Thm (Reidemeister-Singer) Let M be a closed, connected 3-manifold. Any two Heegaard splittings of M share a common stabilization (up to isotopy).

Lemma Stabilizzazioni classificate del genere + isotopie a caso.

Some Morse theory stuff:

Def A map $f: M \to N$ is stable if there exists an open neighborhood $U \subseteq C^\infty(M,N)$ of maps all isotopic to f.

Remark 1) If f is isotopic to a stable map g, f is also stable.
2) The set of stable maps is an open set in $C^\infty(M,N)$.
3) Every PCC of the set of stable maps represents a single isotopy class.
4) A stable function $f: M \to \mathbb{R}$ is just a Morse function.

Take two Morse functions $f, g: M \to \mathbb{R}$; we can construct a map $f \times g: M \to \mathbb{R}^2$ in the obvious way.

Thm (Mather) The set of stable maps $f: M \to \mathbb{R}^2$ is dense.

Using this, we can show the following

Lemma If f, g are Morse functions, $f \times g$ is stable, after arbitrarily small isotopies.

Def Let $F = f \times g$ be a stable map; define the discriminant set $\mathcal{Z}$ as
$\mathcal{Z} = \{p \in M \mid rk(dF_p) = 1\} = \{p \in M \mid \nabla f_p \text{ and } \nabla g_p \text{ are dependent}\}$

Def The graphic of F is the set $F(\mathcal{Z}) \subseteq \mathbb{R}^2$.

How do we get Heegaard Splittings from Morse functions?

Remark If M is compact, there are finitely many critical points.

Def A proper Morse function on M is a Morse function on int M such that the level sets consist of boundary parallel surfaces in some neighborhood of ∂M, and f extends uniquely on M.

Remark If a, b are regular values for f, f is a proper Morse function on $f^{-1}[a,b]$.

Lemma Let M be compact, orientable. If there is a proper Morse function $f: M \longrightarrow \mathbb{R}$ whose critical points only have index 0 or 1, then every component of M is a compression body.

Remark If a component of M has connected boundary, then the indexes are all 0; if a component has n index 0 and m index 1 points, then its genus is $m-n+1$.

Conversely, given a Heegaard splitting on M, one can construct a Morse function on each handlebody, and make them agree on the boundaries, with the index thing still holding.

Remark It is not always true that such a b exists. However, we can partition the index 0 and 1 and 2 and 3 critical points with a sequence $b_1, \ldots, b_n$ such that $\forall j$, $[b_{j-1}, b_j]$ contains index 0 and 1 critical values, and $[b_j, b_{j+1}]$ only contains index 2 and 3 critical values.

The surfaces $f^{-1}(b_j)$ define a generalized Heegaard splitting.

By amalgamation, a gen HS can be turned in a unique (up to isotopy) HS.

We can state this as follows:

Fact Every Morse function on M determines a unique (up to isotopy) HS on M. If M is closed, then the genus of the splitting is $\#\{\text{index } 1\} - \#\{\text{index } 0\} + 1$.

Remark If H is a handlebody in a Heegaard splitting, the Heegaard surface Σ is determined by the spine of H ($\Sigma = \partial H$).

We will construct a HS from a Morse function using the spine for a handlebody.

Def Let $p \in M$ be an index 1 critical point for a Morse function f. A descending arc is an arc starting at p, st $\alpha(1)$ is an index 0 critical point and $f \circ \alpha$ is monotonically decreasing.

Def For every point p of index 1, consider the set of index 0 points in the same PCC. For each of them, pick two transverse descending arcs that connect them with p. The union of all such pair of arcs is called a descending spine for f.

<u>Thm</u> A descending spine for f is isotopic to the spine of a handlebody of a HS for M.

Let us now consider $\alpha = \{ f\sin t + g\cos t \} : [0, \frac{\pi}{2}] \longrightarrow C^\infty(M, \mathbb{R})$. The set of Morse functions in α is open, and every component determines an isotopy class of Morse functions, so if there are finitely many components, α determines a finite sequence of HS on M.

<u>Thm (Mather)</u> If $F : M \to \mathbb{R}^2$ is a stable map, any critical point admits a chart in which the F looks either like
1) $F(u,x,y) = (u, x^2 + y^2)$ definite fold point;
2) $F(u,x,y) = (u, x^2 - y^2)$ indefinite " ";
3) $F(u,x,y) = (u, y^2 + ux - \frac{x^3}{3})$ cusp point

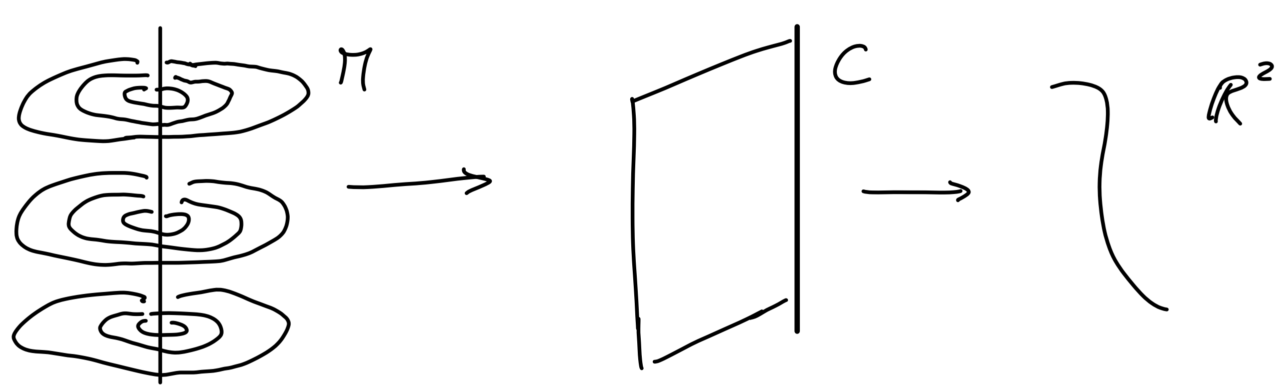
Moreover, no cusp point is a double point of the graphic, and on the complement of the cusps, the graphic is immersed, with normal crossings.

<u>Def</u> The Reeb complex for a stable function $F : M \to \mathbb{R}^2$ is $C = M/\sim$, where $x \sim y$ if they are in the same component of a level set for F.

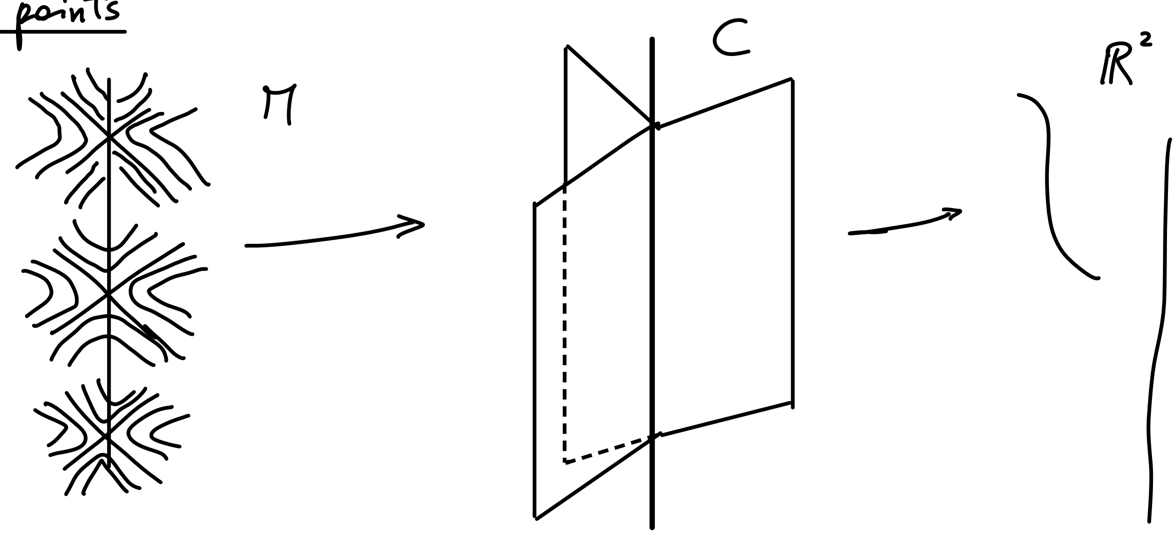
<u>Fact</u> Every stable map factorizes in a unique way through C.

The factorization looks like the following:

<u>Definite fold points</u>



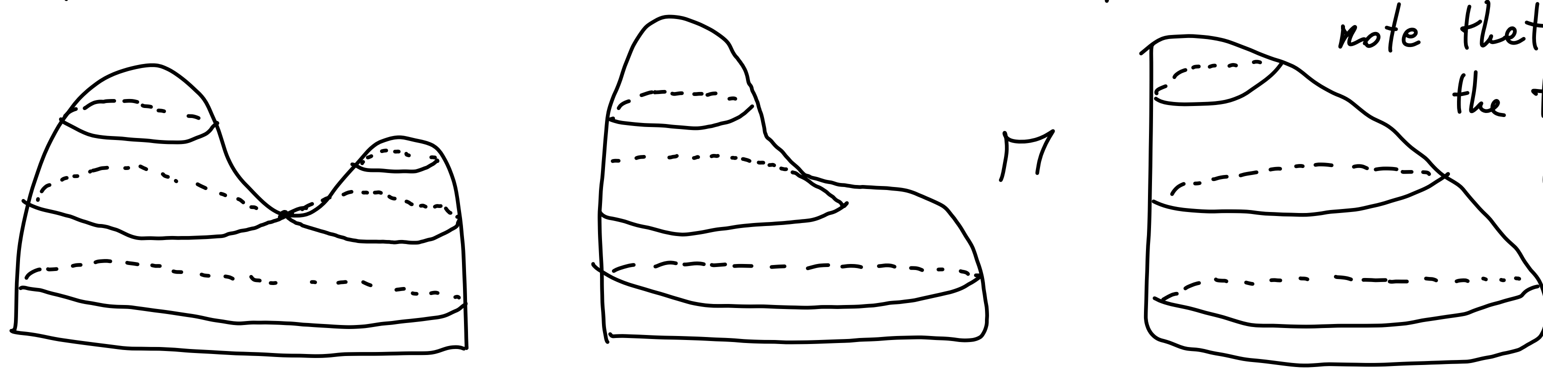
<u>Indefinite fold points</u>

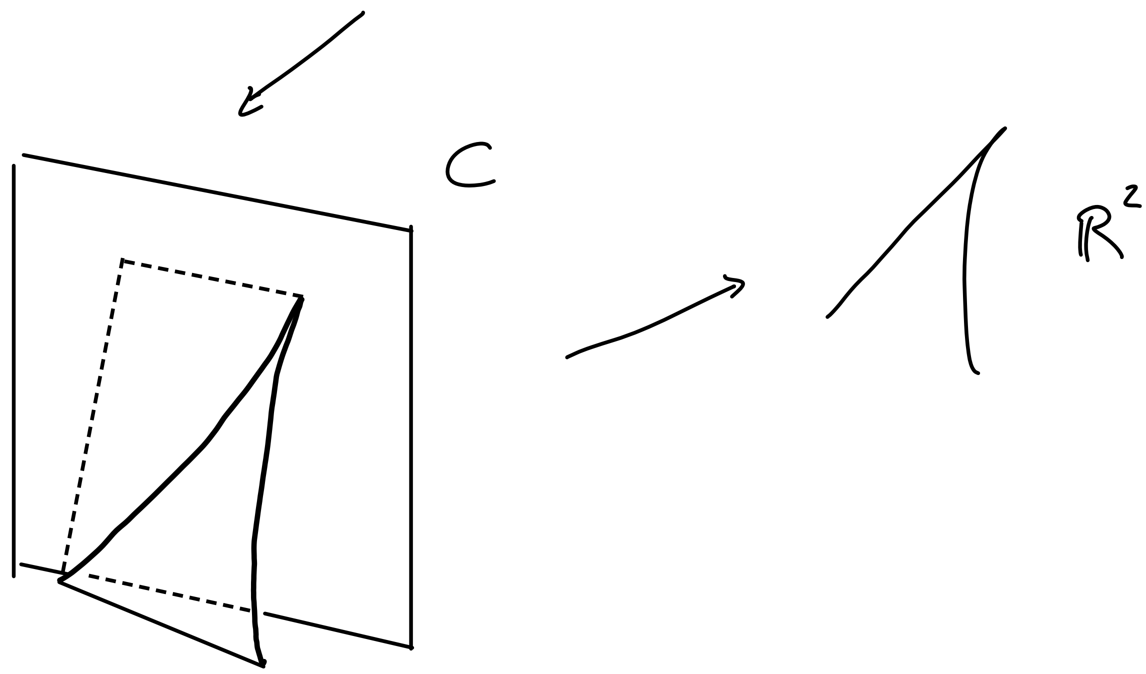


If there is a double pt in the graphic, we have two figure 8s, unentangled crossing or tangled crossing.

Cusps

These sit on top of each other in M, note that in the cusp point, the two arcs come from one definite and one indefinite fold points.





Let us look at the path of smooth functions constructed by projecting a stable function onto a line through the origin. After a rotation, we can assume that such a line is the y axis.

Some facts 1) If there are no horizontal tangents at cusp points of the graphic, there is a bijective correspondence between critical points of $f = \pi_y F$ and points with horizontal tangent.

2) In a neighborhood of a horizontal tangency point, we can use Dini to identify this neighborhood with the graph of some function $\mathbb{R} \to \mathbb{R}$.

3) If a point $p \in \Sigma$ is critical for $g = \pi_y F$ and not a cusp for F, p is non-degenerate iff the second derivative of the implicit function is non-zero.

4) If there are finitely many points as of 3) + finitely many horizontal tangents of $F(\Sigma)$, the path $\alpha(t) = f\cos t + g\sin t$ will pass through finitely many non-Morse functions.

We now only need to prove that when $\varphi(t)$ passes through a non-Morse function, its isotopy class changes in a way that corresponds to a stabilization or destabilization at an inflection point or a type two cusp, or does not change otherwise.
We can do this following some steps:

1) Rotating an inflection point or a type two cusp, the number of horizontal tangencies increases or decreases by two, which either increases or decreases the genus of the HS by one, or does nothing, depending on the type of crit points created/removed.

2) $F^{-1}(\mathbb{R} \times \{y\}) := \Sigma_y$ is a surface, and $F|_{\Sigma_y} := f_y$ is a Morse function.
If $F = f \times g$ is the product of Morse functions, each slice is a Reeb graph for $f|_{\{g=y\}}$.

3) If a line $\mathbb{R} \times \{y\}$ intersects n def and m indefinite fold points, the Reeb graph R_y has $n+m$ vertices, and $\frac{n}{2} + \frac{3m}{2}$ edges, so $\chi(R_y) = \frac{m-n}{2} = \frac{1}{2}\chi(\Sigma_y)$.

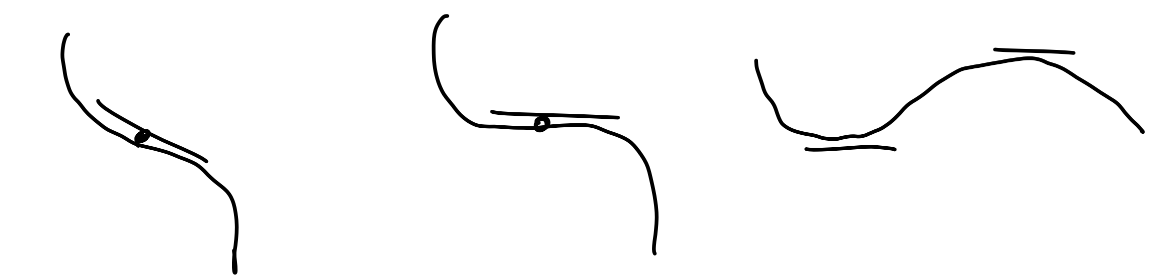
As φ passes through a horizontal tangent value, the number of intersections with a type of edge increases or decreases by two:
- at a horizontal edge of DFP, two leaves are added or removed; this either adds or removes a sphere component of Σ_y, or increases its genus by 1. This depends on whether the 2-cell is above or below the edge.
- at a horizontal edge of IFP, the genus of a component goes ± 1. (This is exactly the familiar behaviour of the level sets of a Morse function passing through a crit point)

Let us take a look at how the projections of F change at the non-generic angles.
There are three cases in which φ_t may be non-Morse: horizontal inflection point, two horizontal tangencies at the same level, horizontal cusp.

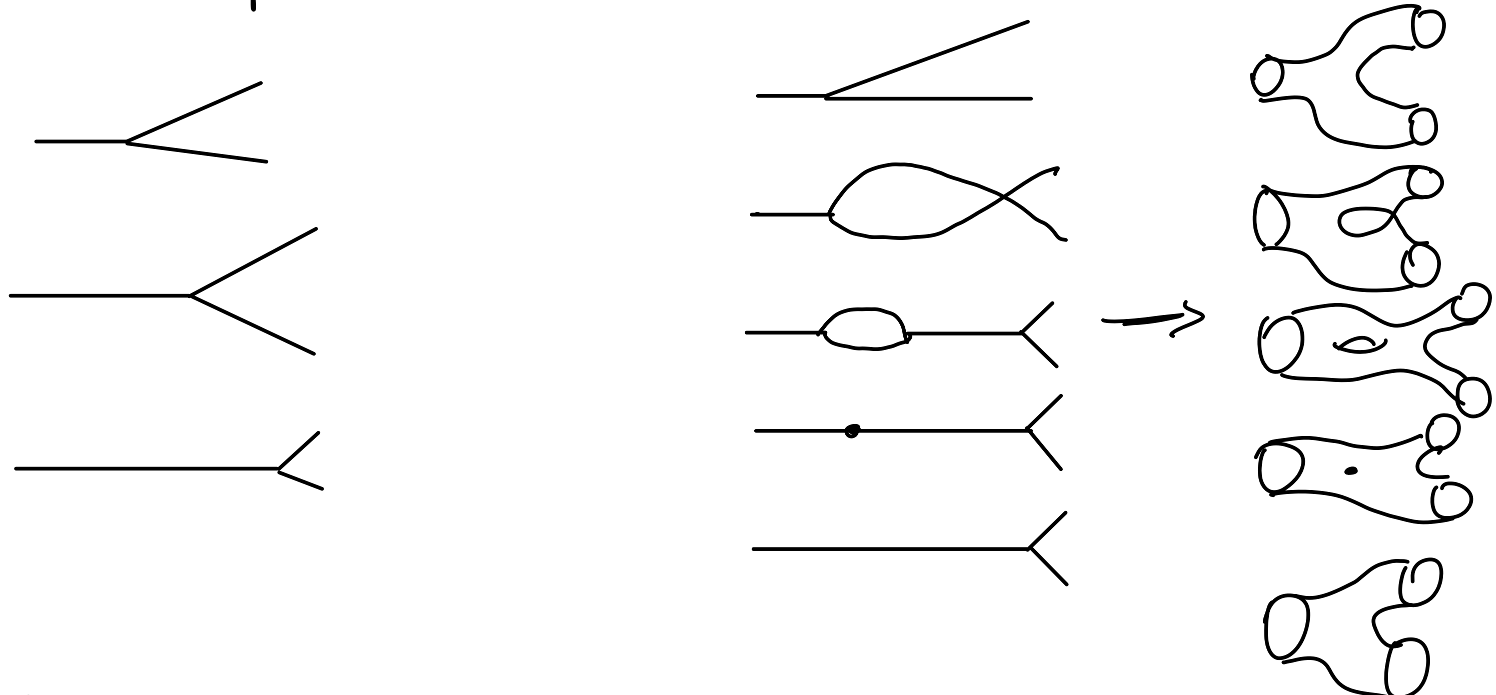
There are thus 8 cases;

f'': $+ \to -$	edge of IFP ①	RC has more sheets above edge
f'': $- \to +$	edge of DFP ②	" " " " " " " " below "

In case ②, the HS does not change, in case ① it gets stabilized or destabilized.

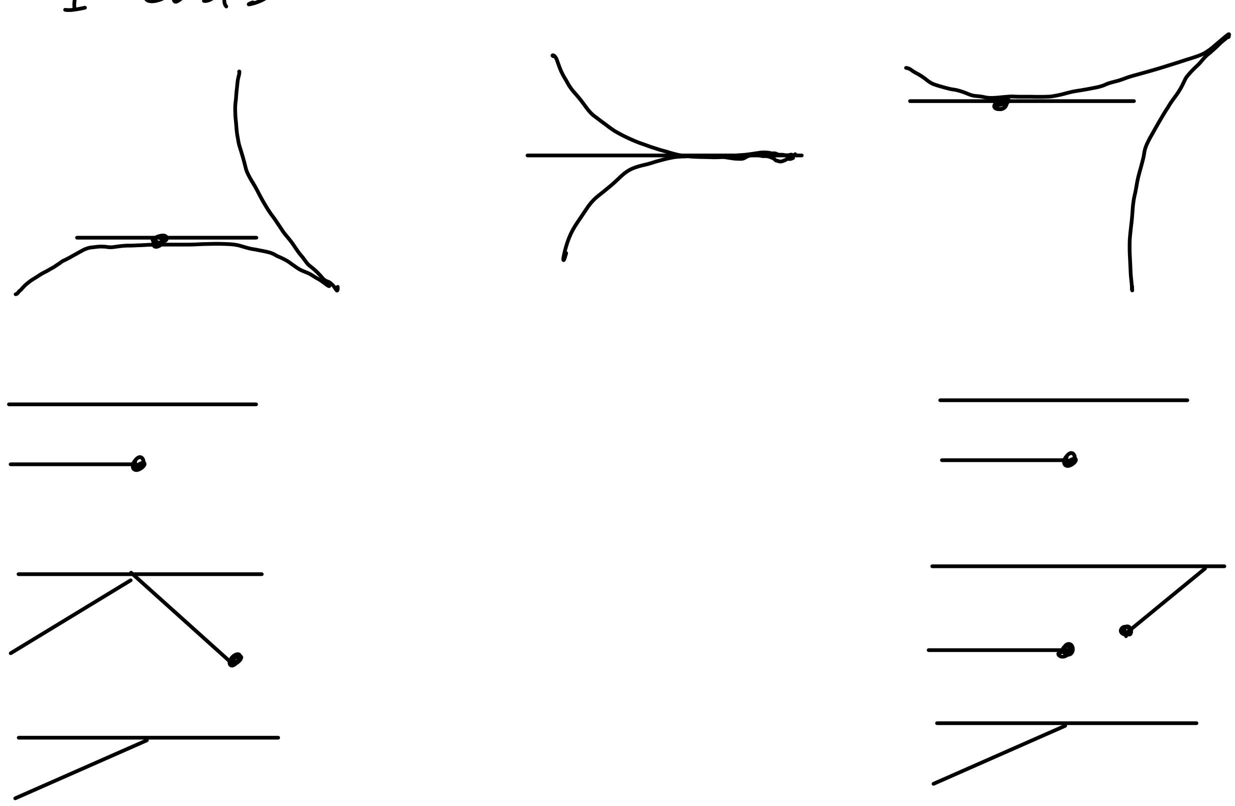


INFLECTION POINT THROUGH HORIZONTAL TANGENCY.



For cusps, we divide in type one and two:

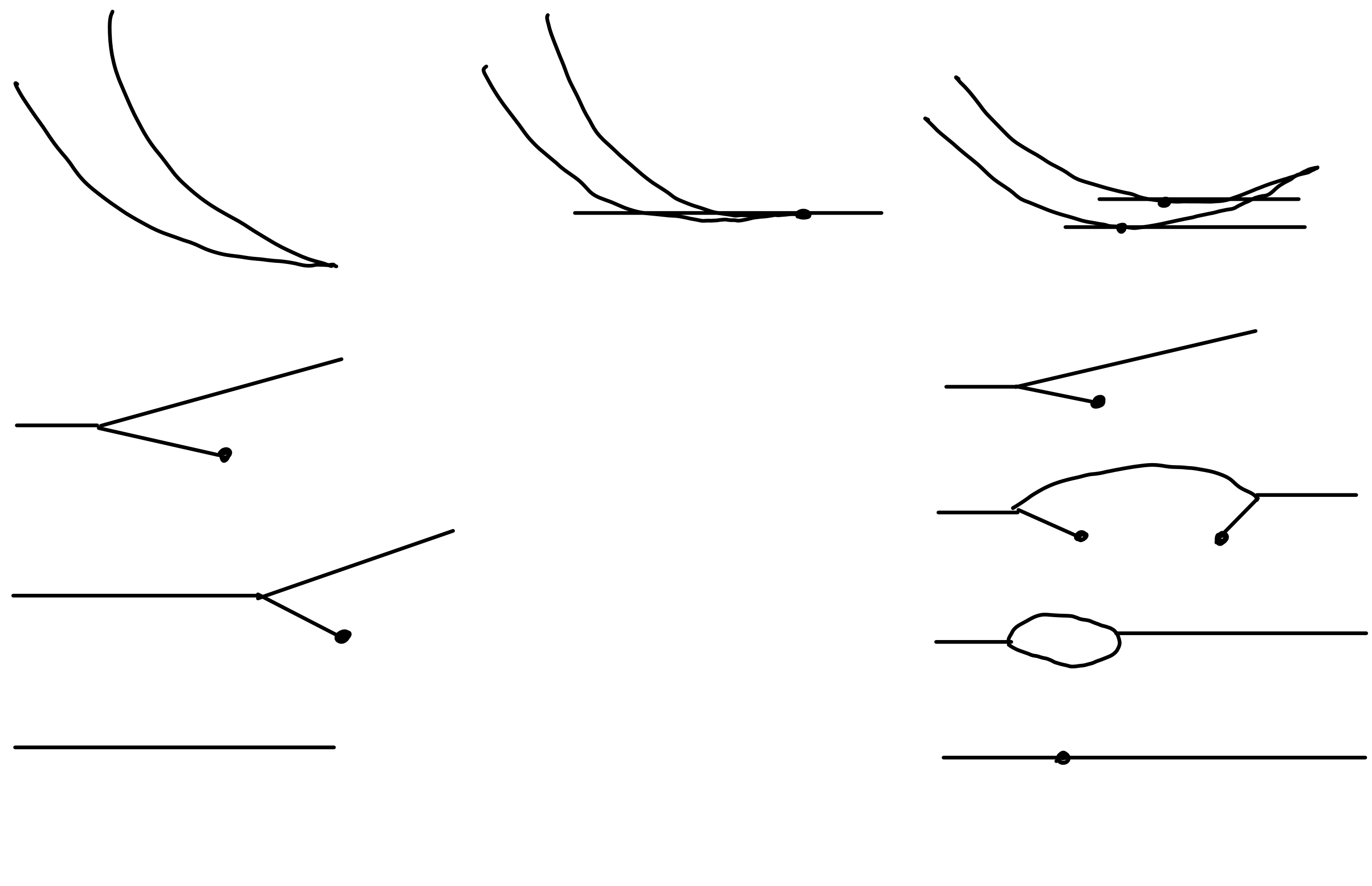
TYPE 1 CUSPS



Rotating passes the horizontal tangency from one edge to the other. In the graphs, the upper edge is of OFPs.

Rotating trough a type one cusp does not change the HS.

TYPE 2 CUSPS



Rotating either creates or removes two horizontal tangencies. In the graphs, the upper edge is of OFPs.

Rotating through a type two cusp induces a stabilization or destabilization of the HS.

<u>Thm (Reidemeister-Singer)</u> Let M be a closed, connected 3-manifold. Any two Heegaard splittings of M share a common stabilization (up to isotopy).

<u>Pf</u> Let f and g be Morse functions that induce Σ_1 and Σ_2. $\varphi(t) = g\cos t + f\sin t$.

Isotope f and g so that in the graphic of $f \times g$ there are finitely many points where the second derivative is zero, and finitely many doubly tangent straight lines.

Then, there are finitely many angles $t_1 < \dots < t_n < \frac{\pi}{2}$ st rotating the graph by t_i creates a horizontal inflection point, a horizontal cusp or two horizontal tangents at the same level.

Now, for $t \in (t_i, t_{i+1})$, $\varphi(t)$ is a Morse function, and they are all isotopic.

The HSs induced by $\varphi|[0,t_1)$ are all isotopic to Σ_2. If at t_1 we produce a horizontal inflection point in an IFE or a type 2 cusp, the HSs induced by $\varphi|(t_1,t_2)$ are a single stabilization or destabilization. We iterate this, and when we get to $\varphi|(t_n, \frac{\pi}{2}]$, we get to the isotopy class of Σ_1. $\square$

<u>Corollary</u> If c is the number of negative slope inflection points and type 2 cusps, then there is a common stabilization of genus $\dfrac{g(\Sigma_1) + g(\Sigma_2) + c}{2}$.