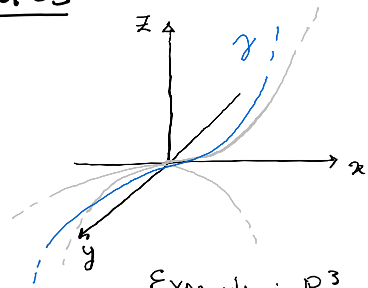
# Decoupling inequalities

## 1. The moment curve

$$\gamma(t) = \left(t, \frac{t^2}{2}, \dots, \frac{t^k}{k!}\right)$$

$$\gamma : [0,1] \longrightarrow \mathbb{R}^k$$




Example in $\mathbb{R}^3$

Cubic on the $z$–$x$

parabola on the $x$–$y$

### 1.1 Affine self-similarity

Let $J = [a, a+\delta] = \{a + \delta t,\ t \in [0,1]\} \subseteq [0,1]$

This interval is the image of $[0,1]$ via the affine map which dilates by $h$ and translates by $a$.

Does the same hold for the sets $\gamma(J)$ and $\gamma([0,1])$ ?

**Lemma 1** For any $a, \delta \in [0,1]$ there exists a linear map $A_{a,\delta} : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$A_{a,\delta}\big(\gamma([0,1])\big) + \gamma(a) = \gamma\big([a, a+\delta]\big)$$

## Proof of Lemma 1

Idea: expand $\gamma$ in (full) Taylor series:

$$\gamma(a+\delta t) = \gamma(a) + \sum_{j=1}^{k} \gamma^{(j)}(a)\, \delta^j \frac{t^j}{j!}$$

where $\gamma^{(j)}(a) = \left(\gamma_1^{(j)}(a), \dots, \gamma_k^{(j)}(a)\right)$.

$$\sum_{j=1}^{k} \gamma^{(j)} \delta^j \frac{t^j}{j} = \begin{bmatrix} \gamma_1^{(1)} & \gamma_1^{(2)} & & \gamma_1^{(k)} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_k^{(1)} & \gamma_k^{(2)} & & \gamma_k^{(k)} \end{bmatrix} \begin{bmatrix} \delta\, t \\ \delta^2 t^2/2 \\ \vdots \\ \delta^k\, t^k/k! \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1^{(1)} & \gamma_1^{(2)} & & \gamma_1^{(k)} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_k^{(1)} & \gamma_k^{(2)} & & \gamma_k^{(k)} \end{bmatrix} \begin{bmatrix} \delta & & & \\ & \delta^2 & & \\ & & \ddots & \\ & & & \delta^k \end{bmatrix} \gamma(t)$$

$$=: \quad M_a \quad D_\delta \quad \gamma(t) \quad =: A_{a,\delta}\, \gamma(t)$$

then $A_{a,\delta}\gamma(t) + \gamma(a) = \gamma(a+\delta t) = A_{a,\delta}\gamma(t)$

$A_{a,\delta}(\gamma([0,1])) = \gamma(J)$ $\square$

Moral: given an interval $J$ at scale $\delta \in (0,1)$

the affine transformation $A_{a,\delta} : \gamma(J) \to \gamma([0,1])$.

What if we use this transformation with a different scale?

We can map a covering of $\gamma(I)$ for an interval $I \in P(\eta)$

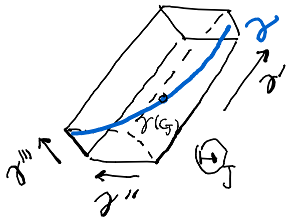
made with $\gamma(J)$ for $J \in P(\delta, I)$, $0<\delta<\eta<1$,

to a covering of the full $\gamma([0,1])$.

## 1.2 Covering $\gamma(I)$ with parallelepipeds

Fix a scale $\delta \in (0,1)$, let $P(\delta)$ be a partition of $[0,1]$ in intervals $J$ of lenght $\delta$.

Given $J \in P(\delta)$ of centre $C_J$ consider the parallelepiped $\Theta_J$ with centre $\gamma(C_J)$ and sides $\delta \times \delta^2 \times \dots \times \delta^k$ parallel to the vectors $\{\gamma^{(j)}(C_J)\}_{j=1,\dots,k}$




We need to understand how the map $A_{a,\delta}$ acts on on $\Theta_J$, and so on the vectors $\gamma^{(j)}(\cdot)$.

How does self-similarity work on derivatives $\gamma^{(j)}$?

$$\gamma(t) = \left(t, \frac{t^2}{2}, \dots, \frac{t^k}{k!}\right)$$

$$\gamma'(t) = \left(1, t, \frac{t^2}{2}, \dots, \frac{t^{k-1}}{(k-1)!}\right)$$

$$\gamma^{(j)}(t) = (\underbrace{0,\dots,0}_{(j-1)\text{ zeros}}, \underset{j\text{th}}{\underset{\uparrow}{1}}, \underbrace{t, \dots, \frac{t^{k-j}}{(k-j)!}}_{(k-j)\text{ components of } \gamma(t)})$$

Consequences: Let $M_a = \left[\gamma^{(1)}(a) \middle| \dots \middle| \gamma^{(k)}(a)\right]$ $(k \times k)$ matrix

(•) $\det M_a = \det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} = 1$

Consequences: Let $M_a = \left[\gamma^{(1)}(a) \mid \dots \mid \gamma^{(k)}(a)\right]$ $(k\times k)$ matrix

(·) $\det M_a = \det \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \\ \Delta & & 1 \end{bmatrix} = 1$

$$(\cdot\cdot) \quad M_a\, \gamma^{(j)}(t) = M_a \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ t \\ \vdots \end{bmatrix} = \sum_{i=j}^{k} \gamma^{(i)}(a) \left(\gamma^{(j)}(t)\right)_i$$

proof of (··)

$$M_a \gamma^{(j)}(t) = \begin{bmatrix} \gamma_1^{(1)}(a) & \gamma_1^{(2)}(a) & \dots & \gamma_1^{(j)}(a) & \dots & \gamma_1^{(k)}(a) \\ \vdots & \vdots & \dots & \vdots & & \vdots \\ \gamma_k^{(1)}(a) & \gamma_k^{(2)}(a) & \dots & \gamma_k^{(j)}(a) & \dots & \gamma_k^{(k)}(a) \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ t \\ \vdots \\ \frac{t^{k-j}}{(k-j)!} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1^{(j)}(a)\,1 + \gamma_1^{(j+1)}(a)\,t + \dots \\ \vdots \\ \gamma_k^{(j)}(a)\,1 + \gamma_k^{(j+1)}(a)\,t + \dots \end{bmatrix} = \sum_{i=j}^{k} \begin{bmatrix} \gamma_1^{(i)}(a) \\ \vdots \\ \gamma_k^{(i)}(a) \end{bmatrix} \left(\gamma^{(j)}(t)\right)_i$$

□

This implies the following

Lemma 2 (self-similarity of derivatives)

$$\gamma^{(j)}(a+\delta t) = \delta^{-j}\, M_a\, D_\delta\, \gamma^{(j)}(t)$$

proof

$$\gamma^{(j)}(a+\delta t) = \gamma^{(j)}(a) + \sum_{m=1}^{k-j} \gamma^{(j+m)}(a)\, \frac{(\delta t)^m}{m!}$$

$(m = i-j)$

$$= \sum_{i=j}^{k} \gamma^{(i)}(a)\, \frac{(\delta t)^{i-j}}{(i-j)!}$$

$$= \sum_{i=j}^{k} \gamma^{(i)}(a)\, \frac{1}{\delta^j} \left(D_\delta\, \gamma^{(j)}(t)\right)_i$$

$$\overset{(\cdot\cdot)}{=} \delta^{-j}\, M_a\, D_\delta\, \gamma^{(j)}(t)$$

□

## 1.3 Transforming $\Theta_J$

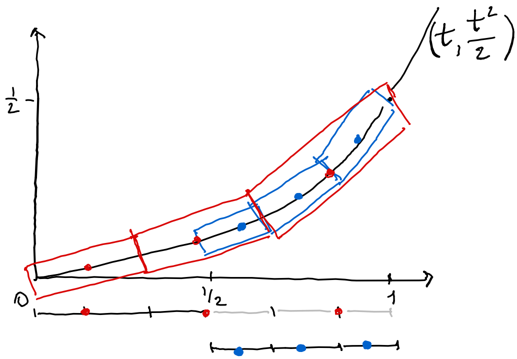
Fix scales $0<\delta<\eta<1$. Let $I\in P(\eta)$ and $J\in P(\delta,I)$.

Recall that $A_\eta^{-1}(\gamma(I)) = \gamma([0,1])$.

**Lemma 3** Consider $\Theta_J$ the parallelepiped over $\gamma(J)$. Then

$$\{A_\eta^{-1}\Theta_J\}_{J\in P(\delta,I)} = \{\Theta_{\tilde J}\}_{\tilde J\in P(\delta/\eta)}$$

**Example** $(k=2)$




$\eta=\frac{1}{2}$ , $\delta=\frac{1}{6}$

$I=[\frac{1}{2},1]$

$c_J$ = center of $J$

$s_J = c_J/\eta$

centers of the new $\tilde J$.

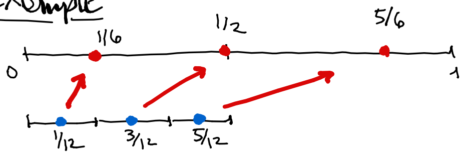
## Proof of Lemma 3

Let $C_J$ be the center of $J$. They are a $\delta$-net in $I$,

$$d(C_{J_1}, C_{J_2}) \geq \delta$$

Then $\{ S_J := C_J/\eta \}$ are a $\delta/\eta$-net in $[0,1]$.

Example




$I = [0, 1/2]$, $\eta = \frac{1}{2}$

$\delta = \frac{1}{6}$

$C_J = \{ \frac{1}{12}, \frac{3}{12}, \frac{5}{12} \}$ $\qquad$ $S_J = \{ \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \}$

$$\xi \in \Theta_J, \quad \xi = \gamma(C_J) + \sum_{j=1}^{k} \gamma^{(j)}(C_J)\, r_j(\xi), \quad |r_j| \leq \delta^j$$

$C_J = a + S_J \eta$, sub in & use $\gamma^{(j)}(a + S_J\eta) = \eta^{-j} M_a D_\eta \gamma^{(j)}(S_J)$

$$= \eta^{-j} A_{a,\eta} \gamma^{(j)}(S_J)$$

$$\xi = \gamma(a + S_J\eta) + \sum_{j=1}^{k} \frac{r_j(\xi)}{\eta^j} M_a D_\eta \gamma^{(j)}(S_J)$$

then

$$A_{a,\eta}^{-1} \xi = \underbrace{A_{a,\eta}^{-1} \gamma(a + S_J\eta)}_{\gamma(S_J)} + \sum_{j=1}^{k} \underbrace{e_j(\xi)}_{\text{s.t. } |e_j| \leq \left(\frac{\delta}{\eta}\right)^j} \gamma^{(j)}(S_J)$$

define

$$\Theta_{\tilde{J}} := \left\{ \gamma(S_J) + \sum_{j=1}^{k} e_j \gamma^{(j)}(S_J), \ |e_j| \leq \left(\frac{\delta}{\eta}\right)^j \right\}$$

$\square$

## 2. Decoupling constants

**Def** (Decoupling constants)

$$\Big\| \sum_{J \in P(\delta)} f_J \Big\|_{L^p(\mathbb{R}^k)} \le C \Big( \sum_{J \in P(\delta)} \| f_J \|^2_{L^p(\mathbb{R}^k)} \Big)^{1/2}$$

$$\mathcal{D}_{k,p}(\delta) = \inf \{ C \text{ above s.t. } \le \text{ holds } \forall f \in \mathcal{S}(\mathbb{R}^k) \text{ w/ } \operatorname{supp} \widehat{f_J} \subseteq \Theta_J \}.$$

**Thm** (Bourgain-Demeter-Guth, Annals 2016)

For all $k \in \mathbb{N}$, and all $\delta > 0$

$$\mathcal{D}_{k,p}(\delta) \lesssim_\varepsilon \frac{1}{\delta^\varepsilon} \quad \forall \varepsilon > 0, \forall p \in [2, \underbrace{k(k+1)}_{=: p_k}]$$

**Lemma 4** (Rescaling of $\mathcal{D}$)

Fix $0 < \delta < \eta < 1$, consider $I \in P(\eta)$ and $J \in P(\delta, I)$.

Let $\{f_J\}_{J \in P(\delta, I)}$ such that $\operatorname{supp} \widehat{f_J} \subseteq \Theta_J$, then

$$\Big\| \sum_{J \in P(\delta, I)} f_J \Big\|_{L^p(\mathbb{R}^k)} \le \mathcal{D}_{k,p}(\delta/\eta) \Big( \sum_{J \in P(\delta, I)} \| f_J \|^2_{L^p} \Big)^{1/2}, \forall p \in [2, \infty).$$

Consequence:

Given $I \in P(\eta)$ we have decoupling at scale $\eta$:

$$\Big\| \sum_{I \in P(\eta)} f_I \Big\|_{L^p} \leq \mathcal{D}_{k,p}(\eta) \Big( \sum_{I \in P(\eta)} \| f_I \|_{L^p}^2 \Big)^{1/2}$$

by decomposing $f_I = \sum_{J \in P(\delta, I)} f_J$ and applying rescaling

$$\ldots \leq \mathcal{D}_{k,p}(\eta) \Big( \sum_{I \in P(\eta)} \mathcal{D}_{k,p}(\delta/\eta)^2 \sum_{J \in P(\delta, I)} \| f_J \|_{L^p}^2 \Big)^{1/2}$$

which gives that, for $0 < \delta < \eta < 1$

$$\mathcal{D}_{k,p}(\delta) \leq \mathcal{D}_{k,p}(\eta)\, \mathcal{D}_{k,p}(\delta/\eta).$$

Example This implies that for $\delta = \frac{1}{2^{n+1}}$, $\eta = \frac{1}{2}$ we have

$$\mathcal{D}(1/2^{n+1}) \leq \mathcal{D}(1/2)\, \mathcal{D}(1/2^n) \leq \ldots \leq \mathcal{D}(1/2)^n\, \mathcal{D}(1/2)$$

if $\mathcal{D}(1/2) \lesssim_\varepsilon (1/2)^{-\varepsilon}$ $\forall \varepsilon > 0$ then

$$\mathcal{D}(1/2^{n+1}) \leq (C_\varepsilon 2^\varepsilon)^n\, C_{\varepsilon_0} 2^{\varepsilon_0}$$