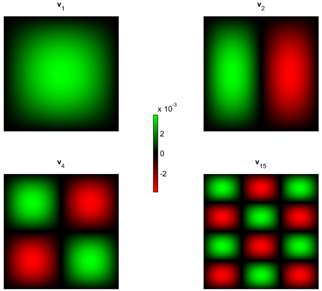
Why does the error term dominate?

$$A^{-1}\mathbf{e} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^T\mathbf{e}}{\sigma_i}\mathbf{v}_i$$

- vectors $\mathbf{u}_i$ are such that $|u_j^{(i)}| \leq 1$
- quantities $|\mathbf{u}_i^T\mathbf{e}|$ are small for all i
- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$, implies $\frac{1}{\sigma_1} \leq \cdots \leq \frac{1}{\sigma_{N-1}} \leq \frac{1}{\sigma_N}$
- the coefficient $\frac{1}{\sigma_N}$ will greately magnifying the component of the error $\mathbf{u}_N^T\mathbf{e}$
- the computed solution will have a large contribution in the direction of $\mathbf{v}_N$.

Looking at singular vectors

- Typically we have that singular vectors associated to small singular values represent higher frequency information.
- $\mathbf{u}_i$ and $\mathbf{v}_i$ will have more and more sign changes



Why does the error term dominate?

$$A^{-1}\mathbf{b} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\mathbf{v}_i$$

- When $\frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}$ is small, the solution has a little contribution from $\mathbf{v}_i$
- When σ_i is small hight contribution from highly oscillating vectors, the error is amplified and the reconstructed image are dominated by high frequencies

A first improvement

We can obtain a better reconstruction of the true image leaving out high frequencies

We can truncate the SVD to the $k-$th term

$$\mathbf{x}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

for some choice of $k < N$

Better than the computed reconstruction before but far from good

For the properties of Kronecker products, in our simple example

$$A = Ar \otimes Ac = (U_r \otimes U_c)(\Sigma_r \otimes \Sigma_c)(V_r \otimes V_c)^T$$

less expensive

```
>> k=50;
>> [Uc, Sc, Vc]=svds(Ac, k);
>> [Ur, Sr, Vr]=svds(Ar, k);
>> Arkinv=Vr*inv(Sr)*Ur';
>> Ackinv=Vc*inv(Sc)*Uc';
>> Xk = Ackinv*B*(Arkinv)';
```

The model

In the general case we have to solve

$$A\mathbf{x} = \mathbf{b}$$

problem of this kind are called "inverse problems".

Several difficulties

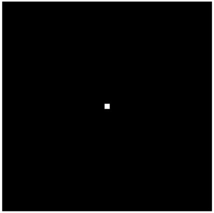
- A is large
- A usually ill conditioned
- A may be an imprecise model of the blurring

The Point spread function (PSF)

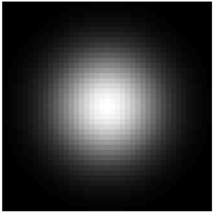
How do we get the blurring matrix A?

We can perform the following experiment

Point source
A single white pixel



Point spread function
Image of point source



The Point spread function (PSF)

- Mathematically the point source is described by the vector $\mathbf{x} = \mathbf{e}_i$.
- The columns of A can be obtained as follows

$$A\mathbf{e}_i = \mathbf{a}_i$$

- We can assemble A moving the point source from the top left corner to the bottom right, obtaining experimentally all the columns of A.

Typically PSF is local $\implies$ the nonzero entries of the PSF are a few $\implies$ A has a sparse structure

References

- P. C. Hansen, J. G. Nagy, D. P. O'Leary. *Deblurring Images: Matrices, Spectra, and Filtering*, SIAM 2006.
- M. Bertero and P. Boccacci. *Introduction to Inverse Problems in Imaging*. IOP Publishing Ltd., London, 1998.