Symbols for Matrix-Sequences

Advisor:
Stefano Serra-Capizzano
Università degli Studi dell'Insubria

Candidate:
Giovanni Barbarino
Scuola Normale Superiore

Supervisor:
Michele Benzi
Scuola Normale Superiore

Intuition and Definition

Motivation

$$\begin{cases} \mathscr{L} u = f \\ BC \end{cases}$$

Motivation

$$\begin{cases} \mathscr{L}u = f \\ BC \end{cases} \xrightarrow[\text{FE, FD}]{\text{IgA, Multigrid}} A_n u_n = f_n$$

$$A_n u_n = f_n \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} u_n$$

$$\uparrow$$

$$\Lambda(A_n)$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

Motivation

$$\begin{cases} \mathscr{L}u = f \\ BC \end{cases} \xrightarrow[\text{FE, FD}]{\text{IgA, Multigrid}} A_n u_n = f_n$$

$$A_n u_n = f_n \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} u_n$$

$$\uparrow$$

$$\Lambda(A_n)$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

Motivation

$$\begin{cases} \mathscr{L}u = f \\ BC \end{cases} \quad \xrightarrow[\text{FE, FD}]{\text{IgA, Multigrid}} \quad A_n u_n = f_n$$

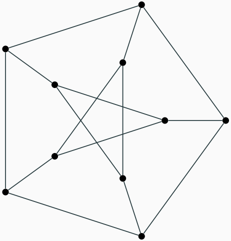
$$A_n u_n = f_n \quad \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} \quad u_n$$

$$\uparrow$$

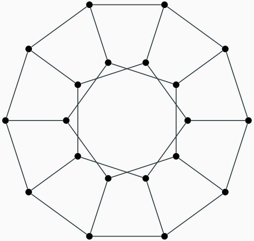
$$\Lambda(A_n)$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

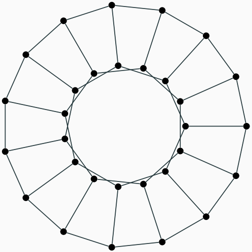
Petersen Graphs



GPG(5,2)



GPG(10,2)



GPG(15,2)

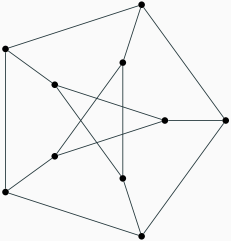
$\downarrow$ A_5

$\downarrow$ A_{10}

$\downarrow$ A_{15}

$$\{A_n\}_n \longrightarrow \{\Lambda(A_n)\}_n$$

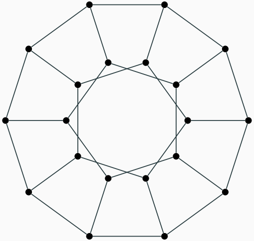
Petersen Graphs



GPG(5,2)

$\downarrow$

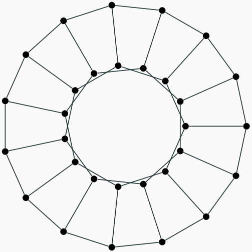
A_5



GPG(10,2)

$\downarrow$

A_{10}



GPG(15,2)

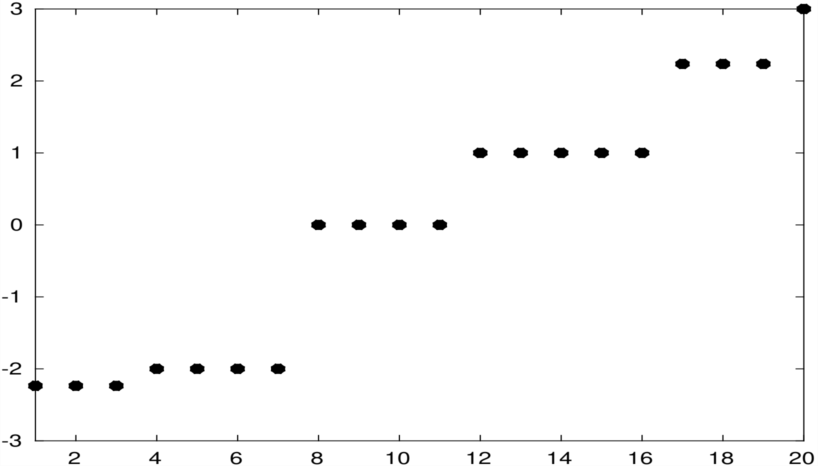
$\downarrow$

A_{15}

$$\{A_n\}_n \longrightarrow \{\Lambda(A_n)\}_n$$

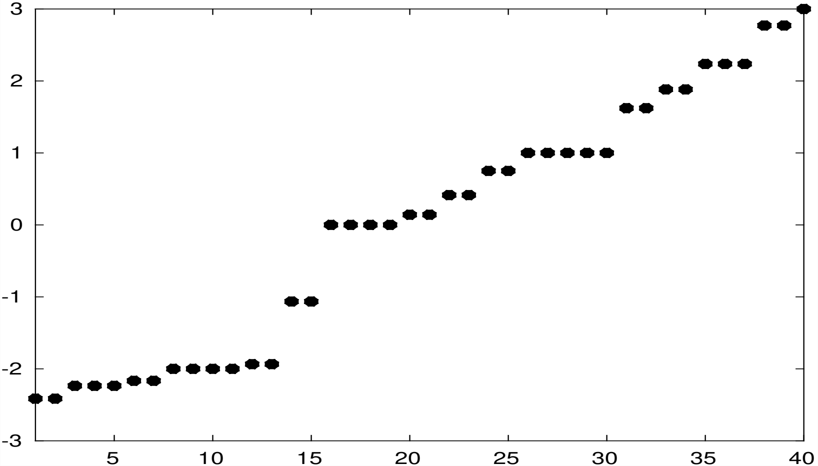
Petersen Graphs

$n = 10$



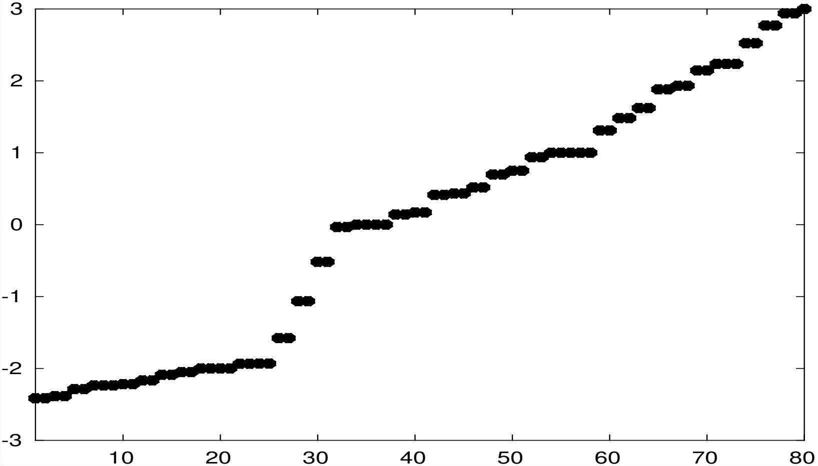
Petersen Graphs

$n = 20$



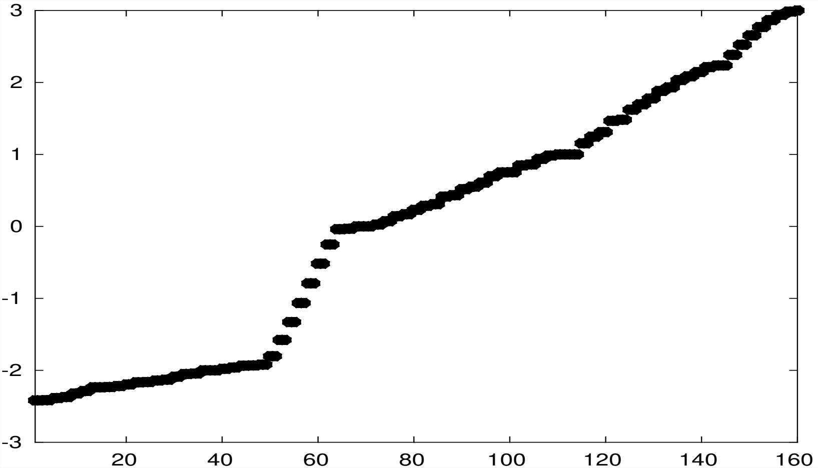
Petersen Graphs

$n = 40$



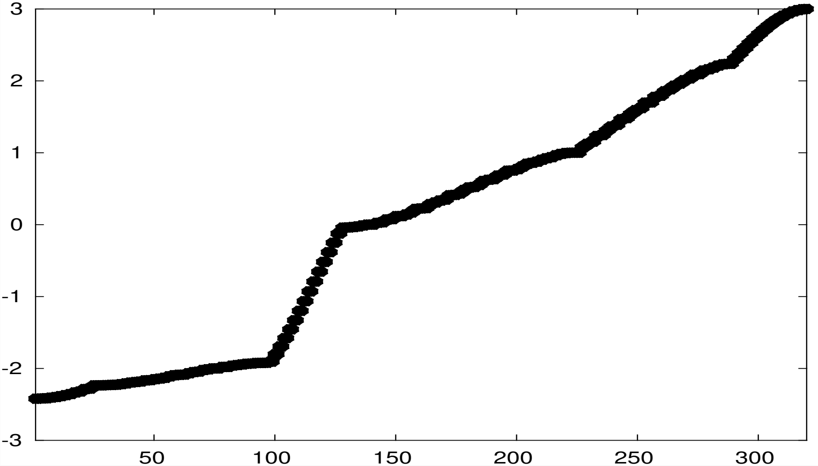
Petersen Graphs

$n = 80$



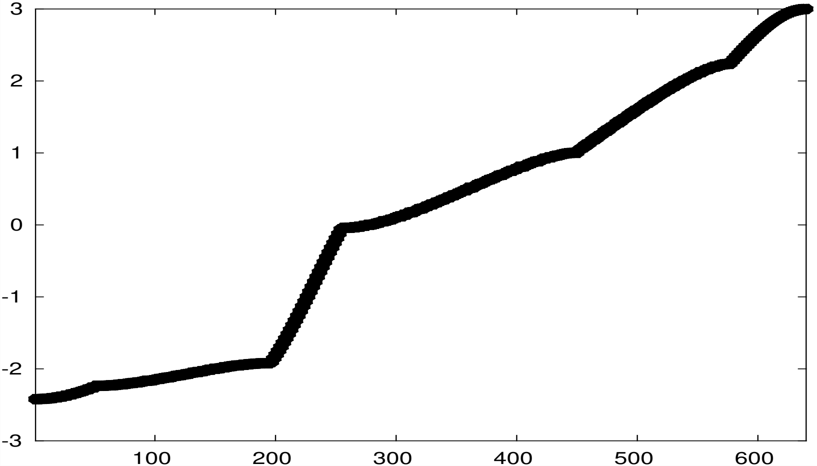
Petersen Graphs

$n = 160$



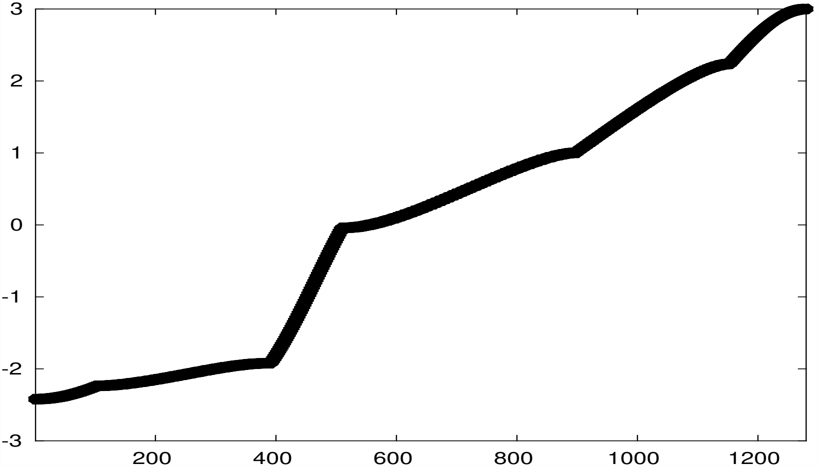
Petersen Graphs

$n = 320$



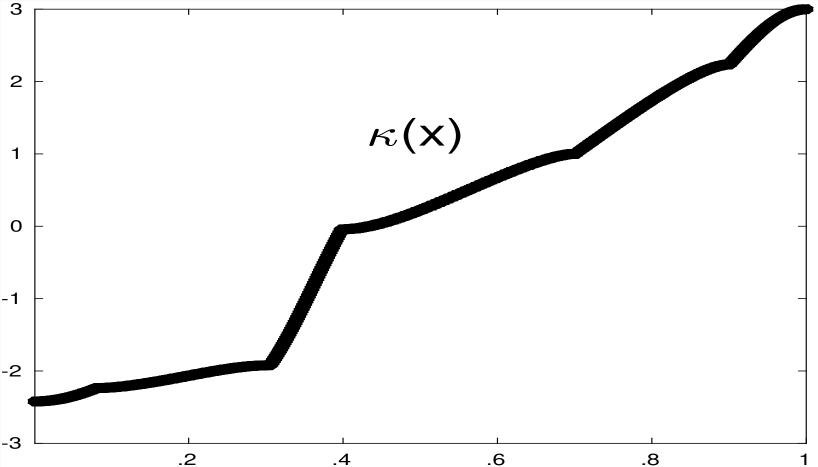
Petersen Graphs

$n = 640$



Petersen Graphs

$$\{A_n\}_n \sim \kappa(x) \qquad x \in [0,1]$$

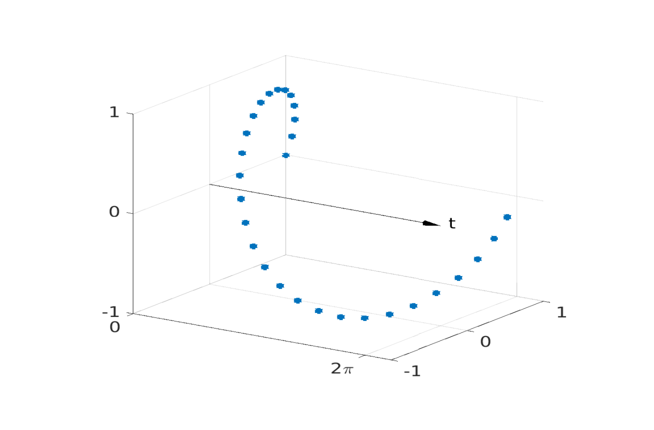


Circulant Sequence

$$C_n = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \qquad \longrightarrow$$

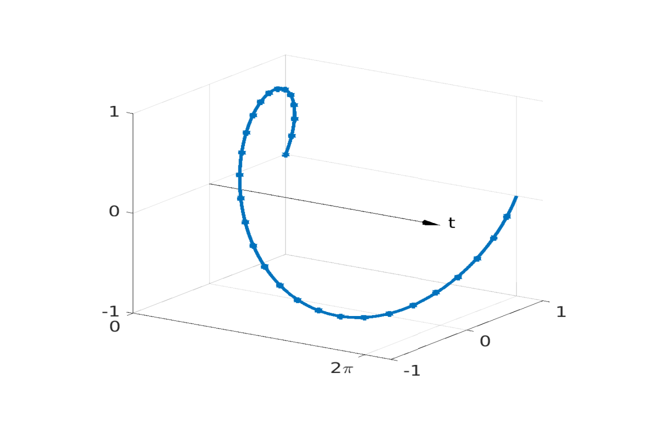
Circulant Sequence

$$C_n = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \qquad \longrightarrow \qquad \lambda_k(C_n) = exp\left(\frac{2\pi k i}{n}\right)$$



Circulant Sequence

$$C_n = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \qquad \longrightarrow \qquad \{C_n\}_n \sim e^{t\mathrm{i}} \quad t \in [0, 2\pi]$$



Spectral Measure

$$S_n = \begin{pmatrix} 1/n & & & \\ & 2/n & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad A_n = S_n \otimes C_n$$

$$\lambda_{a,b} = \frac{a}{n} e^{2b\pi \mathrm{i}/n} \qquad \longrightarrow \qquad \{A_n\}_n \sim x e^{\mathrm{i}\theta}$$

$$a = 1:n, \quad b = 1:n \qquad\qquad x \in [0,1], \quad \theta \in [0, 2\pi]$$

Spectral Measure

$$S_n = \begin{pmatrix} 1/n & & & \\ & 2/n & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad A_n = S_n \otimes C_n$$

$$\lambda_{a,b} = \tfrac{a}{n} e^{2b\pi \mathrm{i}/n} \qquad \longrightarrow \qquad \{A_n\}_n \sim x e^{\mathrm{i}\theta}$$

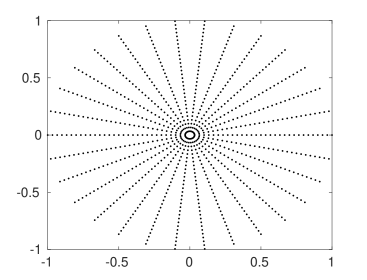
$$a = 1:n, \quad b = 1:n \qquad\qquad x \in [0,1], \quad \theta \in [0, 2\pi]$$

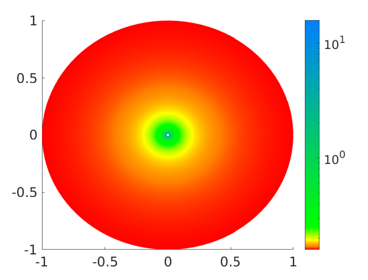
Spectral Measure

$$S_n = \begin{pmatrix} 1/n & & & \\ & 2/n & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad A_n = S_n \otimes C_n$$

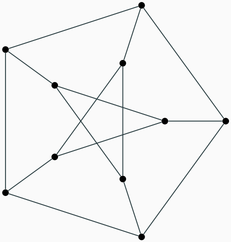
$$\lambda_{a,b} = \frac{a}{n} e^{2b\pi \mathrm{i}/n} \qquad \longrightarrow \qquad \{A_n\}_n \sim \mu$$

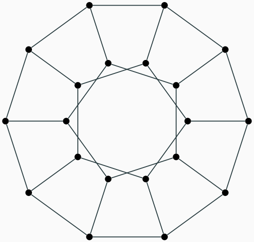
$$a = 1:n, \quad b = 1:n \qquad \qquad \mu(U) = \int_U \frac{1}{2\pi|z|} \mathrm{d}z$$

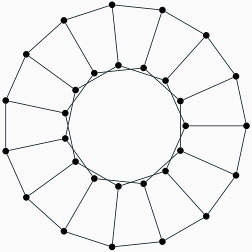




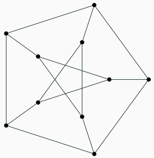
Petersen pt.2


GPG(5,2)


GPG(10,2)


GPG(15,2)

Petersen pt.2



$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

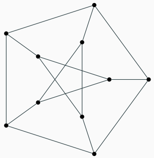
$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi k}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi k}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi k}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi k}{n}\right) \end{pmatrix}\right)$$

$$\lambda_{k,1}(A_n) = \cos(2\pi k/n) + \cos(4\pi k/n) + \sqrt{[\cos(2\pi k/n) - \cos(4\pi k/n)]^2 + 1}$$

$$\lambda_{k,2}(A_n) = \cos(2\pi k/n) + \cos(4\pi k/n) - \sqrt{[\cos(2\pi k/n) - \cos(4\pi k/n)]^2 + 1}$$

Petersen pt.2



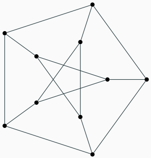
$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi k}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi k}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi k}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi k}{n}\right) \end{pmatrix}\right)$$

$$\begin{aligned} \lambda_{k,1}(A_n) &= \cos(2\pi k/n) + \cos(4\pi k/n) \\ &\quad + \sqrt{[\cos(2\pi k/n) - \cos(4\pi k/n)]^2 + 1} \\ \lambda_{k,2}(A_n) &= \cos(2\pi k/n) + \cos(4\pi k/n) \\ &\quad - \sqrt{[\cos(2\pi k/n) - \cos(4\pi k/n)]^2 + 1} \end{aligned}$$

Petersen pt.2



$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

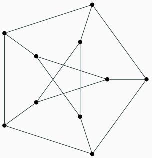
$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi k}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi k}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi k}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi k}{n}\right) \end{pmatrix}\right)$$

$$\lambda_1(\theta) = \cos(\theta) + \cos(2\theta) + \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$\lambda_2(\theta) = \cos(\theta) + \cos(2\theta) - \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

Petersen pt.2



$$\lambda_1(\theta) = \cos(\theta) + \cos(2\theta) + \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

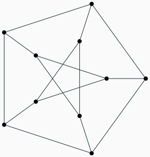
$$\lambda_2(\theta) = \cos(\theta) + \cos(2\theta) - \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(2\theta) \end{pmatrix} = \Upsilon(\theta)$$

$$\{A_n\}_n \sim \Upsilon(\theta) \qquad \theta \in [0, 2\pi]$$

A symbol is a **compact way to describe the overall spectral distribution of a matrix-sequence**

Petersen pt.2



$$\lambda_1(\theta) = \cos(\theta) + \cos(2\theta) + \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

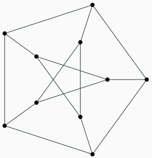
$$\lambda_2(\theta) = \cos(\theta) + \cos(2\theta) - \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(2\theta) \end{pmatrix} = \Upsilon(\theta)$$

$$\{A_n\}_n \sim \Upsilon(\theta) \qquad \theta \in [0, 2\pi]$$

A symbol is a **compact way to describe the overall spectral distribution of a matrix-sequence**

Petersen pt.2



$$\lambda_1(\theta) = \cos(\theta) + \cos(2\theta) + \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

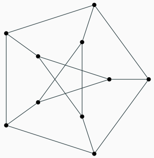
$$\lambda_2(\theta) = \cos(\theta) + \cos(2\theta) - \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(2\theta) \end{pmatrix} = \Upsilon(\theta)$$

$$\{A_n\}_n \sim \Upsilon(\theta) \qquad \theta \in [0, 2\pi]$$

A symbol is a compact way to describe the overall spectral distribution of a matrix-sequence

Petersen pt.2



$$\lambda_1(\theta) = \cos(\theta) + \cos(2\theta) + \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$\lambda_2(\theta) = \cos(\theta) + \cos(2\theta) - \sqrt{[\cos(\theta) - \cos(2\theta)]^2 + 1}$$

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(2\theta) \end{pmatrix} = \Upsilon(\theta)$$

$$\{A_n\}_n \sim \Upsilon(\theta) \qquad \theta \in [0, 2\pi]$$

A symbol is a **compact way to describe the overall spectral distribution of a matrix-sequence**

Definition

A functional $\phi : C_c(\mathbb{C}) \to \mathbb{R}$ is a **spectral symbol** for $\{A_n\}_n$ if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \phi(G) \quad \forall\, G \in C_c(\mathbb{C})$$

- A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathrm{x}))\mathrm{d}\mathrm{x} \quad \forall\, G \in C_c(\mathbb{C})$$

- A measurable function $\Upsilon : D \to \mathbb{C}^{s\times s}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D \frac{\sum_{i=1}^{s} G(\lambda_i(\Upsilon(\mathrm{x})))}{s}\mathrm{d}\mathrm{x} \quad \forall\, G \in C_c(\mathbb{C})$$

- A positive measure μ of mass $|\mu| \le 1$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \int_{\mathbb{C}} G\,\mathrm{d}\mu \quad \forall\, G \in C_c(\mathbb{C}).$$

Definition

A functional $\phi : C_c(\mathbb{C}) \to \mathbb{R}$ is a **spectral symbol** for $\{A_n\}_n$ if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \phi(G) \quad \forall\, G \in C_c(\mathbb{C})$$

- A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

- A measurable function $\Upsilon : D \to \mathbb{C}^{s\times s}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D \frac{\sum_{i=1}^{s} G(\lambda_i(\Upsilon(\mathbf{x})))}{s}\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

- A positive measure μ of mass $|\mu| \le 1$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \int_{\mathbb{C}} G\mathrm{d}\mu \quad \forall\, G \in C_c(\mathbb{C}).$$

Definition

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

A sequence $\{A_n\}_n$ usually have **infinitely many symbols**

Definition

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

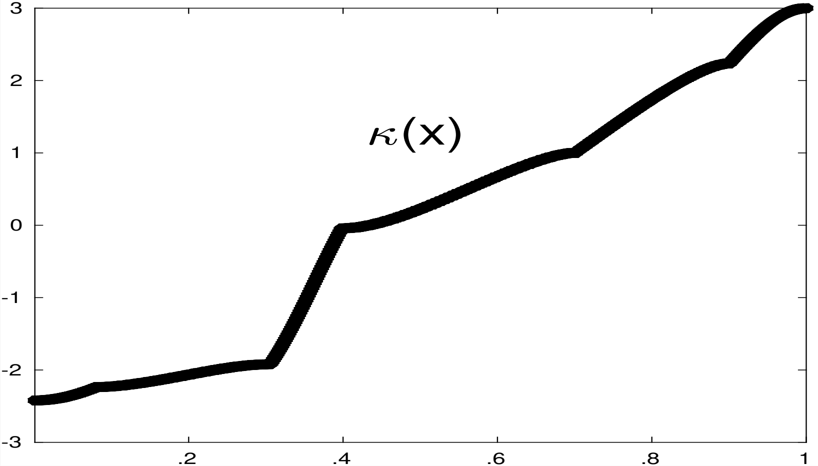
$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

A sequence $\{A_n\}_n$ usually have **infinitely many symbols**

Definition

$$\{A_n\}_n \sim \kappa(x) \qquad x \in [0,1]$$



Definition

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

A sequence $\{A_n\}_n$ usually have **infinitely many symbols**

Definition

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

A sequence $\{A_n\}_n$ usually have **infinitely many symbols**

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right) \qquad \kappa(t) = 2 - 2\cos(t)$$

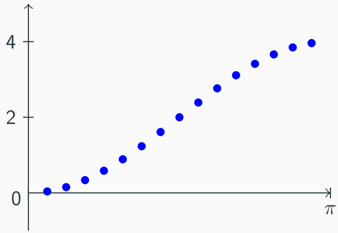
$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \leq n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases} \qquad \widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

$\rightarrow$ **The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$**

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$



$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right)$$

$$\kappa(t) = 2 - 2\cos(t)$$

$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases}$$

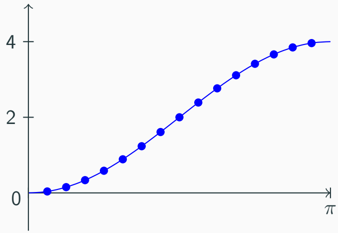
$$\widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

$\rightarrow$ **The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$**

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$



$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right)$$

$$\kappa(t) = 2 - 2\cos(t)$$

$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases}$$

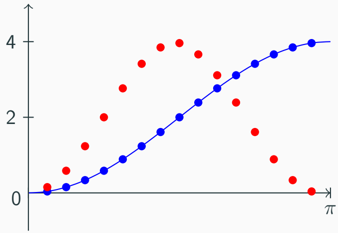
$$\widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

$\to$ **The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$**

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$



$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right) \qquad \kappa(t) = 2 - 2\cos(t)$$

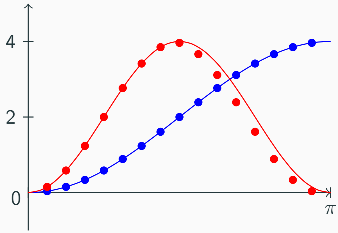
$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases} \qquad \widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

$\rightarrow$ **The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$**

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$



$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right) \qquad \kappa(t) = 2 - 2\cos(t)$$

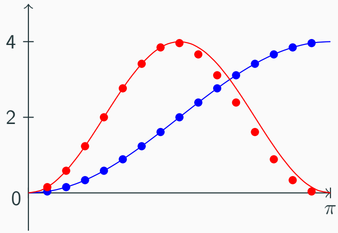
$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases} \qquad \widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

→ The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$

Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

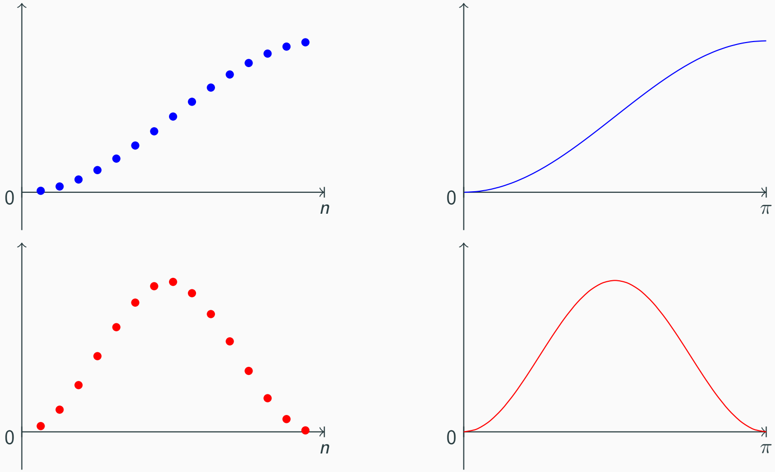
$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$



$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right) \qquad \kappa(t) = 2 - 2\cos(t)$$

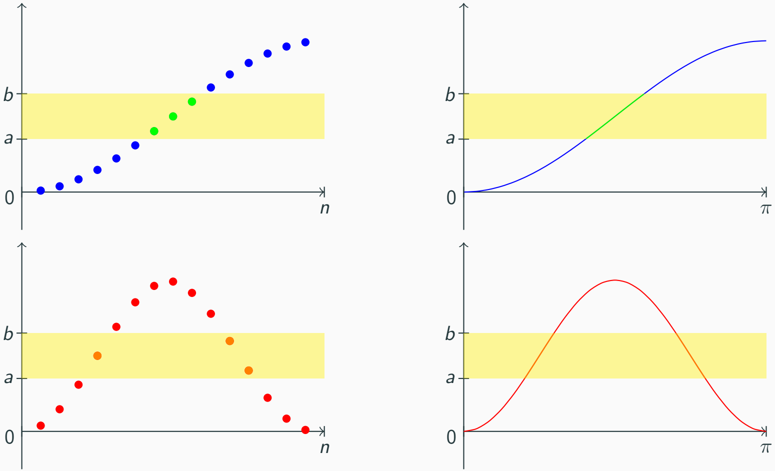
$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \leq n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases} \qquad \widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

$\rightarrow$ **The sequence $\{A_n\}_n$ has Spectral Symbols $\kappa(t), \widetilde{\kappa}(t), \ldots$**



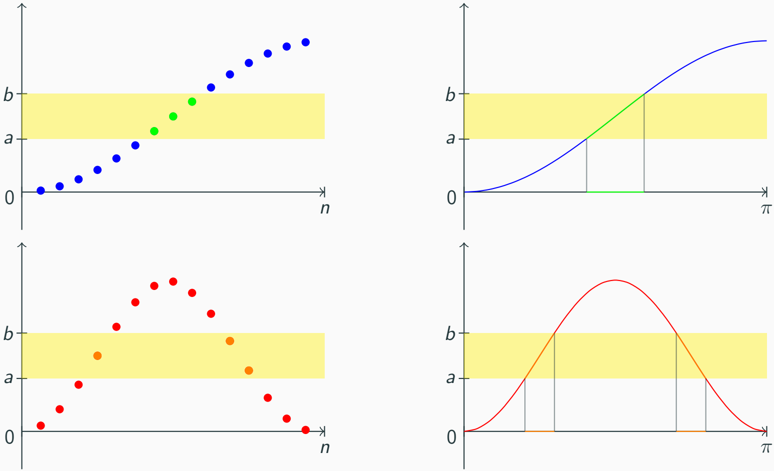
$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < \kappa(t) < b\}}{\pi}$$

$$\{\boldsymbol{A_n}\}_{\boldsymbol{n}} \sim \boldsymbol{\kappa} \iff \textbf{it holds } \widetilde{\forall}[\boldsymbol{a}, \boldsymbol{b}]$$



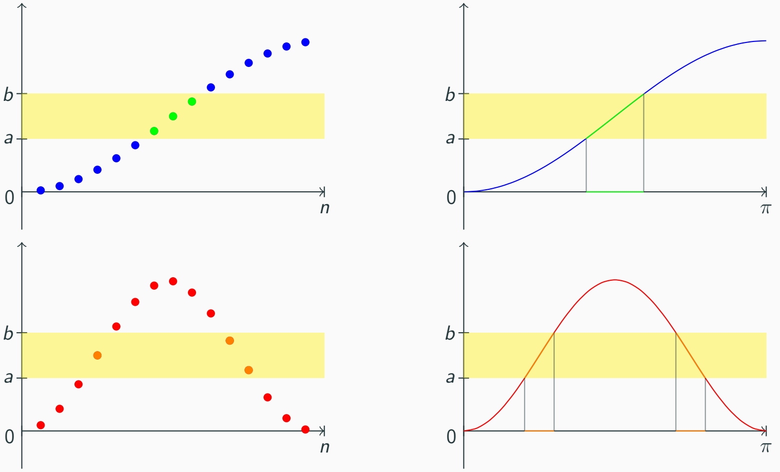
$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < \kappa(t) < b\}}{\pi}$$

$\{A_n\}_n \sim \kappa \iff$ **it holds** $\widetilde{\forall}[a, b]$



$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < \kappa(t) < b\}}{\pi}$$

$$\{A_n\}_n \sim \kappa \iff \text{it holds } \widetilde{\forall}[a,b]$$



$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < \kappa(t) < b\}}{\pi}$$

$\{A_n\}_n \sim \kappa \iff$ it holds $\widetilde{\forall}[a, b]$

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

Theorem [B. '19]
Given a measurable function $\kappa : D \to \mathbb{C}$, $\{A_n\}_n \sim \kappa$ if and only if

$$E_{z_0} := \left\{ r \in \mathbb{R}^+ \middle| \frac{\#\{i : \lambda_i(A_n) \in B(z_0, r)\}}{s_n} \not\to \frac{\mu\{\mathbf{x} : \kappa(\mathbf{x}) \in B(z_0, r)\}}{\mu(D)} \right\}$$

has Lebesgue measure zero for every $z_0 \in \mathbb{C}$.

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

Theorem [B. '19]
Given a measurable function $\kappa : [0,1] \to \mathbb{C}$, then $\{A_n\}_n \sim \kappa$ if and only if the sequence $\{\kappa_n(x)\}_n$ of piecewise linear function interpolating $\{\Lambda(A_n)\}_n$ **in some order** over $[0,1]$ converges in measure to $\kappa(x)$.

Theorem [B. '19]
Given a measurable function $\kappa : D \to \mathbb{C}$, $\{A_n\}_n \sim \kappa$ if and only if

$$E_{z_0} := \left\{ r \in \mathbb{R}^+ \middle| \frac{\#\{i : \lambda_i(A_n) \in B(z_0, r)\}}{s_n} \not\to \frac{\mu\{\boldsymbol{x} : \kappa(\boldsymbol{x}) \in B(z_0, r)\}}{\mu(D)} \right\}$$

has Lebesgue measure zero for every $z_0 \in \mathbb{C}$.

Singular Value Symbol

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

$$\{A_n\}_n \sim_\lambda \kappa$$

A measurable function $\kappa : D \to \mathbb{C}$ is a **singular value symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(|\kappa(\mathbf{x})|)\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{R})$$

$$\{A_n\}_n \sim_\sigma \kappa$$

Singular Value Symbol

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

$$\{A_n\}_n \sim_\lambda \kappa$$

A measurable function $\kappa : D \to \mathbb{C}$ is a **singular value symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(|\kappa(\mathbf{x})|)\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{R})$$

$$\{A_n\}_n \sim_\sigma \kappa$$

Singular Value Symbol

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathsf{x}))\mathrm{d}\mathsf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

$$\{A_n\}_n \sim_\lambda \kappa$$

A measurable function $\kappa : D \to \mathbb{C}$ is a **singular value symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(|\kappa(\mathsf{x})|)\mathrm{d}\mathsf{x} \quad \forall\, G \in C_c(\mathbb{R})$$

$$\{A_n\}_n \sim_\sigma \kappa$$

Singular Value Symbol

A measurable function $\kappa : D \to \mathbb{C}$ is a **spectral symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{C})$$

$$\{A_n\}_n \sim_\lambda \kappa$$

A measurable function $\kappa : D \to \mathbb{C}$ is a **singular value symbol** if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(|\kappa(\mathbf{x})|)\mathrm{d}\mathbf{x} \quad \forall\, G \in C_c(\mathbb{R})$$

$$\{A_n\}_n \sim_\sigma \kappa$$

GLT World

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathscr{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$Z_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$a \in C[0,1] \quad (R.I.)$$

$$D_n(a) := \begin{pmatrix} a(1/n) & & & & \\ & a(2/n) & & & \\ & & a(3/n) & & \\ & & & \ddots & \\ & & & & a(1) \end{pmatrix}$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$f \in L^1[-\pi, \pi] \to \widehat{f}_n = \int_{-\pi}^{\pi} f(\theta) e^{-\mathrm{i}n\theta} d\theta$$

$$T_n(f) := \begin{pmatrix} \widehat{f}_0 & \widehat{f}_1 & \widehat{f}_2 & \dots & \widehat{f}_{n-1} \\ \widehat{f}_{-1} & \widehat{f}_0 & \ddots & \ddots & \vdots \\ \widehat{f}_{-2} & \ddots & \ddots & \ddots & \widehat{f}_2 \\ \vdots & \ddots & \ddots & \widehat{f}_0 & \widehat{f}_1 \\ \widehat{f}_{-n+1} & \dots & \widehat{f}_{-2} & \widehat{f}_{-1} & \widehat{f}_0 \end{pmatrix}$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \widetilde{\mathcal{G}} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$

$$\implies \{A_n\}_n \sim_\sigma a(x)(2 - 2\cos(\theta)) + 0$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \widetilde{\mathcal{G}} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$
$$\implies \{A_n\}_n \sim_\sigma a(x)(2 - 2\cos(\theta)) + 0$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \widetilde{\mathcal{G}} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$

$$\implies \{A_n\}_n \sim_\sigma a(x)(2 - 2\cos(\theta)) + 0$$

Special Choice

Special Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \widetilde{\mathcal{G}} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\begin{cases} (a(x)u'(x))' = f(x) \quad x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$
$$\implies \{A_n\}_n \sim_\sigma a(x)(2 - 2\cos(\theta)) + 0$$

a.c.s. Convergence

- A sequence is **Zero Distributed** if $\{Z_n\}_n \sim_\sigma 0$, that is

$$Z_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

- $\{\{B_{n,m}\}_n\}_m$ is an **Approximating Class of Sequence** for $\{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

$$\frac{\mathrm{rk}\, R_{n,m}}{n} \le c(m) \to 0 \qquad \|N_{n,m}\| \le \omega(m) \to 0 \qquad \forall n > n_m$$

Theorem [S-C '01, Garoni '17]

- $d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$

- $\{\{B_{n,m}\}_n\}_m \sim_\sigma k_m, \quad k_m \to k, \quad \{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$

$$\Longrightarrow \{A_n\}_n \sim_\sigma k$$

a.c.s. Convergence

- A sequence is **Zero Distributed** if $\{Z_n\}_n \sim_\sigma 0$, that is

$$Z_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

- $\{\{B_{n,m}\}_n\}_m$ is an **Approximating Class of Sequence** for $\{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

$$\frac{\mathrm{rk}\, R_{n,m}}{n} \leq c(m) \to 0 \qquad \|N_{n,m}\| \leq \omega(m) \to 0 \qquad \forall n > n_m$$

Theorem [S-C '01, Garoni '17]

- $d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$

- $\{\{B_{n,m}\}_n\}_m \sim_\sigma \kappa_m, \quad \kappa_m \to \kappa, \quad \{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$

$$\implies \{A_n\}_n \sim_\sigma \kappa$$

a.c.s. Convergence

- A sequence is **Zero Distributed** if $\{Z_n\}_n \sim_\sigma 0$, that is

$$Z_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

- $\{\{B_{n,m}\}_n\}_m$ is an **Approximating Class of Sequence** for $\{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

$$\frac{\mathrm{rk}\, R_{n,m}}{n} \le c(m) \to 0 \qquad \|N_{n,m}\| \le \omega(m) \to 0 \qquad \forall n > n_m$$

Theorem [S-C '01, Garoni '17]

- $d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$

- $\{\{B_{n,m}\}_n\}_m \sim_\sigma \kappa_m, \quad \kappa_m \to \kappa, \quad \{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$

$$\implies \{A_n\}_n \sim_\sigma \kappa$$

a.c.s. Convergence

- A sequence is **Zero Distributed** if $\{Z_n\}_n \sim_\sigma 0$, that is

$$Z_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

- $\{\{B_{n,m}\}_n\}_m$ is an **Approximating Class of Sequence** for $\{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

$$\frac{\mathrm{rk}\, R_{n,m}}{n} \le c(m) \to 0 \qquad \|N_{n,m}\| \le \omega(m) \to 0 \qquad \forall n > n_m$$

Theorem [S-C '01, Garoni '17]

- $d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_i \left\{\frac{i-1}{n} + \sigma_i(A_n - B_n)\right\}$

- $\{\{B_{n,m}\}_n\}_m \sim_\sigma \kappa_m, \quad \kappa_m \to \kappa, \quad \{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$

$$\Longrightarrow \{A_n\}_n \sim_\sigma \kappa$$

Generalized Locally Toeplitz Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \widetilde{\mathcal{G}} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\{A_n\}_n \sim_{GLT} \kappa(x,\theta)$$

Theorem [B. '17]
The space of GLT sequences is **Isomorphic** and **Isometric** to the space of measurable functions on $[0,1] \times [-\pi,\pi]$.

Generalized Locally Toeplitz Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]} \qquad \text{(GLT)}$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \mathcal{G} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\{A_n\}_n \sim_{GLT} \kappa(x,\theta)$$

Theorem [B. '17]
The space of GLT sequences is **Isomorphic** and **Isometric** to the space of measurable functions on $[0,1] \times [-\pi,\pi]$.

Generalized Locally Toeplitz Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]} \qquad \text{(GLT)}$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \mathcal{G} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\{A_n\}_n \sim_{GLT} \kappa(x,\theta)$$

Theorem [B. '17]
The space of GLT sequences is **Isomorphic** and **Isometric** to the space of measurable functions on $[0,1] \times [-\pi,\pi]$.

Generalized Locally Toeplitz Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathscr{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathscr{Z}, \mathcal{D}, \mathcal{T}]} \qquad \text{(GLT)}$$

Theorem [S-C '03]

$$(\{A_n\}_n, \kappa(x,\theta)) \in \mathcal{G} \implies \{A_n\}_n \sim_\sigma \kappa(x,\theta)$$

$$\{A_n\}_n \sim_{GLT} \kappa(x,\theta)$$

Theorem [B. '17]
The space of GLT sequences is **Isomorphic** and **Isometric** to the space of measurable functions on $[0,1] \times [-\pi, \pi]$.

Connection with Spectral Symbols

Theorem [B. '18]
$\exists\{U_n\}_n$ unitary sequence such that for any $(\{A_n\}_n, \kappa) \in \mathcal{G}$

$$\{A_n\}_n = \{U_n D_n U_n^H\}_n + \{Z_n\}_n \qquad \{Z_n\}_n \sim_{GLT} 0 \qquad \{D_n\}_n \rightharpoonup \kappa$$

If $\{A_n\}_n \sim_{GLT} \kappa$, then $\{A_n\}_n$ is **close to a normal sequence that has κ as spectral symbol**

Theorem [B. '19]
If X_n are Hermitian matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(\sqrt{n}) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

Theorem [B. '19]
If X_n are normal matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(1) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

Connection with Spectral Symbols

Theorem [B. '18]
$\exists \{U_n\}_n$ unitary sequence such that for any $(\{A_n\}_n, \kappa) \in \mathcal{G}$

$$\{A_n\}_n = \{U_n D_n U_n^H\}_n + \{Z_n\}_n \qquad \{Z_n\}_n \sim_{GLT} 0 \qquad \{D_n\}_n \rightharpoonup \kappa$$

If $\{A_n\}_n \sim_{GLT} \kappa$, then $\{A_n\}_n$ is **close to a normal sequence that has κ as spectral symbol**

Theorem [B. '19]
If X_n are Hermitian matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(\sqrt{n}) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

Theorem [B. '19]
If X_n are normal matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(1) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

Connection with Spectral Symbols

Theorem [B. '18]
$\exists \{U_n\}_n$ unitary sequence such that for any $(\{A_n\}_n, \kappa) \in \mathcal{G}$

$$\{A_n\}_n = \{U_n D_n U_n^H\}_n + \{Z_n\}_n \qquad \{Z_n\}_n \sim_{GLT} 0 \qquad \{D_n\}_n \rightharpoonup \kappa$$

If $\{A_n\}_n \sim_{GLT} \kappa$, then $\{A_n\}_n$ is **close to a normal sequence that has κ as spectral symbol**

Theorem [B. '19]
If X_n are Hermitian matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(\sqrt{n}) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

Theorem [B. '19]
If X_n are normal matrices,

$$\{X_n\}_n \sim_{GLT} \kappa \quad \|Y_n\|_2 = o(1) \implies \{X_n + Y_n\}_n \sim_\lambda \kappa$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$B_n = \frac{1}{2n} \begin{pmatrix} 0 & b_1 & & & \\ -b_2 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & 0 & b_{n-1} \\ & & & -b_n & 0 \end{pmatrix}$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$B_n = \frac{1}{2n} \begin{pmatrix} 0 & b_1 & & & \\ -b_2 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & 0 & b_{n-1} \\ & & & -b_n & 0 \end{pmatrix}$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$B_n = \frac{1}{2n} \begin{pmatrix} 0 & b_1 & & & \\ -b_2 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & 0 & b_{n-1} \\ & & & -b_n & 0 \end{pmatrix}$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$D_n(a)T_n(f) = \begin{pmatrix} 2a(1/n) & -a(1/n) & & & \\ -a(2/n) & 2a(2/n) & -a(2/n) & & \\ & -a(3/n) & 2a(3/n) & -a(3/n) & \\ & & \ddots & \ddots & \ddots \\ & & & -a(1) & 2a(1) \end{pmatrix}$$

$$B_n = \frac{1}{2n} \begin{pmatrix} 0 & b_1 & & & \\ -b_2 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & 0 & b_{n-1} \\ & & & -b_n & 0 \end{pmatrix}$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$D_n(a)T_n(f) = \begin{pmatrix} 2a(1/n) & -a(1/n) & & & \\ -a(2/n) & 2a(2/n) & -a(2/n) & & \\ & -a(3/n) & 2a(3/n) & -a(3/n) & \\ & & \ddots & \ddots & \ddots \\ & & & -a(1) & 2a(1) \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$A_n + B_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta)) + 0$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$D_n(a)T_n(f) = \begin{pmatrix} 2a(1/n) & -a(1/n) & & & \\ -a(2/n) & 2a(2/n) & -a(2/n) & & \\ & -a(3/n) & 2a(3/n) & -a(3/n) & \\ & & \ddots & \ddots & \ddots \\ & & & -a(1) & 2a(1) \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$A_n + B_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta)) + 0$$

How to compute a Symbol?

Step 1: Compute the GLT symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} \qquad A_n u_n = f_n$$

$$D_n(a)T_n(f) = \begin{pmatrix} 2a(1/n) & -a(1/n) & & & \\ -a(2/n) & 2a(2/n) & -a(2/n) & & \\ & -a(3/n) & 2a(3/n) & -a(3/n) & \\ & & \ddots & \ddots & \ddots \\ & & & -a(1) & 2a(1) \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$A_n + B_n = D_n(a(x))T_n(2 - 2\cos(\theta)) + Z_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta)) + 0$$

How to compute a Symbol?

Step 2: Compute the Spectral Symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} \qquad A_n u_n = f_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$\{A_n\}_n = \{A_n + B_n\}_n - \{B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$\{A_n + B_n\}_n \sim_{\lambda} a(x)(2 - 2\cos(\theta))$$

How to compute a Symbol?

Step 2: Compute the Spectral Symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} \quad A_n u_n = f_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$\{A_n\}_n = \{A_n + B_n\}_n - \{B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$\{A_n + B_n\}_n \sim_{\lambda} a(x)(2 - 2\cos(\theta))$$

How to compute a Symbol?

Step 2: Compute the Spectral Symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$\{A_n\}_n = \{A_n + B_n\}_n - \{B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$\{A_n + B_n\}_n \sim_{\lambda} a(x)(2 - 2\cos(\theta))$$

How to compute a Symbol?

Step 2: Compute the Spectral Symbol

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$\{A_n + B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix}$$

$$\|B_n\|_2 = o(1)$$

$$\{A_n\}_n = \{A_n + B_n\}_n - \{B_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

$$\{A_n + B_n\}_n \sim_{\lambda} a(x)(2 - 2\cos(\theta))$$

Applications

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD/FE} a(x)(2-2\cos(\theta)) \qquad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

$$\xrightarrow{IgA\ Coll./Gal.(p)} a(x)f_p(\theta) \qquad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $a(x)u^{(4)}(x) + b(x)u^{(2)}(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD(1,-4,6,-4,1)} a(x)(6-8\cos(x)+2\cos(2x)) \quad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{Prec.\ FD} \{T_n(2-2\cos(\theta))^{-1}A_n\}_n \sim \frac{a(x)(2-2\cos(\theta))}{2-2\cos(\theta)} = a(x)$$

Applications

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD/FE} a(x)(2 - 2\cos(\theta)) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

$$\xrightarrow{IgA\ Coll./Gal.(p)} a(x)f_p(\theta) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

Applications

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD/FE} a(x)(2-2\cos(\theta)) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

$$\xrightarrow{IgA\ Coll./Gal.(p)} a(x)f_p(\theta) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

- $a(x)u^{(4)}(x) + b(x)u^{(2)}(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD(1,-4,6,-4,1)} a(x)(6-8\cos(x)+2\cos(2x)) \quad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{Prec\ FD} \{T_n(2-2\cos(\theta))^{-1}A_n\}_n \sim \frac{a(x)(2-2\cos(\theta))}{2-2\cos(\theta)} = a(x)$$

Applications

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD/FE} a(x)(2-2\cos(\theta)) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

$$\xrightarrow{IgA\ Coll./Gal.(p)} a(x)f_p(\theta) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$$

- $a(x)u^{(4)}(x) + b(x)u^{(2)}(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD(1,-4,6,-4,1)} a(x)(6-8\cos(x)+2\cos(2x)) \quad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{Prec\ FD} \{T_n(2-2\cos(\theta))^{-1}A_n\}_n \sim \frac{a(x)(2-2\cos(\theta))}{2-2\cos(\theta)} = a(x)$$

Applications

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD/FE} a(x)(2-2\cos(\theta)) \qquad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

$$\xrightarrow{IgA\ Coll./Gal.(p)} a(x)f_p(\theta) \qquad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $a(x)u^{(4)}(x) + b(x)u^{(2)}(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{FD(1,-4,6,-4,1)} a(x)(6-8\cos(x)+2\cos(2x)) \quad (x,\theta) \in [0,1]\times[-\pi,\pi]$$

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

$$\xrightarrow{Prec\ FD} \{T_n(2-2\cos(\theta))^{-1}A_n\}_n \sim \frac{a(x)(2-2\cos(\theta))}{2-2\cos(\theta)} = a(x)$$

Multilevel Generalized Locally Toeplitz Sequences

- $\{Z_n\}_n \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim_\sigma a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
- $\{T_n(f)\}_n \sim_\sigma f(\theta) \to \mathcal{T} = \{(\{T_n(f)\}_n, f(\theta))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]} \qquad \text{(GLT)}$$

Theorem [S-C '03]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})$$

$$\boldsymbol{x} \in [0,1]^d \qquad \boldsymbol{\theta} \in [-\pi, \pi]^d$$

Multilevel Generalized Locally Toeplitz Sequences

- $\{Z_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_{\boldsymbol{n}}\}_{\boldsymbol{n}}, 0)\}$
- $\{D_{\boldsymbol{n}}(a)\}_{\boldsymbol{n}} \sim_\sigma a(\boldsymbol{x}) \to \mathcal{D} = \{(\{D_{\boldsymbol{n}}(a)\}_{\boldsymbol{n}}, a(\boldsymbol{x}))\}$
- $\{T_{\boldsymbol{n}}(f)\}_{\boldsymbol{n}} \sim_\sigma f(\boldsymbol{\theta}) \to \mathcal{T} = \{(\{T_{\boldsymbol{n}}(f)\}_{\boldsymbol{n}}, f(\boldsymbol{\theta}))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]}$$ (Multilevel GLT)

Theorem [S-C '03]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})$$

$$\boldsymbol{x} \in [0,1]^d \qquad \boldsymbol{\theta} \in [-\pi, \pi]^d$$

Multilevel Generalized Locally Toeplitz Sequences

- $\{Z_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma 0 \to \mathcal{Z} = \{(\{Z_{\boldsymbol{n}}\}_{\boldsymbol{n}}, 0)\}$
- $\{D_{\boldsymbol{n}}(a)\}_{\boldsymbol{n}} \sim_\sigma a(\boldsymbol{x}) \to \mathcal{D} = \{(\{D_{\boldsymbol{n}}(a)\}_{\boldsymbol{n}}, a(\boldsymbol{x}))\}$
- $\{T_{\boldsymbol{n}}(f)\}_{\boldsymbol{n}} \sim_\sigma f(\boldsymbol{\theta}) \to \mathcal{T} = \{(\{T_{\boldsymbol{n}}(f)\}_{\boldsymbol{n}}, f(\boldsymbol{\theta}))\}$

$$\mathcal{G} := \overline{\mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}]}$$ (Multilevel GLT)

Theorem [S-C '03]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})$$

$$\boldsymbol{x} \in [0,1]^d \qquad \boldsymbol{\theta} \in [-\pi, \pi]^d$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

  $$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u = \lambda cu \qquad \boldsymbol{x} \in \Omega$

  $$\Longrightarrow A_n = M_n^{-1} K_n \qquad \{K_n\}_n \sim a(\boldsymbol{x}) f(\boldsymbol{\theta}) \quad \{M_n\}_n \sim c(\boldsymbol{x}) h(\boldsymbol{\theta})$$

  $$\Longrightarrow \{A_n\}_n \sim \frac{a(\boldsymbol{x}) f(\boldsymbol{\theta})}{c(\boldsymbol{x}) h(\boldsymbol{\theta})} \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

  $$\xrightarrow{(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} h_p(\theta)\right)$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

  $$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u = \lambda c u \qquad x \in \Omega$

  $$\Longrightarrow A_n = M_n^{-1} K_n \qquad \{K_n\}_n \sim a(x) f(\theta) \qquad \{M_n\}_n \sim c(x) h(\theta)$$

  $$\Longrightarrow \{A_n\}_n \sim \frac{a(x) f(\theta)}{c(x) h(\theta)} \qquad (x, \theta) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad x \in \Omega$, irregular grid

  $$\xrightarrow{(G)} \mathbf{1}(A_G(x) \circ H_p(\theta))\mathbf{1}^T \qquad (x, \theta) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

$$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

$$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u = \lambda cu \qquad \boldsymbol{x} \in \Omega$

$$\Longrightarrow A_n = M_n^{-1} K_n \qquad \{K_n\}_n \sim a(\boldsymbol{x}) f(\boldsymbol{\theta}) \quad \{M_n\}_n \sim c(\boldsymbol{x}) h(\boldsymbol{\theta})$$

$$\Longrightarrow \{A_n\}_n \sim \frac{a(\boldsymbol{x}) f(\boldsymbol{\theta})}{c(\boldsymbol{x}) h(\boldsymbol{\theta})} \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{\dots(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

  $$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u = \lambda c u \qquad \boldsymbol{x} \in \Omega$

  $$\xrightarrow{\cdots} A_{\boldsymbol{n}} = M_{\boldsymbol{n}}^{-1} K_{\boldsymbol{n}} \qquad \{K_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim a(\boldsymbol{x}) f(\boldsymbol{\theta}) \quad \{M_n\}_n \sim c(\boldsymbol{x}) h(\boldsymbol{\theta})$$

  $$\Longrightarrow \{A_n\}_n \sim \frac{a(\boldsymbol{x}) f(\boldsymbol{\theta})}{c(\boldsymbol{x}) h(\boldsymbol{\theta})} \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

  $$\xrightarrow{\cdots(G)} \mathbf{1}(A_G(x) \circ H_p(\theta))\mathbf{1}^T \qquad (x, \theta) \in [0,1]^d \times [-\pi, \pi]^d$$

  $$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

$$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

$$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u = \lambda c u \qquad \boldsymbol{x} \in \Omega$

$$\xrightarrow{\cdots} A_{\boldsymbol{n}} = M_{\boldsymbol{n}}^{-1} K_{\boldsymbol{n}} \qquad \{K_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim a(\boldsymbol{x}) f(\boldsymbol{\theta}) \quad \{M_n\}_n \sim c(\boldsymbol{x}) h(\boldsymbol{\theta})$$

$$\Longrightarrow \{A_n\}_n \sim \frac{a(\boldsymbol{x}) f(\boldsymbol{\theta})}{c(\boldsymbol{x}) h(\boldsymbol{\theta})} \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{\dots(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

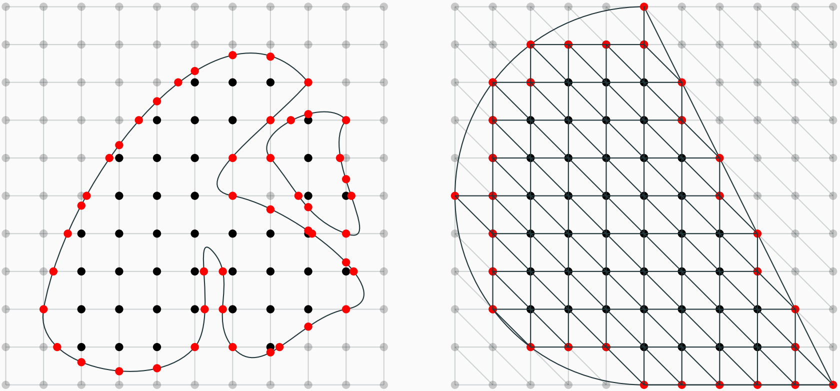
$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Reduced GLT

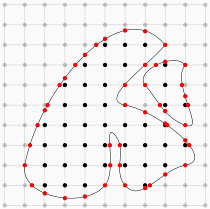
$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$



Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies \chi_\Omega$ R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies \{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x},\boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x},\boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

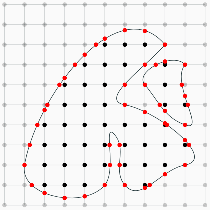
Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x},\boldsymbol{\theta})) \in \mathcal{G} \implies \{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x},\boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega}$$

Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies \chi_\Omega$ R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies \{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

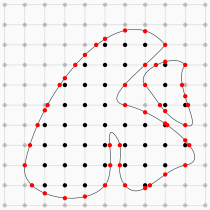
Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})|_{\boldsymbol{x} \in \Omega}$$

Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies \chi_\Omega$ R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies \{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

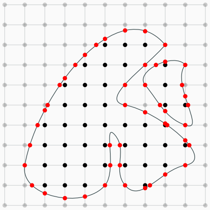
Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega}$$

Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies \chi_\Omega$ R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies \{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

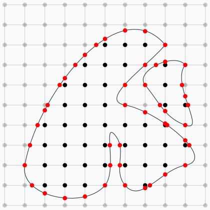
Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A_{\boldsymbol{n}}^\Omega\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})|_{\boldsymbol{x} \in \Omega}$$

Reduced GLT

$$-\nabla \cdot A\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies \chi_\Omega$ R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies \{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

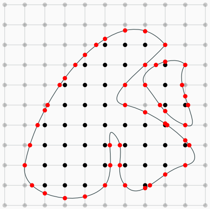
Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})|_{\boldsymbol{x} \in \Omega}$$

Reduced GLT

$$-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$$

Ω bounded, Peano-Jordan measurable $\implies$ χ_Ω R.I., $\mu(\partial\Omega) = 0$



- $\mu(\partial\Omega) = 0 \implies$ there are $o(n)$ border conditions
- χ_Ω R.I. $\implies$ $\{D_{\boldsymbol{n}}(\chi_\Omega)\}_{\boldsymbol{n}} \sim_{GLT} \chi_\Omega$

$$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta}) \implies D_{\boldsymbol{n}}(\chi_\Omega)\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_\Omega) \sim_{GLT} \kappa(\boldsymbol{x}, \boldsymbol{\theta})\chi_\Omega(\boldsymbol{x})$$

Restriction Operator: $\{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} := R_\Omega(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}})$ (Ω-submatrix)

Theorem [B. '19]

$$(\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}}, \kappa(\boldsymbol{x}, \boldsymbol{\theta})) \in \mathcal{G} \implies \{A^\Omega_{\boldsymbol{n}}\}_{\boldsymbol{n}} \sim_\sigma \kappa(\boldsymbol{x}, \boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega}$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$

$$\xrightarrow{\text{IgA Gal., Coll. }(p)} \mathbb{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbb{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{\dots(G)} \mathbb{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbb{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega' \times [-\pi, \pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

- $$\frac{\partial u(x,t)}{\partial t} = d_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + \qquad x \in \Omega$$
$$c_+(x,t)\frac{\partial^\beta u(x,t)}{\partial_+ y^\beta} + c_-(x,t)\frac{\partial^\beta u(x,t)}{\partial_- y^\beta} + f(x,t)$$

$$\xrightarrow{\text{Grünwald}} d_+(x) f_\alpha(\theta_1) + d_-(x) f_\alpha(-\theta_1) + c_+(x) f_\beta(\theta_2) + c_-(x) f_\beta(-\theta_2) \dots$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$

$$\xrightarrow{IgA\ Gal.,Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$

$$\xrightarrow{IgA\ Gal.,Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{...(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega' \times [-\pi, \pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

- $$\frac{\partial u(\boldsymbol{x},t)}{\partial t} = d_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ x^\alpha} + d_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- x^\alpha} + \qquad \boldsymbol{x} \in \Omega^\circ$$

$$c_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ y^\alpha} + c_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- y^\alpha} + f(\boldsymbol{x},t)$$

$$\xrightarrow{\text{Grünwald}} d_+(\boldsymbol{x})f_\alpha(\theta_1) + d_-(\boldsymbol{x})f_\alpha(-\theta_1) + c_+(\boldsymbol{x})f_\alpha(\theta_2) + c_-(\boldsymbol{x})f_\alpha(-\theta_2)|_{\boldsymbol{x}\in\Omega}$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$

$$\xrightarrow{IgA\ Gal.,Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x},\boldsymbol{\theta}) \in \Omega \times [-\pi,\pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{\ldots(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x},\boldsymbol{\theta}) \in \Omega' \times [-\pi,\pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta)\right)$$

- $$\begin{aligned}\frac{\partial u(\boldsymbol{x},t)}{\partial t} =& d_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ x^\alpha} + d_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- x^\alpha} + \qquad \boldsymbol{x} \in \Omega^\circ \\ & c_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ y^\alpha} + c_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- y^\alpha} + f(\boldsymbol{x},t)\end{aligned}$$

$$\xrightarrow{\text{Grünwald}} d_+(\boldsymbol{x})f_\alpha(\theta_1) + d_-(\boldsymbol{x})f_\alpha(-\theta_1) + c_+(\boldsymbol{x})f_\alpha(\theta_2) + c_-(\boldsymbol{x})f_\alpha(-\theta_2)|_{\boldsymbol{x}\in\Omega}$$

What Else?

- **Block GLT**
  - symbols are **matrix-valued functions**
  - multilevel/reduced variants
  - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality

- Diverging Spectrum

- Perturbations, Semiseparable Structures, $*$-Algebras, etc
- Graph families, Stochastic PDE, Signal analysis etc

- Block GLT
  - symbols are matrix-valued functions
  - multilevel/reduced variants
  - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality
  - all algebraic structures can be embedded in GLT?
  - algebraic relations are linked to distance from normality?
- Diverging Spectrum
  - partial functions as symbols
  - associated to measures μ with mass < 1
- Perturbations, Semiseparable Structures, τ-Algebras, etc.
- Graph families, Stochastic PDE, Signal analysis etc.

- Block GLT
  - symbols are matrix-valued functions
  - multilevel/reduced variants
  - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality
  - all algebraic structures can be embedded in GLT?
  - algebraic relations are linked to distance from normality?
- Diverging Spectrum
  - partial functions as symbols
  - associated to measures μ with mass < 1
- Perturbations, Semiseparable Structures, τ-Algebras, etc.
- Graph families, Stochastic PDE, Signal analysis etc.

- Block GLT
  - symbols are matrix-valued functions
  - multilevel/reduced variants
  - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality
  - all algebraic structures can be embedded in GLT?
  - algebraic relations are linked to distance from normality?
- Diverging Spectrum
  - partial functions as symbols
  - associated to measures μ with mass < 1
- Perturbations, Semiseparable Structures, τ-Algebras, etc.
- Graph families, Stochastic PDE, Signal analysis etc.

- Block GLT
    - symbols are matrix-valued functions
    - multilevel/reduced variants
    - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality
    - all algebraic structures can be embedded in GLT?
    - algebraic relations are linked to distance from normality?
- Diverging Spectrum
    - partial functions as symbols
    - associated to measures μ with mass < 1
- Perturbations, Semiseparable Structures, τ-Algebras, etc.
- Graph families, Stochastic PDE, Signal analysis etc.

- Block GLT
  - symbols are matrix-valued functions
  - multilevel/reduced variants
  - systems of linear PDE, Higher order FE (Splines), PToFE, etc.

Future Works:

- GLT Universality
  - all algebraic structures can be embedded in GLT?
  - algebraic relations are linked to distance from normality?
- Diverging Spectrum
  - partial functions as symbols
  - associated to measures μ with mass < 1
- Perturbations, Semiseparable Structures, τ-Algebras, etc.
- Graph families, Stochastic PDE, Signal analysis etc.

References

- G. Barbarino. **Equivalence between glt sequences and measurable functions.** *Linear Algebra Appl. 529*, 2017.
- G. Barbarino. **Spectral measures.** *Proceedings of Cortona Meeting*, 2018.
- G. Barbarino and C. Garoni. **From convergence in measure to convergence of matrix-sequences through concave functions and singular values.** *Electron. J. Linear Al. 32*, 2017.
- G. Barbarino, C. Garoni, and Serra-Capizzano S. **Block generalized locally toeplitz sequences: Theory and applications in the (multi-)unidimensional case.** *Electron. T. Numer. Ana.*, 2020.
- G. Barbarino and S. Serra-Capizzano. **Non-hermitian perturbations of hermitian matrix-sequences and applications to the spectral analysis of approximated pdes.** *Numer. Linear Algebra Appl.*, 2020.
- C. Garoni and S. Serra-Capizzano. ***Generalized Locally Toeplitz Sequences: Theory and Applications*, volume I-II.** Springer.

That's All,
Folks!