Minimax methods and geodesics

Seminar of Calculus of Variations

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We are interested in finding a solution to a (also nonlinear) differential equation

$$F[u] = 0 \tag{1}$$

where F is a functional on some space of function V. Using variational methods, if F could be represented as a differential of another functional A, this means

$$A' = F$$

we can now look at *critical points* of A for a solution of our problem (1).
Here, instead of looking for critical points of A that are minimals, we look for critical points that are saddle points.

1 Mountain Pass Theorem and generalizations

To be more clear; we start from a finite dimensional case.
Let V be a finite vector space with a norm $\|\cdot\|$, and $f\colon V \to \mathbb{R}$ a function.

Definition 1. A function f is said to be *coercive* if $\lim f(x) = +\infty$ as $\|x\| \to +\infty$.

Notation 1. In the following, we will indicate the *sublevel* of f with

$$\{f < c\} = \{x \in V \text{ such that } f(x) < c\}.$$

On a finite dimensional space, the coercivity of f has implication on the topology of sublevels, in fact:

Proposition 1. *Let V be a finite dimensional normed vector space and $f \in \mathcal{C}^1(V)$. If f is coercive the sublevels $\{f < c\}$ are precompact.*

Proof. We want to show that for all $c \in \mathbb{R}$ the closed set $\overline{\{f < c\}}$ is compact in V.
Let $\{v_1, \ldots, v_n\}$ be a unit basis of V, and $B_r = \{x \in V : \|x\| < r\}$. Given $c \in \mathbb{R}$, by coercivity of f, there exist $\bar{r} = \min\{r \in \mathbb{R} : f(rv_i) > c \text{ for } i = 1, \ldots, n\}$. It follows that the sublevel $\{f < c\} \subseteq B_{\bar{r}}$, so its closure is contained in the closed ball $\overline{B}_{\bar{r}}$. This shows that the closure of $\{f < c\}$ is bounded, so it is compact in V. □

Now, we can deduce the existence of a critical point for f with some topological hypothesis on a sublevel $\{f < c\}$.

Notation 2. Given two open sets $A, B \subset V$, we indicate with Γ the family of differential paths from A to B, that is

$$\Gamma = \{\gamma\colon [0,1] \to V,\ \gamma(0) \in A,\ \gamma(1) \in B\}\,.$$

Notation 3. If V is endowed with a dot product $\langle \cdot\,, \cdot \rangle$ we can identify the linear application $\mathrm{d}f_x$ as the dot product with an element of V depending from the point x, that we call $\nabla f(x)$. We mean that

$$\mathrm{d}f_x(v) = \langle \nabla f(x)\,, v\rangle \qquad \forall\, v \in V.$$

Theorem 1 (Mountain Pass)**.** *Let V be a finite dimensional vector space endowed with a dot product $\langle \cdot\,, \cdot \rangle$ and let $f\colon V \to \mathbb{R}$ be a smooth [1] function. Suppose that f is coercive and that the sublevel $\{f < c\}$ is disconnected, i.e. there exist $A, B \subset V$ such that*

$$\{f < c\} = A \cup B,\ A \cap B = \varnothing.$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{x\in\gamma} f(x)$$

is a critical value for f.

Proof. We argue by contradiction. Assume c is not a critical value for f, so $\nabla f(x) \neq 0$ for every x in $\{f = c\}$. Moreover, the sublevel $\{f = c\}$ is compact, because the coercivity of f. The vector field ∇f is continuous differentiable and it is never zero on a *compact* set, so ∇f will be bounded away from zero[2] on $\{c - \epsilon \leqslant f \leqslant c + \epsilon\}$ for some $\epsilon > 0$.
The the Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(u,t) & = -\nabla f(\phi(u,t)) \\ \phi(u,0) & = u \end{cases}$$

defines a flow $\Phi(u,t) = \psi_t(u)$ that moves the sublevel $\{f \leqslant c + \epsilon\}$ into the sublevel $\{f \leqslant c - \epsilon\}$ in a finite time. In fact, suppose $f(u) - f(\psi_t(u)) < 2\epsilon$, then

$$f(u) - f(\psi_t(u)) = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\psi_s(u))\,\mathrm{d}s = \int_0^t \|\nabla f(\psi_s(u))\|^2\,\mathrm{d}s \geqslant t\delta^2,$$

thus, starting from u in $\{c - \epsilon \leqslant f \leqslant c + \epsilon\}$, the image $\psi_t(u)$ remains in $\{c - \epsilon \leqslant f \leqslant c + \epsilon\}$ only for $t < 2\epsilon/\delta^2$. For larger t, the flow ψ_t moves down u in the sublevel $\{f < c - \epsilon\}$. In the same way, every point of a path $\gamma(s)$ is moved down in the sublevel $\{f < c - \epsilon\}$ by the flow ψ_t, while $\gamma(0)$ and $\gamma(1)$ remain in A and B respectively. So, the new path $\tilde{\gamma} = \psi_t(\gamma)$ belongs to Γ yet, but

$$c < \max_{x\in\tilde{\gamma}} f(x) < c - \epsilon < c$$

and this is absurd. Then exists a critical point x at level c. □

[1] For our scope, it is enough take $f \in \mathcal{C}^2(V)$, to have Lipschitz continuity for the gradient and so generate a flow.

[2] i.e. $\|\nabla f(x)\|_V \geqslant \delta > 0$

1.1 On Hilbert Space

Now we want to generalize this result to an infinite dimensional vector space.
In order to bounded away from zero the gradient ∇f on a open set containing the preimages $\{f = c\}$, in the last proof we exploited the compactness of the sublevel of f. But, in an infinite dimensional space, coercivity is not enough to have compact sublevel:

Counterexample 1. The norm $\|\cdot\|$ on a ∞-dimensional vector space H is coercive, but its sublevels

$$\overline{B}_r = \{x \in H \colon \ \|x\| \leqslant r\}$$

are never closed in the strong topology of H.

In order to have our implication, we will use a new assumption on f, due to Richard Palais and Stephen Smale.

Definition 2 (Palais-Smale condition)**.** A smooth function $f \colon H \to \mathbb{R}$ on a Hilbert space H satisfies Palais-Smale condition if any sequence $\{u_m\}_{m\in\mathbb{N}}$ in H such that:

(i) $f(u_m) \leqslant c \;$ as $m \to \infty$

(ii) $\nabla f(u_m) \to 0 \;$ in H

is a *precompact* sequence.

This means that $\overline{\{u_m\}_{m\in\mathbb{N}}}$ has a convergent subsequence.

Remark 1. We want to stress item (ii): the $\nabla f \to 0$ as element in H. Equivalently $\|\nabla f\|_H \to 0$ in $\mathbb{R}$.

If f satisfies P.S., we have again the implication:

$$\nabla f \neq 0 \;\text{ on } A \quad \Rightarrow \quad \nabla f \neq 0 \;\text{ on an open set } A_\epsilon \supsetneq A$$

We can now state the Mountain Pass Theorem for Hilbert infinite dimensional vector space.

Theorem 2 (Mountain Pass, Hilbert ∞-dimension)**.** *Let H be an Hilbert space and $f \colon H \to \mathbb{R}$ a smooth functional satisfying P.S. condition. If a sublevel $\{f < a\}$ is not connected, then*

$$f(u) = c = \inf_{\gamma\in\Gamma} \sup_{x\in\gamma} f(x)$$

is a critical value for f.

Remark 2. Again, the weaker condition $f \in \mathcal{C}^2(H)$ it is enough to have $\nabla f \in \mathcal{C}^1(H)$, so that ∇f is Lipschitz continuous and we can use the flow generated by ∇f.

1.2 On Banach Space

We try to generalize again the result on a Banach space V.
Here, in general, we can no longer identify $\mathrm{d}f \in V^\star$ as the inner product with a vector field ∇f in a canonical way, so we have to adapt a bit the Palais-Smale condition and to use a different, locally Lipschitz, vector field.

Definition 3 (Palais-Smale condition for Banach space)**.** A functional $f \in \mathcal{C}^2(V)$ on a Banach space V satisfies Palais-Smale condition if any sequence $\{u_m\}_{m\in\mathbb{N}}$ such that:

(i) $f(u_m) \leqslant c$ as $m \to \infty$

(ii) $\mathrm{d}f_{u_m} \to 0$ **in** $V^\star$

is a *precompact* sequence.

This means that $\mathrm{d}f \to 0$ in the dual space $V^\star$, or, equivalently, $\|\mathrm{d}f\|_{V^\star} \to 0$ in $\mathbb{R}$.

Definition 4 (Pseudo-gradient vector field)**.** Given a function $f\colon V \to \mathbb{R}$, a *pseudo-gradient vector field* for f is a vector field defined on the complement of critical points

$$X\colon V\backslash\mathrm{Crit}(f) \to V$$

such that for every $u \in V$ the two conditions hold:

(a) $\|X(u)\|_V < 2\min\{\|\mathrm{d}f_u\|_{V^\star}, 1\}$

(b) $\mathrm{d}f_u(X(u)) > \|\mathrm{d}f_u\|_{V^\star}\ \min\{\|\mathrm{d}f_u\|_{V^\star}, 1\}$

Remark 3. The meaning of the bound above is the following:

$\|\mathrm{d}f_u\|_{V^\star}\ \min\{\|\mathrm{d}f_u\|_{V^\star}, 1\} < \mathrm{d}f_u(X(u))$, by (b)
and by (a): $\mathrm{d}f_u(X(u)) \leqslant \|\mathrm{d}f_u\|_{V^\star}\ \|X(u)\|_V \leqslant \|\mathrm{d}f_u\|_{V^\star}\, 2\min\{\|\mathrm{d}f_u\|_{V^\star}, 1\}$,

so when $\mathrm{d}f_u \neq 0$, this leads to

$$\min\{\|\mathrm{d}f_u\|_{V^\star}, 1\} < \|X(u)\|_V < 2\min\{\|\mathrm{d}f_u\|_{V^\star}, 1\}$$

Note 1. The number 2 is not really important, for example other sources use

$$\frac{1}{2}\min\{\|\mathrm{d}f_u\|_{V^\star}, 1\} < \|X(u)\|_V < \min\{\|\mathrm{d}f_u\|_{V^\star}, 1\}.$$

What really matters it is that X turns out to be a locally Lipschitz vector field.

A remarkable result on a Banach space V, but also for only $\mathcal{C}^1$ functional on Hilbert space, is the following:

Lemma 1. *Any $f \in \mathcal{C}^1(V)$ admits a pseudo-gradient vector field.*

Proof. Consider $\tilde{V} = V \backslash \mathrm{Crit}(f)$ and a vector field $w\colon \tilde{V} \to V$. We take a cover $\{W(u)\}_{u \in \tilde{V}}$ made up by neighbourhoods $W(u)$ of u such that (a) and (b) holds for the field w for every $x \in W(u)$. The space $\tilde{V}$ is a (complete) metric space, so it is *paracompact*, and for every cover $\{W(u)\}$ there is a locally finite refinement $\{W_i\}_{i \in I}$ such that $W_i \subset W(u_i)$. We consider a Lipschitz partition of unity $\{\varphi_i\}_{i \in I}$ subordinate to the cover $\{W_i\}_{i \in I}$.
To sum up, we have the new pseudo-gradient field:

$$v(u) = \sum_{i \in I} \varphi_i(u) w(u_i).$$

□

Remark 4. Considering a cut-off function $\eta\colon V \to \mathbb{R}$ that is zero in a neighbourhood of $\mathrm{Crit}(f)$. We can extend the pseudo-gradient field X to a globally defined vector field

$$\begin{aligned} \tilde{X}\colon V &\to V \\ u &\mapsto \eta(u)\, X(u). \end{aligned}$$

Given a vector field v, we can also consider the bounded vector field:

$$w(u) = \eta(u) \frac{v(u)}{\sqrt{1 + \|v(u)\|^2}} \qquad \text{or} \qquad w(u) = \eta(u)\, \varphi(f(u))\, v(u)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function, i.e. $0 \leqslant \varphi \leqslant 1$. This two fields are globally defined and *bounded* on V, thus the flow generated by them is complete.

Moreover, we never use the fact that Γ was a family of curves, but only that Γ contains *invariant sets* respect to the flow. This leads to the following

Definition 5 (Invariant set)**.** Given a flow $\Phi\colon V \times \mathbb{R} \to V$, a family Γ of subsets of V is *positively invariant* for Φ if

$$\Phi(\gamma, t) \in \Gamma, \quad \forall\, \gamma \in \Gamma, \forall\, t \geqslant 0.$$

Theorem 3 (Minimax principle)**.** *Let f be a $\mathcal{C}^1$ functional satisfy P.S. on a Banach space V. Suppose Γ is a positive invariant set for the flow $\Phi\colon V \times \mathbb{R} \to V$ generated by a pseudo-gradient vector field related to f. Then if*

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} f(u)$$

is finite, then c is a critical value for f.

Example 1. Changing the family Γ we obtain classical inf and sup on V:

- If we take $\Gamma = \{V\}$ as the whole set V, then

$$\inf_{V \in V} \sup_{u \in V} f(u) = \sup_{u \in V} f(u).$$

- If we take $\Gamma = \{\{u\} : u \in V\}$ as the set of all point in V, then

$$\inf_{u \in V} \sup_{u \in u} f(u) = \inf_{u \in V} f(u).$$

1.3 On Hilbert Manifolds

An Hilbert manifold M is a manifold modeled on an Hilbert space H. More precisely

Definition 6. An Hilbert manifold M is an Hausdorff topological space with a countable base endowed with differentiable atlas where the charts take values in a fixed *separable* Hilbert space.

See Klingenberg [Kli12] for further details.

Example 2. An Hilbert vector space is an Hilbert manifold with the only chart (id, H).

Example 3. An open subset $U \subset H$ has a natural Hilbert manifold structure.

If H is a vector space, for every $p \in H$ the tangent space $T_pH \cong H$, thus we can endow an Hilbert space with a Riemann metric g. This metric is simply the inner product of H on every tangent space:

$$g_p(u, v) := \langle u\,, v\rangle_H \quad \forall p \in H,\, \forall u, v \in T_pH.$$

1.4 On Finsler Manifolds

In order to generalize again, we consider a Banach manifold M, that is a manifold in which the charts have their image in a Banach space V.
We take a Banach vector bundle F over M. A Finsler structure on F is a map

$$\|\cdot\| : F \to \mathbb{R}$$

such that:

- $\|\cdot\|_u = \|\cdot\| \restriction_{F_u} : F_u \to \mathbb{R}$ is an admissible norm for the Banach space $\pi^{-1}(u) = F_u$,
- taking $u_0 \in M$, the trivialisation on a neighbourhood U of u_0

  $$\chi \colon \pi^{-1}(U) \to U \times F_{u_0}$$

  the $\|\cdot\|_u$ is a norm on F_{u_0} for every $u \in U$.

Consider a manifold M with a Finsler structure on the tangent bundle TM. The map

$$\|\cdot\| : TM \to \mathbb{R}$$

induces a Finsler structure on the co-tangent bundle $T^\star M$. In fact, given $\varphi \in T^\star M$, there exist $u \in M$ such that $\varphi \in (T^\star M)_u = (T_uM)^\star$, so

$$\begin{aligned} \|\cdot\|^\star : T^\star M &\to \mathbb{R} \\ \varphi &\mapsto \|\varphi\|^\star = \|\varphi\|_u^\star \end{aligned}$$

where $\|\cdot\|_u^\star$ is the norm induced on the dual space $(T_uM)^\star$ by the norm $\|\cdot\|_u$ on $T_uM \cong V$, namely

$$\|\varphi\|_u^\star = \sup_{\substack{v \in T_uM \\ \|v\|_u \leqslant 1}} |\varphi(v)|\,.$$

Finally, the map $\|\cdot\|$ induces a distance on M:

$$d(u,v) = \inf_{\gamma\in\Gamma} \int_0^1 \left\| \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \right\|_{\gamma(t)} \mathrm{d}t$$

where $\Gamma = \{\gamma\colon [0,1] \to M\,,\ \gamma(0) = u, \gamma(1) = v\}$. A sort of "infimum length" among paths connecting u and v. We will say that M is *complete* if M is complete regard to this metric.

There is an analogous of pseudo-gradient vector field also on a M Finsler manifold:

Definition 7 (Pseudo-gradient vector field on manifolds)**.** Given a function $f\colon M \to \mathbb{R}$, a *pseudo-gradient vector field* for f is a vector field

$$v\colon M\backslash\mathrm{Crit}(f) \to TM$$

such that for every $u \in \tilde{M} = M\backslash\mathrm{Crit}(f)$ the two conditions hold:

(a) $\|v(u)\|_u < 2\min\{\|\mathrm{d}f_u\|_u^\star, 1\}$

(b) $\mathrm{d}f_u(v(u)) > \min\{\|\mathrm{d}f_u\|_u^\star, 1\}\ \|\mathrm{d}f_u\|_u^\star$

Finsler manifolds are paracompact sets, so

Lemma 2. *Any functional $f \in \mathcal{C}^1(M)$ admits a pseudo-gradient vector field*

$$v\colon \tilde{M} \to TM.$$

Theorem 4 (Minimax principle)**.** *Let f be a $\mathcal{C}^1$ functional satisfy P.S. on a Finsler manifold M. Suppose Γ is a positive invariant set for the flow $\Phi\colon M\times\mathbb{R} \to M$ generated by a pseudo-gradient vector field v related to f. Then if*

$$c = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma} f(u)$$

is finite, then c is a critical value for f.

2 Closed geodesics on sphere

It's time to apply the theory above on the manifold of closed curves on a manifold M in order to show the existence of a closed geodesic. This space is sometimes indicated with ΛM (see for example [Kli12, KB10]), but here, to specify the use of Sobolev regularity for our maps, we will use the notation:

$$H^1(S^1, M) := \left\{u\colon S^1 \to M, \|u\|_{H^1} < \infty\right\}.$$

More precisely, consider a manifold S and a Riemann manifold (M, g). The space $H^1(S, M)$ is made up by all absolutely continuous maps $u\colon S \to M$ such that the quantity

$$\int_S g_{u(t)}(\dot{u}(t), \dot{u}(t))\,\mathrm{d}t$$

is finite. We can also indicate $g_{u(t)}(\cdot\,,\cdot)$ with $\|\cdot\|^2_{u(t)}$ or with $\langle\cdot\,,\cdot\rangle_{u(t)}$.

On this manifold we will consider the energy functional

$$E(u) = \frac{1}{2}\int_S \|\dot{u}\|^2_{u(t)}\ \mathrm{d}t.$$

Critical points of this functional will be geodesics on our Riemann manifold (M, g). First, a more precise definition:

Definition 8 (Geodesic)**.** Let (M, g) be a Riemann manifold and ∇ be the Levi–Civita connection related to g on M. A curve σ on a manifold M is a *geodesic* on M if its vector field is parallel along σ, that is

$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) \equiv 0.$$

2.1 A theorem by Birkhoff

We now consider the 2-sphere S^2 in $\mathbb{R}^3$ with a generic Riemann metric g. For example the one given by the restriction of the inner product in $\mathbb{R}^3$ on every tangent space T_pS^2. Our energy functional E on the Riemann manifold (S^2, g) is

$$E(u) = \frac{1}{2}\int_0^{2\pi} g_{u(t)}(\dot{u}(t), \dot{u}(t))\,\mathrm{d}t.$$

We indicate with M the ∞-dimensional manifold $H^1(S^1, S^2)$, then:

1. The set $\mathrm{Crit}(E)$ in M are closed geodesics on S^2;
2. There exists a closed non-trivial geodesic on S^2.

We begin by specifying what we mean by the tangent space of M.

Tangent space Given a point $u \in M$, the tangent space T_uM is the space of derivation of the germs space $\mathcal{C}^\infty_M(u)$, and because u is a curve, we can see at T_uM as a vector field along $u(t)$, so

$$T_uM = \left\{\varphi \in H^1(S^1, \mathbb{R}^3) \mid \varphi(\vartheta) \in T_{u(\vartheta)}S^2\right\}.$$

Critical points Since $E\colon M \to \mathbb{R}$, the Fréchet differential $\mathrm{d}E$ in the point u in the direction φ is

$$\begin{aligned} \mathrm{d}E_u\colon T_uM &\to T_{E(u)}\mathbb{R} \cong \mathbb{R} \\ \varphi &\mapsto \mathrm{d}E_u(\varphi). \end{aligned}$$

By definition, taking a curve γ in M, $\gamma(t) = u(\vartheta, t)$ such that $\gamma(0) = u(\vartheta, 0) = u(\vartheta)$ and $\dot{\gamma}(0) = \varphi$, then

$$\mathrm{d}E_u(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}E(u(\vartheta, t))\restriction_{t=0} = \int_0^{2\pi} \frac{\partial^2 u}{\partial t \partial \vartheta}\cdot\frac{\partial u}{\partial \vartheta}\,\mathrm{d}\vartheta = \int_0^{2\pi} \frac{\partial^2 u}{\partial \vartheta \partial t}\cdot\frac{\partial u}{\partial \vartheta}\,\mathrm{d}\vartheta = \int_0^{2\pi} \dot{\varphi}\,\dot{u}\ \mathrm{d}\vartheta.$$

Now if u is a critical point for E, integrating by parts,

$$\mathrm{d}E_u(\varphi) = 0 = -\int_0^{2\pi} \ddot{u}\,\varphi\ \mathrm{d}\vartheta,$$

since $\varphi\,\dot{u}$ vanishes, because both φ and u are periodic. Thus, for a critical point u, the function $\ddot{u}$ is orthogonal to φ for every $\varphi \in T_uM$. This means that the tangent component of $\ddot{u}$ is zero and $\nabla_{\dot{u}}\dot{u} \equiv 0$, so u is a geodesic.

Existence of a closed geodesic To apply the Minimax principle (Theorem 4) we need:

(i) a functional E satisfying the Palais-Smale condition on M;

(ii) a class of invariant sets for the gradient flow generated by E;

(iii) the quantity $\inf_{\sigma \in \Gamma} \sup_{u \in \sigma} E(u) = c$ being greater than 0.

For details of item (i) see [Str00]. For the item (ii), we wish that the class of invariant sets would contain closed loops on S^2. Consider the latitude θ and the longitude ϕ as coordinates (θ, ϕ) on S^2, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\phi \in [0, 2\pi]$.

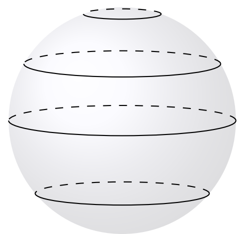
A map $f \colon S^2 \to S^2$ induces a curve in $H^1(S^1, S^2)$. In fact, for any fixed θ, the map

$$t \mapsto f(\theta, 2\pi t)$$

is a closed loop on S^2. Thus, the map

$$\theta \overset{\sigma}{\mapsto} f(\theta, \cdot)$$

is a "curve" in our manifold $H^1(S^1, S^2)$.



This curve joins the two constant loops $f\left(\pm\frac{\pi}{2}, \cdot\right) = \sigma\left(\pm\frac{\pi}{2}\right) =$ constant, the two poles. So, from every path that fixes poles we can obtain a map on S^2:

$$\sigma(\theta) \xrightarrow{\Psi} f(\theta, \phi).$$

Let

$$\Gamma = \left\{\sigma \colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to M \, , \, \sigma\left(\pm\frac{\pi}{2}\right) \equiv \text{constant}\right\},$$

and let

$$\mathcal{F} = \{\sigma \in \Gamma \, ; \Psi(\sigma) = f \sim \mathrm{id}_{S^2}\}$$

the family of curves in $H^1(S^1, S^2)$ induced by maps on S^2 with topological degree 1. The family $\mathcal{F}$ in not empty and it is invariant under the action of any diffeomorphism ψ of M homotopic to identity which fixes constant maps. The diffeomorphisms $\psi_t(\cdot) = \Phi(\cdot, t)$ obtained by the negative flow generated by E are homotopic to the identity and the constant loops.

In conclusion, by the Theorem 4, the value

$$c = \inf_{\sigma \in \mathcal{F}} \sup_{u = \sigma(\theta)} E(u)$$

is a critical value for E. The respective critical point will be a closed geodesic on the sphere. Now we have to check that this geodesic is not a trivial constant loop. We will show that such critical point has positive energy.

Points with bounded energy We start showing that points in M (that actually are "curves") with bounded energy are uniformly $\frac{1}{2}$-Hölder continuous.
Take a point $u \in M$, this is a curve

$$\begin{aligned} u\colon S^1 &\to (S^2, g) \\ \vartheta &\mapsto u(\vartheta). \end{aligned}$$

We can see $u(t)$ as a periodic function in $\mathbb{R}^3$ with values in S^2, then

$$\|u(s) - u(\vartheta)\| = \left\| \int_s^\vartheta \dot{u}(r)\, \mathrm{d}r \right\| \leqslant \int_s^\vartheta \|\dot{u}\|\, \mathrm{d}r \leqslant |\vartheta - s|^{1/2} \left(\int_s^\vartheta \dot{u}\, \mathrm{d}r \right)^{1/2} \leqslant \left(|\vartheta - s|\ 2E(u) \right)^{1/2}.$$

The difference $|\vartheta - s|$ is bounded by π, so, if $E(u) \leqslant c$, we have that

$$\|u(s) - u(\vartheta)\| \leqslant \sqrt{2\pi c}.$$

Positive energy Recall that a point with small energy $E(u) \leqslant c$ is a Hölder continuous bounded curve in $\mathbb{R}^3$. Since u is a loop, if u has small energy α, then u has also a small diameter, where

$$\operatorname{diam}(u) := \sup_{\vartheta, \theta \in [0, 2\pi]} \|u(\vartheta) - u(\theta)\| \leqslant \alpha.$$

For small α, we can consider a small neighbourhood of S^2 containing all u such that their energy $E(u)$ is smaller than α. We can shrink u to a constant loop $u(\phi_0)$:

$$u_s(\phi) = (1 - s)\, u(\phi) + s\, u(\phi_0)\,,\ s \in [0, 1].$$

We can also make a homotopy between $\sigma(\theta)$ and $\sigma(0) = u(\phi_0)$. In fact

$$\sigma_r(\theta) = \sigma((1 - r)\, \theta)\,,\ r \in [0, 1]$$

brings $\sigma(\theta)$ in the single point $\sigma(0)$ and so in a single loop u. The composition of the two homotopies is a homotopy, so

$$\sigma_{r,s}(\theta, \phi) = (1 - s)\, \sigma((1 - r)\, \theta, \phi) + s\, \sigma((1 - r)\, \theta, \phi_0)\,,\ s, r \in [0, 1].$$

Now we have a homotopy between the corresponding maps $f_{r,s}(\theta, \phi) = \Psi(\sigma_{r,s}(\theta))$, but

$$\mathrm{id}_{S^2} \sim f(\theta, \phi) = f_{0,0} \quad \text{and} \quad f_{1,1} = f(0, \phi_0) = u(\phi_0) \equiv \text{constant} \in S^2.$$

This leads to an absurd, because we took $\sigma \in \mathcal{F}$ and we found an homotopy between $\Psi(\sigma) = f \sim \mathrm{id}$ and a constant map, that it is known to have different degree on S^2.

2.2 Open Problems

It is proved that every compact manifold has a non-trivial closed geodesic. For example, by minimising the energy functional E on a non-trivial homotopy class of loops. It is conjectured that every compact manifold would have infinitely many closed geodesics, and this had been proved for several manifolds, but not for higher dimensional spheres yet.

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