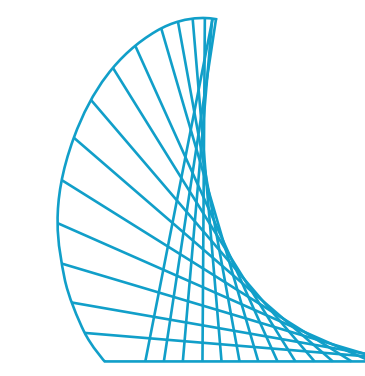


Untwisting the twisted character variety via stacky curves

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Motivation

For a smooth projective complex curve X , the non-abelian Hodge correspondence can be understood as relating the geometry of the generic fiber and the special fiber inside the moduli space of λ -connections on X

$$\begin{array}{ccccccc} \mathcal{M}_{n,Dol}^0(X) & \longleftrightarrow & \mathcal{M}_{n,Hdg}^0(X) & \longleftrightarrow & \mathcal{M}_{n,DR}^0(X) & \xrightarrow{\cong} & \mathcal{M}_{n,B}^0(X) = Loc_n(X) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\ \{0\} & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \{1\} & & \end{array}$$

Problem. Degree d vector bundles don't admit flat connections. How to get a good notion of $\mathcal{M}_{n,DR}^d(X)$ that fits into a similar description?

The stacky curve $\mathcal{X}$

Fix a point $p \in X$. We define the stacky curve $\mathcal{X}$ to be the n -th root stack of X at p

$$\begin{array}{ccc} \mathcal{X} = \sqrt[n]{p/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \pi \downarrow & \lrcorner & \downarrow \wedge n \\ X & \xrightarrow{(\mathcal{O}(p), s_p)} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Degree d on a curve as degree 0 on $\mathcal{X}$

Vector bundles on $\mathcal{X}$ have a third numerical invariant beyond rank and degree, namely the representation obtained as stalk at the stacky point. We denote by $w(-d)$ the weight $-d$ character of μ_n and by $n \cdot w(-d)$ the corresponding n -dimensional representation.

Theorem. The twisted pullback and pushforward along the moduli map $\mathcal{X} \rightarrow X$ define equivalences of algebraic stacks

$$\mathcal{M}_{n,Dol}^d(X) \cong \mathcal{M}_{n \cdot w(-d), Dol}^0(\mathcal{X}).$$

Upshot. Degree d Higgs bundles on X can be understood as degree 0 Higgs bundles on $\mathcal{X}$.

Twisted character variety as Betti moduli space on $\mathcal{X}$

Computing the fundamental group of $\mathcal{X}$, we see that the Betti moduli space on $\mathcal{X}$ is

$$\mathcal{M}_{n,B}(\mathcal{X}) = \left[\left\{ ((A_i, B_i)_{i=1}^g, C) \in GL_n^{2g+1} \mid \prod_{i=1}^g [A_i, B_i] = C, C^n = Id \right\} / GL_n \right].$$

It decomposes with respect to the conjugacy class of C , and the component corresponding to $\mathcal{M}_{n \cdot w(-d), Dol}^0(\mathcal{X})$ is

$$\mathcal{M}_{n \cdot w(-d), B}(\mathcal{X}) = \left[\left\{ (A_i, B_i)_{i=1}^g \in GL_n^{2g} \mid \prod_{i=1}^g [A_i, B_i] = e^{-\frac{2\pi i}{n} d} Id \right\} / GL_n \right].$$

This is exactly the twisted character variety of X .

Upshot. The twisted character variety of X is a connected component of the character variety of $\mathcal{X}$.

Consequence

We can identify $\mathcal{M}_{n,DR}^d(X)$ with $\mathcal{M}_{n \cdot w(-d), DR}^0(\mathcal{X})$. Therefore, the inclusions $\mathcal{M}_{n,Dol}(\mathcal{X}) \hookrightarrow \mathcal{M}_{n,Hdg}(\mathcal{X})$ and $\mathcal{M}_{n,DR}(\mathcal{X}) \hookrightarrow \mathcal{M}_{n,Hdg}(\mathcal{X})$ and the (analytic) Riemann-Hilbert correspondence $\mathcal{M}_{n,DR}(\mathcal{X}) \cong \mathcal{M}_{n,B}(\mathcal{X})$ induce isomorphisms in cohomology

$$H^*(\mathcal{M}_{n,Dol}^d(X)) \cong H^*(\mathcal{M}_{n,DR}^d(X)) \cong H^*(\mathcal{M}_{n,B}^d(X)).$$

Upshot. The non-abelian Hodge correspondence for coprime n, d on X can be understood from the non-abelian Hodge correspondence for 0-degree bundles on $\mathcal{X}$.

What about the Hodge structure?

The equivalence between the De Rham and the Betti moduli space is analytic, so it doesn't preserve the Hodge structure.

Theorem (Shende [1]). The tautological class $e_{i,j}$ (i.e. the j -th Künneth component of $c_i(\mathcal{E}_{univ})$) on $\mathcal{M}_{n,Dol}^0(X)$ has pure weight j . Under the isomorphism $\mathcal{M}_{n,DR}^0(X) \cong \mathcal{M}_{n,B}(X)$ it is mapped to a cohomology class of weight $2i$.

Computing the Hodge weights

To compute the Hodge weights we replicate Shende's argument presented in [1]. We construct a simplicial set $\Delta_{\mathcal{X}}$ realizing a triangulation of $\mathcal{X}$. We make it into a simplicial scheme and use the equivalence

$$\mathcal{M}_{n,B}(\mathcal{X}) = [\mathrm{Hom}(\pi_1(\mathcal{X}, v_0), GL_n) / GL_n] \cong \mathrm{Hom}_{SSch}(\Delta_{\mathcal{X}}, BGL_n)$$

and the universal object on $\mathrm{Hom}_{SSch}(\Delta_{\mathcal{X}}, BGL_n)$ to carry out the computation of the weights.

Theorem. The tautological class $e_{i,j}$ on $\mathcal{M}_{n,Dol}^0(\mathcal{X})$ has pure weight j . Under the isomorphism $\mathcal{M}_{n,DR}^0(\mathcal{X}) \cong \mathcal{M}_{n,B}^0(\mathcal{X})$ it is mapped to a class of weight $2i$.

References

- [1] Vivek Shende. "The weights of the tautological classes of character varieties". In: *Int. Math. Res. Not. IMRN* 22 (2017), pp. 6832–6840. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnv363. URL: <https://doi.org/10.1093/imrn/rnv363>.
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