ON THE FLAT-FOLDABILITY OF A CREASE PATTERN

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ABSTRACT. A crease pattern is the fingerprint that an origami leaves on the paper after being unfolded. A very natural question about the mathematics of origami is if it is possible to read on the crease pattern whether or not it belongs to a flat origami (i.e., an origami that has only 2 dimensions, if we do not consider the thickness of the paper). Necessary conditions for a crease pattern to fold flat have been given by T. Kawasaki [5] and T. Hull [2]. In this paper we give a criterion for flat-foldability of a crease pattern in the case the creases are "not too short" (Theorem 4).

1. INTRODUCTION

1.1. **The flat-foldability problem.** Let consider the square

$$Q = [0,1] \times [0,1] \subset \mathbb{R}^2.$$

Definition 1. A *crease pattern* is the data $\mathscr{C} = (\mathscr{V}, \mathscr{E})$, where

(1) $\mathscr{E}$ is a finite set of edges contained in Q,
(2) $\mathscr{V}$ is the set of all endpoints of edges in $\mathscr{E}$ which are contained in $(0,1) \times (0,1)$,

subject to the conditions

(1) if e and f are two elements of $\mathscr{E}$, then their intersection is either empty or a point of $\mathscr{V}$,
(2) every point in $\mathscr{V}$ is the endpoint of an even number of edges in $\mathscr{E}$.

The last condition will be clear after we state Maekawa's Theorem (see Remark 3).

We call *vertices* the elements of $\mathscr{V}$ and *creases* the elements of $\mathscr{E}$. The creases of a crease pattern $\mathscr{C}$ divide the square Q in a finite number of polygons, that we call *faces*. We say that a crease e is *incident* to a vertex v if v is an endpoint of e; two vertices are *adjacent* if they are the endpoints of the same crease; two creases e and f are *adjacent* if they are incident to the same vertex; finally, we say that two creases are *consecutive* if they are incident to a vertex v and at least one of the two angles between them is not crossed by any other crease incident to v.

REMARK 1. If $\mathscr{C} = (\{v\}, \mathscr{E})$ is a one-vertex crease pattern, then we write

$$\mathscr{E} = \{e_1, \ldots, e_{2n}\}$$

and we mean that the creases are consecutive and ordinated counterclockwise. Moreover, we denote by $\alpha_1, \ldots, \alpha_{2n}$ the angles between the creases, so that α_i is the angle between e_i and e_{i+1}, for $i = 1, \ldots, 2n$ (where $e_{2n+1} = e_1$).

Definition 2. Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern. A *folding map* for $\mathscr{C}$ is a function

$$\varphi\colon \mathscr{E} \to \mathbb{Z}/2\mathbb{Z}.$$

We denote by $\eta_0(\varphi)$ and $\eta_1(\varphi)$ respectively the number of creases of $\mathscr{C}$ mapped to 0 by φ and the number of creases mapped to 1.

A folding map tells us how to fold each crease: we fix an orientation of the square Q (i.e., we view Q embedded in $\mathbb{R}^3$ and lying in the plane $\{z = 0\}$, where x, y, z are the coordinates of $\mathbb{R}^3$), now if $\varphi(e) = 0$ then e is a valley crease, otherwise e is a mountain crease.

Figure 1. Mountain and valley creases.

Definition 3. Let φ be a folding map for a crease pattern $\mathscr{C}$. An *injective deformation* of $\mathscr{C}$ with respect to φ is a continuous map

$$\Phi\colon Q \times [0,1] \to \mathbb{R}^3$$

such that

(1) $\Phi(q, 0) = q$, for all $q \in Q$,
(2) for all $t \in [0,1]$, the image of $\Phi(-, t)$ does not contain transversal self-intersections;
(3) $\Phi(-, t)$ is an isometry on each face of $\mathscr{C}$, for all $t \in [0,1]$,
(4) $\Phi(-, t)$ preserves the orientation, for all $t \in [0,1]$,
(5) if F_1 and F_2 are two adjacent faces and $e \in \mathscr{E}$ is a common crease, then
$$0 + \varphi(e)\pi \leq \beta(t) \leq \pi + \varphi(e)\pi,$$
for all $t \in [0,1]$, where $\beta(t)$ is the angle between $\Phi(F_1, t)$ and $\Phi(F_2, t)$.

The angle $\beta(t)$ is well-defined once we fixed an embedding of the square Q in $\mathbb{R}^3$. We think at an injective deformation as an invisible origamist folding the square Q, following the given crease pattern; so $\Phi(Q, t)$ is a shot of the origami at the istant $t \in [0,1]$. The condition on the transversal self-intersections formalizes the fact that the paper cannot intersect itself.

Definition 4. Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern. A folding map φ for $\mathscr{C}$ is a *flat-folding map* if there exists an injective deformation Φ of φ such that

$$\Phi(Q, 1) \subset H$$

for some plane $H \subset \mathbb{R}^3$. In this case we also say that φ *folds flat*.

Remark 2. If φ is a flat-folding map, then also $1 - \varphi$ folds flat, where

$$(1 - \varphi)(e) = 1 - \varphi(e).$$

In fact, if Φ is an injective deformation of φ, then $\Phi \circ R_\pi$ is an injective deformation of $1 - \varphi$, where R_π is the rotation of angle π.

Definition 5. A crease pattern $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ is *flat-foldable* if there exists a flat-folding map φ for $\mathscr{C}$.

1.2. **One-vertex crease patterns.** The flat-foldability problem is completely understood in the case there is only one vertex ([4], [5]).

Theorem 1 (Maekawa). *Let $\mathscr{C} = (\{v\}, \mathscr{E})$ be a one-vertex crease pattern. If φ is a flat-folding map for $\mathscr{C}$, then*

$$\eta_0(\varphi) - \eta_1(\varphi) = \pm 2. \tag{1}$$

Theorem 2 (Kawasaki). *Let $\mathscr{C} = (\{v\}, \{e_1, \ldots, e_{2n}\})$ be a one-vertex crease pattern. Then $\mathscr{C}$ is flat-foldable if and only if*

$$\alpha_1 - \alpha_2 + \cdots + \alpha_{2n-1} - \alpha_{2n} = 0. \tag{2}$$

If v is a vertex of a crease pattern $\mathscr{C}$, then we will refer to equation (2) as to the *Kawasaki's condition* at v. Moreover if φ is a folding map for $\mathscr{C}$, then we will refer to equation (1) as to the *Maekawa's condition* at v. Notice that Kawasaki and Maekawa's conditions are still necessary in the case of a crease pattern with more vertices, but in general they are not sufficient (see examples in Section 2).

Remark 3. Here the last condition of Definition 1 becomes clear. Indeed, Maekawa's Theorem does not need the assuption on the degree of each vertex, instead it implies that if $\mathscr{C}$ is flat-foldable then $\mathscr{E}$ contains an even number of creases. In fact, let r be the number of creases in $\mathscr{E}$, then

$$\eta_0(\varphi) + \eta_1(\varphi) = r;$$

moreover, by Maekawa's Theorem,

$$\eta_0(\varphi) - \eta_1(\varphi) = \pm 2,$$

hence $r = 2(\eta_0(\varphi) \mp 1)$.

1.3. **From local to global flat-foldability.** The case of a crease pattern with more than one vertex has been treated by T. Kawasaki ([5]), T. Hull ([2], [3]), M. Bern and B. Hayes ([1]).

Kawasaki firstly observed that local flat-foldability (i.e., Kawasaki's condition) does not imply global flat-foldability in general (Remark 4). Hull proposed to associate to a given crease pattern a graph in such a way that it is possible to check some properties of the crease pattern directly on the graph by looking if it is 2-vertex-colorable ([2]). This seems to be an interesting idea, because checking the 2-vertex-colorability of a graph is very easy. The hard part is to construct a graph so that flat-foldability can be completely verified on it.

In this paper we start investigating what kind of obstructions to global flat-foldability can occur. It comes out that there are two types of problems:

(1) lenght-related obstructions, regarding the lenght of the creases (Example 4);
(2) forced creases, regarding local conditions that force creases to fold in a certain way and which do not agree globally (Example 2 and Example 3).

We do not want to deal with the first type here, so we give just some ideas on a way to fix it. Instead we discuss the second type of obstructions, studing the conditions that force two creases to fold equal or different (Lemma 2 and Lemma 1).

Finally we characterize the folding maps which fold flat (Theorem 3) and use this to prove our main result on the flat-foldability of a crease pattern (Theorem 4).

Acknowledgments. I would like to thank Thomas Hull for careful reading. He pointed out some mistakes and his comments helped to improve the exposition.

2. Obstructions to global flat-folding

Since Kawasaki's condition is not sufficient in the general case, we look for additional conditions for flat-foldability. Therefore, we try to understand what kind of problems can occur.

Remark 4 (Kawasaki [5]). If $(\{v\}, \{e_1, \ldots, e_{2n}\})$ is a one-vertex flat-foldable crease pattern and $\alpha_{i-1} > \alpha_i < \alpha_{i+1}$, then $\varphi(l_i) \neq \varphi(l_{i+1})$, for all flat-folding maps φ (that means that e_i and e_{i+1} cannot be both mountain neither valley folds).

Hull observed that, in the case of a multi-vertex crease pattern, the conditions given by Remark 4 give rise to non trivial global conditions as we see in the following example ([2]).

Example 1 (Hull [2]). The crease pattern in Figure 2 can't fold flat, by Remark 4. In fact, if φ is a flat-folding map, then

$$\varphi(e_1) \neq \varphi(e_2) \neq \varphi(e_3) \neq \varphi(e_1),$$

but each crease has only two possible values, so we get an absurd.

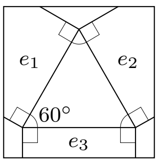

Figure 2.

Example 2. Consider the crease pattern in Figure 3. By Remark 4 and Maekawa's Theorem, if φ is a flat-folding map, then

$$\varphi(e_1) = \varphi(e_2) = \varphi(e_3) = \varphi(e_4) \neq \varphi(e_1),$$

hence this crease pattern is not flat-foldable.

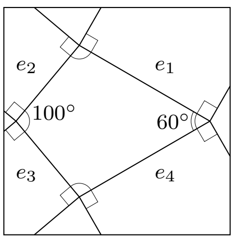

Figure 3.

Example 2 suggests that in order to study flat-foldability for a crease pattern we have to take in consideration all the conditions that force a crease to fold in a fixed way.

EXAMPLE 3. Consider the crease pattern in Figure 4. By Remark 4 and Maekawa's Theorem applied at every vertex except v, we get that, if φ is a flat-folding map, then

$$\begin{gathered}\varphi(e_1) = \varphi(e_2) = \varphi(e_3) = \varphi(e_4) = \varphi(e_5),\\ \varphi(f_1) = \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = \varphi(f_5).\end{gathered}$$

But this implies that Maekawa's condition at v cannot be satisfied.

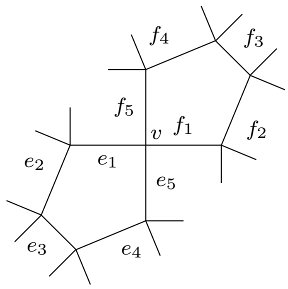

FIGURE 4.

Hull provided also the following example, which shows that something is still missing.

EXAMPLE 4 (Hull [3]). The crease pattern in Figure 5 doesn't fold flat unless d becomes longer, hence this time the problem is lenght-related.

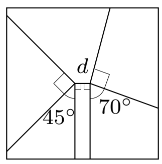

FIGURE 5.

We will see that, if we assume that Kawasaki's condition holds, then the only kinds of obstructions to flat-foldability that can occur are the ones sketched in Example 2, Example 3 and Example 4.

3. THE NON-COLLISION CONDITION

In this paper we do not want to deal with the kind of obstruction of Example 4, so we define a "non-collision" condition as follows. More explicitly, given a crease pattern $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ and a collection $\{\, \varphi_v \mid v \in \mathscr{V} \,\}$ of flat-folding maps that agree on the creases of $\mathscr{C}$ (where every φ_v is defined on the creases incident at v), we want to find conditions that ensure that we can glue these maps together to get a global flat-folding map.

Let $(\{v,w\}, \mathscr{E}_v \cup \mathscr{E}_w \cup \{d\})$ be a crease pattern, with

$$\mathscr{E}_v = \{e_2, \ldots, e_{2m}\}$$
$$\mathscr{E}_w = \{f_2, \ldots, f_{2n}\},$$

where we mean that $\mathscr{E}_v$ (respectively, $\mathscr{E}_w$) are creases incident at v (respectively, at w), and d is a crease incident at both v and w. Moreover let $\alpha_1, \ldots, \alpha_{2m}$ be the angles between the creases incident at v, and $\beta_1, \ldots, \beta_{2n}$ the angles between creases incident at w (Figure 6). We define

$$\sigma_i = \alpha_1 - \alpha_2 + \cdots + \alpha_{2i-1} - \alpha_{2i}$$
$$\sigma^h = \alpha_1 - \alpha_2 + \cdots + \alpha_{2h-1},$$

where $1 \leq i \leq m-1$ and $1 \leq h \leq m$. Similarly, we define

$$\tau_i = \beta_1 - \beta_2 + \cdots + \beta_{2i-1} - \beta_{2i}$$
$$\tau^h = \beta_1 - \beta_2 + \cdots + \beta_{2h-1},$$

where $1 \leq i \leq n-1$ and $1 \leq h \leq n$. If e is a crease, we denote by $l(e)$ the length of e.

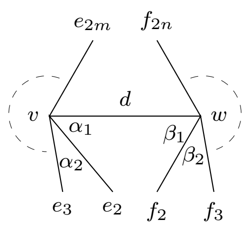

FIGURE 6.

Definition 6. We say that v and w satisfy the *non-collision condition* if

(1) there exists $1 \leq j \leq n$ such that, for all $1 \leq i \leq m-1$,
$$(l(d) - l(e_{2i+1}) \cos \sigma_i) \tan \tau^j \geq l(e_{2i+1}) \sin \sigma_i;$$

(2) there exists $1 \leq i \leq m$ such that, for all $1 \leq j \leq n-1$,
$$(l(d) - l(f_{2j+1}) \cos \tau_j) \tan \sigma^i \geq l(f_{2j+1}) \sin \tau_j.$$

REMARK 5. Assume that Kawasaki's condition holds at v and w. Then we can fold the creases incident at v and w separately, that means that there exist two flat-folding maps φ_v and φ_w. We can assume that $\varphi_v(d) = \varphi_w(d)$ (otherwise, we consider $1 - \varphi_w$ instead of φ_w). The non-collision condition assures that the folding map obtained by gluing φ_v and φ_w folds flat. More explicitly, the first condition of Definition 6 implies that, gluing φ_v and φ_w, we can put all the creases in $\mathscr{E}_v$ inside the crease f_j without ripping the paper.

Definition 7. We say that a crease pattern $\mathscr{C}$ satisfies the *non-collision condition* if every couple of adjacent vertices does.

Remark 6. Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern that satisfies the non-collision condition. Let $\{ \varphi_v \mid v \in \mathscr{V} \}$ be a collection of flat-folding maps that agree on the creases of $\mathscr{C}$ (where every φ_v is defined on the creases incident at v). Then it follows by Remark 5, using an inductive argument, that we can glue them together to get a flat-folding map φ for $\mathscr{C}$.

In the following we assume that all crease patterns satisfy the non-collision condition.

Remark 7. Notice that Definition 6 is too strict for our aim. For example, in the case of a crease pattern with only two vertices, it is enough to require that at least one of the two conditions of Definition 6 holds in order to ensure that local flat-folding maps which agree can be glued together to give a global flat-folding map. So, assuming that the non-collision condition holds, we are throwing away some flat-foldable crease patterns. In order to be accurate, one should require in Definition 6 that, given a crease pattern $\mathscr{C}$, at least one condition holds, then one derives an oriented graph associated to $\mathscr{C}$ and finally one has to look at the conditions on this graph that imply the flat-foldability. However, this is not our purpoise, so we put ourselves in the case the "strong" non-collision condition holds.

4. Flat-foldability of two consecutive creases

We want to characterize the crease patterns $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ for which there exists a collection $\{ \varphi_v \mid v \in \mathscr{V} \}$ of flat-folding maps that agree on the creases of $\mathscr{C}$. Notice that the existence of a collection $\{ \varphi_v \mid v \in \mathscr{V} \}$ of flat-folding maps is ensured by (actually equivalent to) requiring that Kawasaki's condition holds at every vertex (Theorem 2), hence the problem is to make them agree on $\mathscr{E}$.

We saw in Section 2 that sometimes the creases are "forced" to fold in a certain way, so now we want to give conditions for two consecutive creases to be forced to be equal or different.

Definition 8. We say that two creases e and f are *forced to be equal* (respectively, *different*) if for every flat-folding map φ, we have $\varphi(e) = \varphi(f)$ (respectively, $\varphi(e) \neq \varphi(f)$). Moreover, we say that a crease e is *forced* if for every vertex of e there exists at least one crease $f \neq e$ such that e and f are consecutive and forced to be equal or different.

The results of this section are quite technical, so we need some notations. Let $(\{v\}, \mathscr{E})$ be a flat-foldable crease pattern with $\mathscr{E} = \{e_1, \dots, e_{2n}\}$. We define the following

$$\sigma_s = \alpha_2 - \alpha_3 + \cdots + \alpha_{2s},$$
$$\sigma^s = \alpha_{2n} - \alpha_{2n-1} + \cdots + \alpha_{2s+2},$$

where $s \in \{1, \dots, n-1\}$, and

$$I = \{ 1 \leq i \leq n-1 \mid \sigma_i > \alpha_1 \}$$
$$J = \left\{ 1 \leq j \leq n-1 \,\middle|\, \sigma^j > \alpha_1 \right\}.$$

Notice that, by Kawasaki's condition, we have

$$\sigma_s = \alpha_1 - \alpha_{2n} + \cdots + \alpha_{2s+1},$$
$$\sigma^s = \alpha_1 - \alpha_2 + \cdots + \alpha_{2s+1}.$$

Lemma 1. *The creases e_1 and e_2 are forced to be different if and only if there exist $1 \le i < j \le n-1$ such that $\sigma_i > \alpha_1 < \sigma^j$ and*

(1) $\sigma_l < \alpha_1$, for all $j < l \le n-1$,
(2) $\sigma^r < \alpha_1$, for all $1 \le r < i$.

Proof. Suppose that $\varphi(e_1) \ne \varphi(e_2)$, for all flat-folding maps φ. By contraddiction, if $\sigma_i \le \alpha_1$ for every $1 \le i \le n-1$, then we define

$$\varphi(e_h) = \begin{cases} 1 & \text{if } h \equiv 0 \quad (2) \\ 1 & \text{if } h = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 7 we see a transversal section of Q, being folded according to φ, at a certain instant.

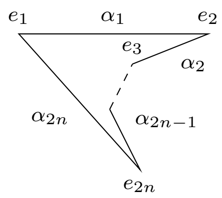

FIGURE 7.

In particular, we can cut Q along e_1 and fold the creases $e_2, \dots, e_{2n}$ in such a way that the creases with even indices are mountain and those with odd indices are valley. After that, we see that we can glue along e_1 and we get a flat origami, since $\sigma_i \le \alpha_1$ for all $1 \le i \le n-1$. It follows that φ folds flat. A similar argument holds if $\sigma^j \le \alpha_1$ for all $1 \le j \le n-1$.

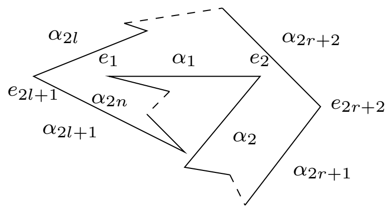

FIGURE 8.

Otherwise, if $i = \min I$, $j = \max J$ and there exists $j < l \le n-1$ such that $\sigma_l \ge \alpha_1$, then we can assume that l realizes the maximum of σ_s for $j < s \le n-1$. Moreover, let $1 \le r \le j$ be an index which realizes the maximum of σ^s for $1 \le s \le n-1$. We define

$$\varphi(e_h) = \begin{cases} 0 & \text{if } h = 2k,\ k = 1, \dots, r, r+2, \dots, l \\ 0 & \text{if } h = 2k+1,\ k = 0, l+1, \dots, n-1 \\ 1 & \text{otherwise.} \end{cases}$$

As before, we can cut along e_{2r+2} and fold the other creases according to φ (where 0 corresponds to valley folds and 1 corresponds to mountain folds), and we see that by assumptions we can glue along e_{2r+2} and get a flat origami (Figure 8). Again, a similar argument holds if $\sigma^r \geq \alpha_1$ for some $1 \leq r < i$.

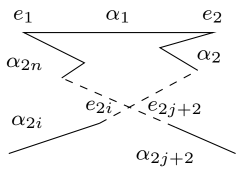

FIGURE 9.

Assume now that the conditions of the statement hold (Figure 9). Note that the creases e_{2i} and e_{2j+2} cannot lie on the same side with respect to α_1, because by hypothesis,

$$\sigma_i > \sigma_l, \text{ for all } j < l \leq n,$$
$$\sigma^j > \sigma^r, \text{ for all } 0 \leq r < i,$$

and so we cannot put the crease e_{2i+1} inside any of the creases in the set $\{e_1, e_{2n}, \ldots, e_{2j+2}\}$, neither we can put the crease e_{2j+1} inside any of the creases in $\{e_1, e_2, \ldots, e_{2i}\}$. Then the creases e_{2i} and e_{2j+2} must lie on opposite sides with respect to α_1, and the only way to make it happens is to fold e_1 and e_2 in different ways, since

$$\sigma_l < \alpha_1 > \sigma^r,$$

for all $j < l \leq n-1$ and $1 \leq r < i$. □

Lemma 2. *The creases e_1 and e_2 are forced to be equal if and only if*

$$\sigma_i < \alpha_1 > \sigma^j,$$

for all $1 \leq i, j \leq n-1$.

Proof. Assume that $\varphi(e_1) = \varphi(e_2)$, for every flat-folding map φ.

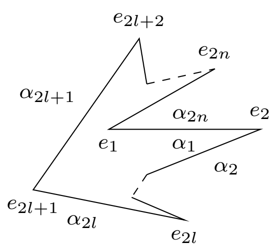

FIGURE 10.

By contraddiction, if there exists $1 \leq i \leq n-1$ such that $\sigma_i \geq \alpha_1$, then let $i \leq l \leq n-1$ be such that $\sigma_s \leq \sigma_l$, for all $l \leq s \leq n-1$. We define the following folding map

$$\varphi(e_h) = \begin{cases} 1 & \text{if } h \equiv 0 \quad (2) \\ 1 & \text{if } h = 2l+1 \\ 0 & \text{otherwise,} \end{cases}$$

and we see easily that φ folds flat (Figure 10). In particular, we can cut along e_{2l+1} and fold the other creases according to φ. Then, by hypothesis, we can glue along e_{el+1} and get a flat origami. A similar argument holds if there exists $1 \leq j \leq n-1$ such that $\sigma^j \geq \alpha_1$.

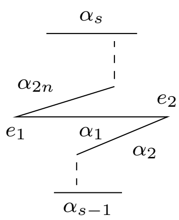

Figure 11.

Suppose now that

$$\sigma_i < \alpha_1 > \sigma^j,$$

for all $1 \leq i, j \leq n-1$, and let φ be a folding map such that $\varphi(e_1) \neq \varphi(e_2)$. We cut along e_s for some $s \neq 1, 2$ and we fold according to φ. It follows from the hypothesis that the creases $\{e_3, \dots, e_{s-1}\}$ and the creases $\{e_{s+1}, \dots, e_{2n}\}$ lie on opposite sides with respect to α_1, hence we cannot glue along e_s (Figure 11).

Since this is true for all $s \neq 1, 2$, we get that φ is not flat-foldable. □

Definition 9. We denote by $P_{\neq}(e_1, e_2)$ the conditions of Lemma 1. Similarly, we denote by $P_{=}(e_1, e_2)$ the conditions of Lemma 2.

Definition 10. With the notation as before, if $\alpha_1 \leq \alpha_h$ for all h, then we can consider the crease pattern

$$\mathscr{C}' = (\{v\}, \{e_3, \dots, e_{2n}\}),$$

where the angle between the creases e_{2n} and e_3 is $\alpha_{2n} - \alpha_1 + \alpha_2$. The crease pattern $\mathscr{C}'$ is said to be *derived*. We can iterate this construction and get more derived crease patterns.

We think at $\mathscr{C}'$ as if we folded the creases e_1 and e_2 in different ways and then we identified the layers of the folded square Q together.

Remark 8. Note that $\mathscr{C}'$ is not flat anymore; however, since we didn't use flatness hypothesis in the previous results, we can define the properties $P_{=}$ and $P_{\neq}$ for the pair (e_{2n}, e_3).

5. Flat-folding maps

Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern and let φ be a folding map for $\mathscr{C}$.

Definition 11. We say that $\mathscr{C}$ and φ are *compatible* if the following two conditions hold for every pair of consecutive creases (e, f),

(1) if $P_=(e,f)$ is satisfied then $\varphi(e) = \varphi(f)$,
(2) if $P_{\neq}(e,f)$ is satisfied then $\varphi(e) \neq \varphi(f)$.

Remark 9. It follows from Lemma 1 and Lemma 2 that if φ folds flat then $\mathscr{C}$ and φ are compatible.

Definition 12. Let $v \in \mathscr{V}$ be a vertex and let $\alpha_1 \leq \alpha_h$ for all h, where $\{\, \alpha_h \,|\, h = 1, \dots, 2n \,\}$ are the angles between the creases $\{\, e_h \,|\, h = 1, \dots, 2n \,\}$ incident at v. If $\varphi(e_1) \neq \varphi(e_2)$, then we can define a crease pattern

$$\mathscr{C}' = (\mathscr{V}, \mathscr{E} \setminus \{e_1, e_2\}),$$

where the angle between e_{2n} and e_3 is $\alpha_{2n} - \alpha_1 + \alpha_2$. We say that this crease pattern is *derived* at v via φ. We can also iterate this construction and get more derived crease patterns.

Remark 10. Notice that if φ folds flat, then, for every vertex $v \in \mathscr{V}$ of degree at least 4, there is an angle α_1 such that $\alpha_1 \leq \alpha_i$, for all i, and $\varphi(e_1) \neq \varphi(e_2)$. In fact, if for all minimal angles α_i, we have $\varphi(e_i) = \varphi(e_{i+1})$ then, by Remark 9, we get that $\alpha_i = \alpha_{i+1}$ or $\alpha_i = \alpha_{i-1}$, hence α_{i-1} and α_{i+1} are minimal angles too, so, iterating this argument, we find that all the angles are equal. However, by Maekawa's Theorem, there are two consecutive creases mapped to different values by φ, therefore we get an absurd.

Definition 13. Let v be a vertex of $\mathscr{C}$. We define $\mathscr{C}$ and φ to be *strictly compatible* at v as follows

(1) if v has degree 2, then $\mathscr{C}$ and φ are strictly compatible at v if they are compatible;
(2) if v has degree at least 4, then $\mathscr{C}$ and φ are strictly compatible at v if they are compatible, the map φ induces a derived crease pattern at v, and for every derived crease pattern $\mathscr{C}'$, we have that $\mathscr{C}'$ and φ are strictly compatible at v.

Definition 14. We say that $\mathscr{C}$ and φ are *strictly compatible* if they are strictly compatible at every vertex.

We want to provide a criterion to establish if φ folds flat or not. We start with the case of a one-vertex crease pattern, and then we prove the general result.

Lemma 3. *Let $(\{v\}, \{e_1, \dots, e_{2n}\})$ be a one-vertex crease pattern and let φ be a folding map. Then φ folds flat if and only if*

(1) Kawasaki's condition holds,
(2) Maekawa's condition holds,
(3) $\mathscr{C}$ and φ are strictly compatible.

Proof. We have already seen that the three conditions are necessary. Now we prove by induction on $n \geq 1$ that they are sufficient.

If $n = 1$ then Maekawa's condition implies that $\varphi(e_1) = \varphi(e_2)$, and by Kawasaki's condition $\alpha_1 = \alpha_2$. Hence there are only two possibilities for φ, and both of them fold flat.

If $n > 1$, let $\alpha_1 \leq \alpha_h$ for all h. By Remark 10, we can assume that $\varphi(e_1) \neq \varphi(e_2)$. If $n = 2$ then, by Maekawa's condition,

$$\varphi(e_1) \neq \varphi(e_2) = \varphi(e_3) = \varphi(e_4),$$

and by Kawasaki's condition,

$$\alpha_4 - \alpha_1 + \alpha_2 = \alpha_3 > 0,$$

and so φ folds flat, by compatibility condition.

If $n > 2$, we consider the derived crease pattern

$$\mathscr{C}' = (\{v\}, \mathscr{E}' = \{e_3, \dots, e_{2n}\}),$$

where the angle between e_3 and e_{2n} is $\alpha_1' = \alpha_{2n} - \alpha_1 + \alpha_2 > 0$. Note that $\mathscr{C}'$ and the restriction of φ to $\mathscr{E}'$ satisfy the three conditions of the statement, so by induction we get the result. □

Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern, with $\mathscr{V} = \{v_1, \dots, v_r\}$. We want to construct a polygonal decomposition of $\mathscr{C}$ as follows. For every face F of $\mathscr{C}$, we take an internal point p_F. Then for every crease f in the boundary of F, we take its middle point p_f, and we consider the edge whose endpoints are p_F and p_f. We do the same with the edges of the boundary of F that are contained in the boundary of Q. Let D be the set of all edges constructed in this way, then D divides the square Q in a finite number of polygons $\{P_1, \dots, P_r\}$ such that

(1) $P_i \subset Q$ for all i, and $\cup_i P_i = Q$,
(2) $P_i \cap P_j$ is contained in the boundary of P_i (and P_j),
(3) $v_i \in \mathscr{V}$ is contained in the interior of P_i, for every $i = 1, \dots, r$,
(4) if e is a common edge of the boundary of P_i and P_j, then e is trasversal to the edge through v_i and v_j; moreover e does not intersect any crease incident to v_i, except eventually for the one incident to v_j.

Theorem 3. *Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern and let φ be a folding map for $\mathscr{C}$. Then φ folds flat if and only if*

(1) Kawasaki's condition is satisfied at every vertex,
(2) Maekawa's condition is satisfied at every vertex,
(3) $\mathscr{C}$ and φ are strictly compatible.

Proof. We only need to prove that the three conditions are sufficient. Let $\{P_1, \dots, P_r\}$ be a decomposition of $\mathscr{C}$ as above. By Lemma 3, we can fold each P_i separately, that means that the restriction of φ to each P_i folds flat. Moreover, since the non-collision condition holds, we can glue these pieces and obtain a flat origami. □

6. Flat-folding crease patterns

6.1. **Construction of the associated graph.** Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern. The following definition generalizes the notion of *origami line graph* given by Hull [2], including all conditions that force creases to fold in a certain way.

Definition 15. The *graph associated to $\mathscr{C}$* is the graph $G(\mathscr{C})$ constructed as follows:

(1) the vertices of $G(\mathscr{C})$ are the creases in $\mathscr{C}$; we identify two vertices e and f of $G(\mathscr{C})$ if they are adjacent creases in $\mathscr{C}$ for which $P_=(e, f)$ is defined and holds;
(2) if $e, f \in \mathscr{E}$, then (e, f) is an edge in $G(\mathscr{C})$ if and only if e and f are adjacent creases in $\mathscr{C}$ for which $P_{\neq}(e, f)$ is defined and holds.

Remark 11. Recall that the properties $P_{=}(e,f)$ and $P_{\neq}(e,f)$ are defined only for the pairs (e,f) of adjacent creases which are consecutive as creases in some derived crease pattern. Notice that if $\mathscr{C}$ is flat-foldable then $G(\mathscr{C})$ is 2-vertex-colorable (any flat-folding map for $\mathscr{C}$ gives a 2-colouring of $G(\mathscr{C})$). Moreover, the graph $G(\mathscr{C})$ can contain loops (since we identified some vertices), and if it does then obviusly it is not 2-vertex-colorable. For example, the graph associated to the crease pattern of Example 2 contains a loop.

6.2. **Maekawa's condition for a crease pattern.** Let $v \in \mathscr{V}$ be a vertex of $\mathscr{C}$, and assume that $G(\mathscr{C})$ is 2-vertex-colorable. Then $G(\mathscr{C})$ gives conditions on the set $\mathscr{E}_v = \{e_1, \dots, e_{2n}\}$ of the creases incident at v. In particular, let $G_1, \dots G_r$ be the connected components of $G(\mathscr{C})$ which involves creases incident at v, such that none of these components corresponds to a single crease of $\mathscr{C}$ (which means that it is a vertex which corresponds to only one crease). Let λ be a 2-colouring of $G(\mathscr{C})$. We denote by $\lambda_1, \dots, \lambda_r$ the restrictions of λ to $G_1 \cap \mathscr{E}_v, \dots, G_r \cap \mathscr{E}_v$ respectively. Then we get non negative integers $\eta_0(\lambda_i)$ and $\eta_1(\lambda_i)$, for $i = 1, \dots, r$.

Definition 16. Let $\mathscr{C}$ be a crease pattern whose associated graph $G(\mathscr{C})$ is 2-vertex-colorable. We say that $\mathscr{C}$ satisfies Maekawa's condition at v if, for every 2-colouring λ of $G(\mathscr{C})$, there exists a map $\epsilon\colon \{1, \dots, r\} \to \mathbb{Z}/2\mathbb{Z}$ such that the following two disequalities hold

$$\sum_{i=1}^{r} \eta_{\epsilon(i)}(\lambda_i) \leq n+1,$$
$$\sum_{i=1}^{r} \eta_{1-\epsilon(i)}(\lambda_i) \leq n-1.$$

Remark 12. Notice that if $\mathscr{C}$ is flat-foldable, then it satisfies Maekawa's condition at every vertex. For istance, the crease pattern in Example 3 satisfies Kawasaki's condition at every vertex, but it does not satisfy Maekawa's condition at v (see Figure 4).

Remark 13. With the notations as in Definition 16, if $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ is a crease pattern which satisfies Kawasaki's and Maekawa's conditions at $v \in \mathscr{V}$, then, by Lemma 3 there exists a flat-folding map φ_v for $(\{v\}, \mathscr{E}_v)$ (we write $\mathscr{E}_v$ for the set of creases in $\mathscr{E}$ incident at v) such that

$$\varphi(e) = \epsilon(i)\lambda_i(e) + (1-\epsilon(i))(1-\lambda_i(e)),$$

for $e \in G_i \cap \mathscr{E}_v$. More explicitly, Kawasaki's condition at v allows us to choose a flat-folding map φ_v defined over $\mathscr{E}_v$, that "respects" the forced creases.

6.3. **The main result.**

Theorem 4. *Let $\mathscr{C} = (\mathscr{V}, \mathscr{E})$ be a crease pattern which satisfies the non-collision condition. Then $\mathscr{C}$ is flat-foldable if and only if*

(1) Kawasaki's condition is satisfied at every vertex,
(2) the associated graph $G(\mathscr{C})$ is 2-vertex-colorable,
(3) $\mathscr{C}$ satisfies Maekawa's condition at every vertex.

Proof. By Remark 11 and Remark 12, it is enough to prove that the three conditions are sufficient.

Let λ' be a 2-colouring of $G(\mathscr{C})$. Notice that λ' induces in a natural way a folding map φ' for $\mathscr{C}$. Moreover, we can change the color of λ' at a vertex of $G(\mathscr{C})$ which corresponds to only one crease in $\mathscr{C}$ and get an other 2-colouring λ''.

Since $\mathscr{C}$ satisfies Maekawa's condition at every vertex, we can change the color of λ' at some vertices of $G(\mathscr{C})$, each of which corresponds to only one crease of $\mathscr{C}$, so to get a 2-colouring λ, which induces a folding map φ that satisfies Maekawa's condition at every vertex (see Remark 13).

Furthermore, the map φ and $\mathscr{C}$ are strictly compatible, since φ is induced by a 2-colouring of the associated graph $G(\mathscr{C})$. Hence the pair $(\mathscr{C}, \varphi)$ verifies the hypothesis of Theorem 3 and it follows that φ is a flat-folding map for the crease pattern $\mathscr{C}$. $\square$

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