Quadratic Sparse Domination

Gianmarco Brocchi
University of Birmingham

LMS MEETING
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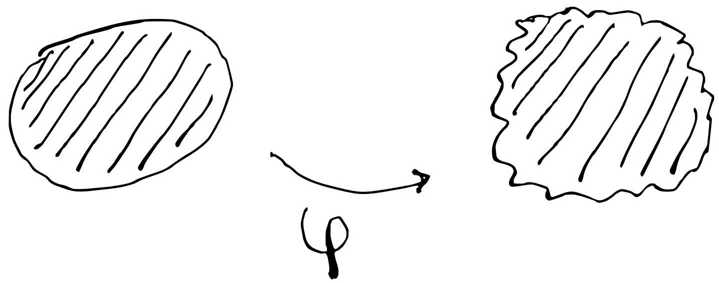
Plan

- what is Sparse domination ?
  What does it give us ?
- Quadratic sparse domination
  is it better ? When ?
- Beyond the classical theory
  Non-integral operators

Weighted inequalities

$$\int_{B} f(x)\,dx \qquad \int_{\varphi(B)} f(y)\,dy = \int_{B} f(\varphi(x))\,|\varphi'(x)|\,dx$$



$$\int_{B} h(x)\,w(x)\,dx$$

Sublinear operator T acting on L^p, $1<p<\infty$

$$\int |Th(x)|^p\,w(x)\,dx \le C(T,w)\int |h(x)|^p w(x)\,dx \qquad \forall h \in L^p$$

$$\sup_{h\in L^p\setminus\{0\}} \frac{\int |Th(x)|^p\,w(x)\,dx}{\int |h(x)|^p w(x)\,dx} = C(T,w) =: \|T\|^p_{L^p(w)}$$

Examples of Operators

$$\mathcal{M}f(x) := \sup_{r>0} \frac{c_d}{r^d}\int_{B_r(x)} |f(u)|\,du \;\tilde{=}\; \sup_{t>0} |f*\varphi_t|(x)$$

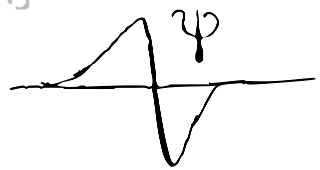
$$\varphi_t(\cdot) := \frac{1}{t^d}\varphi(\cdot/t)$$

Maximal operators

$$\mathcal{H}f(x) := \text{p.v.}\left(f * \frac{1}{y}\right)(x) \quad , \quad Sf(x) = \left(\int_0^\infty |f*\psi_t|^2(x)\,\frac{dt}{t}\right)^{1/2}$$

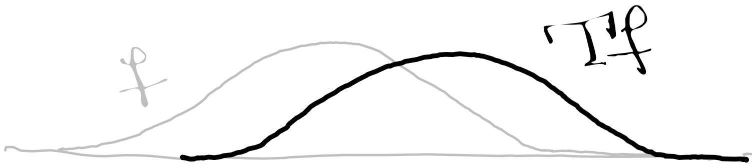
Singular operators

square functions

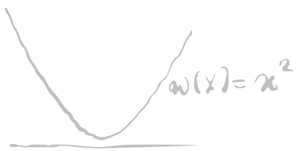


Question : How does T "move" functions ?

Let $\tau h(\cdot) = h(\cdot - t)$. Then $\| Tf \|_{L^p} = \| \tau(Tf) \|_{L^p}$



Replace dx with $w(x)\,dx$



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Singular operators

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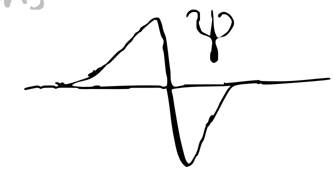

Question : for which weights

$$\| \mathcal{M} : L^p(w) \longrightarrow L^p(w) \| < +\infty \; ?$$

$$\| \mathcal{H} : L^p(w) \longrightarrow L^p(w) \| < +\infty \; ?$$

$$\| S : L^p(w) \longrightarrow L^p(w) \| < +\infty \; ?$$

Examples of Operators

$$\mathcal{M}f(x) := \sup_{r>0} \frac{c_d}{r^d} \int_{B_r(x)} |f(u)|\,du \;\overset{\sim}{=}\; \sup_{t>0} |f * \varphi_t|(x)$$

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Singular operators

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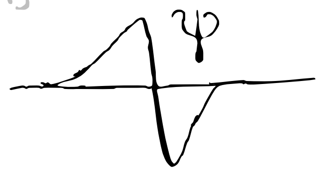


Theorem (Hunt, Muckenhoupt, Weeden et al. '70s)

$$\left.\begin{array}{l} \| \mathcal{M} : L^p(w) \longrightarrow L^p(w) \| < +\infty \\ \| \mathcal{H} : L^p(w) \longrightarrow L^p(w) \| < +\infty \\ \| S : L^p(w) \longrightarrow L^p(w) \| < +\infty \end{array}\right\} \iff w \in A_p$$

Muckenhoupt weights

$$\left([w]_{A_p} < +\infty \right)$$

How do $\|M\|_{L^p(w)}$, $\|H\|_{L^p(w)}$, $\|S\|_{L^p(w)}$ depend on $[w]_{A_p}$?

Theorem (Extrapolation)

Fix $r_0 \in (1,\infty)$.

If

$$\|T : L^{r_0}(w) \to L^{r_0}(w)\| \lesssim [w]_{A_{r_0}}^{\alpha} \qquad \forall w \in A_{r_0}$$

then for all $1<r<\infty$

$$\|T : L^{r}(w) \to L^{r}(w)\| \lesssim [w]_{A_r}^{\alpha \max\{\frac{r_0-1}{r-1}, 1\}} \qquad \forall w \in A_r$$

Example extrapolation from $r_0 = 2$ gives

$$\|T\|_{L^r(w)} \lesssim [w]_{A_r}^{\alpha \max\{\frac{1}{r-1}, 1\}}$$

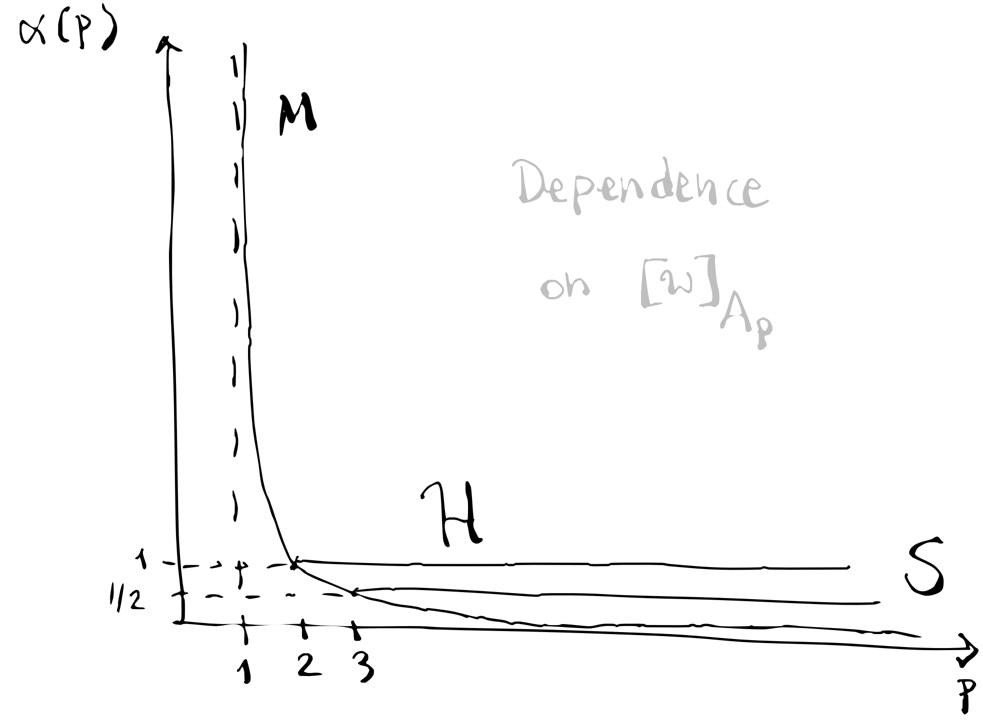
Remark for $r > r_0$, $\|T\|_{L^r(w)} \lesssim [w]_{A_r}^{\alpha}$. (The smaller the α the better.)

How do $\|\mathcal{M}\|_{L^p(w)}, \|\mathcal{H}\|_{L^p(w)}, \|S\|_{L^p(w)}$ depend on $[w]_{A_p}$?

Buckley '90s $\qquad \|\mathcal{M}\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$

Lerner 2007 $\qquad \|S\|_{L^3(w)} \lesssim [w]_{A_3}^{1/2}$

Petermichl 2007 $\qquad \|\mathcal{H}\|_{L^2(w)} \lesssim [w]_{A_2}$



Dependence on $[w]_{A_p}$

Which by extrapolation give:

$$\|\mathcal{H}\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{p-1}, 1\}}$$

$$\|S\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{2}\max\{\frac{2}{p-1}, 1\}}$$

Control a generic $|Tf(x)|$ by $\mathcal{M}f(x)$?

Usually NO :(

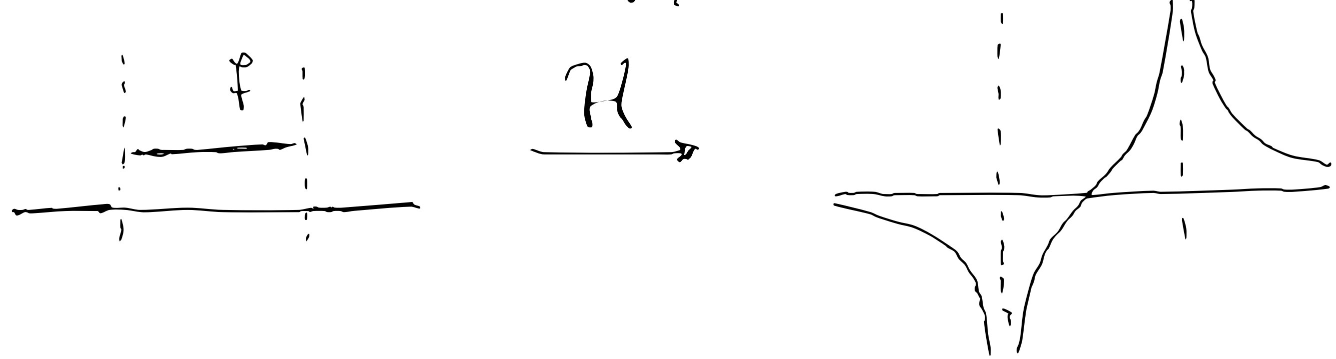
$\mathcal{M}$ is bounded on bounded f : $\|\mathcal{M}f\|_{L^\infty(\mathbb{R})} < +\infty$

but

The Hilbert transform

$$\mathcal{H}f(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x-y) \, \frac{dy}{y} \quad , \quad f \in C_c^\infty(\mathbb{R})$$

is NOT!



Can we (still) understand T in terms of averages?

$$\mathcal{M}f(x) = \sup_{I \in \mathcal{D}} \left(\fint_I |f| \, dy \right) \mathbb{1}_I(x)$$

$$|Tf(x)| \lesssim \sum_{I \in \mathcal{S}} \left(\fint_I |f(y)| \, dy \right) \mathbb{1}_I(x)$$

if $\mathcal{S}$ disjoint:

$$\| Tf \|_{L^\infty} \leq \| Mf \|_{L^\infty} = \sup_I \fint_I |f| \leq \| f \|_{L^\infty}$$

if cubes in $\mathcal{S}$ overlap (∞-many times):

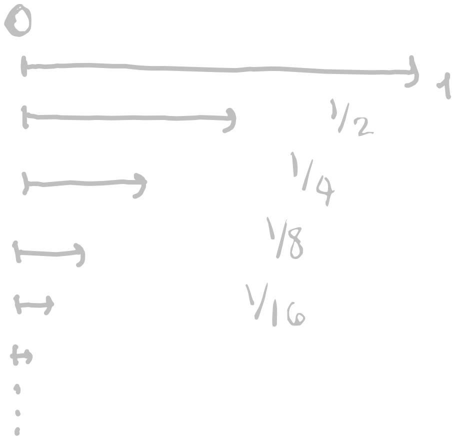
$$\| Tf \|_{L^\infty} \leq \sum_{I \in \mathcal{S}} \left(\fint_I |f| \, dy \right) = +\infty$$

Can we (still) control $\| Tf \|_{L^p} \lesssim \| Mf \|_{L^p}$?

Sparse collection

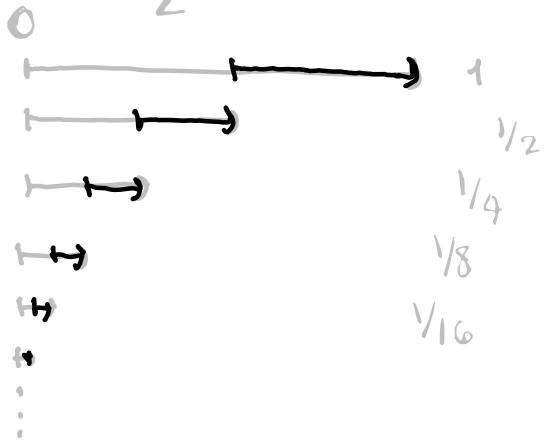
Collection w/ a disjoint, comparable subfamily

$[0, 2^{-n}) =: I_n$



∞ - overlapping

$[\frac{2^{-n}}{2}, 2^{-n}) =: E_n$



$\{E_n\}_n$ disjoint "comparable" subfamily

$$|I_n| \leq 2|E_n| \quad \forall n \in \mathbb{N}$$

$$\int \left(\fint_I |f(y)|\, dy \right) \mathbb{1}_I(x)\, dx$$

$$= \left(\fint_{I_n} |f| \right) |I_n| \leq 2 \left(\fint_{I_n} |f| \right) |E_n| = 2 \int_{E_n} \left(\fint_{I_n} |f| \right) dx \leq 2 \int_{E_n} Mf(x)\, dx$$

What is Sparse domination?

$$|Tf(x)| \lesssim \sum_{I \in \mathcal{S}} \left(\fint_I |f(y)|\,dy \right) \mathbb{1}_I(x)$$

SPARSE

$$|\langle Tf, g \rangle| \lesssim \sum_{I \in \mathcal{S}} \langle |f| \rangle_I \langle |g| \rangle_I |I|$$

<u>Remark</u> Both these dominations imply

$$\|T\|_{L^2(w)} \lesssim [w]_{A_2}$$

What does a sparse domination for S look like?

Sparse domination for Square functions

$$\|S\|_{L^3(w)} \lesssim [w]_{A_3}^{1/2}$$

$$\|Sf\|_{L^3}^2 = \|(Sf)^2\|_{L^{3/2}} = \sup_{\substack{g \in L^3 \\ \|g\|=1}} |\langle (Sf)^2, g\rangle|$$

Weighted estimates at $p=3$ follow from a Quadratic sparse domination:

$$(1) \qquad |\langle (Sf)^2, g\rangle| \lesssim \sum_{I \in \mathcal{S}} \langle |f|\rangle_I^2 \langle |g|\rangle_I |I|$$

Theorem (B. 2020)

Bound (1) holds if $\exists C>0 : \ \forall I \in \mathcal{D}$

$$\langle (S\mathbb{1}_I)^2, \mathbb{1}_I \rangle \leq C|I|$$

Beyond Classical Operators

Consider $T : L^p \to L^p$ ONLY for $p_0 < p < q_0$, $\quad 1 < p_0 < 2 < q_0 < \infty$.

Exemple

$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -\operatorname{div}(\nabla) = -\operatorname{div}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\nabla\right)$$

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_i \in L^\infty(\mathbb{R}^2)$ complex valued, such that $\exists\, \Lambda, \lambda > 0$

$$\lambda |z|^2 \leq \operatorname{Re}(\langle Az, z\rangle) \quad , \quad |\langle Az, \xi\rangle| \leq \Lambda |z||\xi| \quad \forall\, z, \xi \in \mathbb{C}$$

$$\boxed{L := -\operatorname{div}(A\nabla)}$$

Riesz transforms:

$$R_L f := \nabla L^{-1/2} f$$

Square functions:

$$S_L f := \left(\int_0^\infty |Q_t f|^2 \frac{dt}{t}\right)^{1/2}, \quad Q_t = (tL)^{1/2} e^{-tL}$$

Beyond Classical Operators

Consider $T : L^p \to L^p$ ONLY for $p_0 < p < q_0$, $\quad 1 < p_0 < 2 < q_0 < \infty$.

Then $T : L^2(w) \not\to L^2(w) \quad \forall w \in A_2$.

$\Rightarrow$ Restricted class of weights [Auscher, Martell '06]

$$w \in \left(A_{2/p_0} \cap RH_{(q_0/2)'} \right) \subset A_2$$

$$[\![w]\!]_p := [w]_{A_{\frac{p}{p_0}}} \cdot [w]_{RH_{\left(\frac{q_0}{p}\right)'}}$$

where

$$[w]_{RH_s} := \sup_I \left(\fint_I w^s \right)^{1/s} \left(\fint_I w \right)^{-1}$$

is the best constant in

$$\left(\fint_I w^s \right)^{1/s} \lesssim c \fint_I w$$

Remark

$1 < p < q$

$A_1 \subset A_p \subset A_q$

$RH_{1'} \subset RH_{p'} \subset RH_{q'}$

Theorem (Restricted Extrapolation)

Fix $r_0 \in (1,\infty)$. If $\forall w \in A_{r_0/p_0} \cap RH_{(q_0/r_0)'}$

$$\| T : L^{r_0}(w) \to L^{r_0}(w) \| \lesssim [\![w]\!]_{r_0}^{\alpha}$$

then for all $p_0 < r < q_0$

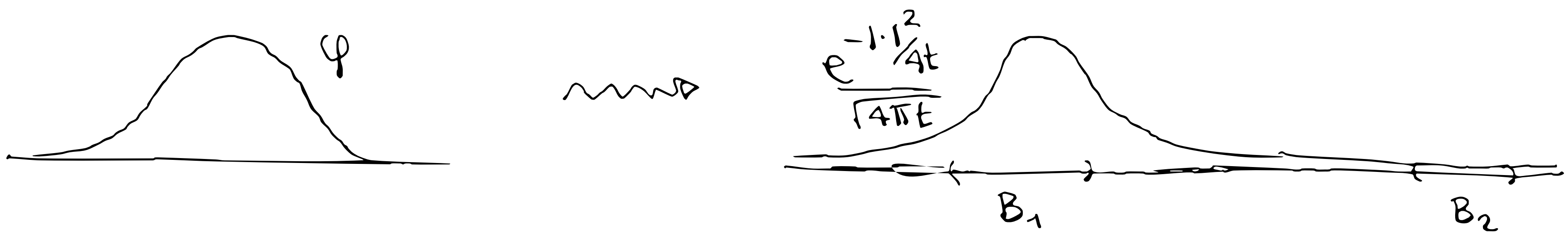
$$\| T : L^{r}(w) \to L^{r}(w) \| \lesssim [\![w]\!]_{r}^{\alpha \max\left\{\frac{q_0 - r}{q_0 - r_0} \cdot \frac{r_0 - p_0}{r - p_0},\, 1\right\}\left(\frac{q_0}{r}\right)'}$$

Remark for $r > r_0$, $\|T\|_{L^r(w)} \lesssim [\![w]\!]_r^{\alpha \left(\frac{q_0}{r}\right)'}$

Beyond Classical Operators

"Heat flow" $e^{-tL}f$ is "smoothing" at scale $\sqrt{t}$

$\langle f \rangle_{B(x,\sqrt{t})} \sim (f * \varphi_{\sqrt{t}})(x)$ $\qquad$ $\left(f * \frac{e^{-|\cdot|^2/4t}}{\sqrt{4\pi t}} \right)(x) = e^{t\Delta} f(x)$

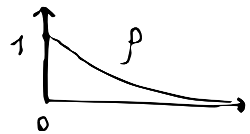


Off-diagonal estimates at scale $\sqrt{t}$

Let f spt on B_1

$$\left(\fint_{B_2} |e^{-tL} f|^{q_0} \, dx \right)^{1/q_0} \lesssim \rho\left(\frac{\operatorname{dist}(B_1, B_2)}{\sqrt{t}} \right) \left(\fint_{B_1} |f|^{p_0} \, dx \right)^{1/p_0}$$

$B_j := B(x_j, \sqrt{t})$, $j = 1,2$, $1 \le p_0 < q_0 < \infty$,



$\lim_{x \to +\infty} |x|^n \rho(x) = 0$ $\qquad$ $\rho(x) = e^{-|x|^2}$ $\qquad$ $\rho(x) = \frac{1}{(1+x^2)^m}$

Theorem (Bernicot, Frey, Petermichl 2016)

Let $T : L^p \to L^p$ for $p \in (p_0, q_0)$, $1 < p_0 < 2 < q_0 < \infty$.

- $\{T \circ (tLe^{-tL})\}_{t>0}$ has (p_0, q_0) <u>off-diagonal estimates</u> (at scale $\sqrt{t}$)

$$\left(\fint_{B_2} |T\, tLe^{-tL} f_{B_1}|^{q_0} \, dx \right)^{1/q_0} \lesssim \rho\left(\frac{\mathrm{dist}(B_1, B_2)}{\sqrt{t}} \right) \left(\fint_{B_1} |f|^{p_0} \, dx \right)^{1/p_0} \qquad \text{(decay)}$$

- (Cotlar type est.) exists $p_1 \in [p_0, 2)$

$$\left(\fint_{B_{\sqrt{t}}(x)} |T e^{-tL} f|^{q_0} \, dy \right)^{1/q_0} \lesssim \inf_{y \in B_{\sqrt{t}}(x)} \left(\mathcal{M}_{p_1}(Tf) + \mathcal{M}_{p_1} f \right) \qquad \text{(regularity)}$$

$$\mathcal{M}_p f = \left(\mathcal{M}(|f|^p) \right)^{1/p}$$

then

$$|\langle Tf, g \rangle| \lesssim \sum_{I \in \mathcal{S}} \langle |f|^{p_0} \rangle_I^{1/p_0} \langle |g|^{q_0'} \rangle_I^{1/q_0'} |I|$$

Remark the sparse domination

$$|\langle Tf, g\rangle| \lesssim \sum_{I\in\mathcal{S}} \langle |f|^{p_0}\rangle_I^{1/p_0} \langle |g|^{q_0'}\rangle_I^{1/q_0'} |I|$$

implies

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim [w]_p^{\alpha(p)}$$

for the critical exponent $p := 1 + \frac{p_0}{q_0'}$, $\alpha(p) = \left(\frac{q_0}{p_0}\right)'$

Remark

In the classical (C-Z) theory $p_0 = q_0' = 1$, so $p = 2$, $\alpha(p) = 1$.

... while for square functions $p_0 = q_0' = 1$, $p_S = 3$, $\alpha_S(p) = \frac{1}{2}$.

Can we improve the power for square functions?

__Theorem__ (Bailey, B., Reguera 2020)

Let $S : L^p \to L^p$ for $p \in (p_0, q_0)$, $1 < p_0 < 2 < q_0 < \infty$.

$$Sf := \left(\int_0^\infty |Q_t f|^2 \frac{dt}{t} \right)^{1/2}$$

- $\{S \circ (tL e^{-tL})\}_{t>0}$ and Q_t have (p_0, q_0) __off-diagonal estimates__ (at scale $\sqrt{t}$)

- (Cotlar type est.) exists $p_1 \in [p_0, 2)$

$$\left(\fint_{B_{\sqrt{t}}} |S e^{-tL} f|^{q_0} dy \right)^{1/q_0} \lesssim \inf_{y \in B_{\sqrt{t}}} \left(M_{p_1}(Sf) + M_{p_1} f \right)$$

then

$$|\langle (Sf)^2, g \rangle| \lesssim \sum_{I \in \mathcal{S}} \langle |f|^{p_0} \rangle_I^{2/p_0} \langle |g|^{(q_0/2)'} \rangle_I^{1/(q_0/2)'} |I|$$

Remark The Quadratic sparse domination

$$\left|\left\langle (Sf)^2, g\right\rangle\right| \lesssim \sum_{I\in\mathcal{S}} \langle |f|^{p_0}\rangle_I^{2/p_0} \, \langle |g|^{(q_0/2)'}\rangle_I^{1/(q_0/2)'} \, |I|$$

implies

$$\|S\|_{L^{\mathfrak{p}_S}(w)\to L^{\mathfrak{p}_S}(w)} \lesssim [\![w]\!]_{\mathfrak{p}_S}^{\alpha_S(\mathfrak{p}_S)}, \quad \mathfrak{p}_S := 2\left(1+\frac{p_0/2}{(q_0/2)'}\right)$$

COMPARISON

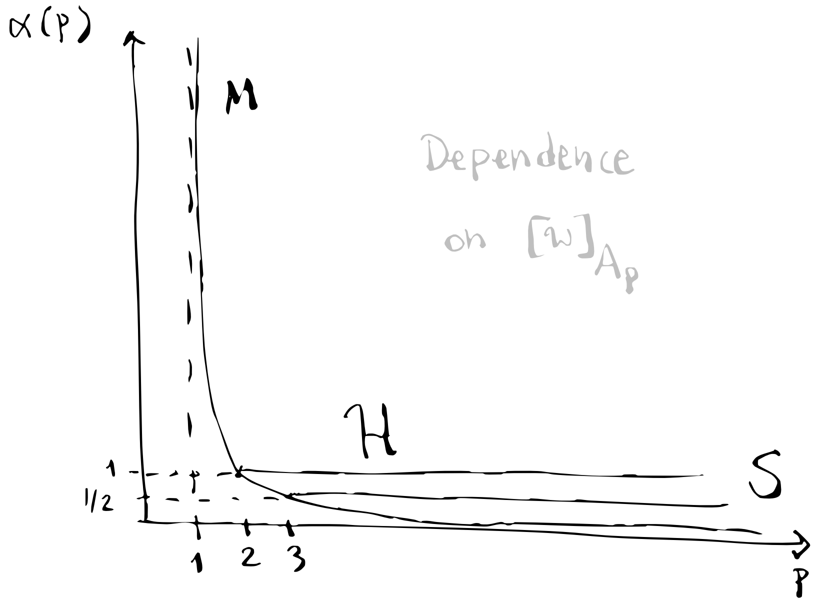
	$\alpha(p)$	critical $\mathfrak{p}$	$\alpha(\mathfrak{p})$
T	$\max\left\{\frac{1}{p-p_0}, \left(\frac{q_0}{p}\right)'\frac{1}{q_0'}\right\}$	$1+\frac{p_0}{q_0'}$	$\left(\frac{q_0}{p_0}\right)'$
S	$\max\left\{\frac{1}{p-p_0}, \left(\frac{q_0}{p}\right)'\frac{1}{2}\frac{1}{(q_0/2)'}\right\}$	$2+\frac{p_0}{(q_0/2)'}$	$\frac{1}{2}\left(\frac{q_0}{p_0}\right)'$

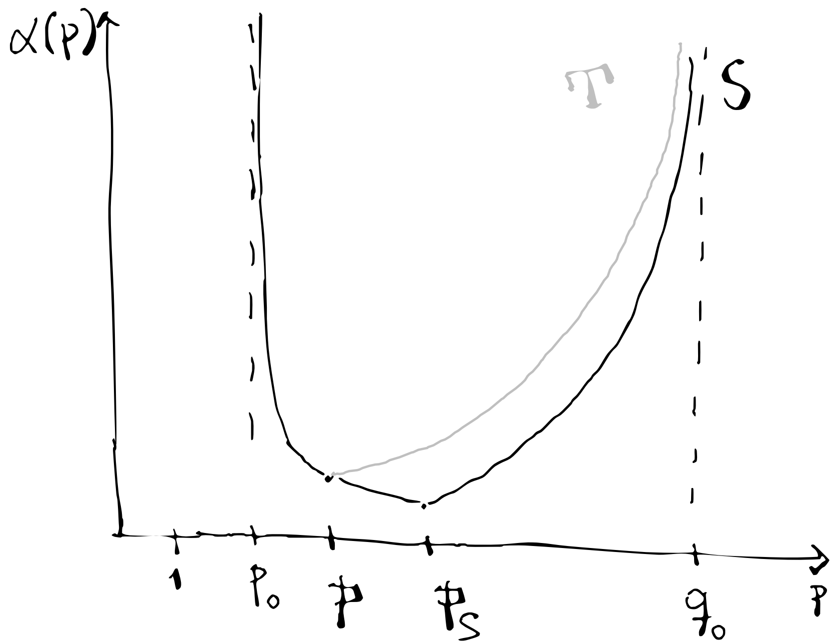
Optimal dependence

	$\alpha(p)$	Extrap. from
$\mathcal{H}$	$\max\{\frac{1}{p-1}, 1\}$	$\mathbf{p}=2$
$\mathcal{S}$	$\max\{\frac{1}{p-1}, \frac{1}{2}\}$	$\mathbf{p}=3$
$\mathcal{M}$	$\frac{1}{p-1}$	/

$(p > \mathbf{p})$

$\mathcal{T}$	$\left(\frac{q_0}{p}\right)' \frac{1}{q_0'}$	$1 + \frac{p_0}{q_0'} = \mathbf{p}$
$\mathcal{S}$	$\left(\frac{q_0}{p}\right)' \frac{1}{2} \frac{1}{(q_0/2)'}$	$2 + \frac{p_0}{(q_0/2)'} = \mathbf{p}_S$





Idea of the proofs (sublinear case)

$\mathrm{supp}\, f, \mathrm{supp}\, g \subseteq 5\Omega$

$$\int_\Omega Tf \cdot g = \int_{\Omega \setminus E(\Omega)} \text{not large} + \int_{E(\Omega)} \text{large}$$

$$E(\Omega) \subseteq \bigcup_{I \in \mathcal{E}_0} I$$ maximal dyadic interval

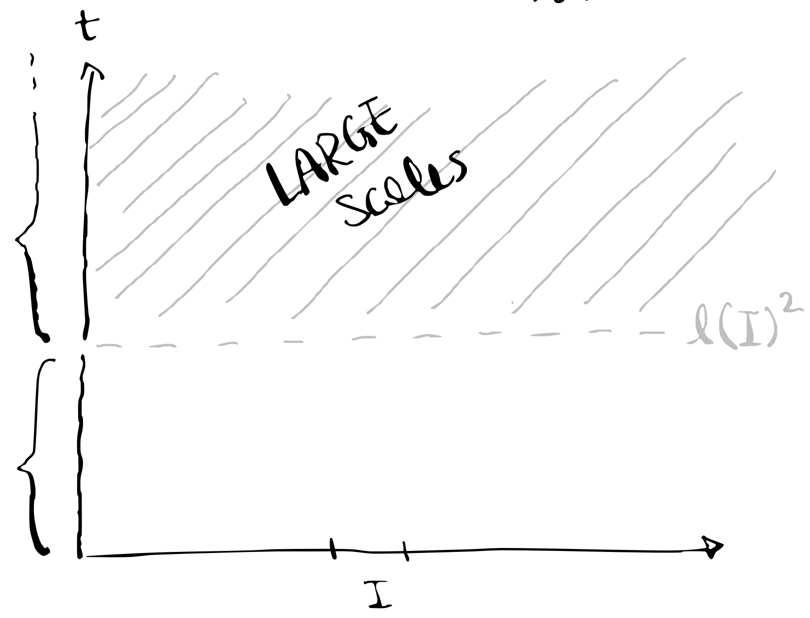
& Split scales: $|Tf| \leq \int_0^\infty \underbrace{|TQ_t f|}_{T_t f} \frac{dt}{t}$

large part:

$$\sum_{I \in \mathcal{E}_0} \int_I \int_0^\infty |T_t f| \frac{dt}{t}\, g$$

$$\int_{\ell(I)^2}^\infty |T_t f| \frac{dt}{t}$$

$$\int_0^{\ell(I)^2} |T_t f| \frac{dt}{t}$$

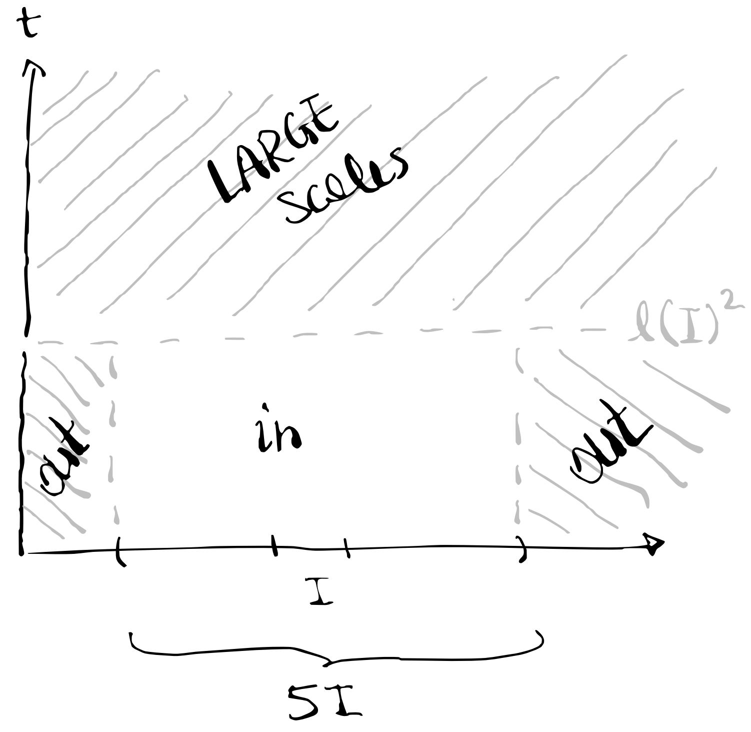


Idea of the proofs (sublinear case)

$\operatorname{supp} f, \operatorname{supp} g \subseteq 5\Omega$

$$\int_\Omega Tf \cdot g = \int_{\Omega \setminus E(\Omega)} \text{not large} + \sum_I \int_I \text{LARGE scales} \cdot g + \sum_I \int_I \text{small scales} \cdot g$$

$$|I| \fint_I |T_t f| \cdot g \le |I| \left(\fint_I |T_t f|^{q_0} \right)^{1/q_0} \left(\fint_I |g|^{q_0'} \right)^{1/q_0'}$$



Localise in space

$$f = f 1\!\!1_{5I} + f 1\!\!1_{\mathbb{R} \setminus 5I} =: f_{in} + f_{out}$$

$\boxed{\text{out}}$: exploit off-diag. est.

$\boxed{\text{in}}$: $\int_I |T_t f 1\!\!1_{5I}| \cdot g$ iterate □

Idea of the proofs (Quadratic case)

$\operatorname{supp} f, \operatorname{supp} g \subseteq 5\Omega$

$$\int_\Omega (Sf)^2 \cdot g = \int_{\Omega \setminus E(\Omega)} \text{not large} + \int_{E(\Omega)} \text{large}$$

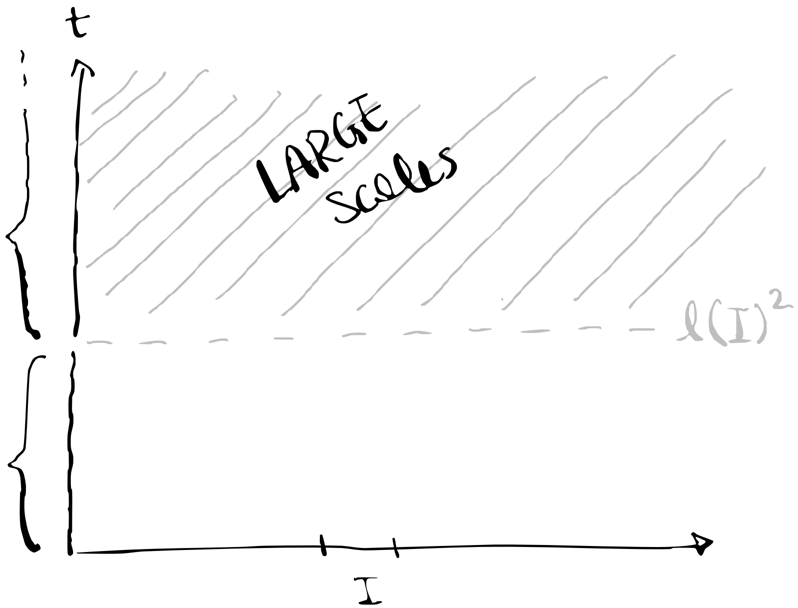
$E(\Omega) \subseteq \bigcup_{I \in \mathcal{E}_0} I$ maximal dyadic interval & Split scales: $(Sf)^2 = \int_0^\infty |Q_t f|^2 \frac{dt}{t}$

large part:

$$\sum_{I \in \mathcal{E}_0} \int_I \int_0^\infty |Q_t f|^2 \frac{dt}{t}\, g$$

$$\int_{\ell(I)^2}^\infty |Q_t f|^2 \frac{dt}{t}$$

$$\int_0^{\ell(I)^2} |Q_t f|^2 \frac{dt}{t}$$



Idea of the proofs

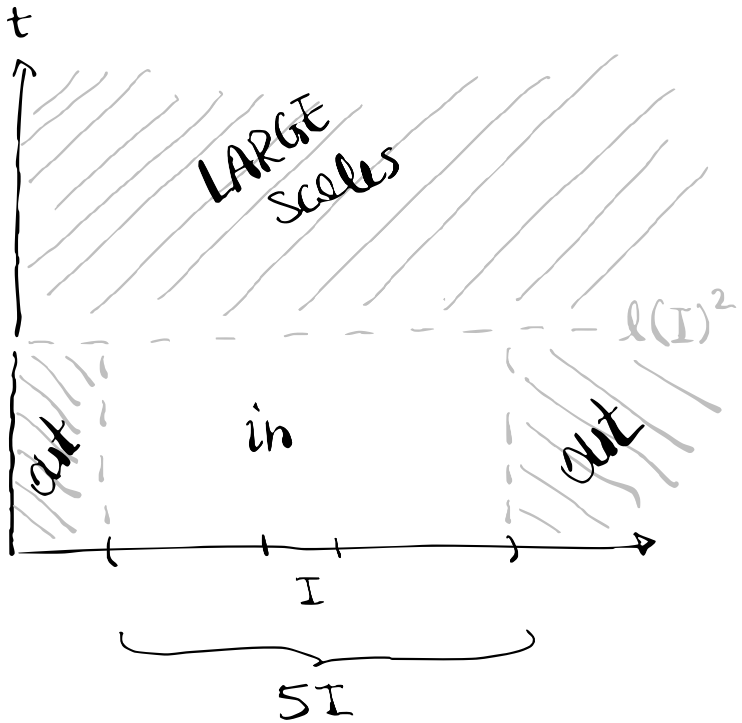
$\operatorname{supp} f, \operatorname{supp} g \subseteq 5\Omega$

$$\int_\Omega (Sf)^2 \cdot g = \int_{\Omega \setminus E(\Omega)} \text{not large} + \sum_I \int_I \text{LARGE scales} \cdot g + \sum_I \int_I \text{small scales} \cdot g$$

$$|I| \fint_I |Q_t f|^2 \cdot g \leq |I| \left(\fint_I |Q_t f|^{q_0} \right)^{2/q_0} \left(\fint_I |g|^{(q_0/2)'} \right)^{1/(q_0/2)'}$$

Localise in space: $f = f_{in} + f_{out}$

$$|Q_t f|^2 = |Q_t f_{in}|^2 + |Q_t f_{out}|^2 + 2 \boxed{in} \boxed{out}$$



$\boxed{out}$: exploit off-diag. est.

$\boxed{in}$: $\int_I |Q_t f \mathbb{1}_{5I}|^2 \cdot g$ iterate $\square$

Open Question

Find the best $\gamma > 0$ such that

$$\lim_{p \to q_0^-} \|S\|_{L^p} \sim \frac{1}{(q_0 - p)^\gamma}$$

Quadratic sparse domination implies

naive guess

$$\gamma \le \frac{1}{2} \frac{1}{(q_0/2)'} \le \frac{1}{2} \frac{1}{q_0'} \le \frac{1}{q_0'}$$

[BFP16]

Question: Is it sharp?

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[BFP16]

Question: Is it sharp?

Thank you for listening

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