Summer/Fall School on
Decoupling and Polynomial Methods in Analysis

Behaviour of the Schrödinger evolution for initial data near $H^{\frac{1}{4}}$

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after

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Background

Consider the Schrödinger equation in 1 dimension:

$$\begin{cases} i\partial_t \Psi(x,t) + \Delta\Psi(x,t) = 0\,, & x, t \in \mathbb{R} \\ \Psi(x,0) = f(x) \end{cases}$$

The solution is given by

$$\Psi(t,x) = e^{it\Delta} f(x) := \mathcal{F}^{-1}\left(e^{it\xi^2}\hat{f}\right).$$

The operator $e^{it\Delta}$ is bounded on L^2, in particular

$$\lim_{t\to 0} e^{it\Delta} f \;=\; f \quad \text{in } L^2(\mathbb{R}).$$

What about $\lim_{t\to 0} e^{it\Delta} f(x)$?

Question

When $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ for almost every $x \in \mathbb{R}$?

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When $\quad \lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad$ for almost every $x \in \mathbb{R}$?

Let $(T_t)_{t\in[0,1]}$ be the family of operators

$$T_t f(x) := e^{it\Delta} f(x) = \mathcal{F}^{-1}\left(e^{it\xi^2}\hat{f}\right).$$

Theorem (Carleson, 1980)

Let $\alpha > \frac{1}{4}$ and f is α-Hölder and compactly supported , then

$$\lim_{t\to 0} T_t f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}.$$

Furthermore, when $\alpha < \frac{1}{8}$, there exists a α-Hölder function f such that

$$\limsup_{t\to 0} |T_t f(x)| = \infty \qquad \text{for almost every } x \in \mathbb{R}.$$

Theorem (Dahlberg & Kenig, 1982)

If $s < \frac{1}{4}$ there exists $f \in H^s(\mathbb{R})$ and a set E, $|E| > 0$, such that

$$\limsup_{t\to 0} |T_t f(x)| = +\infty \qquad \text{for almost every } x \in E.$$

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Let $s \geqslant \frac{1}{4}$ *and* f *is in* $H^s(\mathbb{R})$, *then*

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Carleson's result

We want to bound the maximal operator

$$T^*f(x) := \sup_{0<t<1} |T_t f(x)| = \sup_{t\in(0,1)} |e^{it\Delta} f(x)|$$

Remark

For $s > \frac{1}{4}$, $H^s(\mathbb{R}) \subset H^{\frac{1}{4}}(\mathbb{R})$. It is enough to show $s = \frac{1}{4}$.

Proposition (A priori estimate)

Let $f \in \mathscr{S}(\mathbb{R})$. Then there exists $C > 0$ such that

$$\|T^*f\|_{L^4(\mathbb{R})} \leqslant C\|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

It's enough to prove a *local* estimate:

$$\|T^*f\|_{L^4(\mathbf{B}_r)} \leqslant C\|f\|_{H^{\frac{1}{4}}(\mathbb{R})}$$

for $r > 0$ with a constant C independent of r.

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Idea of the proof

Step 1 **Linearize**

For each $x \in \mathbb{R}$ there exists a time $t(x) > 0$ such that

$$|e^{it(x)\Delta} f(x)| \geqslant \frac{1}{2} \sup_{t>0} |e^{it\Delta} f(x)|$$

so that

$$T^* f(x) \leqslant 2\,|e^{it(x)\Delta} f(x)|.$$

Step 2 **Dualize**

There exists $w \in L^{\frac{4}{3}}(\mathbf{B}_r) \cong L^{4'}(\mathbf{B}_r)$, with $\|w\|_{\frac{4}{3}} = 1$, such that

$$\|e^{it(\cdot)\Delta} f(\cdot)\|_{L^4(\mathbf{B}_r)} = \int_{\mathbf{B}_r} e^{it(x)\Delta} f(x) w(x)\, dx$$

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assuming $\mathrm{supp}(w) \subset \mathbf{B}_r$.

Step 3 **Split**

Expand, use Fubini and Cauchy-Schwarz.

$$\int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x)\, dx = \iint_{\mathbb{R}^2} \hat{f}(\xi) e^{i(x\xi - t(x)\xi^2)}\, đ\xi\, w(x)\, dx$$

$$= \int_{\mathbb{R}} \hat{f}(\xi) |\xi|^{\frac{1}{4}} \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx\, đ\xi$$

$$\leqslant \left\| \hat{f} |\xi|^{\frac{1}{4}} \right\|_{L^2} \quad \left\| \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx \right\|_{L^2}$$

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$$\leqslant \left\| \hat{f} \right\|_{L^2\left(|\xi|^{\frac{1}{2}}\, đ\xi\right)} \left\| \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx \right\|_{L^2}$$

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&= \int_{\mathbb{R}} \hat{f}(\xi) |\xi|^{\frac{1}{4}} \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx\, đ\xi \\
&\leqslant \left\| \hat{f} \right\|_{L^2\left(|\xi|^{\frac{1}{2}}\, đ\xi\right)} \left\| \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx \right\|_{L^2} \\
&\leqslant \|\hat{f}\|_{L^2\left((1+|\xi|^2)^{\frac{1}{4}}\, đ\xi\right)} \underbrace{\left\| \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}}\, dx \right\|_{L^2}}_{A}
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&\leqslant \|f\|_{H^{\frac{1}{4}}(\mathbb{R})} A.
\end{aligned}$$

Step 4 **Estimate A.**

$$A^2 = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i(x\xi - t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}} \, dx \right|^2 đ\xi.$$

Bound the oscillatory integral inside

$$\begin{aligned} A^2 &= \int \left(\int \cdot \, dx \right) \overline{\left(\int \cdot \, dy \right)} đ\xi \\ &= \int_{\mathbb{R}} \iint_{\mathbb{R}^2} e^{i\left((x-y)\xi - (t_x - t_y)\xi^2\right)} w(x)\overline{w(y)} \, dx \, dy \frac{đ\xi}{|\xi|^{\frac{1}{2}}}. \end{aligned}$$

Lemma (Carleson)

Let $a, b \in (-2, 2)$, and $\gamma \in (0, 1)$. Then

$$\int_{\mathbb{R}} e^{i(a\xi + b\xi^2)} \frac{đ\xi}{|\xi|^{\gamma}} \leqslant C_\gamma \left(|b|^{\gamma - \frac{1}{2}} |a|^{-\gamma} + |a|^{\gamma - 1} \right).$$

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$$\int_{\mathbb{R}} e^{i(a\xi + b\xi^2)} \frac{d\xi}{|\xi|^{\gamma}} \leqslant C'_{\gamma} |a|^{-\frac{1}{2}}.$$

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$$\leqslant C \iint_{\mathbb{R}^2} \frac{|w(x)|\,|w(y)|}{|x-y|^{\frac{1}{2}}}\,dx\,dy.$$

Use Hölder and Hardy-Littlewood-Sobolev inequalities:

$$A^2 \leqslant C\|w\|_{L^{\frac{4}{3}}} \left\| \int_{\mathbb{R}} \frac{|w(y)|}{|x-y|^{\frac{1}{2}}}\,dy \right\|_{L^4} \leqslant C\|w\|^2_{L^{\frac{4}{3}}(\mathbb{R})}.$$

Summing up:

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4(\mathbf{B}_r)} \leqslant 2 \left\| e^{it(\cdot)\Delta} f(\cdot) \right\|_{L^4(\mathbf{B}_r)} \leqslant C\|w\|_{L^{\frac{4}{3}}(\mathbb{R})} \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

Take the limit as $r \to \infty$ to conclude. □

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Carleson's positive result

Theorem (Carleson, 1980, revised)

Let $s \geqslant \frac{1}{4}$ *and* f *is in* $H^s(\mathbb{R})$, *then*

$$\lim_{t\to 0} T_t f(x) = f(x) \quad \textit{for almost every } x \in \mathbb{R}.$$

A glimpse at the proof.

By density of $\mathscr{S}(\mathbb{R})$ in $H^{\frac{1}{4}}(\mathbb{R})$, the local estimate holds in $H^{\frac{1}{4}}(\mathbb{R})$.

$$T^* : H^s(\mathbb{R}) \to L^4(\mathbb{R}) \quad \text{is bounded for} \quad s \geqslant \frac{1}{4}.$$

This gives pointwise convergence a.e. for $(T_t)_{t\in[0,1]}$, so

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}.$$

□

Negative result: Dahlberg and Kenig

Let $s < \frac{1}{4}$. We look for $f \in H^s(\mathbb{R})$ such that

$$\lim_{t \to 0} e^{it\Delta} f(x) \neq f(x)$$

for every $x \in E$, where $E \subset \mathbb{R}$ is a set of positive measure.

Let $f \in \mathscr{M}(0,1) = \{f \text{ on } (0,1), \text{ measurable}, , f(x) < \infty \text{ a.e.}\}$.

We are happy with f such that for every $x \in E$

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$$\text{Bound for } T^* \quad \Rightarrow \quad \lim_{t\to 0} T_t f(x) = f(x) \text{ a.e.}$$

$$\|T^* f\|_{L^{p,\infty}} \gtrsim \|f\| \quad \overset{?}{\Rightarrow} \quad \lim_{t\to 0} T_t f(x) \neq f(x)$$

$$\Uparrow$$

$$\boxed{\textit{Weak} \text{ bound for } T^* \quad \Leftarrow \quad \lim_{t\to 0} T_t f(x) = f(x) \text{ a.e.}}$$

Reduce to a countable family.

Remark

Let $(T_t)_{t\in I}$, with $I \subset \mathbb{R}$, then

$$\sup_{t\in I} |T_t f| = \sup_{t\in I\cap\mathbb{Q}} |T_t f| .$$

Let $t = \frac{1}{n}$. Let $(T_n)_{n\in\mathbb{N}}$ be the family of operators

$$T_n f(x) = e^{i\frac{\Delta}{n}} f(x) , \qquad T^* f(x) = \sup_{n\in\mathbb{N}} |T_n f(x)| .$$

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$$T_n f(x) = e^{i\frac{\Delta}{n}} f(x) , \qquad T^* f(x) = \sup_{n\in\mathbb{N}} |T_n f(x)| .$$

$\lim_{t\to 0} T_t f(x) = f(x) \quad \Rightarrow \quad$ Weak bound for T^*

Notice that

$$\lim_{n\to\infty} T_n f(x) = f(x) \quad \text{a.e.} \quad \Rightarrow \quad \limsup_{n\to\infty} T_n f(x) < \infty \quad \text{a.e.}$$

		Range of p	Conditions
1961	**Stein**	$1 \leqslant p \leqslant 2$	$\limsup T_n f(x) < \infty$ on $E \subset X$, $\mu(E) > 0$, $(T_n)_n$ commuting with G compact group, (X, μ) G-homogeneous
1966	**Sawyer**	$1 \leqslant p < \infty$	ergodic theory setting
1970	**Nikišin**	$1 \leqslant p < \infty$	T *hyperlinear* and continuous in measure

Let (X, μ) and (Y, ν) two σ-finite measure spaces.

Definition (Continuity in measure)

A linear operator $T\colon L^p(X, \mu) \to \mathscr{M}(Y, \nu)$ is continuous in measure if

$$\nu\left(\{y \in Y : Tf(y) > \lambda\}\right) \to 0 \quad \text{as} \quad \|f\|_{L^p(X)} \to 0.$$

Theorem (Banach, 1926)

Let $(T_n)_{n\in\mathbb{N}}$ be linear operators from $L^p(X, \mu)$ to $\mathscr{M}(Y, \nu)$, continuous in measure. If for every $f \in L^p(X, \mu)$

$$\limsup_{n\to\infty} T_n f(y) < \infty \text{ a.e. } \Rightarrow \ T^* \textit{ continuous in measure.}$$

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$$\nu(\{y \in Y : Tf(y) > \lambda\}) \to 0 \quad \text{as} \quad \|f\|_{L^p(X)} \to 0.$$

Theorem (Banach, 1926)

Let $(T_n)_{n \in \mathbb{N}}$ be linear operators from $L^p(X, \mu)$ to $\mathscr{M}(Y, \nu)$, continuous in measure. If for every $f \in L^p(X, \mu)$

$$\limsup_{n \to \infty} T_n f(y) < \infty \text{ a.e.} \Rightarrow T^* \textit{ continuous in measure.}$$

Let (X,μ) and (Y,ν) two σ-finite measure spaces.

Definition (Hyperlinearity)

An operator $T\colon L^p(X,\mu) \to \mathscr{M}(Y,\nu)$ is *hyperlinear* if for each $f_0 \in L^p(X)$ there exist a *linear* operator T_{f_0} such that

(i) $|T_{f_0} f_0| = |Tf_0|$ ν- a.e. and

(ii) $|T_{f_0} g| \leqslant |Tg|$ ν- a.e. for all $g \in L^p(X)$.

Example (Truncated maximal operator)

Given a sequence of operators $(T_n)_n\colon L^p(X,\mu) \to \mathscr{M}(Y,\nu)$, then

$$T_N^* f := \sup_{1\leqslant n\leqslant N} |T_n f|$$

is hyperlinear.

Given $f \in L^p(X)$, exists $n_f\colon Y \to \{1,\ldots,N\}$ such that

$$T_N^* f(y) = \left|T_{n_f(y)} f(y)\right|.$$

Nikišin's theorem

Take $Y = [0, 1]$, and let ν be the Lebesgue measure.

Theorem (Nikišin, 1970)

Let $1 \leqslant p < \infty$, *and* $T^* \colon L^p(X, \mu) \to \mathscr{M}[0, 1]$ *such that*

- *hyperlinear,*
- *continuous in measure.*

Then $\forall \epsilon > 0$ *there exists* $E_\epsilon \subset [0, 1]$ *with* $|E_\epsilon| \geqslant 1 - \epsilon$ *such that*

$$\|T^* f\|_{L^{q,\infty}(E)} \lesssim_\varepsilon \|f\|_{L^p(X)}$$

with $q = \min\{p, 2\}$.

Equivalently, there exists $C_\varepsilon > 0$ such that

$$|\{y \in E_\epsilon \, : \, T^* f(y) > \lambda\}| \leqslant \frac{C_\epsilon}{\lambda^q} \|f\|_{L^p}^q ,$$

for all $\lambda > 0$.

Apply Nikišin's theorem

Our maximal operator is hyperlinear and continuous in measure

$$T^*\colon H^s(\mathbb{R}, d\xi) \to \mathscr{M}([0,1])$$

Apply Nikišin with $p = 2$, $X = \mathbb{R}$. There exists $E \subset [0,1]$ such that

$$\|T^* f\|_{L^{2,\infty}(E)} \lesssim \|f\|_{L^2(\mathbb{R}, \langle \xi \rangle^{2s} d\xi)} .$$

Equivalently, there exists $C > 0$ such that $\forall \lambda > 0$

$$|\{y \in E : T^* f(y) > \lambda\}| \leqslant \frac{C}{\lambda^2} \|f\|_{H^s}^2 .$$

Take f_n such that $\lim_{n \to \infty} \|f_n\|_{H^s} = 0$, and for some λ_0, $E \subset \{T^* f_n > \lambda_0\}$,

$$0 < |E| \leqslant |\{T^* f_n > \lambda_0\}| \lesssim \|f_n\|_{H^s} \searrow 0.$$

Contradiction $\lightning$.

Apply Nikišin's theorem

Our maximal operator is hyperlinear and continuous in measure

$$T^*\colon L^2(\mathbb{R}, (1+\xi^2)^s \, d\xi) \to \mathscr{M}([0,1])$$

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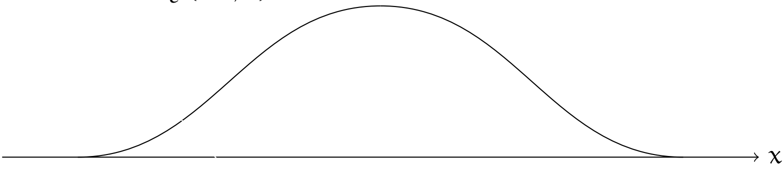
Consider $f \in C_c^\infty(-1, 1)$.



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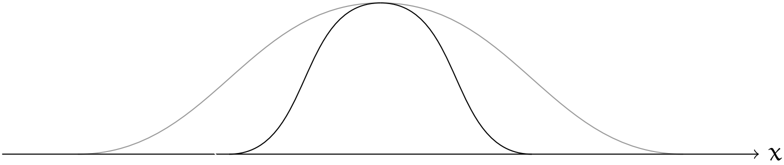

$$f(x) \mapsto f(nx)$$

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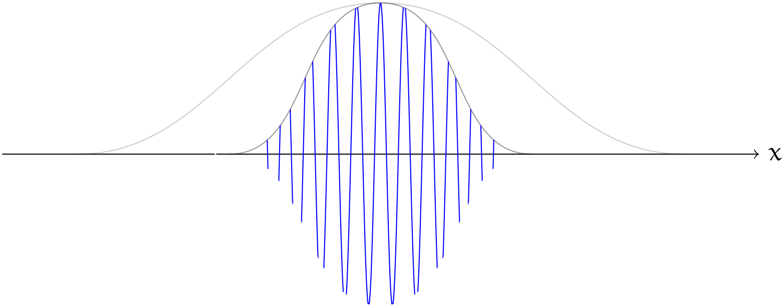

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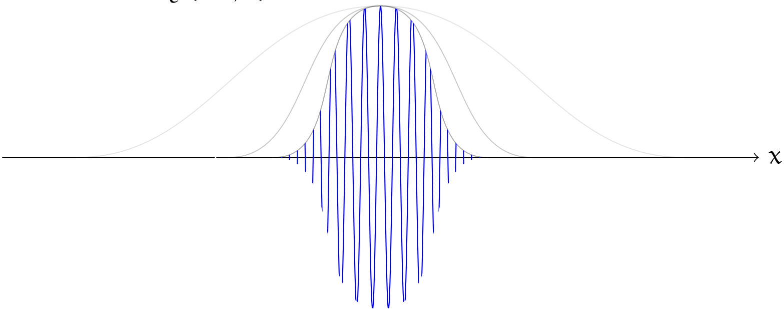

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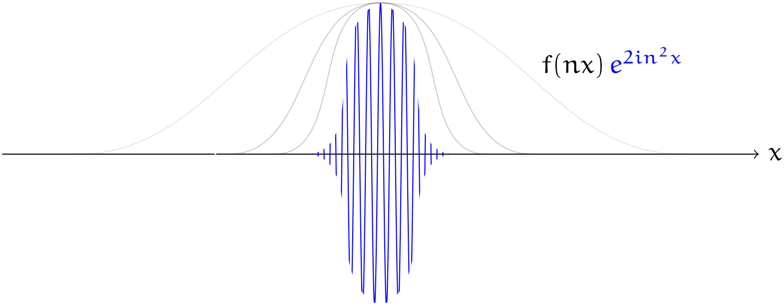

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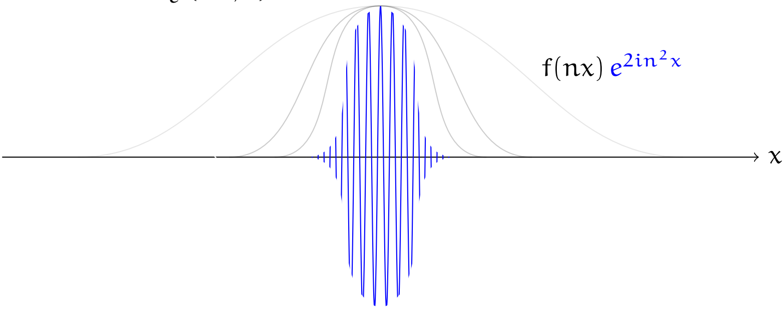

$$\|f_n\|^2_{\dot{H}^s(\mathbb{R})} = \int_{\mathbb{R}} \frac{1}{n^2} \left|\hat{f}\left(\frac{\xi}{n} - 2n\right)\right|^2 |\xi|^{2s} \, d\xi$$

$$= n^{2s-1} \int_{\mathbb{R}} \left|\hat{f}(\xi - 2n)\right|^2 |\xi + 2n|^{2s} \, d\xi$$

$$\lesssim n^{4s-1} \, \|f\|^2_{\dot{H}^s(\mathbb{R})} \sim \left(\frac{1}{n}\right)^{1-4s} \xrightarrow{n\to\infty} 0.$$

Let $n(x) = \frac{x}{n^2}$ and consider

$$T_{n(x)} f_n(x) = e^{ix\frac{\Delta}{n^2}} f_n(x) = \frac{1}{\sqrt{x}} \int_{\mathbb{R}} f(y) e^{i\frac{y^2}{x}} dy.$$

$$\lambda_0 := \min_{x \in E} |g(x)| .$$

Since $|g(x)| = \left| T_{n(x)} f_n(x) \right| \leqslant T^* f_n(x)$

$$|E| \leqslant |\{x \in E : T^* f_n > \lambda_0\}| \lesssim \|f_n\|_{H^s(\mathbb{R})}^2 \lesssim \frac{1}{n^{1-4s}}.$$

Contradiction, since for $s < \frac{1}{4}$

$$0 < |E| \lesssim \frac{1}{n^{1-4s}} \to 0 \quad \text{as } n \to \infty.$$

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Theorem (Carleson, Dahlberg & Kenig)

$$\lim_{t\to 0} T_t f(x) = f(x) \quad \textit{for almost every } x \in \mathbb{R},$$

for $f \in H^s(\mathbb{R})$ *if and only if* $s \geqslant \frac{1}{4}$.

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