# Higher order spectral symbols and eigenvalues approximation for Toeplitz matrices

March 11, 2018

# Zeroth Order

## Spectral Symbol

Given $A_n \in \mathbb{C}^{n\times n}$ and $f : [0,1] \to \mathbb{C}$ measurable,

$$\{A_n\}_n \sim_\lambda f(x)$$

$$\Lambda(A_n) = \{\lambda_{\sigma_n(1),n}, \dots, \lambda_{\sigma_n(n),n}\}$$

$$\lambda_{1,n} \le \lambda_{2,n} \le \cdots \le \lambda_{n,n}$$




$$\lambda_{\sigma_n(i),n} \sim f\left(\frac{i}{n+1}\right) \qquad \tilde{\forall} i$$

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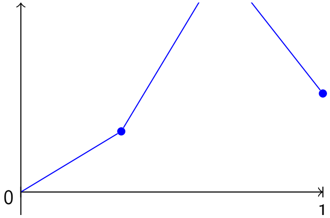


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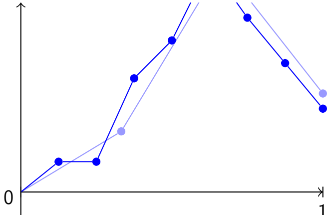


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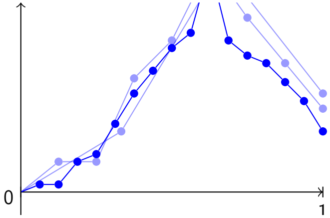


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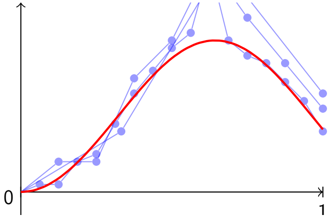


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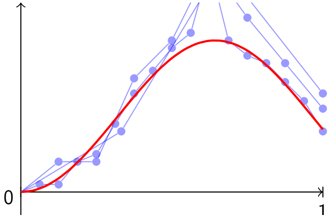


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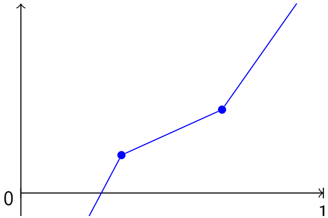


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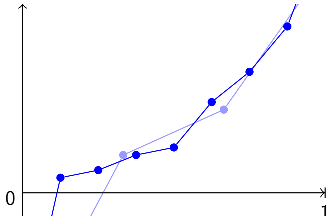


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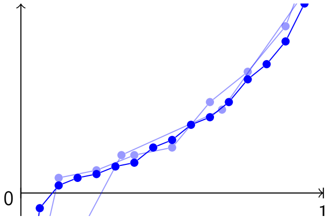


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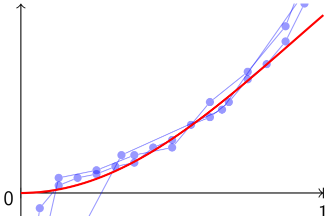


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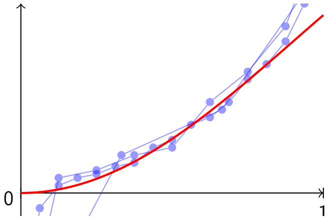


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**Szegő Theorem**

Let $f \in L^1[-\pi, \pi]$ real-valued function

$$T_n(f) = [f_{i-j}]_{i,j} \implies \{T_n(f)\}_n \sim_\lambda f$$

$$\Lambda(T_n(f)) \subseteq [\inf \operatorname{ess} f, \sup \operatorname{ess} f]$$

If $f$ is bounded and its quantile $k$ on $[0, \pi]$ is continuous,

$$\lambda_{i,n} - k\left(\frac{i\pi}{n+1}\right) = o(1) \qquad n \to \infty$$

**Hope**

$$\lambda_{i,n} = k\left(\frac{i\pi}{n+1}\right) + \frac{c_1\left(\frac{i\pi}{n+1}\right)}{n+1} + \frac{c_2\left(\frac{i\pi}{n+1}\right)}{(n+1)^2} + \dots$$

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Given $f$ continuous, let $k$ be the quantile on $[0, \pi]$ and compute

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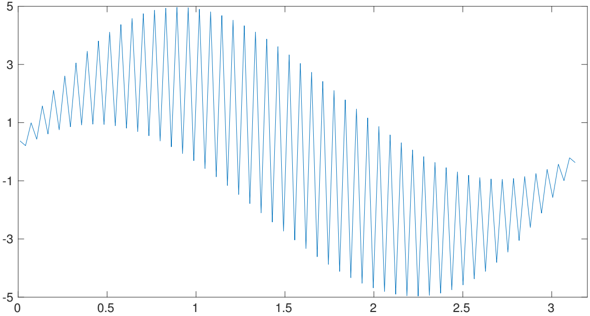


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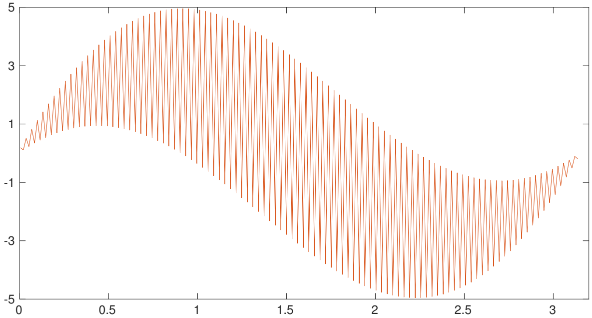


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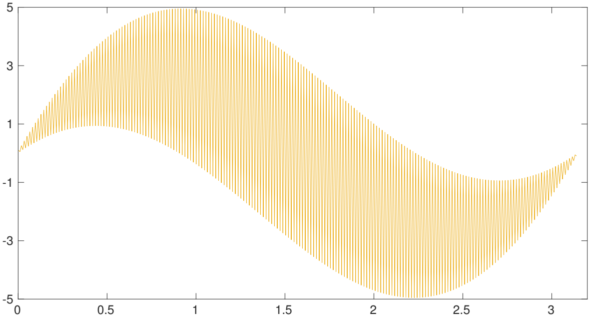


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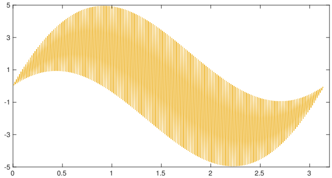


$$f(x) = 2cos(2x) \qquad T_n(f) = \begin{pmatrix} 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ 1 & 0 & 0 & 0 & 1 & \\ & 1 & 0 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \end{pmatrix}$$

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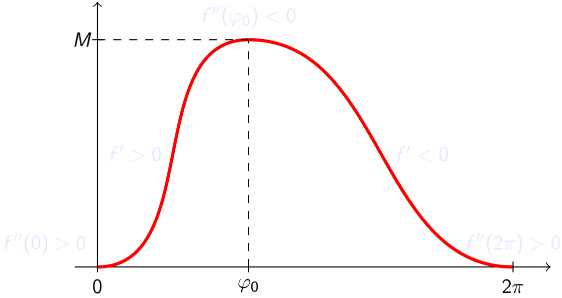
Suppose the function $f : [0, 2\pi] \to \mathbb{R}$ is $C^m_{per}$ and




## BBGM

If $\alpha = m - 3$ and $k$ is the quantile of $f$ on $[0, \pi]$, then

$$\lambda_{i,n} = k\left(\frac{i\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{i\pi}{n+1}\right)}{(n+1)^s} + o\left(\frac{1}{n^\alpha}\right)$$

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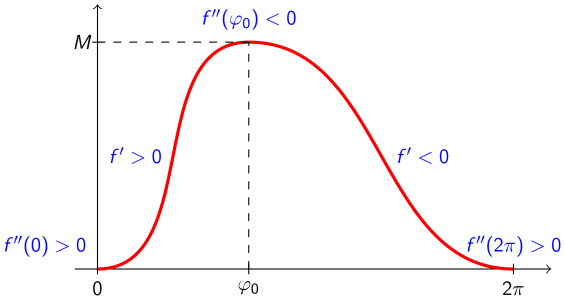


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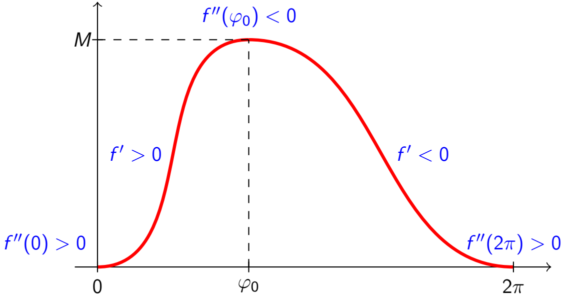


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# Toeplitz Operator on $L^2_{(n)}$

## Aims

Suppose $f(x) = f(2\pi - x)$ ($\implies f|_{[0,\pi]} \equiv k$ quantile)

- Find a characterization of $\Lambda(T_n(f))$
- Find an expansion of $f^{-1}(\lambda_{i,n}) - \frac{i\pi}{n+1}$

$$f(\theta) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta} = \sum_{k=-\infty}^{\infty} f_k t^k = f(t) \qquad P_n\left(\sum_{k=-\infty}^{\infty} a_k t^k\right) = \sum_{k=0}^{n-1} a_k t^k$$

$$L^2_{(n)} := P_n(L^2)$$

## Toeplitz

Given $f \in L^2$, then $T_n(f) : L^2_{(n)} \to L^2_{(n)}$ is a linear operator

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$$\lambda \in \Lambda(T_n(f)) \iff \exists g \in L^2_{(n)} : T_n(f(t)-\lambda)g(t) = 0$$

$$b(t,\widetilde{\theta}) := \frac{f(t)-f(\widetilde{\theta})}{(t-e^{i\widetilde{\theta}})(e^{-i\widetilde{\theta}}-t)} t \in \mathbb{R}^+$$

$$h(t) := T_{n+2}(f(\cdot)-f(\widetilde{\theta}))(tg(t)) \neq 0$$




$$h(t) = P_{n+2}\left[b(t,\widetilde{\theta})\frac{(t-e^{i\widetilde{\theta}})(e^{-i\widetilde{\theta}}-t)}{t}tg(t)\right] = T_{n+2}(b(\cdot,\widetilde{\theta}))\widetilde{g}(t)$$

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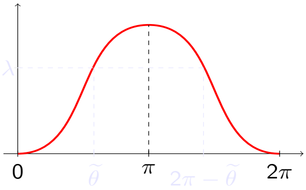


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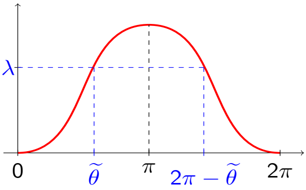


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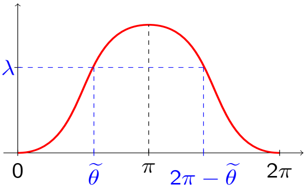


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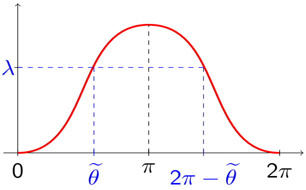


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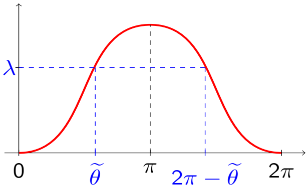


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# Toeplitz Operator on $L^2_+$

$$T(f) = [f_{i-j}]_{i=1,\dots,\infty}^{j=1,\dots,\infty} \qquad P\left(\sum_{k=-\infty}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} a_k t^k$$

## Toeplitz

Given $f \in L^2$, then $T_n(f) : L^2_+ \to L^2_+$ is a linear operator

$$T(f)g := P(fg)$$

## Wiener Hopf

Suppose $a(t) \in L^2$ where $a(t) \neq 0$ and $wind(a,0) = 0$. Then

$$a = a_+ a_- \qquad a_+ \in L^2_+, \quad a_- \in L^2_-$$

$$T(a)^{-1} = T(a_+^{-1})T(a_-^{-1})$$

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$$\Longrightarrow \cdots \Longrightarrow$$

$$\exp(2i\eta(\widetilde{\theta}) + 2iR^{(n)}(\widetilde{\theta})) = \exp(2(n+1)i\widetilde{\theta})$$

## $\widetilde{\theta}$ Approximation

$\lambda$ is an eigenvalue of $T_n(f)$ iff there exists $j \in \mathbb{Z}$ such that $\widetilde{\theta} = f^{-1}(\lambda) \in (0,\pi)$ and satisfies

$$(n+1)\widetilde{\theta} - \eta(\widetilde{\theta}) - R^{(n)}(\widetilde{\theta}) = j\pi$$

- $\eta(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\log b(\tau,x)}{\tau - e^{ix}} d\tau - \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\log b(\tau,x)}{\tau - e^{-ix}} d\tau$, $\eta(0) = \eta(\pi) = 0$
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$$\theta_{j,n} = \frac{j\pi}{n+1} + \frac{\eta(\theta_{j,n})}{n+1} + o(n^{3-m}) \qquad f(\theta_{j,n}) = \lambda_{j,n}$$

$$\begin{aligned}
\theta_{j,n} &= \frac{j\pi}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)}{n+1} + o(n^{-1}) \\
&= \frac{j\pi}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)\eta'\left(\frac{j\pi}{n+1}\right)}{(n+1)^2} + o(n^{-2}) \\
&= \dots \\
&= \frac{j\pi}{n+1} + \sum_{s=1}^{m-3} \frac{d_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + o\left(\frac{1}{n^{m-3}}\right)
\end{aligned}$$

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$$\begin{aligned}
\theta_{j,n} &= \frac{j\pi}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)}{n+1} + o(n^{-1}) \\
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# Conjecture

**RCTP**

$u(\theta)$ is a Real Cosine Trigonometrical Polynomial if

$$u(\theta) = u_0 + 2\sum_{i=1}^{m} u_i \cos(k\theta)$$

**Assumption**

Let $f = u/v$ increasing on $[0,\pi]$, where $u, v$ are RCTPs and $v \neq 0$

$$\lambda_j(T_n(f)) = f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + E_{j,n,\alpha}$$

- $c_k \in C^{\alpha-k+1}[0,\pi]$
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Focus on $\lambda_j(T_n(f))$. Let

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Given $p_{k,j,n}$ the resulting polynomials,

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## $c_k \sim p_k$

$$|c_k(d_{j,n}) - p_{k,j,n}(d_{j,n})| \leq B_\alpha h_1^{\alpha-k+1}$$

If $q_{k,j,n}$ interpolates

$$(d^{(1)}, c_k(d^{(1)})), (d^{(2)}, c_k(d^{(2)})), \ldots, (d^{(\alpha-k+1)}, c_k(d^{(\alpha-k+1)}))$$

then

$$|c_k(d_{j,n}) - q_{k,j,n}(d_{j,n})| \leq \|c_k\|_\infty (h_1\pi)^{\alpha-k+1} \frac{(\alpha-k+1)^{\alpha-k+1}}{(\alpha-k+1)!}$$

$$|p_{k,j,n}(d_{j,n}) - q_{k,j,n}(d_{j,n})| \leq \sum_{r=1}^{\alpha-k+1} \prod_{s\neq r} \frac{|d_{j,n} - d^{(s)}|}{|d^{(r)} - d^{(s)}|} |c_k(d^{(r)}) - \tilde{c}_k(d^{(r)})|$$

$$\leq A_\alpha h_1^{\alpha-k+1} (\alpha-k+1)^{\alpha-k+1}$$

$$\implies |c_k(d_{j,n}) - p_{k,j,n}(d_{j,n})| \leq B_\alpha h_1^{\alpha-k+1}$$

**$c_k \sim p_k$**

$$|c_k(d_{j,n}) - p_{k,j,n}(d_{j,n})| \leq B_\alpha h_1^{\alpha-k+1}$$

If $q_{k,j,n}$ interpolates

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then

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# Convergence

$$\lambda_j(T_n(f)) = f\left(d_{j,n}\right) + \sum_{s=1}^{\alpha} c_s\left(d_{j,n}\right) h^s + E_{j,n,\alpha}$$

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- The error decreases if $n \to \infty$
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**It is better to keep $\alpha$ low also for the computational cost**

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Input: $n > n_1 > \alpha$, $S \subseteq \{1, \dots, n\}$, $f \in C^{\infty}_{per}[-\pi, \pi]$

**for** $k = 1, \dots, \alpha$ **do**

Compute $eig(T_{n_k}(f))$

**end for**

**for** $j_1 = 1, \dots, n_1$ **do**

Compute $\widetilde{\boldsymbol{c}}_{j_1} = V^{-1}[\lambda_{j_k}(T_{n_k}(f)) - f(d_{j_1,n_1})]_k$

**end for**

**for** $j \in S$ **do**

**for** $k = 1, \dots, \alpha$ **do**

Determine $\alpha - k + 1$ points $d^{(s)}$ closest to $d_{j,n}$.

Compute $p_{k,j,n}(d_{j,n})$ where $p_{k,j,n}$ interpolates $(d^{(s)}, \widetilde{c}_k(d^{(s)}))$ implicitly

**end for**

$\widetilde{\lambda}_j(T_n(f)) = f(d_{j,n}) + \sum_{s=1}^{\alpha} p_{s,j,n}(d_{j,n})h^s$

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$$\text{Cost} : \sum_k O(eig(T_{n_k}(f))) + O(\alpha^2 n_1) + O(\alpha^3 |S|)$$

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    **end for**
    $\widetilde{\lambda}_j(T_n(f)) = f(d_{j,n}) + \sum_{s=1}^{\alpha} p_{s,j,n}(d_{j,n})h^s$
**end for**

Parallel Cost : $\boldsymbol{O(eig(T_{n_\alpha}(f))) + O(\alpha^3)}$

# Experiments

Compute all $\widetilde{\lambda}_j(T_n(f))$ with $n = 5000$ and

$$f(\theta) = \frac{40 - 15\cos(\theta) - 24\cos(2\theta) - \cos(3\theta)}{1208 + 1191\cos(\theta) + 120\cos(2\theta) + \cos(3\theta)}$$

| Method | CPU time | max error |
| --- | --- | --- |
| Algorithm with $n_1 = 50$, $\alpha = 4$ | 1.69 | $\sim 10^{-7}$ |
| Algorithm with $n_1 = 50$, $\alpha = 4$ | 2.77 | $\sim 10^{-8}$ |
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Compute the first order symbol of

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The previous results apply on the intervals of $[0, \pi]$ where $f$ is **injective**.

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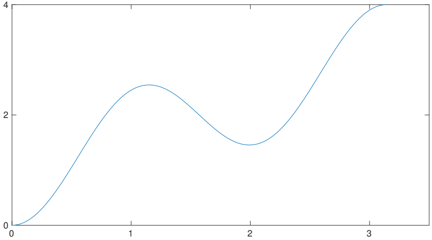


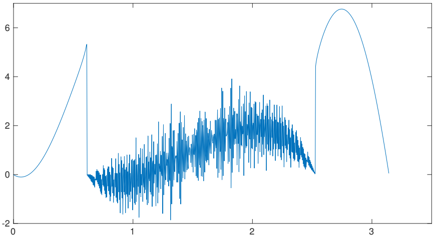


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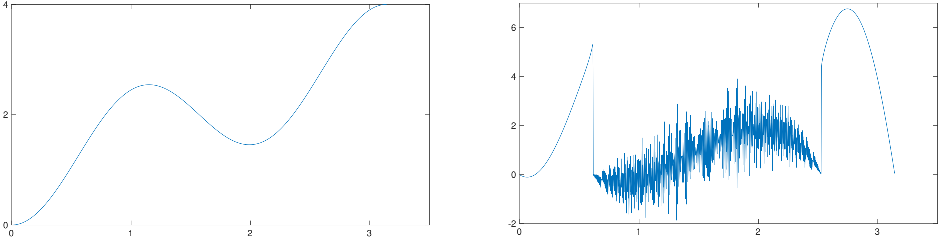


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# Counterexample

## Assumption

Let $f = u/v$ increasing on $[0, \pi]$, where $u, v$ are RCTPs and $v \neq 0$

$$\lambda_j(T_n(f)) = f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + E_{j,n,\alpha}$$

- $c_k \in C^{\alpha-k+1}[0, \pi]$
- $E_{j,n,\alpha} = O(n^{-\alpha-1})$

## BBGM

$f(\theta) = (\sin(\theta/2))^4$ (pentadiagonal) respects the hypothesis, but fails for $\alpha = 5$.

$$f''(0) = 0 \implies b(1,0) = 0 \implies \log(b), b_\pm^{-1} \text{singular}$$

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