Generalized Locally Toeplitz Sequences: a Link between Measurable Functions and Spectral Symbols

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Spectral Symbols

Our Aim

$$\begin{cases} \mathscr{L}u = f \\ BC \end{cases} \xrightarrow[\text{FE, FD}]{\text{IgA, Multigrid}} A_n u_n = f_n$$

$$A_n u_n = f_n \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} u_n$$

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$$\Lambda(A_n)$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

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$$\begin{cases} u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right)$$

$\rightarrow$ **The sequence $\{A_n\}_n$ has Spectral Symbol $k(t)$**

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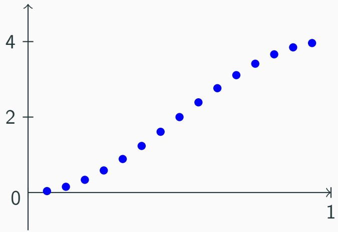

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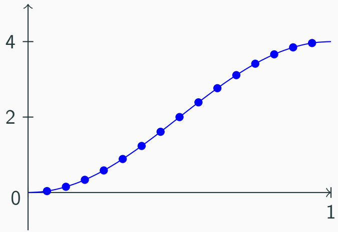

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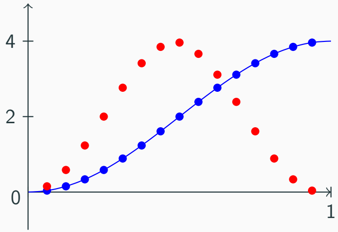

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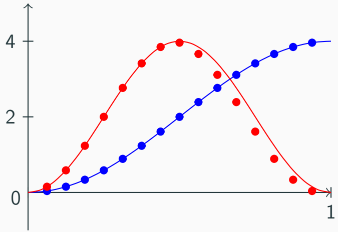

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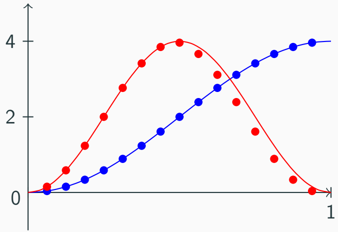

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Asymptotic Distribution

Spectral Symbol

Let $\{A_n\}_n$ a matrix sequence, and $k : D \subseteq \mathbb{R}^m \to \mathbb{C}$ measurable.

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t))\mathrm{dt}$$

$$\{A_n\}_n \sim_\sigma k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{\mu(D)} \int_D F(|k(t)|)\mathrm{dt}$$

for all $F \in C_c(\mathbb{C})$.

Every sequence may have infinite Spectral Symbols

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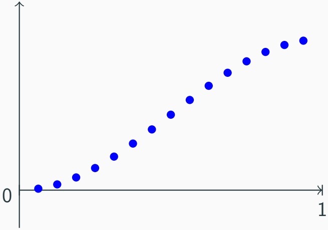

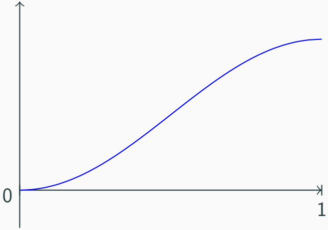

$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < k(t) < b\}}{\mu(D)}$$

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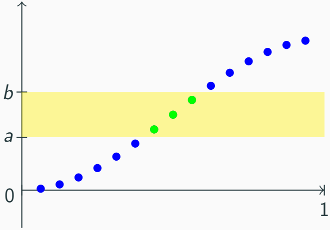

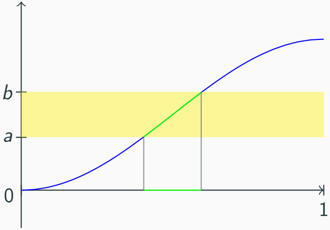

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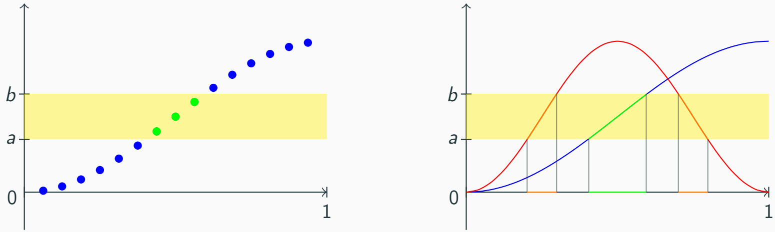

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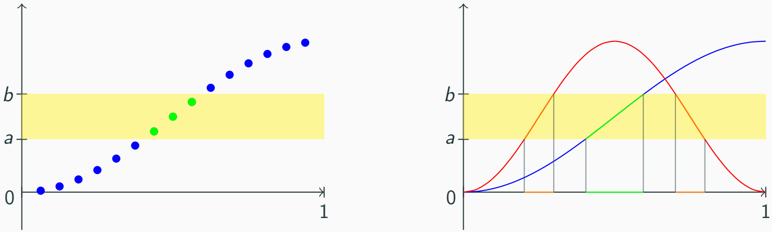

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Three Classes of Matrices

Examples of Symbol

- $Z_n \sim_\sigma 0$
- $\{D_n(a)\}_n \sim_{\lambda,\sigma} a(x)$ where $x \in [0,1]$
- $\{T_n(f)\}_n \sim_\sigma f(\theta)$ where $\theta \in [-\pi,\pi]$

$$a \in C[0,1]$$

$$D_n(a) := \begin{pmatrix} a(1/n) & & & & \\ & a(2/n) & & & \\ & & a(3/n) & & \\ & & & \ddots & \\ & & & & a(1) \end{pmatrix}$$

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$$f \in L^1[-\pi,\pi] \to \widehat{f}_n = \int_{-\pi}^{\pi} f(\theta) e^{-\mathrm{i}n\theta} d\theta$$

$$T_n(f) := \begin{pmatrix} \widehat{f}_0 & \widehat{f}_1 & \widehat{f}_2 & \dots & \widehat{f}_{n-1} \\ \widehat{f}_{-1} & \widehat{f}_0 & \ddots & \ddots & \vdots \\ \widehat{f}_{-2} & \ddots & \ddots & \ddots & \widehat{f}_2 \\ \vdots & \ddots & \ddots & \widehat{f}_0 & \widehat{f}_1 \\ \widehat{f}_{-n+1} & \dots & \widehat{f}_{-2} & \widehat{f}_{-1} & \widehat{f}_0 \end{pmatrix}$$

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They appear frequently in PDEs

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = D_n(a)T_n(2 - 2\cos(\theta)) + Z_n$$

- The sequence $\{A_n\}_n$ has a spectral symbol?
- How do we compute it?

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Space of Matrix Sequences

a.c.s. Convergence

$$\widehat{\mathscr{E}} := \{\{A_n\}_n \mid A_n \in \mathbb{C}^{n\times n}\}$$

Approximating Class of Sequence [Serra-Capizzano, LAA01]

$\{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

for which exist $c(m), \omega(m), n_m$ such that

$$\frac{\operatorname{rk} R_{n,m}}{n} \le c(m) \qquad \|N_{n,m}\| \le \omega(m) \qquad \forall n > n_m$$

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0$$

→ The difference is a sum of **small rank** and **small norm** matrices.

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Metric Spaces

$$\{A_n\}_n, \{B_n\}_n \in \widehat{\mathscr{E}}$$

The a.c.s. convergence is **metrizable**

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) =$$
$$\limsup_{n\to\infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$$

$$i \le j \implies \sigma_i \ge \sigma_j$$
$$\{\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{n-1}, \sigma_n\}$$

$$f(x), g(x) \in \mathscr{M}_D$$

The convergence in measure is metrizable

$$d_m(f,g) =$$
$$\inf_{z\in\mathbb{R}^+} \left\{ \frac{\mu\{x : |f(x) - g(x)| > z\}}{\mu(D)} + z \right\}$$

Theorem [Barbarino, LAA17]

- d_{acs}, d_m are complete pseudometrics
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$$d_m(f,g) = \inf_{z\in\mathbb{R}^+} \left\{ \frac{\mu\{x : |f(x) - g(x)| > z\}}{\mu(D)} + z \right\}$$

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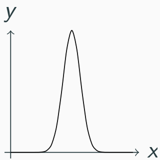

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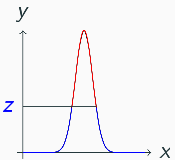

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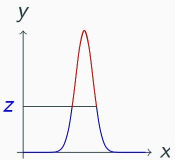

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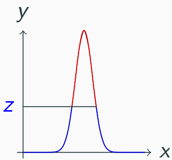

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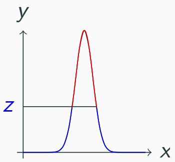

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Measurable Functions

Closure Property

Let $\{B_{n,m}\}_n \sim_\sigma k_m(x)$. Given

1. $k_m(x) \xrightarrow{\mu} k(x)$
2. $\{A_n\}_n \sim_\sigma k(x)$
3. $\{B_{n,m}\}_n \xrightarrow{a.c.s.} \{A_n\}_n$

we have $(1), (3) \implies (2)$.

$$\begin{array}{ccc} \{B_{n,m}\} & \xrightarrow{\text{a.c.s.}} & \{A_n\} \\ \downarrow \sim_\sigma & & \\ k_m & \xrightarrow{\mu} & k \end{array}$$

Problems

- $\{A_n\}_n \sim_\sigma f, \{B_n\}_n \sim_\sigma g \not\implies d_{acs}(\{A_n\}_n, \{B_n\}_n) = d_m(f, g)$
- Stronger: $(1), (2) \not\implies (3)$ $\quad (3), (2) \not\implies (1)$

The Spectral Symbol is **Not Unique**

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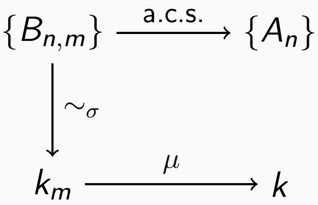

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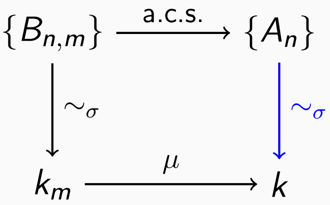

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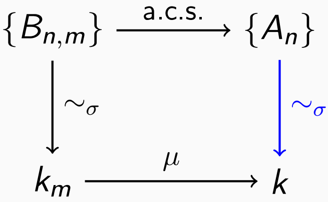

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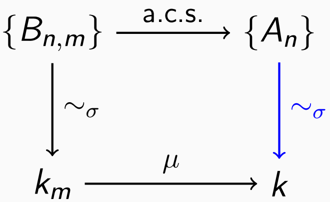

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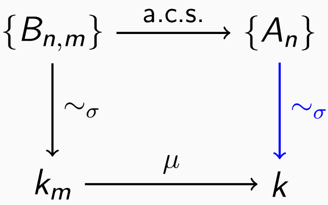

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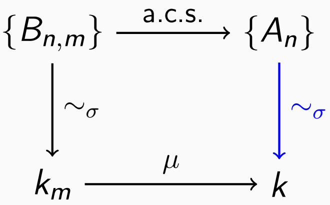

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GLT Sequences

GLT Space

$$\widehat{\mathscr{G}} \subseteq \widehat{\mathscr{E}} \times \mathscr{M}_D$$

where $D = [0,1] \times [-\pi, \pi]$

- $\{T_n(f)\}_n \sim f(\theta) \qquad f(\theta) \in L^1[-\pi,\pi]$
- $\{D_n(a)\}_n \sim a(x) \qquad a(x) \in C([0,1])$
- $Z_n \sim 0$

The algebra generated by $L^1[-\pi,\pi]$ and $C([0,1])$ is dense in $\mathscr{M}_D$.

[Serra-Capizzano, LAA03]

The GLT Space is the smallest closed algebra with respect to $d_{acs} \sim d_m$ that contains

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Three Classes of Matrices

Examples of Symbol

- $Z_n \sim_{GLT} 0$
- $\{D_n(a)\}_n \sim_{GLT} a(x)$ where $a(x) \in C([0,1])$
- $\{T_n(f)\}_n \sim_{GLT} f(\theta)$ where $f(\theta) \in L^1[-\pi,\pi]$

They appear frequently in PDEs

$$\begin{cases} (a(x)u'(x))' = f(x) \quad x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} \qquad A_n u_n = f_n$$

$$A_n = D_n(a)T_n(2 - 2\cos(\theta)) + Z_n$$

- The sequence $\{A_n\}_n$ has a spectral symbol?
- How do we compute it?

Three Classes of Matrices

Examples of Symbol

- $Z_n \sim_{GLT} 0$
- $\{D_n(a)\}_n \sim_{GLT} a(x)$ where $a(x) \in C([0,1])$
- $\{T_n(f)\}_n \sim_{GLT} f(\theta)$ where $f(\theta) \in L^1[-\pi,\pi]$

They appear frequently in PDEs

$$\begin{cases} (a(x)u'(x))' = f(x) \quad x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \qquad \xrightarrow{FD} \qquad A_n u_n = f_n$$

$$A_n = D_n(a)T_n(2 - 2\cos(\theta)) + Z_n$$

$$\{A_n\}_n \sim_{GLT} a(x)(2 - 2\cos(\theta))$$

GLT properties

$$\widehat{\mathscr{E}} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n\times n}\} \qquad \mathscr{M}_D = \{k : D \to \mathbb{C}, k \text{ measurable } \}$$

$$\begin{array}{ccc} \widehat{\mathscr{E}} & & \mathscr{M}_D \\ \cup\!| & & \cup\!| \\ P_1(\widehat{\mathscr{G}}) & & P_2(\widehat{\mathscr{G}}) \end{array}$$

Main Properties

1. $\widehat{\mathscr{G}}$ is an algebra
2. $\widehat{\mathscr{G}}$ is closed as a pseudometric space into $\widehat{\mathscr{E}} \times \mathscr{M}_D$
3. GLT symbols are spectral symbols
   ($\widehat{\mathscr{G}}$ contains $\mathscr{Z}$ the set of zero-distributed sequences)

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More?

Identification

Let $\{A_n\}_n, \{C_n\}_n \in P_1(\mathscr{G})$.

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$$\implies S(\{A_n\}_n - \{C_n\}_n) = S(\{A_n\}_n) - S(\{C_n\}_n) = k_A - k_C$$

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Th2. $\{A_n\}_n \sim_\sigma f \implies d_{acs}(\{A_n\}_n, \{0_n\}_n) = d_m(f, 0)$

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S is an isometry

Let $k \in \mathscr{M}_D$

Let $k \in \mathscr{M}_D$ and $k_m \xrightarrow{\mu} k$ such that exist $S(\{B_{n,m}\}) = k_m$

Iso. S is an isometry

$$\implies d_{acs}(\{B_{n,s}\}, \{B_{n,r}\}) = d_m(k_s, k_r) \implies \{B_{n,m}\} \text{ Cauchy}$$

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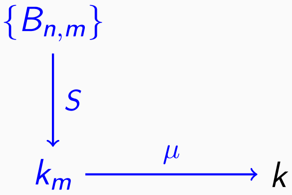

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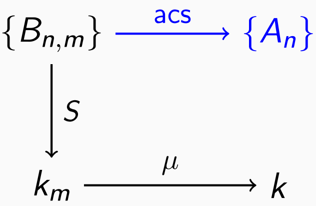

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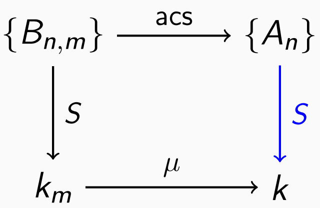

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More?

We know that, for GLT, $\widetilde{Im}(S)$ is dense in $\mathscr{M}_D$, so

$\mathscr{G} \cong \mathscr{M}_D$ [Barbarino, LAA17]

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$\{A_n\}$

- given $\{A_n\}_n$
- find $\{B_{n,m}\}_{n,m}$ GLT sequences with symbols k_m
- if k_m converges, then also $\{B_{n,m}\}_{n,m}$ converges
- if $\{B_{n,m}\}_{n,m}$ converges to $\{A_n\}_n$
- Then $\{A_n\}_n$ has spectral symbol k

$\longrightarrow$ proving the acs convergence is difficult

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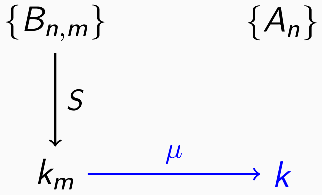

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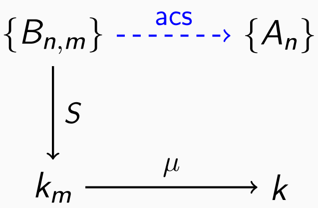

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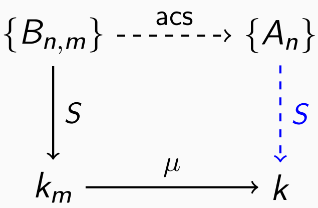

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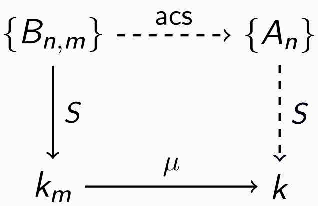

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Metrics on $\mathscr{M}_D$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be an increasing bounded concave and continuous function with $\varphi(0) = 0$

We can define corresponding metrics on $\mathscr{E}$ and $\mathscr{M}_D$

$$p_m^\varphi(f) := \frac{1}{|D|}\int_D \varphi(|f|) \qquad p^\varphi(\{A_n\}_n) := \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \varphi(\sigma_i(A_n))$$

$$d_m^\varphi(f, g) := p_m^\varphi(f - g) \qquad d^\varphi(\{A_n\}_n, \{B_n\}_n) := p^\varphi(\{A_n - B_n\}_n)$$

Theorem 3 [Barbarino, Garoni, '17]
d^φ is a complete metric on $\mathscr{E}$ inducing the acs convergence.

$$\{A_n\}_n \sim_\sigma f \implies p^\varphi(\{A_n\}_n) = p_m^\varphi(f)$$

$$\{A_n\}_n \sim_{GLT} k, \{B_n\}_n \sim_{GLT} h \implies d^\varphi(\{A_n\}_n, \{B_n\}_n) = d_m^\varphi(k, h)$$

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We can define corresponding metrics on $\mathscr{E}$ and $\mathscr{M}_D$

$$p_m^\varphi(f) := \frac{1}{|D|}\int_D \varphi(|f|) \qquad p^\varphi(\{A_n\}_n) := \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \varphi(\sigma_i(A_n))$$

$$d_m^\varphi(f,g) := p_m^\varphi(f-g) \qquad d^\varphi(\{A_n\}_n, \{B_n\}_n) := p^\varphi(\{A_n - B_n\}_n)$$

Theorem 3 [Barbarino, Garoni, '17]

d^φ is a complete metric on $\mathscr{E}$ inducing the acs convergence.

$$\{A_n\}_n \sim_\sigma f \implies p^\varphi(\{A_n\}_n) = p_m^\varphi(f)$$

$$\{A_n\}_n \sim_{GLT} k, \{B_n\}_n \sim_{GLT} h \implies d^\varphi(\{A_n\}_n, \{B_n\}_n) = d_m^\varphi(k,h)$$

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Concave functions

- $\varphi_1(x) = \min\{x, 1\}$
- $\varphi_2(x) = \frac{x}{x+1}$

$$d_1^{\varphi}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \min\{\sigma_i(A_n - B_n), 1\}$$

$$d_2^{\varphi}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_i(A_n - B_n)}{\sigma_i(A_n - B_n) + 1}$$

$\longrightarrow$ New ways to test the acs convergence

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