A Nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the Inverse boundary value problem of Calderón

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$$L_q u := \bar{\partial} u + q\bar{u} = f, \qquad \boxed{q \in L^2(\mathbb{C})} \qquad (\wp)$$

Theorem (Main result)

For $f \in \dot{H}^{-\frac{1}{2}}$, there exists an unique solution $u \in \dot{H}^{\frac{1}{2}}$ to $(\wp)$ with

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \lesssim C(\|q\|_{L^2})\,\|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

Needed in the next episode

$$|\bar{\partial}^{-1}(e^{-i(zk+\overline{zk})}q(z))(x)| \lesssim (M\hat{q}(k))^{\frac{1}{2}}(Mq(x))^{\frac{1}{2}}$$

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Theorem

For any $f \in L^2(\mathbb{R}^2)$ *we have:*

a) $|(-\Delta)^{-\frac{1}{2}}f(x)| \lesssim \lambda M\hat{f}(0) + \lambda^{-1}Mf(x) \quad \forall \lambda > 0$

b) $|(-\Delta)^{-\frac{1}{2}}f(x)| \lesssim \sqrt{M\hat{f}(0)\, Mf(x)}.$

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Rewrite b) as

b) $|(-\Delta)^{-\frac{1}{2}}\big(e^{iy\xi}f(y)\big)(x)| \lesssim \sqrt{M\hat{f}(\xi)\, Mf(x)}.$

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$$\|\bar\partial^{-1}(e^{-i(zk+\overline{zk})}q(z))\|_{L^4_x} \lesssim \|q\|_{L^2}^{\frac{1}{2}}\left(M\hat{q}(k)\right)^{\frac{1}{2}}$$

Needed in the next episode II

$$a(x, D)f(x) := \int_{\mathbb{R}^2} e^{ix\xi} a(x, \xi)\hat{f}(\xi)\, đ\xi, \qquad a \in L^4_{x,\xi}$$

$$|a(x, D)f(x)| \lesssim (Mf(x))^{\frac{1}{2}} \|\partial_\xi a(x, \cdot)\|_{L^{\frac{4}{3}}_\xi} \|f\|^{\frac{1}{2}}_{L^2}, \quad \text{if } \partial_\xi a \in L^4_x L^{\frac{4}{3}}_\xi$$

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Theorem

Let $a(x,\xi)$ *be a symbol on* $\mathbb{R}^2 \times \mathbb{R}^2$ *such that*
i) $a \in L^4(\mathbb{R}^2 \times \mathbb{R}^2)$, *ii)* $\|(-\Delta_\xi)^{\frac{1}{2}} a(x,\xi)\|_{L_\xi^{\frac{4}{3}}} \in L^4_x$
then

$$|a(x,D)f(x)| \lesssim \Big(Mf(x)\Big)^{\frac{1}{2}} \|(-\Delta_\xi)^{\frac{1}{2}} a(x,\cdot)\|_{L_\xi^{\frac{4}{3}}} \|f\|_{L^2}^{\frac{1}{2}} \quad \textit{a.e. } x$$

Proof.

$$
\begin{aligned}
|a(x,D)f(x)| &\leqslant \int \left|\partial_\xi^{-1}\left(e^{ix\xi}\hat{f}(\xi)\right)\right| |\partial_\xi a(x,\xi)| \, đ\xi \\
&\lesssim \left(Mf(x)\right)^{\frac{1}{2}} \int \left(M\hat{f}(\xi)\right)^{\frac{1}{2}} |\partial_\xi a(x,\xi)| \, đ\xi \\
&\leqslant \left(Mf(x)\right)^{\frac{1}{2}} \|(M\hat{f})^{\frac{1}{2}}\|_{L^4} \|\partial_\xi a(x,\cdot)\|_{L_\xi^{\frac{4}{3}}} \\
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Integration gives: $\|a(x,D)f\|_{L^2} \lesssim \|(-\Delta_\xi)^{\frac{1}{2}} a(x,\xi)\|_{L_x^4 L_\xi^{\frac{4}{3}}} \|f\|_{L^2}$

□

Estimates for a $\bar{\partial}$-problem

$$L_q u := \bar{\partial} u + q\bar{u}, \quad q \in L^2(\mathbb{C})$$

Theorem (Main theorem)

The operator $L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ *is invertible and*

$$\boxed{\|L_q{}^{-1}\| \leqslant C(\|q\|_{L^2})}$$

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Plan:

1. **Injectivity on a larger spaces** $L_q^{-1} : L^{\frac{4}{3}} \to L^4$
2. **Restrict** $\dot{H}^{-\frac{1}{2}} \xrightarrow{L_q^{-1}} \dot{H}^{\frac{1}{2}}$

$$\|L_q^{-1} f\|_{\dot{H}^{\frac{1}{2}}} \leqslant C(q)\|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

3. **Dependence** $q \mapsto L_q^{-1}$ is analytic for in $B(q_0, \epsilon)$
4. **Uniform bound** on L^2 balls

Estimates for a $\bar{\partial}$-problem

Lemma (Cauchy transform)

i) $\|\bar{\partial}^{-1} f\|_{L^4} \lesssim \|f\|_{L^{\frac{4}{3}}}$

ii) *Let* $1 < p_1 < 2 < p_2$ *and* $f \in L^{p_1} \cap L^{p_2}$, *then*

$$\|\bar{\partial}^{-1} f\|_\infty \lesssim_{p_1,p_2} \|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}.$$

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Estimates for a $\bar\partial$-problem

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Estimates for a $\bar{\partial}$-problem

Lemma

The operator $L_q : L^4 \to L^{\frac{4}{3}}$ *is invertible.*

Idea of the proof. Write

$$L_q = \bar{\partial}(I + \bar{\partial}^{-1}(q\,\bar{\cdot}\,)) =: \bar{\partial} \circ \mathcal{B}.$$

If $\mathcal{B} \colon L^4 \to L^4$ is invertible, then $u = \mathcal{B}^{-1}\bar{\partial}^{-1}f$.

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If $\mathcal{B} \colon L^4 \to L^4$ is invertible, then $u = \mathcal{B}^{-1}\bar\partial^{-1} f$. $\mathcal{B}$ is Fredholm.
Take $u \in \ker \mathcal{B}$, so $\boxed{\bar\partial u = -q\bar u}(\diamond)$.
Split $q = q_n + q_s$, with $q_n \in (L^{p_1} \cap L^{p_2})$[1] and $\|q_s\|_2 \ll 1$.
We can choose $v \in L^\infty$ such that

$$\bar\partial(uv) = (\bar\partial u)v + u\bar\partial v \overset{(\downarrow)}{=} (\bar\partial u + q_n\bar u)v \overset{(\diamond)}{=} (-q_s\bar u)v$$

$$\|uv\|_{L^4} \leqslant c\|\bar\partial(uv)\|_{L^{\frac{4}{3}}} = c\|q_s\bar u v\|_{L^{\frac{4}{3}}} \leqslant c\|q_s\|_{L^2}\|uv\|_{L^4} \leqslant \frac{1}{2}\|uv\|_{L^4}$$

□

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The same result holds on Sobolev spaces.

Lemma

The operator $L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ *is invertible and*

$$\|L_q^{-1} f\|_{\dot{H}^{\frac{1}{2}}} \leqslant C(q) \|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

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$$L^{\frac{4}{3}} \hookrightarrow \dot{H}^{-\frac{1}{2}} \xrightarrow{L_q^{-1}} \dot{H}^{\frac{1}{2}} \hookrightarrow L^4.$$

$$\mathcal{B}u = \bar{\partial}^{-1} f \in \dot{H}^{\frac{1}{2}}$$

Claim: $\mathcal{B} : \dot{H}^{\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}$ is invertible.

- injectivity: $\mathcal{B} : \dot{H}^{\frac{1}{2}} \hookrightarrow L^4 \to L^4$
- surjectivity: $\exists ! u \in L^4$

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$$q\bar{u} \in L^{\frac{4}{3}} \hookrightarrow \dot{H}^{-\frac{1}{2}} \overset{\bar{\partial}^{-1}}{\to} \dot{H}^{\frac{1}{2}}$$

Smooth dependence

Lemma

The constant $C(q)$ *has a local Lipschitz dependence on* q. *Given* $q_0 \in L^2$, *there exists* $\epsilon > 0$ *such that for every* $q_1, q_2 \in B(q_0, \epsilon)$.

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\| \lesssim C(q_0)^2 \|q_1 - q_2\|_{L^2}$$
$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2 \|q_1 - q_2\|_{L^2}.$$

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Idea of the proof.

$$L_{q_0} u = (q_0 - q)\bar{u} + f$$
$$(I - X)u := \left(I - L_{q_0}^{-1}((q_0 - q)\bar{\cdot})\right)u = L_{q_0}^{-1} f$$

Solve by Neumann series, under the blue assumption

$$\|L_{q_0}^{-1}\left((q_0 - q)\bar{\cdot}\right)\| \leqslant \|L_{q_0}^{-1}\| \, \|q_0 - q\|_2 \ll \|L_{q_0}^{-1}\| \, \frac{1}{C(q_0)} < 1$$

□

Recap

Step 1 **Enlarged spaces**

$$L_q^{-1} : L^{\frac{4}{3}}(\mathbb{C}) \to L^4(\mathbb{C}) \quad \text{invertible}$$

Step 2 **Restrict**

$$L^{\frac{4}{3}} \hookrightarrow \dot{H}^{-\frac{1}{2}} \xrightarrow{L_q^{-1}} \dot{H}^{\frac{1}{2}} \hookrightarrow L^4, \quad \text{still invertible and}$$

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Step 4 **Uniform bound** on L^2 balls

$$C(R) := \sup\{C(q) \;:\; \|q\|_2 \leqslant R\}, \qquad C\colon \mathbb{R}_+ \to [0,\infty].$$

Want to show: $C(R)$ finite $\forall R > 0$.

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By contradiction:

$$R_0 := \inf\{R \in \mathbb{R}_+ \,:\, C(R) = +\infty\}.$$

$\lim_{R\to R_0} C(R) = +\infty$, and exists $\{q_n\}_{n\in\mathbb{N}} \subset B_{R_0}$ such that $\|q_n\|_2 \to R_0$, with $\|L_{q_n}^{-1}\| \xrightarrow{n\to\infty} +\infty$.
Extract a *convergent* subsequence:

$$q_{n_k} \xrightarrow{L^2} q \;\Rightarrow\; \|L_{q_{n_k}}^{-1}\| \xrightarrow{k\to\infty} \|L_q^{-1}\| < +\infty$$

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Problem: $\overline{\{q_n\}}_{n\in\mathbb{N}}$ is *not* compact!

Symmetries: obstruction to compactness

The $\|q\|_{L^2(\mathbb{C})}$ is preserved by

Translations $\tau_h q(x) = q(x-h)$

Dilations $\delta_\lambda q(x) = \lambda q(\lambda x)$

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$$S(\lambda, y)q(x) := \lambda q(\lambda(\,\cdot\, - y))$$

Definition (compactness up to symmetries)

There exists $\{(\lambda_n, y_n)\}_{n\in\mathbb{N}}$ such that $\{S(\lambda_n, y_n)q_n\}_{n\in\mathbb{N}}$ is precompact in L^2.

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Idea: split $\{q_n\}$ in pieces driven by different symmetries

$$q_n = \sum_{k=1}^{N} S(\lambda_n^k, y_n^k)q^k + q_n^N, \quad \text{with} \lim_{N\to\infty} \limsup_{n\to\infty} \|q_n^N\|_2 = 0$$

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This is still too much to hope for.

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Remark: $L^2 \hookrightarrow \dot{B}_{\infty}^{-\frac{1}{3},3}$.

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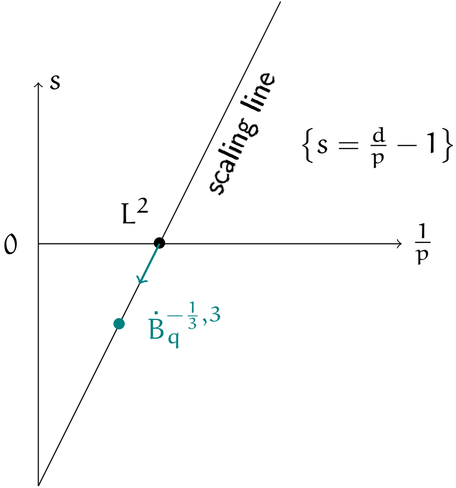

Thank you

Dankeschön Gracias Obrigado Grazie
Dziękuję Ci Хвала вам Kiitos Спасибо Eskerrik asko
Mulțumesc Xièxiè

Adrian Nachman, Idan Regev, and Daniel Tataru, *A Nonlinear Plancherel Theorem with Applications to Global Well-Posedness for the Defocusing Davewy-Stewartson Equation and to the Inverse Boundary Value Problem of Calderon.*, arXiv preprint arXiv:1708.04759 (2017).

Why is $\nu \in L^\infty$?

Consider the case $\nu \neq 1$, otherwise both ν and $1/\nu$ are clearly in L^∞. Then $\nu := e^{\bar{\partial}^{-1}(q_n \frac{\bar{u}}{u})}$, with $q_n \in L^{p_1} \cap L^{p_2}$. Then

$$\|e^{\bar{\partial}^{-1}(q_n \frac{\bar{u}}{u})}\|_\infty \cong \left\| \sum_{k \geqslant 0} \frac{(\bar{\partial}^{-1}(q_n))^k}{k!} \right\|_\infty$$

$$\leqslant \sum_{k \geqslant 0} \frac{\|\bar{\partial}^{-1} q_n\|_\infty^k}{k!} \lesssim_{p_1, p_2} \sum_{k \geqslant 0} \frac{(\|q_n\|_{p_1} + \|q_n\|_{p_2})^k}{k!}$$

that is finite and it equals $e^{\|q_n\|_{p_1} + \|q_n\|_{p_2}}$.