Some Problems of Functional Analysis

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1 The Bipolar Theorem.

Let E be a complex Hilbert space. If A is a nonempty subset of E, the polar of A is defined as

$$A^0 = \{x \in E : \Re(x \mid y) \leq 1 \text{ for all } y \in A\}. \tag{1}$$

The set A^{00} is called the bipolar of A.

a. Prove that the polar of any nonempty subset of E is a closed convex set containing 0.

b. Deduce that, if A is a nonempty subset of E, the closed convex hull of $A \cup \{0\}$ is contained in A^{00}.

c. We now want to show the reverse inclusion. Let C be the closed convex hull of $A \cup \{0\}$ and take $x \in A^{00}$.

i. Prove that

$$\Re((x - P_C(x) \mid P_C(x)) \geq 0. \tag{2}$$

ii. Prove that, for all $\epsilon > 0$,

$$\frac{1}{\epsilon + \Re((x - P_C(x) \mid P_C(x))}(x - P_C(x)) \in A^0. \tag{3}$$

Deduce that $\|x - P_C(x)\|^2 \leq \epsilon$, and so that $x \in C$.

d. Let A be a convex subset of E containing 0. Prove that $\bar{A} = A^{00}$.

e. Let A be a vector subspace of E. Prove that $A^0 = A^\perp$.

Solution

a.

- $\forall y \in A, \Re(0 \mid y) = 0 \leq 1 \Rightarrow 0 \in A^0$.
- $\forall x, y \in A^0,\ \forall z \in A,\ \forall t \in [0,1]$,
  $\Re(tx + (1-t)y \mid z) = t\Re x \mid z + (1-t)\Re y \mid z \leq t + 1 - t = 1 \Rightarrow A^0$ is convex.
- Let $\{x_n\}_n \subseteq A^0$ a sequence, convergent to an element $x \in E$. Using the continuity of the real part and of the scalar product,

$$1 \geq \lim_{n\to\infty} \Re(x_n \mid y) = \Re \lim_{n\to\infty} (x_n \mid y) = \Re(\lim_{n\to\infty} x_n \mid y) = \Re x \mid y \Rightarrow x \in A^0,$$

so A^0 is closed.

b. It's sufficient to prove that $A \subseteq A^{00}$, which is obvious by definition of A^0.

c. **i.** By the projection theorem we have, $\forall z \in A^{00}$,

$$\Re(x - P_C(x) \mid z - P_C(x)) \leq 0.$$

Taking $z = 0$ we immediately get (2).

ii. Let $x \in A^{00}, y \in A, \epsilon > 0$. We have

$$\Re\left(\frac{x - P_C(x)}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} \mid y \right) = \frac{\Re(x - P_C(x) \mid y)}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} =$$

$$= \frac{\Re(x - P_C(x) \mid P_C(x)) + \Re(x - P_C(x) \mid y - P_C(x))}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} \leq \frac{\Re(x - P_C(x) \mid P_C(x))}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} \leq 1,$$

and (3) follows. Consequently

$$\frac{\|x - P_C(x)\|^2}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} = \frac{\Re(x - P_C(x) \mid x - P_C(x))}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} =$$

$$\frac{\Re(x \mid x - P_C(x)) - \Re(P_C(x) \mid x - P_C(x))}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} \leq \frac{\Re(x \mid x - P_C(x))}{\epsilon + \Re(x - P_C(x) \mid P_C(x))}$$

Since $x \in A^{00}$,

$$\Re\left(\frac{x - P_C(x)}{\epsilon + \Re(x - P_C(x) \mid P_C(x))} \mid x \right) \leq 1 \Rightarrow$$

$$\Re(x - P_C(x) \mid x) \leq \epsilon + \Re(x - P_C(x) \mid P_C(x)) \Rightarrow$$

$$\Rightarrow \Re(x - P_C(x) \mid x) - \Re(x - P_C(x) \mid P_C(x)) = \Re(x - P_C(x) \mid x - P_C(x)) = \|x - P_C(x)\|^2 \leq \epsilon.$$

This implies $\|x - P_C(x)\| = 0$, i.e. $x = P_C(x) \in C$.

d. $\bar{A}$ is closed, but also convex: as a matter of fact, if we take $x, y \in \bar{A}$, there exist two sequences $\{x_n\}_n, \{y_n\}_n \in A$ such that $x_n \to n, y_n \to n$, as n tends to ∞. Since A is convex, we have that, $\forall t \in [0, 1]$, $tx_n + (1 - t)y_n \to tx + (1 - t)y$, thus $tx + (1 - t)y \in \bar{A}$ $\forall t \in [0, 1]$. Consequently, $\bar{A} = C = A^{00}$.

e. The inclusion $A^\perp \subseteq A^0$ is obvious, since if $x \in A^\perp$, then $(x \mid y) = 0$ $\forall y \in A$. On the other hand, if $x \in A^0$ and $y \in A$, we have

$$\Re(x \mid y) \leq 1.$$

But A is a vector space, so we also have $-y \in A$ and

$$\Re(x \mid -y) = -\Re(x \mid y) \leq 1.$$

This is possible if and only if $\Re(x \mid y) = 0$. This implies $A^0 \subseteq A^\perp$, and in conclusion $A^0 = A^\perp$.

2 Theorem of Lax-Milgram.

Let E be a real Hilbert space and a a bilinear form on E. Assume that a is continuous and coercive: this means that there exist constants $C > 0$ and $\alpha > 0$ such that

$$|a(x,y)| \leq C\|x\|\|y\| \qquad \text{for all } x, y \in E, \tag{4}$$

$$a(x,x) \geq \alpha\|x\|^2 \qquad \text{for all } x \in E, \tag{5}$$

a. **i.** Show there exists a continuous linear operator T on E such that

$$a(x,y) = (Tx \mid y) \qquad \text{for all } x, y \in E. \tag{6}$$

ii. Prove that $T(E)$ is dense in E.

iii. Prove that $\|Tx\| \geq \alpha\|x\|$ for all $x \in E$. Deduce that T is injective and that $T(E)$ is closed.

iv. Deduce that T is an isomorphism from E onto itself.

b. Let L be a continuous linear form on E.

i. Deduce from the preceding questions that there exists a unique $u \in E$ such that

$$a(u,y) = L(y) \qquad \text{for all } y \in E. \tag{7}$$

ii. Now suppose that the bilinear form a is symmetric and define, for $x \in E$,

$$\Phi(x) = \frac{1}{2}a(x,x) - L(x).$$

Prove that the point u is characteriszed by the condition

$$\Phi(u) = \min_{x \in E} \Phi(x) \tag{8}$$

Solution

a. **i.** Fixed $x \in E$, the map

$$\begin{aligned} a_x : E &\to \mathbb{R} \\ y &\mapsto a(x,y) \end{aligned}$$

is linear and continuous. By the Riesz Representation Theorem there exist a unic element, that we call Tx, such that associates

$$a(x,y) = a_x(y) = (y \mid Tx) = (Tx \mid y).$$

We call $T : E \to E$ the operator that associates the corresponding Tx to each $x \in E$. We now prove that T is linear and continuous.

- **Linearity**
  - $\forall y \in E$, $a(0,y) = (T0 \mid y) = (0 \mid y)$. For the uniqueness, we deduce that $T0 = 0$.

- $\forall x, x', y \in E$,

$$(T(x+x') \mid y) = a(x+x' \mid y) = a(x \mid y)+a(x' \mid y) = (Tx \mid y)+(Tx' \mid y) = (Tx+Tx' \mid y).$$

  Again for the uniqueness, we deduce that $Tx + Tx' = T(x + x')$.
- $\forall c \in \mathbb{R}, \forall x, y \in E$, we have $(T(cx) \mid y) = a(cx, y) = ca(x, y)c(Tx \mid y) = (cTx \mid y)$. Again for the uniqueness, we deduce that $T(cx) = cTx$.

- **Continuity** We will prove that, $\forall x \in E$

$$\forall \epsilon > 0\ \exists \delta > 0 \mid \forall y \in E,\ \|x - y\| < \delta \Rightarrow \|Tx - Ty\| < \epsilon.$$

  Fix $x \in E$, $\epsilon > 0$. we have

$$\|Tx - Ty\|^2 = \|T(x - y)\|^2 = (T(x - y) \mid T(x - y)) = a(x - y, T(x - y)) =$$

$$|a(x - y, T(x - y))| \leq C\|x\|\|y\|,$$

  which implies

$$\|T(x - y)\| \leq C\|x - y\|.$$

  If we take $\delta < \frac{\epsilon}{C}$, we have the thesis.

ii. It's immediate to prove that $T(E)$ is a vector subspace of E. Consequently we have decomposition $E = \overline{T(E)} \oplus T(E)^{\perp}$, and $T(E)$ is dense in E if and only if $T(E)^{\perp} = \{0\}$.

$$T(E)^{\perp} = \{z \in E \mid (z \mid x) = 0\, \forall x \in T(E)\} = \{z \in E \mid (z \mid T(y)) = (T(y) \mid z) = 0\, \forall y \in E\} =$$

$$\{z \in E \mid a(y, z) = 0\ \forall x \in T(E)\}.$$

If $z \in T(E)^{\perp}$, then $a(z, z) = 0$. But

$$0 = a(z, z) \geq \alpha\|z\|^2 \Rightarrow z = 0,$$

which implies $T(E)^{\perp} = \{0\}$, so $T(E)$ is dense in E.

iii. Using the fact that a is coercive and Cauchy-Schwarz's inequality, we have

$$\alpha\|Tx\|^2 \leq a(x, x) = (Tx \mid x) \leq \|Tx\|\ \|x\| \Rightarrow \|Tx\| \geq \alpha\|x\|.$$

Consequently, T is injective: if $x \neq y$, then

$$\|Tx - Ty\| = \|T(x - y)\| \geq \alpha\|x - y\| \neq 0 \Rightarrow Tx - Ty \neq 0.$$

Furthermore, $T(E)$ is closed: let $\{y_n\}_n \subseteq T(E)$ be a sequence such that $y_n \to y$ as n tends to infinity. Since $\{y_n\}_n \subseteq T(E)$, we can consider the sequence $\{x_n\}_n \subseteq E$ such that $y_n = T(x_n)\ \forall n \in \mathbb{N}$. The convergence of $\{y_n\}_n$ means that

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \mid \forall m, n > N\ \|y_n - y_m\| < \epsilon.$$

Since $y_n - y_m = T(x_n - x_m)$, we have that

$$\epsilon > T(x_n - x_m) \geq \alpha\|x_n - x_m\|.$$

Thus $\{x_n\}_n$ is Cauchy, and since E is complete it converges to an element $x \in E$. Inasmuch T is continuous, we have that

$$Tx = T(\lim_{n\to\infty} x_n) \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y,$$

which means $T(E)$ is closed.

iv. From the preceding results, we have that

$$E = \overline{T(E)} = T(E),$$

thus T is surjective. Being T injective and surjective, it is an isomorphism from E onto itself.

b. **i.** By the Riesz Representation Theorem, there exist a unic element $v \in E$ such that

$$L(y) = (y \mid v) \qquad \forall y \in E.$$

Being T an isomorphism, there exists a unic element $u \in E$ such that $v = Tu$, thus

$$L(y) = (y \mid v) = (y \mid Tu) = (Tu \mid y) = a(u, y) \qquad \forall y \in E.$$

ii. We have that

$$\Phi(x) - \Phi(u) = \frac{a(x,x)}{2} - L(x) - \frac{a(u,u)}{2} + L(u) = \frac{a(x,x)}{2} - a(u,x) - \frac{a(u,u)}{2} + a(u,u)$$

$$= \frac{a(x,x) + a(u,u)}{2} - a(u,x).$$

Since

$$|a(u,x)| \leq \|x\| \; \|u\| \leq \frac{\|x\|^2 + \|y\|^2}{2} = \frac{a(x,x) + a(u,u)}{2},$$

we have the thesis.

3 Lions-Stampacchia Theorem (symmetric case).

Consider a real Hilbert space E, a nonempty, closed, convex set C in E, a continuous and coercive bilinear symmetric form a on E, and a continuous linear form L on E. Let J be the function defined on E by

$$J(u) = a(u,u) - 2L(u) \qquad \text{for all } u \in E. \tag{9}$$

Prove that there exists a unique $c \in C$ such that $J(c) \leq J(v)$ for all $v \in C$, and that c is characterized by the following condition:

$$a(c, v - c) \geq L(v - c) \qquad \text{for all } v \in C. \tag{10}$$

Solution By the Lax-Milgram theorem, there exist a unic element $u \in E$ such that

$$L(v) = a(u, v) \ \forall v \in E.$$

Because of this fact, we have that

$$a(c, v - c) \geq L(v - c) \ \forall v \in C \Leftrightarrow$$

$$\Leftrightarrow a(c, v - c) \geq a(u, v - c) \ \forall v \in C \Leftrightarrow$$
$$a(u - c, v - c) \leq 0 \ \forall v \in C, \tag{11}$$

and the thesis (the existence and the uniqueness of the element c and the inequality (11)) immediately follows from the Projection Theorem.