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## Chapter 1

## An Introduction to the Spectrum of a Ring

In the classical algebraic geometry, we usually work on algebraically closed fields $k$ and we have the affine spaces $\mathbb{A}^{n}=k^{n}$ with the Zariski topology. So in this case

Definition 1.1. An affine algebraic set is a closed subset of $\mathbb{A}^{n}$.
We usually consider regular functions on this sets; these are functions that locally behaves like a quotient of polynomials:

Definition 1.2. A regular function on an affine algebraic set $X \subseteq \mathbb{A}^{n}$ is a map $f: X \rightarrow \mathbb{A}^{1}$ such that for all $x \in X$ there exists an open neighbourhood $U_{x}$ and two polynomials $g, h \in k\left[x_{1}, \ldots, x_{n}\right]$ such that for all $y \in U_{x} h(y) \neq 0$ and

$$
f(y)=\frac{g(y)}{h(y)}
$$

Given two algebraic sets $X, Y$, a morphism is a continuous map $f: X \rightarrow Y$ such that, for all regular functions $\varphi$ on an open subset $U \subseteq Y, f \circ \varphi: f^{-1}(U) \longrightarrow \mathbb{A}^{1}$ is regular.

Affine sets and morphism form a category; furthermore given an affine set $X$ we can consider the reduced $k$-algebra $k[X]$ given by the regular function on $X$. Such rings are strictly connected to the affine set; given a morphism $f: X \rightarrow Y$ we get a homomorphism of $k$-algebras $f^{*}: k[Y] \rightarrow k[X]$. Viceversa, given a homomorphism of $k$-algebras $g: k[Y] \rightarrow k[X]$, there exists a unique morphism $f: X \rightarrow Y$ such that $f^{*}=g$. This fact gives an equivalence of categories between the category of reduced $k$-algebras and the category of affine sets.
Notice that there is a correspondance between points $p \in X$, homomorphism $k[X] \rightarrow k$ and maximal ideals of $k[X]$. Therefore, in order to extend this theory to non-algebraically closed fields, it's convenient to consider the maximal spectrum:

Definition 1.3. Let $A$ be a ring. We call the maximal spectrum as the set

$$
\operatorname{Spec} M(A)=\{p \subseteq A \mid p \text { is maximal }\}
$$

We will consider only commutative rings with 1 . Given a subset $S \subseteq A$, we call $V(S)$ the set of maximal ideals that contain $S$.

Lemma 1.4. Let $A$ be a ring. Then

- $V(0)=\operatorname{Spec} M(A)$
- Given a set of ideals $\left\{I_{\alpha}\right\}_{\alpha}, V\left(\sum_{\alpha} I_{\alpha}\right)=\cap_{\alpha} V\left(I_{\alpha}\right)$
- Given two ideals $I, J, V(I \cap J)=V(I) \cup V(J)$

Theorem 1.5. There is a unique topology on $\operatorname{Spec} M(A)$ such that every closed subset is of the kind of $V(I)$.

If we consider $k$-algebras, where $k$ is a ring, given a morphism $f: X \rightarrow Y$ we get a morphism $\varphi: \operatorname{Spec} M(k[Y]) \rightarrow \operatorname{Spec} M(k[X])$ that sends a maximal ideal to its inverse image.
Observation 1.6. It's not true that $V(I)=V(J)$ implies $\sqrt{I}=\sqrt{J}$. For example, in the local ring $\mathbb{Z}_{(2)}$, the ideals (0) and (2) define the same closed subset $V(0)=V(2)=(2)$, but they are both radical.

Notice that to make things work, we have considered only the case of $k$ algebras: given a homomorphism of rings $\varphi: A \rightarrow B$ the contraction of a maximal ideal doesn't need to be maximal. Therefore, to consider a wider class of rings, we can consider the prime spectrum:

Definition 1.7. Let $A$ be a ring. The (prime) spectrum of $A$ is the set

$$
\operatorname{Spec}(A):=\{p \subseteq A \mid p \text { is a prime ideal }\}
$$

As in the case of the maximal spectrum, we can give a topology to this set defining $V(I)=\{p \in \operatorname{Spec}(A) \mid I \subseteq p\}$.
With these definitions, given a homomorphism of rings $f: A \rightarrow B$ we get an induced map

$$
f^{*}: \operatorname{Spec}(B) \rightarrow \operatorname{Spec}(A)
$$

that sends a prime ideal $p \in \operatorname{Spec}(B)$ to its contraction. This is continuous since

$$
f^{*-1}(V(I))=\{p \in \operatorname{Spec}(B) \mid p \supseteq f(I)\}=V(I B)
$$

Notice that the topology we have just defined is not $T_{1}$; the set of closed points coincide exactly with the maximal spectrum. There is a bijection between radical ideals and closed subset of $\operatorname{Spec}(A)$ that reverses inclusions, given by $I \leftrightarrow V(I)$.

Definition 1.8. Let $X$ be a topological space. $X$ is irreducible if given two closed subsets $C_{1}, C_{2}$ such that $C_{1} \cup C_{2}=X$, then either $C_{1}=X$ or $C_{2}=X$.

Proposition 1.9. Let $I \subseteq A$ be an ideal. Then $I$ is prime if and only if $V(I)$ is irreducible.

Lemma 1.10. Let $X$ be a topological space and let $Y \subseteq X$ be a subspace. The following are equivalent:

1. Every closed subset of $Y$ is the intersection of a unique closed subset in $X$ and $Y$.
2. If $A \subseteq X$ is closed then $Y \cap A$ is dense in $A$

If those conditions hold then $Y$ contains all closed points of $X$.
Proof.
$(1) \Rightarrow(2)$ Let $A \subseteq X$ be a closed subset. We have to show that $Y \cap A$ is dense in $A$. Assume by contradiction that $\overline{Y \cap A} \neq A$. Then there exists a proper closed subset $Z \subsetneq A$ such that $Y \cap A \subseteq Z$. By hypotesis, $Z$ is the intersection of a unique closed subset $C \subseteq X$ and $A$. Therefore $Y \cap A \subseteq C \cap A$, which implies that $Y \subseteq C$. However, $Y=Y \cap C=Y \cap X$; by the uniqueness of the closed subset $X=C$ and this gives a contradiction.
$(2) \Rightarrow(1)$ Let $C \subseteq Y$ be a closed subset and assume by contradiction that $C=$ $Y \cap A=Y \cap B$. By hypotesis, $Y \cap A$ is dense in $A$. Therefore, taking closure,

$$
A=\overline{Y \cap A}=\bar{Y} \cap A=\overline{Y \cap B} \subseteq \bar{Y} \cap \bar{B}=\bar{Y} \cap B
$$

and therefore $\bar{Y} \cap A \supseteq \bar{Y} \cap B$ which implies $A \supseteq B$. Since it is symmetric in $A, B$, we get the thesis.

Definition 1.11. Let $A$ be a ring. $A$ is Jacobson if Spec $M(A)$ is dense in every closed subset of $\operatorname{Spec}(A)$.

Example. If $A$ is a PID, $A$ is Jacobson since $\operatorname{Spec}(A)=\operatorname{Spec} M(A) \cup\{(0)\}$. If $A$ is local and $\# \operatorname{Spec}(A) \geq 2$, then $A$ is not Jacobson.

Proposition 1.12. Let $A$ be a ring. The following are equivalent:

1. $A$ is Jacobson
2. Every prime ideal of $A$ is an intersection of maximal ideals
3. Every radical ideal of $A$ is the intersection of maximal ideals

Proof.
$(1) \Rightarrow(2)$ Let $p \in \operatorname{Spec}(A)$. Assume by contradiction that

$$
q=\bigcap_{\substack{m \in \operatorname{Spec} M(A) \\ m \supseteq p}} m \supsetneq p
$$

Then $V(q) \subsetneq V(p)$ and $V(q) \cap \operatorname{Spec} M(A)=V(p) \cap \operatorname{Spec} M(A)$. By the lemma, $V(p)=V(p)$ which is absurd.
$(2) \Rightarrow(3)$ Trivial.
$(3) \Rightarrow(1)$ Let $V(I), V(J)$ be closed subsets of $\operatorname{Spec}(A)$; by the lemma, we have to show that if $\operatorname{Spec} M(A) \cap V(I)=\operatorname{Spec} M(A) \cap V(J)$ then $V(I)=V(J)$. We can assume that $I, J$ are radical; by hypotesis $I=\cap M_{i}$ and $J=\cap N_{j}$ where $M_{i}, N_{j}$ are maximal ideals. We notice that by the prime avoidance lemma in this intersection all the maximal ideals containing $I$ and $J$ appear and only them. Therefore $V(I)=V(J)$.

Notice that $\operatorname{Spec} M(A)=V(\mathcal{J}(A))$ and $\operatorname{Spec}(A)=V(\mathcal{N}(A))$. Therefore Spec $M(A)$ is dense in $\operatorname{Spec}(A)$ if and only if $\mathcal{J}(A)=\mathcal{N}(A)$.

Proposition 1.13. The following are equivalent:

- $A$ is Jacobson
- For all $I \subseteq A, \mathcal{J}(A / I)=\mathcal{N}(A / I)$
- For all radical ideals $I, \mathcal{J}(A / I)=0$.

Theorem 1.14 (Nullstellensatz). Let $A$ be a Jacobson ring and let $B$ a finitely generated $A$-algebra.

- $B$ is Jacobson
- If $M \in \operatorname{Spec} M(A)$ then $M \cap A \in \operatorname{Spec} M(A)$ and

$$
A / M \cap A \longrightarrow B / M
$$

is a finite extension of fields.
Definition 1.15. A discrete valuation ring (DVR) $R$ is a domain such that exists $t \in R \backslash\{0\}$ such that every $f \in R \backslash\{0\}$ can be written uniquely as $f=u t^{n}, u \in R^{*}$ and $n \in \mathbb{N}$.
Example. The localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ for a maximal ideal are DVR. The ring of power series over a field $k \llbracket x \rrbracket$ is a DVR.

It follows from the definition that the decomposition of an element $f=u t^{n}$ is unique. This implies that the only ideals of $R$ are ( 0 ) of $\left(t^{k}\right), k \in \mathbb{N}$ and in particular $R$ is a local PID (but not Jacobson). Therefore if we invert $t$ we get the quotient field of $R Q(R)=R_{t}$. Every element of this field can be written uniquely as $u t^{k}, k \in \mathbb{Z}$ and this defines a discrete valuation on $Q(R)=k$ such that $v_{R}\left(u t^{k}\right)=k$ :

Proposition 1.16. Let $a, b \in k \backslash\{0\}$. Then $v_{R}(a+b) \geq \min \left(v_{R}(a), v_{R}(b)\right)$. Conversely, given a discrete valuation $v: k^{*} \rightarrow \mathbb{Z}$, the set

$$
R=\left\{a \in k^{*} \mid v(a) \geq 0\right\} \cup\{0\}
$$

is a DVR.
Consider now $A=R[x]$, where $R$ is a DVR, and let $k$ be the residue field $k=R /(t)$. We get a projection $A \rightarrow k[x]$ and every maximal ideal of $k[x]$ gives a maximal ideal of $A$. In particular, we know that $k[x]$ is a PID and if $m=(p(x))$ is maximal, $\pi^{-1}(m)=(t, p(x))$. However, they are not the only maximal ideals. For example, $I=(x t-1)$ is maximal since the quotient is exactly $k$.
Observation 1.17. Notice that $A$ is not Jacobson but $\operatorname{Spec} M(A)$ is dense in $\operatorname{Spec}(A)$.

Definition 1.18. Let $R$ be a ring. We define $\mathbb{A}_{R}^{n}=\operatorname{Spec}\left(R\left[x_{1}, \ldots, x_{n}\right]\right)$.

Proposition 1.19. Let $A$ be a ring and let $S$ be a multiplicative subset. The homomorphism $A \rightarrow S^{-1} A$ induces a map $\operatorname{Spec}\left(S^{-1} A\right) \rightarrow \operatorname{Spec}(A)$ which is a homeomorphism with the subspace $\{p \in \operatorname{Spec}(A) \mid p \cap S=\emptyset\}$.

This proposition gives a way to identify easily some open subset of $\operatorname{Spec}(A)$. Let $f \in A$ and consider the open subset $\operatorname{Spec}(A) \backslash V(f)$. Then $\operatorname{Spec}\left(A_{f}\right) \simeq$ $\operatorname{Spec}(A) \backslash V(f)$.

Let now $A$ be a Jacobson ring. Then we have a bijection between irreducible closed subsets of $\operatorname{Spec} M(A)$ and irreducible closed subsets of $\operatorname{Spec}(A)$ and these can be identified as the closure of points of $\operatorname{Spec}(A)$.
Definition 1.20. Let $X=\operatorname{Spec}(A)$ and let $p \in X$. The residue field of $p$ is the quotient

$$
k(p):=A_{p} / p A_{p}
$$

Notice that the following diagram is commutative:


Given $f \in A$, we can evaluate it in a point $p \in \operatorname{Spec}(A)$

$$
f(p):=\varphi(f) \in k(p)
$$

Notice that $(f+g)(p)=f(p)+g(p)$ and $(f g)(p)=f(p) g(p)$. Furthermore, $f(p)=0$ if and only if $f \in p$ and this implies $f(p)=0$ for all $p \in \operatorname{Spec}(A)$ if and only if $f \in \mathcal{N}(A)$.
Definition 1.21. Let $X$ be a topological space. We say that $X$ is quasi-compact if for all open cover of $X$ there exists a finite subcover.

Notice that it it equivalent to say that given a family of closed subsets $C_{i}$ such that $\cap_{i \in I} C_{i}=\emptyset$, there exists a finite subfamily such that $\cap_{j=1}^{n} C_{i_{j}}=0$
Observation 1.22 . Usually, we say that $X$ is compact when it is compact and Hausdorff.
Proposition 1.23. The open subsets of the form $X_{f}=X \backslash V(f)$ form a basis of open sets for the topology on $X$.
Proof. Let $U \subseteq X$ be an open subset. We have to show that there exists $f \in A$ such that $X_{f} \subseteq U$. Notice that $X \backslash U$ is closed and therefore $X \backslash U=V(I)$. Let $f \in I$. Then $X_{f}=X \backslash V(f) \subseteq X \backslash V(I)=U$ and this gives the thesis.

Theorem 1.24. Let $A$ be a ring. Then $\operatorname{Spec}(A)$ is quasi-compact.
Proof. Consider a family of closed subsets $V\left(I_{k}\right), k \in K$ and assume that the intersection is empty $\cap_{k \in K} V\left(I_{k}\right)=\emptyset$. This means $V\left(\sum_{k \in K} I_{k}\right)=\emptyset$ and therefore $1 \in \sum_{k \in K} I_{k}$. So there exists a finite combination of elements of these ideals such that

$$
1=\sum \underbrace{a_{i}}_{\in A} \underbrace{s_{k_{i}}}_{\in I_{k_{i}}}
$$

Therefore $1 \in \sum_{i=1}^{n} I_{k_{i}}$ and so $\cap_{i=1}^{n} V\left(I_{k_{i}}\right)=V\left(\sum_{i=1}^{n} I_{k_{i}}\right)=\emptyset$.

## Chapter 2

## Sheaves

Definition 2.1. Let $X$ be a topological space. A presheaf $P$ of abelian groups on $X$ is a functor from the set of open sets of $X$ to abelian groups, so for each pair of open subsets $U \subseteq V$ there exists a homomorphism

$$
\rho_{V U}: P(V) \longrightarrow P(U)
$$

called restriction homomorphism, such that $\rho_{U U}=\mathrm{Id}$ and if $U \subseteq V \subseteq W$ then $\rho_{W U}=\rho_{V U} \circ \rho_{W V}$.
A subpresheaf $Q$ of a presheaf $P$ is a collection of subgroups $Q(U) \subseteq P(U)$ that commutes with restrictions.

Example. If we associate to each open set $U \subseteq X$ the continuous functions from $U$ to $X$, we get a presheaf. If $A$ is an abelian group, $P_{A}$ given by the $P_{A}(U)=A$ and $\rho_{U V}=\operatorname{Id}_{A}$ is a presheaf.

Definition 2.2. Let $P, Q$ be presheaves on $X$. A homomorphism of presheaves is a natural transformation between $P, Q$. This means that for every open subset $U$ there exists a homomorphism $f_{U}: P(U) \rightarrow Q(U)$ such that for all $U \subseteq V$ the following commutes


The kernel of a homomorphism is the subpresheaf

$$
\operatorname{Ker}(\varphi):=\left\{U \longmapsto \operatorname{Ker}\left(\varphi_{U}\right) \mid \varphi_{U}: P(U) \rightarrow Q(U)\right\}
$$

Example. If $X \subseteq \mathbb{C}$ is open and $\mathcal{O}_{X}$ is the presheaf of nowhere zero holomorphic functions (with multiplication), then $\exp : \mathcal{O}_{X} \rightarrow \mathcal{O}_{X}$ is a homomorphism of presheaves.
Let $A$ be an abelian group and $X$ a toplogical space. We have the presheaf $A_{X}$ of locally constant functions $f: U \rightarrow A$. There exists a homomorphism $\varphi: P_{A} \rightarrow A_{X}$ such that

$$
\begin{array}{llll}
P_{A}(U): & A & \longrightarrow & A_{X}(U) \\
& a & \longmapsto(u \mapsto a)
\end{array}
$$

If $Q \subseteq P$ is a presheaf, we define the quotient presheaf as

$$
P / Q^{(U)}:=P(U) / Q(U)
$$

For example, if $R$ is a subpresheaf of $P$ that goes to zero through $\varphi: P \rightarrow Q$, then it factors through

$$
P / R \longrightarrow Q
$$

Consider now the quotient $Q=\mathcal{C}_{X} / \mathcal{C} B_{X}$, which is the quotient of all continuous function for all the bounded continuous function. Then if $s \in Q(U)$ there exists an open cover $U=\cup V_{i}$ such that $\left.s\right|_{V_{i}}=0$ for all $i$. We now take $P$ as the bounded continuous functions and consider an open cover $U=\cup V_{i}$. Let $s_{i} \in P\left(V_{i}\right)$ such that $\left.s_{i}\right|_{V_{i j}}=\left.s_{j}\right|_{V_{i j}}$. Then we can't lift lift these section to a global section $s \in P(U)$. These observations give rise to the following:

Definition 2.3. A presheaf $P$ is separated if for all $U \subseteq X$ open subset and for all $s \in P(U)$, if there exists an open cover $U=\cup V_{i}$ such that $\left.s\right|_{V_{i}}=0$ then $s=0$.
A sheaf $F$ is a separated presheaf such that for all open subsets $U$ and for all open covers $U=\cup V_{i}$, given sections $s_{i} \in F\left(V_{i}\right)$ such that $\left.s_{i}\right|_{V_{i j}}=\left.s_{j}\right|_{V_{i j}}$, there exists $s \in F(U)$ such that $\left.s\right|_{V_{i}}=s_{i}$.
A morphism of sheaves $\varphi: P \rightarrow Q$ is a morphism of presheaves where $P, Q$ are sheaves.

Proposition 2.4. The kernel of a morphism of sheaves is a sheaf.
Proof. Let $\varphi: F \rightarrow G$ be a morphism of sheaves. Clearly, $\operatorname{Ker}(\varphi)$ is a presheaf; we have to show that it is a sheaf. Let $U$ be an open subset and consider an open cover $U=\cup V_{i}$. Let $s_{i} \in \operatorname{Ker}(\varphi)\left(V_{i}\right)$; we want to show that they lift to a section in $\operatorname{Ker}(\varphi)(U)$. By the property of sheaves, we can find $s \in F(U)$ such that $\left.s\right|_{V_{i}}=s_{i}$. Then $\left.\varphi(s)\right|_{V_{i}}=\varphi\left(s_{i}\right)=0$ and therefore $\varphi(s)$ is locally zero. Since $G$ is a sheaf, $\varphi(s)=0$ and this concludes the proof.

Definition 2.5. Let $X$ be a topological space and let $P$ be a presheaf on $X$. Given $x \in X$, we consider the set

$$
\{(U, \varphi) \mid U \text { neighbourhood of } x, \varphi \in P(U)\}
$$

and the relation

$$
(U, s) \sim(V, t) \Longleftrightarrow \exists W \subseteq U \cap V \text { neighbourhood of } x \text { s.t. }\left.s\right|_{W}=\left.t\right|_{W}
$$

We call the quotient as the stalk $P_{x}$ of $P$ at $x$.
The stalk can also be identified as the direct limit:

$$
P_{x}=\underset{x \in U}{\lim } P(U)
$$

and it is an abelian group. We call the element of this group as germs. If $s \in P(U)$, we denote its germ as $s_{p}=[(U, s)]$.
Proposition 2.6. Let $X$ be a topological space and $P$ a presheaf on $X . P$ is separated if and only if if $s \in P(U)$ and $s_{p}=0$ for all $p \in U$, then $s=0$.

Let $\varphi: P \rightarrow Q$ be a homomorphism of presheaves. It induces a map between the stalks:

$$
\begin{array}{cccc}
\varphi_{p}: & P_{x} & \longrightarrow & Q_{x} \\
{[(U, s)]} & \longmapsto & {\left[\left(U, \varphi_{U}(s)\right)\right]}
\end{array}
$$

Notice that if $F$ is a sheaf on a topological space $X$, then $F(\emptyset)=0$. $\emptyset$ is a covering of the empty set and $F(\emptyset)$ is the equalizer of the maps

$$
F(\emptyset) \rightarrow \prod_{i \in \emptyset} F\left(U_{i}\right) \rightrightarrows \prod_{i, j \in \emptyset} F\left(U_{i j}\right)
$$

Since the product are over an empty set of indexes, they are zero, so $F(\emptyset)=0$. Example. Let $X$ be an irreducible toplogical space and let $A$ be an abelian group $A \neq 0$. We define the constant presheaf $P_{A}(U)=A$ for all open subsets $U$. This is not a sheaf; indeed, $P_{A}(\emptyset)=A \neq 0$.

### 2.1 Sheafification

Let $P$ be a presheaf on $X$. We want to find in a certain sense the smallest sheaf containing $P$. Given an open subset $U \subseteq X$, we define

$$
\tilde{P}(U)=\left\{\varphi: U \rightarrow \sqcup_{x \in U} P_{x} \mid \varphi(x) \in P_{x} \forall x \in X\right\}
$$

$\tilde{P}$ is a presheaf since we are considering all the function ignoring continuity; given $U \subseteq V \subseteq W$ we naturally get the restriction maps

$$
\tilde{P}(W) \xrightarrow{\rho_{W V}} \tilde{P}(V) \xrightarrow{\rho_{V U}} \tilde{P}(U)
$$

obtained just by ignoring the points. In a certain sense, this presheaf ignores the continuity of the maps. We now define the sheafification as the biggest subsheaf $P^{s h} \subseteq \tilde{P}$ just by forcing the lifting property:

$$
P^{s h}(U)=\left\{\varphi \in \tilde{P}(U) \mid \exists\left\{U_{i}\right\}_{\text {cover }}^{\text {open }}, s_{i} \in P\left(U_{i}\right) \text { s.t. } \forall x \in U_{i} \varphi(x)=\left(s_{i}\right)_{x}\right\}
$$

This is a subsheaf of $\tilde{P}$. First of all, we have to show that the restriction map is well-defined

$$
\left.\begin{aligned}
\rho: \quad P^{s h}(U) & \longrightarrow P^{s h}(V) \\
\phi & \longmapsto
\end{aligned} \phi\right|_{V}
$$

Let $\phi \in P^{s h}(U)$; then, since the restriction map of a subpresheaf is the restriction of the one of the presheaf, $\left.\phi\right|_{V} \in \tilde{P}(V) . \phi \in P^{s h}(U)$ and by definition there exist an open cover $U_{i}$ and sections $s_{i} \in P\left(U_{i}\right)$ such that for all $i, \phi(x)=\left(s_{i}\right)_{x}$ for all $x \in U_{i}$. Let now $V_{i}=V \cap U_{i}$ and $\tilde{s}_{i}=\rho_{U_{i} V_{i}}\left(s_{i}\right)$; then $\left.\phi\right|_{V} \in P^{s h}(V)$.
We have now to show that coherent sections lift. Let $U \subseteq X$ be an open set and consider an open cover $U=U V_{i}$ and sections $s_{i} \in P^{s h}\left(V_{i}\right)$ such that $\left.s_{i}\right|_{V_{i j}}=\left.s_{j}\right|_{V_{i j}}$. We have to show that there exists $s \in P^{s h}(U)$ such that $\left.s\right|_{V_{i}}=s_{i}$. Clearly, there exists $s \in \tilde{P}(U)$ such that $\left.s\right|_{V_{i}}=s_{i}$. Since $s_{i} \in P^{s h}\left(V_{i}\right)$, we can find open covers $V_{i j}$ and $s_{i j} \in P\left(V_{i j}\right)$ such that $s_{i}(x)=\left(s_{i j}\right)_{x}$. Then $V_{i j}$ is an open cover of $U$ and the $s_{i j} \in P\left(V_{i j}\right)$ have the property such that $s(x)=s_{i j}(x)$. Therefore by definition $s \in P^{s h}(U)$ and furthermore it is unique. This shows that $P^{s h}$ is a sheaf.
We notice that there exists a canonical map

$$
\begin{array}{rllc}
\eta_{p}: & P & \longrightarrow & P^{s h} \\
s & \longmapsto & \left(x \mapsto s_{x}\right)
\end{array}
$$

Lemma 2.7. $\eta_{p}$ is injective if and only if $P$ is separated.
Proof. Given $s \in P(U)$,

$$
\eta_{p}(s)=0 \Longleftrightarrow s_{x}=0 \quad \forall x \in U
$$

Assume $\eta_{p}$ is injective: then if $s_{x}=0$ for all $x \in U$ we get $s=0$.
Conversely, if $P$ is separated and $s_{x}=0$ for all $x \in U, s=0$ and therefore $\eta_{p}$ is injective.

Lemma 2.8. $P$ is a sheaf if and only if $\eta_{p}$ is an isomorphism.
Proof. Assume that $P$ is a sheaf. Then by the lemma $\eta_{p}$ is injective and we have only to show surjectivity as a map of presheaves. Let $\varphi \in P^{s h}(U)$ : by definition there exists an open cover $U=\cup V_{i}$ and sections $s_{i} \in P\left(V_{i}\right)$ such that $\varphi(x)=s_{i}(x)$ for all $x \in U_{i}$. Then $\left.s_{i}\right|_{U_{i j}}=\left.s_{j}\right|_{U_{i j}}$ and since $P$ is a sheaf there exists $s \in P(U)$ such that $\left.s\right|_{U_{i}}=s_{i}$ and we get the thesis.

Lemma 2.9. For all $x \in X,\left(\eta_{p}\right)_{x}: P_{x} \rightarrow P_{x}^{s h}$ is an isomorphism.

$$
\begin{array}{cccc}
\left(\eta_{p}\right)_{x}: & P_{x} & \longrightarrow & P_{x}^{s h} \\
& {[(s, U)]} & \longmapsto & {\left[\left(\eta_{p}(s), U\right)\right]}
\end{array}
$$

Proof. First, we show surjectivity. Let $[(\varphi, V)] \in P_{x}^{s h}$. Then $\varphi \in P^{s h}(V)$ and by definition there exist an open cover $A_{i}$ and sections $s_{i} \in A_{i}$ such that $\varphi(y)=s_{i}(y)$ for all $y \in A_{i}$. Assume that $x \in A_{\bar{i}}$ then $[(\varphi, V)]=\left[\left(\eta_{p}\left(s_{i}\right), A_{i}\right)\right]$. We now show that $\left(\eta_{p}\right)_{x}$ is injective. Let $[(s, U)] \in P_{x}$ such that $\left[\left(\eta_{p}(s), U\right)\right]=0$. Since $\eta_{p}(s) \in P^{s h}(U)$, there exist an open cover $A_{i}$ and sections $s_{i} \in P\left(A_{i}\right)$ such that $\eta_{p}(s)(y)=s_{i}(y)=0$. Therefore

$$
[(s, U)]=\left[\left(s_{i}, A_{i}\right)\right]=\left[\left(0, A_{i}\right)\right]
$$

Lemma 2.10. Let $\varphi: P \rightarrow Q$ be a homomorphism of presheaves. Then there exists a unique $\varphi^{s h}: P^{s h} \rightarrow Q^{s h}$ such that the following commutes:


Proof. It is enough to define

$$
\varphi^{s h}(\alpha)(x)=\varphi_{x}(\alpha)(x)
$$

Therefore we have shown that the sheafification has the following universal property:

Let $P$ be a presheaf and $F$ a sheaf. Given a morphism of presheaves $\varphi: P \rightarrow F$, there exists a unique $\varphi^{s h}: P^{s h} \rightarrow F$ such that


Example. Let $A$ be an abelian group with the discrete topology and we consider the presheaves $P_{A}(U)=A$ for every $U \subseteq X$ open set. Then

$$
\tilde{P}(U)=\{f: U \rightarrow A\} \quad P^{s h}(U)=\{f: U \rightarrow A \mid f \text { is locally constant }\}
$$

Let now $F$ be a sheaf (of abelian groups) and $G \subseteq F$ be a presheaf. Then

$$
\widetilde{F / G}(U)=F(U) / G(U)
$$

is a separated presheaf but in general not a sheaf. We define

$$
F / G:=\widetilde{F / G}^{s h}
$$

The composition

$$
G \rightarrow F \rightarrow \widetilde{F / G}
$$

is zero and so

$$
G \rightarrow F \rightarrow F / G
$$

is zero too. In general, if $f: F \rightarrow H$ is a homomorphism of sheaves such that $G \rightarrow F \rightarrow H$ is zero, we get $F / G \rightarrow H$.

Definition 2.11. Let $\varphi: F \rightarrow G$ be a morphism of sheaves. We define

$$
\operatorname{Im}(\varphi):=\widetilde{\operatorname{Im}}(\varphi)^{s h}
$$

$\operatorname{Ker}(\varphi)$ is a sheaf, so we don't need to sheafify it. In particular,
$\varphi$ is injective $\Longleftrightarrow \varphi$ is injective as a morphism of presheaves $\varphi: F \rightarrow G$ is surjective $\Longleftrightarrow \quad \operatorname{Im}(\varphi)=G$
and therefore $\varphi$ doesn't need to be surjective as a homomorphism of presheaves. However, if $\varphi$ is injective, $\widetilde{\operatorname{Im}}(\varphi)=\operatorname{Im}(\varphi)$ and so $\varphi$ is a homomorphism if and only if $\varphi$ is injective and surjective.

Lemma 2.12. Let $F, G$ be sheaves and let $\varphi: F \rightarrow G$ be a homomorphism of sheaves.

1. If $\varphi_{x}$ is surjective for all $x \in X$, then $\varphi$ is surjective.
2. If $\varphi_{x}$ is injective for all $x \in X$, then $\varphi$ is injective.
3. If $\varphi_{x}$ is an isomorphism for all $x \in X$, then $\varphi$ is an isomorphism.

Proof. First we notice that the third point follows from the others. We only have to show the first two.

1. Assume $\varphi_{x}$ is surjective for all $x \in X$. Let $g \in G(U)$; by the surjectivity on the stalks, for all $x \in U[(g, U)]=\varphi_{x}\left(\left[\left(f_{x}, V_{x}\right)\right]\right)$. Notice that $U=$ $\cup_{x \in U}\left(U \cap V_{x}\right)$ and $f_{x} \in F\left(V_{x} \cap U\right)$. Since $F$ is a sheaf, there exists $\tilde{f} \in F(U)$ such that $\left.\tilde{f}\right|_{V_{x}}=f_{x}$ for all $x$. Then $\varphi(\tilde{f})=g$, as desired.
2. Assume $\varphi_{x}$ is injective for all $x \in X$. Let $f \in F(U)$ such that $\varphi_{U}(f)=0$. Then for all $x \in X$,

$$
\left(\varphi_{U}\right)_{x}[(f, U)]=\left[\left(\varphi_{U}(f), V_{x}\right)\right]=\left[\left(0, V_{x}\right)\right]
$$

Since the map is injective on the stalks, $f$ is locally zero and since $F$ is a sheaf $f=0$.

Observation 2.13. In the proof of this lemma we didn't use the fact that $G$ is a sheaf; therefore if $F$ is a sheaf and $G$ is a presheaf and we have a map $\varphi: F \rightarrow G$ that induces isomorphisms on the stalks, $G$ is a sheaf and the map is an isomorphism.

Definition 2.14. A sequence $F^{\prime} \xrightarrow{\alpha} F \xrightarrow{\beta} F^{\prime \prime}$ is exact in $F$ if $\operatorname{Im}(\alpha)=\operatorname{Ker}(\beta)$.
Proposition 2.15. $F^{\prime} \rightarrow F \rightarrow F^{\prime \prime}$ is exact if and only if $F_{x}^{\prime} \rightarrow F_{x} \rightarrow F_{x}^{\prime \prime}$ is exact for all $x \in X$.

Example. Since surjectivity of a morphism of sheaves is different from surjectivity of a morphism of presheaves, even if $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ is exact the global section can lose this property

$$
0 \rightarrow F^{\prime}(X) \rightarrow F(X) \rightarrow F^{\prime \prime}(X)
$$

### 2.2 Operations on Sheaves

In this section, we suppose that $X$ is a topological space and $\left\{P_{i}\right\}_{i \in I}$ is a collection of presheaves

Product We define the product of presheaves as

$$
\left(\prod_{i \in I} P_{i}\right)(U):=\prod_{i \in I} P_{i}(U)
$$

The product has the following universal property:

If $Q$ is a presheaf, given $\forall i \in I$ homomorphisms $\varphi_{i}: Q \rightarrow P_{i}$, then there exists a unique homomorphism $\varphi: Q \rightarrow \prod_{i} P_{i}$ such that $\pi_{i} \circ \varphi=\varphi_{i}$, where $\pi_{i}: \prod_{j} P_{j} \rightarrow P_{i}$ is the canonical projection.

Notice that if the $P_{i}$ 's are sheaves, so is the product $\prod_{i} P_{i}$.

Direct Sum We define the direct sum of presheaves as

$$
\left(\widetilde{\bigoplus_{i \in I} P_{i}}\right)(U):=\bigoplus_{i \in I} P_{i}(U)
$$

It satisfies the following universal property:

If $Q$ is a presheaf, given $\forall i \in I$ homomorphisms $\varphi_{i}: P_{i} \rightarrow Q$, then there exists a unique homomorphism $\varphi: \widetilde{\oplus P_{i}} \rightarrow Q$ such that $\varphi \circ \epsilon_{i}=\varphi_{i}$, where $\epsilon_{i}: P_{i} \rightarrow \widetilde{\oplus P_{i}}$ is the canonical injection.

Remark 2.16. If the $P_{i}$ 's are sheaves, it's not true that $\widetilde{\oplus P}_{i}$ is a sheaf.
If $P_{i}$ 's are sheaves, we define $\oplus_{i \in I} P_{i}=\left({\left.\widetilde{\oplus_{i \in I} P_{i}}\right)^{s h} \text {. Since we have the presheaves }}_{\text {she }}\right.$. inclusion, we obtain

$$
\widetilde{\bigoplus_{i \in I} P_{i}} \subseteq \prod_{i \in I} P_{i} \longrightarrow \bigoplus_{i \in I} P_{i} \subseteq \prod_{i \in I} P_{i}
$$

In particular, we can identify the direct sum in the product in this way:

$$
\left(\bigoplus_{i \in I} P_{i}\right)(U)=\{s \in \prod_{i \in I} P_{i}(U) \mid \forall p \in U \exists \underbrace{V}_{p \in} \subseteq U \text { s.t. } \#\left\{i\left|s_{i}\right|_{V} \neq 0\right\}<\infty\}
$$

In this way, the universal property comes from the presheaves' one. In particular,

we have the injection given by $\varepsilon_{i}$. So given a family of homomorphisms $\varphi_{i}: F_{i} \rightarrow$ $G$, we get unique maps $\overparen{\oplus P}_{i} \rightarrow G$. Since $G$ is a sheaf, there exists a unique $\tilde{\varphi}: \oplus P_{i} \rightarrow G$ (for the $(\cdot)^{s h}$ property) such that $\varepsilon_{i} \circ \tilde{\varphi}=\varphi_{i}$.
Example. Let $X$ be $\mathbb{R}$ and $I=\mathbb{N}$. We consider the functions $f_{i} \in \mathcal{C}_{\mathbb{R}}$

such that $f_{i}(i)=1, f_{i}(x)=0$ if $|x-i|<1$, as in the figure. Then $f=$ $\left(f_{1}, f_{2}, \ldots\right) \in \prod_{\mathbb{N}} \mathcal{C}_{\mathbb{R}}$, while $f \notin \widetilde{\oplus_{\mathbb{N}} \mathcal{C}_{\mathbb{R}}}(\mathbb{R})$. However, we get $f \in \oplus_{\mathbb{N}} \mathcal{C}_{\mathbb{R}}$.
If $U \subseteq \mathbb{R}$ is bounded, $\left.f\right|_{U} \in \widetilde{\oplus_{\mathbb{R}}(U)}$ but $f \notin \widetilde{\oplus_{\mathbb{N}} \mathcal{C}_{\mathbb{R}}}(\mathbb{R})$.
Restriction Let $U$ be an open set of $X$ and let $F$ be a sheaf on $X$. We define the restriction sheaf as

$$
\left(\left.F\right|_{U}\right)(V)=F(V)
$$

where $V$ is an open set of $U$. Since $U$ is open, this is well defined.
In this case, given $p \in U$, we have $\left(\left.F\right|_{U}\right)_{p} \simeq F_{p}$.

Pushforward Lef $f: X \rightarrow Y$ be a continuous function and $F$ be a sheaf on $X$. We define the pushforward to $Y$ as

$$
\left(f_{*} F\right)(V)=F\left(f^{-1}(V)\right)
$$

We notice that this is well defined because the inverse image of an open set is an open set by the definition of continuous function. If we have a composition

$$
X \xrightarrow{f} Y \xrightarrow{g} Z
$$

and $F$ is a sheaf on $X$, then $(f g)_{*} F=g_{*}\left(f_{*} F\right)$. What's more, $f: X \rightarrow Y$ induces a functor $f_{*}$ from sheaves on $X$ to sheaves on $Y$. Infact, if $\varphi: F \rightarrow G$ is a morphism of sheaves on $X$, we have the induced morphism

$$
f_{*} \varphi: f_{*} F \rightarrow f_{*} G
$$

As a consequence we get an equality of functors between $(f g)_{*}$ and $f_{*} g_{*}$. The pushforward also induces a canonical homomorphism between the stalks

$$
\begin{array}{ccc}
\left(f_{*} F\right)_{f(p)} & \longrightarrow & F_{p} \\
{[(V, s)]} & \longmapsto & {\left[\left(f^{-1}(V), s\right)\right]}
\end{array}
$$

Example (Sheaves on a point). There's an equivalence between sheaves on a point and abelian groups. Infact, $F(\emptyset)=0$ because of the sheaf condition, so we can only choose $F(X)$ and every choice is fine.
Example. Suppose given $f: X \rightarrow Y$ (there's only one map!), where $Y$ is a point, and a sheaf on $X$. The pushforward $f_{*} F$ is a sheaf on a point, so $f_{*} F(Y)=F(X)$ and the sheaf is trivial.

Notice that the map

$$
\begin{array}{cll}
F(X) & \longrightarrow F_{p} \\
s & \longmapsto & s_{m}
\end{array}
$$

is neither surjective or injective in general.
For example, let $X$ be equal to $C$ and consider the sheaf $\mathcal{O}_{X}$ of the holomorphic functions on $X$. We choose the point $0 \in \mathbb{C}$. Then we have

$$
\left[\frac{1}{1-z}, \mathbb{C} \backslash\{0\}\right] \in \mathcal{O}_{\mathbb{C}, 0}
$$

but the function is not holomorphic on $\mathbb{C}$. So the map is not surjective.
Considering then the sheaf of continuous functions on $X=\mathbb{R}$, we know that continuous functions can be the same on an proper open set but not everywhere, so in this case the map is non injective.

Let $X$ be a topological space and $Y$ a subspace. Assume that we have a sheaf $F$ on $Y$. Then we can consider the pushforward on $X$ through the canonical embedding

$$
f: Y \longrightarrow X
$$

and we have $\left(f_{*} F\right)_{p} \simeq F_{p}$ for all $p \in Y$. Indeed, $f$ induces the map

$$
\begin{array}{lccc}
\varphi: & \left(f_{*} F\right)_{p} & \longrightarrow & F_{p} \\
& {[(U, s)]} & \longmapsto & {\left[\left(f^{-1}(U), s\right)\right]}
\end{array}
$$

Injectivity. Let $[(U, s)] \in\left(f_{*} F\right)_{p}$ such that $\varphi[(U, s)]=[(W, 0)]$. $W$ is an open subset of $Y$, so there exists an open subset $V$ of $X$ contained in $U$ such that $V \cap Y=W$. Then $[(U, s)]=[(V, 0)]$, which implies injectivity.
Surjectivity. Let $[(V, s)]$ be an element of $F_{p}$; this means that $V$ is an open subset of $Y$, so there exists an open subset $U$ of $X$ such that $U \cap Y=V$. Then $\varphi[(U, s)]=[(V, s)]$, as desired.
Example. We consider the spaces $X=Y=\mathbb{C}$ and the continuous functions

$$
\begin{aligned}
& f: \mathbb{C} \longrightarrow \mathbb{C} \\
& z \longmapsto z^{2}
\end{aligned}
$$

Let $F=\mathbb{Z}_{X}$ be the sheaf of constant function from $X$ to $\mathbb{Z}$. Let $q \in Y$. We want to study the stalk of the pushforward in $q$.

- If $q=0$, then the inverse image of every open set $B\left(0, r^{2}\right)$ is $B(0, r)$; so the stalk is the same and $\left(f_{*} \mathbb{Z}_{X}\right)_{q} \simeq \mathbb{Z}$.
- If $q \neq 0, f^{-1}(B(q, r))$ (if $r$ is sufficiently small) is the disjoint union of 2 open sets containing the two different square roots of $q$. So $\left(f_{*} \mathbb{Z}_{X}\right)_{q} \simeq \mathbb{Z}^{2}$. Chosen $p \in \mathbb{C}$ such that $p^{2}=q$, then the map

$$
\mathbb{Z}^{2} \simeq\left(f_{*} \mathbb{Z}_{X}\right)_{q} \rightarrow\left(\mathbb{Z}_{X}\right)_{p} \simeq \mathbb{Z}
$$

induced by the pushforward is given by the projection.
Observation 2.17. Let $X$ be a topological space and $F$ be a sheaf on $X$. If $\left\{U_{i}\right\}$ are the connected components of $X$, the map

$$
\begin{aligned}
& F(X) \longrightarrow \\
& s \longmapsto \\
&\left(\left.s\right|_{U_{i}}\right)_{i}
\end{aligned}
$$

is an isomorphism. Therefore, given a discrete space, we get an equivalence between sheaves on $X$ and collection of abelian groups $\left(A_{p}\right)_{p \in X}$.
Example. Let $X$ be a topological space and let $\tilde{X}$ be the same set with the discrete topology. Then $f: \tilde{X} \rightarrow X$ is continuous. We consider on $\tilde{X}$ the sheaf $\mathbb{Q}_{\tilde{X}}$ of continuous functions from $\tilde{X}$ to $\mathbb{Q}$. If $X=\mathbb{R}$, then $\operatorname{dim}_{\mathbb{Q}}\left(f^{*} \mathbb{Q}_{\tilde{X}}\right)_{p}$ is very large.

### 2.3 The Structure Sheaf

Let $A$ be a ring and let consider the topological space $X=\operatorname{Spec}(A)$. We want to construct $\mathcal{O}_{X}$ the structure sheaf on $X$ with the following property:

- $\mathcal{O}_{X}(X)=A$
- $\mathcal{O}_{X}\left(X_{f}\right) \simeq A_{f}$

The first question is: does such a sheaf exist?
Observation 2.18. We notice that the requests imply that $\mathcal{O}_{X, p} \simeq A_{p}$. Infact, we know that the set $\left\{X_{f} \mid f \in A\right\}$ is a bases of open sets for $\operatorname{Spec}(A)$. We consider the map

$$
\begin{array}{cccc}
f: & \mathcal{O}_{X, p} & \longrightarrow & A_{p} \\
& {\left[\left(X_{f}, s\right)\right]} & \longmapsto & s
\end{array}
$$

where we identify $s$ as an element of $A_{p}$ with the inclusion map $A_{f} \rightarrow A_{p}$, which is well defined because $f \notin p$.
Injectivity. If $s=0$ in $A_{p}$, then $s=a / t$ where $t \notin p$ and there exists $h \notin p$ such that $a h=0$ in $A$. Then $\left[\left(X_{f}, s\right)\right]=\left[\left(X_{f h}, s\right)\right]=\left[\left(X_{f h}, 0\right)\right]$.
Surjectivity. If $s \in A_{p}$, then $s=\frac{a}{t}$. Then $f\left(\left[\left(X_{t}, s\right)\right]\right)=s$.
Given $U \subset X$ an open set, the idea is to consider, as in the case of sheafification, the ring

$$
\tilde{\mathcal{O}}_{X}(U)=\left\{f: U \rightarrow \sqcup_{p \in U} A_{p} \mid f(p) \in A_{p} \forall p \in U\right\}
$$

Then we obtain, by restriction,
$\mathcal{O}_{X}(U)=\left\{\alpha \in \tilde{\mathcal{O}}_{X}(U) \mid \forall p \in U \exists f \in A \exists s \in A_{f}\right.$ s.t. $\left\{\begin{array}{l}p \in X_{f} \subseteq U \\ \forall q \in X_{f} \alpha(q)=s_{q}\end{array}\right\}$
Theorem 2.19. $\mathcal{O}_{X}$ is a sheaf of rings.
Proof. First, we have to prove that $\mathcal{O}_{X}(U)$ is a ring. Obviously, the map that sends every point of $U$ to the identity in each $A_{p}$ is the identity in $\mathcal{O}_{X}(U)$. We have to show that if $\alpha, \beta \in \mathcal{O}_{X}(U)$ then $\alpha+\beta, \alpha \beta \in \mathcal{O}_{X}(U)$.

Clearly, $\alpha+\beta \in \tilde{\mathcal{O}}_{X}(U)$. Given $p \in U$, by the definition of $\mathcal{O}_{X}$, there exist $f, g \in A$ and $s \in A_{f}, t \in A_{f g}$ such that $p \in X_{f g}=X_{g} \cap X_{f}$ and for all $q \in X_{f g}$ we have $\alpha(q)=s_{q}$ and $\beta(q)=t_{q}$. So the elements $f g \in A$ and $s+t \in A_{f g}$ satisfy the request, and $\alpha+\beta \in \mathcal{O}_{X}(U)$.


The same proof can be done for the product and so $\mathcal{O}_{X}(U)$ is a ring. It is clearly a presheaf with the restriction maps

$$
\begin{aligned}
\rho: \mathcal{O}_{X}(U) & \longrightarrow \mathcal{O}_{X}(V) \\
\alpha & \longmapsto
\end{aligned} \alpha_{\left.\right|_{V}}
$$

We have now to show that it is a sheaf. Let $U=\bigcup_{i} V_{i}$ be an open cover of $U$ and $s_{i} \in \mathcal{O}_{X}\left(V_{i}\right)$ sections such that $\left.s_{i}\right|_{V_{i j}}=\left.s_{j}\right|_{V_{i j}}$. Define $s \in \tilde{\mathcal{O}}_{X}(U)$ as $s(p)=s_{i}(p)$ if $p \in V_{i}$. Clearly, this is well defined: we have to show that $s \in \mathcal{O}_{X}(U)$. Let $p$ be an element of $U$; for all $i$ such that $p \in V_{i}$, there exists $f_{i} \in A, t_{i} \in A_{f_{i}}$ such that $p \in X_{f_{i}} \subseteq V_{i} \subseteq U$ and $s_{i}(q)=t_{i, q}$. So it is enough to choose one of these indexes $i$ to end the proof.

We have a homomorphism $A \rightarrow \mathcal{O}_{X}(X)$ such that $a \rightarrow\left(p \rightarrow a_{p} \in A_{p}\right)$. We rename $\Gamma\left(U, \mathcal{O}_{X}\right):=\mathcal{O}_{X}(U)$, but we will use both notations.

Theorem 2.20. Let $f \in A$. Then $A_{f} \rightarrow \Gamma\left(X_{f}, \mathcal{O}_{X}\right)$ is an isomorphism. In particular, $A \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$ is an isomorphism.

Proof. We have the map

$$
\begin{array}{rllc}
\varphi: \quad A_{f} & \longrightarrow & \mathcal{O}_{X}\left(X_{f}\right) \\
x & \longmapsto & \alpha_{x}
\end{array}
$$

where $\alpha_{x}(p)=x$ in $A_{p}$. The map is well defined, because $f \notin p$ for all $p \in X_{f}$.

Injectivity. Suppose $\alpha_{x}(p)=0$ for all $p \in X_{f}$. This means that $x=0$ in $A_{p}$ for all $p \in X_{f}$. Suppose $x=a / f^{n}$; it means that for all $p \in X_{f}$ it exists $t_{p} \notin p$ such that $t_{p} a=0$ in $A$. We want to show that

$$
f \in \sqrt{(0: a)}=I
$$

If this happens, then $a / f^{n}=0 / 1$ and so injectivity.
Suppose $f \notin I$. Then there exists a prime ideal $p$ such that $f \notin p, a \in p$ and $I \subseteq p$. Then exists $t_{p} \notin p$ such that $t_{p} a=0$. So $t_{p} \in I$ and $t_{p} \notin p$; this is absurd, so $\varphi$ is injective.
Surjectivity. Let $\alpha$ be an element of $\mathcal{O}_{X}\left(X_{f}\right)$. For each $p \in X_{f}$, by definition of $\alpha$, there exist $f_{p} \in A$ and $s_{p} \in A_{f_{p}}$ such that $\alpha(q)=\left(s_{p}\right)_{q}$ for all $q \in X_{f_{p}} \subseteq X_{f}$. This produces an open cover of $X_{f}$; by compactness of $X_{f}$, we can choose $h_{1}, \ldots, h_{k}$ such that $X_{f}=X_{h_{1}} \cup \cdots \cup X_{h_{k}}$. So for each $i \in\{1, \ldots, k\}$, we suppose without loss of generality, $\alpha(q)=a_{i} / h_{i} \in A_{h_{i}}$ for all $q \in X_{h_{i}}$.

$$
\frac{a_{i}}{h_{i}}=\frac{a_{j}}{h_{j}} \in A_{h_{i} h_{j}} \Rightarrow\left(h_{i} h_{j}\right)^{n_{i j}}\left(a_{i} h_{j}-a_{j} h_{i}\right)=0 \text { in } A
$$

We can choose $n$ as the maximum of $n_{i j}$. Then we have $h_{i}^{n} h_{j}^{n+1} a_{i}=a_{j} h_{i}^{n+1} h_{j}^{n}$. We define $a_{i}^{\prime}=a_{i} h_{i}^{n}$ and $h_{i}^{\prime}=h_{i}^{n+1}$. Then we still have $a_{i}^{\prime} h_{j}^{\prime}=a_{j}^{\prime} h_{i}^{\prime}$.
Since $X_{f}=X_{h_{1}} \cup \cdots \cup X_{h_{k}}$, we get $f \in \sqrt{\left(h_{1}, \ldots, h_{k}\right)}=\sqrt{\left(h_{1}^{\prime}, \ldots, h_{k}^{\prime}\right)}$, so

$$
f^{k}=\sum_{i=1}^{k} b_{i} h_{i}^{\prime}
$$

We define $a=\sum b_{i} a_{i}^{\prime}$; then we obtain for all $j$

$$
a h_{j}^{\prime}=\sum b_{i} h_{j}^{\prime} a_{i}^{\prime}=\sum b_{i} h_{i}^{\prime} a_{j}^{\prime}=a_{j}^{\prime} f^{k}
$$

So the element $x=a / f^{k}$ is the desired one.

## Example.

- Let $k$ be a field; then $k[x]$ is a PID. So, if $U$ is an open set of $\mathbb{A}_{k}^{1}=$ $\operatorname{Spec}(k[x]), U=\emptyset$ or $U=\left(\mathbb{A}_{k}^{1}\right)_{f}$. Then we have $\mathcal{O}(U)=\Gamma(U, \mathcal{O})=k[x]_{f}$.
- Let $X=\mathbb{A}_{k}^{2}=\operatorname{Spec}(k[x, y])$ and $U$ the open set $\mathbb{A}_{k}^{2} \backslash\{(x, y)\}$. We want to compute $\Gamma(U, \mathcal{O})$. We consider the open sets

$$
U_{1}=\left(\mathbb{A}_{k}^{2}\right)_{x}=\operatorname{Spec}\left(k[x, y]_{x}\right) \quad U_{2}=\left(\mathbb{A}_{k}^{2}\right)_{y}=\operatorname{Spec}\left(k[x, y]_{y}\right)
$$

We have $U=U_{1} \cup U_{2}$, because $U=X \backslash\{(x, y)\}=X \backslash V(x, y)=X \backslash$ $(V(x) \cap V(y))=(X \backslash V(x)) \cup(X \backslash V(y))$. Furthermore, we get $U_{1} \cap U_{2}=$ $\operatorname{Spec}\left(k[x, y]_{x y}\right)$ and $\Gamma(U, \mathcal{O})$ is the kernel of the map

$$
\begin{array}{rlc}
k[x, y]_{x} \oplus k[x, y]_{y} & \longrightarrow & k[x, y]_{x y} \\
(\varphi, \psi) & \longmapsto \varphi-\psi
\end{array}
$$

Infact, if $\varphi \in \mathcal{O}_{X}\left(U_{1}\right)$ and $\psi \in \mathcal{O}_{X}\left(U_{2}\right)$ coincide on the intersection, because of the sheaf condition they coincide in the union, so an element
of the kernel gives rise to an element of $\Gamma(U, \mathcal{O})$. Viceversa, given an element of $\Gamma(U, \mathcal{O})$, the restriction maps produce elements that coincide on the intersection, so an element of the kernel. We now want to show that $\Gamma(U, \mathcal{O}) \simeq k[x, y]$. Consider an element of the kernel

$$
\varphi(x, y)=\frac{\alpha(x, y)}{x^{m}} \quad \psi(x, y)=\frac{\beta(x, y)}{y^{n}}
$$

From the relation $\varphi=\psi$ we obtain

$$
\frac{\alpha(x, y)}{x^{m}}=\frac{\beta(x, y)}{y^{n}} \Rightarrow \alpha(x, y) y^{n}=\beta(x, y) x^{m}
$$

where the implication is due to the fact that the relation on domains is easier. Noticing that $k[x, y]$ is an UFD and $x, y$ are irreducible, we obtain

$$
\begin{aligned}
y^{n} \mid \beta(x, y) & \Rightarrow \beta(x, y)=y^{n} \bar{\beta}(x, y) \\
x^{m} \mid \alpha(x, y) & \Rightarrow \alpha(x, y)=x^{m} \bar{\alpha}(x, y)
\end{aligned}
$$

So, we have

$$
\varphi(x, y)=\frac{\bar{\alpha}(x, y)}{1} \quad \psi(x, y)=\frac{\bar{\beta}(x, y)}{1}
$$

and so $\bar{\alpha}=\bar{\beta}$ and $\Gamma(U, \mathcal{O})=k[x, y]$. Analogously, if $X=\operatorname{Spec}(\mathbb{Z}[x])$ and $U=X \backslash\{(2, x)\}$, we have $\Gamma(U, \mathcal{O})=\mathbb{Z}[x]$.

### 2.4 Ringed Spaces

Definition 2.21. A ringed space $\left(X, \mathcal{O}_{X}\right)$ is a topological space $X$ with a sheaf of ring $\mathcal{O}_{X}$; a locally ringed space is a ringed space such that $\forall p \in X \mathcal{O}_{X, p}$ is a local ring.

Example.

- If $X=\operatorname{Spec}(A),\left(X, \mathcal{O}_{X}\right)$ is a locally ringed space.
- If $A$ is a ring and $X$ is a topological space, we consider the constant sheaf $A_{X} .\left(X, A_{X}\right)$ is a ringed space and $\left(A_{X}\right)_{p} \simeq A$. So,

$$
\left(X, A_{X}\right) \text { is a locally ringed space } \Longleftrightarrow X=\emptyset \text { or } A \text { is local }
$$

- If ( $X, \mathcal{O}_{X}$ ) is a (locally) ringed space and $U \subseteq X$ is an open subset, the restriction $\left(U, \mathcal{O}_{\left.X\right|_{U}}\right)$ is a (locally) ringed space.
- If $X$ is a topological space, then $\left(X, \mathcal{C}_{X}\right)$ is a locally ringed space
- If $X$ is a $\mathcal{C}^{\infty}$ manifold, $\left(X, \mathcal{C}_{X}^{\infty}\right)$ is a locally ringed space
- If $X \subseteq \mathbb{C}$ is an open set, $\left(X, \mathcal{O}_{X}\right)$ is a locally ringed space, where in this case $\mathcal{O}_{X}$ is the sheaf of holomorphic functions

Definition 2.22. Let $\left(X, \mathcal{O}_{X}\right)$ and $\left(Y, \mathcal{O}_{Y}\right)$ be ringed spaces. A morphism of ringed spaces $f: X \rightarrow Y$ is a pair $\left(f, f^{\#}\right)$ where $f: X \rightarrow Y$ is a continuous map and $f^{\#}: \mathcal{O}_{Y} \rightarrow f_{*} \mathcal{O}_{X}$ is a homomorphism of sheaves of rings.

In the case of locally ringed spaces, we have a different definition. If $A, B$ are local rings, a homomorphism $f: A \rightarrow B$ is local if $f^{-1}\left(\mathfrak{M}_{B}\right)=\mathfrak{M}_{A}$. For example, the inclusion map $i: \mathbb{Z}_{(p)} \rightarrow \mathbb{Q}$ isn't a local homomorphism.

Definition 2.23. Let $X, Y$ be locally ringed spaces. A morphism of locally ringed space is a morphism of ringed spaces such that $\forall p \in X$ the composition

$$
\mathcal{O}_{Y, f(p)} \xrightarrow{f_{f(p)}^{\#}}\left(f_{*} \mathcal{O}_{X}\right)_{f(p)} \longrightarrow \mathcal{O}_{X, p}
$$

is a local homomorphism.
Example. If $X$ is a point and $A, B$ are local rings, we have the locally ringed spaces $(X, A)$ and $(X, B)$. A morphism of ringed spaces in this case is essentially a homomorphism of rings $f^{\#}: B \rightarrow A$. It is a morphism of locally ringed spaces if and only if $B \rightarrow A$ is a local homomorphism.
Example. Let $X, Y$ be topological space and $f: X \rightarrow Y$ a continuous function. We have the function $f^{\#}: \mathcal{C}_{Y} \rightarrow f_{*} \mathcal{C}_{X}$ such that, given $\varphi \in \mathcal{C}_{Y}(V), f^{\#}(\varphi)=$ $\varphi \circ f \in \mathcal{C}_{X}\left(f^{-1} V\right)$. If $p \in X$, we have the composition


We have that $\psi[\varphi]=[\varphi \circ f]$. We notice that $\mathfrak{M}_{p}=\left\{[f] \in \mathcal{C}_{X, p} \mid f(p)=0\right\}$ is the maximal ideal of $\mathcal{C}_{X, p}$ and so the homomorphism is local. So, this is a morphism of locally ringed spaces.
Example. Let $f: X \rightarrow Y$ a $\mathcal{C}^{\infty}$ map between $\mathcal{C}^{\infty}$ manifolds.

$$
\begin{aligned}
f^{\#}: \quad \mathcal{C}_{Y}^{\infty} & \longrightarrow f_{*} \mathcal{C}_{X}^{\infty} \\
\varphi & \longmapsto \varphi \circ f
\end{aligned}
$$

$\left(f, f^{\#}\right)$ is a morphism of locally ringed spaces.
Theorem 2.24. Let $A, B$ be rings. There is a bijective corrispondence between homomorphisms of rings $\varphi: B \rightarrow A$ and morphisms of locally ringed spaces $\varphi^{\#}: \operatorname{Spec}(A) \rightarrow \operatorname{Spec}(B)$.

$$
\{\varphi: B \rightarrow A \mid \underset{\text { of rings }}{\substack{\text { homomorphism }}} \stackrel{\text { 1:1 }}{\longleftrightarrow}\{f: \operatorname{Spec}(A) \rightarrow \operatorname{Spec}(B) \mid \underset{\text { ringed spaces }}{\text { morphism of locally }}\}
$$

Proof. Let $\varphi: B \rightarrow A$ be a homomorphism of rings. The homomorphism induces a continuous map

$$
\begin{array}{cccc}
f: & X & \longrightarrow & Y \\
p & \longmapsto & \varphi^{-1}(p)
\end{array}
$$

So if $q=\varphi^{-1}(p)$ we have the following commutative diagram:


What's more, $\varphi_{p}$ is a local homomorphism. In this way, we have found a way to associate a local homomorphism $\varphi_{p}$ to each point $p$ of the spectrum.
Now, we define a map $f^{\#}: \mathcal{O}_{Y} \rightarrow f_{*} \mathcal{O}_{X}$. By the definition of $f_{*} \mathcal{O}_{X}$, we have to send an element of $\mathcal{O}_{Y}(U)$ to $\mathcal{O}_{X}\left(f^{-1}(U)\right)$. So, given $\beta \in \mathcal{O}_{Y}(U)$, we define

$$
\begin{aligned}
f^{\#}(\beta): \quad f^{-1}(U) & \longrightarrow \bigsqcup_{p \in f^{-1}(U)} A_{p} \\
p & \longmapsto \varphi_{p}(\beta(f(p)))
\end{aligned}
$$

We have the induced maps

$$
\mathcal{O}_{Y, f(p)}^{\longrightarrow\left(f_{*} \mathcal{O}_{X}\right)_{f(p)} \longrightarrow} \mathcal{O}_{X, p}
$$

which is a local homomorphism. So $\left(f, f^{\#}\right)$ is a morphism of locally ringed space.

Conversely, let $\left(f, f^{\#}\right)$ be a morphism of locally ringed space. Taking global sections, we have an homomorphism

$$
f^{\#}(Y): \mathcal{O}_{Y}(Y) \longrightarrow f_{*} \mathcal{O}_{X}(Y) \simeq \mathcal{O}_{X}(X)
$$

We call this homomorphism $\varphi: B \rightarrow A$. We have to show that $\varphi$ induces the morphism $\left(f, f^{\#}\right)$. We get the diagram:


The homomorphism $B_{f(p)} \rightarrow A_{p}$ is a local homomorphism, so the inverse image of $p$ is $f(p)$. By the commutativity of the diagram, the same must happen for $\varphi$. This show that the continuous map induced on the topological spaces $\operatorname{Spec}(A), \operatorname{Spec}(B)$ is $f$. The same is true for $f^{\#}$.

Proposition 2.25. Locally ringed spaces form a category.

Proof. The object of the category are obviously locally ringed spaces; the morphisms are morphisms of locally ringed spaces.
First, we have to show that the composition of morphisms is still a morphism.

$$
X \xrightarrow{f} Y \xrightarrow{g} Z
$$

Clearly, $g \circ f$ is a continuous map on $\operatorname{Spec}(X)$ to $\operatorname{Spec}(Z)$. We have to show that the same hold for the morphisms of sheaves $f^{\#}, g^{\#}$. So we need a map

$$
(g f)^{\#}: \mathcal{O}_{Z} \rightarrow(g f)_{*}\left(\mathcal{O}_{X}\right)=g_{*}\left(f_{*}\left(\mathcal{O}_{X}\right)\right)
$$

The correct map is the composition of

$$
\mathcal{O}_{Z} \xrightarrow{g^{\#}} g_{*} \mathcal{O}_{Y} \xrightarrow{g_{*} f^{\#}} g_{*} f_{*} \mathcal{O}_{X}
$$

Is if $W \subseteq X$ is an open subset, we get

$$
\mathcal{O}_{Z}(W) \xrightarrow{g_{W}^{\#}} \mathcal{O}_{Y}\left(g^{-1}(W)\right) \xrightarrow{f_{g^{-1}(W)}^{\#}} \mathcal{O}_{X}\left(f^{-1}\left(g^{-1}(W)\right)\right)=\mathcal{O}_{X}\left((g f)^{-1}(W)\right)
$$

The composition is also associative (easy exercise). We still need to show that the induced homomorphism on the stalk is local.


It is true because the composition of local homomorphism is a local homomorphism and we obtain the map as the composition:

$$
\mathcal{O}_{Z, g f(p)} \longrightarrow\left(f * \mathcal{O}_{Y}\right)_{f g(p)} \longrightarrow \mathcal{O}_{Y, g(p)} \longrightarrow\left(f_{*} \mathcal{O}_{X}\right)_{p} \longrightarrow \mathcal{O}_{X, p}
$$

Finally, the identity function is the couple

$$
i d_{X}: X \rightarrow X \quad i d_{X}^{\#}: \mathcal{O}_{X} \rightarrow\left(i d_{X}\right)_{*} \mathcal{O}_{X}=\mathcal{O}_{X}
$$

We notice that a morphism $f: X \rightarrow Y$ of locally ringed spaces is an isomorphism if and only if $f$ is an homeomorphism and $f^{\#}$ is an isomorphism of sheaves. So we get an equivalence between the category of rings and a full subcategory of the category of locally ringed spaces

$$
\begin{array}{clc}
\text { Ring }^{o p} & \longrightarrow & \text { LRS } \\
R & \longmapsto & \left(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}\right)
\end{array}
$$

We have shown that this functor is fully faithfully. It is also essentially surjective on the subcategory, but this is not trivial.

Observation 2.26. If ( $X, \mathcal{O}_{X}$ ) is a locally ringed spaces and $U \subseteq X$ is an open subset, we have the restriction sheaf $\mathcal{O}_{U}=\left.\mathcal{O}_{X}\right|_{U}$. So $\left(U, \mathcal{O}_{X}\right)$ is a locally ringed space and we have the inclusion map

$$
j: U \rightarrow X \quad j^{\#}: \mathcal{O}_{X} \rightarrow j_{*} \mathcal{O}_{U}
$$

Seen that $j^{-1}(V)=U \cap V, j^{\#}$ is simply the restriction map of the sheaf $\mathcal{O}_{X}$. Furthermore, the induced maps on the stalks

$$
j_{p}^{\#}: \mathcal{O}_{U, p} \rightarrow \mathcal{O}_{X, p}
$$

is an isomorphism, so $j$ is a morphism of locally ringed space. We have the following

Proposition 2.27. There is a bijection between the morphisms of locally ringed spaces $Y \rightarrow U$ and morphisms $Y \rightarrow X$ such that $g(Y) \subseteq U$.

## Chapter 3

## Schemes

Definition 3.1. An affine scheme is a locally ringed space which is isomorphic to the spectrum of a ring.
A scheme is a locally ringed space which has an open cover by affine schemes.
Theorem 3.2. An open subspace of a scheme is a scheme.
Proof. Let $X$ be a scheme and $U \subseteq X$ an open subset. Since $X$ is a scheme, $X$ is covered by open affine subscheme $X=\cup X_{i}$; then $U=\cup\left(X_{i} \cap U\right)$. We need to show that $U \cap X_{i}$ is covered by open affine schemes. So we can assume that $X$ is affine $X \simeq \operatorname{Spec}(A)$ and $U \subseteq X$ is an open set. Let $p$ be a point of $U$. Then there exists $f \in A$ such that $f \notin P$ and $X_{f} \subseteq U$, because $X_{f}$ form a basis of the topology of $\operatorname{Spec}(A)$. Noticing that $\left(X_{f},\left.\mathcal{O}_{X}\right|_{X_{f}}\right) \simeq\left(\operatorname{Spec}\left(A_{f}\right), \mathcal{O}_{\operatorname{Spec}\left(A_{f}\right)}\right)$, we have that $X_{f}$ is an affine scheme. So we have found a open cover of $U$, as requested.

Example. Let $K$ be a field and $\mathbb{A}_{K}^{2}=\operatorname{Spec}(K[x, y])$. Let $X$ be the open set $\mathbb{A}_{K}^{2} \backslash\{(x, y)\}$. Then $X$ is a scheme, but not an affine scheme. Infact, we have the maps

$$
i: X \rightarrow \mathbb{A}_{K}^{2} \quad i^{\#}: \mathcal{O}\left(\mathbb{A}_{K}^{2}\right) \rightarrow \mathcal{O}(X)
$$

and $i^{\#}$ is an isomorphism. If $X$ was an affine scheme, by Theorem 2.24 , it would induce an isomorphism of rings $\varphi$. However, if $\varphi$ was an isomorphism, $i$ would be a homeomorphism between the topological spaces, but this is absurd.

Definition 3.3. A morphism of schemes is a morphism of locally ringed spaces between schemes.

Usually, we don't want to consider all the possible morphism. In ring theory, given a ring $R$, we consider $R$-algebras and homomorphisms of $R$-algebras $\varphi: A \rightarrow B$, requiring that the following diagram is commutative:


We require the same for schemes. Given a scheme $S$, a scheme over $S$ is a scheme $X$ with a morphism $X \rightarrow S$. A morphism of schemes over $S$ is a morphism of schemes $\varphi: X \rightarrow Y$ such that the following diagram is commutative


If $R$ is a ring, a scheme over $R$ is a scheme over $\operatorname{Spec}(R)$.
Example.

- $S$ is a scheme over itself. The only morphism $S \rightarrow S$ over $S$ is the identity.
- More generally, if $X$ is a scheme over $S$, exists a unique morphism $X \rightarrow S$ over $S$; in other words, $S$ is a terminal object in the category of schemes over $S$.
- Let $R$ be a ring; we consider the scheme $\mathbb{A}_{R}^{n} \simeq \operatorname{Spec}\left(R\left[x_{1}, \ldots, x_{n}\right]\right)$ as a scheme over $R$ with the $\mathbb{A}_{R}^{n} \rightarrow \operatorname{Spec}(R)$ induced by the inclusion $R \rightarrow$ $R\left[x_{1}, \ldots, x_{n}\right]$.
The lifting property of sheaves still holds for morphism:
Theorem 3.4. Let $X, Y$ be locally ringed spaces and let $\left\{X_{i}\right\}$ be an open cover of $X$. Suppose we are given for all $i$ morphisms $f_{i}: X_{i} \rightarrow Y$ such that for all $i,\left.j f_{i}\right|_{X_{i j}}=\left.f_{j}\right|_{X_{i j}}: X_{i j} \rightarrow Y$. Then exists a morphism of locally ringed spaces $f: X \rightarrow Y$ such that $\left.f\right|_{X_{i}}=f_{i}$ for every $i$.

Proof. Clearly, exists a unique $f: X \rightarrow Y$ continuous such that $\left.f\right|_{X_{i}}=f_{i}$. So we have to find a morphism of sheaves

$$
\mathcal{O}_{Y} \rightarrow f_{*} \mathcal{O}_{X}
$$

Clearly, given an open set $V$, we have the maps


We construct a map $f^{\#}: \mathcal{O}_{Y}(V) \rightarrow \mathcal{O}_{X}\left(f^{-1}(V)\right)$ using the fact that $\mathcal{O}_{X}$ is a sheaf. Given $g \in \mathcal{O}_{Y}(V)$, we consider the images $f_{i}^{\#}(g):=s_{i} \in \mathcal{O}_{X}\left(f^{-1}(V \cap\right.$ $\left.X_{i}\right)$ ). Noticing that $\left.s_{i}\right|_{X_{i j}}=\left.s_{j}\right|_{X_{i j}}$, exists a unique $s \in \mathcal{O}_{X}\left(f^{-1}(V)\right)$ such that $\left.s\right|_{X_{i}}=s_{i}$; we define $f^{\#}(g)=s$.

Observation 3.5. The theorem states that the presheaf of sets on $X$ given by

$$
U \rightarrow\{\underset{\text { ringed spaces }}{\substack{\text { morphism of locally } \\ \text { ring }}} U \rightarrow Y
$$

with the restriction map

$$
f:\left.U \rightarrow Y \rightsquigarrow f\right|_{V}: V \rightarrow Y
$$

is a sheaf.

Corollary 3.6. Let $X$ be a scheme and $R$ be a ring. Given a morphism $f: X \rightarrow$ $\operatorname{Spec}(R)$, we get a homomorphism of rings

$$
\Gamma(\operatorname{Spec}(R), \mathcal{O}) \simeq R \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)
$$

This gives a bijection between morphism $X \rightarrow \operatorname{Spec}(R)$ and homomorphisms of rings $R \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$.
Proof. We know this is true when $X$ is affine by Theorem 2.24. So let us suppose that is is a scheme and let $\left\{X_{i}\right\}$ be an affine cover of $X$. First, we show the injectivity of the correspondance. Suppose we are given $f, g: X \rightarrow \operatorname{Spec}(R)$ such that $f^{\#}=g^{\#}: R \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$. We want to show that $f=g$. We have a family of maps

$$
\left.f\right|_{X_{i}},\left.g\right|_{X_{i}}: X_{i} \rightarrow R
$$

Since $\left.f^{\#}\right|_{X_{i}}=\left.g^{\#}\right|_{X_{i}}$ by hypotesis and $X_{i}$ is an affine scheme, again by Theorem 2.24 we obtain $\left.f\right|_{X_{i}}=\left.g\right|_{X_{i}}$. So, by the Pasting Lemma for continuous function, we have $f=g$, as desired.
We now show surjectivity. Let $\varphi: R \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$ be a ring homomorphism. For all $i, \varphi_{i}$ is the composite


Since $X_{i}$ is affine, $\exists!f_{i}: X_{i} \rightarrow \operatorname{Spec}(R)$ such that $f_{i}^{\#}=\varphi_{i}$. We notice that $\left.f_{i}\right|_{X_{i j}}$ and $\left.f_{j}\right|_{X_{i j}}$ correspond to the restriction $\Gamma\left(X, \mathcal{O}_{X}\right) \rightarrow \Gamma\left(X_{i j}, \mathcal{O}_{X}\right)$, so they must coincide on $X_{i j}$. By Pasting Lemma, we have a continuous function $f: X \rightarrow \operatorname{Spec}(R)$ such that $\left.f\right|_{X_{i}}=f_{i}$. We have now to show that $f^{\#}=\varphi$.


The sheaf condition implies that the map

$$
\rho: \Gamma\left(X, \mathcal{O}_{X}\right) \longrightarrow \prod_{i \in I} \Gamma\left(X_{i}, \mathcal{O}_{X}\right)
$$

is injective. Since $\rho \circ \varphi=\rho \circ f^{\#}$, we obtain $\varphi=f^{\#}$, as requested.
Corollary 3.7. $\operatorname{Spec}(\mathbb{Z})$ is a terminal object in the category of schemes. Equivalently, given a scheme $X$, there exists a unique morphism of schemes from $X$ to $\operatorname{Spec}(\mathbb{Z})$.

Observation 3.8. If $R$ is a ring and $A$ is an $R$-algebra, given a homomorphism $\varphi: A \rightarrow B$, there's a unique $R$-algebra structure on $B$ which makes $\varphi$ into a homomorphism of $R$-algebras.

Given $R \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$, for any open subset $U$ of $X$ there's a unique $R$ algebra structure on $\mathcal{O}_{X}(U)$ which makes the restriction $\mathcal{O}_{X}(X) \rightarrow \mathcal{O}_{X}(U)$ a homomorphism of $R$-algebras. This implies that every restriction map is a homomorphism of $R$-algebras, so $\mathcal{O}_{X}$ is a sheaf of $R$-algebras.

Proposition 3.9. Let $X$ be a scheme. There is a bijection between morphisms $X \rightarrow \operatorname{Spec}(R)$ and structure of sheaf of $R$-algebras on $\mathcal{O}_{X}$.

Let $X, Y$ be schemes and $f: X \rightarrow Y$ a morphism of schemes over $R$. The following diagram commutes

if and only if the following is commutative

and this is equivalent to the fact that $f_{Y}^{\#}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is an homomorphism of $R$-algebras. This happens if and only if $f^{\#}: \mathcal{O}_{Y} \rightarrow f_{*} \mathcal{O}_{X}$ is a homomorphism of sheaves of $R$-algebras. So we have found a more convenient way of saying that a morphism of schemes is over $R$.

### 3.1 Gluing Schemes

Let $X_{1}, X_{2}$ be schemes and $U_{1} \subseteq X_{1}, U_{2} \subseteq X_{1}$ be open subschemes. Suppose given an isomorphism of schemes $f: U_{1} \rightarrow U_{2}$. We want to glue the schemes along these open sets. As a topological space, we call

$$
X=X_{1} \sqcup X_{2} / \sim \quad x \sim y \Longleftrightarrow\left\{\begin{array}{l}
x=y \\
x=f(y) \\
y=f(x)
\end{array}\right.
$$

$X$ contains an open subset $U \subseteq X$ whose inverse image in the disjoint union $X_{1} \sqcup X_{2}$ is $U_{1} \sqcup U_{2}$. The maps

$$
U_{1} \rightarrow U \quad U_{2} \rightarrow U
$$

are homeomorphisms, so $X_{1} \rightarrow X$ and $X_{2} \rightarrow X$ are open embeddings. We also have the projection maps

$$
\pi: X_{1} \sqcup X_{2} \longrightarrow X
$$

If we call $\tilde{X}_{1}, \tilde{X}_{2}$ the images of $X_{1}$ and $X_{2}$ respectively, we have

$$
\begin{array}{ll}
\pi^{-1}\left(\tilde{X}_{1}\right)=X_{1} \sqcup U_{2} & \pi^{-1}\left(\tilde{X}_{2}\right)=U_{1} \sqcup X_{2} \\
X=\pi\left(X_{2}\right) \cup \pi\left(X_{2}\right) & \pi\left(X_{1}\right) \cap \pi\left(X_{1}\right)=U
\end{array}
$$

Theorem 3.10. Let $X_{1}, X_{2}$ be schemes, $U_{1} \subseteq X_{1}, U_{2} \subseteq X_{2}$ open subschemes and let $f: U_{1} \rightarrow U_{2}$ be an isomorphism. It exists a unique scheme $X$ with two open subschemes $X_{1}^{\prime}, X_{2}^{\prime}$ and two isomorphisms $f_{1}: X_{1} \rightarrow X_{1}^{\prime}, f_{2}: X_{2} \rightarrow X_{2}^{\prime}$ such that

$$
X=X_{1}^{\prime} \cup X_{2}^{\prime} \quad U=X_{1}^{\prime} \cap X_{2}^{\prime} \quad f_{i}^{-1}(U)=U_{i}
$$

and the following diagram commutes


Proof. We have already defined $X$ as a topological space. If $V \subseteq X$ is an open set, we call $V_{i}=f_{i}^{-1}(V)$. So

$$
\mathcal{O}_{X}(V)=\left\{\left(s_{1}, s_{2}\right) \in \mathcal{O}_{X_{1}}\left(V_{1}\right) \times \mathcal{O}_{X_{2}}\left(V_{2}\right) \mid s_{\left.1\right|_{U_{1} \cap V_{1}}}=\varphi^{\#} s_{\left.2\right|_{U_{2} \cap V_{2}}}\right\}
$$

So $\left(X, \mathcal{O}_{X}\right)$ is a scheme with the required properties. Furthermore, $X$ is unique up to isomorphism.

Example.

- Let $k$ be a field and let $X_{1}$ be $\operatorname{Spec}\left(k\left[x_{1}\right]\right)$ and $X_{2}$ be $\operatorname{Spec}\left(k\left[x_{2}\right]\right)$. We consider $U=U_{1}=U_{2}=\mathbb{A}_{k}^{1} \backslash\{(x)\}$ and $\varphi=i d_{U}$. Then, the scheme $X$ is "the line with two origins". However, $X$ is not affine. By definition,

$$
\mathcal{O}(X)=\left\{\left(s_{1}, s_{2}\right) \in k[x] \times k[x] \mid s_{1}=s_{2} \text { in } k[x]\right\} \simeq k[x]
$$

The map $X \rightarrow \mathbb{A}_{k}^{1}$ induces an isomorphism $\mathcal{O}\left(\mathbb{A}_{k}^{1}\right) \rightarrow \mathcal{O}(X)$. However, the map $X \rightarrow \mathbb{A}_{k}^{1}$ isn't an isomorphism and so $X$ is not affine.

- We can do the same with $X_{1}, X_{2}=\mathbb{A}_{K}^{2}$ and $U_{1}=U_{2}=\mathbb{A}_{K}^{2} \backslash\{(x, y)\}$. So we have $X_{1}, X_{2} \subseteq X$ and $X_{1} \cap X_{2}=\mathbb{A}_{K}^{2}$; this is an example of two open affine subschemes whose intersection is not affine.
- Let $X_{1}=X_{2}=\mathbb{A}_{R}^{1}$ and $U_{1}=U_{2}=\mathbb{A}_{R}^{1} \backslash\{(x)\}$. We consider the open subschemes $U_{1}=\operatorname{Spec}\left(R\left[x_{1}^{ \pm}\right]\right)$and $U_{2}=\operatorname{Spec}\left(R\left[x_{2}^{ \pm}\right]\right)$and the gluing map

$$
\begin{array}{llll}
\varphi: & U_{2} & \longrightarrow & U_{1} \\
& x_{2} & \longmapsto & x_{1}^{-1}
\end{array}
$$

We get a scheme called $\mathbb{P}_{R}^{1}$, which is a scheme over $R$. Furthermore, we have

$$
\Gamma\left(\mathbb{P}_{R}^{1}, \mathcal{O}\right)=\left\{\left(s_{1}, s_{2}\right) \in R\left[x_{1}\right] \times R\left[x_{2}\right] \mid s_{1}(t)=s_{2}(-t)\right\} \simeq R
$$

Disjoint Union Let $I$ be a set of indexes and suppose we are given a family of locally ringed spaces $\left\{X_{i}\right\}_{i \in I}$. We can easily define, as a topological space

$$
X=\bigsqcup_{i \in I} X_{i}
$$

and we have the open embeddings of topological spaces. $X_{i} \subseteq X$. Up to isomorphism, it exists a unique sheaf of rings $\mathcal{O}_{X}$ such that

$$
\left.\mathcal{O}_{X}\right|_{X_{i}} \simeq \mathcal{O}_{X_{i}}
$$

Moreover, if $U \subseteq X$ is an open set, so $U=\sqcup U \cap X_{i}$

$$
\mathcal{O}_{X}(U)=\prod_{i \in I} \mathcal{O}_{X_{i}}\left(U \cap X_{i}\right)
$$

If $f_{i}: X_{i} \rightarrow Y$ is a family of morphisms of locally ringed spaces, it exists a unique $f: X \rightarrow Y$ such that $\left.f\right|_{X_{i}}=f_{i}$. If $X_{i}$ are schemes, $X$ is a scheme too. Let $A_{1}, \ldots, A_{n}$ be rings and let $X_{i}=\operatorname{Spec}\left(A_{i}\right)$. Let $A$ be the product of these rings. We get

$$
\operatorname{Spec}(A)=\bigsqcup_{i=1}^{n} \operatorname{Spec}\left(A_{i}\right)
$$

as schemes. So finite disjoint union of affine schemes is affine. A disjoint union (not necessarily finite) of affine schemes is affine if and only if $X_{i} \neq \emptyset$ for finite indexes, because an affine scheme is quasi-compact.

### 3.2 Proj Construction

Definition 3.11. Let $B$ be a ring. $B$ is graded (in $\mathbb{N}$ ) if

$$
B=\bigoplus_{d \geq 0} B_{d}
$$

where each $B_{i}$ is a subgroup and $B_{d} B_{e} \subseteq B_{d+e}$.
If $\alpha \in B_{d}$, we say that $\alpha$ is homogeneous of degree $d$. We notice that $1 \in B_{0}$. In fact, by the direct sum property, we have

$$
1=e_{0}+e_{1}+\cdots+e_{m}
$$

where each $e_{i}$ lies in $B_{i}$. Then, given $x \in B_{d}$, we have

$$
x=x \cdot 1=x e_{0}+\cdots+x e_{m} \in B_{d}
$$

and this is possible if and only if $e_{0}=1$. As a consequence, $B_{0}$ is a subring of $B$. We also notice that $B_{+}=\oplus_{d \geq 1} B_{d}$ is an ideal of $B$ and

$$
\begin{gathered}
\varphi: \\
\\
\left(x_{0}, x_{1}, \ldots\right)
\end{gathered} \begin{array}{ccc}
B & \longmapsto & B_{0} \\
x_{0}
\end{array}
$$

is a well defined surjective homomorphism whose kernel is $B_{+}$. By the first homomorphism theorem, we obtain

$$
B_{0} \simeq B / B_{+}
$$

## Example.

- If $A$ is a ring, $A\left[x_{1}, \ldots, x_{n}\right]$ is naturally graded by $\operatorname{deg}\left(x_{i}\right)=1$. Furthermore, $A_{d}$ is the set of homogeneous polynomials of degree $d$.
- We can define a graduation on $A\left[x_{1}, \ldots, x_{n}\right]$ in other ways. Given natural numbers $d_{1}, \ldots, d_{n}$, we can define $\operatorname{deg}\left(x_{i}\right)=d_{i}$ so that $\operatorname{deg}\left(x_{1}^{e_{1}} \ldots x_{n}^{e_{n}}\right)=$ $d_{1} e_{1}+\cdots+d_{n} e_{n}$; in this way, $A_{d}$ is generated as an $A$-module by monomials of degree $d$.

Lemma 3.12. Le $B$ be a graded ring and $I$ an ideal of $B$. The following are equivalent:

1. $I$ is generated by homogeneous elements
2. If $a \in I$, every homogeneous component of $a$ lies in $I$
3. $I=\oplus\left(I \cap B_{d}\right)$

Definition 3.13. Let $A$ be a graded ring and $I$ an ideal of $B . ~ I$ is called homogeneous if $I$ satisfies one the the condition of the previous Lemma 3.12.

If $I$ is a homogeneous ideal of $B$, we can define an induced grading on the quotient. In particular,

$$
(B / I)_{d}=B_{d} /\left(I \cap B_{d}\right)
$$

We remark that this equivalence is intended in the sense of groups.
By the definition of homogeneous ideals, we also get that give a family $\left\{I_{\lambda}\right\}_{\lambda \in \Lambda}$ of homogeneous ideals, both $\sum I_{\lambda}$ and $\cap I_{\lambda}$ are homogeneous. The same happens for the product, but only when $\Lambda$ is finite.

Proposition 3.14. Let $I$ be an proper homogeneous ideal of a graded ring $B$. Suppose that for all $a, b$ homogeneous elements, if $a b \in I$ then either $a \in I$ or $b \in I$. Then $I$ is a prime ideal.

Proof. The thesis is equivalent to saying that the quotient is a domain. The hypotesis imply that the cancellation law holds for homogeneous elements in the quotient. Now let $a, b \in B / I$ be nonzero elements. $B / I$ is graded, so we can write

$$
a=a_{0}+\cdots+a_{m} \quad b=b_{0}+\cdots+b_{m}
$$

where $a_{i}, b_{i}$ lies in $(B / I)_{i}$ and $b_{m} \neq 0, a_{n} \neq 0$. By computing the product,

$$
u v=\underbrace{a_{0} b_{0}}_{\in(B / I)_{0}}+\underbrace{\left(a_{1} b_{0}+a_{0} b_{1}\right)}_{\in(B / I)_{1}}+\cdots+\underbrace{a_{n} b_{m}}_{\in(B / I)_{2 m}}
$$

Since cancellation law holds for homogeneous elements, $a_{n} b_{m} \neq 0$ so $a b \neq 0$.
Proposition 3.15. Let $I$ be a homogeneous ideal of a graded ring $B$.

- $\sqrt{I}$ is homogeneous
- If for all $a$ homogeneous element $a^{n} \in I$ implies $a \in I$, then $I$ is radical.

Proof. Clearly it suffices to prove the first point. Let $a$ be an element of $\sqrt{I}$; the $a=a_{0}+\cdots+a_{m}$. By the definition of radical,

$$
a^{n}=a_{m}^{n}+\underbrace{(\ldots)}_{\begin{array}{c}
\text { lower order } \\
\text { terms }
\end{array}} \in I
$$

Since $I$ is homogeneous, $a_{m}^{n} \in I$, so $a_{m} \in I$. We can apply inductively the proof on $a-a_{m} \in I$ to achieve the thesis.

Definition 3.16. Let $B$ be a graded ring. We define

$$
\operatorname{Proj}(B):=\left\{p \in \operatorname{Spec}(B) \mid\left(B_{+} \nsubseteq p\right) \wedge(p \text { homogeneous })\right\}
$$

As in the case of rings, given $I$ a homogeneous ideal of $B$, we define $\mathcal{V}_{+}(I)=$ $\{p \in \operatorname{Proj}(B) \mid I \subseteq p\}$. They satisfy the axioms for closed sets of a topology on $\operatorname{Proj}(B)$, which is called the Zariski Topology on $\operatorname{Proj}(B)$.

Proposition 3.17. Let $I$ be a homogeneous ideal of $B$. Then

$$
\mathcal{V}_{+}(I) \simeq \operatorname{Proj}(B / I)
$$

Proof. The map

$$
\begin{array}{ccc}
\operatorname{Proj}(B / I) & \longrightarrow & \operatorname{Proj}(B) \\
q & \longmapsto & \pi^{-1}(q)
\end{array}
$$

is a homeomorphism on the image, which is $\mathcal{V}_{+}(I)$.
We remark that $\mathcal{V}_{+}(I)=\mathcal{V}_{+}(\sqrt{I})$; unfortunately, this doesn't give a bijection between radical ideals and closed subsets of $\operatorname{Proj}(B)$. In fact, if $B_{+}$is radical,

$$
\mathcal{V}\left(B_{+}\right)=\mathcal{V}(B)=\emptyset
$$

So the corrispondence fails certanly when $B_{+}$is radical; this happens if and only if $B / B_{+} \simeq B_{0}$. So $B_{0}$ is reduced if and only if $B_{+}$is radical. In general, if $I$ is an ideal contained in $B_{+}, \sqrt{I}=\mathfrak{N}\left(B_{0}\right) \oplus \sqrt{I} \cap B_{+}$.

Proposition 3.18. Let $I$ be a homogeneous ideal of a graded ring $B$. Then $\mathcal{V}_{+}(I)=\mathcal{V}_{+}\left(I \cap B_{+}\right)$.
Proof. Since $I \cap B_{+} \subseteq I$, we get $\mathcal{V}_{+}(I) \subseteq \mathcal{V}_{+}\left(I \cap B_{+}\right)$. Viceversa, let $p \in$ $\mathcal{V}_{+}\left(I \cap B_{+}\right)$. Then $I \cap B_{+} \subseteq p$; we want to show that $I \subseteq p$. Since $B_{+} \nsubseteq p$, we can take $f \in B_{+} \backslash p$. Then for all $a \in I$, af $\in I \cap B_{+}$, so $a f \in p$; but $f \notin p$, so $a \in p$. This means that $I \subseteq p$.

Lemma 3.19. Let $I$ be an ideal of a graded ring $B$. Then $I^{h}=\oplus\left(I \cap B_{d}\right)$ is the largest homogeneous ideal contained in $I$. Furthermore, if $I$ is prime so $I^{h}$ is.

Proof. Clearly, $I^{h}$ is homogeneous (it can be seen as the ideal generated by all the homogeneous elements of $I$ ) and it is contained in $I$. Let $J$ be a homogeneous ideal such that $I^{h} \subseteq J \subseteq I$. Since $J$ is homogeneous, it is generated by its homogeneous elements and therefore all its generators lie in $I^{h}$, giving the equality.

Proposition 3.20. Let $I, J$ be homogeneous ideals of a graded ring $B$. Then

$$
\mathcal{V}_{+}(I) \subseteq \mathcal{V}_{+}(J) \Longleftrightarrow J \cap B_{+} \subseteq \sqrt{I}
$$

Proof. The implication $\Leftarrow$ is trivial. By contradiction, let's assume $J \cap B_{+} \nsubseteq \sqrt{I}$; this implies that exists $p \in \operatorname{Spec}(B)$ such that $J \cap B_{+} \nsubseteq p$. By the previous Lemma 3.19, $I \subseteq P^{h}$ and $J \cap B_{+} \nsubseteq P^{h}$. Moreover, $P^{h} \in \operatorname{Proj}(B)$ because is a homogeneous prime and $P^{h} \nsupseteq B_{+}$. So $P^{h} \in \mathcal{V}_{+}(I) \backslash \mathcal{V}_{+}\left(J \cap B_{+}\right)$, absurd.

Corollary 3.21. There's a bijection between closed subsets of $\operatorname{Proj}(B)$ and radical homogeneous ideals $I \subseteq B$ such that $I \cap B_{0}=\mathfrak{N}\left(B_{0}\right)$.

Proof. Let $C$ be a closed subset of $\operatorname{Proj}(B)$. Then

$$
C=\mathcal{V}_{+}(J)=\mathcal{V}_{+}\left(J \cap B_{+}\right)=\mathcal{V}_{+}\left(\sqrt{J \cap B_{+}}\right)
$$

and so surjectivity. Injectivity comes from the previous proposition.
Corollary 3.22. $\operatorname{Proj}(B)=\emptyset \Longleftrightarrow B_{+} \subseteq \mathfrak{N}(B)$
Proof. If $B_{+} \subseteq \mathfrak{N}(B)$, then obviously $\operatorname{Proj}(B)=\emptyset$. Viceversa, if $\operatorname{Proj}(B)=\emptyset$, then $\mathcal{V}_{+}(0) \subseteq \mathcal{V}_{+}(B)$, which implies that $B_{+} \subseteq \mathfrak{N}(B)$.

Corollary 3.23. If $B_{0}$ is a field, there is a bijection between homogeneous proper radical ideals and closed subsets of $\operatorname{Proj}(B)$.

### 3.2.1 Homogeneous Localization

Let $B$ be a graded ring and $S$ a multiplicative subset consisiting of homogeneous elements. We define as usual the ring of fractions $S^{-1} B$ as the ring

$$
S^{-1} B=\left\{\left.\frac{a}{s} \right\rvert\, a \in B \quad s \in S\right\}
$$

However, the condition on the multiplicative subset induces a grading on this ring

$$
\left(S^{-1} B\right)_{d}=\left\{\left.\frac{a}{s} \right\rvert\,(a \text { homogeneous }) \wedge(\operatorname{deg}(a)-\operatorname{deg}(s)=d)\right\}
$$

We also notice that if $B$ was $\mathbb{N}$-graded, $S^{-1} B$ is $\mathbb{Z}$-graded in this way. In particular, we are interested in the degree zero elements, so we define

$$
\left(S^{-1}\right) B:=\left(S^{-1} B\right)_{0}
$$

If $f \in B$, we write $B_{(f)}:=\left(B_{f}\right)_{0}$ and if $p \in \operatorname{Proj}(B), B_{(p)}:=\left(S_{p}^{-1}\right) B$, where $S_{p}$ is the set of homogeneous elements which don't lie in $p$.
Example. Let $K$ be a field and consider $K\left[x_{0}, \ldots, x_{n}\right]$ with the usual grading $\operatorname{deg}\left(x_{i}\right)=1$. The ring of fractions

$$
B_{x_{0}}=K\left[x_{0}^{ \pm 1}, x_{1}, \ldots, x_{n}\right]
$$

is a $\mathbb{Z}$-graded ring and $B_{x_{0}} \simeq \oplus_{d \in \mathbb{Z}}\left(B_{x_{0}}\right)_{d}$ where $\operatorname{deg}\left(x_{0}^{-1}\right)=-1$. We notice that the subring $\left(B_{x_{0}}\right)_{0}$ is isomorphic to $K\left[x_{1} / x_{0}, x_{2} / x_{0}, \ldots, x_{n} / x_{0}\right]$.

### 3.2.2 $\operatorname{Proj}(B)$ as a scheme

Let $X$ be $\operatorname{Proj}(B)$ and consider $f \in B_{+}$a homogeneous element. We notice that the subset $X_{f}:=X \backslash \mathcal{V}_{+}(f)$ is an open set of $X$. Taken a set of homogeneous generators $\left\{f_{i}\right\}$ for $B_{+}$, we have found an open cover of $\operatorname{Proj}(B)$. Infact,

$$
\bigcup X_{f_{i}}=\bigcup X \backslash \mathcal{V}_{+}\left(f_{i}\right) \Longleftrightarrow \bigcap \mathcal{V}_{+}\left(f_{i}\right)=\mathcal{V}_{+}\left(\sum\left(f_{i}\right)\right)=\mathcal{V}_{+}\left(B_{+}\right)=\emptyset
$$

Let $f \in B_{+}$be a homogeneous element and let $p \in X_{f}$. Then $p_{f} \subseteq B_{f}$ is a prime and

$$
p_{(f)}=p_{f} \cap B_{(f)}=\left(p_{f}\right)_{0}
$$

is a prime in $B_{(f)}$. So we get a map

$$
\begin{array}{clc}
X_{f} & \longrightarrow & \operatorname{Spec}\left(B_{(f)}\right) \\
p & \longmapsto & \left(p_{f}\right)_{0}
\end{array}
$$

We can also write the inverse function

$$
\begin{array}{clc}
\operatorname{Spec}\left(B_{(f)}\right) & \longrightarrow & X_{f} \\
q & \longmapsto & \left(\left\{\text { homogeneous elements of } B \mid \exists m, n \in \mathbb{N} \text { s.t. } \frac{a^{n}}{f^{m}} \in q\right\}\right)
\end{array}
$$

Proposition 3.24. The maps written before define a homeomorphism between $X_{f}$ and $\operatorname{Spec}\left(B_{(f)}\right)$.

Guided by the affine case, we want to construct a sheaf on $\operatorname{Proj}(B)$. Given an open set $U \subseteq X$, we define

$$
\tilde{\mathcal{O}}_{X}(U)=\left\{\alpha: U \rightarrow \sqcup_{p \in U} B_{(p)} \mid s(p) \in B_{(p)}\right\}
$$

which is a sheaf of rings on $X$. The right sheaf to be considered is the following

$$
\mathcal{O}_{X}(U)=\left\{s \in \tilde{\mathcal{O}}_{X}(U) \left\lvert\, \forall p \in U \begin{array}{ll}
\exists f \in B_{+} \text {hom. } & \begin{array}{l}
p \in X_{f} \subseteq U \\
\exists a \in B \text { hom. } \\
\exists n \in \mathbb{N} .
\end{array} \\
& \text { s.t. } \begin{array}{l}
\operatorname{deg}(a)=n \operatorname{deg}(f) \\
\forall q \in X_{f} \quad s(q)=\frac{a}{f^{n}}
\end{array}
\end{array}\right.\right\}
$$

Proposition 3.25. For all $p \in X, \mathcal{O}_{X, p} \simeq B_{(p)}$.
Proof. The map

$$
\begin{array}{rll}
\mathcal{O}_{X, p} & \longrightarrow & B_{(p)} \\
{[s]} & \longmapsto & s(p)
\end{array}
$$

is an isomorphism.
In this way, we have found an isomorphism of locally ringed spaces between $X_{f}$ and $\operatorname{Spec}\left(B_{(f)}\right)$. So $\left(X, \mathcal{O}_{X}\right)$ has an affine open cover and that implies it is a scheme.
We notice that if $f, g \in B_{+}$are homogeneous, the following diagram commutes

where the right inclusion is due to

$$
\begin{aligned}
& B_{(f)} \longrightarrow B_{(f g)} \\
& \frac{a}{f n} \longmapsto \\
& \frac{a g^{2}}{}
\end{aligned}
$$

Furthermore, called $a=g^{\operatorname{deg} f} / f^{\operatorname{deg} g}$, we get an isomorphism

$$
\left(B_{(f)}\right)_{a} \simeq B_{(f g)}
$$

Notice that morphisms from $X$ to $\operatorname{Spec}\left(B_{0}\right)$ are in bijective correspondance with $\operatorname{Hom}\left(B_{0}, \mathcal{O}_{X}(X)\right)$ so $X$ is a scheme over $B_{0}$.

Definition 3.26. If $R$ is a ring, we define $\mathbb{P}_{R}^{n}:=\operatorname{Proj}\left(R\left[x_{0}, \ldots, x_{n}\right]\right)$ with $\operatorname{deg}\left(x_{i}\right)=1$. If $d_{0}, \ldots, d_{n}$ are natural numbers, we define $\mathbb{P}_{R}^{n}\left(d_{1}, \ldots, d_{n}\right):=$ $\operatorname{Proj}\left(R\left[x_{0}, \ldots, x_{n}\right]\right)$ with $\operatorname{deg}\left(x_{i}\right)=d_{i}$.

Example.

- Let's consider $B=R\left[x_{1}, \ldots, x_{n}\right]$ with the usual grading. Since in this case $B_{+}=\left(x_{0}, \ldots, x_{n}\right)$ we have the open sets

$$
U_{i}=\left(\mathbb{P}_{R}^{n}\right)_{x_{i}} \simeq \operatorname{Spec}\left(R\left[x_{0}, \ldots, x_{n}\right]_{\left(x_{i}\right)}\right)
$$

Since

$$
R\left[x_{0}, \ldots, x_{n}\right]_{\left(x_{i}\right)} \simeq R\left[u_{1}, \ldots, \hat{u}_{i}, \ldots, u_{n}\right] \quad u_{j}=\frac{x_{j}}{x_{i}}
$$

this gives an affine open cover of $\mathbb{P}_{R}^{n}=U_{0} \cup \cdots \cup U_{n}$.

- $\mathbb{P}_{R}^{0}=\operatorname{Proj}(R[x]) \simeq \operatorname{Spec}(R)$
- $\mathbb{P}_{R}^{1}=\operatorname{Proj}\left(R\left[x_{0}, x_{1}\right]\right)$, so $\mathbb{P}_{R}^{1}=U_{0} \cup U_{1}$. in particular, $U_{0}=\operatorname{Spec}(R[u])$ where $u=x_{1} / x_{0}$ and $U_{1}=\operatorname{Spec}(R[v])$ where $v=x_{0} / x_{1}$. The intersection of these open sets $U_{0} \cap U_{1}=\operatorname{Spec}\left(R\left[x_{0}, x_{1}\right]_{\left(x_{0} x_{1}\right)}\right) \simeq \operatorname{Spec}\left(R[v]_{v}\right)=$ $\operatorname{Spec}\left(R[u]_{u}\right)$.

Proposition 3.27. $\Gamma\left(\mathbb{P}_{R}^{n}\right)$ is isomorphic to $R$ for all $n>0$.
Proof. We have a map

$$
\left.\begin{array}{rl}
\varphi: \quad \prod_{i} \Gamma\left(U_{i}, \mathcal{O}\right) & \longrightarrow \\
\left(\psi_{i}\right)_{i \in I} & \longmapsto
\end{array} \prod_{i, j} \Gamma\left(U_{i j}, \mathcal{O}\right)\right)
$$

Since $\mathcal{O}$ is a sheaf, the function coinciding on the intersection give rise to a function on the union, so we get the equality

$$
\Gamma\left(\mathbb{P}_{R}^{n}, \mathcal{O}\right)=\operatorname{Ker}(\varphi)
$$

We have to show that the kernel is isomorphic to $R$. Let's take $\varphi \in \mathcal{O}\left(U_{i}\right)$ and $\psi \in \mathcal{O}\left(U_{j}\right)$ such that $\psi=\varphi$ in the intersection. So we have

$$
\varphi\left(\frac{x_{0}}{x_{i}}, \ldots, \frac{x_{n}}{x_{i}}\right)=\frac{p\left(x_{0}, \ldots, x_{n}\right)}{x_{i}^{n}} \quad \psi\left(\frac{x_{0}}{x_{j}}, \ldots, \frac{x_{n}}{x_{j}}\right)=\frac{q\left(x_{0}, \ldots, x_{n}\right)}{x_{j}^{m}}
$$

By the equality on the intersection, we obtain

$$
x_{i}^{n}\left|p\left(x_{0}, \ldots, x_{n}\right) \quad x_{j}^{m}\right| q\left(x_{0}, \ldots, x_{n}\right)
$$

This implies

$$
\varphi=\frac{\tilde{p}\left(x_{0}, \ldots, x_{n}\right)}{1} \quad \psi=\frac{\tilde{q}\left(x_{0}, \ldots, x_{n}\right)}{1}
$$

So, since the equality holds in the intersection, we obtain $\tilde{p}=\tilde{q}$. Since we have the equality

$$
\tilde{p}\left(x_{1}, \ldots, x_{n}\right)=\varphi\left(\frac{x_{0}}{x_{i}}, \ldots, \frac{x_{n}}{x_{i}}\right)
$$

$\varphi, \psi$ must be constant. This holds for every $i, j$, so every component must be the same constant.

Corollary 3.28. $\mathbb{P}_{R}^{n}$ is not affine if $R \neq 0$ and $n>0$.
Example. Let $X$ be $\mathbb{P}_{K}(1,1,2)=\operatorname{Proj}(K[x, y, z])$. As usual, we have $X=$ $X_{0} \cup X_{1} \cup X_{2}$, where

$$
X_{0}=X_{x} \quad X_{1}=X_{y} \quad X_{2}=X_{z}
$$

The grading causes a change in the usual structure of these open sets. While for the first two we have the isomorphisms

$$
X_{0} \simeq \operatorname{Spec}\left(K\left[\frac{y}{x}, \frac{z}{x}\right]\right) \quad X_{1} \simeq \operatorname{Spec}\left(K\left[\frac{x}{y}, \frac{z}{y}\right]\right)
$$

About $X_{2}$, we have the isomorphism
$X_{2}=\operatorname{Spec}\left(K[x, y, z]_{(z)}\right) \simeq \operatorname{Spec}\left(K\left[\frac{x y}{z}, \frac{x^{2}}{z}, \frac{y^{2}}{z}\right]\right) \simeq \operatorname{Spec} K[u, v, z] /\left(w^{2}-u v\right)$
So $\mathbb{P}_{K}(1,1,2)$ is singular in a sense that we will clarify later.

### 3.2.3 Rational Points

Let $K$ be a field; we want to understand in which sense the projective space we have just contructed is the natural generalization of the usual projective space obtained as a quotient of $K^{n}$. We know that the structure sheaf $\mathbb{P}_{K}^{n}$ on $\mathcal{O}_{X}$ is a sheaf of $K$-algebras; in particular the stalks $\mathcal{O}_{X, p}$ are $K$-algebras. For each $p \in X$ we have the residue field $k(p)=\mathcal{O}_{X, p} / \mathfrak{M}_{p}$, which is an extension field of $K$.

Definition 3.29. A point $x \in X$ is rational if the residue field $k(p)$ is $K$.
Suppose $X$ is affine, so $X \simeq \operatorname{Spec}(A)$, and let $p \in X$. Since $X$ is affine, $\mathcal{O}_{X, p} \simeq A_{p}$ and $k(p)=A_{p} / p A_{p}$. Since localization and quotient commute, we can also view $k(p)$ as the quotient field of $A / p$. So we get the inclusions

$$
K \subseteq A / p \subseteq k(p)
$$

and saying that a point is rational is equivalent to say that $A / p=K$. So rational points correspond to maximal ideals $\mathfrak{M}$ such that $A / \mathfrak{M}=K$. These correspond to homomorphisms of $K$-algebras $A \rightarrow K$ (they are the kernels); from the topological point of view, they are the same of the sections of the map $X \rightarrow \operatorname{Spec}(K)$ :


In particular, rational points are closed.
Suppose now $X$ is a generic $K$-scheme; in this case rational point are closed too, because locally closed in an open cover implies closed. So we get the correspondance

$$
X(K):=\{\text { rational points }\} \longleftrightarrow\{\text { sections } \operatorname{Spec}(K) \rightarrow X\}
$$

For example, $\mathbb{A}_{K}^{n}(K) \simeq K^{n}$. Indeed, $\mathbb{A}_{K}^{n}=\operatorname{Spec}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and by the correspondance we know

$$
\mathbb{A}_{K}^{n}(K) \longleftrightarrow \operatorname{Hom}_{K}\left(K\left[x_{1}, \ldots, x_{n}\right], K\right) \simeq K^{n}
$$

On the other hand, we can consider the ring

$$
A=K\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{k}\right)
$$

and $X=\operatorname{Spec}(A)$. The we have the correspondance

$$
X(K) \longleftrightarrow \operatorname{Hom}_{K}(A, K)=\left\{\varphi \in \operatorname{Hom}\left(K\left[x_{1}, \ldots, x_{n}\right], K\right) \mid \varphi\left(f_{i}\right)=0 \forall i\right\}
$$

However, this homomorphisms are in bijective correspondance with points of $\mathcal{V}\left(f_{1}, \ldots, f_{k}\right)$.
We want now to show that

$$
\mathbb{P}_{K}^{n}(K) \simeq K^{n+1} \backslash\{0\} / K^{*}
$$

First, we notice that a morphism of $K$-schemes sends rational points into rational points.

Then, let $B$ be a graded ring and let $B_{0}=B / B_{+}$. Then $V\left(B_{+}\right) \simeq \operatorname{Spec}\left(B_{0}\right)$. We get a morphism

$$
\operatorname{Spec}(B) \backslash V\left(B_{+}\right)=Y \rightarrow \operatorname{Proj}(B)=X
$$

which can be constructed as follows. Let $f \in B_{+}$an homogeneous element; so we have the open covers $Y=\cup Y_{f}, X=\cup X_{f}$. Remember that $X_{f}=\operatorname{Spec} B_{(f)}$, while $Y_{f}=\operatorname{Spec} B_{f}$. Since $X_{f} \supseteq Y_{f}$, we have a morphism $Y_{f} \rightarrow X_{f}$ for all $f \in B_{+}$. Furthermore, they patch togheter

because the diagram commutes. By Pasting Lemma, we obtain a function $f: Y \rightarrow X$. In our particular case, $Y=\mathbb{A}_{k}^{n+1} \backslash\{0\}=Y_{0} \cup \cdots \cup Y_{n}$, where $Y_{i}=$ $Y_{x_{i}}=\operatorname{Spec}\left(K\left[x_{0}, \ldots, x_{n}\right]_{x_{i}}\right)$. We know that $Y_{i}(K)=\left\{a \in K^{n+1} \mid a_{i} \neq 0\right\} . X$ is $\mathbb{P}_{K}^{n}$ and $X_{i}=\left(\mathbb{P}_{K}^{n}\right)_{x_{i}}=\operatorname{Spec}\left(K\left[x_{0}, \ldots, x_{n}\right]_{\left(x_{i}\right)}\right)=\operatorname{Spec}\left(K\left[x_{0} / x_{i}, \ldots, x_{n} / x_{i}\right]\right)$.

So we have a map $Y \rightarrow X$ which induces a function on rational points $\phi$ from $Y(K)$ to $X(K)$.

$$
a \in Y_{i}(K) \longrightarrow\left(\frac{a_{0}}{a_{i}}, \ldots, \frac{a_{n}}{a_{i}}\right) \in X_{i}(K) \subseteq P_{K}^{n}(K)
$$

We notice that $\phi$ is surjective and $\phi(a)=\phi(b)$ if and only if it exists $\lambda \in K^{*}$ such that $b=\lambda a$. So, as desired, we obtain

$$
\mathbb{P}_{K}^{n}(K) \simeq K^{n+1} \backslash\{0\} / K^{*}
$$

where the point $\left[\left(a_{0}, \ldots, a_{n}\right)\right]$ correspond to the ideal $\left(a_{i} x_{j}-a_{j} x_{i}\right) \in \mathbb{P}_{K}^{n}$.

### 3.3 Closed Subschemes and Pullback

Definition 3.30. A closed embedding is a morphism of sheaves $f: Y \rightarrow X$ such that

1. it is a closed embedding topologically
2. $f^{\#}: \mathcal{O}_{X} \rightarrow f_{*} \mathcal{O}_{Y}$ is surjective

We notice that the second condition can be reformulated. In fact, given $q \in Y$, the map

$$
\left(f_{*} \mathcal{O}_{Y}\right)_{q} \longrightarrow \mathcal{O}_{Y, q}
$$

is an isomorphism by the definition of pushforward. What's more, $\left.f_{*} \mathcal{O}_{Y}\right|_{X \backslash Y}=$ 0 because $Y \subseteq X$ is closed. So the second condition is equivalent to
2. For all $q \in Y, f_{q}^{\#}: \mathcal{O}_{X, q} \rightarrow \mathcal{O}_{Y, q}$ is surjective

Proposition 3.31. Let $X, Y$ be affine schemes, so $X=\operatorname{Spec}(A), Y=\operatorname{Spec}(B)$ and let $f: Y \rightarrow X$ be a morphism of schemes. Let $\varphi: A \rightarrow B$ be the induced homomorphism of rings. Then

$$
f \text { is a closed embedding } \Longleftrightarrow \varphi \text { is surjective }
$$

Proof.
$\Longleftarrow$ Let $I$ be the kernel of $\varphi$. Then $A / I$ is isomorphic to $B$. Consequently, since we know that $\operatorname{Spec}(A / I)$ is homeomorphic to $\mathcal{V}(I)$, it is a closed embedding topologically. We have now to show that the second condition holds. Let $q$ be an element of $\operatorname{Spec}(B)$; so there exists $p \in \operatorname{Spec}(A)$ such that $f(q)=\varphi^{-1}(p)$. Considering the stalks, we have

$$
A_{p} \simeq \mathcal{O}_{X, p} \rightarrow B_{q} \simeq \mathcal{O}_{Y, q}
$$

which is surjective since surjectivity is a local property.
$\Longrightarrow$ We know that there exists a bijective correspondance between morphisms of affine schemes and ring homomorphisms. In particular, tha induced ring homomorphism is the composition of the map

$$
\Gamma\left(A, \mathcal{O}_{A}\right) \longrightarrow \Gamma\left(B, \mathcal{O}_{B}\right)
$$

with isomorphisms. Since surjectivity is a local property, we have to prove that for every localization the induced map is surjective; this follows from the second condition of the definition of closed embedding.

Definition 3.32. Let $f: Y \rightarrow X$ and $f^{\prime}: Y^{\prime} \rightarrow X$ be closed embeddings. We say that $f, f^{\prime}$ are equivalent if there exists an isomorphism $g: Y \rightarrow Y^{\prime}$ such that the following diagram commutes:


We notice that it is equivalent to say that $f(Y)=f^{\prime}\left(Y^{\prime}\right)$ and there exists an isomorphism of rings $\tilde{g}: f_{*} \mathcal{O}_{Y} \rightarrow f_{*}^{\prime} \mathcal{O}_{Y^{\prime}}$ such that


In fact, if $\mathcal{O}_{Y^{\prime}} \xrightarrow{\sim} g_{*} \mathcal{O}_{Y}$, then $f_{*}^{\prime} \mathcal{O}_{Y} \xrightarrow{\sim} f_{*}^{\prime} g_{*} \mathcal{O}_{Y}=f_{*} \mathcal{O}_{Y}$.
Remark 3.33. $f_{*} \mathcal{O}_{Y}$ determines $f(Y)$ : indeed, we can look to non-zero stalks

$$
\left(f_{*} \mathcal{O}_{Y}\right)_{p} \neq 0 \Longleftrightarrow p \in f(Y)
$$

So if there exists an isomorphism $f_{*}^{\prime} \mathcal{O}_{Y}^{\prime} \xrightarrow{\sim} f_{*} \mathcal{O}_{Y}$, then $f, f^{\prime}$ are equivalent. We notice that if $f: Y \rightarrow X$ is a closed embedding, then we can consider the sheaf of ideals $I_{Y}=\operatorname{Ker}\left(\mathcal{O}_{X} \rightarrow f_{*} \mathcal{O}_{Y}\right)$ and we have

$$
\mathcal{O}_{X / I_{Y}} \rightarrow f_{*} \mathcal{O}_{Y}
$$

Definition 3.34. Let $F$ be a sheaf on $X$. We define

$$
\begin{aligned}
\text { Supp } F & =X \backslash\left\{p \in X \mid \exists U_{p} \text { neighbourhood of } p \text { such that }\left.F\right|_{U}=0\right\} \\
& =\overline{\left\{p \in X \mid F_{p} \neq 0\right\}}
\end{aligned}
$$

The definition implies easily that $\operatorname{Supp} F \subseteq Y$ if and only if $\left.F\right|_{X \backslash Y}=0$. So if $G$ is a sheaf on $Y$ and $j: Y \rightarrow X$ is a closed embedding, then $\operatorname{Supp} j_{*} G=$ Supp $G \subseteq Y$.

Proposition 3.35. Every sheaf with support contained in $Y$ is isomorphic to a pushforward $j_{*} G$

So the functor

$$
j_{*}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)
$$

is fully faithful and it gives an equivalence between $\operatorname{Sh}(Y)$ and the full subcategory of $\operatorname{Sh}(X)$ supported on $Y$.
We want now to take the inverse construction: so given a continuous function $f: Y \rightarrow X$ and a sheaf on $X$, we want to induce a sheaf on $Y$. The idea is to follow the stalk construction. So given a presheaf $P$ on $X$ and $V$ an open set of $Y$, we define

$$
I_{V}=\{U \subseteq X \text { open subsets } \mid f(V) \subseteq U\}
$$

This is a partial ordered set

$$
U \leq U^{\prime} \Longleftrightarrow U \supseteq U^{\prime}
$$

and inductive.
Definition 3.36. We define the pullback of the presheaf $P$ as

$$
f^{\mathrm{p}} P(V)=\lim _{U \in I_{V}} F(U)
$$

Remark 3.37. If $Y=\{q\}$ is a point of $X$, then clearly $\left(f^{\mathrm{p}} P\right)_{q}=P_{q}$.
We notice that if $V \subseteq V^{\prime}$, then $I_{V^{\prime}} \subseteq I_{V}$. Then we get the restriction maps

$$
f^{\mathrm{p}} P\left(V^{\prime}\right) \rightarrow f^{\mathrm{p}} P(V)
$$

So we have a presheaf on $Y$. Notice that if $f$ is an open map, the pullback is $f^{\mathrm{p}} P(V)=P(f(V))$. In particular, if $Y \subseteq X$ is an open embedding, $f^{\mathrm{p}} P=\left.P\right|_{Y}$.

Proposition 3.38. Let $Q$ be a presheaf on $Y$ and $P$ a presheaf on $X$. Then there exists a canonical isomorphism

$$
\operatorname{Hom}_{Y}\left(f^{\mathrm{p}} P, Q\right) \simeq \operatorname{Hom}_{X}\left(P, f_{*} Q\right)
$$

Proof. Let $\varphi: P \rightarrow f_{*} Q$. Given an open set $V \subseteq Y$, every section $s \in f^{\mathrm{p}} P(V)$ comes from some $\bar{s} \in P(U)$, where $f(V) \subseteq U$. Then $\varphi(\bar{s}) \in \varphi_{*} Q(U)=$ $Q\left(f^{-1}(U)\right) \xrightarrow{\text { rest. }} Q(V)$.
Conversely, given $\psi: f^{\mathrm{p}} P \rightarrow Q$ and an open set $U \subseteq X$, given $s \in P(U)$ we can find $s^{\prime} \in f^{\mathrm{p}} P\left(f^{-1}(U)\right)$. Then $\psi\left(s^{\prime}\right) \in Q\left(f^{-1}(U)\right)=\varphi_{*} Q(U)$.
$f^{\mathrm{p}} P$ gives a functor from presheaves on $X$ to presheaves on $Y$. Furthermore, if

$$
Z \xrightarrow{g} Y \xrightarrow{f} X
$$

are continuous maps and $P$ is a presheaf on $X$, then there exists a canonical isomorphism $(g f)^{\mathrm{p}} P \simeq f^{\mathrm{p}} g^{\mathrm{p}} P$. If $x \in X$ is a point, then $f: x \hookrightarrow X$ has the property that $f^{\mathrm{p}} P_{(x)}=P_{x}$ and $f^{\mathrm{p}} P(\emptyset)=P(\emptyset)$. As a consequence, $\left(f^{\mathrm{p}} P\right)_{y} \simeq$ $P_{f(y)}$.
Example. Let $X$ be a point and $f: Y \rightarrow X$ a continuous function. The pullback of the constant sheaf $A_{X}$ is not a sheaf. In fact,

$$
f^{\mathrm{p}} A_{X}(U)=\left\{\begin{array}{l}
0 \text { if } U=\emptyset \\
A \text { if } U \neq \emptyset
\end{array}\right.
$$

So if $A \neq 0$ and $Y \neq \emptyset$, this is a sheaf if and only if $Y$ is irreducible.
Definition 3.39. If $F$ is a sheaf on $X$, we define the pullback of sheaf

$$
f^{-1} F:=\left(f^{\mathrm{p}} F\right)^{s h}
$$

Proposition 3.40. If $f: X \rightarrow Y$ is a continuous map, $F$ is a sheaf on $X$ and $G$ is a sheaf on $Y$, there exists a canonical isomorphism

$$
\operatorname{Hom}_{Y}\left(f^{-1} F, G\right) \simeq \operatorname{Hom}_{X}\left(F, f_{*} G\right)
$$

Proof. It follows immediately from Proposition 3.38 using the fact that

$$
\operatorname{Hom}_{Y}\left(f^{-1} F, G\right) \simeq \operatorname{Hom}_{Y}\left(f^{\mathrm{p}} F, G\right)
$$

for the sheafification property.
Remark 3.41. If $y \in Y$, then $\left(f^{-1} F\right)_{y} \simeq\left(f^{\mathrm{p}} F\right)_{y} \simeq F_{f(y)}$.
The pullback can be considered as a functor

$$
f^{-1}: S h(X) \longrightarrow S h(Y)
$$

once we have shown the following:
Lemma 3.42. If $f: Y \rightarrow X$ is continuous and $P$ is a presheaf on $X$, then

$$
\left(f^{\mathrm{p}} P\right)^{s h} \simeq\left(f^{\mathrm{p}}(P)^{s h}\right)^{s h}
$$

Proposition 3.43. Let $P$ be a sheaf on $X$ and let

$$
Z \xrightarrow{g} Y \xrightarrow{f} X
$$

be continuous maps. Then $g^{-1} f^{-1} F \simeq(f g)^{-1} F$
Notice that if $f: Y \rightarrow X$ is a continuous map, $F$ is a sheaf on $X$ and $G$ is a sheaf on $Y$, then there exists natural homomorphisms

$$
f^{-1} f_{*} G \rightarrow G \quad F \rightarrow f_{*} f^{-1} F
$$

In fact, we have the isomorphism

$$
\operatorname{Hom}_{X}\left(f^{-1} f_{*} G, G\right) \simeq \operatorname{Hom}_{Y}\left(f_{*} G, f_{*} G\right)
$$

and we have the identity map in the second group. So the inverse image of the identity is the canonical homomorphism. The same thing can be done in the other case.

We call the canonical maps just obtained as $\epsilon$ and $\eta$. Summing up, given $j: Y \rightarrow X$ a closed embedding, we have two functors:

$$
j_{*}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X) \quad j^{-1}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)
$$

and $j^{-1} F(V)$ locally comes from $s \in F(U)$ where $U \cap Y \subseteq V$. Now, let $S h_{Y}(X)$ be the full subcategory of sheaves on $X$ with support contained in $Y$. Then we have the functors

$$
j_{*}: S h(Y) \rightarrow S h_{Y}(X) \quad j^{-1}: S h_{Y}(X) \rightarrow S h(Y)
$$

## Proposition 3.44.

1. If $G$ is a sheaf on $Y, j^{-1} j_{*} G \xrightarrow{\epsilon} G$ is an isomorphism.
2. If $F$ is a sheaf on $X$ such that $\operatorname{Supp} F \subseteq Y, F \xrightarrow{\eta} j_{*} j^{-1} F$ is an isomorphism.

Proof. Both $j_{*}$ and $j^{-1}$ preserves stalks.

Let $f: Y \rightarrow X$ be a closed embedding and let $Y^{\prime}=f(Y)$.

$$
\mathcal{O}_{X} \rightarrow f_{*} \mathcal{O}_{Y} \quad \mathcal{O}_{Y^{\prime}}=j^{-1} f_{*} \mathcal{O}_{Y}
$$

So we get a locally ringed space $\left(Y^{\prime}, \mathcal{O}_{Y^{\prime}}\right)$ isomorphic to $\left(Y, \mathcal{O}_{Y}\right)$. In this way we have contructed a scheme, canonical representative among all the equivalent closed embeddings.
In particular, consider $X=\operatorname{Spec}(A)$, let $I$ be an ideal of $A$ and let $Y$ be $\operatorname{Spec}(A / I)$. The inclusion map $Y \rightarrow X$ is a closed embedding; given $I_{Y}$ the sheaf of the kernels of the map, we have the exact sequence

$$
0 \rightarrow I_{Y} \longrightarrow \mathcal{O}_{X} \longrightarrow f_{*} \mathcal{O}_{Y}
$$

We know that taking global section is left exact, so we obtain the sequence

$$
0 \rightarrow I_{Y}(X) \longrightarrow \mathcal{O}_{X}(X) \simeq A \longrightarrow f_{*} \mathcal{O}_{X}(X) \simeq A / I
$$

So $I_{Y}(X)=I$; this means that the subscheme determines $I$.
Theorem 3.45. Every closed subscheme of $\operatorname{Spec}(A)$ is isomorphic to a subscheme of this type.

Corollary 3.46. There exists a bijection between closed subschemes of $\operatorname{Spec}(A)$ and ideals in $A$.

These facts implies that the support of a subschemes is determined uniquely by the radical, but it can have different subscheme structure.

The last thing we want to remark about closed embedding is that, in a certain sense, it has a local nature. Indeed, let $f: Y \rightarrow X$ be a morphism and let $\left\{X_{i} \mid i \in I\right\}$ be an open cover of $X$. Define $Y_{i}:=f^{-1}\left(X_{i}\right)$, which are open subschemes of $Y$. The restriction of the sheaves on $Y_{i}$ determines the restriction morphism of schemes

$$
f_{i}^{\#}: \mathcal{O}_{X_{i}}=\left.\mathcal{O}_{X}\right|_{X_{i}} \longrightarrow f_{*} \mathcal{O}_{Y_{i}}=\left.f_{*} \mathcal{O}_{Y}\right|_{X_{i}}
$$

Proposition 3.47. $Y \rightarrow X$ is a close embedding if and only if $f_{i}$ is a closed embedding for all $i$.

Example. Let $X$ be a scheme over $\operatorname{Spec}(K)$, where $K$ is a field, and let $f: X \rightarrow$ $\operatorname{Spec}(K)$ be a scheme. We know that rational points are equivalent to sections $s: \operatorname{Spec}(K) \rightarrow X$ of $f$. Every section is a closed embedding. We know that if $p \in X$ is a rational point, then $p$ is maximal, in the affine case as in the general case.

### 3.4 Functoriality of Proj

Let $K$ be an algebrically closed field and let $f_{0}, \ldots, f_{m}$ be homogeneous polynomials of degree $d$ in $K\left[x_{0}, \ldots, x_{n}\right]$. The choice of these polynomials induce a homomorphism

$$
\begin{array}{rlc}
g: \quad K\left[y_{0}, \ldots, y_{m}\right] & \longrightarrow & K\left[x_{0}, \ldots, x_{n}\right] \\
y_{i} & \longmapsto & f_{i}
\end{array}
$$

Definition 3.48. Let $\varphi: B \rightarrow C$ be a homomorphism of rings. $\varphi$ is a homomorphism of graded rings of degree $d$ if $\varphi\left(B_{i}\right) \subseteq C_{i d}$.

So in particular $g$ is a homomorphism of graded rings of degree $d$. Moreover, $g$ induces a well defined morphism of schemes

$$
\begin{array}{ccc}
f: \quad \mathbb{P}^{n}(K) \backslash V_{+}\left(f_{0}, \ldots, f_{m}\right) & \longrightarrow & \mathbb{P}^{m}(K) \\
{\left[a_{0}, \ldots, a_{n}\right]} & \longmapsto & {\left[f_{0}\left(a_{1}, \ldots, a_{n}\right), \ldots, f_{m}\left(a_{0}, \ldots, a_{n}\right)\right]}
\end{array}
$$

We notice that if $\varphi: B \rightarrow C$ is a homomorphism of graded rings of degree $d$ and $p \unlhd C$ is a homogeneous prime ideal, then $\varphi^{-1}(p)$ is still homogeneous. Furthermore,

$$
\varphi^{-1}(p) \in \operatorname{Proj}(B) \Longleftrightarrow \varphi\left(B_{+}\right) \subseteq p
$$

Therefore, we get a function $\operatorname{Proj}(C) \backslash V_{+}\left(\varphi\left(B_{+}\right)\right) \rightarrow \operatorname{Proj}(B)$ and we can make it into a morphism of schemes. Let $X$ be $\operatorname{Proj}(B)$ and let $Y$ be $\operatorname{Proj}(C)$. We have $X=\cup X_{b}$, where $b \in B_{+}$are homogeneous elements. Then

$$
f^{-1}\left(X_{b}\right)=Y_{\varphi(b)} \subseteq Y \backslash V_{+}\left(\varphi\left(B_{+}\right)\right)
$$

The map $B \rightarrow C$ induces the maps $B_{(b)} \rightarrow C_{(\varphi(b))}$, which corresponds to morphisms $Y_{\varphi(b)} \rightarrow X_{b}$. These morphisms are coherent, in the sense that the following diagram commutes

so we get a morphism of schemes $Y \backslash V_{+}\left(\varphi\left(B_{+}\right)\right) \rightarrow X$.

## Example.

1. Let $B=K\left[y_{0}, \ldots, y_{m}\right]$ and let $C$ be $K\left[x_{0}, \ldots, x_{n}\right]$. Chosen $f_{1}, \ldots, f_{m} \in$ $K\left[x_{0}, \ldots, x_{n}\right]$, we get a map $F: \mathbb{P}_{K}^{n} \backslash V_{+}\left(f_{0}, \ldots f_{n}\right) \rightarrow \mathbb{P}_{K}^{m}$. Since $\mathbb{P}_{K}^{m}$ is covered by $U_{i}=\left(\mathbb{P}_{K}^{n}\right)_{y_{i}}$, we get an open cover of $X$, given by $F^{-1}\left(U_{i}\right)=$ $\operatorname{Spec}\left(K\left[x_{0}, \ldots, x_{n}\right]_{\left(f_{0}\right)}\right)$.

$$
\begin{aligned}
& B_{\left(y_{0}\right)}=K\left[\frac{y_{1}}{y_{0}}, \ldots, \frac{y_{m}}{y_{0}}\right] \longrightarrow \\
& \frac{y_{i}}{y_{0}} \longmapsto\left[x_{0}, \ldots, x_{n}\right]_{\left(f_{0}\right)} \\
& \hline f_{0}
\end{aligned}
$$

Looking a rational points, this gives a map

$$
\begin{array}{ccc}
\left.\mathbb{P}_{K}^{n} \backslash V_{+}\left(f_{0}, \ldots, f_{n}\right)\right)(K) & \longrightarrow & \mathbb{P}_{K}^{m}(K) \\
{\left[a_{0}, \ldots a_{n}\right]} & \longmapsto & {\left[f_{0}\left(a_{0}, \ldots a_{n}\right), \ldots, f_{m}\left(a_{0}, \ldots, a_{n}\right)\right]}
\end{array}
$$

Notice that particular cases of this functorial map are the Veronese Maps.
2. Let $B$ be a graded ring and $r$ be a positive integer. We define the graded ring $B^{(r)}$ whose homogeneous components are $B_{i}^{(r)}=B_{i r}$. We notice that the embedding is a homomorphism of degree $r$ and the image is a subring of $B$. Furthermore, we get

$$
\sqrt{\varphi\left(B_{+}^{r}\right)}=\mathfrak{N}\left(B_{0}\right) \oplus B_{+}
$$

so $V_{+}\left(B_{+}^{(r)}\right)=\emptyset$. Therefore, there exists a functorial morphism from $Y=\operatorname{Proj}(B)$ to $X=\operatorname{Proj}\left(B^{(r)}\right)$. Surprisingly, this is an isomorphism of schemes. Indeed, let consider an open cover $X_{b}$ of $X$, where $b \in B_{+}^{(r)}$ are homogeneous elements and $f^{-1}\left(X_{b}\right)=Y_{b}$. Moreover, $B_{(b)}^{(r)} \rightarrow B_{(b)}$ is an isomorphism. The following lemma completes the proof

Lemma 3.49. Let $f: Y \rightarrow X$ be a morphism of schemes. Suppose there exists an open cover $X=\cup X_{i}$ and consider the induced open cover of $Y$ given by $Y_{i}=f^{-1}\left(X_{i}\right)$, such that the maps

$$
\left.f\right|_{X_{i}}: Y_{i} \rightarrow X_{i}
$$

are isomorphisms. Then $f$ is an isomorphism.
Proof. It suffices to check the stalks.
3. Let $B$ be a graded ring and let $I \unlhd B$ be a homogeneous ideal. Let $C$ be the quotient ring $C=B / I$. We get the quotient map $\pi: B \rightarrow C$ and $\pi\left(B_{+}\right)=C_{+}$; functorially, this induces a map $\operatorname{Proj}(C) \rightarrow \operatorname{Proj}(B)$. Topologically, this is a closed embedding and gives a homeomorphism between $\operatorname{Proj}(C)$ and $V_{+}(I)$. It also gives a closed subscheme; for all $b \in B_{+}$homogeneous elements, the kernel of the map $B_{(b)} \rightarrow C_{\pi(b)}$ is the ideal $I_{(b)}$. Notice that in general it i hard to compute a set of generators of $I_{(b)}$. An easy example is the following: given an ideal $I=\left(f_{1}, \ldots, f_{m}\right)$ in $R\left[x_{0}, \ldots, x_{n}\right]$, where each $f_{i}$ is homogeneous of degree $d_{i}$, we have

$$
U_{0}=\operatorname{Spec}\left(B_{\left(x_{0}\right)}\right)=\operatorname{Spec}\left(R\left[\frac{x_{1}}{x_{0}}, \ldots, \frac{x_{n}}{x_{0}}\right]\right)
$$

and $\operatorname{Proj}(B / I) \cap U_{0}=\operatorname{Spec}\left(B_{\left(x_{0}\right)} / I_{\left(x_{0}\right)}\right)$. In this case, $I_{\left(x_{0}\right)}=\left(f_{i} / x_{0}^{d_{i}}\right)_{i}$, because $x_{0}$ has degree one.

## Proposition 3.50.

- Let $\varphi: B \rightarrow C$ be a homomorphism of graded rings of degree 1 an let suppose there exists $d>0$ such that $\left.\varphi\right|_{B_{\geq d}}: B_{\geq d} \rightarrow C_{\geq d}$ is an isomorphism. Then this induces an isomorphism between $\operatorname{Proj}(C)$ and $\operatorname{Proj}(B)$.
- If $I, J$ are homogeneous ideals such that $I_{\geq d}=J_{\geq d}$, then $\operatorname{Proj}(B / I)$ is equal to $\operatorname{Proj}(B / J)$ as subschemes of $B$.

Theorem 3.51. Let $R$ be a noetherian ring, let $B$ be $R\left[x_{0}, \ldots, x_{n}\right]$ and $I, J \unlhd B$ be homogeneous ideals. Then $\operatorname{Proj}(B / I)=\operatorname{Proj}(B / J)$ as subschemes if and only if there exists $d>0$ such that $I_{\geq d}=J_{\geq d}$.

### 3.5 Noetherian, Reduced and Integral Schemes

Definition 3.52. A scheme $X$ is locally noetherian if it has an open covering $X=\cup U_{i}$ such that, if $U_{i}=\operatorname{Spec}\left(A_{i}\right)$, then $A_{i}$ is noetherian for all $i . X$ is noetherian if the cover can be chosen to be finite.

First of all, we notice that $X$ is noetherian if and only if it is locally noetherian and quasi-compact. If $X$ is noetherian, it is locally noetherian and given an open cover $U_{i}$, by definition we can find an affine finite cover $V_{j}$ such that $U_{i}=\operatorname{Spec}\left(A_{i}\right)$ where $A_{i}$ is noetherian. Therefore for all $j V_{j} \cap U_{i}$ is an open cover of $V_{j}$ and we can find a finite subcover. Picking all the corresponding $U_{i}$, we get a finite subcover of the $U_{i}$. Viceversa, if $X$ is locally noetherian and quasi-compact, by definition we can find an open cover $U_{i}$ of $X$ such that $U_{i}=\operatorname{Spec}\left(A_{i}\right)$ and $A_{i}$ is noetherian. By the quasi-compactness of $X$, we can find a finite subcover and these open subschemes satisfy the definition of noetherianity.

Proposition 3.53. An open subscheme $Y$ of a locally noetherian scheme $X$ is locally noetherian.

Proof. Let $X=\cap U_{i}$ such that $U_{i}=\operatorname{Spec}\left(A_{i}\right)$; we have to show that $Y \cap U_{i}$ is locally noetherian. This follows immediately from the fact the sets $\left(U_{i}\right)$ form a basis of open sets of $U_{i}$; in particular, we can choose a family of $f_{i, j}$ such that $\cup U_{i, f_{i, j}}=Y \cap U_{i}$. So we get

$$
Y=\bigcup_{i} \bigcup_{j} U_{i, f_{i, j}}
$$

Proposition 3.54. If $X$ is locally noetherian then for every affine open subscheme $U \subseteq X \mathcal{O}(U)$ is noetherian.

Proof. By the previous Proposition 3.53, we have that $U$ is locally noetherian; since it is affine, it is quasi-compact and so it is noetherian. Therefore, we may assume that $X=U=\operatorname{Spec}(A)$. Let $X=\cup_{i=1}^{n} U_{i}$, where $U_{i}=\operatorname{Spec}\left(A_{i}\right)$ is affine for all $i$. For all $i$, we have the homomorphism $\varphi_{i}: A \rightarrow A_{i}$.
We want now to show that every ideal $I \subseteq A$ is finitely generated. Since $A_{i}$ is noetherian for all $i$, the extension $I A_{i}$ is finitely generated, so we can find $a_{i}, \ldots, a_{r} \in I$ such that their images in $A_{i}$ generates $I A_{i}$ for all $i$. We call this ideal as $J$ :

$$
J=\left(a_{1}, \ldots, a_{r}\right) \subseteq I
$$

We notice that if $p \in X$, then there exists $i$ such that $p \in U_{i}$ and a unique $p_{i} \in \operatorname{Spec}\left(A_{i}\right)$ such that $\varphi^{-1}\left(p_{i}\right)=p$. We get the induced homomorphism

and therefore an isomorphism $A_{p} \rightarrow\left(A_{i}\right)_{p_{i}}$. For each $p \in X I A_{p}=\left(I A_{i}\right)_{p_{i}}=$ $\left(J A_{i}\right)_{p_{i}}=J A_{i}$; this means that $I_{p}=J_{p}$ for all $p \in X$. Since being zero is a local property,

$$
I_{p} / J_{p}=0 \forall p \in X \Rightarrow I / J=0
$$

and so $I=J$.

Definition 3.55. A topological space $X$ is noetherian if one of the following equivalent condition holds:

1. Every decreasing sequence of closed subsets stabilizes
2. Every open subset of $X$ is quasi-compact
3. Every subset of $X$ is quasi-compact

It follows immediately from the definition that if $X$ is a finite union of noetherian subspaces, it is noetherian.

Corollary 3.56. The underlying topological space of a noetherian scheme is noetherian.

Proposition 3.57. Every noetherian topological space has finitely many irreducible components.

Corollary 3.58. A noetherian topological space is locally connected.
Proof. Let $X$ be a noetherian topological space. Then $X$ is the union of its irreducible components $X_{1}, \ldots, X_{n}$. We want to show that every $p$ has a connected open neighbourhood. Let $U$ be the subset

$$
U=X \backslash \underbrace{\substack{i=1 \\ p \notin X_{i}}}_{Y} X_{i}^{n}
$$

We notice that $U=\cup_{p \in X_{i}} X_{i} \backslash Y$ is connected to conclude.
Example. Let $k_{i}$ be an infinite set of fields and let $A=\prod_{i \in I} k_{i}$. Consider the discrete topology on $I$; we have an immersion

$$
I \longrightarrow \operatorname{Spec}(A)=X
$$

This can't be an homeomorphism since $X$ is compact; indeed, $X$ is the $\hat{C}$ ech compactification of $I$. The $\hat{C}$ ech compactification is defined by the universal property:

For every $\varphi: X \rightarrow C$ continuous function, where $C$ is compact, there exists a unique extension $\hat{X} \rightarrow C$ to its Cech compactification $\hat{X}$.

Let $B$ be a graded ring. We first recall the following

## Proposition 3.59.

- $f_{1}, \ldots, f_{n} \in B_{+}$homogeneous elements. Then they generates $B_{+}$if and only if they generates $B$ as a $B_{0}$-algebra.
- A graded algebra $B$ is noetherian if and only if $B_{0}$ is noetherian and $B_{+}$ is finitely generated as a $B_{0}$-algebra.
- A graded ring is finitely generated in degree one if and only if $B_{+}$is generated by finitely many elements of degree one. In this case, $B$ is a quotient of a polynomial ring by an homogeneous ideal.
- If $B$ is a noetherian graded ring, there exists $r \in \mathbb{N}$ such that $B^{(r)} \subseteq B$ is finitely generated in degree one.

Proposition 3.60. If $B$ is a noetherian ring, then $\operatorname{Proj}(B)$ is a noetherian scheme.

Proof. If $B=B_{0}\left[x_{1}, \ldots, x_{n}\right]$, then $\operatorname{Proj}(B)=\mathbb{P}_{B_{0}}^{n}$ can be cover by $n+1$ copies of $\mathbb{A}_{B_{0}}^{n}$. In general, choose $n$ such that $B^{(n)}$ is a finitely generated in degree one. We have shown $\left(\right.$ Example 1) that $\operatorname{Proj}\left(B^{(r)}\right) \simeq \operatorname{Proj}(B)$. Therefore, $\operatorname{Proj}\left(B^{(r)}\right)$ is a closed subscheme of $\mathbb{P}_{B_{0}}^{n}$.
Proposition 3.61. Let $R$ be a ring. Then the following are equivalent:

- $A$ is reduced
- $A_{p}$ is reduced for all $p \in \operatorname{Spec}(A)$
- $A_{m}$ is reduced for all $m \in \operatorname{Spec} M(A)$

Definition 3.62. A scheme $X$ is reduced if and only if $\mathcal{O}_{X, p}$ is reduced for all $p \in X$.

Proposition 3.63. The following are equivalent:

- $X$ is reduced
- $X=\cup U_{i}$, where each $U_{i} \simeq \operatorname{Spec}\left(A_{i}\right)$ and $A_{i}$ is reduced
- For all $U$ open set of $X \mathcal{O}(U)$ is reduced

Lemma 3.64. Let $X$ be a scheme and let $X=\cup X_{i}$ be an open cover. Let $Y_{i} \subseteq X_{i}$ be closed subschemes such that $Y_{i} \cap X_{i j}=Y_{j} \cap X_{i j}$. Then there exists a closed subscheme $Y \subseteq X$ such that $Y \cap X_{i}=Y_{i}$ for all $i$.

Proof. It exists a unique closed subspace $Y \subseteq X$ such that $Y \cap X_{i}=Y_{i}$.

$$
Y_{i} \xrightarrow{f_{i}} X \quad \mathcal{O}_{X} \xrightarrow{f_{i}^{\#}} f_{*} \mathcal{O}_{Y_{i}}
$$

Then we get the sheaves of ideals $I_{Y_{i}}=\operatorname{Ker}\left(f_{i}^{\#}\right) \subseteq \mathcal{O}_{X}$. We define the sheaf

$$
I_{Y}(U)=\left\{s \in \mathcal{O}_{Y}(U)|s|_{U \cap X_{i}} \in I_{Y_{i}}\left(U_{i} \cap X_{i}\right)\right\}
$$

Then, the pullback of the quotient sheaf $\mathcal{O}_{Y}=f^{-1}\left(\mathcal{O}_{X} / I_{Y}\right)$ is the desired one.

Theorem 3.65. Let $X$ be a scheme. If $Y \subseteq X$ is a closed subset, there exists a unique reduced subscheme whose support is $Y$. In particular, there exists a unique closed subscheme $X_{\text {red }}$ with support in $X$.
Proof. In the affine case, it suffices to take the reduced subscheme $\operatorname{Spec}(A / \sqrt{I})$, which is unique.

Definition 3.66. A scheme $X$ is integral if for every open affine subscheme $U \subseteq X \mathcal{O}(U)$ is a domain.
Lemma 3.67. If $X$ is a scheme, there is a bijective correspondance between points and irreducible subset.

$$
\begin{array}{ccc}
X & \longrightarrow & \text { Closed irreducible of } X \\
p & \longmapsto & \overline{\{p\}}
\end{array}
$$

Proof. Injectivity. Assume that there exist $p, q \in X$ such that $\overline{\{p\}}=\overline{\{q\}}$ and let $U \subseteq X$ be an open affine subscheme such that $\overline{\{p\}} \cap U \neq \emptyset$. This implies that $p, q \in U$; so $\overline{\{p\}}^{U}=\overline{\{q\}}^{U}$; since we know that the result holds for affine schemes, $p=q$.
Surjectivity. If $V \subseteq X$ is closed and irreducible, there exists an open affine subset $U \subseteq X$ such that $U \cap V \neq \emptyset$. Then $V=\overline{U \cap V}$ and $U \cap V$ is irreducible. Then there exists $p \in U$ such that $\overline{\{p\}}^{U}=U \cap V$ and this implies $\overline{\{p\}}=V$.
Proposition 3.68. The following are equivalent:

1. For every open non-empty subscheme $U, \mathcal{O}(U)$ is a domain
2. $X$ integral
3. $X$ is reduced and irreducible

Proof.
$(1) \Rightarrow(2)$ Obvious
$(2) \Rightarrow(3)$ Clearly, $X$ is reduced. Let's suppose that $X$ is not irreducible. Then there exists two open sets $U, V$ such that $U \cap V=\emptyset$. Then $\mathcal{O}(U \cup V)=$ $\mathcal{O}(U) \times \mathcal{O}(V)$ and both this are nonzero ring. This gives a contradiction since a product of non zero rings can't be a domain.
$(3) \Rightarrow(1)$ Since $\mathcal{O}(U) \neq 0$, we need to prove that given $f, g \in \mathcal{O}(U), f g=0$ means $f=0$ or $g=0$. Let $A$ be the set $\{p \in U \mid f(p)=0\}$ and $B$ be the set $\{p \in U \mid g(p)=0\}$. These are closed and

$$
A \cup B=\{p \in U \mid f(p)=0 \vee g(p)=0\}=\{p \in U \mid f g(p)=0\}=U
$$

Since $U$ is irreducible, $A=U$ or $B=U$. Let's suppose $A=U$; then $f(p)=0$ for all $p \in U$. Since $U$ is reduced, $f=0$. Indeed, let $V_{i}$ be an open affine cover of $U$; we need to show that $f=0$ in $\mathcal{O}\left(V_{i}\right)$ for all $i$. However, in the affine case $f(p)=0$ for all $p$ means that $f$ is contained in all prime ideals; since by hypotesis $V_{i}$ are reduced, $f=0$.

As an immediate corollary, we obtain that $X$ has a unique generic point, which we call $\xi$. The stalk in $\xi$ is a field called $K(X)$, the field of rational function. This is a field: let $U$ be a non-empty open affine subset of $X$. The $U \simeq \operatorname{Spec}(\mathcal{O}(U)) ;$ since $\xi$ is dense, $\xi \in U$ and $\xi=(0) \subseteq \mathcal{O}(U)$. Therefore, $\mathcal{O}_{X, \xi}=\mathcal{O}(U)_{(0)}$, which is a field.
What's more, the inclusion map $\mathcal{O}(U) \rightarrow K(X)$ is injective. This is easy to see in the affine case, since it is the inclusion in the quotient field; since it is locally injective, it is injective. Notice that $\mathcal{O}_{X, p} \hookrightarrow K(X)$; furthermore, $\mathcal{O}(U)=\cap \mathcal{O}_{X, p} \subseteq K(X)$.

### 3.6 Quasi-compact, Affine and Finite Type Morphisms

Definition 3.69. A continuous map $f: X \rightarrow Y$ is quasi-compact if for every $V \subseteq Y$ open quasi-compact subset, $f^{-1}(V)$ is quasi-compact.

Notice that we can check this property on a base; so if $\left\{V_{i}\right\}$ is a basis of open quasi-compact sets of $Y$ and $f^{-1}\left(V_{i}\right)$ is quasi-compact for all $i, f$ is quasi compact. In the case of schemes, a morphism is quasi compact if for every open affine subscheme $f^{-1}(V)$ is quasi compact. However, it is not enough to assume that there is a cover of open quasi-compact subsets whose inverse image is quasi compact. In fact, let $X$ be $\operatorname{Spec}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and let $F_{i}=V\left(x_{1}, \ldots, x_{i}\right)$. Then we have a chain

$$
F_{1} \supsetneq F_{2} \supsetneq F_{3} \supsetneq F_{4} \supsetneq \ldots
$$

The intersection $\cap F_{i}$ is the maximal ideal $\mathfrak{M}=\left(x_{1}, \ldots\right)$; so the open set $U=$ $X \backslash\{\mathfrak{M}\}$ is not quasi compact. Let $Y=X \sqcup_{U} X$ the scheme obtained by gluing two copies of $X$ along $U$. We denote by $X_{1}=f(X)$ be the image of the first copy of $X$ in $Y$ and $X_{2}$ the image of the second copy. Then the inclusion map

$$
f: X_{1} \longrightarrow Y
$$

is a map between quasi compact schemes; however $f^{-1}\left(X_{2}\right)=U$ and $U$ is not quasi compact.

Proposition 3.70. Let $f: X \rightarrow Y$ be a morphism of schemes and let $Y=\cup V_{i}$ be a cover of open affine subsets. If $f^{-1}\left(V_{i}\right)$ is quasi compact for all $i$, then $f$ is quasi compact. If $Y$ is affine, $f: X \rightarrow Y$ is quasi compact if and only if $X$ is quasi compact.
Proof. Let $A$ be the set of open affine subsets $V$ of $Y$ such that $V$ is contained in any $V_{i}$. Then $A$ is a bases of open subsets for $Y$; so we have to check that the inverse image of these subsets are quasi compact. Choose $i \in I$; since $f^{-1}\left(V_{i}\right)$ is quasi-compact, we can cover $f^{-1}\left(V_{i}\right)$ with a finite number of affine open subschemes. So it suffices to show that if $X$ and $Y$ are affine, $f: X \rightarrow Y$ is quasi compact. Let $X$ be $\operatorname{Spec}(A)$ and $Y$ be $\operatorname{Spec}(B)$. We know that a map of affine schemes induces a ring homomorphism $\varphi: B \rightarrow A$. We have $Y=\cup_{b \in B} Y_{b}$; furthermore $f^{-1}\left(Y_{b}\right)=X_{\varphi(b)} \simeq \operatorname{Spec}\left(X_{\varphi(b)}\right)$, which is quasicompact, as desired.

Definition 3.71. A map of schemes $f: X \rightarrow Y$ is affine if the inverse image of every open affine subscheme is affine.

By the previous proposition, an affine map is quasi compact. Furthermore, a similar property holds but we need some lemmas.
Lemma 3.72. Let $X$ be a quasi-compact scheme and let $a, f \in \Gamma\left(X, \mathcal{O}_{X}\right)$. If $a=0$ in $X_{f}$, then there exists $n \in \mathbb{N}$ such that $a f^{n}=0$ in $\Gamma\left(X, \mathcal{O}_{X}\right)$.

Proof. Let $X=\cup_{i=1}^{n} X_{i}$ be a finite open affine cover of $X$. Then we can consider the restrictions $f_{i}=\left.f\right|_{X_{i}}$ and the hypotesis are equivalent to say that $\left.a\right|_{X_{i}}=a_{i}$ is zero in $\left(X_{i}\right)_{f}$. Let $\operatorname{Spec}\left(A_{i}\right)=X_{i}$; then $\left(X_{i}\right)_{f}=\operatorname{Spec}\left(\left(A_{i}\right)_{\left.f\right|_{X_{i}}}\right)$ and $a_{i}$ is in the kernel of the map $A_{i} \rightarrow\left(A_{i}\right)_{\left.f\right|_{X_{i}}}$. This means that there exists $n_{i} \in \mathbb{N}$ such that $a_{i} f_{i}^{n_{i}}=0$. Taking $n=\max n_{i}$, we get the thesis.

Lemma 3.73. Let $X$ be a scheme and assume that $X=\cup_{i=1}^{n} X_{i}$ is an open affine cover such that $X_{i} \cap X_{j}$ is quasi compact. Let $f \in A=\Gamma\left(X, \mathcal{O}_{X}\right)$.

1. Let $b \in \Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right)$. There exists $n \in \mathbb{N}$ such that $f^{n} b$ is the restriction of an element in $A$.
2. $A_{f} \simeq \Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right)$

Proof.

1. First, we show that it is true in the affine case. So assume $X=\operatorname{Spec}(A)$, $f \in A$ and $b \in A_{f}$. We have to show that there exists $n \in \mathbb{N}$ such that $f^{n} b$ comes from an element of $A$. This is trivial; indeed, since $b \in A_{f}, b=c / f^{k}$ where $c \in A$ and therefore $f^{k} b \in A$. We now consider the general case. For all $i=1, \ldots, n$, we can find $k_{i} \in \mathbb{N}$ such that $\left.\left.f^{k_{i}}\right|_{X_{i}} b\right|_{X_{i}}$ comes from an element of $A$. We can take $k=\max k_{i}$ and these condition still holds. We want now to show that these elements lift to a global section. So we consider $i, j$ and we have to show that the restrictions of these elements to the intersection is the same. This is trivial and therefore they lift to a global section.
2. The restriction map $A \rightarrow \Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right)$ factors through $A_{f} \rightarrow \Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right)$ and this is surjective by the previous point. Therefore we only have to show injectivity, which comes from the previous lemma.

Lemma 3.74. Let $X$ be a scheme and assume there exist $f_{1}, \ldots, f_{k} \in \Gamma\left(X, \mathcal{O}_{X}\right)$ such that $\left(f_{1}, \ldots, f_{k}\right)=1$ and $X_{f_{i}}$ is affine. Then $X$ is affine.

Proof. Let $A=\Gamma\left(X, \mathcal{O}_{X}\right)$. We want to show that $\operatorname{Spec}(A) \simeq X$. Notice that using the previous lemma we get $A_{f_{i}} \simeq \Gamma\left(X_{f_{i}}, \mathcal{O}_{X_{f_{i}}}\right)$ for all $i$. The thesis comes from the lemma 3.49.

We are now ready to show the following:
Proposition 3.75. Let $f: X \rightarrow Y$ be a morphism of schemes and let $Y=\cup Y_{i}$ be an affine open cover. If $f^{-1}\left(Y_{i}\right)$ is affine for all $i, f$ is affine. If $f: X \rightarrow Y$ is a morphism and $Y$ is affine, $f$ is affine if and only if $X$ is affine.

Definition 3.76. A morphism of schemes $f: X \rightarrow Y$ is locally of finite type if whenever $U \subseteq X$ and $V \subseteq Y$ are open affine subschemes and $f(U) \subseteq V$, the induced homomorphism $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$ is of finite type.

Proposition 3.77. Let $f: X \rightarrow Y$ be a morphism of schemes and $X=\cup U_{i}$ be an affine open cover. Let suppose that for all $i \in I V_{i}$ are open affine such that $f\left(U_{i}\right) \subseteq V_{i}$ and $\mathcal{O}\left(V_{i}\right) \rightarrow \mathcal{O}\left(U_{i}\right)$ is of finite type. Then $f$ is of locally finite type.

Corollary 3.78. Let $f: X \rightarrow Y$ be a morphism of schemes and suppose $Y$ is affine, so $Y=\operatorname{Spec}(B)$. Let $X=\cup U_{i}$ be an open affine cover. Then $f$ is locally of finite type if and only if $\mathcal{O}\left(U_{i}\right)$ is a finitely generated $B$-algebra.

Corollary 3.79. If $X, Y$ are affine schemes, then the following are equivalent:

- $f: X \rightarrow Y$ is locally of finite type
- $\mathcal{O}(X)$ is a finitely generated $\mathcal{O}(Y)$ algebra
- $f$ factors through a closed embedding $X \subseteq \mathbb{A}_{B}^{n} \rightarrow Y$, where $\operatorname{Spec}(B)=Y$.

Definition 3.80. A morphism $f: X \rightarrow Y$ is of finite type if it is locally of finite type and quasi compact.

For example, the map $\mathbb{P}_{R}^{n} \rightarrow \operatorname{Spec}(R)$ is of finite type.

## Chapter 4

## Fibered Products and Base Change

### 4.1 Fibered Products: First Properties

We want now to define a sort of product in the category of schemes: this is called the fibered product, which is defined by the following universal property. We are considering a category $\mathcal{C}$ and two arrows $f: X \rightarrow S$ and $g: Y \rightarrow S$

The fibered product of $f, g$ consists of an object $X \times_{S} Y \in O b(\mathcal{C})$ and two arrows $p_{1}: X \times_{S} Y \rightarrow X, p_{2}: X \times_{S} Y \rightarrow Y$ such that whenever $\varphi: T \rightarrow X$ and $\psi: T \rightarrow Y$ are arrows such that $f \circ \varphi=g \circ \psi$, there exists a unique $\theta$ such that the following commutes


For example, let $\mathcal{C}$ be the category of sets and let $S$ the set consisting of a single point, which is a terminal objects. Then, given $X$ and $Y$, there exists unique maps $f: X \rightarrow S$ and $g: Y \rightarrow S$. Then, the fibered product $X \times_{S} Y$ is the classical product of sets.
More in general, $X \times_{S} Y=\{(x, y) \in X \times Y \mid f(x)=g(y)\}$. We can also give to the fiber the structure of a fibered product. In fact, consider a map $f: X \rightarrow S$ and let $s$ be in $S$. Then $f^{-1}(s) \simeq X \times_{S}\{s\}$; if $B$ is a subset of $S$, then $f^{-1}(B) \simeq X \times_{S} B$ in the same way.

Consider now the case of affine schemes. So, let $X, Y$ be $\operatorname{Spec}(A), \operatorname{Spec}(B)$ respectively and let $S$ be $\operatorname{Spec}(S)$. Since we have a bijection between ring
homomorphisms and morphisms of affine schemes, we get the diagrams:

$A \otimes_{R} B$ has a ring structure, given by

$$
(a \otimes b)\left(a^{\prime} \otimes b^{\prime}\right)=a a^{\prime} \otimes b b^{\prime}
$$

Furthermore, we have the canonical maps

$$
\alpha: A \rightarrow A \otimes_{R} B \quad \beta: B \rightarrow A \otimes_{R} B
$$

If $C$ is an $R$-algebra, given $A \xrightarrow{\varphi} C$ and $B \xrightarrow{\psi} C$ homomorphisms of $R$-algebras, there exists a unique morphism of $R$-algebras $\theta: A \otimes_{R} B \rightarrow C$ such that the following diagram commutes:


Therefore, we can define $X \times_{S} Y=\operatorname{Spec}\left(A \otimes_{R} B\right)$ in the category of affine schemes and this satisfies the universal property. The projection maps are the schemes morphisms induced by $\alpha$ and $\beta$.

Definition 4.1. Let $\mathcal{C}$ be a category. A commutative diagram in $\mathcal{C}$ is cartesian

if $T$ is isomorphic to the fibered product $X \times_{S} Y$.
Proposition 4.2. Consider the diagram and assume that the square on the right is cartesian:


Then the square on the left is cartesian if and only if the one on the left is cartesian:


For the tensor product, this proposition implies that given a diagram like this,

then $A^{\prime \prime} \otimes\left(A^{\prime} \otimes B\right) \simeq A^{\prime \prime} \otimes B$.
Example.

- $\mathbb{A}_{R}^{m} \times_{\operatorname{Spec}(R)} \mathbb{A}_{R}^{n}=\operatorname{Spec}\left(R\left[x_{1}, \ldots, x_{m}\right] \otimes R\left[y_{1}, \ldots y_{n}\right]\right) \simeq \mathbb{A}_{R}^{n+m}$
- Let $X, Y$ be the schemes

$$
\operatorname{Spec}\left(R\left[x_{1}, \ldots, x_{m}\right] /\left(f_{1}, \ldots, f_{k}\right)\right) \quad \operatorname{Spec}\left(R\left[y_{1}, \ldots, y_{n}\right] /\left(g_{1}, \ldots, g_{s}\right)\right)
$$

respectively. Then the fibered product $X \times_{R} Y$ is the scheme

$$
\operatorname{Spec}\left(R\left[x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right] /\left(f_{1} \otimes 1, \ldots f_{r} \otimes 1,1 \otimes g_{1}, \ldots, 1 \otimes g_{s}\right)\right)
$$

Let $p \in X=\operatorname{Spec}(A)$. Then we can identify $p=\operatorname{Spec}(k(p)) \rightarrow X$. Let $Y=\operatorname{Spec}(B)$ and let $\varphi: A \rightarrow B$ be a homomorphism of rings. Then we get the diagram


Proposition 4.3. $p \times_{X} Y \rightarrow Y$ induces a homeomorphism between $p \times_{X} Y$ and the fiber $f^{-1}(p)$

Proof. If $p \in A$ is maximal, then $K(p)$ is isomorphic to $A / p$ and $p \otimes_{X} Y=$ $\operatorname{Spec}\left(K(p) \otimes_{A} B\right)$, which is isomorphic to $\operatorname{Spec}(B / p B)$. We know that there exists a homeomorphism between $f^{-1}(p)$ and $\operatorname{Spec}(B / p B)$, so we have shown this case. In general, we can localize at $p$ and obtain the diagram


We know that $\operatorname{Spec}\left(A_{p}\right) \times_{X} Y=\operatorname{Spec}\left(A_{p} \otimes_{A} B\right)=\operatorname{Spec}\left(B_{p}\right)$, which is homeomorphic to the set of primes of $q \in Y$ whose inverse image is contained in $p$. Since $p=\operatorname{Spec}(K(p))$ is maximal in $A_{p}$, we get

$$
A_{p} \times_{X} Y \simeq \operatorname{Spec}\left(B_{p} \otimes_{A_{p}} K(p)\right) \simeq f^{-1}(p)
$$

Base Change A particular case of fibered product is the base change, which is the geometric version of the extension of scalar. First, we need to define a notion of injectivity in the category of schemes.

Definition 4.4. An arrow $S \rightarrow S^{\prime}$ is a monomorphism if, for any two maps $T \xrightarrow{g_{1}, g_{2}} S$ such that $f \circ g_{1}=f \circ g_{2}$, then $g_{1}=g_{2}$.

For example, the map $\operatorname{Spec}(\mathbb{C}) \rightarrow \operatorname{Spec}(\mathbb{R})$ is injective but it isn't a monomorphism in the category of schemes. In fact,


More in general, we can consider a field $\mathbb{F}$ and a Galois extension $\mathbb{E} / \mathbb{F}$. The scheme map induced by the inclusion can't be a monomorphism since a nonidentical element $\sigma$ of $G a l(\mathbb{E} / \mathbb{F})$ give rise to the same composition as the identity $i d$ but $\sigma \neq i d$.
We now want to give some functorial examples of monomorphisms in the category of schemes. Consider the diagram


This induces a map between the fibered products


When $S=S^{\prime}$, we get a functorial map $X \times{ }_{S} Y \rightarrow X^{\prime} \times{ }_{S} Y^{\prime}$; when $X=X^{\prime}$ and $Y=Y^{\prime}$, we get a map $X \times_{S} Y \rightarrow X \times_{S^{\prime}} Y$.

## Proposition 4.5.

1. The functorial map $X \times{ }_{S} Y \rightarrow X \times{ }_{S^{\prime}} Y$ is a monomorphism.
2. If $S \rightarrow S^{\prime}$ is a monomorphism, then $X \times{ }_{S} S \rightarrow X \times{ }_{S^{\prime}} Y$ is an isomorphism.

Example. Closed embedding are monomorphisms in the category of schemes.

Existence of Fibered Products We have shown some examples and properties of fibered product, but still we haven't discussed the existence. We now give a sketch of the proof. Consider the diagram

and take affine open covers $X=\cup X_{i}, Y=\cup Y_{i}$ and $S=\cup W_{i}$ such that $f\left(X_{i}\right) \subseteq W_{i}$ and $g\left(Y_{i}\right) \subseteq W_{i}$. Since we have shown the affine case, every $X_{i} \times_{W_{i}} Y_{i}$ exists; furthermore, $X_{i} \times_{W_{i}} Y_{i} \simeq X_{i} \times_{S} Y_{i}$ because $W_{i} \rightarrow S$ is a monomorphism. We notice that $X_{i j} \times_{S} Y_{i j} \rightarrow X_{i} \times{ }_{S} Y_{i}$ is an open embedding.

Lemma 4.6. Let $X, Y$ be affine schemes and let $U \subseteq X$ and $V \subseteq Y$ be open affine subschemes. Then $U \times{ }_{S} V$ is an open subscheme of $X \times_{S} Y$.

Proof. We can complete the diagram on the left with the fibered products and the map given by the universal property to get a diagram as the one on the right:


We want to show that the image of $\psi$ is an open subset; this follows from the fact that $\operatorname{Im} \psi=p_{1}^{-1}(V) \cap p_{2}^{-1}(U)$ by the commutativity of the diagram.

As a consequence, gluing these open subschemes along their intersection gives the fibered product.

As an example, consider a ring $R$ and let $B$ be a graded $R$-algebra (so we have a map $R \rightarrow B_{0}$ ). We have the maps

$$
\operatorname{Proj}(B) \rightarrow \operatorname{Spec}\left(B_{0}\right) \rightarrow \operatorname{Spec}(R)
$$

and $B \otimes_{R} S$ is a graded $S$-algebra, where $(B \otimes S)_{i}=B_{i} \otimes S$. The canonical homomorphism

$$
\varphi: B \rightarrow S \otimes_{R} B
$$

is of degree one and $\varphi(B+)(S \otimes B)=(S \otimes B)_{+}$. By the functoriality of Proj, we get the commutative diagram


We now want to show that this is cartesian. Because of the universal property, we get a map $\operatorname{Proj}(S \otimes B) \rightarrow \operatorname{Spec}(S) \times{ }_{R} \operatorname{Proj}(B)$.


We can cover $X=\operatorname{Proj}(B)$ with affine open subschemes $X_{b}=\operatorname{Spec}\left(B_{(b)}\right)$, where $b \in B_{+}$are homogeneous. So we get

$$
\operatorname{Spec}(S) \times_{R} \operatorname{Proj}(B) \simeq \bigcup_{b} \operatorname{Spec}(S) \times_{R} X_{b}
$$

For every $b, \varphi^{-1}\left(X_{b}\right)=\operatorname{Spec}\left(S \otimes_{R} B_{(\varphi(b))}\right)$. So we need to show that the induced map $S \otimes\left(B_{(b)}\right) \rightarrow(S \otimes B)_{1 \otimes b}$ is an isomorphism, which is immediate.
Example. Let $S, R$ be rings and let $\varphi: R \rightarrow S$ be a homomorphism. Then $\operatorname{Spec}(S) \times_{R} \mathbb{P}_{R}^{n} \simeq \mathbb{P}_{S}^{n}$. In particular, $\operatorname{since} \operatorname{Spec}(\mathbb{Z})$ is a terminal object in the category of schemes, we have $\mathbb{P}_{R}^{n} \simeq \operatorname{Spec}(R) \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^{n}$.
Definition 4.7. Let $S$ be a scheme. We define

$$
\mathbb{P}_{S}^{n}:=S \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^{n}
$$

Example. Let $X$ be the scheme $\operatorname{Proj}\left(R\left[x_{0}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{r}\right)\right)$, where the polynomials $f_{1}, \ldots, f_{r}$ are homogeneous of positive degree. Let $S$ be a ring; then

$$
S \otimes_{R} R\left[x_{0}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{r}\right) \simeq S\left[x_{0}, \ldots, x_{n}\right] /\left(\bar{f}_{1}, \ldots, \bar{f}_{r}\right)
$$

So we obtain

$$
\operatorname{Spec}(S) \times_{R} X \simeq \operatorname{Proj}\left(S\left[x_{0}, \ldots, x_{n}\right] /\left(\bar{f}_{1}, \ldots, \bar{f}_{r}\right)\right)
$$

## Invariance under base change

Proposition 4.8. Consider the following cartesian diagram


Then

1. If $f$ is quasi compact, so is $f^{\prime}$
2. If $f$ is locally of finite type, so is $f^{\prime}$
3. If $f$ is of finite type, so is $f^{\prime}$

Proof.

- We can assume $Y, Y^{\prime}$ affine. Then $X$ is quasi compact, so $X=\cup_{i=1}^{n} X_{i}$, where each $X_{i}$ is open affine. Then $\psi^{-1}\left(X_{i}\right)=Y^{\prime} \times_{Y} X_{i}$ is affine and $Y^{\prime} \times_{Y} X=\cup\left(Y^{\prime} \times_{Y} X_{i}\right)$ is an open affine cover.

Let $X$ be a scheme and let $p \in X$ be the point $p=\operatorname{Spec}(k(p))$, where $k(p)=\mathcal{O}_{X, p} / \mathfrak{M}_{p}$. There exists a natural morphism $\operatorname{Spec}\left(\mathcal{O}_{X, p}\right) \rightarrow X$; given an affine open neighbourhood $U$ of $p$ in $X$, the projection map give rise to this map. Clearly, it doesn't depend on the choice of $U$. Indeed, if $V$ is another affine open neighbourhood, we get the commutative diagram


So we get a a commutative diagram of schemes:


Proposition 4.9. Let $f: \operatorname{Spec}(K) \rightarrow X$ be a morphism of schemes and suppose that the image of $f$ is exactly $p \in X$ a point. Then $f$ factors uniquely through a map $p \rightarrow X$, so giving rise to the following diagram:


Proof. It's enough to show the affine case; the map $\operatorname{Spec}(K) \rightarrow X$ induces the commutative diagram

which proves the statement.

Proposition 4.10. Let $f: Y \rightarrow X$ be a morphism of schemes and let $p \in X$. Then the map $p \times_{X} Y \rightarrow Y$ induces a homeomorphism between $p \times_{X} Y$ and $f^{-1}(p)$.
Proof. We can reduce to the affine case and the use Proposition 4.3.
Definition 4.11. Let $X, Y$ be schemes. $f: X \rightarrow Y$ is an embedding if it factors through

$$
X \rightarrow U \rightarrow Y
$$

where $X \rightarrow U$ is a closed embedding and $U \rightarrow Y$ is an open embedding.
Summing Up We have discussed some different property of morphisms:

1. Closed embeddings
2. Open embeddings
3. Embeddings
4. Quasi-compact morphisms
5. Affine morphisms
6. Locally of finite type morphisms
7. Of finite type morphisms

Furthermore, they have some common properties:

1. The composite of morphisms with property $\mathcal{P}$ has property $\mathcal{P}$
2. They are local on $Y$, so if $Y=\cup Y_{i}$ is an open cover and for all $i$ the morphism $f_{i}=\left.f\right|_{f^{-1}\left(Y_{i}\right)}$ has property $\mathcal{P}$, then $f$ has property $\mathcal{P}$
3. If $f$ has property $\mathcal{P}$,

so $f^{\prime}$ does.
There is another property:
4. They are local on the domain, so given $X=\cup X_{i}$ an open cover of $X$, if $f_{i}=\left.f\right|_{X_{i}}$ has property $\mathcal{P}$ for all $i$, so $f$ does.

For example, being locally of finite type is local on the domain, while being quasi compact is not. For example, let $X, Y$ be affine schemes and let $I$ be an infinite set of indexes. Let $f: X \rightarrow Y$ be a quasi compact map. Then the map

$$
\cup f: \bigsqcup_{i \in I} X \longrightarrow Y
$$

is not quasi compact while it is locally quasi compact.

### 4.2 Separated Morphisms

We know that for the topological product, the following holds:
Proposition 4.12. Let $X$ a topological space and let

$$
\begin{aligned}
\delta: \quad X & \longrightarrow \\
x & \longmapsto \\
& (x, x)
\end{aligned}
$$

Then $X$ is Hausdorff if and only if $\delta$ is a closed embedding.
This property is rarely satisfied by schemes, since they rarely are Hausdorff spaces. However, the fibered product allows to give this proposition a sense even in this case

Definition 4.13. Let $f: X \rightarrow Y$ be a morphism of schemes. We define the diagonal map

$$
\delta: X \longrightarrow X \times_{Y} X
$$

to be a map such that $p r_{1} \circ \delta=p r_{2} \circ \delta=i d$, where $p r_{1}$ and $p r_{2}$ are the projection map given by the fibered product.


We notice that such a map always exists and it is unique. In fact, from the universal property of fibered product, there exists a unique $\varphi$ such that the following commutes and $\varphi$ satisfies all the property of a diagonal map. Since every other diagonal map make the same diagram commutative, we have the uniqueness. First, we discuss the affine case, so we assume $X=\operatorname{Spec}(A)$ and $Y=\operatorname{Spec}(B)$. Then we know that $X \times_{Y} X \simeq$ $\operatorname{Spec}\left(A \otimes_{B} A\right)$ and the homomorphism of rings $\tilde{\delta}$ that induces the diagonal map $\delta$ has the property that

$$
\tilde{\delta}\left(a \otimes a^{\prime}\right)=\tilde{\delta}(a \otimes 1) \tilde{\delta}\left(1 \otimes a^{\prime}\right)=a a^{\prime}
$$

because of the request $p r_{1} \circ \delta=p r_{2} \circ \delta=i d$. So $\tilde{\delta}$ is surjective which implies that $\delta$ is a closed embedding.

Proposition 4.14. $\delta: X \rightarrow X \times_{Y} X$ is a locally closed embedding.
Proof. Let $Y=\cup Y_{i}$ be an open affine cover. Then for all $i$ we get the open embedding

$$
f^{-1}\left(Y_{i}\right) \times_{Y_{i}} f^{-1}\left(Y_{i}\right) \rightarrow X \times_{Y} X
$$

which corresponds to the inverse image of $Y_{i}$ in $X \times_{Y} X$. Let's denote with $U$ the union $\cup f^{-1}\left(Y_{i}\right) \times_{Y} f^{-1}\left(Y_{i}\right)$. Then $\delta(X) \subseteq U$ and $U \rightarrow X \times_{Y} X$ is an open embedding. So it is enough to prove that

$$
\delta: f^{-1}\left(Y_{i}\right) \longrightarrow f^{-1}\left(Y_{i}\right) \times_{Y} f^{-1}\left(Y_{i}\right)
$$

is an embedding. We can assume that $Y$ is affine; suppose $X=\cup X_{i}$ is an open affine cover. We know that $X_{i} \times{ }_{Y} X_{j}$ is an open subscheme of $X \times_{Y} X$; furthermore, $\delta^{-1}\left(X_{i} \times_{Y} X_{j}\right)=X_{i j}$ and we get the closed embedding $\delta: X_{i} \rightarrow$
$X_{i} \times_{Y} X_{i}$ for the affine case (which we have already dealt with). Therefore, we get an embedding

$$
X \xrightarrow{\text { closed }} \cup X_{i} \times_{Y} X_{i} \xrightarrow{\text { open }} X \times_{Y} X
$$

Definition 4.15. A morphism $f: X \rightarrow Y$ is separated if $\delta: X \rightarrow X \times_{Y} X$ is a closed embedding.

We have already shown that every morphism between affine schemes is separated.

Proposition 4.16. Lef $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes.

1. If $g \circ f$ is separated, $f$ is separated.
2. If $f, g$ are separated, $g \circ f$ si separated.

Proof.

1. We consider the canonical map $\psi: X \times_{Y} X \rightarrow X \times_{Z} X$; we get the diagram

which is commutative. Therefore $\psi^{-1}\left(\delta_{X / Z}(X)\right) \supseteq \delta_{X / Y}(X)$. If we show the other inclusion, we are done. Let $s \in \psi^{-1}\left(\delta_{X / Y}(X)\right)$ and let $x \in X$ such that $\psi(s)=\delta_{X / Z}(x)$. Let now $t=\delta_{X / Y}(x)$. We want to show that $s=t$. We can take affine open neighbourhood $U, V, W$ of $x, f(x), f(g(x))$ respectively, such that $U \subseteq f^{-1}(V)$ and $V \subseteq g^{-1}(W)$. Then $\left.\psi\right|_{U \times_{V} U}: U \times_{V} U \rightarrow U \times_{W} U$ is a closed immersion since these are affine subschemes and $s=t$, as desired.
2. The diagonal morphism $\delta_{X / Y}$ can be seen as a map $\delta_{X / Y}: X \rightarrow X \times_{Y} X \simeq$ $X \times_{Y} Y \times_{Y} X$. The other diagonal $\delta_{Y / Z}$ can be considered in a product with the identity maps:
$\operatorname{Id}_{X} \times \delta_{Y / Z} \times \operatorname{Id}_{X}: X \times_{Y} Y \times_{Y} X \longrightarrow X \times_{Y}\left(Y \times_{Z} Y\right) \times_{Y} X \simeq X \times_{Z} X$
The composite $\left(\operatorname{Id}_{X} \times \delta_{Y / Z} \times \operatorname{Id}_{X}\right) \circ \delta_{X / Y}$ is exactly $\delta_{X / Z}$. Since these map are closed by hypotesis (being closed immersion is invariant under base change) and composition of closed map are closed, we get that $\delta_{X / Z}$ is closed. Since the composite of $\delta_{X / Z}$ (seen as the composition of the diagonal morphisms) with the projection is the identity, we get injectivity.

Corollary 4.17. Lef $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Suppose $g$ is a separated morphism. Then $g \circ f$ is separated if and only if $f$ is separated.

Corollary 4.18. Let $X, Y, Z$ be schemes and suppose $Y, Z$ are affine. Then $X \rightarrow Y$ is separated if and only if $X \rightarrow Z$ is separated.

Definition 4.19. A scheme is separated if it is separated over $\operatorname{Spec}(\mathbb{Z})$.
Corollary 4.20. Let $f: X \rightarrow Y$ be a morphism of schemes. If $X$ is a separated scheme then $f$ is separated.

Proof. Remembering that every scheme has a unique map to $\operatorname{Spec}(\mathbb{Z})$, we get the commutative diagram

and by the previous corollary, $X \rightarrow Z$ is separated if and only if $X \xrightarrow{f} Y$ is separated.

Lemma 4.21. If $X$ is a separated scheme and $U, V$ are open affine subschemes of $X, U \cap V$ is affine.

Proof. Let consider the diagonal map $\delta: X \rightarrow X \times_{\mathbb{Z}} X$; we know that $\delta^{-1}\left(U \times_{\mathbb{Z}}\right.$ $V)=U \cap V$. By hypotesis, $\delta$ is a closed embedding and $U \times_{\mathbb{Z}} V$ is affine. So the inclusion $\delta: U \cap V \rightarrow U \times_{\mathbb{Z}} V$ is a closed embedding. Since any closed subscheme of an affine scheme is affine, we get the thesis.

Proposition 4.22. Let $f: X \rightarrow Y$ be a morphism of scheme and let $X=\cup X_{i}$ be an open affine cover. Then $X$ is separated if and only if $X_{i} \cap X_{j} \rightarrow X_{i} \times{ }_{Y} X_{j}$ is a closed embedding.

We now give an example of an affine scheme which is not separated. Let $k$ be a field and let $X$ be the scheme

$$
\mathbb{A}_{k}^{1} \bigsqcup_{\mathbb{A}_{k}^{1} \backslash\{(x)\}} \mathbb{A}_{k}^{1}
$$

By the gluing theorem, $X=X_{1} \cup X_{2}, X_{1}, X_{2} \simeq \mathbb{A}_{k}^{1}$ and $X_{1} \cap X_{2} \simeq \mathbb{A}_{k}^{1} \backslash\{(x)\}$. $X_{1} \cap X_{2}$ is affine but not separated; the inclusion

$$
X_{1} \cap X_{2} \longrightarrow X_{1} \times_{\mathbb{A}_{k}^{1}} X_{2} \simeq \mathbb{A}_{k}^{1}
$$

is not a closed embedding. Let's now give an example of a separated scheme: the projective space $\mathbb{P}_{R}^{n}$. Let's denote with $A$ the ring $R\left[x_{0}, \ldots, x_{n}\right]$. We know that $\mathbb{P}_{R}^{n}$ is covered by affine open set $X_{i}$ such that $X_{i} \simeq \operatorname{Spec}\left(A_{\left(x_{i}\right)}\right)$. What's more $X_{i} \cap X_{j} \simeq \operatorname{Spec}\left(A_{\left(x_{i} x_{j}\right)}\right) \subseteq X_{i} \times_{\operatorname{Spec}(R)} X_{j}$. So we need to show that the map

$$
A_{\left(x_{i}\right)} \otimes_{R} A_{\left(x_{j}\right)} \longrightarrow A_{\left(x_{i} x_{j}\right)}
$$

is surjective, which is obvious. So the inclusion $X_{i} \cap X_{j} \rightarrow X_{i} \times_{\operatorname{Spec}(R)} X_{j}$ is closed.

Proposition 4.23. Being separated is invariant under base change. In other words, given a cartesian diagram

$f$ is separated if and only if $f^{\prime}$ is separated.
Proof. Assume $f$ is separated and notice that $X^{\prime}=Y^{\prime} \times_{Y} X$. The diagonal $\operatorname{map} \delta_{X^{\prime} / Y^{\prime}}: X^{\prime} \longrightarrow X^{\prime} \times_{Y^{\prime}} X^{\prime}=Y^{\prime} \times_{Y} X \times_{Y^{\prime}} Y^{\prime} \times_{Y} X \simeq\left(X^{\prime} \times_{Y} X^{\prime}\right) \times_{Y} Y^{\prime}$ can be seen as $\delta_{X^{\prime} / Y} \times \mathrm{Id}$, which is closed and injective.

Valuative Criterion for Separation In a certain sense, being separated is the analogous of being Hausdorff. We know that two continuous function coinciding on a dense subspace with value in an Hausdorff space must coincide everywhere. Here we have a similar statement:

Theorem 4.24. Let $R$ be a discrete valuation ring and let $K$ be its quotient field. Let $X, Y$ be schemes and let $f: X \rightarrow Y$ be a separated morphism. Let $g_{1}, g_{2}$ be morphisms of schemes from $\operatorname{Spec}(R)$ to $X$ such that $f \circ g_{1}=f \circ g_{2}$ and their restriction to $\operatorname{Spec}(K)$ coincide. Then $g_{1}=g_{2}$.

Proof. The maps $g_{1}, g_{2}$ give rise to a map $g: \operatorname{Spec}(R) \rightarrow X \times_{Y} X$ by the universal property of fibered product. By the diagonal map, we have a closed embedding of $X$ into $X \times_{Y} X$ and $g^{-1}(X)$ is a closed subscheme of $\operatorname{Spec}(R)$ which contains the point $\operatorname{Spec}(K)$.


Since $\operatorname{Spec}(K)$ is dense in $\operatorname{Spec}(R)$, his closure must be a closed subscheme supported on $\operatorname{Spec}(R)$. However $\operatorname{Spec}(R)$ is reduced and therefore $g^{-1}(X)=$ $\operatorname{Spec}(R)$. This means that the image of $g$ is contained in the diagonal of the fibered product, so $g_{1}=g_{2}$.


More generally, suppose given a diagram like the one on the left and assume that

1. $g: Y \rightarrow Z$ is separated
2. There exists an open dense subscheme $U \subseteq X$ such that $\left.f_{1}\right|_{U}=\left.f_{2}\right|_{U}$
3. $X$ is reduced
then $f_{1}=f_{2}$. Indeed, we can consider the map $s: X \rightarrow Y \times_{Z} Y$. The diagonal is closed by hypotesis and therefore the fiber of the diagonal must be a closed subscheme containing the points $p$ such that $f_{1}(p)=f_{2}(p)$. Since these points are dense in $X$, it must be a closed subscheme having $X$ as support. Since $X$ is reduced, there is a unique structure of closed subcheme on $X$, and this gives the thesis.
These hypotesis are essential. For example, we can consider $X$ to be the affine $\operatorname{scheme} \operatorname{Spec}\left(K[x, y] /\left(x y, y^{2}\right)\right)$. We have

$$
\sqrt{\left(x y, y^{2}\right)}=(y) \quad A_{\mathrm{red}} \simeq K[x, y] /(y) \simeq K[x] \quad X_{\mathrm{red}} \simeq \mathbb{A}_{k}^{1}
$$

Furthermore, $X \backslash\{(x, y)\} \simeq \operatorname{Spec}\left(A_{x}\right) \simeq K[x]_{x}$. We get the map

$$
\begin{aligned}
f: & A
\end{aligned} \quad \longrightarrow A
$$

and $\varphi$ induced by the inclusion map $K[x] \subseteq A$. This give rise to the following diagram, which is commutative.


As requested, $g \circ i d=g \circ f$ and, called $U=X \backslash\{(x, y)\},\left.f\right|_{U}=i d_{U}$. However, $f \neq i d$ and so the property doesn't hold.

Given some more hypotesis, the converse hold:
Theorem 4.25. Let $X, Y$ be locally noetherian schemes and let $f: X \rightarrow Y$ be a morphism. Suppose that, for all choice of a discrete valuation ring $R$, two maps $g_{1}, g_{2}: \operatorname{Spec}(R) \rightarrow X$ such that $f \circ g_{1}=f \circ g_{2}$ and $\left.g_{1}\right|_{\operatorname{Spec}(K)}=\left.g_{2}\right|_{\operatorname{Spec}(K)}$, $g_{1}=g_{2}$. Then $f$ is separated.

### 4.3 Proper and Finite Maps

Consider a diagram like the one on the right and suppose it is cartesian. If $Y^{\prime}, Y, X$ are locally compact Hausdorff spaces, so $X^{\prime}$ is. In fact, $X^{\prime}=X \times_{Y} Y^{\prime} \subseteq Y \times Y^{\prime}$ (the last is the product in the category of topological spaces) and the embedding is closed. Since a subspace of a locally compact Hausdorff space is a locally compact Hausdorff
 space, $X^{\prime}$ has the desired properties. In the case of these spaces, it makes sense define the notion of a proper map, which is a map such that the inverse image of a compact subspace is compact. We notice that a proper map is universally closed, so it stays closed under base change to any locally compact space.
Example. The map $f: \mathbb{R} \rightarrow\{p t\}$ is closed but not universally closed. In fact,

the map $p r_{1}$ is not closed (the image of the hyperbola is open) while $f$ is closed.
Furthermore, a continuous map on a locally compact Hausdorff space is proper if and only if it is universally closed. This suggest the following definitions

Definition 4.26. Let $X, Y$ be schemes. $f: X \rightarrow Y$ is universally closed if for every morphism $Y^{\prime} \rightarrow Y$ the projection map $p r_{1}: Y^{\prime} \times_{Y} X \rightarrow Y^{\prime}$ is closed.

Being closed is a local property on the codomain, so it is enough to check it on an affine open cover. So being universally closed is a local property on $Y$. Furthermore, being universally closed is invariant under base change and composition.

Definition 4.27. Let $X, Y$ be schemes and let $f: X \rightarrow Y$ be a morphism of scheme. We say that $f$ is proper if

1. it is of finite type
2. it is separated
3. it is universally closed

The definition may seem strange and it isn't clear why such a proper map is the equivalent of the notion of topological proper map. As an example, consider map $X \rightarrow \operatorname{Spec}(\mathbb{C})$ locally of finite type. We denote as $X^{\text {an }}$ the set of rational point over $\mathbb{C}$. Suppose $X$ is affine; then locally $X \subseteq \mathbb{A}_{\mathbb{C}}^{n}$ and this give an embedding of $X^{\text {an }}$ into $\mathbb{C}^{n}$. This embedding induces a topology on $X^{\text {an }}$ : we say that $U \subseteq X^{\text {an }}$ is open if and only if $U$ is open in the euclidean topology. For example, $\left(\mathbb{P}_{C}^{n}\right)^{\text {an }} \simeq \mathbb{P}^{n}(\mathbb{C})$ has the usual euclidean topology. The following property give a reason why the definition of proper morphism should be the one given:

Proposition 4.28. Let $f: X \rightarrow Y$ be a morphism of schemes over $\mathbb{C}$. Then

1. $f^{\text {an }}: X^{\text {an }} \rightarrow Y^{\text {an }}$ is continuous.
2. $X$ is connected if and only if $X^{\text {an }}$ is connected
3. $X$ is separated if and only if $X^{\text {an }}$ is Hausdorff
4. If $X, Y$ are separated, $f$ is proper if and only if $f^{\text {an }}$ is proper (in the topological sense)

The class of proper morphism satisfies the usual properties, such as invariance under base change, composition and being local on the codomain. Obviously, not all maps are proper. Taking inspiration from the hyperbola in the case of the projection map, consider the morphism $f: \mathbb{A}_{k}^{1} \rightarrow \operatorname{Spec}(K)$ induced by the inclusion $K \rightarrow K[x]$. Then we obtain


Called $V=V(x y-1)$, we get $p r_{1}(V)=\mathbb{A}_{k}^{1} \backslash\{0\}$, which is open.
Definition 4.29. A morphism of schemes $f: X \rightarrow Y$ is finite if it is affine and for every affine open set $V \subseteq Y$ the ring homomorphism $\mathcal{O}(V) \rightarrow \mathcal{O}\left(f^{-1}(V)\right)$ is finite $\left(\mathcal{O}\left(f^{-1}(V)\right)\right.$ is finitely generated as a $\mathcal{O}(V)$-module.)

Every morphism of affine scheme induced by a finite homomorphism of rings is finite. $f^{-1}(V)$ is affine since it is isomorphic to $\operatorname{Spec}\left(\mathcal{O}(V) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y)\right)$, this tensor product is finitely generated and this shows that it is finite.


Proposition 4.30. Let $f: Y \rightarrow X$ be a morphism of schemes. Let $X=\cup X_{i}$ be an open affine cover and assume that $f^{-1}\left(X_{i}\right) \rightarrow X_{i}$ is finite for all $i$. Then $f$ is finite.

Proof. $f$ is affine by Proposition 3.75. So let $V$ be an affine open subset of $Y$ and let $V=\cup_{i=1}^{n} V_{i}$ be an affine open cover ( $V$ is quasi compact) such that for all $j$ there exists an index $i$ such that $V_{j} \subseteq X_{i}$. We have to show that $\mathcal{O}\left(f^{-1}(V)\right)$ is finite over $\mathcal{O}(V)$. Since we have already dealt with the case of an affine scheme, $f^{-1}\left(V_{j}\right)$ is affine and $\mathcal{O}\left(f^{-1}\left(V_{j}\right)\right)$ is finite over $\mathcal{O}\left(V_{j}\right)$ for all $j$. Furthermore, $\mathcal{O}\left(f^{-1}\left(V_{j}\right)\right)=\mathcal{O}\left(V_{j}\right) \otimes_{\mathcal{O}(V)} \mathcal{O}\left(f^{-1}(V)\right)$ because the diagram

is cartesian and so the sub-squares are. Choose $s_{1}, \ldots, s_{r} \in \mathcal{O}\left(f^{-1}(V)\right)$ such that $1 \otimes s_{i} \in \mathcal{O}\left(V_{j}\right) \otimes_{\mathcal{O}(V)} \mathcal{O}\left(f^{-1}(V)\right)$ generate $\mathcal{O}\left(f^{-1}\left(V_{j}\right)\right)$ as an $\mathcal{O}\left(V_{j}\right)$-module for all $j$. We claim that these elements generate $\mathcal{O}\left(f^{-1}(V)\right)$ as an $\mathcal{O}(V)$-module. We can check this locally, so we have to show that for all $p \in V$ the elements $\left(s_{i}\right)_{p}$ generate $\mathcal{O}\left(f^{-1}(V)\right)_{p}$. Since $V_{j}$ is open in $V$, for all $p \in V_{j} \subseteq V$ there exists a unique $q \in \operatorname{Spec}\left(V_{j}\right)$ which correspond to $p$. Notice that

$$
\mathcal{O}(V)_{p} \simeq \mathcal{O}_{V, p} \simeq \mathcal{O}_{V_{j}, q} \simeq \mathcal{O}\left(V_{j}\right)_{q}
$$

So we get

$$
\begin{aligned}
\mathcal{O}\left(f^{-1}(V)\right)_{p} & \simeq \mathcal{O}(V)_{p} \otimes_{\mathcal{O}(V)} \mathcal{O}\left(f^{-1}(V)\right) \\
& \simeq \mathcal{O}\left(V_{j}\right)_{q} \otimes_{\mathcal{O}(V)} \mathcal{O}\left(f^{-1}(V)\right)
\end{aligned}
$$

which gives what desired.

Proposition 4.31. A finite morphism of schemes is proper.
Proof. We can reduce to the affine case; so we need to show that if $\varphi: A \rightarrow B$ is a finite morphism of rings, the induced map is closed. The finiteness of $B$ over $A$ is equivalent to say that the estension is integral; so in this case, by the Going Up Theorem

$$
f(V(J))=V\left(\varphi^{-1}(V)\right)
$$

which shows that the map is closed.
Proposition 4.32. If an affine map is proper, then it is finite.

## Valuative Criterion of Properness

Theorem 4.33. Let $X, Y$ be locally noetherian schemes and let $f: X \rightarrow Y$ be a morphism of schemes of finite type. Then $f$ is proper if and only if given any discrete valuation ring $R$ with quotient field $k$ there exists a unique morphism $\operatorname{Spec}(R) \rightarrow X$ making the following into a commutative diagram:


Notice that the uniqueness implies that $f$ is separated. We now give an example when such a map doesn't exists. First, we have to consider a map $f$ which is not proper; for example, the projection

$$
f: \mathbb{A}_{k}^{1} \longrightarrow \operatorname{Spec}(k)
$$

We consider then the discrete valuation ring $R=k[1 / t]_{1 / t}$. We get the diagram


However, such a map doesn't exists.
Theorem 4.34. The map $\mathbb{P}_{R}^{n} \rightarrow \operatorname{Spec}(R)$ is proper.
Proof. Since $\mathbb{P}_{R}^{n} \simeq \operatorname{Spec}(R) \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^{n}$ and properness is invariant under base change, it is enough to prove that tha map $\mathbb{P}_{\mathbb{Z}}^{n} \rightarrow \operatorname{Spec}(\mathbb{Z})$ is proper. We use the valuative criterion; so consider a discrete valuation ring and a diagram


We can cover $X=\mathbb{P}_{\mathbb{Z}}^{n}$ by open affine subscheme $X_{i}$, which are isomorphic to $\operatorname{Spec}\left(\mathbb{Z}\left[x_{0}, \ldots, x_{n}\right]_{\left(x_{i}\right)}\right)$. By induction, $\operatorname{since} \operatorname{Spec}(k)$ is a point, we can assume that the image is contained in the intersection of the $X_{i}$ 's, since $\mathbb{P}_{R}^{n} \backslash X_{i}$ is isomorphic to $\mathbb{P}^{n-1}$. We know that there exists a correspondance

$$
\operatorname{Spec}(k) \rightarrow X_{i} \longleftrightarrow \mathbb{Z}\left[t_{1}^{i}, \ldots, t_{n}^{i}\right] \rightarrow k \longleftrightarrow a_{1}^{i}, \ldots a_{n}^{i} \in k
$$

Notice that $a_{j}^{i}=a_{j}^{k} a_{k}^{i}$ for all $i, j, k$. Let $v_{R}$ be the valuation of $R$, defined on $K$. We choose the index $i$ such that $v_{R}\left(a_{i}^{0}\right) \leq v_{R}\left(a_{j}^{0}\right)$ for all $j=1, \ldots, n$. Therefore $v\left(a_{j}^{k}\right)=v\left(a_{j}^{0}\right)-v\left(a_{0}^{k}\right) \geq 0$ and this means that $a_{j}^{k} \in R$. Therefore we can define

$$
\begin{array}{cc}
f: \mathbb{Z}\left[t_{1}^{k}, \ldots, t_{n}^{k}\right] & \longrightarrow \\
t_{j}^{k} & \longmapsto a_{j}^{k}
\end{array}
$$

and this is well define and makes the diagram into a commutative one.
Observation 4.35. If $A$ is a noetherian ring, the following are equivalent:

- $A$ is artinian
- All prime ideals are maximal
- $\operatorname{Spec}(A)$ is finite and discrete
- $A$ has finite lenght

In particular, a finite morphism has finite fiber.
Theorem 4.36 (Chevalley). Let $f: X \rightarrow Y$ be a proper morphism of schemes and suppose it has finite fibers. The $f$ is finite.

Definition 4.37. Let $f: X \rightarrow Y$ be a morphism of schemes.

- $f$ is projective if it factors through

$$
X \rightarrow \mathbb{P}_{Y}^{n} \rightarrow Y
$$

a closed embedding and the canonical projection.

- $f$ is quasi-projective if it factors through

$$
X \rightarrow \mathbb{P}_{Y}^{n} \rightarrow Y
$$

a locally closed embedding and the canonical projection.
In particular, a projective morphism is proper. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then if $g$ is not separated and $g \circ f$ is proper, it is not true that $f$ is proper. In fact,

$$
\mathbb{A}^{1} \rightarrow \mathbb{A}^{1} \sqcup \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}
$$

the composition is the identity while the first map is not closed.
Lemma 4.38. Let $\mathcal{P}$ be a property of morphism of schemes such that closed immersion verify $\mathcal{P}$ and the property is stable under composition and base change. Then if $f: X \rightarrow Y, g: Y \rightarrow Z$ are morphism such that $g$ is separated and $f \circ g$ verifies $\mathcal{P}$, then $f$ satisfies $\mathcal{P}$.

Proof. Let pr: $X \times_{Z} Y \rightarrow Y$ be the second projection. By hypotesis, $p r$ verifies $\mathcal{P}$. Notice that $g$ is separated and therefore $(\mathrm{Id}, f): X \rightarrow X \times_{Z} Y$ is a closed immersion. Then $f=q \circ(\mathrm{Id}, f)$ verifies $\mathcal{P}$.

Proposition 4.39. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Suppose $g$ is separated and $g \circ f$ is proper. Then $f$ is proper.

Proof. By the lemma, it is enough to show that closed immersion are proper, that properness is stable under base change and under composition.

## Chapter 5

## Local Properties

### 5.1 Dimension Theory

Definition 5.1. Let $X$ be a topological space. The Krull dimension of $X$ is the supremum of the lenght of the chains

$$
Y_{0} \subsetneq Y_{1} \subsetneq \cdots \subsetneq Y_{n}
$$

where $Y_{i}$ is closed and irreducible. If $p \in X$ is a point, we define

$$
\operatorname{dim}_{P} X=\inf \{\operatorname{dim}(U) \mid p \in U \subseteq X \text { and } U \text { is open }\}
$$

If $A$ is a ring, $\operatorname{dim}(A)=\operatorname{dim}(\operatorname{Spec}(A))$.
Proposition 5.2. If $Y \subseteq X$ then $\operatorname{dim}(Y) \leq \operatorname{dim}(X)$
Proposition 5.3. $\operatorname{dim}(X)=\sup _{p \in X} \operatorname{dim}_{p}(X)$. Equivalently, if $X=\cup X_{i}$ $\operatorname{dim}(X)=\sup \operatorname{dim} X_{i}$.

Definition 5.4. If $Y \subseteq X$ is closed and irreducible, the codimension of $Y$ in $X$ is the supremum

$$
\operatorname{codim}_{Y}(X)=\sup \left\{n \mid Y=Y_{0} \subsetneq Y_{1} \subsetneq \cdots \subsetneq Y_{n}=X\right\}
$$

If $Y$ is closed,

$$
\operatorname{codim}_{Y}(X)=\inf \left\{\operatorname{codim}_{V}(X) \mid V \subseteq Y \text { is closed and irreducible }\right\}
$$

Proposition 5.5. If $Y \subseteq X$ is closed, $\operatorname{dim}(X) \geq \operatorname{dim}(Y)+\operatorname{codim}_{Y}(X)$.
If $X$ is a scheme and $x \in X$ is a point, let $Y$ be the closure of the point. Then we get

$$
\operatorname{codim}_{Y} X=\operatorname{codim}_{Y} U
$$

and so, by definition, $\operatorname{codim}_{Y} X=\operatorname{dim} \mathcal{O}_{X, p}$. So we get $\operatorname{dim}(X)=\sup \operatorname{dim} \mathcal{O}_{X, p}$.
Proposition 5.6. Let $X$ be an integral scheme locally of finite type over a field $k$. Then $\operatorname{dim}(X)=\operatorname{trdeg}_{k} k(x)$. If $Y \subseteq X$ is closed, $\operatorname{dim}(Y)+\operatorname{codim}_{Y}(X)=$ $\operatorname{dim}(X)$.

### 5.2 Normal Schemes

First of all, we recall the basic algebraic facts: an integral domain is integrally closed (or normal) if every $a \in K$ in the quotient field integral over $A$ belong to $A$. Since a finite extension is integral, we get the following equivalence:

Lemma 5.7. $A$ is normal if and only if every ring $B$ contained in $K$ and finite over $A$ is equal to $A$

Furthermore, being a unique factorization domain is a stronger property that being normal:

Proposition 5.8. If $A$ is a UFD, then $A$ is normal.
Normality is invariant under localization:
Proposition 5.9. Let $A$ be a domain and let $K$ be its quotient field.

- If $S \subseteq A \backslash\{0\}$ is a multiplicative subset and $A$ is normal, then $S^{-1} A$ is normal.
- Being normal is a local property.

The algebraic properties give rise to the following definition:
Definition 5.10. Let $X$ be an irreducible scheme. We say that $X$ is normal if $\mathcal{O}_{X, p}$ is normal for all $p \in X$. A scheme is normal if it is a disjoint union of normal irreducible schemes.

The second definition comes naturally from the first. In fact, let $X$ be a noetherian scheme with irreducible components $X_{1}, \ldots, X_{r}$. Let $p \in X_{i} \cap X_{j}$ be a point of $X$; then $\mathcal{O}_{X, p}$ has at least 2 distinct minimal primes, so $\mathcal{O}_{X, p}$ in not a domain. In order to be domains, we have to assume that $X$ is a disjoint union of its irreducible components and this gives rise to the second definition.

We recall this lemma:
Lemma 5.11. Let $R$ be a normal domain. Then $R\left[x_{1}, \ldots, x_{n}\right]$ is normal.
Example.

- If $R$ is a normal ring, $\mathbb{A}_{R}^{n}, \mathbb{P}_{R}^{n}$ are normal schemes.
- Let $A$ be the ring

$$
A=K[x, y] /\left(y^{2}-x^{3}\right)
$$

Then its quotient field is contained in $K(t)$, where $t=y / x$. However, $t$ is integral over $A$ but $t \notin A$. Furthermore, the inclusion $A \subseteq K[t]$ induces a homeomorphism $\mathbb{A}_{k}^{1} \rightarrow \operatorname{Spec}(A)$. The same holds for the ring

$$
K[x, y] /\left(y^{2}-x^{2}(x+1)\right)
$$

Again, let $t$ be $y / x$ in the quotient field. Then $t$ is integral over $A$ but $t \notin A$.

- Let $A$ be the ring

$$
A=K[x, y, z] /\left(z^{2}-x y\right)
$$

In fact, we have $A_{x}=K[x, z]$ and $A_{y}=K[y, z]$, which are UFD. Denote $U=\operatorname{Spec}\left(A_{x}\right) \subseteq \operatorname{Spec}(A)$ and $V=\operatorname{Spec}\left(A_{y}\right) \subseteq \operatorname{Spec}(A)$; then

$$
U \cup V=(\operatorname{Spec}(A) \backslash V(x)) \cup(\operatorname{Spec}(A) \backslash V(y))=\operatorname{Spec}(A) \backslash V(x, y)
$$

Since an ideal containing $x, y$ must contain also $z$, we get $U \cup V=\operatorname{Spec}(A) \backslash$ $\{(x, y, z)\}$.
Lemma 5.12. $A=A_{x} \cap A_{y}=\mathcal{O}(U \cup V)$
Proof. Let $K=K(x, z)=K(y, z)$ and $y=z^{2} / x$. So if $f \in A_{x} \cap A_{y}$, then

$$
\begin{aligned}
f & =\sum_{\substack{i, j \\
j \geq 0}} a_{i j} x^{i} z^{j}=\sum_{\substack{i, j \\
j \geq 0}} b_{i j} y^{i} z^{j} \\
& =\sum_{\substack{i, j \\
j \geq 0}} b_{i j} x^{-i} z^{2 i+j}
\end{aligned}
$$

So we get $a_{i j}=0$ unless $2 i+j \geq 0$, therefore $f$ is the sum of a polynomial in $x, z$ and a polynomial in $y, z$, as desired.

Since $A$ is intersection of two normal rings, it is normal, as we wish.
If the dimension of the scheme is low, we get the following:
Proposition 5.13. Let $X$ be a locally noetherian integral scheme of dimension $\leq 1$. Then $X$ is normal if and only if for every closed points $p \in X \mathcal{O}_{X, p}$ is a discrete valuation ring.

Proof. If $X$ is normal, then $\mathcal{O}_{X, p}$ is a local normal ring of dimension 1, so a DVR. On the other hand, a DVR is in particular a PID and therefore it is normal.

Dedekind domains take an important role in this part of the theory:
Definition 5.14. Let $A$ be a noetherian domain. $A$ is a Dedekind domain if $A_{p}$ is a discrete valuation ring for all $p \in \operatorname{Spec}(A)$.

Lemma 5.15. Let $A$ be a normal noetherian domain. Let $I$ be an ideal of $A$, $g$ be an element of the quotient field $K$ and suppose that $g I \subseteq I$. Then $g \in I$.
Proof. Let $B$ be the set $\{g \in K \mid g I \subseteq I\} \subseteq K$. If $g \in B$, then we get the map

$$
\begin{array}{cllc}
f_{g}: & I & \longrightarrow & I \\
& & \longmapsto & g x
\end{array}
$$

This gives a homomorphism

$$
\begin{array}{ccc}
B & \longrightarrow & \operatorname{Hom}_{A}(I, I) \\
g & \longmapsto & f_{g}
\end{array}
$$

The map is injective since $I \neq 0$ and $A$ is a domain. What's more, $\operatorname{Hom}_{A}(I, I)$ is a finitely generated $A$-module. Since $A$ is noetherian, $B$ is finite over $A ; A$ is normal, so $A=B$.

Theorem 5.16. A noetherian domain is Dedekind if and only if it is normal of dimension 1.

Proof. Since being normal is a local property, we can check it on $A_{p}, p \in$ $\operatorname{Spec}(A)$. Since $A$ is a Dedekind domain, $A_{p}$ is a DVR, so it is a PID and so it is normal. Moreover, it is of dimension 1, since every localization is a DVR (so it has dimension 1 and $\operatorname{dim}(A)=\sup \operatorname{dim}\left(A_{p}\right)$ ).
Let's now suppose that $A$ is a noetherian normal domain of dimension 1 . We have to show that $A_{p}$ is a DVR; equivalently, we have to prove that the maximal ideal $\mathfrak{M}$ is generated by one element. By Nakayama's Lemma, $\mathfrak{M} \neq \mathfrak{M}^{2}$, otherwise $A_{p}$ would be a field. So let $f \in \mathfrak{M} / \mathfrak{M}^{2}$; then $A /(f)$ is artinian and the nilradical is nilpotent. Therefore, there exists $r \in \mathbb{N}$ such that $\mathfrak{M}^{r} \subseteq(f)$. If $r=1$ then $(f)=\mathfrak{M}$, concluding the proof. So let's suppose that $r \geq 2$. In this case, we want ot show that $(f) \supseteq \mathfrak{M}^{r-1}$. Let $a \in \mathfrak{M}^{r-1}$. Then $a \mathfrak{M} \subseteq \mathfrak{M}^{r} \subseteq(f)$. There are two possibilities:

- If $\left(f^{-1} a\right) \mathfrak{M}=A$, then there exists $m \in \mathfrak{M}$ such that $a m=f$ and this gives a contradiction since $f \notin \mathfrak{M}^{2}$.
- If $\left(f^{-1} a\right) \mathfrak{M} \subsetneq A$, then $\left(f^{-1} a\right) \mathfrak{M} \subseteq \mathfrak{M}$. By the lemma $5.15, f^{-1} a \in A$.

Theorem 5.17. Let $A$ be a normal noetherian domain. Then

$$
A=\bigcap_{\substack{p \in \operatorname{Spec}(A) \\ h t(p)=1}} A_{p}
$$

Proof. Let $B=\cap_{h t(p)=1} A_{p}$; then we have the inclusions $A \subseteq B \subseteq K$, where $K$ is the fraction field of $A$. By contradiction, assume $B \neq A$. Given $f \in B \backslash A$, we define the ideal

$$
I_{f}:=\{a \in A \mid a f \in A\}
$$

$I_{f}$ is proper for any choice of $f$, since $f \notin A$; we can suppose that $I_{f}$ is maximal in the set of such ideals by the noetherianity of $A$. Then $p=I_{f}$ is prime; in fact, if $a b \in p$, then $a b f \in A$. If $b f \in A$ then $b \in p$; if $b f \notin A, a \in I_{b f}=I_{f}$ by maximality of $p$ and then $a \in I_{f}=p$.
We notice that $f \notin A_{p}$; in fact if $f \in A_{p}$ then $f=a / s$ where $s \notin p$; by assumptions, we would have $s f \in A$ which is absurd. So $f \notin A_{p}$ and $f p \subseteq A$ is an ideal. There are two possibilities:

- If $f p A_{p} \subseteq p A_{p}$, we notice that $A_{p}$ is normal and noetherian. Applying 5.15 we get $f \in A_{p}$ and this is absurd.
- If $f p A_{p}=A_{p}$, then $p A_{p}=\left(f^{-1}\right)$. Therefore the maximal ideal of $A_{p}$ is principal; then $h t(p)=1$ and $A_{p}$ is a DVR. Since $f \in B$ and $A_{p}$ is a DVR, then $f \in A_{p}$ and this is absurd.

Corollary 5.18. Let $X$ be a locally noetherian normal scheme and let $Z \subseteq X$ be a closed subset of codimension at least 2. Then $\mathcal{O}(X) \rightarrow \mathcal{O}(X \backslash Z)$ is an isomorphism.

Proof. Since the scheme is normal, every connected component is irreducible and clearly it is enough to show the statement when $X$ is connected. Normality implies that the scheme is reduced, so $X$ is integral. Then we know that the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(X \backslash Z)$ is injective

$$
\mathcal{O}(X) \subseteq \mathcal{O}(X \backslash Z)=\cap_{p \in X \backslash Z} A_{p}
$$

Since $Z$ has codimension at least 2 , every $p \in Z$ has at least height 2 . So

$$
\mathcal{O}(X \backslash Z) \subseteq \cap_{h t(p)=1} A_{p}=\mathcal{O}(X)
$$

proving the equality.
Example. We now give a counterexample to the theorem. We consider the set $A=\left\{f=\sum a_{i} x^{i} \in K[x, y] \mid a_{1}=0\right\}$ of polynomials with no linear terms; we can view $A$ as a $K$-algebra generated by

$$
A=K\left[x^{2}, x y, y^{2}, x^{3}, x^{2} y, x y^{2}, y^{3}\right]
$$

$A$ is not normal since $A \subseteq K[x, y]$ is an integral extension. Let $X$ be the spectrum of $A$ and consider the morphism of schemes $\pi: \mathbb{A}^{2} \rightarrow X$ induced by the inclusion $A \hookrightarrow K[x, y]$. We notice that $A_{x^{2}}=K[x, y]_{x^{2}}=K[x, y]_{x}$. If we call $U=X_{x^{2}}$, we obtain an isomorphism $\pi^{-1}(U) \rightarrow U$. The same can be obtained inverting $y^{2}$; we get an isomorphism $\pi^{-1}(V) \rightarrow V$, where $V=X_{y^{2}}$. Let $p$ the maximal ideal $(x, y) \cap A$; then $U \cap V=X \backslash\{p\} . \pi^{-1}(X \backslash\{p\}) \rightarrow X \backslash\{p\}$ is an isomorphism and so $\pi$ defines a homeomorphism between the topological spaces. Moreover, since $A \subseteq K[x, y]$ is finite, $\pi$ is closed (as a map between schemes). Let $q \in X$ such that $h t(q)=1$; then $q \neq p$ since $h t(p)=2$. Therefore

$$
\bigcap \mathcal{O}_{X, q}=\bigcap \mathcal{O}_{\mathbb{A}_{K}^{2}, q}=K[x, y] \supsetneq A
$$

Definition 5.19. Let $X$ be an integral scheme. A normalization of $X$ is a dominating morphism $\nu: \bar{X} \rightarrow X$ such that $\bar{X}$ is integral and normal and for all $f: Y \rightarrow X$ dominating (with $Y$ integral and normal) factors uniquely through $\bar{X}$


Clearly, if a normalization exists, it is unique up to a unique isomorphism.
Lemma 5.20. Let $X$ be an affine integral scheme. Then $X$ has a normalization $\bar{X}$.

Proof. Let $X=\operatorname{Spec}(A)$ and let $\bar{A}$ be the integral closure of $A$ in its quotient field. The inclusion $A \rightarrow \bar{A}$ induces a morphism of schemes $\operatorname{Spec}(\bar{A}) \rightarrow \operatorname{Spec}(A) ;$ we want ot show that it is the normalization. Clearly, $\operatorname{Spec}(\bar{A})$ is integral and normal and, as a corollary of the Cohen-Seidenberg theorem, the map $\operatorname{Spec}(\bar{A}) \rightarrow \operatorname{Spec}(A)$ is surjective (in particular dominating). Now, let $Y$ be a normal scheme and let $f: Y \rightarrow X$ be a dominating morphism. Since $X$ is
affine, this corresponds to a morphism $A \rightarrow \mathcal{O}(Y)$ (which is injective because the morphism is dominating). We notice that $\mathcal{O}(Y)=\cap_{q \in Y} \mathcal{O}_{Y, q} \subseteq K(Y)$. $Y$ is normal and $\mathcal{O}_{Y, q}$ is normal for all $q$ and so the same holds for $\mathcal{O}(Y)$. By the injectivity of the map $A \rightarrow \mathcal{O}(Y)$, we get an inclusion map $A \rightarrow K \rightarrow K(Y)$ and so the inclusions $A \subseteq \bar{A} \subseteq \mathcal{O}(Y)$. This induces the diagram

concluding the proof.
Lemma 5.21. Let $X$ be an integral scheme and let $\nu: \bar{X} \rightarrow X$ be a normalization. Let $U \subseteq X$ be a non-empty open subscheme. Then the restriction $\nu^{-1}(U) \rightarrow U$ is a normalization.

Proof. Let $f: Y \rightarrow U$ be a dominating morphism ( $Y$ normal), let $\bar{X}$ be the normalization of $X$ and $\bar{U}=\nu^{-1}(U)$. The composite of $f$ with the inclusion $U \rightarrow X$ gives a dominating morphism $Y \rightarrow X$ since $X$ is irreducible ( $U$ is dense). Then we can factor it through $\bar{X}$

and since the image is contained in $\bar{U}$ we get the desired factorization.
In general, let $X$ be an integral scheme and let $U_{i}$ be an affine open cover. Then it is possible to glue the normalization of the open sets in order to obtain the normalization scheme of $X$. So we obtain the following:

Theorem 5.22. Every integral scheme has a normalization $\bar{X} \xrightarrow{\nu} X$. Moreover, $\nu$ is affine and $\bar{X}$ is integral.

Example.

- Let $X$ be the scheme

$$
\operatorname{Spec} \underbrace{\left(K[x, y] /\left(y^{3}-x^{3}\right)\right)}_{=A}
$$

Then the map

$$
\begin{array}{ccc}
A & \longrightarrow & K[t] \\
x & \longmapsto & t^{2} \\
& & t^{3}
\end{array}
$$

induces the normalization.

- Let $X$ be the scheme

$$
\operatorname{Spec} \underbrace{\left(K[x, y] /\left(y^{2}-x^{2}(x+1)\right)\right)}_{=A}
$$

Then the map

$$
\begin{array}{ccc}
A & \longrightarrow & K[t] \\
x & \longmapsto & t^{2}-1 \\
y & \longmapsto & t^{3}-t
\end{array}
$$

induces the normalization.

- In $\mathbb{A}^{2}$, we glue the points $(0, y) \sim(0,-y)$. So let $A$ be the set

$$
A=\{f(0, y)=f(0,-y)\} \subseteq K[x, y]
$$

We obtain the ring $B=K\left[x, y^{2}, x y\right]$, which is isomorphic to

$$
K[u, v, w] /\left(u^{2} v-w^{2}\right)
$$

Then $\mathbb{A}^{2}=\operatorname{Spec}(K[x, y])$ is its normalization

$$
\mathbb{A}^{2} \xrightarrow{\nu} X
$$

Let $\mathbb{A}^{1}=\operatorname{Spec}(K[u, v, w] /(u, w)) \simeq \operatorname{Spec}(K[v])$. Then we have an inclusion map $\mathbb{A}^{1} \rightarrow X$; however, it isn't dominating, so it is not obvious that a lift exists.


Indeed, it can't exists; the image of $y^{2}$ in $K[v]$ is $v$ and $v$ is not a square in $K[v]$.

Theorem 5.23. Let $X$ be a scheme of finite type over a field. Then $\bar{X}$ is finite over $X$.

Clearly, it is an algebraic fact; it is enough to prove that if $A$ is a finitely generated domain over a field $k, \bar{A}$ is finite over $A$. We will prove the following more general statement:

Theorem 5.24. Let $A$ be a finitely generated domain over a field $k$; let $K$ be the quotient field of $A$. Let $L$ be a finite extension of $K$. The normalization of $A$ in $L$ is finite over $A$.

We need some preliminary results:
Lemma 5.25. Let $L / K$ be a finite extension of field. Then $\operatorname{tr}_{L / K}: L \rightarrow K$ is not zero.

Proof. Let $R$ be a finite $K$-algebra and let $K^{\prime}$ be an extension field of $K$. Let's call $R^{\prime}=K^{\prime} \otimes_{K} R$. Then, the trace map becomes $\operatorname{tr}_{R^{\prime} / K^{\prime}}=\operatorname{tr}_{R / K} \otimes \operatorname{Id}_{K^{\prime}}$. Since $L \otimes_{K} \bar{K}=\bar{K}^{n}$, we can take the first vector of the canonical basis $e_{1}$. Then $\operatorname{tr}_{\bar{K}^{n} / \bar{K}}\left(e_{1}\right)=1$; this implies that $\operatorname{tr}_{L / K}$ is not the zero map.

Lemma 5.26. Let $A$ be a finitely generated domain over a field $k$ and let $K$ be its quotient field. Let $L / K$ be a finite extension and let $x$ be an element of $L$. Then there exists an element $a \in A$ such that $a x \in \bar{A}^{L}$.

Lemma 5.27. Let $A$ be a finitely generated domain over a field $k$ and call $K$ its quotient field. Let $L / K$ be a finite separable extension of field and $B$ the integral closure of $A$ in $L$. Then, given $b \in B, \operatorname{tr}_{L / K}(b) \in A$.

Proof. Let $n=[L: K]$; fixed an algebraic closure $\bar{K}$, there exist $n$ linearly independent immersions $\sigma_{1}, \ldots, \sigma_{n}: L \rightarrow \bar{K}$ that fix $K$. Then $\operatorname{tr}_{L / K}(b)=\sum \sigma_{i}(b)$ is an element of $K$ because it is fixed by these automorphism and it is integral over $A$

Proposition 5.28. Let $A$ be an integral normal noetherian domain, let $K$ be its fraction field and let $L$ be a finite separable extension of $K$. Then the integral closure of $A$ in $L$ is finite over $A$.

Proof. By Noether's Normalization, there exist $x_{1}, \ldots, x_{n} \in A$ algebraically independent such that $A$ is finite over $k\left[x_{1}, \ldots, x_{n}\right]$. In particular, $K$ is finite over $k\left(x_{1}, \ldots, x_{n}\right)$. By the transitivity of integral extension, the normalization of $A$ in $L$ coincides with the normalization of $k\left[x_{1}, \ldots, x_{n}\right]$ in $L$. So we may assume that $A$ is a polynomial ring and we denote the integral closure of $A$ in $L$ with $B$
We consider the bilinear simmetric form on $L$ :

$$
\begin{array}{rccc}
\beta: & L \times L & \longrightarrow & K \\
(x, y) & \longmapsto & \operatorname{tr}_{L / K}(x y)
\end{array}
$$

which is non-degenerate. We can choose a basis for $L$ as a $K$-vector space given by elements of $B$; in fact, given $x \in L$, there exists an element $a \in A$ such that $a x \in B$ and this keeps the set linearly independent. So let $e_{1}, \ldots, e_{n}$ be a basis of $L$ over $K$ of elements of $B$. The choice of these elements gives us the dual basis $e_{i}^{*} \in L$ and these elements have the property that $\beta\left(e_{i}, e_{j}^{*}\right)=\delta_{i j}$. Given $b \in B$, we can express it as a linear combination of the element of the dual basis

$$
b=\sum_{i=1}^{n} u_{i} e_{i} \quad u_{i} \in K
$$

Then $\operatorname{tr}\left(b e_{j}\right)=u_{i j} \in A$; this means that $B \subseteq \sum_{i=1}^{n} A e_{i}^{*} \subseteq L$; since $A$ is noetherian, $B$ is a submodule of a finite $A$-module an therefore it is finite over $A$.

We are ready for the proof in the general case:

Proof of the Theorem 5.24. We have already seen the separable case; so we assume that $L$ is not separable over $K$. Then we can find a subfield $E \subseteq L$ such that

$$
K \stackrel{\substack{\text { purely } \\ \text { inseparable }}}{\subseteq} E \stackrel{\text { separable }}{\subseteq} L
$$

Then we can assume that $L$ is purely inseparable over $K$. Let $p=\operatorname{char}(K)$. Let $e_{1}, \ldots e_{s} \in B$ be a basis of $L$ over $K$. There exists $m \in \mathbb{N}$ such that $e_{i}^{p^{m}} \in K$ for all $i$. Since they are integral over $A$, we have $e_{i}^{p^{m}} \in A=K\left[x_{1}, \ldots, x_{n}\right]$. The Frobenius map gives an automorphism of $\bar{L}\left(q=p^{m}\right)$

$$
\begin{array}{ccc}
\bar{L} & \longrightarrow & \bar{L} \\
x & \longmapsto & x^{q}
\end{array}
$$

so there exist $f_{1} \ldots, f_{s} \in A$ such that $e_{i}=f_{i}\left(x_{1}, \ldots, x_{n}\right)^{\frac{1}{q}}=g_{i}\left(x_{1}^{\frac{1}{q}}, \ldots x_{n}^{\frac{1}{q}}\right)$. Let $K^{\prime}$ be the subfield of $\bar{L}$ generated by the coefficient of the $g_{i}$ 's. Then $K^{\prime}$ is finite over $K$; let $b \in B$. Then $b$ can be expressed as

$$
b=\sum_{i=1}^{n} a_{i} e_{i}
$$

The $q^{\text {th }}$ power of $b$ is $b^{q}=\sum a_{i}^{q} f_{i}$ which lies in $K^{\prime}\left[x_{1}^{\frac{1}{q}}, \ldots, x_{n}^{\frac{1}{q}}\right]$. This implies that $A \subseteq B \subseteq K^{\prime}\left[x_{1}^{\frac{1}{q}}, \ldots, x_{n}^{\frac{1}{q}}\right]$ and so $B$ is finite over $A$.

Proposition 5.29. Let $X$ be an integral scheme of finite type over $K$. Then the set

$$
\left\{p \in X \mid \mathcal{O}_{X, p} \text { is normal }\right\}
$$

is open in $X$.
Proof. Since it is a local property, we may assume that $X$ is affine, so $X=$ $\operatorname{Spec}(A)$. Let $K$ the quotient field of $A$ and let $\bar{A}$ be the integral closure. We consider the $A$-module $M=\bar{A} / A$; we get the equality

$$
\left\{p \in X \mid \mathcal{O}_{X, p} \text { is normal }\right\}=\left\{p \in X \mid M_{p}=0\right\}
$$

Since $M$ is finite, the set $\left\{p \in X \mid M_{p}=0\right\}$ is open.
Corollary 5.30. Every reduced scheme of finite type over a field has a dense open subset of points whose stalk is normal.

Proof. It is enough to apply the proposition to every irreducible component.

### 5.3 Regularity and Smoothness

Let $A$ be a local noetherian domain and let $m$ be its maximal ideal. We know that as a corollary of Nakayama's Lemma, the dimension of $m / m^{2}$ over the reisidue field coincides with the minimum number of generators of $m$. The Krull's Principal Ideal Theorem implies then that $h t(m) \leq \operatorname{dim}\left(m / m^{2}\right)$.

Definition 5.31. A noetherian local ring is regular if the equality holds.

It's not hard to show that every regular ring is a domain; we assume the following theorem:

## Theorem 5.32.

- Any regular local ring is a UFD.
- Any localization of a regular local ring is a regular ring.

As a consequence, we can generalize the definition:
Definition 5.33. A noetherian ring is regular if $A_{p}$ is regular for all $p \in$ $\operatorname{Spec}(A)$.

Clearly, we could have required the regularity of the localization for maximal ideals; it would have been the same. The idea of regularity is equivalent to the one given in classical algebraic geometry. For example, let $R$ be the localization of the polynomial ring of $n$ variables at the maximal ideal $\left(x_{1}, \ldots, x_{n}\right)$. Let $A$ be the ring

$$
A:=R /(f)
$$

where $f$ lies in $\left(x_{1}, \ldots, x_{n}\right) \backslash\{0\}$. We know that $\operatorname{dim}(A)=n-1$; let's call $M$ the maximal ideal of $R$ and $m$ the maximal ideal of $A$. Clearly $m=M /(f)$; so we get the exact sequence

$$
M \xrightarrow{\cdot f} M \rightarrow m \rightarrow 0
$$

Computing the tensor product $-\otimes_{R} R / M$ we get

$$
M / M^{2} \xrightarrow{\cdot f} M / M^{2} \rightarrow m / m^{2} \rightarrow 0
$$

Since this is a sequence of $k$-vector spaces, looking at dimensions we obtain that $A$ is regular if and only if $f \notin M^{2}$ if and only if it exists $i$ such that $\frac{\partial f}{\partial x_{i}}(0) \neq 0$. Let's generalize this reasoning. Let $k$ be a field and let $R$ be $k\left[x_{1}, \ldots, x_{n}\right]$. Let $A$ be the quotient $R / I$, where $I=\left(f_{1}, \ldots, f_{m}\right)$. Let $p \in$ $\operatorname{Spec}(A)$ a maximal ideal such that $A / p=k$. So we can write $p=M / I$, where $M=\left(x_{1}-a_{1}, \ldots, x_{n}-a_{n}\right) \subseteq R_{p}$ is the maximal ideal. We have the homomorphism

$$
\begin{array}{clc}
R^{m} & \longrightarrow & M \\
\left(a_{1}, \ldots, a_{m}\right) & \longrightarrow & \sum a_{i} f_{i}
\end{array}
$$

which give rise to the exact sequence

$$
R^{m} \rightarrow M \rightarrow p
$$

Again, the tensor product by $k$ produces the sequence

$$
k^{n} \rightarrow M / M^{2} \rightarrow m / m^{2} \rightarrow 0
$$

The matrix which represents the first map is $\left(\partial f_{j} / \partial x_{i}\right)_{i, j}$, which is the Jacobian in $a$. So the dimension of $m / m^{2}$ is $n-r k(\mathcal{J}(a))$. Then $A_{p}$ is regular if and only if $n-r k(\mathcal{J}(a))=\operatorname{dim}_{p}(X)$.
Let's consider now the affine case. Let $X=\operatorname{Spec}(A) \subseteq \mathbb{A}_{k}^{n}$ and let $p \in X(k)$ be a rational point. Let $X_{1}, \ldots, X_{r}$ be the irriducible components of $X$. Clearly, if
$p$ lies in the intersection of two irreducible components, then $A_{p}$ is not a domain; in particular, it can't be regular. Notice that the set

$$
\left\{p \in X_{i}(k) \mid \mathcal{O}_{X, p} \text { is regular }\right\} \subseteq X_{i} \backslash\left(\cup_{j \neq i} X_{j}\right)
$$

so it is an open subset of $X_{i}$. If $d$ is the dimension of $X_{i}$, we get

$$
\left\{p \in X_{i} \mid \mathcal{O}_{X, p} \text { is regular }\right\}=\left\{p \in X_{i} \backslash\left(\cup_{j \neq i} X_{j}\right) \mid \operatorname{rk}\left(\mathcal{J}_{f}(p)\right) \geq n-d\right\}
$$

and this is open since is defined by non-vanishing of some minors of the Jacobian matrix. So the set of regular rational points is open in the set of rational points.

Definition 5.34. A locally noetherian scheme $X$ is regular if $\mathcal{O}_{X, p}$ is regular for all $p \in X$.

We have just proved the following:
Proposition 5.35. Let $X$ be a scheme locally of finite type over an algebraically closed field $K$. Let $X_{l}$ be the set of closed point. Then the set

$$
\left\{p \in X_{l} \mid \mathcal{O}_{X, p} \text { is regular }\right\}
$$

is open in $X_{l}$.
Theorem 5.36. Let $A$ be a finitely generated $K$-algebra, where $K$ is an algebraically closed field, and let $X=\operatorname{Spec}(A)$. Then the set of regular points is open in $X$.

Proposition 5.37. Let $X$ be a reduced scheme of finite type over an algebraically closed field $K$. Then $X$ contains a regular point.

However, things are much more difficult if the field is not perfect. Let $K$ be a non perfect field and let $\alpha \in K \backslash K^{p}$. Then we know that $x^{p}-\alpha \in K[x]$ is irreducible but the field

$$
K^{\prime}=K[x] /\left(x^{p}-\alpha\right)=K(\sqrt[p]{\alpha})
$$

is purely inseparable over $K$. Moreover, let $X$ be $\operatorname{Spec}\left(K^{\prime}\right)$; since $K^{\prime}$ is a field, $X=\{\xi\}$ and $\mathcal{O}_{X, \xi}$ is regular. However, the Jacobian matrix in (0) is the zero matrix.

More in general, let $X$ be $\operatorname{Spec}(A)$, where $A=K\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{r}\right)$ and let $p \in X$ be a closed point. Then the extension $k(p) / k$ is finite (Nullstellensatz); chosen $\bar{K}$ an algebraic closure of $K$, we consider the fibered product

$$
X_{\bar{K}}:=\operatorname{Spec}(\bar{K}) \times_{K} X=\operatorname{Spec}\left(\bar{K} \otimes_{K} A\right)
$$

We notice that

$$
\bar{K} \otimes_{K} A \simeq \bar{K}\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{r}\right)
$$

and that the map $X_{\bar{K}} \rightarrow X$ is surjective. There are two different ways to see this:

- The map correspond to the integral extension $A \rightarrow \bar{A}$ which is surjective by the Going Up Theorem.
- Being surjective is stable under base change:

Lemma 5.38. Given a cartesian diagram,

if $f$ is surjective then $f^{\prime}$ is surjective.
Proof. Let $q^{\prime} \in Y$ and let $q$ be its image in $Y$ through $\psi$. Since $f$ is surjective, there exists $p \in X$ such that $f(p)=q$. We get the diagram

and since $p \times_{q} q^{\prime}=\operatorname{Spec}\left(k(p) \otimes_{k(q)} k\left(q^{\prime}\right)\right)$ and this is a tensor product over a field, the fiber of $q^{\prime}$ is non-empty by the commutativity of the diagram.

Let $\bar{p} \in X_{\bar{K}}$ be a closed point that maps to the closed point $p$. Then $p \in X_{\bar{K}}(\bar{K})$, because $\bar{K} \otimes_{K} k(p)$ is a finitely generated $\bar{K}$-algebra and it is isomorphic to $\bar{K}$.

Definition 5.39. Let $X$ be a scheme over $K . X$ is smooth at the closed point $p$ if $X_{\bar{K}}$ is regular at the closed point $\bar{p}$.

Although the definition seems to depend on the choice of $\bar{p}$, we can show that it is independent on it. Let $A=K\left[x_{1}, \ldots, x_{n}\right]$ and $X=\operatorname{Spec}(A)$. Let $f=\left(f_{1}, \ldots, f_{r}\right) \in K\left[x_{1}, \ldots, x_{n}\right]$ and let $\mathcal{J}_{f}=\left(\partial f_{i} / \partial x_{j}\right)$ be the Jacobian matrix. Then, evaluating $\mathcal{J}_{f}$ at $p$, we get a matrix with values in $k(p)$. If $\bar{p}$ be a closed point in the fiber, then $\bar{p}$ is regular in $X_{\bar{K}}$ if and only if $\operatorname{rk}\left(\mathcal{J}_{f}(p)\right)=\operatorname{rk}\left(\mathcal{J}_{f}(\bar{p})\right)$. Since $\operatorname{rk}\left(\mathcal{J}_{f}(p)\right)=n-\operatorname{dim}_{p} X$ and $A \rightarrow \bar{A}$ is an integral extension, we get $\operatorname{dim}_{p}(X)=\operatorname{dim}_{\bar{p}} \bar{X}$ and so the equality. So we have shown that smoothness doesn't depend on the choice of the point.

Theorem 5.40. Let $X$ be a scheme over $k$ of finite type and let $p \in X$ be a closed point.

1. If $X$ is smooth at $p$ then $\mathcal{O}_{X, p}$ is regular
2. If $\mathcal{O}_{X, p}$ is regular and $k(p) / k$ is separable then $X$ is smooth at $p$

Corollary 5.41. If $k$ is a perfect field, $p$ is regular if and only if $p$ is smooth.

Example. Let $k$ be a non-perfect field of characteristic $p$ and let $\alpha$ be an element of $k \backslash k^{p}$. We consider the quotient

$$
k^{\prime}=k[x] /\left(x^{p}-\alpha\right)
$$

and let $\xi$ be the only point of $\operatorname{Spec}\left(k^{\prime}\right)$. Then $\operatorname{Spec}\left(k^{\prime}\right)$ is regular at $\xi$ but

$$
X_{\bar{k}}=\operatorname{Spec}\left(\bar{k}[x] /\left(x^{p}-\alpha\right)\right) \simeq \operatorname{Spec}\left(\bar{k}[x] /(x-u)^{p}\right) \simeq \operatorname{Spec}\left(\bar{k}[x] /\left(x^{p}\right)\right)
$$

which is not a domain and therefore it can't be regular.
Summing up, being smooth depends on the field; while regularity is a property of the scheme, smoothness is a property of morphisms $(X \rightarrow \operatorname{Spec}(k))$.

### 5.4 Flat Morphisms

Proposition 5.42 (Baer's Criterion). Let $A$ be a ring and $M$ an $A$-module. Then $M$ is flat if and only if for all $I$ ideals $I \otimes_{A} M \rightarrow M$ is injective.

Let $A$ be a domain, let $f \in A$ be a non-zero element.

$$
0 \rightarrow(f) \rightarrow A
$$

Computing the tensor product and using the isomorphism $(f) \simeq A$, we obtain

$$
M \simeq(f) \otimes_{A} M \rightarrow M
$$

and so being flat implies being torsion free.
Proposition 5.43. Let $A$ be a principal ideal domain and let $M$ be an $A$ module. Then $M$ is flat if and only if it is torsion free.

Proposition 5.44. Let $M$ be a flat $A$-module and let $B$ be an $A$-algebra. Then $B \otimes_{A} M$ is flat over $B$.

Proposition 5.45. If $S \subseteq A$ is a multiplicative system and $M$ is an $S^{-1} A$ module, then $M$ is flat over $S^{-1} A$ if and only if $M$ is flat over $A$.

Proposition 5.46. Let $M$ be an $A$-module. The following are equivalent:

- $M$ is flat over $A$
- $M_{p}$ is flat over $A_{p}$ for all maximal ideals $p$

Corollary 5.47. Let $A$ be a Dedekind domain. Then $M$ is flat if and only if it is torsion free.

Proposition 5.48. Let $B$ be a flat $A$-algebra and let $M$ be a flat $B$-module. Then $M$ is flat over $A$.

Definition 5.49. Let $M$ be an $A$-module. $M$ is faithfully flat if, given a homomorphism $N^{\prime} \rightarrow N$,

$$
N^{\prime} \otimes_{A} M \rightarrow N \otimes_{A} M \text { is injective } \Longleftrightarrow N^{\prime} \rightarrow N \text { is injective }
$$

Proposition 5.50. Let $M$ be a flat $A$-module. The following are equivalent:

1. $M$ is faithfully flat
2. If $N$ is an $A$-module, $M \otimes_{A} N=0$, then $N=0$
3. For all maximal ideals $\mathfrak{M}, M \neq \mathfrak{M} M$

Proof.
$(1) \Rightarrow(2)$ Given the zero homomorphism $N \rightarrow 0$, tensorizing by $M$ we obtain the homomorphism $M \otimes_{A} N \rightarrow 0$ which is injective. Then $N=0$.
$(2) \Rightarrow(1)$ Let $\varphi: N^{\prime} \rightarrow N$ be a homomorphism of ring and assume $N^{\prime} \otimes M \rightarrow N \otimes M$ is injective. $\operatorname{Ker}(\varphi)$ becomes zero if tensorized for $M$ and then $\operatorname{Ker}(\varphi)=0$.
$(2) \Rightarrow(3)$ We consider the $A$-module $M / \mathfrak{M} M$. We know it is isomorphic to the tensor product $M \otimes A / \mathfrak{M}$; since $A / \mathfrak{M} \neq 0$, we obtain $\mathfrak{M} M \neq M$.
$(3) \Rightarrow(2)$ Since $M$ is flat, a submodule $N^{\prime} \subseteq N$ corresponds to the submodule $N^{\prime} \otimes M \subseteq N \otimes M$ of the tensor product. So it is enough to prove that for all the ideals $I \subseteq A$ hold $M \otimes_{A} A / I \neq 0$. Let $I$ be an ideal and let $\mathfrak{M}$ be a maximal ideal containing $I$. Then the homomorphism

$$
A / I \rightarrow A / \mathfrak{M}
$$

correspond to the homomorphism

$$
M / I M \rightarrow M / \mathfrak{M} M
$$

and $M / I M=M \otimes A / I \neq 0$.

Proposition 5.51. Let $A \rightarrow B$ be a flat homomorphism. The following are equivalent:

1. $A \rightarrow B$ is faithfully flat
2. $\operatorname{Spec}(B) \rightarrow \operatorname{Spec}(A)$ is surjective
3. The image of $\operatorname{Spec}(B)$ in $\operatorname{Spec}(A)$ contains all the maximal ideals of $A$

Proof. Let $\varphi: A \rightarrow B$ be a homomorphism of rings and let $f: \operatorname{Spec}(B) \rightarrow$ $\operatorname{Spec}(A)$ be the corresponding morphism of schemes. Given $p \in \operatorname{Spec}(A)$, we know that

$$
f^{-1}(p)=\operatorname{Spec}\left(B \otimes_{A} k(p)\right)
$$

is the fiber. If $f$ is surjective, then $B \otimes k(p) \neq 0$ for all maximal ideals $p$ and so $\varphi$ is faithfully flat. Viceversa, if $\varphi$ is faithfully flat, the $\operatorname{ring} B \otimes k(p)$ is non-zero and so the fiber is non-empty. It is clear that we can check these condition on maximal ideals.

Proposition 5.52. Let $\varphi: A \rightarrow B$ be a ring homomorphism. Then $\varphi$ is flat if and only if for all $q \in \operatorname{Spec}(B)$ the map $A_{q^{c}} \rightarrow B_{q}$ is flat.

Proposition 5.53. A flat local homomorphism of local rings is faithfully flat.

Proposition 5.54. A faithfully flat homomorphism is injective.
Proof. Let $\operatorname{Ker}(\varphi) \subseteq A$. Then we get the homomorphism $\operatorname{Ker}(\varphi) \rightarrow 0$ and the induced $\operatorname{Ker}(\varphi) \otimes B \rightarrow 0$. We notice that $\operatorname{Ker}(\varphi) \otimes_{A} B=\operatorname{Ker}\left(B \rightarrow B \otimes_{A} B\right)=0$, so $\operatorname{Ker}(\varphi)=0$

Proposition 5.55. Let $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime \prime} \rightarrow 0$ be an exact sequence of $A$-module. Let $N$ be an $A$-module and assume that $M^{\prime \prime}$ is flat. Then the sequence

$$
0 \rightarrow N \otimes_{A} M \rightarrow N \otimes_{A} M \rightarrow N \otimes_{A} M^{\prime \prime} \rightarrow 0
$$

is exact.
Proof. It follows from the properties of the derived functor Tor. There's also a more elementary proof using the Snake Lemma.

Proposition 5.56. Let $M$ be a finitely generated flat module over a local noetherian ring. Then $M$ is free.

Proposition 5.57. Let $A$ be a noetherian ring and let $M$ be a finitely generated $A$-module. The following are equivalent:

1. $M$ is projective
2. $M$ is flat
3. $M_{p}$ is free $\forall p \in \operatorname{Spec}(A)$
4. $M_{p}$ is free $\forall p \in \operatorname{Spec} m(A)$

We can prove the following theorem using the concept of faithfully flat modules. We recall that given $X=\operatorname{Spec}(A)$, the structure sheaf is defined as

$$
\mathcal{O}_{X}(U)=\left\{s: U \rightarrow \sqcup A_{p} \mid s(p) \in A_{p} \forall p \in U \text { and } s \underset{\text { an element of } A_{f}}{\text { comes locally from }}\right\}
$$

Theorem 5.58. Let $A$ be a commutative ring. Then the map

$$
\begin{array}{lllc}
\phi: & A & \longrightarrow & \Gamma\left(X, \mathcal{O}_{X}\right) \\
a & \longmapsto & \left(p \mapsto a / 1 \in A_{p}\right)
\end{array}
$$

is an isomorphism.
Proof. Clearly, the map $A \rightarrow \Gamma\left(X, \mathcal{O}_{X}\right)$ is injective: if $a / 1=0$ in $A_{p}$ for all $p$, then $a=0$. Let's show surjectivity. Let $s \in \Gamma(X, \mathcal{O})$. Given an open cover $X=\cup X_{i}$, we get an injective homomorphism $\Gamma(X, \mathcal{O}) \rightarrow \prod \Gamma\left(X_{i}, \mathcal{O}\right)$; we can assume that each open set is of the form $X_{f}, f \in A$, and, by definition of the sheaf, we can choose the $f_{i}$ 's to be elements such that $s$ comes locally from an element of one of the $A_{f_{i}}$. We get the diagram


We can continue the sequence and obtain the diagram


By the requirement made on the open cover, is enough to show that the sequence $0 \rightarrow A \rightarrow \sum A_{f_{i}} \rightarrow \sum A_{f_{i} f_{j}}$ is exact. Let $B=\prod A_{f_{i}}$; the homomorphism $A \rightarrow B$ is faithfully flat since it is flat and the map on the spectra is surjective. Moreover, $A_{f_{i}} \otimes_{A} A_{f_{j}}=A_{f_{i} f_{j}}$ and so $\prod A_{f_{i} f_{j}}=B \otimes_{A} B$. The corresponding map is

$$
\begin{array}{cccc}
\psi: & B & \longrightarrow & B \otimes_{A} B \\
b & \longmapsto & b \otimes 1-1 \otimes b
\end{array}
$$

The thesis follows from the following proposition:
Proposition 5.59. Let $\varphi: A \rightarrow B$ be a faithfully flat homomorphism. Then the sequence

$$
0 \rightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} B \otimes_{A} B
$$

is exact.
Proof. First, assume that there exist $\xi: B \rightarrow A$ such that $\xi \circ \varphi=\operatorname{Id}_{A}$; then $\varphi$ is injective. Moreover, let $b \in B$ such that $b \otimes 1=1 \otimes b$. Then $\xi$ induces the map

$$
\begin{array}{cccc}
\xi \otimes \mathrm{Id} & B \otimes_{A} B & \longrightarrow & A \otimes_{A} B \simeq B \\
& b_{1} \otimes b_{2} & \longmapsto & \xi\left(b_{1}\right) \otimes b_{2} \simeq \varphi\left(\xi\left(b_{1}\right)\right) b_{2}
\end{array}
$$

Therefore, we obtain

$$
b \otimes 1=1 \otimes b \Rightarrow \xi \otimes \operatorname{Id}(b \otimes 1)=\xi \otimes \operatorname{Id}(1 \otimes b) \Rightarrow \varphi(\xi(b))=b
$$

and $b \in \operatorname{Im}(\varphi)$. we have shown in this way that $\operatorname{Im}(\varphi) \supseteq \operatorname{Ker}(\psi)$; since the other inclusion is trivial, we get the equality and the exactness of the sequence.
We now want to show that we can always reduce to this case. Let $f: A \rightarrow A^{\prime}$ be a homomorphism of rings and let $B^{\prime}$ be the tensor product $A^{\prime} \otimes_{A} B$. Then

$$
A^{\prime} \otimes_{A}\left(B \otimes_{A} B\right) \simeq\left(A^{\prime} \otimes_{A} B\right) \otimes_{A^{\prime}}\left(A^{\prime} \otimes_{A} B\right) \simeq B^{\prime} \otimes_{A^{\prime}} B^{\prime}
$$

This means that, from the homomorphism $B \rightarrow B \otimes_{A} B$, we obtain the homomorphism

$$
\begin{array}{ccc}
B^{\prime} & \longrightarrow & B^{\prime} \otimes_{A^{\prime}} B^{\prime} \\
b^{\prime} & \longmapsto & b^{\prime} \otimes 1-1 \otimes b^{\prime}
\end{array}
$$

by applying the functor $A^{\prime} \otimes_{A}$. The consequence of this reasoning is that we can reduce to the previous case if we can find a faithfully flat map $A \rightarrow A^{\prime}$ such that $A^{\prime} \rightarrow B^{\prime}$ admits a section. We have an easy choice: $A^{\prime}=B$.

So $s \in \operatorname{Im}(\phi)$ and this concludes the proof.
Definition 5.60. Let $f: X \rightarrow Y$ be a morphism of schemes. Then $f$ is flat if for all $p \in X$ the map $\mathcal{O}_{Y, f(p)} \rightarrow \mathcal{O}_{X, p}$ is flat. We say that $f$ is faithfully flat if it is flat and surjective.

In the affine case, the definition implies that the corresponding homomorphism of rings is flat.

Proposition 5.61. Let $f: X \rightarrow Y$ be a morphism of schemes. The following are equivalent:

1. $f$ is flat
2. For every $U \subseteq X, V \subseteq Y$ open affine subsets such that $f(U) \subseteq V$, $\mathcal{O}(V) \rightarrow \mathcal{O}(V)$ is flat
3. There exist open affine covers $X=\cup U_{i}, Y=\cup V_{i}$ such that $f\left(U_{i}\right) \subseteq V_{i}$ and $\mathcal{O}\left(V_{i}\right) \rightarrow \mathcal{O}\left(U_{i}\right)$ is flat.

Corollary 5.62. Let $f: X \rightarrow Y$ be a morphism of finite type. Then $f$ is flat if and only if for every closed point $q \in Y$ and for every closed point $p \in X_{q} f$ is flat at $p$.

Proof. We can assume that $X, Y$ are affine and $f: X \rightarrow Y$ corresponds to a homomorphism of rings $\varphi: A \rightarrow B$. By hypotesis, for every maximal ideal $m \in \operatorname{Spec} M(A)$ the localization $\varphi_{m}: A_{m} \rightarrow B_{m}$ is flat; since being flat is local, we get the thesis.

Proposition 5.63. Flatness is closed under composition, base change and it is local on the domain.

Proof. First, we show the statement about composition. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be flat morphism; then

$$
\mathcal{O}_{Z, g f(p)} \longrightarrow \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X, p}
$$

is a flat homomorphism of ring since the composition of flat homomorphism is flat.
We now want to deal with the base change.


We want to show that if $f$ is flat, so $f^{\prime}$ is. We can assume that $X, Y, Y^{\prime}$ are affine, so

$$
X=\operatorname{Spec}(A) \quad Y=\operatorname{Spec}(B) \quad Y^{\prime}=\operatorname{Spec}\left(B^{\prime}\right) \quad X^{\prime}=\operatorname{Spec}\left(A \otimes_{A} B^{\prime}\right)
$$

Since by hypotesis the homomorphism $B \rightarrow A$ is flat, $B^{\prime} \rightarrow A \otimes_{B} B^{\prime}$ is flat and this is true.

Corollary 5.64. Being faithfully flat is stable under base change.
Lemma 5.65. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be homomorphisms of rings and assume that $g \circ f$ is flat and $g$ is faithfully flat. Then $f$ is flat.

Proof. Let $M, M^{\prime}$ be $A$-modules and let $\varphi: M \rightarrow M^{\prime}$ be an injective homomorphism. We want to show that $\varphi \otimes \mathrm{Id}: M \otimes_{A} B \rightarrow M^{\prime} \otimes_{A} B$ is injective. We call $N=\operatorname{Ker}(\varphi \otimes \mathrm{Id})$. Tensoring by $C$, we get the sequence

$$
0 \rightarrow N \otimes_{B} C \longrightarrow M \otimes_{A} C \longrightarrow M^{\prime} \otimes_{A} C
$$

Since $g \circ f$ is flat, the second map is injective and therefore $N \otimes_{B} C=0$. Since $g$ is faithfully flat, $N=0$ and therefore $\varphi$ is injective, as wished.

Proposition 5.66. Given the cartesian diagram


If $\psi$ is faithfully flat and $f^{\prime}$ is flat, then $f$ is flat.
Proof. Let $p \in Y^{\prime}$. Since $\psi$ is surjective, there exists $q \in X$ such that $\psi(q)=$ $f(p)$. Then we get the following diagram:


Let $x \in p \times q$; considering the stalk, we get


Then $\varphi^{\#} \circ f^{\#}$ is flat and $\varphi^{\#}$ is faithfully flat: this implies that $f^{\#}$ is flat, as desired.

Proposition 5.67. Let $f: X \rightarrow Y$ be a morphism of integral schemes. If $f$ is flat then $f$ is dominant.

Proof. Let $\xi$ be the generic point of $X$. Then

$$
\mathcal{O}_{Y, f(\xi)} \rightarrow \mathcal{O}_{X, \xi}=K(X)
$$

is flat and local and therefore injective. This implies that $\mathcal{O}_{Y, f(\xi)}$ is a field and so $f(\xi)$ is the generic point of $Y$.

Proposition 5.68. Let $f: X \rightarrow Y$ be a flat morphism of schemes and suppose that $X$ is noetherian and $Y$ irreducible. Then each component of $X$ dominates $Y$.

Proof. By deleting the other components, we can assume that $X$ is irreducible. Let $\xi$ be the generic point.

$$
\mathcal{O}_{Y, f(\xi)} \longrightarrow \mathcal{O}_{X, \xi}
$$

The homomorphism is flat and local and therefore injective; furthermore, $\mathcal{O}_{X, \xi}$ is artinian. This implies that $m_{\xi}$ is nilpotent and therefore $m_{f(\xi)}$ is nilpotent. Then $\operatorname{dim}\left(\mathcal{O}_{Y, f(\xi)}\right)=0$ and so $f(\xi)$ is the generic point.

Proposition 5.69. Let $f: X \rightarrow Y$ be a morphism of scheme. Suppose $X$ is noetherian and reduced and $Y$ be regular, irreducible of dimension 1. Then $f$ is flat if and only if every irreducible component dominates $Y$.

Proof. We have already proved one implication. So, assume that every irreducible component of $X$ dominates $Y$ and let $p \in X$. Denoting by $q \in Y$ the image of $x$, we get

$$
\varphi: \mathcal{O}_{Y, q} \longrightarrow \mathcal{O}_{X, p}
$$

If $q$ has dimension 0 , then $\mathcal{O}_{Y, q}$ is a field and so the morphism is flat.
If $q$ has dimension 1, then $\mathcal{O}_{Y, q}$ is a DVR. Call $R=\mathcal{O}_{Y, q}$ and $A=\mathcal{O}_{X, p}$. It is enough to prove that $A$ is a torsion free $R$-module. Let $p_{1}, \ldots, p_{r}$ be the minimal primes of $A$. Then $\varphi^{-1}\left(p_{i}\right)=0$ since every irreducible component of $X$ dominates $Y$ and therefore $\varphi$ is injective. It is enough to show that every non-zero element of $R$ maps to a non-zero divisor in $A$. By hypotesis, $A$ is reduced and the zero divisors are the union of the minimal primes and so $A$ is flat, since it is torsion free over a PID.

Example. We consider the morphism of schemes induced by

$$
\begin{array}{ccc}
K[x] & \longrightarrow & A=K[x] /\left(y^{2}, x y\right) \\
x & \longmapsto & x
\end{array}
$$

$A$ is not torsion free over $k[x]$ since $y \neq 0$ and $x y=0$ and therefore it isn't flat. Example. We consider the normalization morphism induced by

$$
\begin{array}{rlc}
A=K[x, y] /\left(y^{2}-x^{2}(x+1)\right) & \longrightarrow & K[t] \\
x & \longmapsto & t^{2}-1 \\
y & \longmapsto t^{3}-t
\end{array}
$$

This map is not flat; in fact, let $m_{p}=(x, y)$. The map

$$
A_{m_{p}} \longrightarrow K[x]_{m_{p}}
$$

is finite and

$$
f^{-1}(p)=\operatorname{Spec}\left(K[t] /\left(t^{2}-1\right)\right)=\operatorname{Spec}(K) \sqcup \operatorname{Spec}(K)
$$

We now consider $\mathbb{N}$ with the lower-semicontinuous topology, so that the sets $(n, \infty)$ are open.

Proposition 5.70. Let $A$ be a noetherian ring and let $M$ be a finite flat module. Then the map

$$
\begin{array}{clc}
\operatorname{Spec}(A) & \longrightarrow & \mathbb{N} \\
p & \longmapsto & \operatorname{dim}_{k(p)} M \otimes_{A} k(p)
\end{array}
$$

is locally constant.
In order to show this result, we need to do some work:
Proposition 5.71. Let $A$ be a noetherian ring and let $M$ be a finitely generated $A$-module. For all $p \in \operatorname{Spec}(A)$, there exists an open neighbourhood $U$ of $p$ such that $r k_{q}(M) \leq r k_{p}(M)$ for all $q \in U$.

Proof. Let $n=r k_{p}(M)$; this means that there exist $x_{1}, \ldots, x_{n} \in M$ such that $\left(x_{i}\right)_{p}, \ldots,\left(x_{n}\right)_{p}$ generate $M_{p}$ as an $A_{p}$-module. Let $\varphi$ be the map

$$
\begin{aligned}
& \varphi: A^{n} \\
& \longrightarrow \\
& e_{i} \longmapsto \\
& x_{i}
\end{aligned}
$$

and let $Q$ be the cokernel. Since $Q_{p}=0$ and $Q$ is finitely generated, there exists $f \in A \backslash p$ such that $f Q=0$; therefore $Q_{f}=0$. The map $A_{f}^{n} \rightarrow M_{f}$ is surjective; so for all $q \in X_{f} A_{q}^{n} \rightarrow M_{q}$ is surjective and $r k_{q}(M) \leq n$.

Proposition 5.72. Let $A$ be a noetherian ring and let $M$ be a finitely generated $A$-module. The following are equivalent:

1. $M$ is flat
2. $M$ is projective
3. $M_{p}$ is a free $A_{p}$-module for all $p \in \operatorname{Spec}(A)$
4. There exist $f_{1}, \ldots, f_{r} \in A$ that generate the unit ideal such that $M_{f_{i}}$ is a free $A_{f_{i}}$ module for all $i$

Proof.
$(3) \Rightarrow$ (4) Let $p \in X=\operatorname{Spec}(A)$ and assume $M_{p}=A_{p}^{n}$ and choose $x_{1}, \ldots, x_{n} \in M$ such that $\left(x_{1}\right)_{p}, \ldots,\left(x_{n}\right)_{p}$ is a basis for $M_{p}$ (Nakayama's Lemma). We get the following exact sequence

$$
\begin{aligned}
0 \longrightarrow K \longrightarrow & A^{n} \\
e_{i} & \longmapsto
\end{aligned} \quad \longrightarrow \quad x_{i} \quad l \longrightarrow 0
$$

we know that $K, Q$ are finitely generated and $K_{p}=Q_{p}=0$. This means that there exists $f \in A \backslash p$ such that $K_{f}=Q_{f}=0$ and so $A_{f}^{n} \simeq M_{f}$. By the quasi-compactness of the spectrum, we can cover $X$ with finitely many $X_{f}$.

Corollary 5.73. Let $A$ be a noetherian ring and let $M$ be a finitely generated flat $A$-module. Then $p \mapsto r k_{p}(M)$ is locally constant.

Unfortunately, the converse doesn't hold. For example, consider the ring $A=K[x] /\left(x^{2}\right)$ an $M=K$ as an $A$-module. Since it is local of dimension zero, there is only one prime ideal and so the function is locally constant; however, is isn't flat.

Lemma 5.74. Let $A \rightarrow B$ be a homomorphism of ring and let $q \in \operatorname{Spec}(B)$. Then $r k_{q^{c}}(M)=r k_{q}\left(M \otimes_{A} B\right)$.

Proposition 5.75. Let $A$ be a reduced noetherian ring and let $M$ be a finitely generated $A$-module. If the map $p \mapsto r k_{p}(M)$ is locally constant, $M$ is flat.

Proof. Let $p \in X=\operatorname{Spec}(A)$. We want to show that $M_{p}$ is free over $A_{p}$. We can assume that $A$ is local reduced and $p$ is its maximal ideal. Let $n=r k_{p}(M)$; we get the following exact sequence:

$$
0 \rightarrow K \rightarrow A^{n} \rightarrow M \rightarrow 0
$$

Since $A$ is reduced, the intersection of the minimal primes $p_{1}, \ldots, p_{r}$ is zero and $A_{p_{i}}$ are fields, since they are artinian and reduced. For all $i$, localization give rise to the exact sequence

$$
0 \rightarrow K_{p_{i}} \rightarrow A_{p_{i}}^{n} \rightarrow M_{p_{i}} \rightarrow 0
$$

and $M_{p_{i}}$ is free of rank $n$ on $A_{p_{i}}$; therefore $K_{p_{i}}=0$ for all $i$. Clearly, for all $i$ we have an inclusion map

$$
\varphi_{i}: A \rightarrow A_{p_{i}}
$$

and the kernel is contained in $p_{i}$; in fact, the kernel is the set $\operatorname{Ker}\left(\varphi_{i}\right)=$ $\left\{a \in A \mid \exists s \in A \backslash p_{i}\right.$ s.t. $\left.s a=0\right\} ;$ since $0 \in p_{i}$ and $s \notin p_{i}$, this implies $a \in p_{i}$. So the function

$$
A \rightarrow A_{p_{1}} \oplus A_{p_{2}} \oplus \ldots A_{p_{n}}
$$

is injective. In particular, we get a map $A^{n} \rightarrow \oplus A_{p_{i}}^{n}$; by restriction,

$$
K \hookrightarrow \underbrace{K_{p_{1}} \oplus K_{p_{2}} \oplus \cdots \oplus K_{p_{n}}}_{=0}
$$

and so $K=0$.
Proposition 5.76. Let $A$ be a Jacobson noetherian ring and let $M$ be a finite $A$-module. If the map

$$
\begin{array}{ccc}
\text { Spec } M(A) & \longrightarrow & \mathbb{N} \\
p & \longmapsto & \operatorname{dim}_{k(p)} M \otimes k(p)
\end{array}
$$

is locally constant, then the extension to the prime spectrum is locally constant.
Proof. First of all, we notice that $\operatorname{Spec}(A)$ is connected if and only if $\operatorname{Spec} M(A)$ is connected. We can assume that $\operatorname{Spec}(A)$ is connected. Let $p \in \operatorname{Spec} M(A)$ and let $n \in \mathbb{N}$ be its image. Then the set $S=\left\{q \in X \mid r k_{q}(M) \leq n-1\right\}$ is closed in $X$; since $\operatorname{Spec} M(A)$ is dense in every closed subset, either $S=\emptyset$ or there exists $q \in \operatorname{Spec} M(A)$ such that $\operatorname{rk}_{q} M=n-1$. Since the rank map is constant on Spec $M(A)$, the second is impossible and therefore $S=\emptyset$.

Let $f: X \rightarrow Y$ be a finite map and suppose that $Y$ is locally noetherian and that $X, Y$ are affine. Then for all $q \in Y f^{-1}(q)=\operatorname{Spec}\left(A_{q}\right)$ and $A_{q}$ is a finitely generated $k(q)$-algebra; in particular, $A_{q}=A \otimes_{B} k(q)$ and in a certain sense
this counts the number of points in the fiber, counted with multiplicity. For example, let $p \in K[t]$ and consider the homomorphism of rings

$$
\begin{array}{clc}
K[x] & \longrightarrow & K[t] \\
x & \longmapsto p(t)
\end{array}
$$

It induces a morphism $f: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$; in this case, $K[t]$ is free over $K[x]$ and so $f$ is flat. If $q=(x-a) \in \mathbb{A}^{1}(K)$

$$
f^{-1}(q)=\operatorname{Spec} K[t] /(p(t)-a)
$$

which has dimension $\operatorname{deg}(p)$ over $K$. Summing up, we have shown that if $f: X \rightarrow$ $Y$ is a morphism of schemes, where $Y$ is locally noetherian and $f$ is flat, then the $\operatorname{map} q \mapsto \operatorname{dim}_{k(q)} A_{q}$ is locally constant. Viceversa, if $X$ is reduced and the map is locally constant, $f$ is flat. Note that a normalization is almost never flat. For example, let $Y$ be the spectrum of $K[x, y] /\left(x^{3}-y^{2}\right)$ and let $X$ be its normalization. Then $f^{-1}(x, y)=\operatorname{Spec}\left(K[t] /\left(t^{2}, t^{3}\right)\right)=\operatorname{Spec}\left(K[t] /\left(t^{2}\right)\right)$ which has dimension two over $K$; however, the other fibers have dimension one.
Another example of a non-flat morphism is the morphism induced by the map

$$
\begin{array}{clc}
K[u, v] & \longrightarrow & K[x, y] \\
u & \longmapsto & x \\
v & \longmapsto & x y
\end{array}
$$

which is a dominant morphism of integral, noetherian, regular schemes. This can't be flat since the restriction to the horizontal line is not flat:

$$
f^{-1}(\operatorname{Spec}(K[u, v] /(v))) \simeq \operatorname{Spec}(K[x, y] /(x y))
$$

and the map

$$
\begin{array}{ccc}
K[u] & \longrightarrow & K[x, y] /(x y) \\
u & \longmapsto & x
\end{array}
$$

is not flat (it isn't torsion free).
Theorem 5.77. Let $f: X \rightarrow Y$ be a morphism of locally noetherian schemes. Given $p \in X$, let $q=f(p)$ and let $X_{q}=f^{-1}(q)=\operatorname{Spec}(k(q)) \times_{Y} X$ be the fiber. Then

$$
\operatorname{dim} \mathcal{O}_{X_{q}, p} \geq \operatorname{dim} \mathcal{O}_{X, p}-\operatorname{dim} \mathcal{O}_{Y, q}
$$

If $f$ is flat, equality holds.
Let $A$ be $\mathcal{O}_{X, q}$ and $B$ be $\mathcal{O}_{Y, p}$; then

$$
\mathcal{O}_{f^{-1}(q), p}=\mathcal{O}_{Y, p} \otimes_{\mathcal{O}_{X, p}} k(q) \simeq B / \mathfrak{M} A
$$

So the proof of the theorem is equivalent to the one of the following lemma
Lemma 5.78. Let $f: A \rightarrow B$ be a local homomorphism of local noetherian rings. Then

$$
\operatorname{dim}(B / \mathfrak{M} B) \geq \operatorname{dim}(A)-\operatorname{dim}(B)
$$

If $f$ is flat, equality holds.

Theorem 5.79. Let $f: X \rightarrow Y$ be a morphism of locally noetherian regular schemes. If

$$
\operatorname{dim} \mathcal{O}_{X_{q}, p} \geq \operatorname{dim} \mathcal{O}_{X, p}-\operatorname{dim} \mathcal{O}_{Y, q}
$$

for all $p \in X$, then $f$ is flat.
Theorem 5.80. Let $f: A \rightarrow B$ be a local flat homomorphism of noetherian local rings.

- If $B$ is regular, then $A$ is regular
- If $B$ is normal, then $A$ is normal

Proposition 5.81. Let $f: X \rightarrow Y$ be a faithfully flat morphism of locally noetherian schemes.

- If $X$ is regular, $Y$ is regular
- If $X$ is normal, $Y$ is normal.

Proposition 5.82. Let $f: X \rightarrow Y$ be a morphism of locally noetherian schemes and assume

$$
\operatorname{dim} \mathcal{O}_{f^{-1}(p), q}=\operatorname{dim} \mathcal{O}_{X, q}-\operatorname{dim} \mathcal{O}_{Y, p}
$$

then $f$ is faithfully flat.
Recall that the map $f: X_{\bar{k}} \rightarrow X$ is always faithfully flat; furthermore, if $p \in X$ is closed, every $q \in f^{-1}(q)$ is closed.

Proposition 5.83. Let $X \rightarrow K$ be a zero dimensional scheme locally of finite type. Then $X$ has the discrete topology $X=\sqcup X_{i}$ and each $X_{i}$ is the spectrum of an Artinian local ring $A_{i}$.

Proof. Let $U=\operatorname{Spec}(A)$ be an open affine subset; then $A$ is 0-dimensional noetherian $K$-algebra and therefore $U$ has the discrete topology. Since $U$ is open, every point is open in $X$ and $X$ has the discrete topology. Therefore $X=\sqcup_{p \in X} \operatorname{Spec}\left(\mathcal{O}_{X, p}\right)$ and this gives the thesis.

Proposition 5.84. Let $K^{\prime}$ be an extension field of $K$ and assume $X \rightarrow \operatorname{Spec}(K)$ is locally of finite type. Then $X_{K^{\prime}}$ is smooth over $K^{\prime}$ if and only if $X$ is smooth over $K$.

Proof. Let $\bar{K}$ be the algebraic closure of $K$ in $\bar{K}^{\prime}$. We get the diagram

and we know that $X_{K^{\prime}}$ is smooth if and only if $X_{\bar{K}^{\prime}}$ is regular. This implies that $X_{\bar{K}}$ is regular by faithfully flatness and this is equivalent to say that $X$ is smooth.
Viceversa, assume that $X$ is smooth and affine, so $X=\operatorname{Spec}(A)$. Since $X$ is regular, $\mathcal{O}_{X, p}$ is a domain for all $p \in X$ and therefore $X=\sqcup X_{i}$, where every $X_{i}$ is integral. As a consequence, we can reduce to the case of $X$ reduced and irreducible, so $A$ is a domain of dimension $d$. Since it is locally of finite type,

$$
A=K\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{r}\right)
$$

and therefore $X \subseteq \mathbb{A}_{K}^{n}$. The Jacobian Criterion gives us the equivalence

$$
X \text { smooth } \Longleftrightarrow \forall p \in X \text { closed, } \operatorname{rk} J_{f}(p)=n-d
$$

We have to prove that the dimension of every irreducible component of $X_{K^{\prime}}$ is $d$. We call $\varphi$ the projection $X_{\bar{k}^{\prime}} \rightarrow X$. By base change, we know that $\varphi^{-1}(p)=\operatorname{Spec}\left(\bar{k}^{\prime} \otimes k(p)\right)$. If $p$ is closed, we know that $k(p) / k$ is finite and therefore $\varphi^{-1}(p)$ is artinian: this implies that all points $q \in \varphi^{-1}(p)$ are closed. So $\operatorname{dim} \mathcal{O}_{\varphi^{-1}(p), q}=0$ and since the map is flat we get the equality $\operatorname{dim} \mathcal{O}_{X, p}=$ $\operatorname{dim} \mathcal{O}_{X_{K^{\prime}}, q}$. We want now to prove that $\operatorname{dim} \mathcal{O}_{X_{K^{\prime}}, q}=d$ for all $p \in X$ closed. $\bar{K}^{\prime}$ is trascendent over $\bar{K}$ and so $\bar{K}^{\prime} \otimes A$ is a finitely generated domain over $\bar{K}^{\prime}$ and therefore every maximal ideal has the same height. This gives the thesis.

Definition 5.85. Let $f: X \rightarrow Y$ be a morphism of schemes and assume $Y$ is locally noetherian. We say that $f$ is smooth if $f$ is locally of finite type, flat and for all $q \in X, f^{-1}(q)$ is smooth over $k(q)$.

Proposition 5.86. Being smooth is stable under base change, composition and it is local on $X$.

Definition 5.87. A morphism of schemes $f: X \rightarrow Y$ is ètale if it is smooth of relative dimension zero (the fibers are zero-dimensional).

For example, let $X=\operatorname{Spec}(K[t])$ and let $Y=\operatorname{Spec}(K[x])$. Let $f(t)$ be a non constant polynomial and consider the morphism of schemes induced by the homomorphism

$$
\begin{array}{cll}
K[x] & \longrightarrow & K[t] \\
x & \longmapsto f(t)
\end{array}
$$

This morphism is flat since $K[t]$ is free of $\operatorname{rank} \operatorname{deg}(f)$ over $K[x]$. We can see $K[t]$ as the ring $K[x, t] /(x-f(t))$; then, given $p \in Y$,

$$
f^{-1}(p)=\operatorname{Spec}\left(K[t, x] /(x-f(t))^{\left.\otimes_{K[x]} k(p)\right)}\right.
$$

If $p$ is the generic point of $X$, then $k(p)=K(X)$; so

$$
f^{-1}(p)=\operatorname{Spec}\left(K[t, x] /(x-f(t)) \otimes_{K[x]} K(X)\right) \simeq \operatorname{Spec}(K(x)[t] /(x-f(t)))
$$

Since $x-f(t)$ is irreducible, $f^{-1}(p)$ is smooth if and only if $\left(f^{\prime}(t), x-f(t)\right)=(1)$ if and only if $f^{\prime}(t) \neq 0$ (we are assuming that $\operatorname{char}(K)=0$ or $\operatorname{char}(K)>0$ and $f$ separable) if and only if $k(t) / k(x)$ is separable. The morphism $\mathbb{A}_{k}^{1} \backslash V\left(f^{\prime}(t)\right) \rightarrow$ $\mathbb{A}_{K}^{1}$ is ètale.

Theorem 5.88. Let $f: X \rightarrow Y$ be a locally of finite type morphism and assume that $Y$ is locally noetherian. The map

$$
\begin{array}{ccc}
X & \longrightarrow & \mathbb{N} \\
p & \longmapsto & \operatorname{dim}_{p}\left(f^{-1}(f(p))\right)
\end{array}
$$

is upper semicontinuous.
Proposition 5.89. Let $f: X \rightarrow Y$ be a morphism of schemes and assume that $X, Y$ are locally of finite type over $k$. Suppose that for all closed points $q \in Y$ the fiber $X_{q}=f^{-1}(q)$ is smooth at its closed points. Then $f$ is smooth.

Proof. We have shown that under these hypotesis the map is flat. Furthermore, we know that $X_{q}$ is smooth if $\left(X_{q}\right)_{\bar{k}}$ is regular at its closed points. Therefore $X_{q}$ is smooth for every closed point $q \in Y$. We can assume that $X, Y$ are affine. So we can consider a homomorphism $g: A \rightarrow B$. Let $q \in \operatorname{Spec}(A)$. We have to show that $f^{-1}(q)=B \otimes k(p)$ is smooth over $k(p)$, or equivalently that

$$
f^{-1}(q)_{\overline{k(p)}}=B \otimes k(p) \otimes \overline{k(p)}=B \otimes_{A} \overline{k(p)}
$$

is regular. By hypotesis, $B \otimes \bar{k}$ is regular and $\overline{k(p)} / \bar{k}$ is not algebraic; by the previous proposition, $f^{-1}(q)$ is regular, as wished.

Suppose that $K$ is an algebraically closed field and let $X$ be a scheme locally of finite type over $K$. Let $p \in X(K)$ be a rational point.

Definition 5.90. The tangent space of $X$ at $p$ is

$$
T_{p} X=\left(m_{p} / m_{p}^{2}\right)
$$

Let $f: X \rightarrow Y$ be a morphism and let $q$ be the image of $p$. This induces a morphism over the cotangent space:

$$
f^{*}: m_{q} / m_{q}^{2} \longrightarrow m_{p} / m_{p}^{2}
$$

So we get a map between the tangent spaces, the differential:

$$
d_{p}(f):=\left(f^{*}\right)^{v}: T_{p} X \longrightarrow T_{p} Y
$$

For example, if $X=\mathbb{A}_{k}^{n}$ and $p=\left(x_{1}-a_{1}, \ldots, x_{n}-a_{n}\right), T_{p} \mathbb{A}_{k}^{n}$ has a basis formed by $\left(\partial / \partial x_{i}\right)(a)$
Observation 5.91. In the case of a composite, $d(f g)_{p}=d g_{f(p)} \circ d f_{p}$.
Proposition 5.92. Let $f: X \rightarrow Y$ be a morphism of schemes and let $p \in X(K)$. Let $q$ be the image of $p$ and let $X_{q}=f^{-1}(q) \stackrel{j}{\subseteq} X$. Then we get an exact sequence

$$
0 \rightarrow T_{p} X_{q} \longrightarrow T_{p} X \xrightarrow{d f_{p}} T_{p} Y
$$

Proof. Let $A=\mathcal{O}_{Y, q}, B=\mathcal{O}_{X, p}$ and $C=\mathcal{O}_{X_{q}, p}=B / m_{A} B$. So it is enough to show that the sequence

$$
m_{A} / m_{A}^{2} \longrightarrow m_{B} / m_{B}^{2} \longrightarrow m_{C} / m_{C}^{2} \rightarrow 0
$$

is exact.

Let $X \subseteq \mathbb{A}^{n}$ be a closed subscheme. Then

$$
X=\operatorname{Spec}\left(K\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{m}\right)\right)
$$

and let $p$ be a rational point of $X$. We can consider the map $\phi: \mathbb{A}^{n} \rightarrow \mathbb{A}^{m}$ induced by the homomorphism

$$
\begin{array}{ccc}
K\left[x_{1}, \ldots, x_{m}\right] & \longrightarrow & K\left[x_{1}, \ldots, x_{n}\right] \\
x_{i} & \longmapsto & f_{i}
\end{array}
$$

Then $f^{-1}(0)=X$ and $\operatorname{dim} T_{p} X=n-r k\left(d f_{p}\right)=n-r k\left(\mathcal{J}_{f}(p)\right)$.
Proposition 5.93. Let $K$ be an algebraically closed field and let $f: X \rightarrow Y$ be a morphism of schemes over $K$. Assume that $X, Y$ are regular and for all $p \in X$ consider the map $d f_{p}: T_{p} X \rightarrow T_{f(p)} Y$. Then

- $f$ is smooth if and only if $\forall p \in X$ the map $d f_{p}$ is surjective
- $f$ is étale if and only if $\forall p \in X$ the map $d f_{p}$ is an isomorphism

Proof. We know that $f$ is flat if and only if for all closed points $p \in X$ we have the equality $\operatorname{dim}_{p} f^{-1}(f(p))=\operatorname{dim}_{p} X+\operatorname{dim}_{f(p)} Y$. Therefore, $f$ is smooth if and only if the fibers are regular and

$$
\operatorname{dim}_{p} f^{-1}(f(p))=\operatorname{dim}_{p} X+\operatorname{dim}_{f(p)} Y
$$

We know that $\operatorname{dim} T_{p} f^{-1}(f(p)) \geq \operatorname{dim}_{p} f^{-1}(f(p))$ and we have the equality if and only if $f^{-1}(f(p))$ is regular at $p$; since the field is algebraically closed, this holds if and only if $f^{-1}(f(p))$ is smooth at $p$. Therefore, $f$ is smooth if and only if

$$
\operatorname{dim} T_{p} f^{-1}(f(p))=\operatorname{dim}_{p} X-\operatorname{dim}_{f(p)} Y=\operatorname{dim} T_{p} X-\operatorname{dim} T_{p} Y
$$

and this is equivalent to say that $d f_{p}$ is surjective.
Notice now that a map is etalé if and only if the fibers have dimension zero and therefore the second statement holds.

## Chapter 6

## Coherent Sheaves and Cohomology

### 6.1 Quasi-Coherent Sheaves

Definition 6.1. Let $\left(X, \mathcal{O}_{X}\right)$ be a ringed space. A sheaf of $\mathcal{O}_{X}$-modules is a sheaf of abelian groups $F$ togheter with a structure of $\mathcal{O}_{X}(U)$-module on $F(U)$ for all $U \subseteq X$ open subsets such that if $V \subseteq U$ is an open set and $s \in \mathcal{O}(U)$, $f \in F(U)$, we get $\left.(f s)\right|_{V}=\left.\left.f\right|_{V} s\right|_{V}$.

Example.

1. Given a sheaf $\mathcal{O}_{X}$, every sheaf of ideal is an $\mathcal{O}_{X}$-module.
2. Let $X$ be a topological space, $Y$ a closed subspace and let $\mathcal{C}_{X}$ be the sheaf of continuous functions. For every open set $U$, we define the ideal $I_{Y}(U)=\left\{f \in \mathcal{C}_{X}|f|_{U \cap Y}=0\right\} ;$ this gives a sheaf of $\mathcal{C}_{X}$-module.
3. Let $X$ be a $\mathcal{C}^{\infty}$ manifold. The sheaf $\Omega_{X}^{i}$ of $i$-forms on $X$ is a sheaf of $\mathcal{C}_{X}^{\infty}$-modules.

We notice that if $f:\left(X, \mathcal{O}_{X}\right) \rightarrow\left(Y, \mathcal{O}_{Y}\right)$ is a morphism of ringed spaces and $F$ is a sheaf of $\mathcal{O}_{X}$-modules, $f_{*} F$ is naturally a sheaf of $\mathcal{O}_{Y}$-modules. We can also define presheaves of $\mathcal{O}_{X}$-modules and they form a category. Furthermore, they are an abelian category: in fact, given a morphism of presheaves of $\mathcal{O}_{X}$-module $\varphi: F \rightarrow G$, the kernel, the image and the cokernel have a natural structure of presheaves of $\mathcal{O}_{X}$-modules. Sheaves of $\mathcal{O}_{X}$-modules define a full subcategory; moreover, the following holds:

Proposition 6.2. Let $F$ be a presheaf of $\mathcal{O}_{X}$-modules. The sheafification $F^{s h}$ has a unique structure of sheaf of $\mathcal{O}_{X}$-modules such that $F \rightarrow F^{s h}$ is $\mathcal{O}_{X}$-linear. If $F \rightarrow G$ is an $\mathcal{O}_{X}$-linear morphism and $G$ is a sheaf of $\mathcal{O}_{X}$-modules, then $F^{s h} \rightarrow G$ is $\mathcal{O}_{X}$-linear.

As a corollary, the category of $\mathcal{O}_{X}$-modules is abelian.
Let $X=\operatorname{Spec}(A)$ be an affine scheme and let $M$ be an $A$-module. We want to construct a sheaf $\tilde{M}$ of $\mathcal{O}_{X}$-modules on $X$. We want this construction to be in
a certain sense natural, so that if $M=A$ we obtain $\tilde{A}=\mathcal{O}_{X}$. As in the case of the structure sheaf on the spectrum of a ring, we consider the presheaf

$$
\tilde{\tilde{M}}(U):=\left\{s: U \rightarrow \sqcup_{p \in U} M_{p} \mid s(p) \in M_{p} \forall p \in U\right\}
$$

We define

$$
\tilde{M}:=\left\{s \in \tilde{\tilde{M}} \mid \forall p \in U \exists f \in A, m \in M_{f} \text { s.t. }\left\{\begin{array}{l}
X_{f} \subseteq U \\
\forall q \in X_{f} s(q)=m_{q} \in M_{q}
\end{array}\right\}\right.
$$

This is a subsheaf of $\tilde{\tilde{M}}(U)$ of $\mathcal{O}_{X}$-modules.
Proposition 6.3. Let $p \in X$. Then the map

$$
\begin{array}{ccc}
\tilde{M}_{p} & \longrightarrow & M_{p} \\
{[s]} & \longmapsto & s(p)
\end{array}
$$

is an isomorphism.
Lemma 6.4. Let $f: A \rightarrow B$ be a faithfully flat morphism and let $M$ be an $A$-module. Then the sequence

$$
M \otimes_{A} A \longrightarrow M \otimes_{A} B \longrightarrow M \otimes_{A} B \otimes_{A} B
$$

is exact.
Proposition 6.5. The map

$$
\begin{array}{lll}
M & \longrightarrow & \Gamma(X, \tilde{M}) \\
M & \longmapsto & \left(p \mapsto m_{p}\right)
\end{array}
$$

is an isomorphism.
Operations on $\mathcal{O}_{X}$-modules Let $\left(F_{i}\right)_{i \in I}$ be a collection of $\mathcal{O}_{X}$-modules. Then the product $\prod F_{i}$ is an $\mathcal{O}_{X}$-module. The direct sum $\sum F_{i} \subseteq \prod F_{i}$ is an $\mathcal{O}_{X}$-submodule. They have the usual universal property.
We can also define the tensor product. Let $F, G$ be sheaves of $\mathcal{O}_{X}$-modules. For every open set $U$, we define

$$
\left(F \otimes_{\mathcal{O}_{X}} G\right)(U):=F(U) \otimes_{\mathcal{O}_{X}(U)} G(U)
$$

If $V \subseteq U$, then we have a bilinear map

$$
F(U) \times G(U) \longrightarrow F(V) \times G(V) \longrightarrow F(V) \otimes_{\mathcal{O}_{X}(V)} G(V)
$$

By the universal property of the tensor product, we get the restriction map

$$
F(U) \otimes_{\mathcal{O}_{X}(U)} G(U) \longrightarrow F(V) \otimes_{\mathcal{O}_{X}(V)} G(V)
$$

This gives a presheaf $F \tilde{\otimes}_{\mathcal{O}_{X}} G$; we define the tensor product $F \otimes_{\mathcal{O}_{X}} G$ to be its sheafification.

Proposition 6.6. Let $p$ be a point of $X$ and let $F, G$ be $\mathcal{O}_{X}$-modules. Then

$$
\left(F \otimes_{\mathcal{O}_{X}} G\right)_{p}=F_{p} \otimes_{\mathcal{O}_{X, p}} G_{p}
$$

Proposition 6.7. Let $A$ be a ring and let $M$ be an $A$-module. If $F$ be a sheaf of $\mathcal{O}_{X}=\mathcal{O}_{\mathrm{Spec}(A)}$-modules then there is an isomorphism

$$
\operatorname{Hom}_{\mathcal{O}_{X}}(\tilde{M}, F) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X}(X)}(M, F(X))
$$

Proof. First, we want to show injectivity. Let $\varphi: M \rightarrow F(X)$ be an $\mathcal{O}_{X}(X)$ linear map and let $p \in X$. We get the diagram

and since the factorization of $M \rightarrow F_{p}$ to $M_{p} \rightarrow F_{p}$ is unique, we get injectivity. We now want to show surjectivity. Let $U \subseteq X$ be an open subset; we get a $\operatorname{map} \tilde{M}(U) \rightarrow F(U)$. Let $s \in \tilde{M}(U)$; then $s: U \rightarrow \sqcup M_{p} \rightarrow \sqcup F_{p}$. We claim that the image of $s$ is contained in $F(U) \subseteq \sqcup F_{p}$. Since it is a local problem, we can assume $U=X_{f}$ and $s \in M_{f}$. The thesis follows from the commutative diagram:


Definition 6.8. A sheaf of $\mathcal{O}_{X}$-modules is quasi-coherent if it is isomorphic to one of the form $\tilde{M}$.

We have the following criterion to determine if a sheaf of $\mathcal{O}_{X}$-modules is quasi-coherent:

Proposition 6.9. Let $F$ be a sheaf of $\mathcal{O}_{X}$-modules. The following are equivalent:

1. $F$ is quasi-coherent
2. For all $f \in A, F(X)_{f} \rightarrow F\left(X_{f}\right)$ is an isomorphism
3. For all $p \in X, F(X)_{p} \rightarrow F_{p}$ is an isomorphism

We have also shown that
Proposition 6.10. Let $\varphi: M \underset{\sim}{\sim} N$ be a homomorphism of $A$-modules. Then there exists a unique $\tilde{\varphi}: \tilde{M} \rightarrow \tilde{N}$ morphism of $\mathcal{O}_{X}$-modules such that the following diagram commutes


The proposition defines a functor from $A$-module to $\mathcal{O}_{X}$-modules and it gives an equivalence between $A$-modules and quasi-coherent sheaves on $X$.

Proposition 6.11. The functor is exact: if $M^{\prime} \rightarrow M \rightarrow M^{\prime \prime}$ is exact, then $\tilde{M}^{\prime} \rightarrow \tilde{M} \rightarrow \tilde{M}^{\prime \prime}$ is exact.

Proof. It is enough to check on the stalks.
Corollary 6.12. Let $\varphi: F \rightarrow G$ be a morphism of quasi-coherent sheaves on $X$. Then $\operatorname{Ker}(\varphi), \operatorname{Coker}(\varphi), \operatorname{Im}(\varphi)$ are quasi-coherent.
Proposition 6.13. Let $\left(M_{i}\right)_{i \in I}$ be $A$-modules. Then $\widetilde{\oplus M_{i}} \simeq \oplus \tilde{M}_{i}$.
Proof. By sheafification, we get a map $\oplus M_{i} \rightarrow \Gamma\left(X, \oplus \tilde{M}_{i}\right)$. By passing to stalks, we obtain a map $\left(\oplus M_{i}\right)_{p} \rightarrow \oplus\left(M_{i}\right)_{p}$ which is clearly an isomorphism for all $p$.

Corollary 6.14. Direct sums of quasi-coherent sheaves are quasi-coherent.
Proposition 6.15. Let $M, N$ be $A$-modules. Then $\widetilde{M \otimes_{A} N} \simeq \tilde{M} \otimes_{\mathcal{O}_{X}} \tilde{N}$.
Proof. By definition of the tensor product, we get a map $M \otimes N \rightarrow \Gamma\left(X, \tilde{M} \otimes \mathcal{O}_{X}\right.$ $\tilde{N})$. On stalks we obtain a map $(M \otimes N)_{p} \rightarrow\left(\tilde{M} \otimes_{\mathcal{O}_{X}} \tilde{N}\right)_{p}$ and they are both isomorphic to $M_{p} \otimes_{A_{p}} N_{p}$.
Corollary 6.16. Tensor products of quasi-coherent sheaves are quasi-coherent.
Quasi-coherent sheaves are closed under the operations that commutes with localization. For example, it doesn't commute with infinite product.
Example. Let $X=\operatorname{Spec}(k[t])$ be the affine line; then the sheaf of $\mathcal{O}_{X}$-modules $\mathcal{O}_{X}^{\mathbb{N}}=\prod \mathcal{O}_{X}$ is not quasi-coherent. In fact, if we consider the stalk on 0 , we get

$$
\varphi: k[t]_{(t)}^{\mathbb{N}} \longrightarrow\left(\mathcal{O}_{X}^{\mathbb{N}}\right)_{0}
$$

The map is not surjective: the element

$$
\left(\frac{1}{(t-1)}, \frac{1}{(t-1)^{2}}, \ldots\right) \notin \operatorname{Im}(\varphi)
$$

because the powers of the denominators are not bounded (the tensor product allows only finite sums!).

Notice that we have a bijection between quasi-coherent sheaves of ideals and ideals in $A$. Not all sheaves of ideals define a quasi-coherent sheaf:
Example. Let $X=\mathbb{A}_{k}^{1}$ be the affine line an consider the collection of ideals $I_{n}=\left(t^{n}\right) \subseteq k[t]$. Then $\left(\cap \tilde{I}_{j}\right)(U)=\cap\left(I_{j}(U)\right)$ defines a sheaf of ideals; however it is not quasi-coherent. Indeed, $\left(\cap_{n} \tilde{I}_{n}\right)(X)=\cap\left(t^{n}\right)=(0)$, while $\left(\cap \tilde{I}_{n}\right)\left(X_{t}\right)=$ $\mathcal{O}\left(X_{t}\right)$, since $\tilde{I}_{n}\left(X_{t}\right)=\mathcal{O}\left(X_{t}\right)$ for all $n$.

Proposition 6.17. Let $X=\operatorname{Spec}(A)$ be an affine scheme and let $U \subseteq X$ be an open affine subscheme. Let $F$ be a quasi-coherent sheaf on $X$. Then $\left.F\right|_{U}$ is quasi-coherent.

Proof. Let $B=\mathcal{O}(U)$; the inclusion $U \rightarrow X$ induces a homomorphism of rings $A \rightarrow B$. Furthermore, we know that for each $p \in U$ the map on the stalks $A_{p} \rightarrow B_{p}$ is an isomorphism. Called $M=F(X)$, we want to show that $\left.F\right|_{U}=$ $\widetilde{M \otimes_{A} B}$. We have the composite map

$$
M \longrightarrow F(X) \longrightarrow F(U)
$$

By extension of scalar, we obtain a homomorphism of $B$-module

$$
M \otimes_{A} B \rightarrow F(U)
$$

and our purpose is to show that it is an isomorphism. Localizing at $p \in U$, we get

$$
\left(M \otimes_{A} B\right)_{p} \simeq M_{p} \quad F(U)_{p} \simeq F_{p} \simeq M_{p}
$$

and this concludes the proof.
Proposition 6.18. Let $X=\operatorname{Spec}(A)$ be an affine scheme and let $F$ be a sheaf of $\mathcal{O}_{X}$-modules. Let $X=\cup X_{i}$ be an open affine cover and assume that $\left.F\right|_{X_{i}}$ is quasi-coherent. Then $F$ is quasi-coherent.

Proof. Since $X$ is quasi-compact, we can assume that the cover is finite. We can reduce to the case $X=X_{1} \cup X_{2}$ and $\left.F\right|_{X_{0}}=\tilde{M}_{0},\left.F\right|_{X_{1}}=\tilde{M}_{1}$ are quasi coherent. Since the restriction of a quasi-coherent sheaf is quasi-coherent and affine schemes are separated, we can take a module $N$ such that $\tilde{N}=\left.F\right|_{X_{0} \cap X_{1}}$. We get the maps $M_{1} \rightarrow N$ and $M_{2} \rightarrow N$. We call $M$ the pullback of these maps; then $M \simeq F(X)$ and therefore $F=\tilde{M}$.

Corollary 6.19. Let $X$ be a scheme and let $F$ be a sheaf of $\mathcal{O}_{X}$-modules. The following are equivalent:

1. For all $U \subseteq X$ open affine, $\left.F\right|_{U}$ is quasi-coherent
2. There exists an open affine cover $X=\cup X_{i}$ such that $\left.F\right|_{X_{i}}$ is quasicoherent for all $i$.

Definition 6.20. Let $X$ be a scheme and let $F$ be a sheaf of $\mathcal{O}_{X}$-module. $X$ is quasi-coherent if it satisfies one of the equivalent condition of the corollary.

Example. Let $Y \subseteq X$ be a closed subscheme; so the map $\mathcal{O}_{X} \xrightarrow{j^{\#}} j_{*} \mathcal{O}_{Y}$ is surjective and we have the sheaf of ideal $I_{Y}=\operatorname{Ker}\left(j^{\#}\right)$. Suppose that $X$ is affine, so $X=\operatorname{Spec}(A)$ and $Y=\operatorname{Spec}(A / I)$. Then $I_{Y}=\tilde{I}$. Notice that we are using the lemma

Lemma 6.21. Let $A \rightarrow B$ be a homomorphism of rings and let $f$ be the induced map on the schemes. Let $M$ be a $B$-module and let $M_{A}$ be $M$ considered as an $A$-module. Then $\tilde{M}_{A} \simeq f_{*} \tilde{M}$.
$I_{Y}$ is quasi-coherent; this gives a bijection between quasi-coherent sheaves of ideals and closed subschemes. It is trivial when $X$ is affine; in general, we consider an open affine cover $X=\cup X_{i}$. Then the restriction of the sheaf of ideals gives a closed subscheme $Y_{i}$ in every $X_{i}$. Furthermore, $Y_{i} \cap\left(X_{i} \cap X_{j}\right)=$ $Y_{j} \cap\left(X_{i} \cap X_{j}\right)$. This gives a closed subscheme $Y \subseteq X$ such that $Y \cap X_{i}=Y_{i}$.

The graded case Let $A$ be a graded ring $A=\oplus_{i=0}^{\infty} A_{i}$. We recall that a graded $A$-module $M$ is an $A$-module such that

- $M=\oplus_{i \in \mathbb{Z}} M_{i}$
- $A_{i} M_{j} \subseteq M_{i+j}$

For example, every homogeneous ideal gives a graded $A$-module. Given a graded $A$-module $M$ and an integer $d \in \mathbb{Z}$, we can obtain a new graded module $M(d)$ by shifting

$$
M(d)=M \quad M(d)_{i}=M_{i+d}
$$

Similarly to the ring case, we can define the homogeneous localization. Given $S \subseteq A$ a homogeneous multiplicative system, we can consider $\left(S^{-1}\right) M \subseteq$ $\left(S^{-1} M\right)_{0}$; if $f \in A$, then $M_{(f)}=\left(\left\{f^{i}\right\}\right) M$.
Example. Let $A=R\left[x_{0}, \ldots, x_{n}\right]$ a graded $R$-algebra where $\operatorname{deg}\left(x_{i}\right)=1$ for all $i$. Given $d \in \mathbb{Z}$, we consider the graded module $M=A(d)$. Then

$$
A(d)_{\left(x_{0}\right)}=\left\{\left.\frac{p}{x_{0}^{m}} \right\rvert\, p \in A \text { homogeneous of degree } m+d\right\}
$$

is a module over $R\left[x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right]$. In this case, we get the isomorphism

$$
\begin{array}{clc}
A_{\left(x_{0}\right)} & \longrightarrow & A(d)_{\left(x_{0}\right)} \\
f & \longmapsto & x_{0}^{d} f
\end{array}
$$

Given a graded $A$-module $M$, we can construct a sheaf $\tilde{M}$ on $X=\operatorname{Proj}(A)$ in the same manner as the proj. First, we call

$$
\tilde{\tilde{M}}(U):=\left\{s: U \rightarrow \sqcup_{p \in U} M_{(p)} \mid s(p) \in M_{(p)}\right\}
$$

Then we consider the subsheaf

$$
\tilde{M}(U):=\left\{s \in \tilde{\tilde{M}}(U) \mid \text { locally } s \text { comes from some element of } M_{(f)}\right\}
$$

Then $\tilde{M}$ becomes a sheaf of $\mathcal{O}_{X}$-modules; $\tilde{M}_{p} \simeq M_{(p)}$ as an $A_{(p)}$-module and $\left.\tilde{M}\right|_{X_{f}} \simeq M_{(f)}$. This implies that $\tilde{M}$ is a quasi-coherent sheaf on $\operatorname{Proj}(A)$. We can define

$$
\mathbb{P}_{R}^{n}=\operatorname{Proj}\left(R\left[x_{0}, \ldots, x_{n}\right]\right) \quad \mathcal{O}_{\mathbb{P}_{R}^{n}}(d)=R\left[x_{0}, \widetilde{x_{n}}\right](d)
$$

Theorem 6.22. The map $A_{d} \rightarrow \Gamma\left(\mathbb{P}_{R}^{n}, \mathcal{O}(d)\right)$ is an isomorphism, where $A=$ $R\left[x_{0}, \ldots, x_{n}\right]$.

Proof. Let $X=\mathbb{P}_{R}^{n}$ and let

$$
U_{i}=\left(\mathbb{P}_{R}^{n}\right)_{\left(x_{i}\right)}=\operatorname{Spec}\left(R\left[\frac{x_{0}}{x_{i}} \ldots \frac{x_{n}}{x_{i}}\right]\right)
$$

We know that $\left.\mathcal{O}(d)\right|_{U_{i}}=\mathcal{O}_{U_{i}}$. The map

$$
\begin{array}{ccc}
A(d) & \longrightarrow \Gamma\left(\mathbb{P}_{R}^{n}, \mathcal{O}(d)\right) \longrightarrow & \begin{array}{c}
\Gamma\left(U_{0}, \mathcal{O}(d)\right) \\
p\left(x_{0}, \ldots, x_{n}\right)
\end{array} \\
\longmapsto & \frac{p\left(x_{0}, \ldots, x_{n}\right)}{1}
\end{array}
$$

is injective and therefore $A(d) \subseteq \Gamma\left(\mathbb{P}_{R}^{n}, \mathcal{O}(d)\right)$. Let $s \in \Gamma\left(\mathbb{P}_{R}^{n}, \mathcal{O}(d)\right)$. Then the restrictions

$$
\left.s\right|_{U_{i}}=\frac{p_{i}\left(x_{0}, \ldots, x_{n}\right)}{x_{i}^{m}}
$$

and we are assuming that the power doesn't depend on $i$ and $\operatorname{deg}\left(p_{i}\right)=m+d$. On the intersection $U_{i} \cap U_{j}=\operatorname{Spec}\left(A_{\left(x_{i} x_{j}\right)}\right)$ we get

$$
\left.s\right|_{U_{i j}}=\frac{x_{j}^{m} p_{i}\left(x_{0}, \ldots, x_{n}\right)}{x_{i}^{m} x_{j}^{m}}
$$

Therefore

$$
\frac{x_{j}^{m} p_{i}\left(x_{0}, \ldots, x_{n}\right)}{x_{i}^{m} x_{j}^{m}}=\frac{x_{i}^{m} p_{j}\left(x_{0}, \ldots, x_{n}\right)}{x_{i}^{m} x_{j}^{m}} \Rightarrow x_{i}^{m} p_{j}\left(x_{0}, \ldots, x_{n}\right)=x_{j}^{m} p_{i}\left(x_{0}, \ldots, x_{n}\right)
$$

It follows that $p_{i}=x_{i}^{m} q_{i}$ and $p_{j}=x_{j}^{m} q_{j}$; it means that $q_{i}=q_{j}$ for all $i, j$.
We have shown that, given $X=\operatorname{Spec}(A)$ an affine scheme, $U=\operatorname{Spec}(B)$ an open affine subset and $M$ an $A$-module, the restriction of the quasi coherent sheaf $\tilde{M}$ at $U$ is still quasi-coherent and $\left.\tilde{M}\right|_{U}=\widetilde{M \otimes_{A} B}$. Furthermore, if $M$ is finitely generated as an $A$-module, $M \otimes_{A} B$ is finitely generated as a $B$-module. The other implication is not always true; however

Proposition 6.23. Let $X$ be an affine scheme and let $X=\cup U_{i}$ be an affine open cover, where $U_{i}=\operatorname{Spec}\left(B_{i}\right)$. Assume that $M \otimes_{A} B_{i}$ is finitely generated over $B_{i}$ for all $i$. Then $M$ is finitely generated as an $A$-module.

Proof. We can assume that the open affine cover is finite since the spectrum is always quasi-compact. By assumptions, for all $i$ there exists a finite number of elements of $M$ that generate $M \otimes_{A} B_{i}$. So we can find a finite subset $S \subseteq M$ whose image in $M \otimes_{A} B_{i}$ generate $M \otimes_{A} B_{i}$ for all $i$. Let $p \in X$; then there exists an index $i$ such that $p \in U_{i}$. We have the isomorphism

$$
A_{p}=\mathcal{O}_{X, p}=\mathcal{O}_{U_{i}, p}=\left(B_{i}\right)_{p}
$$

and it means that the elements of $S$ generate $M_{p}$ as an $A_{p}$-module for all $p \in X$. This means that $S$ generates $M$.

Proposition 6.24. Let $X$ be a scheme and let $F$ be a quasi-coherent sheaf on $X$. The following are equivalent:

1. For every open affine subset $U \subseteq X F(U)$ is finitely generated as an $\mathcal{O}(U)$-module.
2. There exists an affine open cover $X=\cup U_{i}$ such that $F\left(U_{i}\right)$ is a finitely generated $\mathcal{O}\left(U_{i}\right)$-module.

Definition 6.25. A quasi-coherent sheaf $F$ on $X$ is finitely generated if it satisfies one of the condition of the proposition. $F$ is coherent if it is finitely generated and $X$ is locally noetherian.

Proposition 6.26. Let $X$ be a locally noetherian scheme.

- Quasi-coherent subsheaves and quotient of a coherent sheaf are coherent.
- If $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ is an exact sequence of quasi-coherent sheaves, then $F$ is coherent if and only if $F^{\prime}, F^{\prime \prime}$ are coherent.

Corollary 6.27. Coherent modules over a locally noetherian scheme form an abelian category.

### 6.2 Cohomology

Let $X$ be a topological space and let $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ be an exact sequence of sheaves on $X$. We know that $0 \rightarrow F^{\prime}(X) \rightarrow F(X) \rightarrow F^{\prime \prime}(X)$ is exact; however, the last arrow is often not surjective. We will define abelian groups $H^{i}(X, F)$ for all $i \in \mathbb{N}$ such that

- $H^{i}(X, F)=0$ for all $i<0$
- $H^{0}(X, F)=F(X)$
- given an exact sequence $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$, we get an exact sequence

$$
\begin{aligned}
0 & \longrightarrow H^{0}\left(X, F^{\prime}\right) \longrightarrow H^{0}(X, F) \longrightarrow H^{0}\left(X, F^{\prime \prime}\right) \\
& \longleftrightarrow H^{1}\left(X, F^{\prime}\right) \longrightarrow H^{1}(X, F) \longrightarrow H^{1}\left(X, F^{\prime \prime}\right) \\
& \left.\leftrightarrow H^{2}\left(X, F^{\prime}\right) \longrightarrow H^{2}(X, F) \longrightarrow F^{\prime \prime}\right) \\
& =H^{n}\left(X, F^{\prime}\right) \longrightarrow H^{n}(X, F) \longrightarrow H^{n}\left(X, F^{\prime \prime}\right)
\end{aligned}
$$

- $H^{i}(X, F)$ is functorial in $F$
- The connecting homomorphisms $\delta$ are functorial
- For all sheaves $F$ on $X$, there exists a sheaf $G$ such that $F \subseteq G$ and $H^{i}(X, G)=0$ for all indexes $i$

Definition 6.28. A sheaf $F$ on $X$ is flabby if for all $U \subseteq X$ open subsets the restriction map is surjective

We notice that this implies that all the restriction maps are surjective. In particular, the restriction of a flabby sheaf to an open subset is still flabby.
Example. Let $X$ a topological space and suppose given an abelian group $A_{p}$ for all $p \in X$. We can define a flabby sheaf

$$
U \longmapsto\left\{s: U \rightarrow \bigsqcup_{p \in U} A_{p} \mid s(p) \in A_{p}\right\}
$$

Proposition 6.29. Let $0 \rightarrow F^{\prime} \xrightarrow{\alpha} F \xrightarrow{\beta} F^{\prime \prime} \rightarrow 0$ be an exact sequence of sheaves. If $F^{\prime}$ is flabby, the map $F(X) \rightarrow F^{\prime \prime}(X)$ is surjective.

Proof. Let $s^{\prime \prime} \in F^{\prime \prime}(X)$ be a global section of $F^{\prime \prime}$. We consider the set

$$
\mathcal{S}=\left\{(U, s) \mid U \subseteq X \text { open } s \in F(U) \beta(s)=\left.s^{\prime \prime}\right|_{U}\right\}
$$

We define a partial order on $\mathcal{S}$ by

$$
(U, s) \leq(V, t) \Longleftrightarrow(U \subseteq V) \wedge\left(s=\left.t\right|_{U}\right)
$$

Since $\mathcal{S} \neq \emptyset$ (the pair $(\emptyset, 0)$ lies in $\mathcal{S}$ ), if every chain has an upper bound, we can apply Zorn's lemma. So let $\left\{\left(U_{i}, s_{i}\right)\right\}$ be a chain in $\mathcal{S}$. Then we can take $U=\cup U_{i}$ and there exists a unique $s \in F(U)$ such that $\left.s\right|_{U_{i}}=s_{i}$. So, by Zorn's lemma, $\mathcal{S}$ has a maximal element $(U, s)$.
By contradiction, assume that $U \neq X$. Then we can pick an element $x_{0} \in$ $X \backslash U$ and choose an open neighbourhood $V$ of $x_{0}$. We can find $t \in F(V)$ such that $\beta(t)=\left.s^{\prime \prime}\right|_{V}$. Let $u$ be the difference $\left.s\right|_{U \cap V}-\left.t\right|_{U \cap V} \in F(U \cap V)$; by definition, $\beta(u)=0$ and by the exactness of the sequence, there exists a unique $s^{\prime} \in F^{\prime}(U \cap V)$ such that $\alpha\left(s^{\prime}\right)=s-t$. By the flabbiness of $F^{\prime}$, we can find $s_{1}^{\prime} \in F^{\prime}(V)$ such that $\left.s_{1}^{\prime}\right|_{U \cap V}=s^{\prime}$. Then

$$
\left.\left(t-\alpha\left(s_{1}^{\prime}\right)\right)\right|_{U \cap V}=\left.s\right|_{U \cap V}
$$

and since $F$ is a sheaf there exists a unique $s_{1} \in F(U \cup V)$ such that $\left.s_{1}\right|_{U}=s$ and $\left.s_{1}\right|_{V}=t-\alpha\left(s_{1}^{\prime}\right)$. Therefore $\beta\left(s_{1}\right)=\left.s\right|_{U \cup V}$, contradicting the maximality of $(U, s)$.

Definition 6.30. Let $F$ be a sheaf on $X$. We define the Godement sheaf $G F$ to be

$$
G F(U):=\left\{s: U \rightarrow \sqcup F_{p} \mid s(p) \in F_{p} \forall p \in U\right\}=\prod_{p \in U} F_{p}
$$

The Godement sheaf is flabby and $F$ canonically injects into $G F$ :

$$
\begin{array}{ccc}
F & \longrightarrow & G F \\
s & \longmapsto & \left(p \rightarrow s_{p}\right)
\end{array}
$$

Furthermore, given $\varphi: F \rightarrow F^{\prime}$ a morphism of sheaf, it induces a morphism

$$
\begin{array}{ccc}
G F & \longrightarrow & G F^{\prime} \\
s & \longmapsto & \left(p \mapsto \varphi_{p}(s(p))\right)
\end{array}
$$

We get a commutative diagram


Proposition 6.31. Let $F^{\prime} \rightarrow F \rightarrow F^{\prime \prime}$ be an exact sequence of sheaves. Then

$$
G F^{\prime} \rightarrow G F \rightarrow G F^{\prime \prime}
$$

is exact as a sequence of presheaves.

We now want to construct the Godement Resolution of a sheaf $F$. We define $G^{0} F=G F$ and $G^{1} F=G(\operatorname{Coker}(F \rightarrow G F))$. We notice that

$$
G F \rightarrow \operatorname{Coker}(F \rightarrow G F) \rightarrow G^{\prime} F
$$

and therefore the composition gives a map $G^{0} F \rightarrow G^{\prime} F$. In general, we define for all $i \geq 1$

$$
G^{i} F=G\left(\operatorname{Coker}\left(G^{i-2} F \rightarrow G^{i-2} F\right)\right)
$$

and this gives an exact sequence.

$$
F^{\cdot} F: 0 \rightarrow G^{0} F \rightarrow G^{1} F \rightarrow G^{2} F \rightarrow \ldots
$$

It is exact at all degrees different from zero, where

$$
\operatorname{Ker}\left(G^{0} F \rightarrow G^{1} F\right) \simeq F
$$

Definition 6.32. Let $F$ be a sheaf and let $G \cdot F$ be its Godement resolution. We define the cohomology

$$
H^{i}(X, F):=H^{i}(\Gamma(X, G \cdot F))
$$

By definition,

$$
H^{0}(X, F)=\operatorname{Ker}\left(\Gamma\left(X, G^{0} F\right) \rightarrow \Gamma\left(X, G^{1} F\right)\right) \simeq \Gamma(X, F)=F(X)
$$

Given a morphism of sheaves $\varphi: F \rightarrow F^{\prime}$, we get a morphism of complexes of sheaves $G^{\prime} F \rightarrow G^{\cdot} F^{\prime}$


As a consequence, we get a homomorphism $\Gamma(X, G \cdot F) \rightarrow \Gamma\left(X, G \cdot F^{\prime}\right)$ and so a map $H^{i}(X, F) \rightarrow H^{i}\left(X, F^{\prime}\right)$. This fact gives the functoriality of cohomology. Let $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ be an exact sequence and let

$$
C_{F}^{0}=\operatorname{Coker}\left(F \rightarrow G^{0} F\right) \quad C_{F}^{i}=\operatorname{Coker}\left(G^{i-1} F \rightarrow G^{i} F\right)
$$

As a consequence of this fact, we get the diagram

using the Snake Lemma. Therefore we get an exact sequence $0 \rightarrow G^{1} F^{\prime} \rightarrow$ $G^{1} F \rightarrow G^{1} F^{\prime \prime} \rightarrow 0$ and so on. The sequence

$$
0 \rightarrow G \cdot F^{\prime} \longrightarrow G \cdot F \longrightarrow G^{\cdot} F^{\prime \prime} \rightarrow 0
$$

is an exact sequence of complexes of sheaves, exact at each level as a sequence of presheaves. We get an exact sequence of abelian groups

$$
0 \rightarrow \Gamma\left(X, G \cdot F^{\prime}\right) \longrightarrow \Gamma(X, G \cdot F) \longrightarrow \Gamma\left(X, G \cdot F^{\prime \prime}\right) \rightarrow 0
$$

which gives the long exact sequence of cohomology groups.
Lemma 6.33. Let $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ be an exact sequence of sheaves. If $F^{\prime}, F$ are flabby, then $F^{\prime \prime}$ is flabby.

Proof. Let $U$ be an open subset of $X$. We can consider the diagram


The Five Lemma implies that $F^{\prime \prime}(X) \rightarrow F^{\prime \prime}(U)$ is surjective.
Proposition 6.34. Let $F$ be a flabby sheaf. Then $H^{i}(X, F)=0$ for all $i>0$.
Proof. First of all, we want to show that $C^{i} F$ is flabby for all $i$. We proceed by induction. The sequence

$$
0 \rightarrow F \rightarrow G^{0} F \rightarrow C^{0} F \rightarrow 0
$$

is exact and then $C^{0}$ is flabby. In general, we consider the sequence

$$
0 \rightarrow C^{i-1} F \rightarrow G^{i} F \rightarrow C^{i} F \rightarrow 0
$$

and by inductive hypotesis $C^{i}$ is flabby. Now, we consider the diagram


Passing to the global sections, we get the diagram


The sequence is still exact and therefore every cohomology group is zero.
Definition 6.35. Let $F$ be a sheaf. A resolution $F^{*}$ of $F$ is a sequence

$$
\cdots \rightarrow F^{-1} \rightarrow F^{0} \rightarrow F^{1} \rightarrow F^{2} \rightarrow \cdots
$$

such that

- $F^{i}=0$ for all negative $i$
- $F^{*}$ is exact in positive degree
- $\operatorname{Ker}\left(F^{0} \rightarrow F^{1}\right) \simeq F$

We say that a sheaf $F$ on $X$ is acyclic if $H^{i}(X, F)=0$ for all $i>0$.
Example. Let $X$ be a $\mathcal{C}^{\infty}$-manifold and let $F=\mathbb{R}_{X}$. There is a famous resolution, the De Rham Complex. Every term of the complex is the sheaf of $i$-forms $\Omega_{X}^{i} ; \Omega_{X}^{*}$ is a resolution of $\mathbb{R}_{X}$.
Proposition 6.36. If $F^{*}$ is an acyclic resolution of $F$, then $H^{i}(X, F) \simeq$ $H^{i}\left(\Gamma\left(X, F^{*}\right)\right)$ for all $i \geq 1$.
Proof. First of all, we consider the case $i=1$. We have the sequence of sheaves

$$
0 \rightarrow F \stackrel{f}{\rightarrow} F^{0} \longrightarrow C^{0} \rightarrow 0
$$

where $C^{0}=\operatorname{Coker}(f)$. Passing to the long exact cohomology sequence, we get

$$
0 \rightarrow H^{0}(F) \rightarrow H^{0}\left(F^{0}\right) \rightarrow H^{0}\left(C^{0}\right) \rightarrow H^{1}(F) \rightarrow 0
$$

Therefore we have the equality

$$
H^{1}(X, F)=\operatorname{Coker}\left(H^{0}\left(F^{0}\right) \rightarrow H^{0}\left(C^{0}\right)\right)
$$

On the other hand,
$H^{1}\left(\Gamma\left(X, F^{i}\right)\right)=\operatorname{Ker}\left(F^{1}(X) \rightarrow F^{2}(X)\right) / \operatorname{Im}\left(F^{0}(X) \rightarrow F^{1}(X)\right)=C^{0}(X) / F^{0}(X)$
and the thesis follows for $i=1$.
We now deal with the general case. The long exact sequence gives

$$
H^{i}\left(F^{0}\right) \rightarrow H^{i}\left(C^{0}\right) \rightarrow H^{i+1}(F) \rightarrow H^{i+1}\left(F^{0}\right)
$$

and therefore $H^{i}\left(C^{0}\right) \simeq H^{i+1}(F)$. We now consider the exact sequence

$$
0 \rightarrow C^{i-1} \longrightarrow F^{i} \longrightarrow C^{i} \rightarrow 0
$$

and the cohomology exact sequence gives $H^{i-1}\left(C^{i}\right) \simeq H^{i}\left(C^{i-1}\right)$. Therefore $H^{i}\left(C^{0}\right) \simeq H^{0}\left(C^{i}\right)$ and this is equal to $H^{0}\left(C^{i}\right) \simeq \operatorname{Coker}\left(F^{i+1} \rightarrow C^{i+1}(X)\right)$. As before, this coincides with $H^{i}\left(\Gamma\left(X, F^{i}\right)\right)$ and this gives the thesis.

We will often use the following theorems:
Theorem 6.37 (Grothendieck). If $F$ is a sheaf of abelian groups on $X$ then $H^{i}(X, F)=0$ for all $i>\operatorname{dim} X$.
Theorem 6.38. Let $X$ be an affine scheme and let $F$ be a quasi-coherent sheaf on $X$. Then $H^{i}(X, F)=0$ for all $i>0$.

## 6.3 Čech Cohomology

Let $F$ be a sheaf on a topological space $X$, let $\mathcal{U}=\cup_{i \in I} U_{i}$ be an open cover and assume that $I$ is totally ordered. As usual, we will use the convention

$$
i_{0}<i_{1}<\cdots<i_{p} \quad U_{i_{0} i_{1} \ldots i_{p}}:=U_{i_{0}} \cap \cdots \cap U_{i_{p}}
$$

We define

$$
\check{C}^{p}(\mathcal{U}, F)=\prod_{i_{0}<i_{1}<\cdots<i_{p}} F\left(U_{i_{0} i_{1} \ldots i_{p}}\right)
$$

We want to define the differential; the first one is the following

$$
\delta(f)_{i_{0} i_{1} i_{2}}=\left.f_{i_{1} i_{2}}\right|_{U_{i_{0} i_{1} i_{2}}}-\left.f_{i_{0} i_{2}}\right|_{U_{i_{0} i_{1} i_{2}}}+\left.f_{i_{0} i_{1}}\right|_{U_{i_{0} i_{1} i_{2}}}
$$

In general,

$$
\delta(f)_{i_{0} \ldots i_{p+1}}=\left.\sum_{k=0}^{p+1}(-1)^{k} f_{i_{0} \ldots \hat{i}_{k} \ldots i_{p+1}}\right|_{U_{i_{0} \ldots i_{p+1}}}
$$

It's easy to see that $\delta^{2}=0$; in this way we have built a complex $\check{C}(\mathcal{U}, F)$ and we define the Čech cohomology as

$$
\check{\mathrm{H}}^{i}(\mathcal{U}, F):=H^{i}(\mathcal{U}, F)
$$

As a first application, we consider the exact sequence $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ and let $s^{\prime \prime} \in F^{\prime \prime}(X)$. We want to find (if it exists) $s \in F(X)$ such that $\beta(s)=s^{\prime \prime}$. We can complete the sequence of global sections

$$
F(X) \rightarrow F^{\prime \prime}(X) \rightarrow H^{1}(X, F)
$$

Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be a cover of $X$ such that $\left.s^{\prime \prime}\right|_{U_{i}}=\beta\left(s_{i}\right)$; we call $s_{i j}=s_{i}-s_{j}$. Then $\beta\left(s_{i j}\right)=0$ in $F^{\prime \prime}$ and by the exactness of the sequence, $s_{i j}=\alpha\left(s_{i j}^{\prime}\right)$, where $s_{i j}^{\prime} \in F^{\prime}\left(U_{i j}\right)$. Then, by the definition of the differential,

$$
\delta\left(s_{i j}^{\prime}\right)=s_{j k}^{\prime}-s_{i k}^{\prime}+s_{i j}^{\prime}
$$

and applying $\alpha$ we obtain $\alpha\left(\delta\left(s_{i j}^{\prime}\right)\right)=s_{k}-s_{j}-s_{k}+s_{i}+s_{j}-s_{i}=0$. Since $\alpha$ is injective, it implies that $\delta\left(s_{i j}^{\prime}\right)=0$. We want now to show that the class of $s_{i j}^{\prime}$ doesn't depend on the lifting. If $t_{i} \in F\left(U_{i}\right)$ is an element such that $\beta\left(t_{i}\right)=\left.s^{\prime \prime}\right|_{U_{i}}$, then $t_{i}-s_{i}=\alpha\left(s_{i}^{\prime}\right)$.

$$
\alpha\left(t_{i j}^{\prime}\right)=t_{j}-t_{i}=s_{j}-s_{i}+\alpha\left(s_{j}^{\prime}-s_{i}^{\prime}\right)
$$

and therefore $t_{i j}^{\prime}=s_{i j}^{\prime}+\delta\left(s_{i}^{\prime}\right)$. The class of $s_{i j}^{\prime}$ is zero if and only if $s^{\prime \prime}$ lifts.
Example. Let $M=S^{1}$ be the circle considered as a manifold. We have the De Rham complex for the sheaf $\mathbb{R}_{M}$

$$
0 \rightarrow \mathbb{R}_{M} \rightarrow \mathcal{C}_{S^{1}}^{\infty} \rightarrow \Omega_{S^{1}}^{1} \rightarrow 0
$$

Taking global section, we get

$$
0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}^{\infty}\left(S^{1}\right) \rightarrow \Omega^{1}\left(S^{1}\right)
$$

Consider the open cover $S^{1}=\left(S^{1} \backslash\{N\}\right) \cup\left(S^{1} \backslash\{S\}\right)$. Then $H^{1}\left(\mathcal{U}, \mathbb{R}_{S^{1}}\right) \simeq \mathbb{R}$.

Definition 6.39. A double complex $A^{* *}$ is a $\mathbb{Z}^{2}$-graded abelian group with differentials

$$
d: A^{p q} \rightarrow A^{p+1, q} \quad \delta: A^{p q} \rightarrow A^{p, q+1}
$$

such that $d^{2}=0, \delta^{2}=0, \delta d=d \delta$.
Given a double complex $A^{* *}$, we can associate to it a complex in a natural way. We define

$$
C^{n}=\bigoplus_{p+q=n} A^{p, q}
$$

and we define the differential $\left.D\right|_{A^{p, q}}=d+(-1)^{p} \delta$. We call this complex $\left(\operatorname{Tot}\left(A^{* *}\right), D\right)$.

Theorem 6.40. Let $X$ be a topological space and let $F$ be a sheaf on $X$. Consider an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and assume that $I$ is totally ordered. There exists a canonical group homomorphism

$$
\check{H}^{p}(\mathcal{U}, F) \rightarrow H^{p}(X, F)
$$

Furthermore, if $H^{p}\left(\left.U_{i_{0} \ldots i_{p}} \cdot F\right|_{U_{i_{0}}, \ldots, i_{p}}\right)=0$, it is an isomorphism.
Proof. Let us consider the Godement resolution

$$
0 \rightarrow G^{0} F \xrightarrow{\delta} G^{1} F \xrightarrow{\delta} G^{2} F \xrightarrow{\delta} \ldots
$$

We obtain the double complex $A^{p q}=\check{C}^{q}\left(\mathcal{U}, G^{p} F\right)$ with differentials

$$
\partial: A^{p q} \longrightarrow A^{p+1, q} \quad \delta: A^{p q} \longrightarrow A^{p, q+1}
$$

Let $K^{n}$ be the kernel of $A^{n, 0} \xrightarrow{\delta} A^{n, 1}$; this is a subcomplex of $\operatorname{Tot}^{*}(A)$. Simmetrically, let $L^{n}$ be the kernel of $A^{0, n} \xrightarrow{\partial} A^{1, n}$. Remembering the definition of the double complex, we get

$$
K^{n}=\operatorname{Ker}\left(\check{C}^{0}\left(\mathcal{U}, G^{n} F\right) \rightarrow \check{C}^{1}\left(\mathcal{U}, G^{n} F\right)\right)=\check{H}^{0}\left(\mathcal{U}, G^{n} F\right)=\Gamma\left(X, G^{n} F\right)
$$

and therefore $H^{n}\left(K^{*}\right)=H^{n}(X, F)$. In the same way,

$$
L^{n}=\operatorname{Ker}\left(\check{C}^{n}\left(\mathcal{U}, G^{0} F\right) \xrightarrow{\partial} \check{C}^{n}\left(\mathcal{U}, G^{1} F\right)\right)=\check{C}^{n}(\mathcal{U}, F)
$$

So

$$
\check{H}^{n}(\mathcal{U}, F) \xrightarrow{\sim} L^{n} \rightarrow H^{n}\left(\operatorname{Tot}^{*}\left(A^{* *}\right)\right) \leftarrow H^{n}\left(X, K^{*}\right) \leftarrow H^{n}(X, F)
$$

Assume the following facts:

1. If $F$ is a flabby sheaf, $\check{H}^{n}(\mathcal{U}, F)=0$ for all $n \geq 1$
2. If $A^{p *}$ is exact for all $p$ in $\operatorname{deg}>0$, then $H^{*}\left(K^{*}\right) \xrightarrow{\sim} H^{*}\left(\operatorname{Tot}^{*}\left(A^{* *}\right)\right)$

Moreover, the second condition implies that if $A^{*, q}$ is acyclic in degree $>0$ for all $q>0$, the map $L^{*} \rightarrow T^{*}$ induces an isomorphism in cohomology. By definition

$$
A^{p q}=\check{C}^{q}\left(\mathcal{U}, G^{p} F\right)=\prod_{i_{0}<\cdots<i_{q}} G^{p} F\left(U_{i_{0}, \ldots, i_{p}}\right)
$$

and therefore

$$
H^{p}\left(A^{* q}\right)=\prod_{i_{0}<\cdots<i_{q}} H^{p}\left(U_{i_{0}<\cdots<i_{q}},\left.F\right|_{U_{i_{0}<\cdots<i_{q}}}\right)
$$

The theorem has an important corollary:
Corollary 6.41. Let $F$ be a quasi-coherent sheaf on a scheme $X$ and assume that the intersection of every affine open set is affine (for example, $X$ is separated). Let $\mathcal{U}=\cup U_{i}$ be a cover by affine open sets. Then

$$
\check{H}(\mathcal{U}, F) \simeq H^{p}(X, F)
$$

Example. Let $X=\mathbb{P}_{R}^{1}$ and we consider the sheaf of $\mathcal{O}_{X}$-modules $F=\mathcal{O}_{\mathbb{P}^{1}}(d)$. We have the open cover

$$
U_{0}=\left(\mathbb{P}_{R}^{1}\right)_{x_{0}} \quad U_{1}=\left(\mathbb{P}_{R}^{1}\right)_{x_{1}}
$$

By the previous theorem, $\check{H}^{p}(\mathcal{U}, \mathcal{O}(d)) \simeq H^{p}\left(\mathbb{P}_{R}^{1}, \mathcal{O}(d)\right)$. Since $\operatorname{dim}\left(\mathbb{P}_{R}^{1}\right)=1$, $H^{p}(\mathcal{U}, \mathcal{O}(d))=0$ for all $i>1$. We know that $H^{0}\left(\mathbb{P}_{R}^{1}, \mathcal{O}(d)\right) \simeq R$ so we only have to compute $H^{1}(\mathcal{U}, \mathcal{O}(d))$, which is by definition the cokernel of the map

$$
\begin{array}{rll}
\varphi: A(d)_{\left(x_{0}\right)} \oplus A(d)_{\left(x_{1}\right)} & \longrightarrow & A(d)_{\left(x_{0} x_{1}\right)} \\
\left(\frac{p(x)}{x_{0}^{m}}, \frac{q(x)}{x_{1}^{m}}\right) & \longmapsto & \left(\frac{x_{0}^{m} q(x)-x_{1}^{m} p(x)}{x_{0}^{m} x_{1}^{m}}\right)
\end{array}
$$

where $\operatorname{deg}(p)=\operatorname{deg}(q)=m+d$. We notice that the map is surjective if $d \geq-1$; if $d=-1$, we get

$$
\begin{aligned}
H^{1}\left(\mathbb{P}_{R}^{1}, \mathcal{O}(-2)\right) & \simeq R \\
\left(x_{0} x_{1}\right)^{-1} & \longleftrightarrow 1
\end{aligned}
$$

In general, $H^{1}\left(\mathbb{P}^{1}, \mathcal{O}(d)\right) \simeq R^{-d-1}$ for all $d<0$.
Another way to see this is the following. We consider the exact sequence

$$
0 \rightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^{1}} \rightarrow 0
$$

We get this exact sequence by considering the homomorphism

$$
\begin{array}{cccc}
A(-1) \oplus A(-1) & \longrightarrow & A_{\geq 1} & \longrightarrow 0 \\
\left(f_{0}, f_{1}\right) & \longmapsto & x_{0} f_{0}+x_{1} f_{1} &
\end{array}
$$

The kernel is formed by the pairs of polynomials $\left(f_{0}, f_{1}\right)$ for which there exists a $\varphi$ such that

$$
f_{0}=x_{1} \varphi \quad f_{1}=-x_{0} \varphi
$$

Considering the homomorphism

$$
\begin{aligned}
A(-2) & \longrightarrow A(-1) \oplus A(-1) \\
\varphi & \longmapsto\left(x_{1} \varphi,-x_{0} \varphi\right)
\end{aligned}
$$

we get the desired exact sequence. Considering the cohomology sequence

$$
0 \rightarrow H^{0}\left(\mathbb{P}^{1}, \mathcal{O}\right) \longrightarrow H^{1}\left(\mathbb{P}^{1}, \mathcal{O}(-2)\right) \longrightarrow H^{1}\left(\mathbb{P}^{1}, \mathcal{O}(-1)\right) \oplus H^{1}\left(\mathbb{P}^{1}, \mathcal{O}(-1)\right)
$$

since we know that the first and the last term are zero, we get one again $H^{1}\left(\mathbb{P}^{1}, \mathcal{O}(-2)\right)=0$.

If $F$ is a sheaf of $\mathcal{O}_{X}$-modules, the groups $H^{p}(X, F)$ have naturally the structure of $\mathcal{O}_{X}(X)$-modules. If $a \in \mathcal{O}_{X}(X)$, we have the map

$$
\begin{array}{clc}
F & \longrightarrow & F \\
s & \longmapsto & a s
\end{array}
$$

and by functoriality we get a map $H^{p}(X, F) \xrightarrow{a} H^{p}(X, F)$. Furthermore, if $\varphi: F \rightarrow G$ is a homomorphism of sheaves of $\mathcal{O}_{X}$-modules, it induces a homomorphism of $\mathcal{O}_{X}(X)$-modules $H^{p}(X, F) \rightarrow H^{p}(X, G)$.

Theorem 6.42. Let $X=\mathbb{P}_{R}^{n}$ be the projective space and let us consider the quasi-coherent sheaf $F=\mathcal{O}(d)$. Then

- $H^{0}(X, F)=0$ if $d>0$ and $R^{\binom{d+n}{n}}$ otherwise
- $H^{p}(X, F)=0$ for all $p>0, p \neq n$
- There exists a natural isomorphism $H^{n}(X, F)$ and $H^{0}(X, \mathcal{O}(-d-n-1))$

Definition 6.43. Let $k$ be a field and let $X$ be a scheme over $k$. Given $F$ a quasi-coherent sheaf on $X$ we define

$$
h^{i}(X, F):=\operatorname{dim}_{k} H^{i}(X, F)
$$

We define the Euler Characteristic as

$$
\chi(F)=\sum_{i \geq 0}(-1)^{i} h^{i}(X, F)
$$

Theorem 6.44. Let $R$ be a noetherian ring and let $X$ be a proper scheme over $R$. Let $F$ be a coherent sheaf on $X$. Then $H^{p}(X, F)$ is finitely generated over $R$ for all $p$.

Corollary 6.45. Let $X$ be a proper scheme over $k$ and let $F$ be a coherent sheaf on $X$. Then $\chi(F)$ is well defined.

Observation 6.46. Since $\binom{d+n}{n}=(-1)^{n}\binom{-d-1}{n}$, we get

$$
\chi\left(\mathcal{O}_{\mathbb{P}_{k}^{n}}(d)\right)=\binom{d+n}{n}
$$

Proposition 6.47. Let us consider the exact sequence

$$
0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0
$$

of coherent sheaves on $X$ and assume that $X$ is proper over $k$. Then $\chi(F)=$ $\chi\left(F^{\prime}\right)+\chi\left(F^{\prime \prime}\right)$.

Proof. Is is enough to consider the long exact cohomology sequence and to remember that the same results holds for dimension of vector spaces.

Corollary 6.48. Let $\cdots \rightarrow F_{i-1} \rightarrow F_{i} \rightarrow F_{i+1} \rightarrow \ldots$ be a long exact sequence of coherent sheaves on $X$ and assume that $F_{i} \neq 0$ for finite $i$. Then $\sum(-1)^{i} \chi\left(F_{i}\right)=0$.

Let $F$ be a homogeneous polynomial of degree $d$ and consider the closed subscheme

$$
X=\operatorname{Proj}\left(k\left[x_{0}, \ldots, x_{n}\right] /(F)\right) \subseteq \mathbb{P}_{k}^{n}
$$

We call these closed subschemes hypersurfaces.
Remark 6.49. Let $X \stackrel{i}{\subseteq} Y$ be a closed embedding of topological spaces and $F$ a sheaf on $X$. If $G^{*} F$ is the Godement resolution of $F, i_{*} G^{*} F$ is the Godement resolution of $i_{*} F$.

Notice that the multiplication for a homogeneous polynomial $F$ induces the following exact sequence

$$
0 \rightarrow A(-d) \rightarrow A \rightarrow A /(F) \rightarrow 0
$$

which corresponds to

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^{n}}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^{n}} \rightarrow \mathcal{O}_{X} \rightarrow 0
$$

Therefore we get the cohomology sequence

$$
0 \rightarrow H^{0}\left(\mathcal{O}_{\mathbb{P}^{n}}(-d)\right) \rightarrow H^{0}\left(\mathcal{O}_{\mathbb{P}^{n}}\right) \rightarrow H^{0}\left(\mathcal{O}_{X}\right) \rightarrow H^{1}(\mathcal{O}(-d))
$$

and so $H^{0}\left(X, \mathcal{O}_{X}\right)=k$. In particular, every hypersurface in $\mathbb{P}^{n}$ is connected for $n \geq 2$. Consider now the case of two homogeneous polynomial $F, G$. We get the sequence

$$
\begin{array}{clcl}
A(-d) \oplus A(-e) & \longrightarrow & A & \longrightarrow A /(F, G)
\end{array} \quad \longrightarrow 0
$$

and the kernel is the set of pair such that $\varphi=h G$ and $\psi=-h F$. Analogously, let

$$
C=\operatorname{Proj}(A /(F)) \quad D=\operatorname{Proj}(A /(G))
$$

Then

$$
h^{0}\left(\mathcal{O}_{C \cap D}\right)=1-\binom{-d+2}{2}-\binom{-e+2}{2}+\binom{-d-e+2}{2}
$$

### 6.4 Pullback of sheaves of $\mathcal{O}_{X}$-modules

Let $f: X \rightarrow Y$ be a continuous map. We have already seen that, if $G$ is a sheaf on $Y$, we can pullback the sheaf to get $f^{-1} G$ which is a sheaf on $X$.

$$
\left(f^{-1} G\right)(U)=\left\{s:\left.U \rightarrow \sqcup G_{f(p)}\right|_{s \text { comes locally from a section of } G}\right\}
$$

$f^{-1}$ gives a functor from sheaves on $Y$ to sheaves on $X$. We have shown that if $F$ is a sheaf on $X$ and $G$ is a sheaf on $Y$, there exists a canonical isomorphism

$$
\operatorname{Hom}_{Y}\left(G, f_{*} F\right) \longrightarrow \operatorname{Hom}_{X}\left(f^{-1} G, F\right)
$$

which is functorial in $F$ and $G$.

Observation 6.50. Let $G$ be a sheaf on $Y$ and let $G^{\prime}, G^{\prime \prime}$ be sheaves on $X$. Assume that for all $F$ sheaf on $X$ there exists an isomorphism

$$
\operatorname{Hom}_{X}\left(G^{\prime}, F\right) \simeq \operatorname{Hom}_{Y}\left(G, f_{*} F\right) \simeq \operatorname{Hom}_{X}\left(G^{\prime \prime}, F\right)
$$

that are functorial in $F$. Then there exists a unique isomorphism $\varphi: G^{\prime} \rightarrow G^{\prime \prime}$ such that $\operatorname{Hom}_{X}\left(G^{\prime}, F\right) \simeq \operatorname{Hom}_{X}\left(G^{\prime \prime}, F\right)$. In fact, by hypotesis

$$
\operatorname{Hom}\left(G^{\prime}, G^{\prime \prime}\right) \simeq \operatorname{Hom}\left(G, f_{*} G^{\prime \prime}\right) \simeq \operatorname{Hom}\left(G^{\prime \prime}, G^{\prime \prime}\right)
$$

In the last group, we have a canonical element: the identity. Its inverse image in $\operatorname{Hom}\left(G^{\prime}, G^{\prime \prime}\right)$ gives the desired isomorphism.

Let $X, Y$ be locally ringed spaces, let $G$ be a sheaf of $\mathcal{O}_{Y}$-modules and let $f: X \rightarrow Y$ be a morphism of locally ringed spaces. Then $f_{*}$ carries sheaves of $\mathcal{O}_{X}$ into sheaves of $\mathcal{O}_{Y}$-modules; the same doesn't hold for $f^{-1}$. For example, let $f: X \rightarrow\{p t\}$ and consider the sheaf of locally constant function on the point (so they are constant).

Definition 6.51. Let $X, Y$ be locally ringed spaces and let $G$ be a sheaf of $\mathcal{O}_{Y}$-modules. We define the pullback

$$
f^{*} G:=f^{-1} G \otimes_{f^{-1} \mathcal{O}_{Y}} \mathcal{O}_{X}
$$

which is naturally a sheaf of $\mathcal{O}_{X}$-modules.
First of all, we notice that the construction makes sense: the pullback of the structure sheaf is the structure sheaf

$$
f^{*} \mathcal{O}_{X}=f^{-1} \mathcal{O}_{Y} \otimes_{f^{-1}} \mathcal{O}_{Y} \mathcal{O}_{X} \simeq \mathcal{O}_{X}
$$

Furthermore, if $F$ is a sheaf of $\mathcal{O}_{X}$-modules and $G$ is a sheaf of $\mathcal{O}_{Y}$-modules, there exists a canonical isomorphism

$$
\operatorname{Hom}_{\mathcal{O}_{Y}}\left(G, f_{*} F\right) \simeq \operatorname{Hom}_{\mathcal{O}_{X}}\left(f^{-1} G, F\right)
$$

which is functorial in $F$. If $\varphi: G \rightarrow f_{*} F$ is a homomorphism of $\mathcal{O}_{Y}$-modules, the map $f^{-1} G \rightarrow F$ is $f^{-1} \mathcal{O}_{Y}$-linear. A homomorphism of sheaves $G \rightarrow f_{*} F$ is $\mathcal{O}_{Y}$-linear if and only if $f^{-1} G \rightarrow F$ is $f^{-1} \mathcal{O}_{Y}$-linear. This characterize uniquely $f^{*} G$ as an $\mathcal{O}_{X}$-module.
We now want to consider the stalks. By canonical isomorphisms, we get

$$
\begin{aligned}
\left(f^{*} G\right)_{p} & =\left(f^{-1} G \otimes_{f-1} \mathcal{O}_{Y} \mathcal{O}_{X}\right) \\
& =\left(f^{-1} G\right)_{f(p)} \otimes_{\mathcal{O}_{Y, f(p)}} \mathcal{O}_{X, p} \\
& =G_{f(p)} \otimes_{\mathcal{O}_{Y, f(p)}} \mathcal{O}_{X, p}
\end{aligned}
$$

$f^{*}$ gives a functor from $\mathcal{O}_{Y}$-modules to $\mathcal{O}_{X}$-modules, because $f^{-1}$ gives a functor from $\mathcal{O}_{Y}$-modules to $f^{-1} \mathcal{O}_{Y}$-modules and $\otimes_{f^{-1}} \mathcal{O}_{Y} \mathcal{O}_{X}$ gives a functor from $f^{-1} \mathcal{O}_{Y}$-modules to $\mathcal{O}_{X}$-modules.

Proposition 6.52. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of locally ringed spaces and let $H$ be a sheaf of $\mathcal{O}_{Z}$-modules. Then there exists a canonical isomorphism

$$
f^{*} g^{*} H \simeq(g f)^{*} H
$$

which is functorial in $H$.

Proof. Let $F$ be a sheaf of $\mathcal{O}_{X}$-modules. Then

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{O}_{X}}\left(f^{*} g^{*} H, F\right) & \simeq \operatorname{Hom}_{\mathcal{O}_{Y}}\left(g^{*} H, f_{*} F\right) \\
& \simeq \operatorname{Hom}_{\mathcal{O}_{Z}}\left(H, g_{*} f_{*} F\right) \\
& \simeq \operatorname{Hom}_{\mathcal{O}_{Z}}\left(H,(g f)_{*} F\right) \\
& \simeq \operatorname{Hom}_{\mathcal{O}_{X}}\left((g f)^{*} H, F\right)
\end{aligned}
$$

Since this doesn't depend on $F$, we get $(g f)^{*} H \simeq g^{*} f^{*} H$.
Observation 6.53. Let $\left(Y, \mathcal{O}_{Y}\right)$ be a locally ringed space and let $X \subseteq Y$ be an open subspace.

$$
j: X \longrightarrow Y
$$

If $G$ is a sheaf of $\mathcal{O}_{Y}$-modules, since $j^{-1} \mathcal{O}_{Y}=\left.\mathcal{O}_{Y}\right|_{X}=\mathcal{O}_{X}$ and $\left.j^{-1} G \simeq G\right|_{X}$, then $\left.j^{*} G \simeq G\right|_{X}$.

Proposition 6.54. Let $X=\operatorname{Spec}(A), Y=\operatorname{Spec}(B)$ be affine schemes and let $G=\tilde{N}$ be a quasi-coherent sheaf of $\mathcal{O}_{Y}$-modules, where $N$ is a $B$-module. Let $f: B \rightarrow A$ be a ring homomorphism. Then

$$
f^{*} \tilde{N} \simeq \widetilde{N \otimes_{B} A}
$$

Proof. Clearly, the map

$$
N \simeq H^{0}(Y, \tilde{N}) \longrightarrow H^{0}\left(X, f^{*} \tilde{N}\right)
$$

is $B$-linear and tensoring by $A$ we get an $A$-linear map

$$
N \otimes_{B} A \longrightarrow H^{0}\left(X, f^{*} \tilde{N}\right)
$$

and therefore a map $\widetilde{N \otimes_{B} A} \rightarrow f^{*} \tilde{N}$. We only have to check that it is an isomorphism on the stalks. Let $p \in X=\operatorname{Spec}(A)$ and let $q=f(p)$. Then
$N_{q} \otimes_{B_{q}} A_{p} \simeq\left(N \otimes_{B} A\right)_{p}=\widetilde{N \otimes_{B}} A_{p} \longrightarrow\left(f^{*} \tilde{N}\right)_{p} \simeq \tilde{N}_{p} \otimes_{\mathcal{O}_{Y, q}} \mathcal{O}_{X, p}=N_{q} \otimes_{B_{q}} A_{p}$ and this commutes.

Let $f: X \rightarrow Y$ be a morphism of schemes and let $G$ be a quasi-coherent sheaf on $Y$. We can take affine open covers $\left\{U_{i}\right\}_{i \in I},\left\{V_{i}\right\}_{i \in I}$ of $X, Y$ respectively such that $f\left(U_{i}\right) \subseteq V_{i}$. Let $f_{i}: U_{i} \rightarrow V_{i}$ be the restriction map. Then

$$
\begin{aligned}
\left.f^{*} G\right|_{U_{i}} & \simeq f_{i}^{*}\left(\left.G\right|_{V_{i}}\right) \\
& \simeq f_{i}^{*}\left(\widetilde{\left(G\left(V_{i}\right)\right)}\right) \\
& \simeq G\left(V_{i}\right) \widetilde{\otimes_{\mathcal{O}\left(V_{i}\right)}} \mathcal{O}\left(U_{i}\right)
\end{aligned}
$$

This shows that
Proposition 6.55. Let $f: X \rightarrow Y$ be a morphism of schemes and let $G$ be a quasi-coherent sheaf on $Y$. Then $f^{*} G$ is quasi coherent.
Furthermore, if $X, Y$ are locally noetherian and $G$ is coherent, $f^{*} G$ is coherent.

Observation 6.56. Let $f: X \rightarrow Y$ be a morphism of locally ringed spaces. Then the functor $f^{*}$ is right exact. In fact, the functor $f^{-1}$ is exact (the stalks are the same) while the functor $\otimes_{f^{-1}} \mathcal{O}_{Y} \mathcal{O}_{X}$ is right exact. Notice that if $f$ is flat, it becomes exact (not only right exact).
Example. Let $f: X \rightarrow Y$ be a morphism of schemes and let $Y^{\prime} \subseteq Y$ be a closed subscheme, $\mathcal{O}_{Y^{\prime}}=j_{*} \mathcal{O}_{Y}$. This correspond to the exact sequence

$$
0 \rightarrow I_{Y} \longrightarrow \mathcal{O}_{Y} \longrightarrow \mathcal{O}_{Y^{\prime}} \rightarrow 0
$$

$f^{*}$ gives a closed subscheme $X^{\prime}$ of $X$

$$
f^{*} I_{Y} \longrightarrow \mathcal{O}_{X} \longrightarrow f^{*} \mathcal{O}_{Y^{\prime}} \rightarrow 0
$$

Let $I_{X^{\prime}}=\operatorname{Ker}\left(\mathcal{O}_{X} \longrightarrow f^{*} \mathcal{O}_{Y^{\prime}}\right)$. We claim that $X^{\prime}=Y^{\prime} \times_{Y} X$. We can see this on an open affine cover. Let $X=\operatorname{Spec}(A), Y=\operatorname{Spec}(B), Y^{\prime}=\operatorname{Spec}\left(B^{\prime}\right)=$ $\operatorname{Spec}(B / I)$. Then

$$
f^{*} \mathcal{O}_{Y^{\prime}}=\widetilde{A_{\otimes_{B} B^{\prime}} \simeq \widetilde{A / I A}}
$$

and therefore $I_{X^{\prime}}=\widetilde{I A}$, which is exactly $Y^{\prime} \times_{Y} X$.

### 6.5 Coherent sheaves on Projective Spaces

Lemma 6.57. Let $F$ be a coherent sheaf on $X$ and assume that $X$ is locally noetherian. Then the support of $F$

$$
\operatorname{Supp} F=\left\{p \in X \mid F_{p} \neq 0\right\}
$$

is closed.
Proof. Let $p \in X$. Then

$$
F_{p}=0 \Longleftrightarrow F_{p} / m_{p} F_{p}=0
$$

and this means that there exists an open neighbourhood of $p$ in $X$ such that $F_{q}=0$ for all $q \in U$.
Lemma 6.58. Let $f: A \rightarrow B$ be a local homomorphism of local rings and let $M$ be a finitely generated $A$-module. If $M \otimes_{A} B=0$ then $M=0$.

Proof. Assume $M \otimes_{A} B=0$. We notice that $B / m_{B}$ is a vector space over $A / m_{A}$. Then

$$
0=\left(M \otimes_{A} B\right) \otimes_{A} B / m_{B} B \simeq M / m_{A} M \otimes_{A} / m_{A} B / m_{B}
$$

Therefore $M / m_{A} M=0$ and by Nakayama $M=0$.

Proposition 6.59. Let $f: X \rightarrow Y$ be a morphism of locally noetherian schemes and let $G$ be a coherent sheaf on $Y$. Then

$$
\operatorname{Supp}\left(f^{*} G\right)=f^{-1}(\operatorname{Supp}(G))
$$

Proof. Let $p \in X$ be a point and let $q \in Y$ be its image. Then

$$
\left(f^{*} G\right)_{p}=G_{q} \otimes_{\mathcal{O}_{Y, q}} \mathcal{O}_{X, p}
$$

If $G_{q}=0$ then $\left(f^{*} G\right)_{p}=0$ and therefore $\operatorname{Supp}\left(f^{*} G\right) \subseteq f^{-1}(\operatorname{Supp}(G))$. Viceversa, if $\left(f^{*} G\right)_{p}=0$, by the lemma we get $G_{q}=0$.

Let $X$ be a quasi-compact separated scheme over $\operatorname{Spec}(R)$ and let $R \rightarrow S$ be a homomorphism of rings. We get the cartesian diagram

where $Y=\operatorname{Spec}(S) \times_{R} X$. Let $F$ be a quasi-coherent sheaf on $X$; we have seen that $f^{*} F$ is quasi-coherent on $Y$. Then $H^{i}(X, F)$ is a $R$-module and $H^{i}\left(Y, f^{*} F\right)$ is an $S$-module.

Proposition 6.60. Let $X$ be a quasi compact separated scheme and assume that $R \rightarrow S$ is flat. Then

$$
H^{i}\left(Y, f^{*} F\right) \simeq H^{i}(X, F) \otimes_{R} S
$$

Proof. Let $\mathcal{U}=\left\{U_{i}\right\}_{i=1}^{r}$ be an affine open cover; since the scheme is separated, we know that $H^{i}(X, F)=\check{H}^{i}(\mathcal{U}, F)=H^{i}\left(\check{C}^{*}(\mathcal{U}, F)\right)$. We notice that the hypotesis implies that $Y$ is quasi-compact and separated. We define $V_{i_{0} \ldots i_{p}}=$ $f^{-1}\left(U_{i_{0} \ldots i_{p}}\right)$. Then we get the equality $H^{i}\left(X, f^{*} F\right)=H^{i}\left(\check{C}^{*}\left(V, f^{*} F\right)\right)$. By definition

$$
\check{C}^{p}\left(V, f^{*} F\right)=\bigoplus_{i_{0}<\cdots<i_{p}} H^{0}\left(V_{i_{0} \ldots i_{p}}, f^{*} F\right)
$$

Each term of the sum can be written as

$$
\begin{aligned}
f^{*} F\left(V_{i_{0} \ldots i_{p}}\right) & \simeq F\left(U_{i_{0} \ldots i_{p}}\right) \otimes_{\mathcal{O}_{Y}\left(U_{i_{0} \ldots i_{p}}\right)} \mathcal{O}_{X}\left(V_{i_{0} \ldots i_{p}}\right) \\
& \simeq F\left(U_{i_{0} \ldots i_{p}}\right) \otimes_{\mathcal{O}_{Y}\left(U_{i_{0} \ldots i_{p}}\right)} \mathcal{O}_{Y}\left(U_{i_{0} \ldots i_{p}}\right) \otimes_{R} S \\
& \simeq F\left(U_{i_{0} \ldots i_{p}}\right) \otimes_{R} S
\end{aligned}
$$

Therefore, $\check{C}^{p}\left(V, f^{*} F\right)=\check{C}^{p}(\mathcal{U}, F) \otimes_{R} S$ for all $p$ and since $S$ is flat over $R$ and therefore keep exactness, we get

$$
H^{i}\left(\check{C}^{*}\left(V, f^{*} F\right)\right) \simeq H^{i}\left(\check{C}^{*}(\mathcal{U}, F)\right) \otimes_{R} S
$$

Definition 6.61. Let $F$ be a quasi-coherent sheaf on $\mathbb{P}_{R}^{n}$. We define $F(d)$ as the sheaf

$$
F \otimes_{\mathbb{P}_{R}^{n}} \mathcal{O}(d)
$$

We say it is the sheaf $F$ shifted by $d$.

It follows immediately from the definition that $\left.\left.F(d)\right|_{U_{i}} \simeq F\right|_{U_{i}}$ and, if $F=$ $\tilde{M}$, then $F(d)=\tilde{M(d)}$. In fact,

$$
\left.\left.\widetilde{M(d)}\right|_{U_{0}} \xrightarrow{\sim} \tilde{M}\right|_{U_{0}}=\left.\left.F\right|_{U_{0}} \xrightarrow{\sim} F(d)\right|_{U_{0}}
$$

and all these morphisms patch togheter.
Theorem 6.62 (Noether's Theorem or $A F+B G$ Theorem). Let $F, G, H \in$ $k\left[x_{0}, x_{1}, x_{2}\right]$ be homogeneous polynomials of degree $d, e, m$ respectively and let $C, D, E \subseteq \mathbb{P}_{k}^{2}$ be the corresponding curves. Assume that $C \cap D \subseteq E$, or equivalently, $I_{E} \subseteq I_{C}+I_{D}$. Then there exist $A, B$ homogeneous of degree $m-d$, $m-e$ respectively such that $H=A F+B G$.

Proof. Let $R=k\left[x_{0}, x_{1}, x_{2}\right]$. We have the exact sequence

$$
0 \rightarrow R(-d-e) \xrightarrow{\binom{G}{-F}} R(-d) \oplus R(-e) \xrightarrow{(F G)} R \rightarrow R /(F, G) \rightarrow 0
$$

Shifting by $m$, we get

$$
0 \rightarrow R(m-d-e) \rightarrow R(m-d) \oplus R(m-e) \rightarrow R(m) \rightarrow R /(F, G)(m) \rightarrow 0
$$

and in terms of sheaves,
$0 \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(m-d-e) \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(m-d) \oplus \mathcal{O}_{\mathbb{P}^{2}}(m-e) \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(m) \rightarrow \mathcal{O}_{C \cap D}(m) \rightarrow 0$
Let now $H \in \operatorname{Ker}\left(H^{0}(\mathcal{O}(m)) \rightarrow H^{0}\left(\mathcal{O}_{C \cap D}(m)\right)\right)$. We want to show that $H$ lies in the image of the map $H^{0}(\mathcal{O}(m-d)) \oplus H^{0}(\mathcal{O}(m-e)) \rightarrow H^{0}(\mathcal{O}(m))$ induced by

$$
\begin{array}{clc}
R_{m-d} \oplus R_{m-e} & \longrightarrow & R_{m} \\
(A, B) & \longmapsto & A F+B G
\end{array}
$$

Notice that $\operatorname{Ker}\left(\mathcal{O}(m) \rightarrow \mathcal{O}_{C \cap D}(m)\right)=I_{C \cap D}$; we can split the sequence into two sequences

$$
\begin{gathered}
0 \rightarrow I_{C \cap D} \rightarrow \mathcal{O}(m) \rightarrow \mathcal{O}_{C \cap D}(m) \rightarrow 0 \\
0 \rightarrow \mathcal{O}(m-d-e) \rightarrow \mathcal{O}(m-d) \oplus \mathcal{O}(m-e) \rightarrow I_{C \cap D} \rightarrow 0
\end{gathered}
$$

Noticing that $H^{1}\left(\mathbb{P}^{2}, \mathcal{O}(m-d-e)\right)=0$, we can consider the cohomology sequence of the second short exact sequence that we have written

$$
0 \rightarrow H^{0}(\mathcal{O}(m-d-e)) \rightarrow H^{0}(\mathcal{O}(m-d)) \oplus H^{0}(\mathcal{O}(m-e)) \rightarrow H^{0}\left(I_{C \cap D}\right) \rightarrow 0
$$

and therefore $H$ comes from an element of $H^{0}(\mathcal{O}(m-d)) \oplus H^{0}(\mathcal{O}(m-e))$, as desired.

Example. Let $S \subseteq \mathbb{P}^{2}$ be a closed subscheme; we want to find a way to know if $S$ is contained in a hypersurface of some degree $d$. Denoting $k\left[x_{0}, \ldots, x_{n}\right]$ by $R$ and identifing $I_{S}(d)$ as $\operatorname{Ker}\left(H^{0}(\mathcal{O}(m)) \rightarrow H^{0}\left(\mathcal{O}_{S}(d)\right)\right)$, we have a correspondance

$$
\{F \in R \mid S \subseteq \operatorname{Proj}(R /(F))\} \longleftrightarrow F \in H^{0}\left(I_{S}(d)\right)
$$

given by Noether's theorem.

Definition 6.63. Let $R$ be a noetherian ring and let $F$ be a coherent sheaf on $\mathbb{P}_{R}^{n}$. We say that $F$ is generated by global sections if for all $p \in \mathbb{P}_{R}^{n}$ the map

$$
H^{0}(F) \otimes_{R} \mathcal{O}_{\mathbb{P}_{R}^{n}, p} \longrightarrow F_{p}
$$

is surjective.
Tensoring for $\mathcal{O}_{\mathbb{P}_{R}^{n}, p} / m_{p}$, we get an equivalent condition since tensor product preserves surjectivity:

$$
H^{0}(F) \otimes_{R} k(p) \longrightarrow F_{p} / m_{p} F_{P}
$$

Example. $\mathcal{O}(d)$ is generated by global sections if and only if $d \geq 0$. In particular, in this case, it is generated by the monomials $x_{0}^{d}, \ldots, x_{n}^{d}$.
Remark 6.64.

- Since $\mathcal{O}(0)=\mathcal{O}, F(0)=F$.
- Since $\mathcal{O}(d) \otimes \mathcal{O}(e) \simeq \mathcal{O}(d+e)$, we get the formula $F(d)(e) \simeq F(d+e)$.

Theorem 6.65 (Serre). Let $R$ be a noetherian ring and let $F$ be a coherent sheaf on $\mathbb{P}_{R}^{n}$.

- $F(d)$ is generated by global sections for all $d \gg 0$
- For all $d \gg 0$, the cohomology groups $H^{i}\left(\mathbb{P}_{R}^{n}, F(d)\right)$ are zero for all $i \geq 1$

Example. If $R=k$ is a field, the Euler characteristic $\chi(F(d))$ is equal to $h^{0}(F(d))$ for all $d \gg 0$.

We now want to establish a bijection between coherent sheaf on the projective space and a particular class of graded module. Clearly, if $M$ is a finitely generated graded $R$-module, $\tilde{M}$ is a quasi-coherent sheaf on $\mathbb{P}_{R}^{n}$. Since $M_{\left(x_{i}\right)}$ is finitely generated over $R_{\left(x_{i}\right)}$, it is coherent.
Conversely, let $F$ be a coherent sheaf on $\mathbb{P}_{R}^{n}$ and let $f \in H^{0}(\mathcal{O}(d))$. For every $d \in \mathbb{Z}$, we get a map

$$
\begin{array}{rll}
\mathcal{O} & \longrightarrow \mathcal{O}(d) \\
s & \longmapsto & s f
\end{array}
$$

Tensoring with $F(e)$,

$$
\begin{array}{clc}
F(e) & \longrightarrow & F(e+d) \\
s & \longmapsto & s f
\end{array}
$$

Therefore, $\oplus H^{0}(F(d))$ give rise to a graded $\oplus H^{0}(\mathcal{O}(d))=R$-module.
Proposition 6.66. $\oplus_{d \in \mathbb{Z}} H^{0}(F(d))$ is a finitely generated $R$-module.
Proof. Let $d \in \mathbb{Z}$ be a integer such that $F(d)$ is generated by global section (it exists by Serre's theorem) and let $p \in \mathbb{P}_{R}^{n}$. We can choose $s_{1}, \ldots, s_{N} \in H^{0}(F(d))$ that generate $F_{p}$ as an $\mathcal{O}_{\mathbb{P}_{R}^{n}, p}$-module and find an open neighbourhood $U$ of $p$ such that $s_{1}, \ldots s_{N}$ generate $F_{q}$ for all $q \in U$. By the quasi-compactness of the projective space, we can find finitely many of these $s_{i}$ that generates $F$. We call these sections $s_{1}, \ldots, s_{N}$ by abuse of notation. We get a map

$$
\begin{array}{ccc}
\mathcal{O}^{N} & \longrightarrow & F(d) \\
\left(f_{1}, \ldots, f_{N}\right) & \longmapsto & \sum_{i=1}^{N} f_{i} s_{i}
\end{array}
$$

Shifting, we get a map $\mathcal{O}(-d)^{N} \rightarrow F$; let $K$ be its kernel. For all $e \in \mathbb{Z}$, we get an exact sequence

$$
0 \rightarrow K(e) \longrightarrow \mathcal{O}(e-d)^{N} \longrightarrow F(e) \rightarrow 0
$$

Taking direct sums

$$
0 \rightarrow \oplus_{e \in \mathbb{Z}} H^{0}(K(e)) \longrightarrow R^{N}(-d) \longrightarrow \oplus_{e \in \mathbb{Z}} H^{0}(F(e)) \longrightarrow \oplus_{e \in \mathbb{Z}} H^{1}(K(e))
$$

Since $R(-d)^{N}$ and $\oplus H^{1}(K(e))$ are finitely generated $R$-modules, $\oplus_{e \in \mathbb{Z}} H^{0}(F(e))$ is finitely generated too, as desired.

Let $F$ be a coherent sheaf on $X=\mathbb{P}_{R}^{n}$ and let $M=\oplus H^{0}(F(d))$. We can construct a $\operatorname{map} \tilde{M} \rightarrow F$. In fact, let $s / x_{0}^{n} \in M_{\left(x_{0}\right)}$, where $s \in H^{0}(F(m))$. Since $1 / x_{0}^{n} \in H^{0}\left(U_{0}, \mathcal{O}(-m)\right)$, the element $s \otimes 1 / x_{0}^{n}$ lies in $H^{0}\left(U_{0}, F(m)(-m)\right) \simeq$ $H^{0}\left(U_{0}, F\right)$. This gives a map

$$
\left.\left.\tilde{M}\right|_{U_{0}} \longrightarrow F\right|_{U_{0}}
$$

We can do the same for the other affine subsets $U_{i}$; they patch togheter giving a map $\tilde{M} \rightarrow F$. These maps are isomorphisms, since they are surjective and

$$
\frac{s}{x_{0}^{d}}=0 \Rightarrow \exists n \in \mathbb{N} \text { s.t. } x_{0}^{n} s=0
$$

Therefore every coherent sheaf $F$ comes from the graded finitely generated module $M=\oplus H^{0}(F(d))$. To check that this correspondance is bijective, we have only to show that

Proposition 6.67. The map $M_{d} \rightarrow H^{0}\left(\mathbb{P}^{n}, \widetilde{M(d)}\right)$ is an isomorphism for all $d \gg 0$.

### 6.6 The Hilbert Polynomial

Definition 6.68. Let $A$ be a noetherian ring and let $M$ be an $A$-module. We say that $p \in \operatorname{Spec}(A)$ is associated with $M$ if there exists $m \in M$ such that $\operatorname{Ann}_{A}(m)=p$, or equivalently there exists an injective map of $A$-modules

$$
A / p \longrightarrow M
$$

We call $\operatorname{Ass}_{A}(M)=\{p \in \operatorname{Spec}(A) \mid p$ is associated with $M\}$; in the case $M=A$, we just say $\operatorname{Ass}(A)=\operatorname{Ass}_{A}(A)$.

Proposition 6.69. Let $A$ be a noetherian ring and let $M$ be an $A$-module.

1. If $M \neq 0$, then $\operatorname{Ass}_{A}(M) \neq \emptyset$.
2. $\{a \in A \mid a x=0$ for some $x \in M\}=\bigcup_{p \in \operatorname{Ass}_{A}(M)} p \subseteq A$.
3. The minimal primes of $A$ lie in $\operatorname{Ass}(A)$.
4. If $A$ is reduced, every $p \in \operatorname{Ass}(A)$ is minimal.
5. If $S \subseteq A$ is a multiplicative set, then $\operatorname{Ass}_{S^{-1} A}\left(S^{-1} M\right)$ is the inverse image of $\operatorname{Ass}_{A}(M)$ in $\operatorname{Spec}\left(S^{-1} A\right)$.
6. If $M$ is finitely generated, $\operatorname{Ass}_{A}(M)$ is finite.

## Proof.

1. Since $A$ is noetherian, the set $\{\operatorname{Ann}(m) \mid m \in M\}$ has a maximal element $I$. We want to show that $I=\operatorname{Ann}(m)$ is prime. Let $a, b \in A$ such that $a b \in I$. This means that $a b m=0$; if $b m \neq 0$, it means that $\operatorname{Ann}(b m) \supseteq \operatorname{Ann}(m)=I$ which is maximal in the set of annihilators. Therefore $\operatorname{Ann}(b m)=\operatorname{Ann}(m)$ and therefore $a m=0$.
2. We call $B=\{a \in A \mid a x=0$ for some $x \in M\}$. Clearly, the containment $B \supseteq \bigcup_{p \in \operatorname{Ass}_{A}(M)} p$ holds. On the other hand, let $a \in B$; then we can consider the set of the annihilators $\{\operatorname{Ann}(m) \mid m \in M \quad a \in \operatorname{Ann}(m)\}$ and a maximal element is an associated prime.
3. Let $p$ be a minimal prime ideal; then $A_{p}$ is an artinian ring and it has a unique prime ideal, $p A_{p}$. Therefore, since $A_{p} \neq 0$, it is the only associated prime. Recalling that the annihilators commute with localization, we get the thesis.
4. It follows from the fact that the zero divisors are exactly the elements that lies in a minimal prime ideal.

A consequence of the proposition is the following equivalence:
Corollary 6.70. Let $A$ be a noetherian ring and let $p \in \operatorname{Spec}(A)$. The following are equivalent:

- $p \in \operatorname{Ass}(A)$
- $m_{p} \in \operatorname{Ass}\left(A_{p}\right)$
- There exists $a \in A_{p} \backslash\{0\}$ such that $a m_{p}=0$
- $\operatorname{dim}\left(A_{p}\right)=0$

Definition 6.71. Let $A$ be a local ring. We say that $A$ has depth zero if $m \in \operatorname{Ass}(A)$.
Let $X$ be a locally noetherian scheme. We say that $p \in X$ is an associated point if $\mathcal{O}_{X, p}$ has depth zero.

Assume that $X$ is noetherian. Then $X$ is the union of its irreducible components and $\operatorname{dim}\left(\mathcal{O}_{X, p}\right)=0$ if and only if $p$ is a generic point of an irreducible component.

Corollary 6.72. Let $X$ be a noetherian scheme. Every generic point of the irreducible component of $X$ is associated. If $X$ is reduced, the converse holds.

Definition 6.73. Let $X$ be a noetherian scheme and let $F$ be a coherent sheaf on $X . p \in X$ is associated with $F$ if $m_{p} \in \operatorname{Ass}_{\mathcal{O}_{X, p}} F_{p}$.

Proposition 6.74. Let $X$ be a noetherian scheme and let $F$ be a coherent sheaf on $X$. Then

- $\operatorname{Ass}_{X}(F)$ is finite
- $\operatorname{Ass}_{X}(F) \subseteq \operatorname{Supp}(F)$
- Every generic point of the irreducible component of the support of $F$ lies in $\operatorname{Ass}_{A}(M)$.

Proof. By the proposition, it is clear that $\operatorname{Ass}_{X}(F)$ is finite, since we can cover $X$ with a finite number of affine open subset and for each of these, $\operatorname{Ass}_{U_{i}}\left(\left.F\right|_{U_{i}}\right)$ is finite. The second statement follows immediately from definition. We only need to check the third. We can assume that $X=\operatorname{Spec}(A), F=\tilde{M}$, where $M$ is a finite $A$-module. Let $p \in X$ be a generic point of an irreducible component of $X$. It follows from the proposition that the support $\operatorname{Supp}_{A_{p}} F_{p}$ is the inverse image of the support of $F$ in $\operatorname{Spec}\left(A_{p}\right)$. Since $A_{p}$ is zero-dimensional by hypotesis, we only need to check that $m_{p} \in \operatorname{Ass}_{A_{p}} F_{p} \subseteq \operatorname{Supp}_{A_{p}} F_{p}$. By the definition of support $F_{p} \neq 0$, hence there exists at least an associated prime.

We want now to introduce an algebraic invariant for coherent sheaves on projective spaces: the Hilbert polynomial. In particular, we want to show that given a coherent sheaf $X$ on $\mathbb{P}_{k}^{n}$, the map

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
t & \longmapsto & \chi(F(t))
\end{array}
$$

is a polynomial. To show this, we need to find a hyperplane that doesn't contain a finite set of points.
Let now $A$ be the polynomial ring $k\left[x_{0}, \ldots, x_{n}\right]$ graded in the usual way and let $h \in A_{1} \backslash\{0\}$ be a linear polynomial. We can consider the multiplication for $h$ as a morphism $\mathcal{O}_{\mathbb{P}_{k}^{n}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}_{k}^{n}}$. This give rise to an exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}_{k}^{n}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_{k}^{n}} \longrightarrow \mathcal{O}_{H} \rightarrow 0
$$

where $H=\operatorname{Proj}(A /(h))$. The fact that $h$ is linear gives an isomorphism $A / H \simeq$ $K\left[y_{0}, \ldots, y_{n-1}\right]$ and therefore $H \simeq \mathbb{P}_{k}^{n-1}$. Consider now a coherent sheaf $F$ on $\mathbb{P}_{k}^{n}$ : we want to understand how multiplication for $h$ behaves on $F$. Tensoring the exact sequence $\otimes F$, we get

$$
F(-1) \xrightarrow{\cdot h} F \longrightarrow F \otimes \mathcal{O}_{H} \rightarrow 0
$$

Let $j: H \rightarrow \mathbb{P}_{k}^{n}$ be the closed embedding of $H$ into $\mathbb{P}_{k}^{n}$. Then

$$
F \otimes \mathcal{O}_{H} \simeq F \otimes j^{*} \mathcal{O}_{H} \simeq j_{*}\left(j^{*} F\right)
$$

Let $p \in \mathbb{P}_{k}^{n}$. We know that shifting doesn't change the stalks, so $\mathcal{O}(-1)_{p} \simeq$ $\mathcal{O}_{\mathbb{P}_{k}^{n}, p}$ and, called $h_{p}$ the image of $h$ in $\mathcal{O}(-1)_{p}$, the multiplication map for $h_{p}$ $F(-1)_{p} \simeq F_{p} \rightarrow F_{p}$ is injective if and only if $h_{p}$ doesn't lie in the union of the associated primes of $F_{p}$ over $A_{p}$. We have shown the following
Proposition 6.75. Let $X=\mathbb{P}_{k}^{n}, F$ be aa coherent sheaf on $X$. Let $h$ be a linear homogeneous polynomial and denote by $H$ the corresponding hyperplane. The multiplication map $\varphi_{h}: F(-1) \rightarrow F$ is injective if and only if $\operatorname{Ass}_{\mathbb{P}_{k}^{n}}(F) \cap H=\emptyset$.

Lemma 6.76. Let $k$ be an infinite field and let $S \subseteq \mathbb{P}_{k}^{n}$ be a finite set. Then there exists $h \in A_{1} \backslash\{0\}$ such that $V(h) \cap S=\emptyset$.

Proof. Let $p \in S$; then $p$ corresponds to a homogeneous prime such that $p \nsupseteq$ $\left(x_{0}, \ldots, x_{n}\right)$. In particular, $p \cap A_{1} \neq A_{1}$. We notice that $p \cap A_{1}$ is a $k$-vector subspace of $A_{1}$. Since $k$ is infinite,

$$
\bigcup_{p \in S}\left(p \cap A_{1}\right) \neq A_{1}
$$

Therefore, it is enough to take $h \in A_{1} \backslash \cup_{p \in S} p \cap A_{1}$.
Lemma 6.77. Let $\varphi: A \rightarrow B$ be a homomorphism of graded rings of degree one. $\varphi$ induces a map $f: \operatorname{Proj}(B) \backslash V_{+}\left(\varphi\left(A_{+}\right) B\right) \rightarrow \operatorname{Proj}(A)$. Let $M$ be a graded $A$-module. Then

$$
\left.\widetilde{M \otimes_{A} B}\right|_{\operatorname{Proj}(B) \backslash V_{+}\left(\varphi\left(A_{+}\right) B\right)} \simeq f^{*} \tilde{M}
$$

Proof. We have already shown that this is true in tha affine case and the maps glue togheter. Since it is an isomorphism on the stalks, it is an isomorphism.

Theorem 6.78. Let $k$ be an infinite field and let $F$ be a coherent sheaf on $\mathbb{P}_{k}^{n}$. The function

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
t & \longmapsto & \chi(F(t))
\end{array}
$$

is a polynomial function of degree equal to $\operatorname{dim}(\operatorname{Supp}(F))$.
Proof. We proceed by induction on $n$.
If $n=0$, we have $\mathbb{P}_{k}^{n}=\operatorname{Spec}(K)$ and since it is zero-dimensional the cohomology groups $H^{i}(\operatorname{Spec}(K), F)=0$ for all $i>0$. Hence $\chi(F(t))=h^{0}(F)$ is constant. Assume $n>0$. By the lemma, we can choose $h \in H^{0}(\mathcal{O}(1)) \backslash\{0\}$ such that

$$
\left.0 \rightarrow F(-1) \longrightarrow F \longrightarrow F\right|_{H} \rightarrow 0
$$

is exact. Tensoring for $\mathcal{O}(t)$, we get

$$
\left.0 \rightarrow F(t-1) \longrightarrow F(t) \longrightarrow F(t)\right|_{H} \rightarrow 0
$$

We know that $H \simeq \mathbb{P}_{k}^{n-1}$ and, called $j: \mathbb{P}_{k}^{n-1} \rightarrow \mathbb{P}_{k}^{n}$ the corresponding closed embedding, $\left.F(t)\right|_{H}=j^{*} F(t) \simeq j^{*} F \otimes_{\mathcal{O}_{H}} j^{*} \mathcal{O}(t)$. By lemma 6.77, $j^{*} \mathcal{O}_{\mathbb{P}_{k}^{n}}(t) \simeq$ $\mathcal{O}_{\mathbb{P}_{k}^{n-1}}(t)$ and therefore

$$
\left.F(t)\right|_{H} \simeq\left(\left.F\right|_{H}\right)(t)
$$

on $H \simeq \mathbb{P}_{k}^{n-1}$. Then $\operatorname{Supp}\left(\left.F\right|_{H}\right)=H \cap \operatorname{Supp}(F)$ and $\operatorname{dim}(H \cap \operatorname{Supp}(F))=$ $\operatorname{dim}(\operatorname{Supp}(F))-1$. By inductive hypotesis, $\chi\left(\left.F(t)\right|_{H}\right)$ is a polynomial of degree $\operatorname{dim}(\operatorname{Supp} F)-1$. The addictivity of the Euler characteristic implies that

$$
\chi(F(t))-\chi(F(t-1))=\chi\left(\left.F(t)\right|_{H}\right)
$$

A lemma about this relation concludes the proof.
We want now to obtain the same results over any field. Let $k^{\prime} / k$ be an extension of field. By extension of scalars, we get the diagram


By lemma 6.59, $\operatorname{dim}\left(\operatorname{Supp}\left(\varphi^{*} F\right)\right)=\operatorname{dim}(\operatorname{Supp} F)$ and $\varphi^{*} \mathcal{O}_{\mathbb{P}_{k}^{n}}(t) \simeq \mathcal{O}_{\mathbb{P}_{k^{\prime}}^{n}}(t)$ by lemma 6.77. Then $\varphi^{*}(F(t))=\left(\varphi^{*} F\right)(t)$; since $k^{\prime} / k$ is flat, we can use the theorem 6.60 and obtain the relation $H^{i}\left(\varphi^{*} F(t)\right)=H^{i}(F(t)) \otimes_{k} k^{\prime}$ and therefore $\chi(F(t))=\chi\left(\left(\varphi^{*} F\right)(t)\right)$. We get the following:

Theorem 6.79. Let $k$ be a field and let $F$ be a coherent sheaf on $\mathbb{P}_{k}^{n}$. Then

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
t & \longmapsto & \chi(F(t))
\end{array}
$$

is a polynomial function of degree $\operatorname{dim}(\operatorname{Supp}(F))$.
Summing up, if $k$ is a field and $F$ is a coherent sheaf on $X=\mathbb{P}_{k}^{n}, \chi(F(t))$ is a polynomial function of degree equal to the dimension of the support of $F$.

$$
\chi(F(t))=\frac{d}{m!} t^{m}+\text { lower order terms }
$$

We call $d$ the degree of $F$. If $H \subseteq \mathbb{P}_{k}^{n}$ is a hyperplane that doesn't contain any of the associated points of $F$, then $\operatorname{deg}\left(\left.F\right|_{H}\right)=\operatorname{deg}(F)$.
Let $X \subseteq \mathbb{P}_{k}^{n}$ be a closed subscheme and let $F=\mathcal{O}_{X}$. We say that $\operatorname{deg}(X)=$ $\operatorname{deg}\left(\mathcal{O}_{X}\right)$; if $\operatorname{dim} X=0$ then $\operatorname{deg} X=h^{0}(X, \mathcal{O})$. If $\operatorname{dim} X>0$ and $H \subseteq \mathbb{P}_{k}^{n}$ is a hyperplane not passing through any associated point of $X$, then $\operatorname{deg}(X)=$ $\operatorname{deg}(X \cap H)$ This gives an inductive definition of the degree of $X$.

### 6.7 Locally Free Sheaves

Definition 6.80. Let $X$ be a locally ringed space and let $r \geq 0$. A sheaf of $\mathcal{O}_{X}$-modules $L$ is locally free of rank $r$ if there exists an open cover $X=\cup U_{i}$ such that $\left.L\right|_{U_{i}} \simeq \mathcal{O}_{U_{i}}^{\oplus r}$. We say that a sheaf is invertible if it is locally free of rank one.

Example. $\mathcal{O}_{\mathbb{P}^{n}}(d)$ is invertible on $\mathbb{P}_{k}^{n}$.
It follows from the definition that a locally free sheaf on a scheme $X$ is always quasi-coherent and it is coherent if $X$ is locally noetherian. Furthermore, if $f: X \rightarrow Y$ is a morphism of locally ringed spaces and $F$ is a locally free sheaf of rank $r$ on $X, f^{*} F$ is locally free of rank $r$ since locally pullback corresponds to tensor product and tensor product preserves freeness. In particular, the pullback of an invertible sheaf is invertible.

Proposition 6.81. Let $L$ be a coherent sheaf on a locally noetherian scheme $X$. Then $L$ is locally free of rank $r$ if and only if $L_{p} \simeq \mathcal{O}_{X, p}^{\oplus r}$ for all $p \in X$.

This follows from the algebraic property
Lemma 6.82. Let $A$ be a noetherian ring and let $M$ be a finite $A$-module. Let $p \in \operatorname{Spec}(A)$ such that $M_{p} \simeq A_{p}^{\oplus r}$. Then there exists $f \in A \backslash p$ such that $M_{f} \simeq A_{f}^{\oplus r}$.

Proof. Let $x_{1}, \ldots x_{n} \in M$ such that $\left\{\left(x_{1}\right)_{p}, \ldots,\left(x_{n}\right)_{p}\right\}$ is a basis for $M_{p}$. We consider the map

$$
\begin{aligned}
\varphi: \quad A^{n} & \longrightarrow M \\
e_{i} & \longmapsto
\end{aligned}
$$

Let $K$ be the kernel of this map. Then $K_{p}=0$ by hypotesis and since $K$ is finitely generated there exists $f \in A \backslash p$ such that $f K=0$; therefore $\varphi$ is injective on $A_{f}$. The same holds for the cokernel and the thesis follows.

Proposition 6.83. Let $X$ be a reduced noetherian scheme and let $F$ be a coherent sheaf on $X$. Assume that

$$
\mathrm{rk}_{p} F=\operatorname{dim}_{k(p)} F_{p} / m_{p} F_{p}=r
$$

for all $p \in X$. Then $F$ is locally free of rank $r$.
Proof. It suffices to use the propositions 5.75 and 5.72 , togheter with the previous lemma.

We now focus our attention on invertible sheaves. Let $L$ be an invertible sheaf on a scheme $X$ and let $s \in H^{0}(X, L)$. We can evaluate $s$ at every point of X

$$
s(p)=\left[s_{p}\right] \in L_{p} / m_{p} L_{p} \simeq k(p)
$$

Definition 6.84. We say that $s \in H^{0}(X, L)$ never vanishes or vanishes nowhere if $s(p) \neq 0$ for all $p \in X$ or equivalently $s_{p} \neq m_{p} L_{p}$.

Every $s \in H^{0}(X, L)$ induces a morphism

$$
\begin{array}{clc}
\mathcal{O}_{X} & \longrightarrow & L \\
f & \longmapsto & f s
\end{array}
$$

which is a homomorphism of sheaf of $\mathcal{O}_{X}$-modules. Conversely, given such a map, we can identify $s \in H^{0}(X, L)$ by looking to the image of 1 in the map between global sections. This gives a bijection $\operatorname{Hom}_{\mathcal{O}_{X}}\left(\mathcal{O}_{X}, L\right) \simeq H^{0}(X, L)$.
Proposition 6.85. The map

$$
\begin{array}{clc}
\mathcal{O}_{X} & \longrightarrow & L \\
1 & \longmapsto & s
\end{array}
$$

is an isomorphism if and only if $s$ is never vanishing.
Proof. A morphism of sheaves is an isomorphism if and only if it is an isomorphism on the stalks and this happens if and only if $s$ is never vanishing.

Let $L$ be an invertible sheaf and let $s_{0}, \ldots, s_{n} \in H^{0}(X, L)$. We say that they never vanish if for all $p \in X$ there exists $i$ such that $\left(s_{i}\right)(p) \neq 0$. This is equivalent to say that the map

$$
\begin{array}{ccc}
\mathcal{O}_{X}^{n+1} & \longrightarrow & L \\
\left(f_{0}, \ldots, f_{n}\right) & \longmapsto & \sum_{i=0}^{n} f_{i} s_{i}
\end{array}
$$

is injective. For example, if $X=\mathbb{P}_{k}^{n}$ and $L=\mathcal{O}(d)(d \geq 0)$, we can consider the global sections

$$
\left(x_{0}^{d}, \ldots, x_{n}^{d}\right) \in H^{0}(X, \mathcal{O}(d))^{n+1}
$$

and these never vanish. We now want to relate this $n+1$-uples to morphism to $\mathbb{P}_{k}^{n}$.

Let $X$ be a scheme over a field $k, L$ be an invertible sheaf on $X$ and let $s_{0}, \ldots, s_{n} \in H^{0}(X, L)$. We get an open cover $X=\cup_{i=0}^{n} X_{s_{i}}$, where

$$
X_{s_{i}}=\left\{p \in X \mid s_{i}(p) \neq 0\right\}
$$

We can construct in this way a map $X_{s_{i}} \longrightarrow \mathbb{A}_{k}^{n}$ induced by the homomorphism

$$
k\left[x_{1}, \ldots, x_{n}\right] \underset{x_{j}}{\longrightarrow} H^{0}\left(X_{s_{i}}, L\right) \longrightarrow s_{j} / s_{i}
$$

We can glue these map since they coincide on the intersection and therefore we can define a map $\left(s_{0}, \ldots, s_{n}\right): X \rightarrow \mathbb{P}_{k}^{n}$.
Observation 6.86. Let $L$ is an invertible sheaf and $s_{0}, \ldots s_{n}$ lie in $H^{0}(X, L)$. Assume that there exist two different isomorphism $\varphi: \mathcal{O}_{X} \rightarrow L$ and $\psi: \mathcal{O}_{X} \rightarrow L$. $\left(s_{0}, \ldots, s_{n}\right)$ corresponds to $\left(f_{0}, \ldots, f_{n}\right) \in \mathcal{O}_{X}(X)^{n+1}$ via $\varphi$ and to $\left(g_{0}, \ldots, g_{n}\right) \in$ $\mathcal{O}(X)^{n+1}$ via $\psi$. We notice that $\varphi^{-1} \circ \psi$ is an automorphism of $\mathcal{O}_{X}$ and therefore it must be an isomorphism at the level of global section. Since an automorphism in this case is a multiplication for an invertible element, there exists $s \in \mathcal{O}_{X}(X)^{*}$ such that $s f_{i}=g_{i}$.

An alternative way of constructing maps to $\mathbb{P}_{R}^{n}$ is the following. Consider

$$
\mathbb{G}_{m, R}=\mathbb{A}_{R}^{1} \backslash \operatorname{Spec}(R)=\operatorname{Spec}\left(R[t]_{t}\right)
$$

and the map

$$
\begin{array}{ccc}
\mathbb{A}_{R}^{1} \times \mathbb{A}_{R}^{n+1} & \longrightarrow & \mathbb{A}_{R}^{n+1} \\
\left(t, x_{0}, \ldots, x_{n}\right) & \longmapsto & \left(t x_{0}, \ldots, t x_{n}\right)
\end{array}
$$

induced by

$$
\begin{array}{rlc}
R\left[x_{0}, \ldots, x_{n}\right] & \longrightarrow & R\left[t, x_{0}, \ldots, x_{n}\right] \\
x_{i} & \longmapsto & t x_{i}
\end{array}
$$

We get the restriction

$$
\mathbb{G}_{m, R} \times\left(\mathbb{A}^{n+1} \backslash\{0\}\right) \longmapsto \mathbb{A}^{n+1} \backslash\{0\}
$$

which induces the commutative diagrams


Definition 6.87. Let $L, L^{\prime}$ be invertible sheaves on $X$. Given $s_{0}, \ldots, s_{n}$, $t_{0}, \ldots, t_{n}$ nowhere vanishing section of $L, L^{\prime}$ respectively, we say $\left(L, s_{0}, \ldots, s_{n}\right)$ is isomorphic to $\left(L^{\prime}, t_{0}, \ldots, t_{n}\right)$ if there exists an isomorphism

$$
\begin{array}{llll}
\varphi: \quad L & \longrightarrow L^{\prime} \\
& s_{i} & \longmapsto & t_{i}
\end{array}
$$

If there exists such an isomorphism $\varphi$, then $\left(s_{0}, \ldots, s_{n}\right)$ and $\left(t_{0}, \ldots, t_{n}\right)$ define the same map $X \rightarrow \mathbb{P}_{R}^{n}$. This follows from the diagram

where $f$ is the map induced by $s_{0}, \ldots, s_{n}$ and $g$ is the map induced by $t_{0}, \ldots, t_{n}$. The diagram commutes and hence the map doesn't depend on the trivialization. Notice that if there exists an isomorphism $\varphi: L \rightarrow L^{\prime}$ such that $\varphi\left(s_{i}\right)=t_{i}$, it must be unique by the commutativity of these diagram


Corollary 6.88. Let $X=\cup U_{i}$ be an open affine cover of $X$ and assume

$$
\left.\left.\left(L, s_{0}, \ldots, s_{n}\right)\right|_{U_{i}} \simeq\left(L^{\prime}, t_{0}, \ldots, t_{n}\right)\right|_{U_{i}}
$$

Then $\left(L, s_{0}, \ldots, s_{n}\right) \simeq\left(L^{\prime}, t_{0}, \ldots, t_{n}\right)$.
Let $f: X \rightarrow \mathbb{P}_{k}^{n}$ be a morphism of $k$-schemes. Then $L=f^{*} \mathcal{O}_{\mathbb{P}_{k}^{n}}(1)$ is an invertible sheaf.

$$
\begin{array}{cccc}
f^{*}: & H^{0}(\mathcal{O}(1)) & \longrightarrow & H^{0}(X, L) \\
x_{i} & \longmapsto & s_{i}
\end{array}
$$

The induced map has the property that $X_{s_{i}}=f^{-1}\left(U_{i}\right)$.
Theorem 6.89. Let $X$ be a scheme over $\operatorname{Spec}(R)$. There exists a bijection between morphisms of $R$-schemes $X \rightarrow \mathbb{P}_{R}^{n}$ and isomorphism classes of invertible sheaves $\left(L, s_{0}, \ldots, s_{n}\right)$ as above.

Example. Let $X=\mathbb{P}_{R}^{n}$ and $L=\mathcal{O}(d), d>0$. Then $H^{0}\left(\mathbb{P}_{R}^{n}, \mathcal{O}(d)\right)$ is free over $\binom{n+d}{n}$ monomials. They generated $\mathcal{O}(d)$; the corresponding map

$$
V_{d}: \mathbb{P}^{n} \longrightarrow \mathbb{P}^{\binom{n+d}{n}-1}
$$

which is called the Veronese embedding, is a closed embedding (it is proper!).
Let $X$ be a scheme over $\operatorname{Spec}(R)$ and let $L$ be an invertible sheaf on $X$. We have seen that the choice of an $n$-uple $s_{0}, \ldots, s_{n} \in H^{0}(X, L)$ induces a morphism of $R$-schemes $X \rightarrow \mathbb{P}_{R}^{n}$. Assume that $R=k$ is a field; then an element $h \in A=k\left[x_{0}, \ldots, x_{n}\right]_{(1)}$ corresponds to the hyperplane

$$
H=\operatorname{Proj}(A /(h)) \subseteq \mathbb{P}_{k}^{n-1}
$$

Definition 6.90. A morphism $X \rightarrow \mathbb{P}_{k}^{n}$ is non-degenerate if it doesn't factor through a hyperplane.

Proposition 6.91. Let $L=f^{*} \mathcal{O}(1)$ and let $s_{i}=f^{*} x_{i}$. Then $f$ is non degenerate if and only if $s_{0}, \ldots, s_{n}$ are linearly independent over $k$.

Let $a=\left(a_{0}, \ldots, a_{n}\right)$ be an element of $k^{n+1} \backslash\{0\}$. The point corresponds to the hyperplane

$$
H_{a}=\operatorname{Proj}\left(A /\left(a_{0} x_{0}+\cdots+a_{n} x_{n}\right)\right) \subseteq \mathbb{P}_{k}^{n}
$$

Then $f$ factors through $H_{a}$ if and only if $\sum a_{i} s_{i}=0$ if and only if $f^{*}\left(\sum a_{i} x_{i}\right)=$ 0 . It can be seen reducing to the affine case.
Example. Let $R$ be a ring. Consider the product


Let $L$ be the sheaf of modules $p r_{1}^{*} \mathcal{O}_{\mathbb{P}^{n}}(1) \otimes p r_{2}^{*} \mathcal{O}_{\mathbb{P}^{m}}(1)$. We get a map

$$
H^{0}\left(\mathcal{O}_{\mathbb{P}^{m}}(1)\right) \otimes_{R} H^{0}\left(\mathcal{O}_{\mathbb{P}^{n}}(1)\right) \rightarrow H^{0}\left(p r_{1}^{*} \mathcal{O}_{\mathbb{P}^{m}}(1)\right) \otimes_{R} H^{0}\left(p r_{2}^{*} \mathcal{O}_{\mathbb{P}^{n}}(1)\right) \rightarrow H^{0}(L)
$$

The $(m+1)(n+1)$ sections of $\mathbb{P}^{m} \times \mathbb{P}^{n}$ given by $x_{i} \otimes y_{j}$ are nowhere vanishing and therefore induce an embedding

$$
\mathbb{P}^{m} \times \mathbb{P}^{n} \longrightarrow \mathbb{P}^{m n-n-m}
$$

### 6.7.1 The Picard Group

Let $X$ be a locally ringed space and let $L_{1}, L_{2}$ be invertible sheaves on $X$. Then $L_{1} \otimes_{\mathcal{O}_{X}} L_{2}$ is invertible and this gives the structure of monoid to the set of invertible sheaves on $X$.

Definition 6.92. Let $F, G$ be sheaves of $\mathcal{O}_{X}$-modules. We define the sheaf of $\mathcal{O}_{X}$-modules

$$
\underline{\operatorname{Hom}}_{\mathcal{O}_{X}}(F, G)(U):=\operatorname{Hom}_{\mathcal{O}_{X}}\left(\left.F\right|_{U},\left.G\right|_{U}\right)
$$

We define the dual sheaf $F^{\vee}$

$$
F^{\vee}=\underline{\operatorname{Hom}}_{\mathcal{O}_{X}}\left(F, \mathcal{O}_{X}\right)
$$

For every open subset $U$, we get a map

$$
\begin{array}{clc}
F(U) \times F^{\vee}(U) & \longrightarrow \mathcal{O}(U) \\
(s, \alpha) & \longmapsto \alpha(s)
\end{array}
$$

which induces a map

$$
F \otimes_{\mathcal{O}_{X}} F^{\vee} \longrightarrow \mathcal{O}
$$

Remark 6.93. Let $X$ be a scheme and let $F, G$ be quasi-coherent sheaves. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_{X}}(F, G)$ is not quasi coherent. For example,

$$
\left(\mathcal{O}_{X}^{\mathbb{N}}\right)^{\vee} \simeq \mathcal{O}_{X}^{\mathbb{N}}
$$

which is not quasi-coherent.
If $X$ is locally noetherian and $F$ is coherent, then $\operatorname{Hom}_{\mathcal{O}_{X}}\left(F, \mathcal{O}_{X}\right)$ is quasicoherent.

Proposition 6.94. Let $L$ be an invertible sheaf on $X$. Then $L^{\vee}$ is invertible and the map $L \otimes_{\mathcal{O}_{X}} L^{\vee} \rightarrow \mathcal{O}$ is an isomorphism.

Proof. Notice that it is a local statement. Let $U$ be an open subscheme such that $\left.L\right|_{U} \simeq \mathcal{O}_{U}$. Then $L^{\vee}(U)=\operatorname{Hom}_{\mathcal{O}_{X}}\left(\mathcal{O}_{U}, \mathcal{O}_{U}\right) \simeq \mathcal{O}_{U}$ and therefore $L^{\vee}$ is invertible.

Definition 6.95. The $\operatorname{Picard} \operatorname{Group} \operatorname{Pic}(X)$ is the set of isomorphism classes of invertible sheaves on $X$.

$$
\left[L_{1}\right]+\left[L_{2}\right]=\left[L_{1} \otimes L_{2}\right] \quad 0=\left[\mathcal{O}_{X}\right] \quad-[L]=\left[L^{\vee}\right]
$$

## Chapter 7

## Divisors

### 7.1 Weil Divisors

Definition 7.1. Let $X$ be an integral, noetherian scheme. We say that $X$ is regular in codimension 1 if for all $p \in X$ such that $\operatorname{dim} \mathcal{O}_{X, p}=1$ the stalk is a DVR. We denote the set of point of codimension 1 as $X^{(1)}$.

We will consider during this chapter only schemes that are regular in codimension 1.
Example. If $\operatorname{dim} X=0$, then $X^{(1)}=\emptyset$; therefore $X$ is regular in codimension 1 . If $X$ is normal, $X$ is regular in codimension 1. The converse doesn't hold; we have the equivalence

$$
X \text { normal } \Longleftrightarrow\left\{\begin{array}{l}
X \text { is regular in codimension 1 } \\
\forall U \subseteq X \mathcal{O}_{X}(U)=\cap_{p \in U^{(1)}} \mathcal{O}_{X, p}
\end{array}\right.
$$

Definition 7.2. The group of Weil Divisors is the free abelian group $\operatorname{Div}(X)$ generated by $X^{(1)}$. If $D \in \operatorname{Div}(X)$ then

$$
D=\sum_{p \in X^{(1)}} n_{p} p
$$

where the set $\left\{p \in X^{(1)} \mid n_{p} \neq 0\right\}$ if finite.
Let $p \in X^{(1)}$; then $\mathcal{O}_{X, p} \subseteq K(X)$ is a DVR and it corresponds to a discrete valuation

$$
v_{p}: K(X)^{*} \longrightarrow \mathbb{Z}
$$

Proposition 7.3. Let $f \in K(X)^{*}$. Then the set

$$
\left\{p \in X^{(1)} \mid v_{p}(f) \neq 0\right\}
$$

is finite.
Proof. Since $K(X)$ is a field, $f$ has an inverse $f^{-1}$ and there exists a nonempty open subset $U \subseteq X$ such that $f, f^{-1} \in \mathcal{O}(U) \subseteq K(X)$. This means that $f \in \mathcal{O}(U)^{*}$ and therefore $v_{p}(f)=0$ for all $p \in U^{(1)} \subseteq X^{(1)}$. The complement $Z=X \backslash U$ is a proper closed subset; let $p \in X^{(1)} \cap Z$. Then its closure is an irreducible component of $Z$; by noetherianity, $Z$ has only a finite number of irreducible components.

As a consequence, we get a map

$$
\begin{array}{cccc}
\text { div: } \quad K(X)^{*} & \longrightarrow & \operatorname{Div}(X) \\
f & \longmapsto \sum_{p \in X^{(1)}} v_{p}(f) p
\end{array}
$$

which is a group homomorphism since $v_{p}(f g)=v_{p}(g)+v_{p}(f)$.
Definition 7.4. A Weil Divisor is principal if it lies in the image of the map div. We define the cokernel of the map div as the class group of $X$

$$
\mathrm{Cl}(X)=\operatorname{Div}(X) / \operatorname{Im}(d i v)
$$

If $k$ is a field, since $\operatorname{Div}(k)=0$, we get $\operatorname{Cl}(k)=0$.
The same holds if $R$ is a PID, since div is surjective. Given $p \in R^{(1)}, p$ is a maximal ideal and so $p=(f)$. Clearly, $v_{p}(f)=1$ and $v_{q}(f)=0$ for all $q \in X^{(1)} \backslash\{p\}$ and therefore $\operatorname{div}(f)=p$.

Proposition 7.5. Let $A$ be a noetherian domain. Then $A$ is a UFD if and only if all primes of height one are principal.
Proof. One implication is trivial. Then assume that every prime of height one is principal. Since $A$ is noetherian, every element is a product of irreducible element. So it is enough to prove that if $f \in A$ is irreducible, $(f)$ is prime. Let $p$ be a minimal prime ideal of $(f)$. Then the height of $p$ is 1 by Krull's Hauptidealsatz and therefore $p=(g)$. This means that $f=g h$ since $f \in p ; f$ is irreducible and so $h \in A^{*}$. This means that $p=(g)=(f)$, as desired.

Let $A$ be a noetherian domain regular in codimension one and let $K$ be its fraction field. Then $A$ is normal if and only if the set

$$
\left\{f \in K^{*} \mid v_{p}(f) \geq 0 \forall p \in A^{(1)}\right\}
$$

is contained in $A$, since this is equivalent to say $A=\cap_{h t(p)=1} A_{p}$.
Definition 7.6. We say that a divisor $D$ is effective (and we write $D \geq 0$ ) if $n_{p} \geq 0$ for all $p \in X^{(1)}$.

Let $A$ be a normal domain and let $p \in A^{(1)}$. We notice that in this case $p$ is principal (in the sense that it is generated by one element) if and only if there exists $f \in K^{*}$ such that $\operatorname{div}(f)=p$. In fact, $\operatorname{div}(f)=p$ if and only if $v_{p}(f)=1$ and $v_{q}(f)=0$ for all $q \in A^{(1)} \backslash\{p\}$. Since $A$ is normal, this is equivalent to say that $f \in p \subseteq A$ and that if $g \in p, \operatorname{div}(g) \geq \operatorname{div}(f)$. So $g f^{-1}$ is effective; $A$ is normal, therefore $g f^{-1}$ lies in $A$ and this means that $g \in(f)$ and $(f)=p$

Proposition 7.7. Let $X$ be an integral normal scheme and $f \in K(X)^{*}$. Then $f \in \mathcal{O}(X)$ if and only if $\operatorname{div}(f) \geq 0$.

Proof. Let $f \in \mathcal{O}(X)$. Then for every $U \subseteq X$ open affine subset $\left.f\right|_{U} \in \mathcal{O}(U)$ and since the scheme is normal we get $\operatorname{div}(f) \geq 0$. Viceversa, assume that $\operatorname{div}(f) \geq 0$. Then since $\mathcal{O}(U)$ is normal for every open affine subset, we get $f \in \mathcal{O}(U)$ for all $U$. Then $f \in \mathcal{O}(X)$ for the gluing property.

Corollary 7.8. Let $A$ be a normal noetherian domain. Then $A$ is UFD if and only if $\mathrm{Cl}(A)=0$.

Proof. If $\mathrm{Cl}(A)=0$, every prime $p \in A^{(1)}$ lies in the image of div and since the scheme is normal every prime of height 1 is principal. Therefore $A$ is UFD. Viceversa, if $A$ is a UFD, every prime of height one is principal and therefore it lies in the image of the map div.

Corollary 7.9. Let $R$ be a UFD. Then $\operatorname{Cl}\left(\mathbb{A}_{R}^{n}\right)=0$.
Let $X$ be an integral scheme and let $U$ be an open non-empty subscheme. We can define the restriction map

$$
\begin{aligned}
\operatorname{Div}(X) & \longrightarrow\left\{\begin{array}{cc}
\operatorname{Div}(U) \\
p & p \in U \\
0 & p \notin U
\end{array}\right.
\end{aligned}
$$

It is surjective and since $K(U)=K(X)$ and for every $f \in K(X)^{*}$ holds

$$
\left.\operatorname{div}_{X} f\right|_{U}=\operatorname{div}_{U} f
$$

As a consequence, we get the induced map between the class groups $\mathrm{Cl}(X) \rightarrow$ $\mathrm{Cl}(U)$.
We now consider $Z=X \backslash U$; then $Z=\cup Z_{i}$ is the union of its irreducible components and $\operatorname{codim}_{Z_{i}} X \geq 1$. If the equality holds, we can consider the generic point $z_{i}$ of $Z_{i}$. Then $z_{i} \in \operatorname{Div}(X)$.

Proposition 7.10. In this setting, the kernel of the restriction map is generated by the classes of the $z_{i}$ 's.

$$
\operatorname{Ker}(\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(U))=\left\langle\left[z_{i}\right]\right\rangle_{\operatorname{codim}_{Z_{i}} X=1}
$$

Proof. Let $z_{1}, \ldots, z_{s}$ be the generic points of the irreducible components of $Z$. Then certainly

$$
\operatorname{Ker}(\operatorname{Div}(X) \rightarrow \operatorname{Div}(U))=\left\langle z_{1}, \ldots, z_{s}\right\rangle
$$

We want to show that the same holds for the class group. Let $\alpha \in \mathrm{Cl}(X)$ be an element such that $\left.\alpha\right|_{U}=0$ in $\operatorname{Cl}(U)$. By definition, $\alpha=[D]$, where $D \in \operatorname{Div}(X)$. Then there exists $f \in K(X)^{*}=K(U)^{*}$ such that $\left.D\right|_{U}=d i v_{U} f$. Therefore,

$$
\left.\left(D-\operatorname{div}_{X} f\right)\right|_{U}=0
$$

in $\operatorname{Div}(U)$ and $D-\operatorname{div}_{X} f=\sum n_{i} z_{i}$. Passing to the class group,

$$
\alpha=\sum n_{i}\left[z_{i}\right]
$$

as desired.
Corollary 7.11. If $\operatorname{codim}_{X \backslash U} X \geq 2$, then $\mathrm{Cl}(U) \simeq \mathrm{Cl}(X)$.
Corollary 7.12. Let $k$ be a field; then $\operatorname{Cl}\left(\mathbb{P}_{k}^{n}\right) \simeq \mathbb{Z}$.
Proof. Let $H$ be the hyperplane

$$
H=\operatorname{Proj}\left(k\left[x_{0}, \ldots, x_{n}\right] /\left(x_{0}\right)\right)
$$

Then $H$ is irreducible of codimension 1 and $\mathbb{P}_{k}^{n} \backslash H \simeq \mathbb{A}_{k}^{n}$; since $\operatorname{Cl}\left(\mathbb{A}_{k}^{n}\right)=0$, it holds

$$
\mathrm{Cl}\left(\mathbb{P}_{k}^{n}\right)=\langle[H]\rangle
$$

We want to show that it is torsion free. Assume $d[H]=0$. Then $d H=\operatorname{div}(f)$ where $f \in K\left(\mathbb{P}_{k}^{n}\right)^{*}$. We can assume $d \geq 0$; therefore $f \in H^{0}\left(\mathbb{P}_{k}^{n}, \mathcal{O}\right)=k$ since the scheme is normal. We get $f \in k^{*}$ and $d=v_{H}(f)=0$.

We now want to generalize this corollary. Let $X$ be the scheme $\mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{r}}$ and consider the projections

$$
p r_{i}: X \longrightarrow \mathbb{P}_{k}^{n_{i}}
$$

We call $L_{i}$ an hyperplane in the $i$-th coordinate. Then

$$
H_{i}=p r^{-1}\left(L_{i}\right) \simeq \mathbb{P}^{n_{1}} \times \cdots \times L_{i} \times \cdots \times \mathbb{P}^{n_{r}}
$$

$H_{i}$ is integral over $k$ and $\operatorname{codim}_{H_{i}} X=1$. Since

$$
X \backslash\left(\bigcup_{i=1}^{r} H_{i}\right)=\bigcap_{i=1}^{r} X \backslash H_{i} \simeq \prod_{i=1}^{r} \mathbb{A}_{k}^{n_{i}}
$$

we get as before $\mathrm{Cl}(X)=\left\langle H_{1}, \ldots, H_{r}\right\rangle$.

## Proposition 7.13.

$$
\mathrm{Cl}(X)=\mathbb{Z}\left[H_{1}\right] \oplus \cdots \oplus \mathbb{Z}\left[H_{r}\right]
$$

Proof. Let $a_{1}, \ldots, a_{r} \in \mathbb{Z}$ be elements such that $\left[\sum a_{i} H_{i}\right]=0$. Then there exists $\varphi \in K(X)$ such that $\sum a_{i} H_{i}=\operatorname{div}(\varphi)$. Then $v_{p}(\varphi)=0$ for all $p \in\left(\mathbb{A}_{k}^{\sum_{n} n_{i}}\right)^{(1)}$ and it means that $\varphi \in k\left[x_{0}, \ldots, x_{\sum n_{i}}\right]^{*}=k^{*}$ and therefore $\operatorname{div}(\varphi)=0$.

### 7.2 Cartier Divisors

Definition 7.14. Let $X$ be an integral, noetherian, regular in codimension one scheme, let $\xi$ be its generic point and consider an invertible sheaf $L$ on $X$. We call a rational section an element of $L_{\xi}$.

Notice that if $U \subseteq X$ is open and non-empty, the map

$$
\begin{array}{rlr}
H^{0}(U, L) & \longrightarrow & L_{\xi} \\
s & \longmapsto & s_{\xi}
\end{array}
$$

is injective. We now want to define a valuation: since $L$ is invertible, for all $p \in X^{(1)}$ there exists an open neighbourhood of $p$ such that $\left.L\right|_{U} \simeq \mathcal{O}_{U}$ and this gives an isomorphism $L_{\xi} \stackrel{\varphi}{\simeq} \mathcal{O}_{X, \xi}$.

Definition 7.15. Let $s \in L_{\xi} \backslash\{0\}$ and $p \in X^{(1)}$. We define

$$
v_{p}(s):=v_{p}(\varphi(s))
$$

The valuation is well defined. In fact, if we choose a different isomorphism, they differ from the multiplication for an invertible element and the valuation of an invertible element is 0 . The independence follows from the addictivity of $v_{p}$.
In the same way, we can define

$$
\operatorname{div}(s):=\sum_{p \in X^{(1)}} v_{p}(s) p
$$

The sum is taken over a finite set of indexes, since we can restrict to an open cover where $L \simeq \mathcal{O}_{U}$; by the quasi-compactness of $X$, we get that the sum is finite.
Remark 7.16. If $s \in H^{0}(X, L) \backslash\{0\}, \operatorname{div}(s) \geq 0$; if $X$ is normal, the converse holds.

Definition 7.17. Let $(L, s),\left(L^{\prime}, s^{\prime}\right)$ be invertible sheaves with rational sections. We say they are isomorphic is there exists an isomorphism $\varphi: L \rightarrow L^{\prime}$ such that $\varphi_{\xi}(s)=s^{\prime}$.
A Cartier Divisor of $X$ is an isomorphism class of pairs $(L, s)$ as above. We denote this set by $\operatorname{CaDiv}(X)$.

We want to give to this set a natural group structure. We can consider the tensor product of two pairs $(L, s),\left(L^{\prime}, s^{\prime}\right)$; indeed, $\left(L \otimes \mathcal{O}_{X} L^{\prime}\right)_{\xi} \simeq L_{\xi} \otimes_{\mathcal{O}_{X, \xi}} L_{\xi}^{\prime}$ and here we have the element $s \otimes s^{\prime}$. Therefore the tensor product is the pair $\left(L \otimes L^{\prime}, s \otimes s^{\prime}\right)$. We define the sum as

$$
[L, s]+\left[L^{\prime}, s^{\prime}\right]=\left[L \otimes L^{\prime}, s \otimes s^{\prime}\right]
$$

which is associative and commutative. The zero is clearly the element $\left[\mathcal{O}_{X}, 1\right]$; we have to find the inverse element. Let $[L, s] \in \operatorname{CaDiv}(X)$; then $\left(L^{\vee}\right)_{\xi}=\left(L_{\xi}\right)^{\vee}$ (we are considering the dual as a $K(X)$-vector space). Given $s_{\xi} \in L_{\xi}$, we can identify canonically an element of $s_{\xi}^{\prime} \in L_{\xi}^{\vee}$. We get the maps

\[

\]

Then $\left[L^{\vee}, s^{\vee}\right]$ is the inverse of $[L, s]$.
We easily get the group homomorphism

$$
\begin{array}{clc}
\mathrm{CaDiv}(X) & \longrightarrow & \operatorname{Pic}(X) \\
{[L, s]} & \longmapsto & {[L]}
\end{array}
$$

and obiously it is surjective. By definition, the kernel is given by the elements $[L, s]$ such that $L \simeq \mathcal{O}_{X}$ and $s$ corresponds to some $f \in K(X) \backslash\{0\}$. The homomorphism

$$
\begin{array}{clc}
K(X)^{*} & \longrightarrow & \operatorname{CaDiv}(X) \\
f & \longmapsto & {[\mathcal{O}, f]}
\end{array}
$$

has the kernel as image but it's not injective. The kernel is given by the elements $f \in K(X)$ such that $[\mathcal{O}, f]=[\mathcal{O}, 1]$; it happens if and only if there exists an isomorphism $\phi: \mathcal{O} \rightarrow \mathcal{O}$ that sends $1 \mapsto f$ which is equivalent to say that $f \in \mathcal{O}(X)^{*}$. We obtain the exact sequence

$$
0 \rightarrow \mathcal{O}(X)^{*} \longrightarrow K(X)^{*} \longrightarrow \operatorname{CaDiv}(X) \longrightarrow \operatorname{Pic}(X) \rightarrow 0
$$

We now want to find a relation between Cartier divisors and Weil divisors. For all $p \in X^{(1)}$, we have a trivializing neighbourhood $U_{p}$ which induces an isomorphism $L_{\xi} \rightarrow \mathcal{O}_{X, \xi}$. These isomorphisms induces the map div, which is therefore global on $X$ : We have a map

$$
\text { div: } \begin{aligned}
\operatorname{CaDiv}(X) & \longrightarrow \operatorname{Div}(X) \\
{[L, s] } & \longmapsto
\end{aligned}
$$

It is a homomorphism since $\operatorname{div}\left(s \otimes s^{\prime}\right)=\operatorname{div}(s)+\operatorname{div}\left(s^{\prime}\right)$. It gives rise to a map between the quotients $\mathrm{Cl}(X), \operatorname{Pic}(X)$ and we get the commutative diagram


Under normality hypotesis, we get the injectivity:
Lemma 7.18. Let $X$ be a normal scheme. Then the map div: $\operatorname{CaDiv}(X) \rightarrow$ $\operatorname{Div}(X)$ is injective.

Proof. We have the equivalences:

$$
\begin{aligned}
{[L, s] \in \operatorname{Ker}(\operatorname{div}) } & \Longleftrightarrow \operatorname{div}(s)=0 \\
& \Longleftrightarrow v_{p}(s)=0 \forall p \in X^{(1)} \\
& \Longleftrightarrow s \in H^{0}(X, L) \text { never vanishes } \\
& \Longleftrightarrow \mathcal{O} \ni 1 \rightarrow s \in L \text { is an isomorphism }
\end{aligned}
$$

Theorem 7.19. Let $X$ be a normal scheme. Then the map $\operatorname{Pic}(X) \rightarrow \mathrm{Cl}(X)$ is injective.

Proof. It follows from the five lemma applied to the following diagram with exact rows


Example. Let $A$ be the ring

$$
A=\mathbb{C}[x, y, z] /\left(z^{2}-x y\right)
$$

and let $X=\operatorname{Spec}(A)$. We know that $A$ is normal and then the map $\operatorname{Pic}(X) \rightarrow$ $\mathrm{Cl}(X)$ is injective.
First, we want to compute $\mathrm{Cl}(X)$. Let $L$ be the closed subscheme $L=V(y)$. Then $X \backslash L \simeq \operatorname{Spec}\left(A_{y}\right)=\operatorname{Spec}\left(k[y, z]_{y}\right)$; since $\operatorname{Cl}\left(A_{y}\right)=0$, we get $\mathrm{Cl}(X)=[L]$. We want now to determine the order of $L$; we claim that $2[L]=0$. Indeed, $A_{L}$ has $z$ as a uniformizing parameter and $v_{L}(y)=2$; it follows that $2[L]=0$. So either $\mathrm{Cl}(X)=0$ or $\mathrm{Cl}(X)=\mathbb{Z} / 2 \mathbb{Z}$. But $\mathrm{Cl}(X)=0$ if and only if $A$ is UFD and this is false since $x, y, z$ are irreducible.
We want now to compute $\mathrm{Cl}(X)$ : since $\operatorname{Pic}(X) \hookrightarrow \operatorname{Cl}(X)$, either $\operatorname{Pic}(X)=0$ or $\operatorname{Pic}(X)=\mathbb{Z} / 2 \mathbb{Z}$. We claim that $\operatorname{Pic}(X)=0$. Assume that there exists an invertible sheaf $M$ such that $M \nsucceq \mathcal{O}$. Let $p=(x, y, z) \in \operatorname{Spec}(A)$; there exists $s \in M_{\xi} \backslash\{0\}$ such that $\operatorname{div}(s)=L=V(y)$. In particular, $s \in M_{p}$ and it corresponds to an element $f$ in $A_{p}$

$$
\begin{array}{ccc}
M_{p} & \simeq \mathcal{O}_{X, p}=A_{(x, y, z)} \\
s & \leftrightarrow & f
\end{array}
$$

Then $\operatorname{div}(f)$ is $L$ plus a sum of components not passing through the origin; this means that $\operatorname{div}\left(f^{2}\right)=\operatorname{div}(y)$ in a neighbourhood of $p$. Since $A$ is normal, $f^{2}=u y$ where $u$ is a unit in $A_{p}$. This can't happen since these polynomial have different degree one part and so we get a contradiction. Hence $\operatorname{Pic}(X)=0$.

We now want to study when the map is surjective.
Definition 7.20. A locally noetherian scheme is locally factorial if $\mathcal{O}_{X, p}$ is a UFD for all $p \in X$.

Clearly, it is enough to check this condition on closed points. We notice that every regular scheme is factorial while the converse is false. We now want to prove the following

Theorem 7.21. Let $X$ be a locally factorial scheme. Then $\operatorname{Pic}(X) \simeq \operatorname{Cl}(X)$ and $\operatorname{CaDiv}(X) \simeq \operatorname{Div}(X)$.

Since being locally factorial implies being normal, we only need to show surjectivity. We need to prove that for all $p \in X^{(1)}$ there exists a Cartier Divisor $[L, s]$ such that $\operatorname{div}(L, s)=p$. The closure of $p$ is an irreducible subset $V \subseteq X$ and we can consider the reduced structure on it. Then $V$ is closed, integral of codimension 1 . We need to discuss a correspondance between these closed subschemes and invertible sheaves.

Definition 7.22. A Cartier Divisor $[L, s]$ is effective if $s \in H^{0}(X, L)$
Observation 7.23. As in the Weil Divisors case, it implies that $\operatorname{div}(s) \geq 0$.
Let $[L, s]$ be an effective Cartier Divisor. The map

$$
\begin{array}{rlc}
\mathcal{O}_{X} & \longrightarrow & L \\
f & \longmapsto & f s
\end{array}
$$

is injective and tensoring by $\otimes L^{\vee}$ we get

$$
\begin{array}{clc}
L^{\vee} & \longrightarrow & \mathcal{O} \\
\alpha & \longmapsto & \alpha(s)
\end{array}
$$

which is still injective since $L^{\vee}$ is invertible. Therefore, $L^{\vee}$ is a quasi-coherent sheaf of ideals and we denote by $D$ the corresponding subscheme.
Conversely, let $D \subseteq X$ be a closed subscheme such that $I_{D}$ is an invertible sheaf and call $s: I_{D} \rightarrow \mathcal{O}_{X}$ the corresponding injection. Tensoring by $I_{D}^{\vee}$, we get in injection $\mathcal{O}_{X} \rightarrow I_{D}^{\vee}$ which identify an element $s_{D} \in H^{0}\left(X, I_{D}^{\vee}\right)$ (the image of 1 ). We have shown the following:

Proposition 7.24. There is a bijection between effective Cartier Divisors and closed subschemes $D$ in $X$ such that $I_{D}$ is invertible.
We call these subsets Cartier Divisors by abuse of notation.
Now we come back to our first issue: the surjectivity of the map $\operatorname{CaDiv}(X) \rightarrow$ $\operatorname{Div}(X)$ under the hypotesis of local factoriality. Let $V \subseteq X$ be a closed integral subscheme of codimension 1. We know it corresponds to a quasi-coherent sheaf of ideal $I_{V} \subseteq \mathcal{O}_{X}$; we claim that $I_{V}$ is invertible. In this case, $V$ would be an effective Cartier Divisor and it would come from the group $\operatorname{CaDiv}(X)$. Let $p \in V$ be a point of $V$; then $I_{V, p} \subsetneq \mathcal{O}_{X, p}$ is a prime ideal of height one. It is prime because we have the exact sequence

$$
0 \rightarrow I_{V, p} \longrightarrow \mathcal{O}_{X, p} \longrightarrow \mathcal{O}_{V, p} \rightarrow 0
$$

It is of height one because

$$
\left(\mathcal{O}_{X, p}\right)_{I_{V, p}}=\mathcal{O}_{X, V}
$$

and since the scheme is locally factorial, every prime of height one is principal and $I_{V, p}$ is principal, hence $\mathcal{O}_{X, V}$ is a DVR. This proves that $\operatorname{CaDiv}(X) \rightarrow$ $\operatorname{Div}(X)$ is surjective and therefore, using the five lemma, gives an isomorphism

$$
\operatorname{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X)
$$

We now want to describe the inverse $\operatorname{Div}(X) \rightarrow \operatorname{CaDiv}(X)$ : this will be useful when dealing with curves. So, given a divisor $D$, we want to find a sheaf $\mathcal{O}(D)$ and a rational section $s$ that correspond to $D$. Let $K=K(X)$ and denote by $K_{X}$ the locally constant sheaf

$$
K_{X}(U)= \begin{cases}K & \text { if } U \neq \emptyset \\ 0 & \text { if } U=\emptyset\end{cases}
$$

$K_{X}$ is well defined since the scheme is integral and it is quasi-coherent. We want to find $\mathcal{O}(D)$ as a subsheaf of $K_{X}$. The only tool we can use to define it is the valuation map. The following observation is crucial for the understanding of this:
Observation 7.25. Let $[L, s] \in \operatorname{CaDiv}(X)$ and denote by $D$ the element $\operatorname{div}(L, s)$. For all $U$ open non-empty subset of $X$, we get the map

$$
\begin{aligned}
H^{0}(U, L) & \longrightarrow \\
t & \longmapsto
\end{aligned}
$$

which is an injective homomorphism of sheaves $L \rightarrow K_{X}$. Let $f \in K=K_{X}(U)$; we have the equivalences

$$
\begin{aligned}
f=\frac{t}{s} \text { for some } t \in L(U) & \Longleftrightarrow f s \in L(U) \subseteq L_{\xi} \\
& \Longleftrightarrow v_{p}(f)+v_{p}(s) \geq 0 \forall p \in U^{(1)}
\end{aligned}
$$

Guided by this observation, given a divisor $D=\sum n_{p} p$, we define the sheaf $\mathcal{O}(D) \subseteq K_{X}$

$$
\mathcal{O}(D)(U)=\left\{\begin{array}{l}
0 \text { if } U=\emptyset \\
\left\{f \in K \mid v_{p}(f) \geq-n_{p} \quad \forall p \in U^{(1)}\right\}
\end{array}\right.
$$

It is a sheaf of $\mathcal{O}_{X}$-modules; given $D=\operatorname{div}(L, s)$ we have to show that $L \simeq$ $\mathcal{O}(D)$. We can check this on an open cover; in this case we have a map

$$
\begin{array}{clc}
L(U) & \longrightarrow & \mathcal{O}(D)(U) \\
t & \longrightarrow & t / s
\end{array}
$$

which is an isomorphism by the observation. Since $X$ is locally factorial, $\mathcal{O}(D)$ is invertible, $\mathcal{O}(D)_{\xi} \simeq K . L$ is invertible and we have a map

$$
\begin{array}{ccc}
L_{\xi} & \longrightarrow & K \\
s
\end{array}
$$

Then the pair $(L, s)$ is isomorphic to $(\mathcal{O}(D), 1)$.
We notice that $D \geq 0$ if and only if $1 \in \mathcal{O}(D)(X)$. In this case, the effective Cartier divisors correspond to the effective Weil divisors.
Example. Let $X=\mathbb{P}_{k}^{1}$; since $X$ is locally factorial, $\operatorname{Pic}(X) \simeq \operatorname{Cl}(X) \simeq \mathbb{Z}$. Let $p \in X^{(1)}$ and let $d=[k(p): p]$. We can identify $p$ as the kernel of the valuation map $K[t] \rightarrow k(p)$, where $t=x_{1} / x_{0}, p=(\varphi(t))$ and $\operatorname{deg}(\varphi(t))=d$. Homogenizing, we get $\phi \in K\left[x_{0}, x_{1}\right]_{d}$ and therefore

$$
p=\operatorname{Proj}\left(K\left[x_{0}, x_{1}\right] /\left(\phi\left(x_{0}, x_{1}\right)\right)\right)
$$

This gives rise to the exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^{1}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^{1}} \rightarrow \mathcal{O}_{p} \rightarrow 0
$$

Therefore, $\mathcal{O}(-d)$ is the sheaf of ideals of $p$. By the correspondance, we get $\mathcal{O}(p)=\mathcal{O}(-d)^{\vee}=\mathcal{O}(d)$.

## Chapter 8

## Curves

In this chapter, we will deal with curves and we will require some hypotesis:
Definition 8.1. A curve is a proper, integral, regular, 1-dimensional scheme over a field $k$.

Under these hypotesis, $X^{(1)}$ coincides with the set of closed points of $X$ and we have the map

$$
p \in X^{(1)} \longmapsto \operatorname{deg}(p)=[k(p): k]
$$

This extends to a homomorphism

$$
\begin{array}{lllc}
\text { deg: } & \operatorname{Div}(X) & \longrightarrow & \mathbb{Z} \\
& \sum n_{p} p & \longmapsto & \sum n_{p}[k(p): k]
\end{array}
$$

We want now to prove that the composition $\operatorname{deg}(\operatorname{div}(f))=\sum v_{p}(f)[k(p): k]=$ 0 ; we know that this is true in the projective space $\mathbb{P}_{k}^{1}$, since the degree (as polynomials) of the numerator and the denominator of a rational function is the same.

During all this chapter, we will use the following extension property:
Proposition 8.2. Let $X$ be a proper, integral, regular, 1-dimensional schemes over a field $k$ and let $Y$ be a proper scheme over $\operatorname{Spec}(k)$. Let $U \subseteq X$ be an open subscheme and let $f: U \rightarrow Y$ be a morphism of $k$-schemes. Then $f$ extends to a unique morphism $X \rightarrow Y$.

Proof. We notice that $X \backslash U=\left\{p_{1}, \ldots, p_{r}\right\}$ is finite. For all these points, $\mathcal{O}_{X, p_{i}}$ is a DVR and $K\left(\mathcal{O}_{X, p}\right)=K(X)$. By the valuative criterion of properness, we get the diagram


We call $g_{i}: \operatorname{Spec}\left(\mathcal{O}_{X, p_{i}}\right) \rightarrow Y$ the dashed map. Since the map is of finite type, for all $i$ we can find an extension $g_{i}: U_{p_{i}} \rightarrow Y$, where $U_{p_{i}}$ is an affine open
neighbourhood of $p_{i}$. Let now $W$ be an affine open neighbourhood of $g\left(p_{i}\right)$ and let $U^{\prime}=f^{-1}(W) \cap g^{-1}(W)$. $U^{\prime}$ is non-empty since it contains $\xi$ and the homomorphisms $\mathcal{O}_{Y}(W) \rightarrow \mathcal{O}_{X}\left(U^{\prime}\right)$ corresponding to $\left.f\right|_{U^{\prime}}$ and $\left.g\right|_{U^{\prime}}$ are identical because the coincide on $K(X) \subseteq \mathcal{O}_{X}\left(U^{\prime}\right)$. Since $W$ is affine, we get $\left.f\right|_{U^{\prime}}=\left.g\right|_{U^{\prime}}$. By separatedness, $f, g$ coincide on $U \cap U_{p_{i}}$ and we can glue togheter all these morphism.

## Geometrically Connected Schemes

Definition 8.3. Let $X$ be a scheme over a field $k$. We say that $X$ is geometrically connected if $X_{\bar{k}}$ is connected

We notice that, since surjectivity is stable under base change, the map $X_{\bar{k}} \rightarrow$ $X$ is surjective and being geometrically connected implies being connected.

There is a useful criterion:
Proposition 8.4. Let $X$ be a proper smooth scheme over $k$. Then $X$ is geometrically connected if and only if $H^{0}\left(X, \mathcal{O}_{X}\right) \simeq k$
Proof. Using theorem 6.60, we get $H^{0}\left(X_{\bar{k}}, \mathcal{O}\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\right) \otimes_{k} \bar{k}$. Therefore it is enough to show

$$
k \simeq H^{0}\left(X, \mathcal{O}_{X}\right) \Longleftrightarrow H^{0}\left(X_{\bar{k}}, \mathcal{O}\right) \simeq \bar{k}
$$

If $H^{0}\left(X, \mathcal{O}_{X}\right) \simeq k$, we get $H^{0}\left(X_{\bar{k}}, \mathcal{O}\right) \simeq \bar{k}$ and therefore $X_{\bar{k}}$ is connected. Viceversa, assume $X_{\bar{k}}$ is connected. Then the smoothness hypotesis implies that $X_{\bar{k}}$ is integral and $H^{0}\left(X_{\bar{k}}, \mathcal{O}\right)$ is a domain. Furthermore, from theorem 6.44, we get $h^{0}\left(X_{\bar{k}}, \mathcal{O}\right)$ is finite; since $\bar{k}$ is algebraically closed, we get $H^{0}\left(X_{\bar{k}}, \mathcal{O}\right)=\bar{k}$.

## Genus

Definition 8.5. Let $k$ be a field and let $X$ be a smooth projective and geometrically connected scheme of dimension 1 . We define the genus $g(X)$ as

$$
g(X)=h^{1}\left(X, \mathcal{O}_{X}\right)
$$

Since $h^{0}\left(X, \mathcal{O}_{X}\right)=1$ and $X$ is one-dimensional, we have the relation $g(X)=$ $1-\chi\left(\mathcal{O}_{X}\right)$, because of Grothendieck vanishing theorem.
Example. We have already noticed that a smooth curve $X \subseteq \mathbb{P}_{k}^{2}$ is always geometrically connected. In particular, we have the exact sequence

$$
0 \rightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X} \rightarrow 0
$$

The addictivity of the Euler characteristic implies

$$
1-g(X)=\chi\left(\mathcal{O}_{X}\right)=1-\frac{(d-1)(d-2)}{2} \Rightarrow g(X)=\frac{(d-1)(d-2)}{2}
$$

### 8.1 Degree of a map

Let $X, Y$ be curves and let $f: X \rightarrow Y$ be a morphism of schemes over $k$. Then $f$ is proper by the diagram

$$
\begin{gathered}
X \\
\text { proper } \\
\operatorname{Spec}(k)
\end{gathered}
$$

In particular, $f(X)$ is closed. Since $Y$ is irreducible, we get $f(X)=Y$ or $f$ is constant.

Proposition 8.6. Assume that $f$ is constant. Then $f(X)$ is a rational point.
Proof. Certainly, it is a closed point since the map is proper. We call $q=f(X)$; we have to show that $k(q)=k$. We notice that $f^{-1}(q)=X$ since $X$ is reduced and therefore there exists a factorization

$$
X \longrightarrow \operatorname{Spec} k(q) \longrightarrow Y
$$

Taking global section, we get

$$
k \longrightarrow k(q) \longrightarrow k
$$

Since these are homomorphisms of fields, we get $k(q)=k$.
Assume now that $f(X)=Y$. Then $f$ is dominant and it induces a finite extension of field

$$
K(Y) \longrightarrow K(X)
$$

Definition 8.7. We define $\operatorname{deg}(f)=[K(X): K(Y)]$.
We notice that $f$ is flat (it is dominant so we can apply proposition 5.69). We now want to show that it has finite fibers. Let $p \in Y$ and let $U=\operatorname{Spec}(A)$ be an open affine neighbourhood. We can cover $f^{-1}(U)$ with finitely many affine open subsets; let $V=\operatorname{Spec}(B)$ be one of these. Then $f$ corresponds to a map $\psi: A \rightarrow B$ (injective since $f$ is dominant) and the fiber of $p$ correspond to $\operatorname{Spec}(B / p B)$. Since $B$ is a one-dimensional, $B / p B$ is an artinian ring and therefore $\operatorname{Spec}(B / p B)$ is finite, as desired. As a consequence, $f$ is finite by Chevalley's theorem. Let $V=\operatorname{Spec}(B) \subseteq Y$ be an open affine subset and let $U=f^{-1}(V)=\operatorname{Spec}(A)$ (a finite map is affine). Since $B \rightarrow A$ is flat and finite, we get that $A$ is a projective $B$-module. In particular, the rank function is constant. Let $q \in V$ be a closed point and $\eta$ be the generic point; then

$$
r k_{q}(A)=\operatorname{dim}_{k(q)} A / q A \quad r k_{\eta} A=\operatorname{dim}_{k(Y)} K(X)=\operatorname{deg} f
$$

Therefore $r k_{q} B=\operatorname{deg}(f)$ for all $q \in V$ and $f^{-1}(q)=\operatorname{Spec}(B / q B)$ is finite over $k$.
In particular, let $p \in X$ be a closed point and let $q=f(p)$.

$$
f^{\#}: \mathcal{O}_{Y, q} \longrightarrow \mathcal{O}_{X, p}
$$

Since the scheme is regular and of dimension one, we know that these stalks are DVR. Called $t_{q}$ the uniformizing parameter of $m_{q}$, we get

$$
f^{\#}\left(t_{q}\right) \in m_{p}
$$

Definition 8.8. The ramification index $e_{p}(f)$ is the valuation at $p$ of $f^{\#}\left(t_{q}\right)$

Then, from the equality

$$
f^{-1}(q)=\operatorname{Spec}\left(\prod_{p \in f^{-1}(q)} \mathcal{O}_{X, p} /\left(t_{p}^{e_{p}(f)}\right)\right)
$$

we get

$$
\begin{aligned}
\operatorname{deg} f & =\operatorname{dim}_{k(q)} \prod_{p \in f^{-1}(q)} \mathcal{O}_{X, p} / t_{p}^{e_{p}(f)} \\
& =\sum_{p \in f^{-1}(q)} \operatorname{dim}_{k(q)} \mathcal{O}_{X, p} /\left(t_{p}^{e_{p}(f)}\right) \\
& =\sum_{p \in f^{-1}(q)} \operatorname{dim}_{k(q)}\left(\mathcal{O}_{X, p} /\left(t_{p}\right)\right)^{e_{p}(f)} \\
& =\sum_{p \in f^{-1}(q)} e_{p}(f) \operatorname{dim}_{k(q)} \mathcal{O}_{X, p} /\left(t_{p}\right)
\end{aligned}
$$

Therefore

$$
\operatorname{deg}(f)=\sum_{p \in f^{-1}(q)} e_{p}(f)[k(p): k(q)]
$$

In particular, if $k=\bar{k}$, we get

$$
\operatorname{deg}(f)=\sum_{p \in f^{-1}(q)} e_{p}(f)
$$

Example. Let $f \in k[t] \backslash k$ and consider the map $\mathbb{A}^{1} \longrightarrow \mathbb{A}^{1}$ given by

$$
\begin{array}{clc}
k[x] & \longrightarrow & k[t] \\
x & \longmapsto & f(t)
\end{array}
$$

This extends to a map $\mathbb{P}_{k}^{1} \rightarrow \mathbb{P}_{k}^{1}$ by sending $u=1 / t$ to $1 / f(1 / y)$. Let $q=$ $(x-a) \in \mathbb{A}_{k}^{1}(k)$; then

$$
f^{-1}(q)=\operatorname{Spec}^{k[t]} /(f(t)-a)
$$

Assume that $f(t)-q=g_{1}^{e_{1}} \ldots g_{r}^{e_{r}}$ and let $p_{i}=\left(g_{i}\right)$. Then

$$
e_{p_{i}}(f)=v_{p_{i}}(f(t)-q)=e_{i}
$$

Noticing that

$$
\left[k\left(p_{i}\right): k\right]=\operatorname{deg}\left(g_{i}\right)
$$

we get the formula

$$
\operatorname{deg}(f)=\sum_{i=1}^{r} e_{i} \operatorname{deg}\left(g_{i}\right)
$$

If $q$ is not a rational point, we get

$$
\sum e_{p}(f)[k(p): k(q)]=\#\{\text { points in the fiber counted with multiplicity }\}
$$

Let us consider the point at the infinity $\infty$. Then

$$
k(\infty)=k \quad f^{-1}(\infty)=\infty
$$

Then we obtain that $e_{\infty}(f)$ coincides with the degree of $f$ as a polynomial.
We now want to study the pullback of invertible sheaf in this particular case. So let $f: X \rightarrow Y$ be a non-constant morphism of $k$-schemes between curves. Then we get a map between the Picard Groups:

$$
\begin{aligned}
f^{*}: & \operatorname{Pic}(Y) \\
{[L] } & \longrightarrow
\end{aligned} \quad \operatorname{Pic}(X)
$$

This map lifts to the group of Cartier Divisors. In fact, let $\xi \in X$ and $\eta \in Y$ be the generic points and let $L$ be an invertible sheaf on $Y$. Since the map is dominant, $f(\xi)=\eta$ and this induces a map between the field of rational function $K(Y) \rightarrow K(X)$. We have the isomorphisms $L_{\eta} \simeq K(Y)$ and $f^{*} L_{\xi} \simeq K(X)$ because $L$ and $f^{*} L$ are invertible; therefore we obtain an injection $\iota: L_{\eta} \rightarrow f^{*} L_{\xi}$. This defines a map

$$
\begin{aligned}
f^{*}: \quad \operatorname{CaDiv}(Y) & \longrightarrow \operatorname{CaDiv}(X) \\
{[L, s] } & \longmapsto\left[f^{*} L, \iota(s)\right]
\end{aligned}
$$

It is well defined since the choice of a different isomorphism defines the same class of isomorphism. Since the hypotesis implies that $\operatorname{CaDiv}(X) \simeq \operatorname{Div}(X)$ and $\operatorname{CaDiv}(Y) \simeq \operatorname{Div}(Y)$, we get a map (that we still call $f^{*}$ by abuse of notation) $f^{*}: \operatorname{Div}(Y) \rightarrow \operatorname{Div}(X)$. We want to study this map.
Let $q \in Y^{(1)}$; then $q$ correspond to a Cartier Divisor $[L, s]$ where $s$ has the property that

$$
v_{q^{\prime}}(s)= \begin{cases}0 & q^{\prime} \neq q \\ 1 & q^{\prime}=q\end{cases}
$$

Therefore

$$
\left[f^{*} L, f^{*} s\right] \longleftrightarrow \sum_{p \in X} v_{p}\left(f^{*} s\right) p \longleftrightarrow \sum_{p \in f^{-1}(q)} e_{p}(f) p
$$

and the map becomes

$$
\begin{array}{rllc}
f^{*}: \quad \operatorname{Div}(Y) & \longrightarrow & \operatorname{Div}(X) \\
q & \longmapsto \sum_{p \in f^{-1}(q)} e_{p}(f) p
\end{array}
$$

Passing to the class group, we get $f^{*}: \mathrm{Cl}(Y) \rightarrow \mathrm{Cl}(X)$.
Let now $f \in K(X)^{*}$. We now that there exists an open subscheme $U \subseteq X$ such that $f \in H^{0}(U, \mathcal{O})$; the ring homomorphism

$$
\begin{array}{clc}
K[t] & \longrightarrow & H^{0}(U, \mathcal{O}) \\
t & \longmapsto & f
\end{array}
$$

induces a map $f: U \rightarrow \mathbb{A}_{k}^{1}$. One possible choice for $U$ is for example

$$
U=\left\{p \in X \mid v_{p}(f) \geq 0\right\}
$$

This map extends to a map $X \longrightarrow \mathbb{P}_{k}^{1}$ by proposition 8.2 . We get

$$
\operatorname{div}(f)=\sum_{p \in X^{(1)}} v_{p}(f) p=\sum_{\substack{p \in X^{(1)} \\ v_{p}(f) \geq 0}} v_{p}(f) p-\sum_{\substack{p \in X^{(1)} \\ v_{p}(f) \leq 0}} v_{p}\left(\frac{1}{f}\right) p
$$

Since

$$
v_{p}(f)>0 \Longleftrightarrow f(p)=0 \quad v_{p}(f)<0 \Longleftrightarrow f(p)=\infty
$$

it holds

$$
f^{*}(0)=\sum_{v_{p}(f)>0} v_{p}(f) p \quad f^{*}(\infty)=\sum_{v_{p}(f)<0} v_{p}\left(\frac{1}{f}\right) p
$$

and we get $\operatorname{div}(f)=f^{*}(0)-f^{*}(\infty)$.
Proposition 8.9. $f$ is an isomorphism if and only if $\operatorname{deg}(f)=1$
Proof. Clearly, if $f$ is an isomorphism, $\operatorname{deg}(f)=1$. On the other hand, since $f$ is finite, it is also affine and, given an open affine subset $V=\operatorname{Spec}(B) \subseteq Y$, the restriction

$$
(\operatorname{Spec} A)=f^{-1}(V) \longrightarrow V
$$

is an integral extension. Since $\operatorname{deg}(f)=1, K(X)=K(Y)$ and $B \subseteq A \subseteq K(X)$ implies $B=A$. Since the map is an isomorphism on an affine open cover, it is an isomorphism.

Definition 8.10. We define the degree map

$$
\begin{array}{cccc}
\operatorname{deg}: & \operatorname{Div}(X) & \longrightarrow & \mathbb{Z} \\
p & \longmapsto & {[k(p): k]}
\end{array}
$$

Let now $f: X \rightarrow Y$ be a morphism of curves. Then

$$
\begin{aligned}
\operatorname{deg} f^{*}(q) & =\sum_{p \in f^{-1}(q)} e_{p}(f)[k(p): k] \\
& =\sum_{p \in f^{-1}(q)} e_{p}(f)[k(p): k(q)][k(q): k] \\
& =\operatorname{deg}(q) \operatorname{deg}(f)
\end{aligned}
$$

Therefore, by linearity, the map $f^{*}: \operatorname{Div}(Y) \longrightarrow \operatorname{Div}(X)$ gives the formula

$$
\operatorname{deg}\left(f^{*} D\right)=\operatorname{deg}(f) \operatorname{deg}(D)
$$

Corollary 8.11. $\operatorname{deg}(\operatorname{div}(f))=0$
Proof. Let $f \in K(X) \backslash K$. Then we know that $\operatorname{div}(f)=f^{*}(0)-f^{*}(\infty)$. Computing the degree, we get

$$
\operatorname{deg}(\operatorname{div}(f))=\operatorname{deg}\left(f^{*}(0)\right)-\operatorname{deg}\left(f^{*}(\infty)\right)=\operatorname{deg}(f)(\operatorname{deg}(0)-\operatorname{deg}(\infty))=0
$$

As a consequence, the map

$$
\operatorname{deg}: \operatorname{Cl}(X) \longrightarrow \mathbb{Z}
$$

is well-defined. Notice that if $k=\bar{k}$ or $X(k) \neq \emptyset$ the map deg: $\mathrm{Cl}(X) \rightarrow \mathbb{Z}$ is surjective. Infact, given $p \in X(k), \operatorname{deg}(p)=1$ and and since deg: $\operatorname{Div}(X) \rightarrow \mathbb{Z}$ is a homomorphism the map is surjective. Hence the map deg: $\mathrm{Cl}(X) \rightarrow \mathbb{Z}$ is surjective.

Example. Let $X \simeq \mathbb{P}_{k}^{1}$; we know that $\operatorname{Pic}\left(\mathbb{P}_{k}^{1}\right)=\mathbb{Z}[\mathcal{O}(1)]$. Let 0 be the point corresponding to the origin in $U_{0}=\operatorname{Spec}\left(K\left[x_{i} / x_{0}\right]\right)$. Then

$$
\operatorname{deg}[\mathcal{O}(1)]=\operatorname{deg}[0]=1
$$

Therefore deg in this case is an isomorphism.
Proposition 8.12. Let $X$ be a curve and assume $X \not \not \mathbb{P}_{k}^{1}$. Let $p, q \in X(k)$ be two distinct rational point. Then $[p-q] \neq 0$ in $\mathrm{Cl}(X)$.
Proof. By contradiction, assume $p-q=\operatorname{div}(f)$, where $f \in k(X) \backslash k$. Then we consider the associated map $f: X \rightarrow \mathbb{P}_{k}^{1}$; we know that

$$
p-q=f^{*}(0)-f^{*}(\infty)
$$

Since $f^{*}(0), f^{*}(\infty) \geq 0$ and $f^{*}(0) \cap f^{*}(\infty)=\emptyset$, we get $f^{*}(0)=p$ and $f^{*}(\infty)=q$. Therefore $\operatorname{deg}(f)=\operatorname{deg} f^{*}(0)=\operatorname{deg}(p)=1$ and $f$ is an isomorphism, which is absurd.

Corollary 8.13. If $k=\bar{k}$, $\operatorname{deg}: \operatorname{Cl}(X) \rightarrow \mathbb{Z}$ is injective if and only if $X \simeq \mathbb{P}_{k}^{1}$.

### 8.1.1 Base Change

Let $K^{\prime} / K$ be a field extension and let $X$ be a curve on $K$. Then $X_{K^{\prime}}=$ $X \times_{\operatorname{Spec}\left(K^{\prime}\right)} \operatorname{Spec}(K)$ defines a smooth projective and geometrically connected scheme and therefore a curve on $K^{\prime}$. It is geometrically connected since

$$
K^{\prime}=K^{\prime} \otimes_{k} H^{0}(X, \mathcal{O})=H^{0}\left(X_{K^{\prime}}, \mathcal{O}\right)
$$

We notice that the projection map $\pi: X_{K^{\prime}} \rightarrow X$ gives a map between the Picard Groups $\pi^{*}: \operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(X_{K^{\prime}}\right)$ which induces a map $\pi^{*}: \mathrm{Cl}(X) \rightarrow \mathrm{Cl}\left(X_{K^{\prime}}\right)$. Let $p \in X^{(1)}$; we consider $\pi^{-1}(p) \subseteq X_{K^{\prime}}$. Notice that $p$ is a closed point and $\pi^{-1}(p)=\operatorname{Spec}\left(K^{\prime} \otimes_{K} K(p)\right)$. Since $p$ is an effective Cartier Divisor, $\pi^{-1}(p)$ is an effective Cartier Divisor.

$$
[p] \in \mathrm{Cl}(X) \longleftrightarrow\left[\mathcal{O}_{X}(p)\right]=\left[I_{p}^{\vee}\right] \in \operatorname{Pic}(X)
$$

Remark 8.14. If $f: X^{\prime} \rightarrow X$ is a flat map and $Y \subseteq X$ is a closed subscheme, we have the sequence

$$
0 \rightarrow I_{Y} \rightarrow \mathcal{O}_{X} \rightarrow \mathcal{O}_{Y} \rightarrow 0
$$

Applying $\pi^{*}$ and using flatness, we get

$$
0 \rightarrow \pi^{*} I_{Y} \rightarrow \mathcal{O}_{X^{\prime}} \rightarrow \mathcal{O}_{\pi^{-1}(Y)} \rightarrow 0
$$

and therefore $I_{\pi^{-1}(Y)}=\pi^{*} I_{Y}$
Since $\pi$ is flat, we get $I_{\pi^{-1}}(p)=\pi^{*} I_{p}$ and $\pi^{*} \mathcal{O}(p) \simeq \mathcal{O}_{\pi^{-1}(p)}$; in particular $\pi^{*}[p]=\left[\pi^{-1}(p)\right]$. Let $t_{p}$ be the uniformizing parameter of $\mathcal{O}_{X, p}$ and let $\pi^{-1}(p)=$ $\left\{q_{1}, \ldots, q_{r}\right\}$. Then $I_{\pi^{-1}(p), q_{i}}=\pi^{*} t_{p} \mathcal{O}_{X_{K^{\prime}}, q_{i}}$; if we set $e_{i}=v_{q_{i}}\left(t_{p}\right)$, we get

$$
\begin{aligned}
\operatorname{deg}\left(\pi^{*}(p)\right) & =\sum e_{i}\left[k\left(q_{i}\right): k\right] \\
& =\operatorname{dim}_{K^{\prime}} \prod_{i} \mathcal{O}_{X_{K^{\prime}}, q_{i}} /\left(I_{\pi^{-1}(p), q_{i}}\right) \\
& =\operatorname{dim}_{K^{\prime}} K^{\prime} \otimes_{K} K(p) \\
& =\operatorname{dim}_{K} K(p) \\
& =\operatorname{deg} p
\end{aligned}
$$

We have shown the following:

Proposition 8.15. $\pi^{*}: \mathrm{Cl}(X) \rightarrow \mathrm{Cl}\left(X_{K^{\prime}}\right)$ preserves degree.
Lemma 8.16. Let $L$ be an invertible sheaf on $X$. Assume that $H^{0}(X, L) \neq 0$ and $H^{0}\left(X, L^{\vee}\right) \neq 0$. Then $L \simeq \mathcal{O}_{X}$.

Proof. Let $s \in H^{0}(X, L)$ and $t \in H^{0}\left(X, L^{\vee}\right)$. Then

$$
D=\operatorname{div}(s) \geq 0 \quad E=\operatorname{div}(t) \geq 0
$$

We get $D+E \sim 0$ and

$$
\underbrace{\operatorname{deg}(D)}_{\geq 0}+\underbrace{\operatorname{deg}(E)}_{\geq 0}=0 \Longrightarrow D=E=0
$$

Therefore $[L]=0 \in \operatorname{Pic}(X)$.
Proposition 8.17. $\pi^{*}: \mathrm{Cl}(X) \rightarrow \mathrm{Cl}\left(X_{K^{\prime}}\right)$ is injective.
Proof. Assume that $L$ is an invertible sheaf on $X$ such that $\pi^{*}(L) \simeq \mathcal{O}_{X_{K^{\prime}}}$. Then

$$
K^{\prime} \otimes_{k} H^{0}(X, L)=H^{0}\left(X_{K^{\prime}}, \pi^{*} L\right)=K^{\prime}
$$

Therefore $H^{0}(X, L)=k$. In the same way, $H^{0}\left(X, L^{\vee}\right)=k$ and therefore $L \simeq \mathcal{O}$.

Example. Let $X \subseteq \mathbb{P}_{K}^{2}$ be a smooth conic. If $X(k) \neq \emptyset$, then $X \simeq \mathbb{P}_{k}^{1}$. In fact, assume $p=[1,0,0] \in X(k)$ and consider the map

$$
\begin{array}{ccc}
\mathbb{P}_{k}^{2} \backslash\{p\} & \longrightarrow & \mathbb{P}_{k}^{1} \\
{[x, y, z]} & \longrightarrow & {[y, z]}
\end{array}
$$

This induces an isomorphism $X \simeq \mathbb{P}_{k}^{1}$. Assume now that $X(k)=\emptyset$ and let $L \subseteq \mathbb{P}_{k}^{2}$ be a line. By Bezout's Theorem, $L \cap X$ is an effective Cartier Divisor of degree 2 and therefore

$$
\operatorname{Im}(\operatorname{deg}(\operatorname{Cl}(X))) \supseteq 2 \mathbb{Z}
$$

Notice that deg: $\mathrm{Cl}\left(X_{\bar{K}}\right) \rightarrow \mathbb{Z}$ is injective since $X_{\bar{K}}(\bar{K}) \neq \emptyset$ and therefore by composition the map $\mathrm{Cl}(X) \rightarrow \mathbb{Z}$ is injective. Notice that $\mathrm{Cl}(X) \rightarrow \mathbb{Z}$ is an isomorphism if and only if there exists $p \in X^{(1)}$ such that $[k(p): k]$ is odd. This can't happen by the following theorem:

Theorem 8.18 (Springer). Let $k$ be a field of characteristic $\operatorname{char}(k) \neq 2$. Let $q \in k\left[x_{0}, \ldots, x_{n}\right]$ be a non-degenerate quadratic form and let $k^{\prime} / k$ be a finite extension of odd degree. If $q$ has a zero in $\left(k^{\prime}\right)^{n} \backslash\{0\}$ then it has a zero in $k^{n} \backslash\{0\}$.

Therefore, $[k(p): k]$ must be even for all $p \in X^{(1)}$ and therefore $\mathrm{Cl}(X) \simeq 2 \mathbb{Z}$.
In general,

$$
\operatorname{Im}(\mathrm{Cl}(X) \rightarrow \mathbb{Z})=\operatorname{gcd}\left\{\left[[k(p): k] \mid p \in X^{(1)}\right\} \mathbb{Z}\right.
$$

### 8.2 Differentials

Let $X$ be a differentiable manifold and let $\mathcal{C}_{X}^{\infty}$ be the sheaf of $\mathcal{C}^{\infty}$-function and $\Omega_{X}^{1}$ be the sheaf of 1 -forms, which is a $\mathcal{C}_{X}^{\infty}$-modules. The map

$$
\begin{aligned}
d: \quad \mathcal{C}_{X}^{\infty} & \longrightarrow \Omega_{X}^{1} \\
f & \longmapsto d f
\end{aligned}
$$

is a homomorphism of sheaves but not a homomorphism of $\mathcal{C}_{X}^{\infty}$-modules. Furthermore, it satisfies the Leibniz rule, it is $\mathbb{R}$-linear and the differential of a constant is zero. We want now to adapt this idea in the case of rings:

Definition 8.19. Let $A$ be a commutative ring and let $M$ be an $A$-module. A derivation on $A$ is a homomorphism of groups $d: A \rightarrow D$ such that for all $f, g \in A$

$$
d(f g)=g D(f)+f D(g)
$$

It follows immediately from the definition that $D(1)=0$. In fact,

$$
D(1)=D(1 \cdot 1)=D(1)+D(1)
$$

Definition 8.20. Let $\varphi: R \rightarrow A$ be a ring homomorphism. An $R$-derivation is a derivation $D: A \rightarrow M$ which is $R$-linear too.

The following equivalence holds:
Proposition 8.21. $D: A \rightarrow M$ is a $R$-derivation if and only if $D(\varphi(r))=0$ for all $r \in R$.

Proof. Assume first that $D$ is a $R$-derivation. Then

$$
D(\varphi(r))=\varphi(r) D(1)=0
$$

Viceversa, if $D(\varphi(r))=0$, for all $a \in A$ we get

$$
D(\varphi(r) a)=\varphi(r) D(a)+a D(\varphi(r))=\varphi(r) D(a)
$$

Let now $M$ be an $A$-module. We want to give to $A \oplus M$ an $A$-algebra structure. We define the operations as

$$
(a, m)+\left(a^{\prime}, m^{\prime}\right)=\left(a+a^{\prime}, m+m^{\prime}\right) \quad(a, m)\left(a^{\prime}, m^{\prime}\right)=\left(a a^{\prime}, a m^{\prime}+a^{\prime} m\right)
$$

Then $A \oplus M$ becomes a commutative ring and we get the projection map pr: A $\oplus$ $M \rightarrow A$ which is a homomorphism of $A$-algebras. We want now to show that there exists a correspondance between the section of this projection that are group homomorphisms and the derivations $D: A \rightarrow M$.
Let $D: A \rightarrow D$ be a derivation. This induces the section

$$
\begin{array}{lllc}
S_{D}: & A & \longrightarrow & A \oplus M \\
& a & \longmapsto & (a, D(a))
\end{array}
$$

Viceversa, let $S: A \rightarrow A \oplus M$ be a section. Then restricting to the second component, we get a derivation. The details follow from this lemma:

Lemma 8.22. $D$ is a derivation if and only if $S_{D}$ is a ring homomorphism.
Proof. Assume first that $D$ is a derivation. Then

$$
\begin{aligned}
S_{D}\left(a a^{\prime}\right) & =S_{D}\left(a a^{\prime}, D\left(a a^{\prime}\right)\right) \\
& =\left(a a^{\prime}, a^{\prime} D(a)+a D\left(a^{\prime}\right)\right) \\
& =(a, D(a))\left(a^{\prime}, D\left(a^{\prime}\right)\right) \\
& =S_{D}(a) S_{D}\left(a^{\prime}\right)
\end{aligned}
$$

$$
\begin{aligned}
S_{D}\left(a+a^{\prime}\right) & =\left(a+a^{\prime}, D\left(a+a^{\prime}\right)\right) \\
& =\left(a+a^{\prime}, D(a)+D\left(a^{\prime}\right)\right) \\
& =(a, D(a))+\left(a^{\prime}, D\left(a^{\prime}\right)\right) \\
& =S_{D}(a)+S_{D}\left(a^{\prime}\right)
\end{aligned}
$$

and therefore $S_{D}$ is a homomorphism of rings. Viceversa, if $S: A \rightarrow A \oplus M$ is a ring homomorphism,

$$
\begin{aligned}
S\left(a a^{\prime}\right) & =\left(a a^{\prime}, D\left(a a^{\prime}\right)\right) \\
S(a) S\left(a^{\prime}\right) & =(a, D(a))\left(a^{\prime}, D\left(a^{\prime}\right)\right)=\left(a a^{\prime}, a D\left(a^{\prime}\right)+a^{\prime} D(a)\right)
\end{aligned}
$$

and therefore $D\left(a a^{\prime}\right)=a D\left(a^{\prime}\right)+a^{\prime} D(a)$.
Notice that if $R \rightarrow A$ is a ring homomorphism, the same fact holds for the $R$-derivation; so there exists a bijection

$$
D: A \rightarrow M R \text {-derivation } \leftrightarrow \quad S_{D}: A \rightarrow A \oplus M \text { morphism of } R \text {-algebras }
$$

Proposition 8.23. Let $A=R\left[x_{0}, \ldots, x_{n}\right]$ and let $M$ be an $A$-module. Then for all $m_{1}, \ldots, m_{n} \in M$ there exists a unique $R$-derivation $D: A \rightarrow M$ such that $D\left(x_{i}\right)=m_{i}$
Proof. It is enough to choose

$$
D(f)=\frac{\partial f}{\partial x_{1}} m_{1}+\ldots+\frac{\partial f}{\partial x_{n}} m_{n}
$$

These certainly defines a derivation with the desired property. We have to show that it is unique. Let $F$ be a derivation such that $F\left(x_{i}\right)=m_{i}$. We have to show the it coincides with $D$ on all the monomials; the $R$-linearity implies the uniqueness. Notice that the request and the Leibniz rule imply that $F\left(x_{i}^{h}\right)=x_{i}^{h-1} m_{i}$. Using this fact, we get

$$
\begin{aligned}
F\left(x_{1}^{e_{1}} \ldots x_{n}^{e_{n}}\right) & =x_{1}^{e_{1}} F\left(x_{2}^{e_{2}} \ldots x_{n}^{e_{n}}\right)+x_{1}^{e_{1}-1} x_{2}^{e_{2}} \ldots x_{n}^{e_{n}} m_{1} \\
& =\sum_{i=1}^{n} x_{1} \ldots x_{i}^{e_{i}-1} x_{i+1}^{e_{i+1}} \ldots x_{n}^{e_{n}} m_{i}
\end{aligned}
$$

and therefore such as derivation is unique.
Notice that the set of the $R$-derivation $\operatorname{Der}(A, M)$ has a natural structure of $A$-module. Indeed,

$$
\left(D_{1}+D_{2}\right)(a)=D_{1}(a)+D_{2}(A) \quad(a D)(f)=a D(f)
$$

Furthermore, it is a Lie algebra:

$$
\left[D_{1}, D_{2}\right]=D_{1} D_{2}-D_{2} D_{1}
$$

The common rules hold: if $a \in A^{*}$ then $D\left(\frac{1}{a}\right)=-\frac{D(a)}{a^{2}}$ and

$$
D\left(\frac{a}{b}\right)=\frac{b D(a)-a D(b)}{b^{2}}
$$

Definition 8.24. The module of Kahler differential $\Omega_{A / R}$ is an $A$-module with a $R$-derivation $d: A \rightarrow \Omega_{A / R}$ such that for all $D \in \operatorname{Der}(A, M)$ there exists a unique homomorphism of $A$-modules $\varphi$ such that the following commutes


The universal property gives the homomorphism of $A$-modules

$$
\begin{array}{clc}
\operatorname{Hom}_{A}\left(\Omega_{A / R}, M\right) & \longrightarrow & \operatorname{Der}_{R}(A, M) \\
\varphi & \longmapsto & \varphi \circ d
\end{array}
$$

Remark 8.25. Let $A, B$ be $R$-algebras and let $f_{1}, f_{2}: A \rightarrow B$ be homomorphisms of $R$-algebras. Let $J \subseteq B$ an ideal such that $J^{2}=0$ and assume that, called $\pi: B \rightarrow B / J$ the projection, $\pi \circ f_{1}=\pi \circ f_{2}$. Then $D=f_{1}-f_{2}$ is a derivation $D: A \rightarrow J$, as follows:

$$
\begin{aligned}
D(a b) & =f_{1}(a b)-f_{2}(a b) \\
& =f_{1}(a) f_{1}(b)-f_{2}(a) f_{2}(b) \\
& =f_{1}(a) f_{1}(b)-f_{1}(a) f_{2}(b)+f_{1}(a) f_{2}(b)-f_{2}(a) f_{2}(b) \\
& =f_{1}(a)\left(f_{1}(b)-f_{2}(b)\right)+f_{2}(b)\left(f_{1}(a)-f_{2}(a)\right) \\
& =f_{1}(a) D(b)+f_{2}(b) D(a)
\end{aligned}
$$

Theorem 8.26. $\Omega_{A / R}$ exists and it is unique up to a unique isomorphism.
Proof. Consider the map

$$
\begin{array}{rlll}
\mu: \quad A \otimes_{R} A & \longrightarrow & A \\
a \otimes b & \longmapsto a b
\end{array}
$$

This map is surjective and we call $\operatorname{Ker}(\mu)=J$. We define $\Omega_{A / R}=J / J^{2}$, which is an ideal of $A \otimes A / J^{2}$ with the structure given by the embedding on the first factor

$$
\begin{array}{ccc}
A & \longrightarrow & A \otimes A \\
a & \longmapsto & a \otimes 1
\end{array}
$$

Furthermore, the elements $a \otimes 1-1 \otimes a \in J$. We can define

$$
\begin{array}{lllc}
d: & A & \longrightarrow & J / J^{2} \\
a & \longmapsto a \otimes 1-1 \otimes a
\end{array}
$$

We have to show that $d$ is a derivation. Called

$$
\begin{array}{ccccccc}
f_{1}: \quad A & \longrightarrow & A \otimes A / J^{2} & f_{2}: & A & \longrightarrow & A \otimes A / J^{2} \\
a & \longmapsto & a \otimes 1 & & \longmapsto & 1 \otimes a
\end{array}
$$

it holds $d=f_{1}-f_{2}$ and since they are the same map $(\bmod J)$, using the remark we get that $d$ is a derivation on $J / J^{2}$. Notice that the set $\{d(f) \mid f \in A\}$ generates $\Omega_{A / R}$. Given $\sum a_{i} \otimes b_{i} \in J=\operatorname{Ker}(\mu)$, it immediately follows that $\sum a_{i} b_{i}=0$.

$$
\sum a_{i}\left(b_{i} \otimes 1\right)=0 \Rightarrow \sum a_{i} \otimes b_{i}=\sum a_{i}\left(1 \otimes b_{i}-b_{i} \otimes 1\right)=\sum a_{i} d b_{i}
$$

We now have to verify that $\Omega_{A / R}$ satisfy the universal property. Let $D: A \rightarrow M$ be a derivation. There is a unique $R$-linear map $\varphi A \otimes A \rightarrow M$ such that

$$
a \otimes b \longmapsto a D(b)
$$

and by our structure on $A \otimes A$ it is $A$-linear too. We consider $\varphi$ restricted to $J$; this factor through $J^{2}$ and so it defines a map $\psi: \Omega_{A / R} \rightarrow M$. So

$$
\psi(f(a))=\psi(1 \otimes a-a \otimes 1)=D(a)-a D(1)=D(a)
$$

Uniqueness follows from the fact that $\psi$ is uniquely determined on a set of generators ( $d(a), a \in A)$.

Corollary 8.27. If we call $A=R\left[x_{1}, \ldots, x_{n}\right]$, the module

$$
\Omega_{A / R}=\left\langle d x_{1}, \ldots, d x_{n}\right\rangle
$$

is free of rank $n$.
Let $f: A \rightarrow B$ a homomorphism of $R$-algebras. We have three different modules:

$$
\Omega_{A / R} \quad \Omega_{B / R} \quad \Omega_{B / A}
$$

Let $d_{B / A}: B \rightarrow \Omega_{B / A}$ be the canonical derivation. Then $d$ is an $R$-derivation and therefore we can factor it through $\Omega_{B / R}$. So we get a map

$$
\begin{array}{rllc}
\varphi: & \Omega_{B / R} & \longrightarrow & \Omega_{B / A} \\
d_{B / R}(b) & \longmapsto d_{B / A}(b)
\end{array}
$$

which is surjective since $d_{B / A}=\varphi \circ d_{B / R}$ and so $\varphi$ contains in the image all the elements $d_{B / A}(a)$, which generates $\Omega_{B / A}$.
Observation 8.28. Let $D: B \rightarrow M$ be a derivation and let $\varphi: A \rightarrow B$ be a homomorphism of rings. Then $\varphi$ induces a structure of $A$-module on $M$ and the composite $D \circ \varphi$ is a derivation with this structure:

$$
D \circ \varphi(a b))=D(\varphi(a) \varphi(b))=\varphi(a) D(\varphi(b))+\varphi(b) D(\varphi(a))
$$

Let $d_{B / R}: B \rightarrow \Omega_{B / R}$ be the canonical derivation. By the observation, $d_{B / R} \circ f: A \rightarrow \Omega_{B / R}$ is an $R$-derivation on $A$ and therefore it factors through $\Omega_{A / R}$, giving a map

$$
\begin{aligned}
& \tilde{\psi}: \quad \Omega_{A / R} \quad \longrightarrow \quad \Omega_{B / R} \\
& d_{A / R}(a) \longmapsto d_{B / R}(f(a)) \\
& \text { ( }
\end{aligned}
$$

Since $\Omega_{B / R}$ is also a $B$-module, we get a map

$$
\begin{array}{cccc}
\psi & B \otimes_{A} \Omega_{A / R} & \longrightarrow & \Omega_{B / R} \\
b \otimes d_{A / R}(a) & \longmapsto & b d_{B / R}(f(a))
\end{array}
$$

Theorem 8.29. Let $B$ be an algebra over a ring $A$. The sequence

$$
B \otimes_{A} \Omega_{A / R} \xrightarrow{\psi} \Omega_{B / R} \xrightarrow{\varphi} \Omega_{B / A} \rightarrow 0
$$

is exact.
Proof. We have already shown that $\varphi$ is surjective, therefore we only have to check it is exact in the middle. The composition

$$
\varphi \circ \psi\left(b \otimes d_{A / R}(a)\right)=\varphi\left(b d_{B / R}(f(a))\right)=b d_{B / A}(f(a))=0
$$

is zero and therefore $\operatorname{Im}(\psi) \subseteq \operatorname{Ker}(\psi)$. We have to show that $\operatorname{Ker}(\psi) \subseteq \operatorname{Im}(\psi)$. Let $Q=\operatorname{Coker}(\psi)$ and we call $\tilde{\varphi}: Q \rightarrow \Omega_{B / A}$. The composite

$$
\delta: B \xrightarrow{d_{B / R}} d \Omega_{B / R} \longrightarrow Q
$$

is an $A$-derivation and we get the diagram


This implies the thesis.
Consider the case $B=A / I$. Then $\Omega_{B / A}=0$ and we get the sequence

$$
B \otimes_{A} \Omega_{A / R} \simeq \Omega_{A / R} / I \Omega_{A / R} \rightarrow \Omega_{B / R} \rightarrow 0
$$

We have the composite

$$
\begin{array}{rlll}
I & \longrightarrow A & \longrightarrow & \Omega_{A / R} \\
& \longrightarrow & \Omega_{A / R} / I \Omega_{A / R} \\
f & \longmapsto & & \\
d d f]
\end{array}
$$

Notice that if $f \in I$ and $a \in A$, we get

$$
[d(a f)]=[a d f+f d a]=[a d f]=a[d f]
$$

and therefore the map

$$
I \longrightarrow \Omega_{A / R} / I \Omega_{A / R}
$$

is $A$-linear and $I^{2}$ is contained in the kernel. Thus we can factor the map through $I / I^{2}$.

Theorem 8.30. Let $A$ be a $R$-algebra, $I \subseteq A$ an ideal and let $B=A / I$. The sequence

$$
I / I^{2} \longrightarrow \Omega_{A / R} \otimes_{A} B \longrightarrow \Omega_{B / R} \rightarrow 0
$$

is exact.
We want to use this sequence to give a smoothness criterion.
Let $A=k\left[x_{1}, \ldots, x_{r}\right]$ and let $B=A / I$, where $I=\left(f_{1}, \ldots, f_{s}\right)$. We know that $\Omega_{A / k}$ is a free $A$-module on $d x_{1}, \ldots, d x_{r}$ and

$$
B \otimes_{A} \Omega_{A / k}=\Omega_{A / k /} I \Omega_{A / k}
$$

is a free $B$-module with basis $\left[d x_{1}\right], \ldots,\left[d x_{r}\right] . \quad$ Since $I / I^{2}$ is generated by $\left[f_{1}\right], \ldots,\left[f_{s}\right]$, we get

$$
\left[f_{i}\right] \longmapsto d f_{i}=\sum_{j=1}^{r} \frac{\partial f}{\partial x_{j}} d x_{j} \in B \otimes_{A} \Omega_{A / k}
$$

Therefore

$$
\Omega_{B / k}=\left(B \otimes_{A} \Omega_{A / k}\right) /\left(d f_{1}, \ldots, d f_{r}\right)
$$

Let $p \in \operatorname{Spec}(B) \subseteq \operatorname{Spec}(A)$ be a rational point, $p=\left(x_{i}-a_{i}\right)$.

$$
\Omega_{A / k} \otimes k(p) \simeq k(p)^{r}
$$

is free on $d x_{1}, \ldots, d x_{r}$. So we get the sequence

$$
I / I^{2} \otimes k(p) \longrightarrow k(p)^{r} \longrightarrow \Omega_{B / k} \otimes k(p) \rightarrow 0
$$

Notice that

$$
\operatorname{dim}_{k(p)} \Omega_{B / k}=r-\operatorname{rk} \mathcal{J}_{f}(a)=\operatorname{dim}_{k} m_{p} / m_{p}^{2}
$$

and

$$
m_{p} / m_{p}^{2} \longrightarrow \Omega_{B / k} \otimes k(p) \longrightarrow \underbrace{\Omega_{k(p) / k}}_{=0} \rightarrow 0
$$

and therefore

$$
m_{p} / m_{p}^{2} \simeq \Omega_{B / k} \otimes k(p)
$$

Example. In the case $k=\mathbb{C}$ and

$$
B=\mathbb{C}[x, y, z] /\left(x^{2}+y^{2}+z^{2}\right)
$$

we get

$$
\Omega_{B / \mathbb{C}}=B d x \oplus B d y \oplus B d z /(x d x+y d y+z d z)
$$

If $p=(x-a, y-b, z-c) \in X(\mathbb{C})$, we get

$$
\operatorname{dim}_{\mathbb{C}} \Omega_{B / \mathbb{C}} \otimes \mathbb{C}(p)= \begin{cases}2 & (a, b, c) \neq 0 \\ 3 & \text { otherwise }\end{cases}
$$

Corollary 8.31. Let $A$ be a finitely generated $k=\bar{k}$ algebra and assume that $A$ has pure dimension $d$. Then $\Omega_{A / k}$ is projective of rank $d$ if and only if $A$ is smooth.

Proof. Let $p \in \operatorname{Spec} M(A)$; then

$$
\operatorname{dim}_{k(p)} \Omega_{A / K} \otimes k(p)=\operatorname{dim}_{k(p)} m_{p} / m_{p}^{2} \geq d
$$

and we get the equality if and only if $A$ is regular at $p$, which is equivalent to smoothness.

Base change Let $R \rightarrow A$ be a homomorphism of rings and let $R \rightarrow R^{\prime}$ be an extension. If we call $A^{\prime}=R^{\prime} \otimes_{R} A$, we get the following:

## Theorem 8.32.

$$
\Omega_{A^{\prime} / R^{\prime}} \simeq \Omega_{A / R} \otimes_{R} R^{\prime} \simeq \Omega_{A / R} \otimes_{A} A^{\prime}
$$

Proof. Let $d: B \rightarrow \Omega_{B / A}$ be the canonical derivation. By base change, we get a derivation

$$
d^{\prime}=d \otimes \operatorname{Id}: B^{\prime} \longrightarrow \Omega_{B / A} \otimes_{A} A^{\prime}
$$

Notice that

$$
\Omega_{B / A} \otimes_{A} A^{\prime}=\Omega_{B / A} \otimes_{A} A^{\prime} \otimes_{B} B=\Omega_{B / A} \otimes_{B} B^{\prime}
$$

and $\left(\Omega_{B / A} \otimes_{B} B^{\prime}, d^{\prime}\right)$ verifies the universal property of $\Omega_{B^{\prime} / A^{\prime}}$.
As a consequence, let $A$ be a finitely generated algebra over a field $k$ and let

$$
\bar{A}=\bar{K} \otimes_{K} A
$$

Since $\bar{A}$ is purely $d$-dimensional, we get that $A$ is smooth over $K$ if and only if $\bar{A}$ is regular over $\bar{K}$ if and only if $\Omega_{\bar{A} / \bar{K}}$ is projective of rank $d$. This follows from the fact that $A \rightarrow \bar{A}$ is faithfully flat and from the lemma

Lemma 8.33. Let $A \rightarrow B$ a faithfully flat homomorphism of noetherian rings and let $M$ be a finite $A$-module. Then $M \otimes_{A} B$ is projective of rank $d$ if and only if $M$ is projective of rank $d$.

To prove this lemma, we can reduce to the local case. In fact, $M$ is projective of rank $d$ if and only if $M_{p}$ is free of rank $d$ for all $p \in \operatorname{Spec}(A)$. Since the extension is faithfully flat, there exists $q \in \operatorname{Spec}(B)$ such that $\varphi^{-1}(q)=p$ and

$$
M \otimes_{A} B \otimes B_{q} \simeq M_{p} \otimes_{A_{p}} B_{q}
$$

Then we have to show the following:
Lemma 8.34. If $A \rightarrow B$ is a flat local homomorphism of local noetherian rings and $M$ is an $A$-module, $M$ is free of rank $d$ if and only if $M \otimes_{A} B$ is free of rank $d$.

Proof. Let $K(B)=B / m_{B} B$ and $K(A)=A / m_{A} A$. Then

$$
M \otimes B \otimes K(B)=(M \otimes K(A)) \otimes_{K(A)} K(B)
$$

and then $M \otimes K(A) \simeq K(A)^{d}$. Nakayama's lemma implies that

$$
0 \rightarrow N \rightarrow A^{d} \rightarrow M \rightarrow 0
$$

becomes

$$
0 \rightarrow N \otimes B \rightarrow B^{d} \rightarrow M \otimes B \simeq B^{d} \rightarrow 0
$$

Therefore $N \otimes B=0$ and since $B$ is faithfully flat, $N=0$.
Let now $R \rightarrow A_{1}, R \rightarrow A_{2}$ be ring homomorphisms and consider $A=$ $A_{1} \times A_{2}$. Then every $A$-module decompose into a product $M_{1} \times M_{2}$. Therefore if $D: A \rightarrow M$ is a $R$-derivation, we can split $D$ into $D_{1}: A_{1} \rightarrow M_{1}$ and $D_{2}: A_{2} \rightarrow$ $M_{2}$. Therefore

$$
\Omega_{A / R}=\Omega_{A_{1} / R} \oplus \Omega_{A_{2} / R}
$$

Let now $E / K$ be a finite separable field extension. Then, by the primitive element theorem, we can find $\theta \in E$ such that $E=K(\theta)$. If we call $f$ the minimum polynomial of $\theta$, we get

$$
E=K(\theta) \simeq K[x] / f(x)
$$

and since the extension is sparable, $f^{\prime}(\theta) \neq 0$. We get the sequence

$$
(f(x)) /(f(x))^{2} \longrightarrow \Omega_{K[x] / K} \otimes_{K} E \longrightarrow \Omega_{E / K} \longrightarrow 0
$$

Notice that, if we choose $d x$ as a base of $\Omega_{K[x] / K}$, the image of $[f(x)]$ is $f^{\prime}(\theta) d x$, which is different from zero by hypotesis. Therefore, the first map is injective and surjective and

$$
(f(x)) /(f(x))^{2} \simeq \Omega_{K[x] / K} \otimes_{K} E \quad \quad \Omega_{E / K}=0
$$

On the other hand, if $f$ is irreducible but not separable, we get $\Omega_{E / K} \simeq$ $\Omega_{K[x] / K} \otimes E \simeq E$.
Proposition 8.35. Let $R \rightarrow A$ be a ring homomorphism and let $S \subseteq A$ be a multiplicative system. Then $\Omega_{S^{-1} A / R} \simeq S^{-1} \Omega_{A / R}$.
Proof. Notice that the module on $S^{-1} A$ corresponds to the module on $A$ such that every $s \in S$ acts as an isomorphism. Then if $M$ is an $S^{-1} A$-module, every $D: A \rightarrow M$ extends uniquely to a derivation $S^{-1} A \rightarrow M$. So we get an isomorphism $\operatorname{Hom}_{S^{-1} A}\left(S^{-1} \Omega_{A_{R}}, M\right) \simeq \operatorname{Hom}_{A}\left(\Omega_{A / R}, M\right)$ and therefore an isomorphism

$$
\operatorname{Der}_{R}(A, M) \simeq \operatorname{Der}_{R}\left(S^{-1} A, M\right)
$$

Let $K$ be a field; we can now compute $\Omega_{K\left(x_{1}, \ldots, x_{n}\right) / K}$. If we set $S=$ $K\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\}$, we get

$$
\Omega_{K\left(x_{1}, \ldots, x_{n}\right) / K} \simeq S^{-1} \Omega_{K\left[x_{1}, \ldots, x_{n}\right] / K}
$$

which is therefore a free vector space with base $d \frac{x_{1}}{1}, \ldots, d \frac{x_{n}}{1}$.
The algebraic theory globalizes to schemes. If $f: X \rightarrow S$ is a morphism of schemes, there exists a quasi-coherent sheaf $\Omega_{X / S}$ such that for all open affine subset $U=\operatorname{Spec}(A) \subseteq X$ and $V=\operatorname{Spec}(R) \subseteq S$ such that $f(U) \subseteq V$

$$
\left.\Omega_{X / S}\right|_{U} \simeq \widetilde{\Omega_{A / R}}
$$

If $X$ is a scheme and $F$ is a quasi-coherent sheaf on $X$ one defines a derivation $D: \mathcal{O}_{X} \rightarrow F$ as an additive map satisfing the Leibniz rule. If $f: X \rightarrow S$ is a morphism of schemes, an $S$-derivation $D: \mathcal{O}_{X} \rightarrow F$ is a derivation such that if $u \in \mathcal{O}(V) D\left(\varphi^{\#}(u)\right)=0$. There exists a $S$-derivation $\mathcal{O}_{X} \rightarrow \Omega_{X / S}$ universal among all the $S$-derivations $\mathcal{O}_{X} \rightarrow F$.

Theorem 8.36. $\Omega_{X / S}$ exists.
Proof. We know that in the affine case such a sheaf exists and it corresponds to the module $J / J^{2}$. If $U=\operatorname{Spec}(A)$ is an affine open subset, $\operatorname{Spec}\left(A \otimes_{R}\right.$ $A)=U \times_{S} U$ and the map $A \otimes_{R} A \rightarrow A$ corresponds to the diagonal $\delta: X \rightarrow$ $X \times_{S} X$. We call $I_{X}$ the sheaf of ideal on $X \times_{S} X$ given by the kernel of the morphism of sheaves induced by the diagonal map $\delta: X \rightarrow X \times_{S} X$. If $f: X \rightarrow S$ is separated, we define $\Omega_{X / S}: \delta^{*} I_{X}$, which corresponds locally to $J / J^{2}$; the associated derivation is

$$
\begin{aligned}
d: \quad \mathcal{O}_{X} & \longrightarrow \\
f^{*} & \longmapsto \delta^{*}\left(p r_{2}^{*} f-p r_{1}^{*} f\right)
\end{aligned}
$$

If $X \rightarrow S$ is not separated, we know that $\delta$ is a locally closed embedding and there exist an open subset $U \subseteq X \times_{S} X$ such that $\delta$ factors as

$$
X \xrightarrow{\alpha} U \rightarrow X \times_{S} X
$$

We define $\Omega_{X / S}=\alpha^{*} I_{X}$ and it doesn't depend on the choice of the open subscheme by uniqueness on affine open subsets.

## Proposition 8.37.

1. Let $S$ be a noetherian scheme and let $f: X \rightarrow S$ be locally of finite type. Then $\Omega_{X / S}$ is coherent.
2. Let $S$ be a scheme over $K$. If $X$ is smooth of pure dimension $d, \Omega_{X / K}$ is locally free of rank $d$.

Proof.

1. Since being coherent is local, we can assume $X=\operatorname{Spec}(A)$ and $Y=$ $\operatorname{Spec}(R)$. Then $\Omega_{A / R}$ is a quotient of $\Omega_{R\left[x_{1}, \ldots, x_{n}\right] / R} \otimes A$ and therefore it is finitely generated.

Example. We want to compute $\Omega_{\mathbb{P}_{K}^{1}}$. By the proposition, it is invertible. We can consider the open cover

$$
\mathbb{P}_{K}^{1}=U_{0} \cup U_{1}
$$

On these sets,

$$
\left.\left.\Omega_{\mathbb{P}_{K}^{1} / K}\right|_{U_{0}} \simeq d u \mathcal{O} \quad \Omega_{\mathbb{P}_{K}^{1} / K}\right|_{U_{1}} \simeq d v \mathcal{O}
$$

On the intersection, we get

$$
d u=-\frac{d v}{v^{2}}
$$

Consider the map deg: $\operatorname{Pic}\left(\mathbb{P}_{K}^{1}\right) \rightarrow \mathbb{Z}$. Then

$$
v_{p}(d u)=0 \quad \forall p \in U_{0} \quad v_{\infty}(d u)=-2
$$

Therefore $\operatorname{deg}\left(\Omega_{\mathbb{P}_{K}^{1} / K}\right)=-2$ and it is isomorphic to $\mathcal{O}_{\mathbb{P}_{K}^{1}}(-2)$. We could have shown also the isomorphism; it holds $\Omega_{\mathbb{P}_{R}^{1} / R} \simeq \mathcal{O}_{\mathbb{P}_{R}^{1}}(-2)$ for every ring $R$.
$\Omega_{\mathbb{P}_{K}^{n} / K}$ is locally free of rank $n$; given $f=\frac{\varphi}{\psi} \in \mathcal{O}_{\mathbb{P}_{K}^{n}}(U)$,

$$
\frac{\partial f}{\partial x_{i}}=\frac{\frac{\partial \varphi}{\partial x_{i}} \psi-\varphi \frac{\partial \psi}{\partial x_{i}}}{\psi^{2}} \in \mathcal{O}(-1)(U)
$$

This defines a derivation $\mathcal{O}_{\mathbb{P}_{R}^{n}} \rightarrow \mathcal{O}_{\mathbb{P}_{R}^{n}}(-1)$. The map

$$
\begin{array}{rlr}
\Omega_{\mathbb{P}_{R}^{n}} & \longrightarrow & \mathcal{O}(-1)^{\oplus(n+1)} \\
f & \longmapsto & \left(\frac{\partial f}{\partial x_{0}}, \ldots, \frac{\partial f}{\partial x_{n}}\right)
\end{array}
$$

is injective but not surjective. We can also consider the map

$$
\begin{array}{rlc}
\mathcal{O}(-1)^{\oplus(n+1)} & \longrightarrow & \mathcal{O} \\
\left(f_{0}, \ldots, f_{n}\right) & \longmapsto & \sum x_{i} f_{i}
\end{array}
$$

Notice that if $f \in \mathcal{O}(U)$, the Euler Formula implies that $\sum x_{i} \frac{\partial f}{\partial x_{i}}=0$.
Proposition 8.38. The sequence

$$
0 \rightarrow \Omega_{\mathbb{P}_{K}^{n} / K} \longrightarrow \mathcal{O}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}_{K}^{n}} \rightarrow 0
$$

is exact.
Proof. Since being exact is local, we can reduce to the case of the open affine subset $U_{0}=\left(\mathbb{P}_{K}^{n}\right)_{x_{0}}$.

$$
U_{0}=\operatorname{Spec}\left(k\left[u_{0}, \ldots, u_{n}\right]\right) \quad u_{i}=\frac{x_{i}}{x_{0}}
$$

Then $\Omega_{U_{0} / R}$ is free over $d u_{1}, \ldots, d u_{n}$. It can be verified that the sequence is exact.

### 8.3 Exterior Powers of Sheaves

Let $\left(X, \mathcal{O}_{X}\right)$ be a locally ringed space and let $F$ be a sheaf of $\mathcal{O}_{X}$-modules. For every open affine $U \subseteq X$, we know that there exists the $\mathcal{O}(U)$-module $\Lambda_{\mathcal{O}(U)}^{d} F(U)$ of exterior powers, which has the usual universal property for alternating products. Therefore we can define a presheaf $U \mapsto \Lambda_{\mathcal{O}(U)}^{d} F(U)$, where the restriction maps are given by the universal property. We define the module of exterior powers $\Lambda_{\mathcal{O}}^{d} F$ as the sheafification of this presheaf. It is a universal object among all the $d$-alternating linear maps.

Proposition 8.39. Assume that $F$ is locally free of $\operatorname{rank} n$. Then $\Lambda^{d} F$ is locally free of rank $\binom{n}{d}$.

Proof. Since the fact is local, we can assume that $F$ is free. In this case, we know that $\Lambda^{d} F$ is free over the base

$$
e_{i_{1}} \wedge \cdots \wedge e_{i_{k}} \quad 1 \leq i_{1}<\cdots<i_{k} \leq n
$$

Notice that if $f: F \rightarrow G$ is a homomorphism of sheaves, we get an induced map

$$
\Lambda^{d} f: \Lambda^{d} F \rightarrow \Lambda^{d} G
$$

Definition 8.40. Let $F$ be a locally free sheaf of rank $n$. The determinant of $F$ is the invertible sheaf $\operatorname{det}(F):=\Lambda^{n} F$. If $f: F \rightarrow F$ is a homomorphism of sheaves, we define the map $\operatorname{det} f: \operatorname{det}(F) \rightarrow \operatorname{det}(F)$.

If $L$ is an invertible sheaves, the determinant defines defines an isomorphism End $_{\mathcal{O}}(L) \simeq \mathcal{O}$.

## Lemma 8.41.

1. If $B$ is an $A$-algebra and $M$ is a free $A$-module, we have a canonical isomorphism between $\operatorname{det}\left(M \otimes_{A} B\right)$ and $(\operatorname{det} M) \otimes_{A} B$.
2. Consider an exact sequence of free $A$-modules

$$
0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \rightarrow 0
$$

Then

$$
\operatorname{det}(N) \simeq \operatorname{det}(M) \otimes_{A} \operatorname{det}(P)
$$

Proof.

1. Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be a basis of $M$ and let $e_{i}^{\prime}=e_{i} \otimes 1$. We can construct the homomorphism

$$
\begin{array}{rll}
\psi: \quad(\operatorname{det} M) \otimes_{A} B & \longrightarrow & \operatorname{det}\left(M \otimes_{A} B\right) \\
\left(e_{1} \wedge \cdots \wedge e_{n}\right) \otimes b & \longmapsto & \left(e_{1}^{\prime} \wedge \ldots e_{n}^{\prime}\right) b
\end{array}
$$

which is surjective since $e_{1}^{\prime} \wedge \ldots e_{n}^{\prime}$ is a basis of $M \otimes_{A} B$. This implies that is injective too, and therefore it is an isomorphism.
2. Let $\left\{p_{1}, \ldots, p_{k}\right\}$ and $\left\{m_{1}, \ldots, m_{h}\right\}$ be basis for $P$ and $M$ respectively. Chosen $g_{i} \in \beta^{-1}\left(p_{i}\right)$, and renamed $\alpha\left(m_{i}\right)=s_{i}$, the set $\left\{s_{i}, g_{j}\right\}$ defines a basis of $N$. We define a map

$$
\psi: \begin{array}{ccc}
\operatorname{det}(M) \otimes_{A} \operatorname{det}(P) & \longrightarrow & \operatorname{det}(N) \\
\left(m_{1} \wedge \cdots \wedge m_{h}\right) \otimes\left(p_{1} \wedge \cdots \wedge p_{k}\right) & \longmapsto & s_{1} \wedge \cdots \wedge s_{h} \wedge g_{1} \wedge \cdots \wedge g_{h}
\end{array}
$$

$\psi$ defines an isomorphism (it is surjective since $\left\{s_{i}, g_{j}\right\}$ is a basis of $N$ and therefore it is injective too) and it doesn't depend on the choice of the basis.

## Proposition 8.42.

1. Let $f: X \rightarrow Y$ be a morphism of schemes and let $F$ be a locally free sheaf on $Y$. Then $\operatorname{det}\left(f^{*} X\right)=f^{*} \operatorname{det}(X)$.
2. Let $0 \rightarrow F^{\prime} \rightarrow F \rightarrow F^{\prime \prime} \rightarrow 0$ be an exact sequence of locally free sheaves. Then

$$
\operatorname{det}(F) \simeq \operatorname{det}\left(F^{\prime}\right) \otimes_{\mathcal{O}} \operatorname{det}\left(F^{\prime \prime}\right)
$$

Proof.

1. Let $U \subseteq Y$ be an open subset such that $\left.F\right|_{U} \simeq O_{U}^{n}$. Then, using the lemma,

$$
\begin{aligned}
\left.f\right|_{f^{-1} U} ^{*}\left(\left.\operatorname{det}(F)\right|_{U}\right) & \simeq f^{-1} \operatorname{det}\left(\left.F\right|_{U}\right) \otimes_{f^{-1}(U)} \mathcal{O}_{f^{-1} U} \\
& \simeq \operatorname{det}\left(\left.f^{-1} f\right|_{U} \otimes_{f^{-1}} \mathcal{O}_{U} \mathcal{O}_{f^{-1} U}\right) \\
& \simeq \operatorname{det}\left(f^{*}\left(\left.F\right|_{U}\right)\right)
\end{aligned}
$$

and these isomorphisms glue togheter.
2. Let $\left\{U_{i}\right\}$ be an open cover such that $\left.\left.\left.F^{\prime}\right|_{U_{i}} \simeq F\right|_{U_{i}} \simeq F^{\prime \prime}\right|_{U_{i}} \simeq \mathcal{O}_{U_{i}}$. By the lemma and the first part of this proposition, we have a family of isomorphisms $\psi_{i}:\left.\left.\left.\operatorname{det}\left(F^{\prime}\right)\right|_{U_{i}} \otimes \operatorname{det}\left(F^{\prime \prime}\right)\right|_{U_{i}} \rightarrow \operatorname{det}(F)\right|_{U_{i}}$. These isomorphisms agree on the intersection of these open sets since the isomorphism are independent on the choice of the basis and therefore they lift to a global isomorphism.

Definition 8.43. Let $X$ be a scheme over $K$ smooth of pure dimension $n$. The canonical sheaf or dualizing sheaf of $X$ is $\omega_{X}=\operatorname{det}\left(\Omega_{X / K}\right)$.
Example.

- If $n=1$, we get $\omega_{X}=\Omega_{X / K}$. In particular, $\omega_{\mathbb{P}_{K}^{1}}=\Omega_{\mathbb{P}_{K}^{1} / K}=\mathcal{O}(-2)$
- Using the Euler sequence $0 \rightarrow \Omega_{\mathbb{P}_{K}^{n}} \rightarrow \mathcal{O}(-1)^{\oplus n+1} \rightarrow \mathcal{O} \rightarrow 0$, we get

$$
\mathcal{O}(-n-1) \simeq \bigotimes_{i=1}^{n+1} \operatorname{det}(\mathcal{O}(-1)) \simeq \operatorname{det}\left(\mathcal{O}(-1)^{\oplus n+1}\right) \simeq \omega_{\mathbb{P}_{K}^{n}} \otimes_{\mathcal{O}} \mathcal{O} \simeq \omega_{\mathbb{P}_{K}^{n}}
$$

Suppose that $X$ is smooth over $\operatorname{Spec}(K)$ of dimension $n$ and let $Y \xrightarrow{j} X$ be a smooth closed subscheme of dimension $m$. We have an exact sequence

$$
\left.I_{Y} I_{Y}^{2} \longrightarrow \Omega_{X / K}\right|_{Y} \longrightarrow \Omega_{Y / K} \longrightarrow 0
$$

Proposition 8.44. $I_{Y} / I_{Y}^{2}$ is locally free of rank $n-m$ and the map

$$
\left.I_{Y} I_{Y}^{2} \longrightarrow \Omega_{X / K}\right|_{Y}
$$

is injective.

Sketch of the proof. Notice that if $\operatorname{codim}_{Y} X=1$, then $Y$ is an effective Cartier divisor on $X$. Then $I_{Y}=\mathcal{O}_{X}(Y)$ is invertible on $X$ and therefore

$$
I_{Y} / I_{Y}^{2}=j^{*} I_{Y}
$$

is invertible.
As a consequence, we get the exact sequence

$$
\left.0 \rightarrow I_{Y} I_{Y}^{2} \longrightarrow \Omega_{X / K}\right|_{Y} \longrightarrow \Omega_{Y / K} \longrightarrow 0
$$

Taking determinants, we get the adjunction formula

$$
\left.\omega_{X / K}\right|_{Y}=\omega_{Y / K} \otimes \operatorname{det}\left(I_{Y / I_{Y}^{2}}\right)
$$

Observation 8.45. Notice that if $Y$ is a Cartier divisor, $I_{Y} / I_{Y}^{2}=\left.\mathcal{O}_{X}(-Y)\right|_{Y}$.
Curves in $\mathbb{P}^{2}$ Let $X \subseteq \mathbb{P}_{K}^{2}$ a smooth curve of degree $d$. Then $I_{X}=\mathcal{O}(-d)$ and $\omega_{\mathbb{P}^{3}} \simeq \mathcal{O}(-3)$; therefore

$$
\mathcal{O}_{X}(-3)=\omega_{X} \otimes \mathcal{O}_{X}(-d) \Rightarrow \omega_{X}=\mathcal{O}_{X}(d-3)
$$

Remark 8.46. Notice that $\operatorname{deg}\left(\mathcal{O}_{X}(1)\right)=d$; in fact, if $L \subseteq \mathbb{P}_{K}^{2}$ is a line then $\mathcal{O}_{\mathbb{P}^{2}}(L) \simeq \mathcal{O}(1)$. If $s \in H^{0}\left(\mathbb{P}^{2}, \mathcal{O}(1)\right)$ is the section defining $L$, then

$$
\operatorname{div}\left(\left.s\right|_{X}\right)=X \cap L
$$

and therefore

$$
\operatorname{deg}\left(\mathcal{O}_{X}(L)\right)=\operatorname{deg}\left(\operatorname{div}\left(\left.s\right|_{X}\right)\right)=\chi\left(\mathcal{O}_{X \cap L}\right)=d
$$

As a corollary, $\operatorname{deg}\left(\mathcal{O}_{X}(l)\right)=l d$.
Since $g(X)=h^{1}\left(\mathcal{O}_{X}\right)=\frac{(d-1)(d-2)}{2}=\frac{d(d-3)+2}{2}$, we get $\operatorname{deg}\left(\omega_{x}\right)=2 g(X)-2$ Example.

- If $d=1$, then $X \simeq \mathbb{P}_{K}^{1}$ and $\omega_{X}=\left.\mathcal{O}_{X}(-2)\right|_{X}$ has degree -2 .
- If $d=2$, then $\omega_{X}=\mathcal{O}_{X}(-1)$ has degree -2 . If $X(k) \neq \emptyset$ then $X \simeq \mathbb{P}_{K}^{1}$. If $X(k)=\emptyset$ we can make a base change and we know that the degree is invariant.
- If $d=3, \omega_{X} \simeq \mathcal{O}_{X}$ and $\operatorname{deg}\left(\omega_{X}\right)=0$.
- If $d=4, \omega_{X} \simeq \mathcal{O}_{X}(1)$ and $\operatorname{deg}\left(\omega_{X}\right)=4$.

Example. Let $X \subseteq \mathbb{P}^{2}$ be a smooth curve of degree $d$. Then $\omega_{X}=\mathcal{O}_{X}(d-3)=$ $j^{*} \mathcal{O}(d-3)$. From the exact sequence

$$
0 \rightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X} \rightarrow 0
$$

we get, tensoring for $\mathcal{O}(d-3)$,

$$
0 \rightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(d-3) \longrightarrow \mathcal{O}_{X}(d-3) \rightarrow 0
$$

We can therefore consider the long exact sequence:

$$
0 \rightarrow H^{0}(\mathcal{O}(-3)) \rightarrow H^{0}(\mathcal{O}(d-3)) \rightarrow H^{0}\left(\mathcal{O}_{X}(d-3)\right) \rightarrow 0
$$

and therefore $H^{0}\left(\mathcal{O}_{X}(d-3)\right) \simeq H^{0}(\mathcal{O}(d-3))$.

$$
H^{0}\left(\mathcal{O}_{X}(d-3)\right)=\left\{\begin{array}{l}
0 \text { if } d \leq 2 \\
\binom{d-1}{2} \text { if } d \geq 1
\end{array}\right.
$$

### 8.4 Riemann-Roch Theorem

Using the correspondance bewtween Cartier divisors and some closed subschemes of $X$, we can find some particular divisors:

Definition 8.47. Let $X$ be a smooth proper geometrically connected curve over $k$. A canonical divisor $K$ is a divisor such that $\mathcal{O}_{X}(K)$ corresponds to the class of the canonical sheaf $\omega_{X}$.

Theorem 8.48 (Serre Duality). Let $X$ be a proper, smooth and geometrically connected scheme over a field $k$ of dimension $n$. Let $L$ be a locally free sheaf on $X$. Then for every $i=0, \ldots, n$ there exists a canonical perfect pairing

$$
H^{i}(X, L) \otimes_{k} H^{n-i}\left(X, L^{\vee} \otimes \omega_{X}\right) \longrightarrow H^{n}\left(X, \omega_{X}\right) \simeq k
$$

## Corollary 8.49.

- $h^{i}(L)=h^{n-i}\left(L^{\vee} \otimes \omega_{X}\right)$ and $h^{i}\left(\mathcal{O}_{X}\right)=h^{n-i}\left(\omega_{X}\right)$
- If $X$ is a curve of genus $g$ then $g=h^{0}\left(\omega_{X}\right)$.

Assume now that $\operatorname{dim}(X)=1$ and let $L$ be an invertible sheaf on $X$. We know that $L=\mathcal{O}(D)$ where $D \in \operatorname{Div}(X)$ and $h^{i}(D)=h^{i}(L)$ only depends on the class of $D$. Furthermore, given a canonical divisor $K, \omega_{X}=\mathcal{O}_{X}(K)$ and $L^{\vee} \otimes \omega_{X}=\mathcal{O}_{X}(K-D)$. Serre Duality gives $h^{1}(D)=h^{0}(K-D)$; we want to have control over $h^{0}(D)$. We can do this using the following:

Theorem 8.50 (Riemann-Roch). Let $X$ be a curve and let $K$ be a canonical divisor. For every $D \in \operatorname{Div}(X)$,

$$
h^{0}(D)-h^{0}(K-D)=\operatorname{deg}(D)+1-g(X)
$$

Proof. By Serre Duality, we know that $h^{0}(K-D)=h^{1}(D)$ and therefore

$$
h^{0}(D)-h^{0}(K-D)=\chi(\mathcal{O}(D))
$$

If $D=0$, we know that $\mathcal{O}(D) \simeq \mathcal{O}_{X}$ and therefore $\chi\left(\mathcal{O}_{D}\right)=1-g(X)=$ $\operatorname{deg}(D)+1-g(X)$.
Let now $p \in X^{(1)}$ and consider the exact sequence

$$
0 \rightarrow I_{p} \longrightarrow \mathcal{O}_{X} \longrightarrow \mathcal{O}_{p} \rightarrow 0
$$

We know that $I_{p}^{\vee}=\mathcal{O}(p)$ and therefore $\mathcal{O}(-p)=I_{p}$ since $\operatorname{Pic}(X)$ is a group and the inverse corresponds to the dual.

$$
0 \rightarrow \mathcal{O}(-p) \longrightarrow \mathcal{O}_{X} \longrightarrow \mathcal{O}_{p} \rightarrow 0
$$

Tensoring for $\mathcal{O}(D)$, we get

$$
0 \rightarrow \mathcal{O}_{X}(D-p) \longrightarrow \mathcal{O}_{X}(D) \longrightarrow \mathcal{O}_{X}(D) \otimes \mathcal{O}_{p} \rightarrow 0
$$

Notice that since $\mathcal{O}(D)$ is locally free and $\mathcal{O}_{p}$ is supported on a point, $\mathcal{O}_{X}(D) \otimes$ $\mathcal{O}_{p} \simeq \mathcal{O}_{p}$ and

$$
0 \rightarrow \mathcal{O}_{X}(D-p) \longrightarrow \mathcal{O}_{X}(D) \longrightarrow \mathcal{O}_{X}(D) \otimes \mathcal{O}_{p} \rightarrow 0
$$

We now compute the Euler characteristic of $\mathcal{O}_{p}$; since it has finite support,

$$
\chi\left(\mathcal{O}_{p}\right)=h^{0}(P)=\operatorname{dim}_{k} k(p)=\operatorname{deg}(p)
$$

We can now compute the Euler characteristic of $\mathcal{O}_{X}(D)$ by the addictivity of Euler characteristic:

$$
\chi\left(\mathcal{O}_{X}(D)\right)=\chi\left(\mathcal{O}_{X}(D-p)\right)-\operatorname{deg}(p)
$$

The last formula holds even in the case of $-p$

$$
\chi(\mathcal{O}(D))=\chi\left(\mathcal{O}_{X}(D+p)\right)-\operatorname{deg}(p)
$$

We are now ready to conclude the proof. Given a divisor $D$, we can consider it as a finite sum of points $p \in X^{(1)}$ and the relations we have found show that the theorem still holds after adding a point. Since we have shown that it holds for $D=0$, we get the thesis.
Corollary 8.51 (Riemann formula). $h^{0}(D) \geq \operatorname{deg}(D)+1-g$
Example.

- If $D=0$, then $1-h^{0}(K)=1-g$ and $h^{0}(K)=g$.
- If $D=K, h^{0}(K)-h^{0}(0)=\operatorname{deg}(K)+1-g$ and therefore $\operatorname{deg}\left(\omega_{X}\right)=2 g-2$

Remark 8.52. Notice that if $\operatorname{deg}(D)<0$ then $h^{0}(D)=0$. In fact, $h^{0}(D) \neq 0$ if and only if $D$ is linearly equivalent to an effective divisor.

Corollary 8.53. If $\operatorname{deg}(D)>2 g-2$, then $h^{0}(D)=\operatorname{deg}(D)+1-g$
Example. If $g(X)=0$, we get $h^{0}(D)=\operatorname{deg}(D)+1$ whenever $\operatorname{deg}(D) \geq-1$.
Assume now that $g=0$ and let $p \in X(k)$. Then $\operatorname{deg}(p)=1$ and the corollary implies $h^{0}(\mathcal{O}(p))=2$. So we have

$$
k \subseteq H^{0}\left(\mathcal{O}_{X}\right) \subsetneq H^{0}(\mathcal{O}(p))
$$

Hence there exists $f \in H^{0}(\mathcal{O}(p)) \backslash k$ such that $\operatorname{div}(f)+p \geq 0$. This means that $\operatorname{div}(f)=q-p$, where $q \in X(k)$ and $q \neq p$. Therefore $X \simeq \mathbb{P}_{K}^{1}$.

Corollary 8.54. If $g=0$ and $X(k) \neq \emptyset$, then $X \simeq \mathbb{P}_{k}^{1}$.
Definition 8.55. Let $D \in \operatorname{Div}(X)$. We define the linear system of $D$ as

$$
|D|=\{E \in \operatorname{Div}(X) \mid E \sim D E \geq 0\}
$$

The map

$$
\begin{array}{clc}
H^{0}(\mathcal{O}(D)) \backslash\{0\} & \longrightarrow & |D| \\
f & \longmapsto & \operatorname{div}(f)+D
\end{array}
$$

is trivially surjective and given $f, g \in K(X)^{*}$,

$$
\operatorname{div}(f)=\operatorname{div}(g) \Longleftrightarrow \frac{f}{g} \in k^{*}
$$

Therefore we get a bijection

$$
|D| \longleftrightarrow H^{0}(\mathcal{O}(D)) \backslash\{0\} / k^{*}
$$

Assume now $g(X)=2$; then $h^{0}\left(\omega_{X}\right)=2$ and $\operatorname{deg}\left(\omega_{X}\right)=2$. We can find $s, t \in H^{0}\left(\omega_{X}\right)$ linearly independent elements and they give two distinct effective canonical divisor $D, D^{\prime}$. If $k=\bar{k}, D=p+q$, where $p, q \in X(k)$. Notice that

$$
D \sim D^{\prime} \Rightarrow \exists f \in K(X)^{*} \text { s.t. } \operatorname{div}(f)=D-D^{\prime} \Rightarrow \operatorname{Supp}(D) \cap \operatorname{Supp}\left(D^{\prime}\right)=\emptyset
$$

In fact, if $\operatorname{Supp}(D) \cap \operatorname{Supp}\left(D^{\prime}\right) \neq \emptyset$, since $D \neq D^{\prime}$, there exist $p, q \in X(k)$ such that $p \sim q$ and therefore $X \simeq \mathbb{P}_{K}^{1}$, against the hypotesis. Therefore, there exists $f: X \rightarrow \mathbb{P}_{K}^{1}$.

$$
\operatorname{div}(f)=D-D^{\prime}=f^{*}(0)-f^{*}(\infty)
$$

and $f^{*}(0)=D, f^{*}(\infty)=D^{\prime}$, which means that $\operatorname{deg}(f)=\operatorname{deg}\left(f^{*}(0)\right)=2$.
Corollary 8.56. Every curve of genus 2 has a map to $\mathbb{P}_{K}^{1}$ of degree 2.
Let $X$ be a smooth projective geometrically connected curve over $k$ and let $D \in \operatorname{Div}(X)$. We recall that there exists a bijection between maps to $\mathbb{P}_{K}^{n}$ and classes of isomorphism of tuples $\left(L, s_{0}, \ldots, s_{n}\right)$ where $L$ is an invertible sheaf on $X$ and $s_{0}, \ldots, s_{n}$ generates $L$. Indeed, the set

$$
X_{s_{i}}=\left\{p \in X \mid s_{i}(p) \neq 0\right\}
$$

is open in $X$ and these sets form an open cover of $X$. Therefore, to a map $f: X \rightarrow \mathbb{P}^{n}$ we can associate the tuple $\left(f^{*} \mathcal{O}(1), f^{*} x_{0}, \ldots, f^{*} x_{n}\right)$. Viceversa, given a tuple we can construct a map on the $X_{s_{i}}$ given by gluing the morphisms induced by

$$
\begin{array}{rlc}
R\left[\frac{x_{1}}{x_{i}} \ldots \frac{x_{n}}{x_{i}}\right] & \longrightarrow & H^{0}\left(X_{s_{i}}, \mathcal{O}\right) \\
\frac{x_{j}}{x_{i}} & \longmapsto & \frac{s_{j}}{s_{i}}
\end{array}
$$

Let now $X$ be a curve and $D \in \operatorname{Div}(X)$. Given $p \in X(k)$, we get the exact sequence

$$
0 \rightarrow \mathcal{O}_{X}(-p) \longrightarrow \mathcal{O}_{X} \longrightarrow \mathcal{O}_{p} \rightarrow 0
$$

Tensoring for $\mathcal{O}(D)$ we get

$$
0 \rightarrow H^{0}\left(\mathcal{O}_{X}(D-p)\right) \longrightarrow H^{0}\left(\mathcal{O}_{X}(D)\right) \longrightarrow H^{0}\left(\mathcal{O}_{p}(D)\right) \simeq k
$$

We can distinguish two cases:

1. $h^{0}(D)=h^{0}(\mathcal{O}(D-p))$ and $s(p)=0$ for all $s \in H^{0}(\mathcal{O}(D))$
2. $h^{0}(D-p)=h^{0}(D)-1$ and there exists $s \in H^{0}(\mathcal{O}(D))$ such that $s(p) \neq 0$

Lemma 8.57. If $k=\bar{k}, \mathcal{O}_{X}(D)$ is generated by global sections if and only if $h^{0}(D-p)=h^{0}(D)-1$ for all $p \in X(k)$.

Proof. Assume first that $h^{0}(D-p)=h^{0}(D)-1$ for all $p \in X(k)$. Then for all $p \in X^{(1)}$ there exists $s \in H^{0}(\mathcal{O}(D))$ such that $s(p) \neq 0$ for the second condition, which means exactly that $\mathcal{O}(D)$ is generated by global section. On the other hand, if there exists $p \in X^{(1)}$ such that $h^{0}(D-p)=h^{0}(D), s(p)=0$ for all $s \in H^{0}\left(\mathcal{O}_{X}(D)\right)$; hence $\mathcal{O}_{X}(D)$ is not generated by global section.

If $\mathcal{O}_{X}(D)$ is generated by global section, we can use a basis of $H^{0}(\mathcal{O}(D))$ to define a map $X \rightarrow \mathbb{P}_{K}^{n}$ where $n=h^{0}(\mathcal{O}(D))-1$. This map is unique up to a linear transformation of $\mathbb{P}_{k}^{1}$.
Remark 8.58. Let $X$ be a separated quasi-compact scheme over $k$ and $k^{\prime} / k$ be a field extension. If $L$ is an invertible sheaf on $X$, we get a projection $\pi: X_{k^{\prime}} \rightarrow X$. Then $L$ is generated by global section if and only if $\pi^{*} L$ is generated by global section. To show this, it is useful to use tha fact that $H^{0}\left(X_{k^{\prime}}, \pi^{*} L\right) \simeq k^{\prime} \otimes_{k} H^{0}(X, L)$.

Assume $k=\bar{k}$ and let $D \in \operatorname{Div}(X)$. If $\mathcal{O}_{X}(D)$ is generated by global sections, we get a map $f: X \rightarrow \mathbb{P}^{n}$ and $f^{*} \mathcal{O}(1) \simeq \mathcal{O}_{X}(D)$ since $f^{*}: H^{0}\left(\mathcal{O}_{\mathbb{P}^{n}}(1)\right) \rightarrow$ $H^{0}\left(\mathcal{O}_{X}(D)\right)$ is an isomorphism.

Theorem 8.59. $f$ is an embedding if and only if for all $p, q \in X(k) h^{0}(D-p-$ $q)=h^{0}(D)-2$.

Proposition 8.60. Let $X, Y$ be schemes over $k$ and let $f: X \rightarrow Y$ be a morphism. If $k^{\prime} / k$ is a field extension, then $f$ is a closed embedding if and only if $f_{k^{\prime}}: X_{k^{\prime}} \rightarrow X$ is a closed embedding.

Corollary 8.61. Let $X$ be a curve over $k$ of genus 0 and let $D \in \operatorname{Div}(X)$. If $\operatorname{deg}(D)=d \geq 1, H^{0}(\mathcal{O}(D))$ defines an embedding $X \subseteq \mathbb{P}^{d}$.

Proof. By the proposition, it is enough to do this after base-changing to $\bar{k}$. If $p, q \in X(\bar{k})$, then $h^{0}(D-p-q)=d-1=h^{0}(D)-2$. We get a morphism $f: X_{\bar{k}} \rightarrow \mathbb{P}^{d}$ and by the proposition it is an embedding.

## Proposition 8.62.

1. If $\operatorname{deg}(D) \geq 2 g$, then $\mathcal{O}(D)$ is generated by global sections
2. If $\operatorname{deg}(D) \geq 2 g+1, \mathcal{O}(D)$ defines an embedding.

Observation 8.63. Let $X$ be a curve and $D \in \operatorname{Div}(X)$. Let $f: X \rightarrow \mathbb{P}^{n}$ be the map induced by $H^{0}(\mathcal{O}(D))$, where $n=h^{0}(\mathcal{O}(D))-1$, and assume it is an embedding. So we can identify $\left(X, \mathcal{O}_{X}\right)$ as a closed subscheme of $\mathbb{P}^{n}$; we can relate the degree of the divisor to the degree of $\mathcal{O}_{X}$. The Hilbert polynomial of $\mathcal{O}_{X}$ is

$$
\chi\left(\mathcal{O}_{X}(t)\right)=1-g+d t
$$

where $d=\operatorname{deg}(X)$ in $\mathbb{P}_{k}^{n}$. On the other hand, the Riemann-Roch theorem gives us

$$
\chi\left(\mathcal{O}_{X}(t)\right)=\operatorname{deg}\left(\mathcal{O}_{X}(t)\right)+1-g=t \operatorname{deg}(D)+1-g
$$

Therefore $\operatorname{deg}(D)=\operatorname{deg}(X)$.

Example. Let $X$ be a curve of genus 1 over an algebraically closed field $k$.

- If $X$ has a divisor $D$ of degree 2 , we get a map $f: X \rightarrow \mathbb{P}^{1}$ of degree 2 (not necessarily an embedding).
- If $X$ has a divisor of degree $3, X$ is isomorphic to a cubic in $\mathbb{P}^{2}$
- If $X$ has a divisor of degree $4, X$ corresponds to a quartic in $\mathbb{P}^{3}$.

Curves of genus $\mathbf{0}$ Let $X$ be a curve of genus 0 . We can distinguish two cases:

- If $X(k) \neq \emptyset$, let $D=p \in X(k)$. Then $\operatorname{deg}(D)=1, X \subseteq \mathbb{P}^{1}$ and $X \simeq \mathbb{P}^{1}$.
- If $X(k)=\emptyset$, a canonical divisor $K$ has degree -2 ; therefore $\operatorname{deg}(-K)=2$ and $X \subseteq \mathbb{P}^{2}$ and $X$ is a conic by the previous observation.


## Summing up:

Theorem 8.64. Every curve of genus 0 over a field is either a conic in $\mathbb{P}_{k}^{2}$ or isomorphic to $\mathbb{P}_{k}^{1}$.

### 8.4.1 Complete intersection

Let $S_{1}, S_{2} \subseteq \mathbb{P}^{3}$ be hypersurfaces of degree $d_{1}, d_{2}$ respectively and assume that $S_{1}, S_{2}$ have no common components. We want to study the intersection $X=$ $S_{1} \cap S_{2}$, which has dimension $\operatorname{dim}(X)=1$. We have the exact sequence

$$
0 \rightarrow \mathcal{O}\left(-d_{1}-d_{2}\right) \longrightarrow \mathcal{O}\left(-d_{1}\right) \oplus \mathcal{O}\left(-d_{2}\right) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X} \rightarrow 0
$$

and we get

$$
\chi\left(\mathcal{O}_{X}(t)\right)=d_{1} d_{2} t-\frac{d_{1} d_{2}\left(d_{1}+d_{2}-4\right)}{2}
$$

We could have computed this in a different way. Assume $X$ is smooth: we get the sequence

$$
\left.0 \rightarrow I_{X} I_{X}^{2} \longrightarrow \Omega_{\mathbb{P}^{3} / k}\right|_{X} \longrightarrow \Omega_{X} \rightarrow 0
$$

Then $\omega_{X}=\mathcal{O}_{X}(-4) \otimes \operatorname{det}\left(I_{X} / I_{X}^{2}\right)^{\vee}$. If we consider the exact sequence

$$
0 \rightarrow \mathcal{O}\left(-d_{1}-d_{2}\right) \longrightarrow \mathcal{O}\left(-d_{1}\right) \oplus \mathcal{O}\left(-d_{2}\right) \longrightarrow I_{X} \rightarrow 0
$$

and apply $\otimes \mathcal{O}_{X}$ we get

$$
\mathcal{O}_{X}\left(-d_{1}-d_{2}\right) \xrightarrow{=0} \mathcal{O}_{X}\left(-d_{1}\right) \oplus \mathcal{O}_{X}\left(-d_{2}\right) \longrightarrow I_{X} / I_{X}^{2} \rightarrow 0
$$

Therefore $I_{X} / I_{X}^{2} \simeq \mathcal{O}_{X}\left(-d_{1}\right) \oplus \mathcal{O}_{X}\left(-d_{2}\right)$ and

$$
\operatorname{det}\left(I_{X} / I_{X}^{2}\right)^{\vee}=\mathcal{O}_{X}\left(d_{1}+d_{2}\right)
$$

We have shown that if $X$ is smooth $\omega_{X} \simeq \mathcal{O}_{X}\left(d_{1}+d_{2}-4\right)$. Notice that the degree as coherent sheaf on $\mathbb{P}^{n}$ of $X$ is exactly $\operatorname{deg}(X)=d_{1} d_{2}$. This follows from the fact that we can find an hyperplane not passing through the associated points
of $S_{1}, S_{2}$ and by Bezout's theorem the intersection with $S_{1} \cap S_{2}$ is exactly given by $d_{1} d_{2}$ points.

$$
\operatorname{deg}(X)=d_{1} d_{2} \quad 2 g-2=\operatorname{deg}\left(\omega_{X}\right)=d_{1} d_{2}\left(d_{1}+d_{2}-4\right)
$$

So a smooth intersection of two quadrics in $\mathbb{P}^{2}$ is a curve of genus 1 . In a certain sense, the converse holds:

Lemma 8.65. Let $X, Y$ be closed subschemes of $\mathbb{P}^{n}$ and suppose $X \subseteq Y$. If $\chi\left(\mathcal{O}_{X}(t)\right)=\chi\left(\mathcal{O}_{Y}(t)\right)$ then $X=Y$.

Proof. We consider the exact sequence

$$
0 \rightarrow I \longrightarrow \mathcal{O}_{Y} \longrightarrow \mathcal{O}_{X} \rightarrow 0
$$

and we get

$$
\chi(I(t))=\chi\left(\mathcal{O}_{Y}(t)\right)-\chi\left(\mathcal{O}_{X}(t)\right)=0
$$

and therefore $\operatorname{dim}(\operatorname{Supp}(I))=0$. This means that $I=0$.
Observation 8.66. Let $f \in k\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$ be a homogeneous polynomial of degree $d$ and let $S$ be the corresponding hypersurface. Given a closed subscheme $Y$ of $\mathbb{P}^{3}$, we have a map $H^{0}(\mathcal{O}(d)) \rightarrow H^{0}\left(\mathcal{O}_{Y}(d)\right)$ and $Y \subseteq S$ if and only if $f$ goes to 0 in $H^{0}\left(\mathcal{O}_{Y}(d)\right)$.

Theorem 8.67. Let $X \subseteq \mathbb{P}^{3}$ be a smooth curve of genus 1 and degree 4. Then $X=S_{1} \cap S_{2}$, where $S_{1}, S_{2}$ are quadrics.

Proof. Let $I_{X}$ be the sheaf of ideals associated to $X$.

$$
0 \rightarrow I_{X} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X} \rightarrow 0
$$

We tensor for $\mathcal{O}(2)$ and consider the exact sequence of cohomology

$$
0 \rightarrow H^{0}\left(I_{X}(2)\right) \longrightarrow H^{0}(\mathcal{O}(2)) \longrightarrow H^{0}\left(\mathcal{O}_{X}(2)\right) \longrightarrow \ldots
$$

We get

$$
h^{0}\left(I_{X}(2)\right) \geq \underbrace{h^{0}(\mathcal{O}(2))}_{=10}-h^{0}\left(\mathcal{O}_{X}(2)\right)
$$

We apply Riemann-Roch to estimate $h^{0}\left(\mathcal{O}_{X}(2)\right)$, noticing that $\operatorname{deg}(K-D)=$ $2 g-2-4<0$ :

$$
h^{0}\left(\mathcal{O}_{X}(2)\right)=\operatorname{deg}(2 D)+1-g=8
$$

Therefore $h^{0}\left(I_{X}(2)\right) \geq 2$ and there exist $f_{1}, f_{2} \in H^{0}\left(I_{X}(2)\right)$ linearly independent. Hence we get that the corresponding quadrics $S_{1}, S_{2}$ contain $X$. Notice that $f_{1}, f_{2}$ are irreducible since $X$ is not contained in a plane and a reducible quadric is the union of planes. Indeed, a smooth curve of degree 4 contained in $\mathbb{P}_{k}^{2}$ has genus 3 . Then

$$
\chi\left(\mathcal{O}_{S_{1} \cap S_{2}}(t)\right)=4 t=\chi\left(\mathcal{O}_{X}(t)\right)
$$

and so $X=S_{1} \cap S_{2}$.

### 8.5 Hyperelliptic Curves

Definition 8.68. Let $k$ be an algebraically closed field and let $X$ be a curve over $k$ of genus $g \geq 2$. We say that $X$ is hyperelliptic if there exists a map $f: X \rightarrow \mathbb{P}_{k}^{1}$ of degree 2 .

We saw that every curve of genus 2 is hyperelliptic but we haven't seen an example of an hyperelliptic curve. In fact, every smooth plane curve has genus

$$
g=\frac{(d-1)(d-2)}{2}=0,1,3,6,10 \ldots
$$

and therefore we have no examples in $\mathbb{P}^{2}$.
Let $X$ be a curve of genus $g \geq 2$ and let $p \in X(k)$. Then $h^{0}(p)=h^{0}(\mathcal{O}(p))=1$ and given a rational point $q \in X(k)$ we get $h^{0}(p+q)=1$ or $h^{0}(p+q)=2$.

Lemma 8.69. Let $X$ be a curve of genus $g \geq 1$ and let $E$ be a divisor of degree 1. Then $h^{0}(E) \leq 1$

Proof. Assume by contradiction that $h^{0}(E)>1$. Then the linear system $|E|$ contains more than one effective divisor and therefore there exists $p, q \in X(k)$ such that $p \sim q$, which is absurd by 8.12 .

Proposition 8.70. Let $X$ be a curve over $k=\bar{k}$ of genus $g \geq 2$. The following are equivalent:

1. $X$ is hyperelliptic
2. There exist $p, q \in X(k)$ such that $h^{0}(p+q)=2$
3. There exist $E \in \operatorname{Div}(X)$ of degree 2 such that $h^{0}(D)=2$

Proof. First, observe that

$$
h^{0}(p+q)=2 \Longleftrightarrow|p+q|=\{D \in \operatorname{Div}(X) \mid D \geq 0 D \sim p+q\} \neq\{p+q\}
$$

$(1) \Rightarrow(2)$ Assume that $X$ is hyperelliptic. Then there exists $f: X \rightarrow \mathbb{P}_{k}^{1}$ of degree 2. If $t \in \mathbb{P}_{k}^{1}(k), f^{*}(t)$ is effective of degree 2 and all the fibers are linearly equivalent. We get $f^{*}(0)=p+q$ and $h^{0}(p+q)>1$ and therefore $h^{0}(p+q)=$ 2.
$(2) \Rightarrow(1)$ We call $D=p+q$ and let $r \in X(k)$.By the lemma, $h^{0}(D-r)=1$ and therefore $\mathcal{O}(D)$ is generated by global sections. Then we have seen that $h^{0}(D)$ defines a map $X \rightarrow \mathbb{P}_{k}^{1}$ of degree 2.
$(2) \Rightarrow(3)$ Trivial.
$(3) \Rightarrow(2)$ We can prove this in the same way of $(2) \Rightarrow(1)$.

Corollary 8.71. If $E \in \operatorname{Div}(X)$ is a divisor of degree $\operatorname{deg}(E) \geq 1$ then $h^{0}(E) \leq$ $\operatorname{deg}(E)$.

We want now to apply Riemann-Roch theorem. Let $K$ be a canonical divisor and let $p \in X(k)$.

$$
h^{0}(p)-h^{0}(K-p)=1+1-g \Longrightarrow h^{0}(K-p)=g-1
$$

and therefore $H^{0}\left(\omega_{X}\right)$ is generated by global section and defines a map $f: X \rightarrow$ $\mathbb{P}_{k}^{g-1}$, called the canonical map.
It is natural to ask whether this map defines an embedding. We know that it is an embedding if and only if $h^{0}(K-p-q)=g-2$ for all $p, q \in X(k)$. By Riemann-Roch, we get

$$
h^{0}(p+q)-h^{0}(K-p-q)=3-g \Rightarrow h^{0}(K-p-q)=g-3+h^{0}(p+q)
$$

So $f$ is an embedding if and only if $h^{0}(p+q)=1$ for all $p, q \in X(k)$, which means that $X$ can't be hyperelliptic. However, if $X$ is hyperelliptic, we can factor this map through the Veronese embedding

$$
X \xrightarrow{\operatorname{deg} 2} \mathbb{P}_{k}^{1} \xrightarrow{\text { Veronese }} \mathbb{P}_{k}^{g-1}
$$

Genus 3 Let now $g=3$; we have seen that there are examples of curves having this genus, for example a quartic in $\mathbb{P}_{k}^{2}$. Let $X$ be a smooth curve. If $X$ is not hyperelliptic, $\omega_{X}$ defines an embedding $X \rightarrow \mathbb{P}^{2}$ of degree 4 . Conversely, if $X \subseteq \mathbb{P}^{2}$ is a smooth curve of degree 4 we know that $\omega_{X}=\mathcal{O}_{X}(1)$. We have the exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^{2}}(1) \longrightarrow \mathcal{O}_{X}(1) \rightarrow 0
$$

By the cohomology exact sequence, we get

$$
h^{0}(\mathcal{O}(-3))=0 \quad h^{0}(\mathcal{O}(1))=3 \quad h^{1}(\mathcal{O}(-3))=0
$$

and therefore $h^{0}\left(\mathcal{O}_{X}(1)\right)=3$. Notice that $\operatorname{deg}\left(\omega_{X}\right)=2 g-2=4$ and we get an embedding $X \subseteq \mathbb{P}^{2}$. As a consequence, $X$ is not hyperelliptic since we know that such plane curves are not hyperelliptic.

Genus 4 Let $X$ be a curve of genus 4 and assume that $X$ is not hyperelliptic. Then $X$ admit an embedding $X \rightarrow \mathbb{P}_{k}^{3}$ of degree 6. Let $S_{1}, S_{2} \subseteq \mathbb{P}^{3}$ be two hypersurfaces of degree $d_{1}, d_{2}$. We know that $X=S_{1} \cap S_{2}$ is a smooth curve.

$$
\omega_{X}=\mathcal{O}_{X}\left(d_{1}+d_{2}-4\right) \quad g(X)=\frac{d_{1} d_{2}\left(d_{1}+d_{2}-4\right)}{2}+1
$$

In the case $d_{1}=2$ and $d_{2}=3$, we get a curve of degree 6 of genus 4 which is not hyperelliptic.

Proposition 8.72. Any curve of genus 4 is either hyperelliptic or the intersection of a quadric and a cubic in $\mathbb{P}^{3}$.

Proof. Assume $X$ is not hyperelliptic: then $X \subseteq \mathbb{P}^{3}$ is a curve of degree 6 .

- First, we show that $X$ is contained in a quadric in $\mathbb{P}^{3}$. This happens if and only if $H^{0}(\mathcal{O}(2)) \rightarrow H^{0}\left(\mathcal{O}_{X}(2)\right)$ is not injective. Notice that $h^{0}(\mathcal{O}(2))=$ $\binom{3+2}{2}=10$ and by Riemann-Roch applied on $-K$ we get $h^{0}\left(\mathcal{O}_{X}(2)\right)=9$. Therefore $X$ is contained in a quadric $S_{1}$, which is irreducible since by hypotesis $X$ is not contained in a hyperplane.
- Now, we show that such a quadric is unique. Assume that there exists a different quadric $S_{2}$ such that $S_{2} \supseteq X$. Then $X \subseteq S_{1} \cap S_{2}$ and $\chi\left(\mathcal{O}_{S_{1} \cap S_{2}}(t)\right)=4 t$. We know that $\chi\left(\mathcal{O}_{X}(t)\right)=6 t-3$; since $X \subseteq S_{1} \cap S_{2}$ we get an exact sequence

$$
0 \rightarrow \operatorname{Ker}(\phi) \rightarrow \mathcal{O}_{S_{1} \cap S_{2}} \xrightarrow{\phi} \mathcal{O}_{X} \rightarrow 0
$$

Then $\chi\left(\mathcal{O}_{S_{1} \cap S_{2}}(t)\right)=\chi\left(\mathcal{O}_{X}(t)\right)+\chi(\operatorname{Ker}(\phi)(t))$ and since the leading coefficient of the last polynomial is positive we get a contradiction.

- We now want to find a cubic that contains $X$; so we need to find an element of $H^{0}(\mathcal{O}(3))$ that lies in the kernel of

$$
\psi: H^{0}(\mathcal{O}(3)) \longrightarrow H^{0}\left(\mathcal{O}_{X}(3)\right)
$$

We notice that $h^{0}(\mathcal{O}(3))=20$ and $h^{0}\left(\mathcal{O}_{X}(3)\right)=15$ by Riemann-Roch. Therefore $\operatorname{dim} \operatorname{Ker}(\psi) \geq 5$. Let $f_{1} \in k\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$ such that $S_{1}=V\left(f_{1}\right)$. The map

$$
\begin{array}{cccc}
\eta: \quad H^{0}(\mathcal{O}(1)) & \longrightarrow & H^{0}(\operatorname{Ker}(\psi)) \\
l & \longmapsto & f_{1} l
\end{array}
$$

is linear and injective and therefore $\operatorname{dim} H^{0}(\mathcal{O}(1))=4$. Since dim $\operatorname{Ker}(\psi)=$ 5 , there exists $f_{2} \in \operatorname{Ker}(\psi) \backslash \operatorname{Im}(\eta) . f_{1}, f_{2}$ are relatively prime and, if we call $S_{2}=V\left(f_{2}\right)$ we get $X \subseteq S_{1} \cap S_{2}$. Since they have the same Hilbert polynomial, we get the equality.

Let now $k$ be an algebraically closed field and assume $\operatorname{char}(k) \neq 2$. Let $f \in k[x]$ be a polynomial of degree $d$ with distinct roots.

$$
U=\operatorname{Spec}\left(k[x, y] /\left(y^{2}-f(x)\right)\right) \subseteq \mathbb{A}^{2}
$$

is a smooth integral curve of degree $d$ in $\mathbb{P}^{2}$ and we have a map $U \rightarrow \mathbb{A}^{1}$ finite of degree 2 . We consider the projective closure $\bar{U}$ and its normalization $X$. The inclusion $U \rightarrow \mathbb{A}^{1}$ extends to a morphism $f: X \rightarrow \mathbb{P}^{1}$ of degree 2. Furthermore, $f^{-1}\left(\mathbb{A}^{1}\right)=U$. Indeed, $U$ is proper and $\mathbb{A}^{1}$ is separated; therefore $U$ is both open and closed in $f^{-1}\left(\mathbb{A}^{1}\right)$.
$X$ is hyperelliptic; we would like to find the genus of $X$. This can be computed with the Riemann-Hurwitz formula. Let $X, Y$ be curves over $k=\bar{k}$ and let $f: X \rightarrow Y$ be a morphism of degree $d, d=[K(X): K(Y)]$.

Definition 8.73. $f$ is separable if $K(X) / K(Y)$ is separable.
Lemma 8.74. Let $X$ be an integral scheme and $L, M$ be invertible sheaves on $X$. If $\varphi: L \rightarrow M$ is a morphism then $\varphi=0$ or $\varphi$ is injective.
Proof. We can check this property locally, so assume that $X=\operatorname{Spec}(A)$ is affine and $L \simeq M \simeq \mathcal{O}$; in particular, $L=\tilde{N}$ and $M=\tilde{P}$. Consider the map $\varphi: L \rightarrow M$; these corresponds to a homomorphism of $A$-modules $f: N \rightarrow P$ and both of them are free of rank one. Therefore the map is either zero or injective, as desired.

Observation 8.75. This lemma implies that if a morphism of invertible sheaves is injective on a stalk, it is injective globally.
We now consider the sequence

$$
f^{*} \Omega_{Y / k} \longrightarrow \Omega_{X / k} \longrightarrow \Omega_{X / Y} \rightarrow 0
$$

Assume first that char $k=p$ and $f$ is not separable; notice that $\Omega_{X}, f^{*} \Omega_{Y}$ are invertible, so the first map is either injective or zero by the lemma. If $\xi \in X$ is the generic point,

$$
\left(f^{*} \Omega_{Y / k}\right)_{\xi} \longrightarrow\left(\Omega_{X / k}\right)_{\xi} \longrightarrow\left(\Omega_{X / Y}\right)_{\xi} \rightarrow 0
$$

and since $f$ is not separable, the first map is zero.
Example. We consider the morphism $f: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ given by the homomorphism

$$
\begin{array}{rll}
k[y] & \longrightarrow & k[x] \\
y & \longmapsto & x^{p}
\end{array}
$$

Then $\Omega_{X}=k[x] d x, \Omega_{Y}=k[y] d y$ and the map $f^{*} \Omega_{Y} \rightarrow \Omega_{X}$ corresponds to

$$
\begin{array}{clc}
k[x] d y & \longrightarrow & k[x] d x \\
d y & \longmapsto & d x^{p}=0
\end{array}
$$

Assume now that $f$ is separable. In this case, the first map is injective and we get the sequence

$$
0 \rightarrow f^{*} \Omega_{Y / k} \longrightarrow \Omega_{X / k} \longrightarrow \Omega_{X / Y} \rightarrow 0
$$

Notice that $\Omega_{X / Y}$ is a coherent sheaf supported on finitely many closed points. Remark 8.76. If $X$ is smooth of dimension $d, \Omega_{X}$ is locally free of rank $d$ and given $p \in X(k)$, we have a map

$$
\begin{array}{ccc}
m_{p} / m_{p}^{2} & \longrightarrow & \Omega_{X, p} \otimes k(p) \\
{[f]} & \longmapsto & {[d f]}
\end{array}
$$

If $f_{1}, \ldots, f_{d} \in m_{p}$ generate $m_{p} / m_{p}^{2}$ as a vector space, $\Omega_{X, p}$ is free on $d f_{1}, \ldots, d f_{d}$. In particular, if $\operatorname{dim}(X)=1$ and $t_{p}$ is a uniformizing parameter, $\Omega_{X, p}$ is free on $d t_{p}$.

We now want to use this remark in the case of curves. Let $p \in X(k)$ and $q=$ $f(p) \in Y(k)$. We can choose unformizing parameter $t_{p}, t_{q}$ of $m_{p}, m_{q}$ respectively. Then $f^{*}\left(t_{q}\right)=u t_{p}^{e_{p}(f)}$, where $u \in \mathcal{O}_{Y, p}^{*}$. In particular, we get a map

$$
\left(f^{*} \Omega_{Y}\right)_{p}=\mathcal{O}_{X, p} d t_{q} \longrightarrow \mathcal{O}_{X, p} d t_{p}
$$

and $f^{*}\left(d t_{q}\right)=\varphi d t_{p}$. We define the valuation $e_{p}^{\prime}(f)=v_{p}(\varphi)$. Notice that

$$
f^{*}\left(d t_{q}\right)=d\left(f^{*}\left(t_{q}\right)\right)=d\left(u t_{p}^{e_{p}(f)}\right)=t^{e_{p}(f)} d u+e_{p}(f) u t^{e_{p}(f)-1} d t_{p}
$$

Therefore, if $\operatorname{char}(k)=0$ or $\operatorname{char}(k) \nmid e_{p}(f)$ we get $e_{p}^{\prime}(f)=e_{p}(f)-1$. If $\operatorname{char}(k) \mid e_{p}(f)$, then $e_{p}^{\prime}(f)>e_{p}(f)-1$.
Corollary 8.77. $e_{p}^{\prime}(f)=0 \Longleftrightarrow e_{p}(f)=1$

Notice that $\left(\Omega_{X / Y}\right)_{p} \simeq \mathcal{O}_{X, p} / m_{p}^{e_{p}^{\prime}(f)}$ and the lenght $l_{\mathcal{O}_{X, p}}\left(\Omega_{X / Y, p}\right)=e_{p}^{\prime}(f)$. Since we know that $\Omega_{X / Y}$ has finite support,
Corollary 8.78. If $q \in Y(k), f^{-1}(q)$ has $d$ points with finitely many exceptions.
We know that degrees behave well under pullback, so

$$
\operatorname{deg}\left(\Omega_{Y}\right)=2 g(Y)-2 \Rightarrow \operatorname{deg}\left(f^{*} \Omega_{Y}\right)=d(2 g(Y)-2)
$$

Let $\varphi: L \rightarrow M$ be a non-constant morphism of invertible sheaves on $X$. We know that $\operatorname{Supp}(M / L)$ is finite and given $p \in X(k)$, we get an injective map $\varphi_{p}: L_{p} \rightarrow M_{p}$. Since these sheaves are invertible, this corresponds to a map $\psi: \mathcal{O}_{X, p} \rightarrow \mathcal{O}_{X, p}$. We define $\epsilon_{p}(\varphi)=v_{p}(\psi(1))$ and $\epsilon_{p}(\varphi)=0$ if and only if $\varphi_{p}: L_{p} \rightarrow M_{p}$ is an isomorphism.

$$
\operatorname{Coker}\left(\varphi_{p}\right)=(L / M)_{p} \simeq \mathcal{O}_{X, p} / m_{p}^{e_{p}(\varphi)}
$$

and so

$$
e_{p}(\varphi)=l_{\mathcal{O}_{X, p}}(L / M)_{p}=\operatorname{dim}_{k}(L / M)_{p}
$$

Lemma 8.79. $\operatorname{deg}(M)=\operatorname{deg}(L)+\sum_{p \in X} \epsilon_{p}(\varphi)$
Theorem 8.80 (Riemann-Hurwitz). Let $f: X \rightarrow Y$ be a non-constant separable morphism of curves. Then

$$
2 g(X)-2=d(g(Y)-2)+\sum_{p \in X} e_{p}^{\prime}(f)
$$

Furthermore, if char $k=0$ or char $k>d$ then

$$
2 g(X)-2=d(g(Y)-2)+\sum_{p \in X}\left(e_{p}(f)-1\right)
$$

We now want to see some application of this theorem.
Definition 8.81. $p \in X$ is a ramification point if $e_{p}(f)>1$.
$q \in Y$ is a ramification value if it is the image of a ramification point or equivalently $\left|f^{-1}(q)\right|<\operatorname{deg}(f)$.
If $f$ has no ramification values, we say that $f$ is unramified.
Let $f \in k[x]$ be a polynomial of degree $d$ with distinct roots and let

$$
U=\operatorname{Spec}\left(k[x, y] /\left(y^{2}-f(x)\right)\right) \longrightarrow \mathbb{A}^{1}
$$

We get a map $f: X \rightarrow \mathbb{P}^{1}$ of degree 2 ramified at the $d$ roots of $f(x)$, where $X$ is the normalization of the projective closure of $U$. We don't know if it is ramified at the point at infinity: the Riemann-Hurwitz formula gives us the answer. Indeed, let $Y=\mathbb{P}^{1}$; then $g(Y)=0$ and

$$
\sum_{p \in X}\left(e_{p}(f)-1\right)=d \text { or } d+1
$$

The formula gives us

$$
2 g(X)-2=-4+\left\{\begin{array}{l}
d \\
d+1
\end{array}\right.
$$

and therefore if we know $g(X)$ we can obtain the number of ramification point and viceversa if we know the number of ramification point we can get the genus of the curve. In particular,
Corollary 8.82. There exist hyperelliptic curves of every genus $g \geq 2$.
Corollary 8.83. If $f: X \rightarrow Y$ is separable then $g(X) \geq g(Y)$. Furthermore, if $g(X)=g(Y) \geq 2, f$ is an isomorphism.
Proof. If $d=1$, we know that $f$ is an isomorphism and therefore equality holds. Since $e_{p}^{\prime}(f) \geq 0$, we get

$$
2 g(X)-2=d(2 g(Y)-2)+\sum_{p \in X} e_{p}^{\prime}(f) \geq d(2 g(Y)-2)
$$

- If $g(Y)=0$, then the genus of a curve is always positive and the thesis is trivial.
- If $g(Y)=1$, then we get

$$
2 g(X)-2 \geq 0 \Longrightarrow g(X) \geq 1
$$

- If $g(Y) \geq 2$,

$$
2 g(X)-2 \geq d(2 g(Y)-2)>2 g(Y)-2 \Longrightarrow g(X)>g(Y)
$$

and the equality holds if and only if $d=1$.

Corollary 8.84. Let $f: X \rightarrow \mathbb{P}^{1}$ be a non-constant separable map of degree $d \geq 2$. Assume that $\operatorname{char}(k)=0$ or $\operatorname{char}(k)>d$. Then $f$ has at least 2 ramification values.
Proof. Since

$$
2 g(X)-2=-2 d+\sum_{p \in X} e_{p}^{\prime}(f)
$$

there is at least one ramification point. Since $f$ is separable,

$$
2 g(X)-2=-2 d+\sum_{p \in X}\left(e_{p}(f)-1\right)
$$

By contradiction, assume that $f$ has $q \in \mathbb{P}^{1}(k)$ as unique ramification value. Then

$$
\sum_{p \in X} e_{p}(f)-1=\sum_{p \in f^{-1}(q)} e_{p}(f)-1=d-\left|f^{-1}(q)\right|<d
$$

Hence

$$
2 g(X)-2<-2 d+d=-d \Longrightarrow d<2-2 g(X) \leq 2
$$

and this is absurd.

Example. We now give a counterexample to this corollary, omitting the characteristic hypotesis. Assume $\operatorname{char}(k)=p$ and $k=\bar{k}$. We consider the homomorphism of $k$-algebras

$$
\begin{array}{clc}
k[x] & \longrightarrow & k[t] \\
x & \longmapsto & t^{p}-t
\end{array}
$$

Then we get a map $f: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ of degree $[k(t): k(x)]=p$, since this the minimum polynomial of $t^{p}-t$ is $y^{p}-y-x$. Furthermore, for the derivative criterion, $f$ is unramified. This extends to a separable map $f: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ and $f(\infty)=\infty$. By definition, $\operatorname{deg}(f)=p$ and Riemann Hurwitz implies

$$
-2=-2 p+e_{\infty}^{\prime}(f) \Longrightarrow e_{\infty}^{\prime}(f)=2 p-2>p-1
$$

and there is only one ramification point. We now write $f$ around $\infty$. We get

$$
f(u)=\frac{u^{p}}{1-u^{p-1}}
$$

and the differential gives

$$
d f(u)=\frac{(p-1) u^{p+p-2} d u}{\left(1-u^{p-1}\right)^{2}}
$$

