4-Manifolds and Kirby Calculus

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Abstract. This text is intended to be an introduction and reference for the differential topology of 4-manifolds as it is currently understood. It is presented from a topologist's viewpoint, often from the perspective of handlebody theory (Kirby calculus), for which an elementary and comprehensive exposition is given. Additional topics include complex, symplectic and Stein surfaces, applications of gauge theory, Lefschetz pencils and exotic smooth structures. The text is intended for students and researchers in topology and related areas, and is suitable for an advanced graduate course. Familiarity with basic algebraic and differential topology is assumed.

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Preface

The past two decades represent a period of explosive growth in 4-manifold theory. From a desert of nearly complete ignorance, the theory has flourished into a virtual rain forest of ideas and techniques, a lush ecosystem supporting complex interactions between diverse fields such as gauge theory, algebraic geometry and symplectic topology, in addition to more topological ideas. Numerous books are appearing that discuss smooth 4-manifolds from the viewpoint of other disciplines. The present volume is intended to introduce the subject from a topologist's viewpoint, bridging the gaps to other disciplines and presenting classical but important topological techniques that have not previously appeared in expository literature.

For a better perspective on the rise of 4-manifold theory, it is useful to consider the history of topology. Manifolds have been a central theme of mathematics for over a century. The topology of manifolds of dimensions ≤ 2 (curves and surfaces) has been well understood since the nineteenth century. Although 3-manifold topology is much harder, there has been steady progress in the field for most of the twentieth century. High-dimensional manifold topology was revolutionized by the s-cobordism and surgery theorems, which were developed in the 1960's into powerful tools for analyzing existence and uniqueness questions about manifolds of dimension ≥ 5. The resulting theory has long since matured into a subject with a very algebraic flavor. In dimension 4, however, there was not enough room to apply the fundamental "Whitney trick" to prove these theorems, and as a result, very little was known about 4-manifold topology through the 1970's. The first revolution came in 1981 with Michael Freedman's discovery that the Whitney trick could be performed in dimension 4, provided that we ignore smooth structures and work with the underlying topological manifolds up to homeomorphism (and provided that the fundamental group is suitably "small"). The resulting theory [**FQ**] led quickly to a complete classification of closed, simply connected topological 4-manifolds, and topological 4-manifold theory now seems closely related to the theory of high-dimensional manifolds. Freedman's revolution was immediately followed by the 1982 counterrevolution of Simon Donaldson. Using gauge theory (differential geometry and nonlinear analysis), Donaldson showed that smooth 4-manifolds are much different from their high-dimensional counterparts. In fact, the predictions made by the s-cobordism and surgery conjectures for smooth 4-manifolds failed miserably, resulting in a dramatic clash between the theories of smooth and topological manifolds in this dimension. For example, this is the only dimension in which a fixed homeomorphism type of closed manifold is represented by infinitely many diffeomorphism types, or where there are manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^n$. (In fact, there are uncountably many such "exotic $\mathbb{R}^4$'s".) One might think of dimension 4 as representing a phase transition between low- and high-dimensional topology, where we find uniquely complicated phenomena and diverse connections with other fields. Donaldson's program of analyzing the self-dual Yang-Mills equations [**DK**] was central to smooth 4-manifold theory for 12 years, until it was superseded in 1994 (several revolutions later) by analysis of the Seiberg-Witten equations [**KKM**], [**Mr1**], [**Sa**], which simplifies and expands Donaldson's original approach and results.

The results of gauge theory, from Donaldson through the Seiberg-Witten equations, are primarily in a negative direction, and require balance by positive results. That is, gauge theory proves the nonexistence of smooth manifolds satisfying various constraints, the nonexistence of connected-sum splittings, and the nonexistence of diffeomorphisms between pairs of manifolds. One needs a different approach for the corresponding existence results. While many useful examples come from algebraic geometry [**BPV**] and symplectic topology [**McS1**], perhaps the most powerful general technique for existence results (particularly for manifolds with small Betti numbers) is Kirby calculus. This technique, which allows one to see the internal structure of a 4-manifold (or its boundary 3-manifold) without loss of information, was created and developed into a fine art in the late 1970's by topologists such as Akbulut, Fenn, Harer, Kaplan, Kirby, Melvin, Rourke, Rolfsen and Stern. However, the theory was handicapped by the pre-Donaldson absence of any way to prove negative results. Much time was spent on ambitious goals that gauge theory now shows are impossible. Eventually, the theory was abandoned by all but the most stalwart practitioners. Since the advent of gauge theory, however, Kirby calculus has entered a Renaissance. Armed with the knowledge of what *not* to attempt, topologists are using the calculus to construct new manifolds with novel gauge-theoretic properties, some of which are nonalgebraic or even nonsymplectic, and to show that other examples are diffeomorphic or to decompose them into simple pieces. The insight provided by the calculus into the internal structure of manifolds meshes with gauge theory to create an even more powerful tool for analyzing 4-manifolds. In addition, surprising connections have emerged with affine complex analysis and contact topology [**G13**], [**G14**] since a discovery of Eliashberg led to a theory of Kirby diagrams for representing Stein surfaces.

One of the main goals of the present book is to provide an exposition of Kirby calculus that is both elementary and comprehensive, since there appears to be no previous reference in the literature that satisfies either of these conditions. We have attempted a complete exposition, providing careful proofs of the main theorems and constructions, either directly or through references to the literature (notably to [**M4**] and [**RS**] for careful treatments of handlebody theory in general dimensions). This is at least partly to avoid conveying a false impression of Kirby calculus as being "just pictures and not proofs". For easy reference, we have included an index of important diagrams, following the glossary of notation in Chapter 13. The reader should note that we have included Kirby diagrams representing all of the main types of closed, simply connected 4-manifolds (as viewed from the current perspective of the theory), namely complex surfaces of rational, elliptic and general type, a symplectic but noncomplex manifold and an irreducible nonsymplectic one. (We have also included an example with even $b_2^\pm$ that might be irreducible.) Chapter 13 also provides an index for Kirby moves and related operations such as Rolfsen moves, Gluck twists and logarithmic transformations. The text has been liberally sprinkled with exercises intended to increase the reader's comprehension; many of these are labelled with an asterisk and solved in Chapter 12.

The remaining goal of the book is to introduce 4-manifold theory in its current state. There are many books avaliable on the subject, but ours is almost unique in describing the theory from the point of view of differential topology. The other reference from this viewpoint is Kirby [**K2**]; our text is intended to be complementary to it. Parts of the text were inspired by Harer, Kas and Kirby [**HKK**]; where overlap occurs we have tried to choose a more elementary and leisurely approach. There are many references for gauge theory as applied to 4-manifolds, notably [**DK**] (one of the most recent references from the viewpoint of the self-dual equations), and [**KKM**], [**Mr1**], [**Sa**] on Seiberg-Witten theory. These provide detailed treatments, so our approach to gauge theory is to sketch the main ideas and applications with references for details. Similarly, the theory of complex surfaces is covered in detail in [**BPV**], and symplectic topology is carefully treated in

[**McS1**], so we again focus on the main applications to 4-manifold topology while avoiding unnecessary coverage of other aspects of these theories. For topological 4-manifolds, the reader is referred to [**FQ**] after our brief discussions. Although we treat Rolfsen calculus in some detail, the reader is also referred to [**Ro**] for this 3-dimensional technique related to Kirby calculus. One other noteworthy reference is Kirby's latest list [**K4**] of problems in low-dimensional topology; many of these problems are directly related to 4-manifolds and Kirby calculus.

This book is divided into four parts. The first part covers introductory material and basic techniques for later use, as well as an outline of the current state of the theory of 4-manifolds and surfaces contained in them. Part 2 is our main exposition of Kirby calculus. It is essentially independent of Part 1, except for such elementary notions as intersection forms. The logical dependence of the sections of Part 2 is approximately given by Figure 0.1. (Dashed arrows indicate only occasional or minor dependence.) Part 3 ties together the two previous parts by presenting more advanced applications of Kirby calculus, and consists of five mostly independent chapters intended to cover current research areas within 4-manifold theory and their connections to other disciplines. While we have attempted to include the most recent developments, such a goal is inevitably doomed by the rapid change of the field. Solutions to exercises and the tables described above comprise Part 4. The book can be used as a graduate text, with each of the first two parts providing enough material for nearly a semester. The topics in the third part provide supplementary material intended to introduce a student to research in 4-manifold topology.

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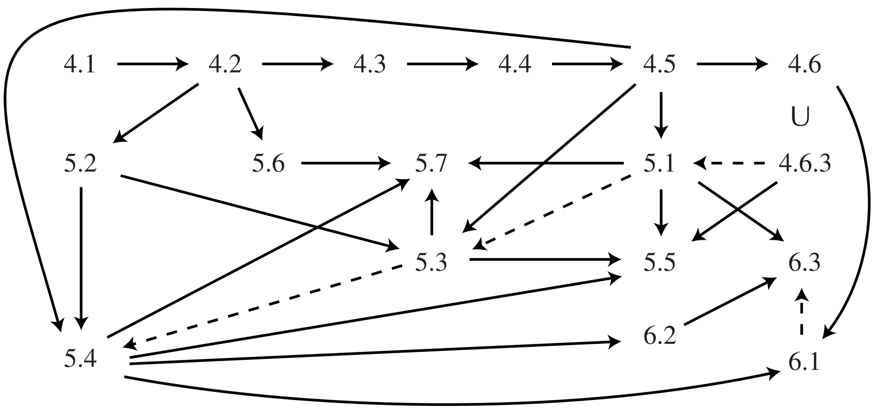


Figure 0.1. Logical dependence of the sections of Part 2

Part 1

4-Manifolds

The first part of these notes is devoted to introducing the fundamental theorems, notions and constructions of 4-manifold theory. The theorems in Chapter 1 are intended to inform the reader about the present status of the classification of topological and smooth 4-manifolds. In Chapter 2 we will pay special attention to *surfaces* in 4-manifolds — information of this kind seems to be increasingly important in the understanding of the smooth structure of a 4-manifold. Finally, Chapter 3 gives further families of examples of 4-manifolds and a short review of the classification of complex surfaces. Throughout the first three chapters we will present various examples of compact 4-manifolds. Most of these examples originate from complex geometric constructions, and a fairly explicit discussion of the smooth topology of these manifolds can be given. More sophisticated examples will be given in Part 3 (e.g., in Chapters 7, 8 and 10).

Chapter 1

Introduction

1.1. Manifolds

We assume that the reader is familiar with the basics of algebraic topology, such as homotopy theory and singular homology and cohomology. We will use the terms *principal G-bundle*, *tangent bundle* of a manifold, *associated vector bundle* and *section* of a bundle without defining them. Similarly, various forms of *Poincaré duality* will be used without explicit description of the theorems. (Detailed treatments of these topics can be found in, e.g., [**GP**], [**MS**] and [**Sp**].) Definitions of *topological* and *smooth manifolds*, *orientability*, *complex structures* and *ambient isotopy* are given to serve as a reference in forthcoming discussions. For similar reasons, we have included two sections (Sections 1.4 and 5.6) introducing *characteristic classes* and *spin structures*. An n-dimensional manifold (with possibly nonempty boundary) is usually denoted by X; we use M in contexts where the manifold is conveniently thought of as a boundary component of some other manifold. If we wish to emphasize that the 4-manifold is equipped with a complex structure, we denote it by S (being a complex *surface*). We define $\mathbb{R}^n_+$ as the upper half space of $\mathbb{R}^n$, in coordinates, $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$; otherwise we use standard notation.

Definition 1.1.1. A separable Hausdorff topological space X is an n-dimensional *topological (or C^0-) manifold* if for every point $p \in X$ there is an open neighborhood U of p in X which is homeomorphic to an open subset of $\mathbb{R}^n_+$. A pair (U_α, ϕ_α) of such a neighborhood and homeomorphism is called a *chart*. A collection of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is an *atlas* if it is a cover of X, that is, $\bigcup\{U_\alpha \mid \alpha \in A\} = X$. The map $\phi_\beta \circ \phi_\alpha^{-1}$ (on $\phi_\alpha(U_\alpha \cap U_\beta)$) is the *transition function* between the charts (U_α, ϕ_α) and (U_β, ϕ_β). The points of

X corresponding to points in $\{(x_1, \dots, x_n) \mid x_n = 0\} = \mathbb{R}^{n-1} \subset \mathbb{R}^n_+$ form a submanifold ∂X of dimension $n-1$, which is called the *boundary* of X. The topological manifold X with an atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is a C^r*-manifold* $(r = 1, 2, 3, \dots, \infty)$ if the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ $(\alpha, \beta \in A)$ are C^r-maps. In the case $r = \infty$, X is called a *smooth* manifold. In the notation, we will not specify the atlas giving the C^r-structure of X. Note that by definition a C^r-structure specifies a C^s-structure on X for all $0 \leq s \leq r$. A map $f\colon X \to X'$ is a C^r*-map* between the two C^r-manifolds X and X' if f is C^r on every chart of the given C^r-atlases. A homeomorphism $f\colon X \to X'$ is a C^r*-diffeomorphism* if both f and f^{-1} are C^r-maps. Two C^r-structures on X are *isotopic* if the identity map id_X is isotopic (homotopic through homeomorphisms) to a C^r-diffeomorphism between the structures. We will not distinguish between isotopic C^r-structures, and frequently use C^r-diffeomorphisms to identify C^r-manifolds with each other. We say that X is *closed* if it is compact and $\partial X = \emptyset$. We call a space X a *singular manifold* if there is a finite subset $Sing \subset X$ such that $X - Sing$ is a smooth manifold. The *n-dimensional disk* $\{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ will be denoted by D^n; in particular, D^n is a compact manifold and the boundary ∂D^n is the $(n-1)$*-dimensional sphere* S^{n-1}.

An *orientation* of the Euclidean vector space $\mathbb{R}^n$ is simply a choice of one orbit of the set of ordered bases under the action of the connected group $GL^+(n;\mathbb{R}) = \{A \in GL(n;\mathbb{R}) \mid \det(A) > 0\}$. The chosen equivalence class is referred to as the set of *positive* bases; the rest of the bases (which form the other equivalence class) are the *negative* bases. An orientation can be given by fixing an ordering (up to even permutations) of a given basis.

Definition 1.1.2. Let X be a smooth n-dimensional manifold, hence it admits a *tangent bundle* $TX \to X$ with fibers isomorphic to $\mathbb{R}^n$ (see, e.g., [**GP**], [**MS**]). A consistent choice of orientation of the tangent space at every point of X is called an *orientation* of X. (By consistent we mean that at each point in X there is a chart (V, φ) mapping into $\mathbb{R}^n$ with its standard orientation, such that $d\varphi_p$ preserves orientation at every point p of V.) This choice, however, cannot be made for every manifold. X is called *orientable* if it admits an orientation, and X is *oriented* if an orientation of X is fixed. The standard convention of "outward normal first" provides an orientation of ∂X induced by an orientation of X [**GP**]: At $p \in \partial X$ the basis $(v_1, \dots, v_{n-1})$ of $T_p\partial X$ is positive if $(\nu, v_1, \dots, v_{n-1})$ is a positive basis of T_pX, where the vector ν stands for the outward normal vector, which is tangent to X but not to ∂X and points out of X. Since orientation plays a key role throughout 4-manifold theory, we will always be careful about specifying a fixed orientation of the manifold X. If we change the orientation on every component of X, the new object will be denoted by $\overline{X}$.

Remark 1.1.3. It can be shown that an orientation specifies a class $[X]$ in $H_n(X, \partial X; \mathbb{Z})$, called the *fundamental class* of the oriented manifold (see, e.g., the appendix of [**MS**]). For X noncompact, we should use "locally finite" homology on infinite chains. In this way, orientability can easily be extended to topological manifolds as well: An n-dimensional connected manifold X is orientable if $H_n(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$, and an orientation of X is simply a choice of a generator of the group $H_n(X, \partial X; \mathbb{Z})$. The induced orientation of ∂X can be seen as the image of $[X] \in H_n(X, \partial X; \mathbb{Z})$ under the map $H_n(X, \partial X; \mathbb{Z}) \to H_{n-1}(\partial X; \mathbb{Z})$ of the long exact sequence of the pair $(X, \partial X)$; moreover $[\overline{X}] = -[X] \in H_n(X, \partial X; \mathbb{Z})$. In terms of bundles, the orientability of X can be rephrased as follows. The smooth n-dimensional manifold X is orientable if the structure group of the tangent bundle $TX \to X$ (which is $GL(n; \mathbb{R})$) can be reduced to its connected component $GL^+(n; \mathbb{R})$; by fixing a reduction we specify an orientation for X. Note that this approach to orientability can easily be extended to arbitrary (real) vector bundles (cf. Lemma 1.4.23).

Definition 1.1.4. An atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A_{\mathbb{C}}\}$ on a (real) $2n$-dimensional manifold X is a *complex structure* if each ϕ_α is a homeomorphism between U_α and an open subset of $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$), and the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic. Complex manifolds are canonically oriented. This is because the connected group $GL(n; \mathbb{C})$ lies in $GL^+(2n; \mathbb{R})$, so any complex n-dimensional vector space is canonically oriented — by choosing a complex isomorphism with $\mathbb{C}^n = \mathbb{C} \times \ldots \times \mathbb{C}$, with $\mathbb{C}$ oriented as a real vector space by the ordered basis $(1, i)$.

Definition 1.1.5. A smooth (resp. topological) *isotopy* between embeddings $\varphi_0, \varphi_1 \colon Y \to X$ is a smooth (resp. topological) homotopy $\varphi_t \colon Y \to X$ $(0 \leq t \leq 1)$ through embeddings. If an isotopy exists, then φ_0 and φ_1 are *isotopic*. By the Isotopy Extension Theorem (Theorem 5.8 of [**M4**]), if Y is compact then any smooth isotopy $Y \to \operatorname{int} X$ can be extended to an *ambient isotopy*, an isotopy $\Phi_t \colon X \to X$ through diffeomorphisms such that $\Phi_0 = \mathrm{id}_X$ and $\varphi_t = \Phi_t \circ \varphi_0$ for each t. Two (possibly singular) submanifolds Y_1, Y_2 in a manifold X are *ambiently isotopic* if there is a diffeomorphism of X homotopic to id_X through diffeomorphisms which maps Y_1 to Y_2.

Having dispensed with the preliminaries, we state a few theorems and conjectures related to the classification problem of n-dimensional manifolds. Theorem 1.1.6 demonstrates the fact that although the notion of C^r-manifold is defined for every integer r $(0 \leq r \leq \infty)$, the $r = 0$ and $r = \infty$ cases are the only interesting ones in terms of classification.

Theorem 1.1.6. ([**Mu**]) *Suppose that X is a C^r-manifold and $1 \leq r \leq k$ (including $k = \infty$). Then there is a C^k-atlas of X for which the induced*

C^r-structure is isotopic to the original C^r-structure of X. (In fact, id_X *is a C^r-diffeomorphism between them.) Moreover, this C^k-structure is unique up to isotopy (through C^r-diffeomorphisms); consequently the C^r-manifold X admits a unique induced C^k-structure for every $k \geq r$. (As we will see, this statement does not hold for $r = 0$.)* □

Our primary aim in manifold theory is to classify topological manifolds, i.e., to give a complete list of n-dimensional (closed) topological manifolds, and to find a way to tell which topological manifolds carry smooth structures (have C^∞-atlases). Furthermore, if there is one such atlas, we would like to determine the total number of these up to diffeomorphism. In most dimensions this aim cannot be achieved for algebraic reasons (cf. Theorem 1.2.33 and Exercise 5.1.10(c)); in those cases we will impose further conditions (like simple connectivity) for the manifolds at hand. For a better understanding of results concerning 4-manifolds, we will conclude this section with theorems concerning manifolds of dimension different from 4. Assume that the manifolds we are working with are closed, connected and oriented. The classification problem is easy in dimension 1 and classical in dimension 2. Up to homeomorphism there is only one topological 1-manifold with the above properties, and this is the circle $S^1 = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$; it admits a unique isotopy class of smooth structures. For $n = 2$, the (oriented) topological 2-manifolds are precisely the surfaces Σ_g with genus g ($g = 0, 1, 2, \dots$); in particular, Σ_0 is the sphere S^2 and Σ_1 is the 2-dimensional torus $T^2 = S^1 \times S^1$. All these topological manifolds carry unique smooth structures (up to isotopy); actually these manifolds carry complex structures as well. The classification problem in dimension 3 is among the most popular in contemporary mathematics. Although we will discuss constructions of 3-manifolds in the present volume, we will not address the classification problem in dimension 3. It is known that every topological 3-manifold admits a unique smooth structure [**Mo**]; the classification problem of topological 3-manifolds is, however, still unsolved. Understanding 3-manifolds homotopy equivalent to the 3-dimensional sphere S^3 would be a major step in this direction; see the conjecture below. (For further discussion of 3-manifolds, see for example [**He**], [**N**], [**Ro**], [**Th2**].)

Conjecture 1.1.7. (Poincaré Conjecture) *A simply connected closed 3-manifold is homeomorphic to S^3.*

For topological manifolds of dimension $n \geq 5$ there is sophisticated machinery for dealing with both the existence problem and the number of nonisotopic smooth structures on a given topological manifold. Parts of this theory will be mentioned later on in this volume (e.g., Theorem 9.2.2), so here we only mention one theorem (which is false in dimension 4):

Theorem 1.1.8. *If X^n is a compact n-dimensional topological manifold and $n \geq 6$ (or $n \geq 5$ and $\partial X = \emptyset$), then there are only finitely many smooth structures on X (up to isotopy). In particular, there are smooth manifolds $Y_1, \ldots, Y_k$ homeomorphic to X such that any smooth manifold homeomorphic to X is diffeomorphic to some Y_i. (Here k might be 0, meaning that there is no smooth structure on X.)* □

Finally, we quote a theorem which emphasizes the special behavior of 4-dimensional manifolds. Let X be a smooth noncompact n-dimensional manifold.

Theorem 1.1.9. *If X is homeomorphic to $\mathbb{R}^n$ and $n \neq 4$ then X is diffeomorphic to $\mathbb{R}^n$. If $n = 4$, this statement is false, there are "exotic" $\mathbb{R}^4$'s; such examples will be given in Sections 9.3 and 9.4.* □

In general, the term *exotic smooth structure* is used to refer to smooth structures not diffeomorphic to the given one on a smooth manifold X. These correspond to manifolds homeomorphic to X but not diffeomorphic to it.

Remark 1.1.10. Another sort of structure frequently used by topologists is a *piecewise linear (PL-) structure*, which is defined by an atlas whose transition functions respect a suitable triangulation of $\mathbb{R}^n$ (e.g., [**RS**]). Any smooth structure determines a PL-structure, and the converse holds for $n \leq 6$ [**HM**], so for our purposes PL-structures are equivalent to smooth structures.

1.2. 4-manifolds

We begin our discussion about 4-dimensional manifolds by defining the *intersection form* of a compact, oriented, topological 4-manifold X. Recall that when X is oriented, it admits a fundamental class $[X] \in H_4(X, \partial X; \mathbb{Z})$.

Definition 1.2.1. The symmetric bilinear form

$$Q_X \colon H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_X(a, b) = \langle a \cup b, [X] \rangle = a \cdot b \in \mathbb{Z}$ is called the *intersection form* of X. Since by Poincaré duality $H_2(X; \mathbb{Z}) \cong H^2(X, \partial X; \mathbb{Z})$, Q_X is defined on $H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z})$ as well.

Note that for this definition of Q_X we only need the topological structure of X. Clearly, $Q_X(a, b) = 0$ if a or b is a torsion element, hence Q_X descends to a pairing on homology mod torsion. By choosing a basis of $H_2(X; \mathbb{Z})/\text{Torsion}$, we can represent Q_X by a matrix. The matrix M of Q_X transforms under a basis transformation C as $C^T M C$. Consequently the determinant $\det M$ is independent of the choice of the basis over $\mathbb{Z}$; we sometimes denote this by $\det Q_X$.

Remarks 1.2.2. **(a)** The above definition of Q_X can be extended to cohomology with arbitrary coefficient ring R. When $R = \mathbb{Z}_2$, the theory generalizes in the obvious way to nonorientable manifolds. For X orientable, the inclusion $r\colon H_2(X;\mathbb{Z}) \otimes \mathbb{Z}_2 \to H_2(X;\mathbb{Z}_2)$ given by the Universal Coefficient Theorem preserves the intersection form; note that r is an isomorphism if $H_1(X;\mathbb{Z})$ has no 2-torsion. The definition also generalizes to noncompact manifolds if we use compactly supported cohomology (which is Poincaré dual to ordinary homology). The intersection form of a compact manifold will not change if we remove its boundary.

(b) Using the same idea, another pairing Q'_X can be defined for the oriented manifold X: The map $Q'_X\colon H^2(X;\mathbb{Z}) \times H^2(X,\partial X;\mathbb{Z}) \to \mathbb{Z}$ is given as $Q'_X(a,b) = \langle a \cup b, [X]\rangle$ (and the second group is compactly supported in the noncompact case). In our discussion we will mainly use Q_X.

(c) By the definition of the intersection form we have $Q_{\overline{X}} = -Q_X$.

If X is a smooth manifold, then $Q_X(a,b)$ can be interpreted as the intersection number of certain submanifolds in X. For a better understanding of this relation we need a little preparation. Let X^n be a smooth n-dimensional manifold. A class $\alpha \in H_2(X^n;\mathbb{Z})$ is *represented by* a closed, oriented surface Σ if there is an embedding $i\colon \Sigma \hookrightarrow X$ such that $i_*([\Sigma]) = \alpha$. (Again, $[\Sigma] \in H_2(\Sigma;\mathbb{Z})$ is the fundamental class of Σ).

Proposition 1.2.3. *Let X be a closed, oriented, smooth 4-manifold. Then every element of $H_2(X;\mathbb{Z})$ can be represented by an embedded surface.*

Proof. Elements of $H^2(X;\mathbb{Z})$ are in 1-1 correspondence with $U(1)$-bundles over X (cf. Proposition 1.4.1). For $\alpha \in H_2(X;\mathbb{Z})$ take its Poincaré dual $a = PD(\alpha) \in H^2(X;\mathbb{Z})$ and denote the corresponding $U(1)$-bundle by $L_\alpha \to X$. The zero set of a generic section of the bundle $L_\alpha \to X$ will be a smooth surface representing α. □

Remark 1.2.4. The proposition is also true if X has boundary or is noncompact or nonorientable, and the analogous statement holds with $\mathbb{Z}_2$-coefficients if we allow Σ to be nonorientable. (See Exercise 4.5.12(b).) Note that if X is simply connected, then by the Hurewicz Theorem $\pi_2(X) \cong H_2(X;\mathbb{Z})$. This implies that for a simply connected 4-manifold every second homology element can be represented by an *immersed* sphere. Such an immersion will not be an embedding in general, but one can assume that an immersion $S^2 \to X^4$ intersects itself only in transverse double points.

Suppose that X^4 is closed and oriented. For $a, b \in H^2(X;\mathbb{Z})$ take surface representatives Σ_α and Σ_β of the Poincaré duals $\alpha = PD(a)$ and $\beta = PD(b)$. Suppose furthermore that Σ_α and Σ_β have been chosen generically, so that their intersections are all transverse. The orientations of Σ_α and Σ_β —

together with the fixed orientation of the ambient 4-manifold X — assign a sign ± 1 to every intersection point of Σ_α and Σ_β in the following way [**GP**]. By concatenating positive bases of the tangent spaces $T_p\Sigma_\alpha$ and $T_p\Sigma_\beta$ at a point $p \in \Sigma_\alpha \cap \Sigma_\beta$ we get a basis of T_pX. The sign of the intersection at p is positive if this basis is positive, and negative otherwise. Note that the sign does not depend on the order of $\{\alpha, \beta\}$, but does depend on the orientations of Σ_α and Σ_β. Now we are in the position to give the geometric interpretation of Q_X — this description explains the name of it.

Proposition 1.2.5. *For $a, b \in H^2(X; \mathbb{Z})$ and $\alpha, \beta \in H_2(X; \mathbb{Z})$ as above, $Q_X(a, b)$ is the number of points in $\Sigma_\alpha \cap \Sigma_\beta$, counted with sign.*

Proof. Assume that η_1 is a smooth 2-form on X representing the image of $a \in H^2(X; \mathbb{Z})$ under the map $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{R})$ induced by $\mathbb{Z} \hookrightarrow \mathbb{R}$. Assume furthermore that η_1 is supported in a small tubular neighborhood of Σ_α. Choose a 2-form η_2 similarly for b and Σ_β. Find a coordinate chart (x, y, u, v) around each intersection point $p \in \Sigma_\alpha \cap \Sigma_\beta$ in which Σ_α is given by $\{x = 0, y = 0\}$ and Σ_β is given by $\{u = 0, v = 0\}$, and assume that $\eta_1 = f(x, y)dx \wedge dy$, $\eta_2 = f(u, v)du \wedge dv$ (where f is a bump function localized around $0 \in \mathbb{R}^2$ with integral 1). Following [**BT**] it is easy to see that $Q_X(a, b) = \int_X \eta_1 \wedge \eta_2$, hence the assertion easily follows. Note that for the last two propositions we assumed that X is a smooth 4-manifold. □

Remarks 1.2.6. (a) Again, the above proposition applies if X^4 has boundary or is noncompact (using suitable versions of Poincaré duality), and a similar statement holds over $\mathbb{Z}_2$ without the orientability hypotheses. In arbitrary dimensions, the same method of counting intersections gives the intersection pairing (including relative versions) for any two homology classes of complementary dimension, and this is again dual to the cup product pairing [**GH**]. The only difference is that high-dimensional homology classes cannot always be represented by submanifolds, so one must allow smooth cycles with singularities.

(b) Recall that a complex structure on X gives an orientation on it, and if Σ_α and Σ_β are complex submanifolds, then they are also canonically oriented. An easy argument shows that the transverse intersection of complex submanifolds is always positive. In particular, $Q_S([C_1], [C_2]) \geq 0$ if $C_1, C_2 \subset S$ are transversely intersecting complex curves in a complex surface S. Applying more delicate arguments, one can prove the same positivity result for any pair of embedded complex curves, provided C_1 and C_2 have no common component. (As we will see, the self-intersection of a complex curve can be negative, cf. Section 2.2.)

We make a short digression and briefly recall the classification of integral forms. (For a more detailed treatment, see [**MH**]). For a given symmetric, bilinear form Q on the finitely generated free abelian group A the rank, signature and parity of Q are defined in the following way: The *rank* $\mathrm{rk}(Q)$ of Q is the dimension of A. Extend and diagonalize Q over $A \otimes_{\mathbb{Z}} \mathbb{R}$. The number of $+1$'s (-1's respectively) on the diagonal is denoted by b_2^+ (resp. b_2^-); the difference $b_2^+ - b_2^-$ is the *signature* $\sigma(Q)$ of Q. Finally, Q is *even* if $Q(\alpha, \alpha) \equiv 0 \pmod 2$ for every $\alpha \in A$; Q is *odd* otherwise.

Exercises 1.2.7. Suppose that Q is an integral form.

(a) Show that Q is even iff $Q(\alpha_i, \alpha_i) \equiv 0 \pmod 2$ $(i = 1, \dots, n)$ for *every* basis $\{\alpha_1, \dots, \alpha_n\}$.

(b) Prove that Q is even iff *every* matrix representing Q has even diagonal.

(c) Prove the above statements after replacing *every* with *at least one*.

(d) Let $\langle n \rangle$ be the bilinear form over $\mathbb{Z}$ represented by the 1×1 matrix $[n]$ $(n \in \mathbb{Z})$. Prove that $\langle n \rangle$ and $\langle m \rangle$ are equivalent iff $m = n$.

Definitions 1.2.8. **(a)** Q is *positive (negative) definite* if $\mathrm{rk}(Q) = \sigma(Q)$ ($\mathrm{rk}(Q) = -\sigma(Q)$ resp.). Q is *indefinite* otherwise.

(b) The *direct sum* $Q = Q_1 \oplus Q_2$ of the forms Q_1 and Q_2 (given on A_1, A_2 respectively) is defined on $A_1 \oplus A_2$ in the following way. If $a, b \in A = A_1 \oplus A_2$ split as $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_i, b_i \in A_i$, then $Q(a, b) = Q_1(a_1, b_1) + Q_2(a_2, b_2)$. If $k > 0$ then kQ denotes the k-fold sum $\oplus_k Q$; for negative k we take kQ to be $|k|(-Q)$; finally if $k = 0$, then the form kQ equals the zero form on the trivial group (represented by the empty matrix $\emptyset$) by definition. The form on $A = \mathbb{Z} \oplus \mathbb{Z}$ represented by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ will be denoted by H.

(c) An element $x \in A$ is called a *characteristic element* if $Q(\alpha, x) \equiv Q(\alpha, \alpha) \pmod 2$ for all $\alpha \in A$. Note that Q is even iff $0 \in A$ is characteristic. An element $\alpha \in A$ is *primitive* if we cannot write α as $d\beta$ ($\beta \in A$, $d \in \mathbb{Z}$) unless $d = \pm 1$. For any $x \in A$ there is a primitive element α such that $x = d\alpha$; the integer $|d|$ is called the *divisibility* of x.

(d) Q is called *unimodular* if $\det Q = \pm 1$.

For an element $x \in A$ define $L_x \in A^*$ by $L_x(y) = Q(x, y)$. In this way we get a homomorphism $L \colon A \to A^*$.

Lemma 1.2.9. *The form Q is unimodular iff L is an isomorphism.*

Proof. Fix a basis $a_1, \dots, a_n \in A$ and take the dual basis $a_i^* \in A^*$ (given by $a_i^*(a_j) = \delta_{ij} \in \mathbb{Z}$). Since $L(a_i) = \sum_j Q(a_j, a_i) a_j^*$, the matrix of L in these bases is $[Q(a_i, a_j)]_{ij}$. A matrix B over $\mathbb{Z}$ is invertible (over $\mathbb{Z}$) iff $\det B = \pm 1$, hence the lemma follows. □

Exercise 1.2.10. * Prove that if X is a closed 4-manifold (so $\partial X = \emptyset$), then Q_X is unimodular. (*Hint*: Apply Poincaré duality.)

Remark 1.2.11. Exercise 1.2.10 can be extended to show that Q_X is unimodular when ∂X is a homology sphere. (A 3-manifold M is a *homology sphere* iff $H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, or equivalently, iff it is closed, orientable and connected with $H_1(M;\mathbb{Z}) = 0$.) One simply observes that by the long exact homology sequence, inclusion induces an isomorphism $H_2(X;\mathbb{Z}) \to H_2(X, \partial X;\mathbb{Z})$. In fact, for a compact, oriented 4-manifold X with $H_1(X;\mathbb{Z}) = 0$, Q_X is unimodular if and only if ∂X is a disjoint union of homology spheres. We consider the relation between Q_X and $H_1(\partial X;\mathbb{Z})$ in more detail in Corollary 5.3.12 and Exercise 5.3.13(f). Note that the pairing Q'_X described in Remark 1.2.2(b) is unimodular for every pair $(X, \partial X)$.

Lemma 1.2.12. *Suppose that the restriction of the symmetric bilinear form Q to the subgroup $A_1 \subset A$ is unimodular. Then (A, Q) can be split as the sum of forms $(A, Q) = (A_1, Q|A_1) \oplus (A_1^\perp, Q|A_1^\perp)$, where $A_1^\perp = \{y \in A \mid Q(x, y) = 0 \ for\ all\ x \in A_1\}$. Moreover, $Q|A_1^\perp$ is unimodular iff Q is.*

Proof. If $0 \neq a \in A_1 \cap A_1^\perp$, then $Q(a, b) = 0$ for all $b \in A_1$, contradicting the fact that $Q|A_1$ is unimodular. For any $x \in A$ we can take the linear function $a \mapsto Q(x, a)$ on A_1; by the unimodularity of $Q|A_1$ there is a unique element $b \in A_1$ with $Q(x, a) = Q(b, a)$ for all $a \in A_1$ (cf. Lemma 1.2.9). Hence $x - b \in A_1^\perp$, so $x = b + (x - b) \in A_1 + A_1^\perp$. We have now proved that $A = A_1 \oplus A_1^\perp$; the unimodularity of $(A_1^\perp, Q|A_1^\perp)$ follows from the fact that $\det Q = \det Q|A_1 \cdot \det Q|A_1^\perp = \pm \det Q|A_1^\perp$. □

As a corollary, the following useful observation can be made:

Corollary 1.2.13. *Suppose that* $\dim A$ *is equal to n and the determinant of the matrix of Q on the set $\{a_1, \ldots, a_n\}$ is ± 1. Then $\{a_1, \ldots, a_n\}$ is a basis of the free group A.* □

For the rest of the section, we only consider unimodular forms. As we will see (cf. Theorem 1.2.30), indefinite forms will be more interesting for our purposes. At the same time, indefinite forms admit a very nice classification scheme.

Theorem 1.2.14. *If indefinite unimodular forms Q_1, Q_2 (defined on A_1, A_2 respectively) have the same rank, signature and parity, then they are equivalent.*

In the following — through a series of exercises — we will give an outline of the proof of Theorem 1.2.14. The proof rests on the following theorem, whose proof requires some difficult algebraic geometry (cf. [**MH**]).

Theorem 1.2.15. *If $A \neq 0$ and $\sigma(Q) = 0$, then there exists a nonzero $\alpha \in A$ with $Q(\alpha, \alpha) = 0$.* □

Using the above result, we can easily classify intersection forms with signature 0.

Lemma 1.2.16. *If $\sigma(Q)=0$, then Q is equivalent to kH if Q is even, and to $l\langle 1\rangle \oplus l\langle -1\rangle$ if Q is odd ($k, l \in \mathbb{N}$).*

Proof. Take $x \in A$ with $Q(x,x)=0$; we can assume that x is primitive. Since Q is unimodular, there is $y \in A$ with $Q(x,y)=1$. Now split A as $\text{span}(x,y) \oplus \text{span}(x,y)^{\perp}$. Since $Q|\text{span}(x,y)$ is unimodular, by Lemma 1.2.12 Q on $\text{span}(x,y)^{\perp}$ is unimodular and obviously has 0 signature. Hence if $\text{span}(x,y)^{\perp}$ is nonzero, the above splitting process can be repeated.

Exercises 1.2.17. Prove that

(a)* If $Q(y,y)$ is even, then $Q|\text{span}(x,y) \cong H$.

(b)* If $Q(y,y)$ is odd, then $Q|\text{span}(x,y) \cong \langle 1\rangle \oplus \langle -1\rangle$.

(c)* $H \oplus \langle -1\rangle \cong 2\langle -1\rangle \oplus \langle 1\rangle$.

The solutions of the above exercises complete the proof of Lemma 1.2.16. □

Proof of Theorem 1.2.14. Note that Lemma 1.2.16 covers the case when $\sigma(Q_1)=0$ in Theorem 1.2.14. Thus, without loss of generality, we can assume that $\sigma(Q_1)>0$, and prove the theorem by induction on $\sigma(Q_1)$. By induction we know that $Q_1 \oplus \langle -1\rangle$ and $Q_2 \oplus \langle -1\rangle$ are both isomorphic to $Q = b_2^+\langle 1\rangle \oplus (b_2^- + 1)\langle -1\rangle$. Assume that $x \in A = A_1 \oplus \mathbb{Z} \cong A_2 \oplus \mathbb{Z}$ is the vector spanning the orthogonal complement of Q_1 and $y \in A$ spans the complement of Q_2 in (A,Q). All we need is an automorphism of (A,Q) mapping x to y — hence Q_1 to Q_2. For the proof of the following proposition see [**W1**].

Proposition 1.2.18. *Suppose that $Q \cong n\langle 1\rangle \oplus m\langle -1\rangle$ ($n,m>1$) on A, and x,y are primitive elements in A such that $Q(x,x)=Q(y,y)$. If both x and y are characteristic elements, then there is an automorphism of (A,Q) mapping x to y. A similar automorphism exists if neither x nor y is characteristic.* □

The above proposition concludes the proof of Theorem 1.2.14, since x,y in A are characteristic iff Q_1 and Q_2 are even, and the equalities $Q(x,x)=Q(y,y)=-1$ show that x,y are primitive elements. □

Remark 1.2.19. The proof of Proposition 1.2.18 given in [**W1**] goes as follows. The case $n=m=2$ is proved first, by explicit construction of automorphisms. In this case it is also shown that if x is characteristic, it can be mapped to a canonical element depending only on $Q(x,x)$. This idea extends to general n and m, and proves Proposition 1.2.18 in the characteristic case. For x,y not characteristic, the proof of the $n=m=2$ case provides an automorphism mapping the noncharacteristic vectors either into the subspace $2\langle 1\rangle \oplus \langle -1\rangle$ or into $\langle 1\rangle \oplus 2\langle -1\rangle$. Now for general n and m, the splitting $n\langle 1\rangle \oplus m\langle -1\rangle = (2\langle 1\rangle \oplus 2\langle -1\rangle) \oplus ((n-2)\langle 1\rangle \oplus (m-2)\langle -1\rangle)$ combined with induction gives the result in the noncharacteristic case. For details see [**W1**].

To finish the description of indefinite forms we must determine the triples (rank, signature, parity) for which a form Q with these invariants exists. Since $\mathrm{rk}(Q) = b_2^+ + b_2^-$ and $\sigma(Q) = b_2^+ - b_2^-$, obviously $|\sigma(Q)| < \mathrm{rk}(Q)$ and $\sigma(Q) \equiv \mathrm{rk}(Q) \pmod 2$. The next lemma gives one further restriction for invariants of even intersection forms.

Lemma 1.2.20. *If $x \in A$ is characteristic, then $Q(x,x) \equiv \sigma(Q) \pmod 8$; in particular, if Q is even, then the signature $\sigma(Q)$ is divisible by 8.*

Proof. Note that if x is characteristic in (A, Q), then $x + e_+ + e_-$ is characteristic in $(A \oplus \mathbb{Z} \oplus \mathbb{Z}, Q \oplus \langle 1\rangle \oplus \langle -1\rangle)$, where $e_\pm$ generate the $\mathbb{Z}$ summands. By Theorem 1.2.14, $Q' = Q \oplus \langle 1\rangle \oplus \langle -1\rangle \cong (b_2^+ + 1)\langle 1\rangle \oplus (b_2^- + 1)\langle -1\rangle$, and a characteristic vector has odd components in this new basis. Since the square of an odd number is congruent to 1 modulo 8, we have that $Q(x,x) = Q'(x+e_++e_-, x+e_++e_-) \equiv (b_2^+ + 1) - (b_2^- + 1) = \sigma(Q) \pmod 8$. If Q is even, then 0 is a characteristic element, which implies that $\sigma(Q) \equiv 0 \pmod 8$. □

To show that all constraints on the triple (rank, signature, parity) have been found, we define a particular 8-dimensional intersection form. Consider the matrix corresponding to the Dynkin diagram of the exceptional Lie algebra E_8:

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

As the matrix of a bilinear form Q on $\mathbb{Z}^8$, E_8 gives a positive definite, even, unimodular form with $\sigma(Q) = 8$. (Check these statements by diagonalizing E_8 over $\mathbb{Q}$; beware that the determinant is not an invariant over $\mathbb{Q}$.) By a slight abuse of notation, from now on E_8 will denote that bilinear form. Recall that H is used for the form corresponding to the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Using E_8 and H as building blocks, for every pair $(\sigma, r) \in \mathbb{Z}\times\mathbb{N}$ with $\sigma \equiv 0 \pmod 8$, $r > |\sigma|$ and $r \equiv \sigma \pmod 2$ one can build up an indefinite unimodular form $Q = aE_8 \oplus bH$ with $\sigma = \sigma(Q)$ and $r = \mathrm{rk}(Q)$. (Take $a = \frac{\sigma}{8}$ and $b = \frac{r-|\sigma|}{2}$.) Consequently Theorem 1.2.14 implies the following.

Theorem 1.2.21. *Suppose that Q is an indefinite, unimodular form. If Q is odd, then it is isomorphic to $b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$; if Q is even then it is isomorphic to $\frac{\sigma(Q)}{8}E_8 \oplus \frac{\mathrm{rk}(Q)-|\sigma(Q)|}{2}H$.* □

Remark 1.2.22. Since the negative definite form $-E_8$ appears more commonly in 4-manifold theory than E_8, some authors use the notation E_8 for the negative definite form. A matrix for the latter form is obtained from the above matrix by reversing all signs on the diagonal. (Check this by a basis change reversing the signs of 4 vectors.) We will call this matrix the $-E_8$-matrix.

Exercises 1.2.23. (a) Let Q be an indefinite, unimodular form. Find a characteristic element $x \in A$ with $Q(x,x) = \sigma(Q)$. (*Hint*: Solve the problem for $\pm E_8, H, \langle \pm 1\rangle$ and apply the Classification Theorem 1.2.21.)

(b) Prove that $H \cong -H$ and $E_8 \oplus (-E_8) \cong 8H$.

In the definite case there is no such nice description of all unimodular forms. For a given rank there are only finitely many definite symmetric unimodular forms (see [**MH**]); this number, however, can be very large. (For example, there are more than 10^{50} definite forms of rank 40.)

Exercise 1.2.24. Prove that the positive definite forms $Q_1 = E_8 \oplus n\langle 1\rangle$ and $Q_2 = (8+n)\langle 1\rangle$ are not equivalent, although they have equal rank, signature and parity for $n > 0$. (*Hint*: Count the number of vectors of length 1 with respect to Q_1 and Q_2.)

We now consider the intersection form of a closed 4-manifold X; recall that this is always unimodular. For the sake of simplicity, let us restrict ourselves to the simply connected case. Since $\pi_1(X) = 0$, the first and the third homologies and cohomologies vanish (by Poincaré duality), and $H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ has no torsion, so Q_X contains all the (co)homological information about X. As the next theorem shows, Q_X classifies topological 4-manifolds up to homotopy.

Theorem 1.2.25. (Whitehead) *The simply connected, closed, topological 4-manifolds X_1 and X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$.*

Proof (sketch). Using homotopy theoretic arguments one can show that a simply connected topological 4-manifold X is homotopy equivalent to a CW-complex of the form $\bigvee_{i=1}^k S_i^2 \cup_g D^4$ — here $\bigvee_{i=1}^k S_i^2$ denotes the wedge (or bouquet) of k 2-spheres and g is the gluing map $S^3 \to \bigvee_{i=1}^k S_i^2$ of the 4-cell, hence defines a class $[g] \in \pi_3(\bigvee_{i=1}^k S_i^2)$. Denoting the additive group of symmetric $k \times k$ matrices with integer entries by $M(k \times k)$, we can obtain an isomorphism $L\colon \pi_3(\bigvee_{i=1}^k S_i^2) \to M(k \times k)$ as follows. Take $x_i \in S_i^2$ and assume that g is transverse to x_i and smooth in a neighborhood of $g^{-1}(x_i)$.

Define the matrix $L([g]) = [\lambda_{ij}(g)]$ as $\lambda_{ij}(g) = \ell k(g^{-1}(x_i), g^{-1}(x_j))$ (and $\lambda_{ii}(g) = \ell k(g^{-1}(x_i), g^{-1}(x_i'))$ for x_i' close to x_i), where $\ell k(L_1, L_2)$ denotes the linking number of the two oriented links L_1 and L_2. (For more about linking numbers see Section 4.5.) It is not hard to see that $L([g])$ represents Q_X in an appropriate basis, hence if $Q_{X_1} \cong Q_{X_2}$, the gluing maps g_1 and g_2 corresponding to X_1 and X_2 are homotopic. The proof of the theorem now easily follows. □

Assume for a moment that X is a finite CW complex with cells of dimension ≤ 4. For a fixed homology element $[X] \in H_4(X;\mathbb{Z})$ the pair $(X, [X])$ is called a (4-dimensional oriented) *Poincaré duality space* if the map $\alpha \mapsto \alpha \cap [X]$ (with $\alpha \in H^i(X;\mathbb{Z})$) defines an isomorphism $H^i(X;\mathbb{Z}) \to H_{4-i}(X;\mathbb{Z})$ for $i = 0, \ldots, 4$. (It can be shown that any oriented topological 4-manifold is homotopy equivalent to a finite CW complex, hence, to a Poincaré duality space.) Note that the formula $Q_X(a,b) = \langle a \cup b, [X]\rangle$ extends the definition of intersection forms to Poincaré duality spaces. Since the solution of Exercise 1.2.10 applies without change, the intersection form Q_X of a Poincaré duality space is unimodular. The proof of Theorem 1.2.25 can be easily modified to show the following result.

Theorem 1.2.26. *Two simply connected (4-dimensional) Poincaré duality spaces X_1 and X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$. Moreover, for each unimodular form Q there exists a (4-dimensional) Poincaré duality space X with $Q_X \cong Q$.*

Proof. The proof of the first statement proceeds verbatim as the proof of Theorem 1.2.25. If Q is represented by the symmetric matrix $B \in M(k \times k)$ in a basis, then take $g \in L^{-1}(B) \in \pi_3(\bigvee_{i=1}^k S_i^2)$ and form $X = \bigvee_{i=1}^k S_i^2 \cup_g D^4$. Because of the unimodularity of Q (i.e., $\det B = \pm 1$), X is a (4-dimensional) Poincaré duality space satisfying $Q_X \cong Q$. □

The following theorem — due to M. Freedman — can be regarded as the topological strengthening of the above homotopy theoretic classification results. Some ideas of the proof of Theorem 1.2.27 will be discussed in later chapters.

Theorem 1.2.27. (Freedman, [**F**], [**FQ**]) *For every unimodular symmetric bilinear form Q there exists a simply connected, closed, topological 4-manifold X such that $Q_X \cong Q$. If Q is even, this manifold is unique (up to homeomorphism). If Q is odd, there are exactly two different homeomorphism types of manifolds with the given intersection form. At most one of these homeomorphism types carries a smooth structure. Consequently, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.* □

One special case (the topological 4-dimensional Poincaré Conjecture) deserves a corollary:

Corollary 1.2.28. *If a topological 4-manifold X is homotopy equivalent to S^4, then X is homeomorphic to the 4-sphere.* □

Regarding smooth structures, the two main questions (existence and uniqueness) can now be formulated as follows.

- **Q1**. Existence: Which simply connected topological manifolds (or equivalently, intersection forms) carry smooth structures?
- **Q2**. Uniqueness: If the intersection form Q does carry a smooth structure, how many nondiffeomorphic smooth manifolds can be found with the same intersection form Q?

The following theorems illustrate what we know about the answers for **Q1** and **Q2**. Assume that X is a simply connected, closed, oriented, smooth 4-manifold.

Theorem 1.2.29. (Rohlin, [**R2**]) *If Q_X is even, then the signature $\sigma(X)$ is divisible by 16.* □

This theorem tells us, for example, that the topological manifold corresponding to E_8 does not carry any smooth structure. Another constraint on the intersection form of a simply connected smooth 4-manifold was found by Donaldson, cf. also Corollary 2.4.29.

Theorem 1.2.30. (Donaldson, [**D1**]) *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is negative definite, then Q_X is equivalent to $n\langle -1\rangle$.* □

Note that by Remark 1.2.2(c) this theorem takes care of manifolds with positive definite intersection forms as well. As we will soon see, Theorem 1.2.30 answers **Q1** for definite and odd intersection forms. For indefinite even intersection forms — besides Theorem 1.2.29 implying that the coefficient of E_8 is even — the following estimate has been proved:

Theorem 1.2.31. (Furuta, [**Fur**]) *If X is a simply connected, closed, oriented, smooth 4-manifold and Q_X is equivalent to $2kE_8 \oplus lH$, then we have $l \geq 2|k| + 1$.* □

The $\frac{11}{8}$-*Conjecture* states that in the above theorem $l \geq 3|k|$ should be the right answer — this conjecture, however, is still open. On the other hand (as we will see in the next section), all intersection forms allowed by Theorems 1.2.29, 1.2.30 and the $\frac{11}{8}$-Conjecture can be represented as intersection forms of simply connected, smooth 4-manifolds. Thus the only remaining question for answering **Q1** in the simply connected case lies in the difference between Furuta's result and the $\frac{11}{8}$-Conjecture.

The next result indicates how much we know about the answer of **Q2**. As a consequence of Theorem 1.2.27, the homeomorphism type of a smooth, simply connected, oriented, closed 4-manifold X is determined by the parity of Q_X and the two numerical invariants $\sigma(X)$ and $b_2(X) = \operatorname{rk} H_2(X;\mathbb{Z})$. In contrast to Theorem 1.1.8, there is no finiteness result on the number of nondiffeomorphic smooth structures on a topological 4-manifold. The known results are of the following type. (An indication of the proof of this result (and similar ones) will be given later on, cf. Corollary 3.3.7 and subsequent text. See also Theorem 10.3.9.)

Theorem 1.2.32. ([**FM1**]) *The (simply connected) topological manifolds corresponding to the intersection forms $2n(-E_8) \oplus (4n-1)H$ $(n \geq 1)$ and $(2k-1)\langle 1\rangle \oplus N\langle -1\rangle$ $(k \geq 2,\ N \geq 10k-1)$ each carry infinitely many distinct (nondiffeomorphic) smooth structures.* □

Throughout the last part of this section we always assumed that the 4-manifolds we considered were simply connected. This assumption can be relaxed in some cases, but the general case (arbitrary fundamental group) is too difficult to study, since:

Theorem 1.2.33. *For every finitely presented group G there is a smooth, closed, oriented 4-manifold X with $\pi_1(X) \cong G$.* □

In Part 2 of this volume we will prove Theorem 1.2.33 from two different points of view (Exercises 4.6.4(b) and 5.2.2(c)) and deduce a theorem of Markov (Exercise 5.1.10(c)) that there can be no algorithm for classifying closed 4-manifolds (or n-manifolds for any fixed $n \geq 4$). Thus, the difficulty of understanding finitely presented groups leads us to focus mainly on simply connected 4-manifolds.

The invariants we have discussed until now — the intersection form Q_X, or more generally the cohomology ring $H^*(X;\mathbb{Z})$, and the fundamental group $\pi_1(X)$ — depend only on the homeomorphism type (in fact, homotopy type) of the manifold. Our ultimate goal, however, is to study *smooth* 4-manifolds. On one hand, we need finer invariants and ways to compute them in order to distinguish nondiffeomorphic 4-manifolds. Seiberg-Witten invariants and Seiberg-Witten basic classes will be introduced in Section 2.4, and we will see some applications of the knowledge of the Seiberg-Witten function to the geometry of the underlying 4-manifold. In this way we will distinguish homeomorphic but nondiffeomorphic 4-manifolds. On the other hand, we also need a method to decide when 4-manifolds given by different constructions result in diffeomorphic manifolds. Part 2 — about

Kirby calculus — will give a way to deal with 4-manifolds defined by various standard constructions. Using Kirby calculus one can (under favorable circumstances) prove that 4-manifolds defined by different constructions are actually diffeomorphic.

In the next section we present some familiar examples of (simply connected) 4-manifolds and determine the corresponding intersection forms. More complicated and more interesting constructions will be shown in Part 3. For our occasional use of characteristic classes, the reader is referred to Section 1.4 for an overview of background material or to [**MS**] for more details.

1.3. Examples

We now present some basic examples of closed, simply connected manifolds. The simplest example of such a 4-manifold is the 4-dimensional sphere $S^4 = \{x \in \mathbb{R}^5 \mid ||x|| = 1\}$; since $H_2(S^4;\mathbb{Z}) = 0$, the intersection form Q_{S^4} is trivial. Other examples are provided by the *complex projective spaces*. Given the obvious free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^{n+1} - \{0\}$ (that is, $\lambda(z_0, \dots, z_n) = (\lambda \cdot z_0, \dots, \lambda \cdot z_n)$ for $\lambda \in \mathbb{C}^*$), one can take the quotient $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. The resulting space is the n-dimensional complex projective space $\mathbb{CP}^n$; $\mathbb{CP}^1 = S^2$ is the *complex projective line* and $\mathbb{CP}^2$ is the *complex projective plane*. Using $\mathbb{R}$ instead of $\mathbb{C}$, one defines the real projective spaces $\mathbb{RP}^n$. If $P \in \mathbb{CP}^n$ and $(z_0, \dots, z_n) \in P$, then we can denote P by its *homogeneous coordinates* $[z_0 : z_1 : \dots : z_n]$, which are defined up to multiplication by $\lambda \in \mathbb{C}^*$. Note that $\mathbb{CP}^n$ can be covered by the *affine coordinate* charts $\psi_i \colon \mathbb{C}^n \to \mathbb{CP}^n$ $(i = 0, \dots, n)$, where $\psi_i(z_1, \dots, z_n) = [z_1 : \dots : z_i : 1 : z_{i+1} : \dots : z_n]$. The 2-dimensional sphere S^2 has a unique complex structure as $\mathbb{CP}^1$, so we can use the symbols S^2 and $\mathbb{CP}^1$ interchangeably.

Exercises 1.3.1. (a)* Prove that $\mathbb{CP}^n$ is compact and $\pi_1(\mathbb{CP}^n) = 1$. Consequently, $\mathbb{CP}^2$ is a closed, simply connected 4-manifold.

(b) Prove that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ if $n > 1$. What is $\mathbb{RP}^1$? For which values of n is $\mathbb{RP}^n$ orientable?

(c) Prove that $H_i(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ if $i = 2d$ $(d = 0, \dots, n)$ and $H_i(\mathbb{CP}^n;\mathbb{Z}) = 0$ otherwise. (For a solution, see Example 4.2.4.)

(d) Determine the functions $\psi_i^{-1} \circ \psi_j$ for the affine coordinate charts of $\mathbb{CP}^n$. (*Hint*: See Example 4.2.4.)

(e)* Show that the homology class $h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ given as the fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0\}$ generates $H_2(\mathbb{CP}^2;\mathbb{Z})$. Show furthermore that $Q_{\mathbb{CP}^2}(h,h) = 1$ and conclude that $Q_{\mathbb{CP}^2} = \langle 1 \rangle$.

(f) We let $\overline{\mathbb{CP}^2}$ denote the manifold $\mathbb{CP}^2$ with the opposite orientation, hence (by Remark 1.2.2(c)) $Q_{\overline{\mathbb{CP}^2}} = -Q_{\mathbb{CP}^2} = \langle -1 \rangle$. (*Caveat*: Do not confuse this notation with complex conjugation, which *preserves* orientation on $\mathbb{CP}^{2k}$.) Prove that there is no orientation-preserving diffeomorphism between $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$.

As in the real case, it is easy to see that any two distinct points of $\mathbb{CP}^2$ lie on a unique projective line ($\approx \mathbb{CP}^1$), and any two (distinct) projective lines in $\mathbb{CP}^2$ intersect each other in exactly one point.

The cartesian product $\mathbb{CP}^1 \times \mathbb{CP}^1$ provides the next example of a simply connected 4-manifold. By the Künneth formula $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z}) = H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and it is not hard to see that $Q_{S^2 \times S^2} = H$: Choose the homology elements $\alpha = [S^2 \times \{\text{pt.}\}]$ and $\beta = [\{\text{pt.}\} \times S^2]$ as a basis for $H_2(S^2 \times S^2; \mathbb{Z})$; the matrix of $Q_{S^2 \times S^2}$ in this basis is equal to $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. To construct other 4-manifolds from the above ones we introduce a general operation for two smooth n-dimensional manifolds with boundary.

Definition 1.3.2. Let X_1, X_2 be oriented n-dimensional manifolds, and assume that $Z_i \subset \partial X_i$ $(i = 1, 2)$ are compact, codimension-zero submanifolds of the boundaries. Assume furthermore that $\varphi \colon Z_1 \to Z_2$ is an orientation-reversing diffeomorphism. By identifying Z_1 with $\overline{Z}_2$ via φ (and smoothing the corners) we get a new oriented manifold, denoted by $X_1 \cup_\varphi X_2$ (or by $X_1 \cup_Z X_2$ if $Z = Z_1 = \overline{Z}_2$ and $\varphi = \mathrm{id}_Z$).

Remark 1.3.3. The operation of smoothing corners is easy in dimension 2: Replace an angular boundary such as $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ or } x_2 \leq 0\}$ by a smooth one, using compactly supported smooth functions; see Figure 1.1. In higher dimensions, the same can be done (canonically) by multiplying the previous model by the extra dimensions of ∂Z_i.

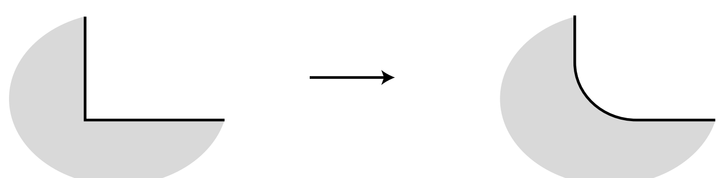


Figure 1.1. Model for smoothing corners in dimension 2.

A special case of the construction of Definition 1.3.2 is the *boundary sum* — when we glue along the $(n-1)$-dimensional ball $Z_1 \approx Z_2 \approx D^{n-1}$. The result is denoted by $X_1 \natural X_2$ and is well-defined (independent of the embeddings of D^{n-1}) whenever each ∂X_i is connected. The boundary sum of m copies of the manifold is denoted by $\natural m X$; if $m = 0$, then $\natural m X = D^n$ by definition. Another special case of the construction given in Definition 1.3.2 is the connected sum of two connected, oriented n-dimensional manifolds X_1 and X_2:

Definition 1.3.4. For $i = 1, 2$, let $D_i^n \subset X_i$ be embedded disks, and let $\varphi\colon D_1^n \to D_2^n$ be an orientation-reversing diffeomorphism. The smooth manifold $(X_1 - \text{int } D_1) \cup_{\varphi|\partial D_1} (X_2 - \text{int } D_2)$ is called the *connected sum* $X_1 \# X_2$ of X_1 and X_2; it does not depend on the choices of D_i^n or φ (since any two orientation-preserving embeddings of a disk are smoothly isotopic). In particular, $\# mX$ denotes the manifold we get by the connected sum of m ($m \geq 0$) copies of the same manifold X. (Again, if $m = 0$, then $\# mX = S^n$ by definition.)

Note that, by definition, the boundary sum of X_1 and X_2 has the connected sum $\partial X_1 \# \partial X_2$ as boundary, so $\partial(X_1 \natural X_2) = \partial X_1 \# \partial X_2$ — this relation might explain the names of the operations. The iterated application of the connected sum operation for $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ and $S^2 \times S^2$ gives other examples of simply connected 4-manifolds.

Exercises 1.3.5. (a)* Given 4-manifolds X_1 and X_2 with intersection forms Q_{X_i}, show that $Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$.

(b)* More generally, prove that if $X = X_1 \cup_N X_2$ and N is a homology 3-sphere, then $Q_X = Q_{X_1} \oplus Q_{X_2}$.

Remark 1.3.6. The converse of the above exercise is also true ([**FT1**], Theorem 1), namely that if X is a closed, smooth, simply connected 4-manifold and Q_X splits as $Q_1 \oplus Q_2$, then there are $X_1, X_2 \subset X$ such that $X = X_1 \cup_N X_2$ giving the splitting $Q_1 \oplus Q_2$ for Q_X (as $Q_i = Q_{X_i}$); moreover N is a (smoothly embedded) homology sphere. Note that by applying the result of Exercises 1.3.1(e) and 1.3.5(a) one can easily prove that the intersection form of $\# n\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ is equivalent to $n\langle 1\rangle \oplus m\langle -1\rangle$ $(n, m \geq 0)$, so these intersection forms — which cover all possible definite (cf. Theorem 1.2.30) and odd indefinite candidates — can be realized by smooth manifolds. Note also that the intersection form of $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$ is isomorphic to the intersection form of $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (cf. Exercise 1.2.17(c)). By Theorem 1.2.27, this implies that these manifolds are homeomorphic. As we will see later, $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$ is, in fact, diffeomorphic to $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.

Polynomials in the variables $\{z_0, \ldots, z_n\}$ are not well-defined functions on $\mathbb{CP}^n$, but for a *homogeneous* polynomial p of degree d, i.e., a polynomial satisfying $p(\lambda z) = \lambda^d p(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^{n+1}$, the *zero set* of p is well-defined. (If p vanishes on a point $z \in \mathbb{C}^{n+1} - \{0\}$ then it vanishes on its entire equivalence class $[z] \in \mathbb{CP}^n$.)

Definition 1.3.7. If p is a homogeneous polynomial of degree d then the set $V_p = \{[z] \in \mathbb{CP}^n \mid p(z) = 0\}$ is called the *hypersurface* corresponding to the polynomial p. The complex submanifolds of $\mathbb{CP}^n$ are called *complex projective* manifolds.

It has been proved [**GH**] that any complex projective manifold can be written as the zero set of a collection of homogeneous polynomials. Not every complex manifold, however, can be embedded in $\mathbb{CP}^n$, so not all complex manifolds are projective.

By considering hypersurfaces in $\mathbb{CP}^3$, we will provide further examples of 4-manifolds with even Q_X; in particular, we will show that if the indefinite even form Q satisfies the constraints posed by Theorem 1.2.29 and the $\frac{11}{8}$-Conjecture, then there exists a smooth 4-manifold X with $Q \cong Q_X$ (cf. Exercise 1.3.12(a)). Consider the hypersurface

$$S_d = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid \sum z_i^d = 0\} \subset \mathbb{CP}^3,$$

where d is a positive integer.

Theorem 1.3.8. (See also [**McS1**].) *The hypersurface S_d is a smooth, simply connected, complex surface. If d is odd, then Q_{S_d} is equivalent to $\lambda_d\langle 1\rangle \oplus \mu_d\langle -1\rangle$, where $\lambda_d = \frac{1}{3}(d^3 - 6d^2 + 11d - 3)$ and $\mu_d = \frac{1}{3}(d-1)(2d^2 - 4d + 3)$; if d is even, then Q_{S_d} is equivalent to $l_d(-E_8) \oplus m_d H$, where $l_d = \frac{1}{24}d(d^2 - 4)$ and $m_d = \frac{1}{3}(d^3 - 6d^2 + 11d - 3)$.*

Proof. The Implicit Function Theorem shows that $S_d \subset \mathbb{CP}^3$ is a smooth 4-manifold; the fact that $\pi_1(S_d) = 1$ follows from the Lefschetz Hyperplane Theorem 1.4.22 (see Exercise 8.1.1(b)). For determining Q_{S_d} we must compute its parity, rank and signature (cf. Theorem 1.2.21). Note that S_d is a complex surface, hence it admits Chern classes $c_1(S_d) \in H^2(S_d; \mathbb{Z})$ and $c_2(S_d) \in H^4(S_d; \mathbb{Z})$. Since $c_2[S_d] = \chi(S_d) = 2 + \mathrm{rk}(Q_{S_d})$ and $c_1^2[S_d] = 3\sigma(S_d) + 2\chi(S_d)$, the classes $c_2(S_d)$ and $c_1(S_d)$ determine the rank and the signature of S_d. (Here $\chi(S_d)$ denotes the topological Euler characteristic of S_d.) Moreover $c_1(S_d) \equiv w_2(S_d) \pmod 2$, and (since $\pi_1(S_d) = 1$) Q_{S_d} is even iff $w_2(S_d) = 0$; hence the parity of Q_{S_d} is determined by $c_1(S_d)$. Consequently we only need to determine $c_1(S_d)$ and $c_2(S_d)$ in order to compute Q_{S_d}. Recall that the total Chern class of $\mathbb{CP}^3$ is $c(\mathbb{CP}^3) = (1+g)^4 \in H^*(\mathbb{CP}^3; \mathbb{Z})$, where g denotes the generator of $H^2(\mathbb{CP}^3; \mathbb{Z})$ satisfying $\langle g, [\mathbb{CP}^1]\rangle = 1$ [**MS**]. In the next lemma, x will denote the pullback $i^*(g) \in H^2(S_d; \mathbb{Z})$ of $g \in H^2(\mathbb{CP}^3; \mathbb{Z})$ via the embedding $i\colon S_d \hookrightarrow \mathbb{CP}^3$.

Lemma 1.3.9. *The Chern classes of S_d are given by $c_1(S_d) = (4-d)x$ and $c_2(S_d) = (d^2 - 4d + 6)x^2$. Moreover, $\langle x^2, [S_d]\rangle = d$, hence $\chi(S_d) = (d^2 - 4d + 6)d$ and $c_1^2[S_d] = (4-d)^2 d$. In addition, Q_{S_d} is even iff d is even.*

Proof. Restrict the tangent bundle $T\mathbb{CP}^3$ of $\mathbb{CP}^3$ to S_d. It splits as the tangent bundle TS_d of S_d and the normal bundle νS_d: $T\mathbb{CP}^3|S_d = TS_d \oplus \nu S_d$.

By the Whitney product formula we have that $c(T\mathbb{CP}^3|S_d) = (1+x)^4 = (1+c_1(S_d)+c_2(S_d))\cdot(1+c_1(\nu S_d))$, so

$$\begin{aligned} 1+c_1(S_d)+c_2(S_d) &= (1+x)^4(1+c_1(\nu S_d))^{-1} \\ &= (1+4x+6x^2)(1-c_1(\nu S_d)+c_1^2(\nu S_d)). \end{aligned}$$

Hence, to prove Lemma 1.3.9 we only need to determine $c_1(\nu S_d)$.

Lemma 1.3.10. *The first Chern class of the normal bundle νS_d equals dx.*

Proof. Suppose that S_d' is a hypersurface of degree d (defined by another homogeneous polynomial of degree d) intersecting S_d transversally in $V = S_d \cap S_d'$. Since $c_1(\nu S_d) = e(\nu S_d)$ and S_d' can be chosen to be a section of $\nu S_d \to S_d$, we get that $c_1(\nu S_d) = PD([V])$. Since $[S_d] = [S_d'] = d\cdot[S_1] \in H_4(\mathbb{CP}^3;\mathbb{Z})$, and in $H^2(S_d;\mathbb{Z})$ one has $PD[S_1\cap S_d] = i^*(PD[S_1]) = i^*(g) = x$, the lemma follows. □

Consequently,

$$\begin{aligned} 1+c_1(S_d)+c_2(S_d) &= (1+4x+6x^2)(1-dx+d^2x^2) \\ &= 1+(4-d)x+(d^2-4d+6)x^2, \end{aligned}$$

and this implies the first statement of Lemma 1.3.9. The term $\langle x^2,[S_d]\rangle$ can be computed in the following way: $\langle x^2,[S_d]\rangle = \langle (i^*g)^2,[S_d]\rangle = \langle g^2, i_*[S_d]\rangle = \langle g^2\cup PD(i_*[S_d]),[\mathbb{CP}^3]\rangle = \langle g^2\cup dg,[\mathbb{CP}^3]\rangle = d\cdot\langle g^3,[\mathbb{CP}^3]\rangle = d$. Note that for odd d the term $\langle x^2,[S_d]\rangle = Q_{S_d}(x,x)$ is odd, hence Q_{S_d} is odd as well. If d is even, then $c_1(S_d) = (4-d)x \equiv 0 \pmod 2$, so $w_2(S_d) = 0$. Consequently, Proposition 1.4.18 implies that Q_{S_d} is even iff d is even, which concludes the proof of Lemma 1.3.9. □

With the results of Lemma 1.3.9, the proof of Theorem 1.3.8 is just a simple computation. □

Our particular choice of the homogeneous polynomial in the definition of S_d has no importance, since

Claim 1.3.11. *If p_1 and p_2 are two homogeneous polynomials with equal degree (and not powers of other polynomials) and the hypersurfaces $F_i = \{P\in\mathbb{CP}^n \mid p_i(P)=0\}$ are smooth submanifolds of $\mathbb{CP}^n$ ($i=1,2$), then F_1 is diffeomorphic to F_2.*

Proof. By taking the coefficients of the monomials in a (homogeneous) polynomial of degree d in $n+1$ variables, one defines a point in $\mathbb{C}^N$ (where N depends on n and d). Conversely, a point $z\in\mathbb{C}^N$ defines a polynomial p_z by specifying the coefficients. The set $V_z = \{P\in\mathbb{CP}^n \mid p_z(P)=0\}$ is a hypersurface unless $z=0$, and clearly $V_{\lambda z} = V_z$ for all $\lambda\in\mathbb{C}^*$, hence the hypersurfaces of degree d in $\mathbb{CP}^n$ are parametrized by the points of

$(\mathbb{C}^N - \{0\})/\mathbb{C}^* = \mathbb{CP}^{N-1}$. Singular hypersurfaces correspond to points of a complex codimension-1 subspace of $\mathbb{CP}^{N-1}$, since the singular objects can be described by equations (specifying that the Implicit Function Theorem fails). Since this subspace has real codimension 2, the points of $\mathbb{CP}^{N-1}$ corresponding to smooth hypersurfaces form a connected subset, which means that one smooth hypersurface can be smoothly deformed to any other smooth one, and this proves that F_1 is diffeomorphic to F_2. The above proof, in fact, shows that F_1 is ambiently isotopic to F_2 in $\mathbb{CP}^n$. □

In the light of Claim 1.3.11, it is easy to see that $S_1 = \mathbb{CP}^2$. (Define $S_1' = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_3 = 0\}$.) By taking the quadric surface $S_2' = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 z_3 = z_1 z_2\}$ (which is diffeomorphic to S_2 by Claim 1.3.11), we see that $S_2 \approx \mathbb{CP}^1 \times \mathbb{CP}^1$. (Notice that the map $([s_0 : s_1], [t_0 : t_1]) \mapsto [s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1]$ gives an isomorphism between $\mathbb{CP}^1 \times \mathbb{CP}^1$ and S_2'. See also Exercise 3.2.1.) We need additional tools to show that $S_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ (Lemma 3.1.17 and subsequent text). The case $d = 4$ gives an example of a simply connected complex surface with $c_1 = 0$; such a surface is called a *K3-surface*. By algebraic geometric methods it can be shown that all $K3$-surfaces are diffeomorphic (cf. Theorem 3.4.9), so from the differential topological point of view we can call S_4 *the* $K3$-surface. By the previous formula $Q_{S_4} = 2(-E_8) \oplus 3H$.

Exercises 1.3.12. (a)* Realize all the (indefinite) even unimodular forms allowed by Theorem 1.2.29 and the $\frac{11}{8}$-Conjecture as intersection forms of simply connected, *smooth* 4-manifolds.

(b)* Show that if X is a simply connected, smooth 4-manifold with even intersection form and $b_2^+(X) = 0$, then X is homeomorphic to S^4. (*Hint*: Apply Theorems 1.2.30 and 1.2.27.)

More examples of simply connected 4-manifolds can be given by generalizing the above construction of S_d. Take homogeneous polynomials p_i of degree d_i in $n+1$ variables ($i = 1, \dots, n-2$). Note that each p_i defines a hypersurface in $\mathbb{CP}^n$. Suppose that their intersection, $S = S(d_1, \dots, d_{n-2}) = \{P \in \mathbb{CP}^n \mid p_i(P) = 0 \ (i = 1, \dots, n-2)\}$, is a smooth submanifold of complex dimension 2 in $\mathbb{CP}^n$. In this case S is called a *complete intersection* surface of multidegree $(d_1, \dots, d_{n-2})$. By generalizing Claim 1.3.11 it can be proved that the diffeomorphism type of $S(d_1, \dots, d_{n-2})$ depends only on the multidegree $(d_1, \dots, d_{n-2})$. Note that without loss of generality we can assume that each $d_i \geq 2$. By the Lefschetz Hyperplane Theorem 1.4.22 (cf. Exercise 8.1.1(b)), we have that $\pi_1(S(d_1, \dots, d_{n-2})) = 1$.

Exercises 1.3.13. For $S = S(d_1, \dots, d_{n-2})$ prove that

(a) $c_2[S] = (\frac{n(n+1)}{2} - (n+1)\sum d_i + \sum d_i^2 + \sum_{i<j} d_i d_j)\prod d_i$;

(b) $c_1^2[S] = (\sum d_i - (n+1))^2 \prod d_i$;

(c) $\sigma(S) = \frac{1}{3}((n+1) - \sum d_i^2) \prod d_i$;

(d) the second Stiefel-Whitney class $w_2(S)$ vanishes (hence Q_S is even) iff $\sum d_i - (n+1)$ is even.

(e) Show that $c_1(S(d_1, \ldots, d_{n-2})) = n_S h_S$, where $n_S = \sum d_i - (n+1)$ and $h_S \in H^2(S(d_1, \ldots, d_{n-2}); \mathbb{Z})$ is a primitive class. (*Hint*: Let $x = i^* g$ in $H^2(S; \mathbb{Z})$, where g generates $H^2(\mathbb{CP}^n; \mathbb{Z})$ and $i\colon S \hookrightarrow \mathbb{CP}^n$ is the embedding. Show that $\nu S = L_1 \oplus \ldots \oplus L_{n-2}$, where $L_i \to S$ is a complex line bundle with $c_1(L_i) = d_i x$; moreover $\langle x^2, [S] \rangle = \prod d_i$ and x is primitive. Characteristic class computations and easy arithmetic yield the solution.)

The surfaces $S(2,3)$ and $S(2,2,2)$ are $K3$-surfaces (hence diffeomorphic to $S(4) = S_4$); $S(2,2)$ is diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. (Recall also that $S(1) = S_1 = \mathbb{CP}^2$, $S(2) = S_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $S(3) = S_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$.) All other complete intersection surfaces are surfaces of general type. (For the definition of surfaces of general type see Section 3.4.)

Other complex surfaces can be given by taking $n-2$ hypersurfaces of $\mathbb{CP}^{i_1} \times \ldots \times \mathbb{CP}^{i_k}$ ($\sum i_j = n$) intersecting each other in a smooth complex surface. These surfaces will be complex projective manifolds (since the product $\mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_k}$ embeds holomorphically in some $\mathbb{CP}^N$) but usually not complete intersections, and they need not be simply connected. A hypersurface of $\mathbb{CP}^{i_1} \times \ldots \times \mathbb{CP}^{i_k}$ can be defined by a multi-homogeneous polynomial of degree $(d_1, \ldots, d_k)$: such a polynomial is homogeneous of degree d_1 in the first (i_1+1) variables, homogeneous of degree d_2 in the next (i_2+1) variables, and so on. The computation of the characteristic numbers of a surface given by the above construction is left to the reader.

1.4. Appendix

In this appendix we will present the very basics of characteristic classes — for a more detailed treatment see [**MS**]. For alternative definitions see [**St**], [**We**] or Section 5.6 of this volume. At the end of the section we will give a quick review of spin structures and Dirac operators (see also Sections 2.4 and 5.6). In the following, X^m will denote an m-dimensional manifold. We will spell out the special relations among the characteristic classes of tangent bundles of 4-manifolds.

1.4.1. Characteristic classes. The set of isomorphism classes of $U(1)$-bundles ($O(1)$-bundles resp.) over X will be denoted by $\mathcal{L}_X$ ($\mathcal{R}_X$ resp.). Obviously $\mathcal{L}_X$ and $\mathcal{R}_X$ admit group structures with the tensor product of line bundles as multiplication.

Proposition 1.4.1. *The groups $\mathcal{L}_X$ and $H^2(X; \mathbb{Z})$ are canonically isomorphic; similarly, $\mathcal{R}_X$ and $H^1(X; \mathbb{Z}_2)$ are canonically isomorphic groups.*

Proof. We only give a hint for proving that $\mathcal{L}_X \cong H^2(X;\mathbb{Z})$; the proof of the other statement follows the same pattern. From algebraic topology we know that $H^2(X;\mathbb{Z}) = [X, K(\mathbb{Z},2)]$, where $[X,Y]$ is the set of homotopy classes of maps from X to Y, and $K(\mathbb{Z},2)$ is the Eilenberg-MacLane space with $\pi_i(K(\mathbb{Z},2)) = 0$ for $i \neq 2$ and $\pi_2(K(\mathbb{Z},2)) = \mathbb{Z}$. Bundle theory tells us that $\mathcal{L}_X$ and $[X, BU(1)]$ are isomorphic (where $BU(1)$ is the classifying space for $U(1)$-bundles). An easy argument shows that both $K(\mathbb{Z},2)$ and $BU(1)$ are homotopy equivalent to $\mathbb{CP}^\infty$, and this gives the desired isomorphism. The proof that $\mathcal{R}_X \cong H^1(X;\mathbb{Z}_2)$ rests on the fact that both $K(\mathbb{Z}_2,1)$ and $BO(1)$ are homotopy equivalent to $\mathbb{RP}^\infty$. □

The isomorphism $\mathcal{L}_X \to H^2(X;\mathbb{Z})$ (with suitably chosen sign) is usually called c_1, and $c_1(L)$ is the *first Chern class* of the complex line bundle L. Similarly, $w_1\colon \mathcal{R}_X \to H^1(X;\mathbb{Z}_2)$ is the isomorphism given by Proposition 1.4.1, and $w_1(R)$ is the *first Stiefel-Whitney class* of the real line bundle $R \to X$. An alternative — obstruction theoretic — description of c_1 (and of w_1) can be given in the following way. (Compare with Section 5.6.) Suppose that X has a CW-decomposition and $L \to X$ is a $U(1)$-bundle; note that for each cell $f\colon D^k \to X$, the bundle f^*L over D^k is canonically trivial. Obviously L is trivial on the 0-skeleton of X, and since $U(1)$ is connected, such a trivialization can be extended over the 1-skeleton. Comparing this trivialization with the canonical trivialization over each 2-cell defines a map $\varphi\colon \partial D^2 \to U(1) = S^1$ for every 2-cell D^2, hence associates a number (the degree of φ) to every 2-cell. In this way we define a cochain c.

Claim 1.4.2. *The cochain c is a cocycle, and the class $[c] \in H^2(X;\mathbb{Z})$ depends only on the bundle $L \to X$ (and is independent of the CW-decomposition and the trivialization).* □

Note that if $[c] = 0$, so the trivialization can be changed over the 1-skeleton in such a way that it extends over the 2-skeleton, then L is trivial. This follows from the fact that when we want to extend the trivialization to higher dimensional cells, we do not find any more obstructions, since all maps $\partial D^k \to U(1) = S^1$ are nullhomotopic once $k > 2$.

Theorem 1.4.3. *The above-defined class $[c] \in H^2(X;\mathbb{Z})$ of $L \to X$ coincides with the Chern class $c_1(L)$ defined by the isomorphism of Proposition 1.4.1.*

Proof. Both c_1 and the above obstruction class $[c]$ are natural with respect to continuous maps. Since each line bundle can be regarded as the pull-back of the tautological line bundle $\tau \to \mathbb{CP}^\infty$, Theorem 1.4.3 has to be proved only for τ. That proof, however, is essentially contained in the proof of Claim 2.2.1. □

A similar argument gives an obstruction theoretic description for $w_1(R)$ of a real line bundle $R \to X$. (For more about obstruction theory see also [**St**] or [**FFG**].)

Next we outline the definition of the other Chern classes $c_i(E)$ in $H^{2i}(X;\mathbb{Z})$ $(i = 2, \dots, n)$ for a complex n-plane bundle E. (For a real n-plane bundle $F \to X$ one can proceed similarly and get the Stiefel-Whitney classes $w_i(F) \in H^i(X;\mathbb{Z}_2)$, $i = 1, \dots, n$.) If $E = L_1 \oplus \dots \oplus L_n$ is a sum of complex line bundles L_i, take $c(E) \in H^*(X;\mathbb{Z})$ (the *total Chern class*) to be the cup product $c(E) = (1 + c_1(L_1)) \cup \dots \cup (1 + c_1(L_n)) \in H^*(X;\mathbb{Z})$. The component of $c(E)$ in $H^{2i}(X;\mathbb{Z})$ is called the $i^{\underline{th}}$ *Chern class* $c_i(E)$ of E, so $c(E) = 1 + c_1(E) + \dots + c_n(E)$. Hence $c_i(E)$ is the value of the $i^{\underline{th}}$ elementary symmetric polynomial of n variables evaluated on $c_1(L_1), \dots, c_1(L_n)$. Not all n-plane bundles are sums of complex line bundles, however. For the definition of Chern classes in those cases we need a theorem.

Theorem 1.4.4. (Splitting Principle) *For a given complex n-plane bundle $E \to X$ there is a space Y and a map $g\colon Y \to X$ such that g^*E splits as $L_1 \oplus \dots \oplus L_n$ (where $L_i \to Y$ are complex line bundles), $g^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is a monomorphism and the elementary symmetric polynomials of the classes $c_1(L_i)$ are in* Im g^*. □

For the proof and an analogous statement for real bundles see [**Sha**], [**Hu**]. Here we only give the inductive step of the construction of Y and g; the properties should be checked by the reader. Projectivize $\pi\colon E \to X$ (replace the fiber $\mathbb{C}^n$ with $\mathbb{CP}^{n-1}$), and get the new bundle $p\colon \mathbb{CP}(E) \to X$. Pull $E \to X$ back via the map $p\colon \mathbb{CP}(E) \to X$ and get the $\mathbb{C}^n$-bundle $p^*(E) \to \mathbb{CP}(E)$. It is easy to see that $p^*(E) = L \oplus E_1$, where L is a line bundle and $E_1 \to \mathbb{CP}(E)$ is a $\mathbb{C}^{n-1}$-bundle. Applying the Leray-Hirsch Theorem [**Hu**] we get that $p^*\colon H^*(X;\mathbb{Z}) \to H^*(\mathbb{CP}(E);\mathbb{Z})$ is a monomorphism. Repeat this inductive step and split E into line bundles; the composition of the corresponding projections will give the desired g. The classes $c(g^*(E)) = 1 + c_1(g^*(E)) + \dots + c_n(g^*(E)) \in H^*(Y;\mathbb{Z})$ are defined, and since these classes are elementary symmetric polynomials of the $c_1(L_i)$'s, they are in Im g^*. The homomorphism g^* is injective, so $g^{*-1}(c_i(g^*(E)))$ is a well-defined cohomology class, which is, by definition, the $i^{\underline{th}}$ Chern class of $E \to X$. The real analogue of the above process defines Stiefel-Whitney classes $w_i(F) \in H^i(X;\mathbb{Z}_2)$ of a given $\mathbb{R}^n$-bundle $F \to X$. For a real n-plane bundle $F \to X$ the *Pontrjagin classes* $p_i(F) \in H^{4i}(X;\mathbb{Z})$ can be defined by the formula $p_i(F) = (-1)^i c_{2i}(F \otimes_{\mathbb{R}} \mathbb{C})$. The next proposition summarizes the most important properties of the Chern, Stiefel-Whitney and Pontrjagin classes. Let $E \to X$ denote a $\mathbb{C}^n$-bundle and $F \to X$ an arbitrary $\mathbb{R}^n$-bundle.

Proposition 1.4.5. **(a)** *The above-defined characteristic classes* $c_i(E) \in H^{2i}(X;\mathbb{Z})$, $w_i(F) \in H^i(X;\mathbb{Z}_2)$ *and* $p_i(F) \in H^{4i}(X;\mathbb{Z})$ *are well-defined cohomology elements. (They depend only on the bundles, not on the particular splitting used in the definition.)*

(b) *These classes are natural with respect to continuous maps:* $c_i(f^*E) = f^*(c_i(E))$, $w_i(f^*F) = f^*(w_i(F))$ *and* $p_i(f^*F) = f^*(p_i(F))$ *for a continuous map* $f\colon X' \to X$.

(c) (Whitney product formula) *For direct sums of bundles we have the identities* $c(E \oplus E') = c(E) \cup c(E')$, $w(F \oplus F') = w(F) \cup w(F')$, *and* $2(p(F \oplus F') - p(F) \cup p(F')) = 0$. □

Exercises 1.4.6. **(a)** Show that if $E \to X$ is a complex n-plane bundle, then $E \otimes \mathbb{C} \cong E \oplus \overline{E}$. Here $\overline{E}$ stands for the conjugate bundle of $E \to X$. (See [**MS**] for a solution.)

(b) Prove that $c_i(\overline{E}) = (-1)^i c_i(E)$.

There is one more characteristic class that will be used in our arguments, namely, the Euler class of an oriented real n-plane bundle. We define it only in the case where X^m is an m-dimensional closed, smooth manifold and $F_1 \to X$ is a smooth, oriented $\mathbb{R}^n$-bundle. Let $s\colon X \to F_1$ be a generic smooth section of $F_1 \to X$ and $Z = s^{-1}(0)$ be its zero set. (By generic we mean that the image $s(X)$ intersects the image of the zero section of $F_1 \to X$ transversally.) The fundamental class of the zero set Z defines a homology class $[Z] \in H_{m-n}(X;\mathbb{Z})$, so its Poincaré dual $PD([Z])$ gives rise to an element in $H^n(X;\mathbb{Z})$.

Claim 1.4.7. *The class* $e(F_1) = PD([Z]) \in H^n(X;\mathbb{Z})$ *depends only on the bundle* $F_1 \to X$*, and by definition this cohomology class is the Euler class of* $F_1 \to X$. □

For a manifold X, the $i^{\underline{th}}$ Stiefel-Whitney class of its tangent bundle TX is denoted by $w_i(X)$. Similarly, for an oriented manifold one defines $p_i(X)$ and $e(X)$. If X is a complex manifold, so TX admits a canonical complex structure, $c_i(X)$ is defined as well. For the following computations see [**MS**].

Examples 1.4.8. **(a)** The total Chern class of the complex projective space $\mathbb{CP}^n$ is given as $c(\mathbb{CP}^n) = (1+g)^{n+1}$, where g is the standard generator of $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$. For example, $c_1(\mathbb{CP}^n) = (n+1)g$ and $c_1(\mathbb{CP}^2) = 3g$, $c_2(\mathbb{CP}^2) = 3g^2$.

(b) Using Exercises 1.4.6(a) and (b) we can see that $p(\mathbb{CP}^n) = (1+g^2)^{n+1}$; in particular, $p_1(\mathbb{CP}^2) = 3g^2$.

(c) The total Stiefel-Whitney class of $\mathbb{RP}^n$ is $w(\mathbb{RP}^n) = (1+a)^{n+1}$, where $a \in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2$.

The following proposition describes the most important relations among characteristic classes.

Proposition 1.4.9. *If $E \to X$ is an n-dimensional complex bundle, then the relations $c_n(E) = e(E)$ and $c_i(E) \equiv w_{2i}(E) \pmod 2$ hold for all $i \leq n$; moreover $w_{2i+1}(E) = 0$. For a smooth, closed, oriented n-dimensional manifold X we have $e(X) \equiv w_n(X) \pmod 2$, while $\langle e(X), [X]\rangle$ is equal to the Euler characteristic $\chi(X)$ of X.* □

Exercise 1.4.10. Find the relation among the Pontrjagin and Chern classes of a complex bundle $E \to X$. In particular, show that $p_1(E)$ is given by $c_1^2(E) - 2c_2(E)$. (*Hint*: Use Exercises 1.4.6 or see [**MS**].)

Note that we denote the Euler characteristic of a manifold X by $\chi(X)$ — while the Euler class of a bundle E is denoted by $e(E)$. (The *holomorphic* Euler characteristic of a complex manifold will be denoted by χ_h.) If X is a closed, complex n-dimensional manifold, and $\{i_1, \ldots, i_k\}$ is a partition of n (that is, i_j are positive integers and $i_1 + \ldots + i_k = n$), then the product $c_{i_1}(X) \cup \ldots \cup c_{i_k}(X)$ can be evaluated on the fundamental class $[X]$, defining the *Chern number* corresponding to the partition $\{i_1, \ldots, i_k\}$. A similar definition gives Stiefel-Whitney (and Pontrjagin) numbers of closed, smooth (oriented) manifolds. If S is a complex surface, the two Chern numbers are denoted by $c_2[S]$ and $c_1^2[S]$. (By slight abuse of notation we will often confuse the Chern numbers $c_2[S]$ and $c_1^2[S] \in \mathbb{Z}$ with the corresponding Chern classes $c_2(S), c_1^2(S) \in H^4(S; \mathbb{Z})$. In this introductory chapter, however, we would like to make the distinction clear.)

Exercises 1.4.11. (a) Prove that if X is a closed, orientable manifold of odd dimension, then $\chi(X) = 0$.

(b)* Show that if Σ is a closed, oriented surface embedded in an arbitrary oriented 4-manifold X, then $e(\nu\Sigma)[\Sigma] = Q_X([\Sigma], [\Sigma]) = [\Sigma]^2$.

Now we turn our attention to the 4-dimensional case, so X denotes a 4-dimensional (closed, oriented, smooth) manifold. The next theorem gives a relation between the signature of X and the first Pontrjagin class of its tangent bundle TX.

Theorem 1.4.12. (Hirzebruch signature theorem for 4-manifolds) *If X is a smooth, closed, oriented 4-dimensional manifold, then its signature $\sigma(X)$ is equal to $\frac{1}{3}\langle p_1(X), [X]\rangle$.* □

To emphasize that a certain smooth 4-manifold X admits a complex structure (so it is a *complex surface*), we will denote it by S. For a complex surface S we have that $p_1(S) = c_1^2(S) - 2c_2(S)$ (cf. Exercise 1.4.10), and so from Theorem 1.4.12 (and from the identity $c_2[S] = e[S] = \chi(S)$) it follows that the Chern number $c_1^2[S]$ equals $3\sigma(S) + 2\chi(S)$. Below, we will show that $c_1(S) \in H^2(S;\mathbb{Z})$ is characteristic for the intersection form (because it reduces mod 2 to $w_2(S)$), so Lemma 1.2.20 implies that $c_1^2[S] \equiv \sigma(S) \pmod 8$, and thus $\sigma(S) + \chi(S) \equiv 0 \pmod 4$. We obtain the following theorem by expressing this in terms of Chern numbers, and also observing that for any closed, oriented 4-manifold we have $\sigma(X) + \chi(X) = b_2^+(X) - b_2^-(X) + (b_2^+(X) + b_2^-(X) - 2b_1(X) + 2) = 2(1 - b_1(X) + b_2^+(X))$.

Theorem 1.4.13. (Noether formula) *For a complex surface S the integer $c_1^2[S] + c_2[S] = 3(\sigma(S) + \chi(X))$ is divisible by 12, or equivalently, $1 - b_1(S) + b_2^+(S)$ is even. In particular, if S is a simply connected complex surface then $b_2^+(S)$ is odd.* □

Note that for defining $c_i(S)$ we do not really need S to be a complex manifold. If $TX \to X$ is a $\mathbb{C}^n$-bundle, c_i already makes sense. Of course, if X is a complex manifold then $TX \to X$ has a natural complex structure. However, a $\mathbb{C}^n$-structure on the fibers (defining multiplication by i fiberwise) can be defined for a much wider class of manifolds. Such a structure on a manifold is called an *almost-complex structure*. Formally, we have

Definition 1.4.14. An *almost-complex structure* on the bundle $TX \to X$ is a smooth, fiberwise linear map $J\colon TX \to TX$ covering id_X such that $J^2 = -\mathrm{id}_{TX}$.

An almost-complex structure defines a natural orientation on the smooth manifold X, since the choice of J reduces the structure group of the tangent bundle to $GL(n;\mathbb{C}) \subset GL^+(2n;\mathbb{R})$. Once an almost-complex structure is specified, Chern classes c_i make sense. In this latter case the Chern classes will also depend on the almost-complex structure chosen; for an oriented manifold X we only consider almost-complex structures generating the given orientation. For a given J the corresponding Chern classes of (TX, J) are denoted by $c_i(X, J) \in H^{2i}(X;\mathbb{Z})$. The following theorem provides a necessary and sufficient condition for the existence of an almost-complex structure on the 4-manifold X.

Theorem 1.4.15. (Wu [**Wu**], see also [**HH**]) *For a given 4-manifold X and almost-complex structure J on X we have $c_2(X, J) = e(X) \in H^4(X;\mathbb{Z})$, $c_1(X, J) \equiv w_2(X) \pmod 2$ and $c_1^2[X, J] = 3\sigma(X) + 2\chi(X)$. Conversely, if for $h \in H^2(X;\mathbb{Z})$ the equation $h^2 = 3\sigma(X) + 2\chi(X)$ and the congruence $h \equiv w_2(X) \pmod 2$ hold, then there is an almost-complex structure J on TX with $h = c_1(X, J)$.* □

(For the proof see Exercise 1.4.21(c); cf. also Exercise 10.1.3(a).) Since the proof of the Noether formula only used properties of $c_1(S)$ satisfied by the first Chern class $c_1(X, J)$ of an almost-complex structure J — namely that $c_1^2[X,J] = 3\sigma(X) + 2\chi(X)$ and $c_1(X,J) \equiv w_2(X) \pmod 2$ — Theorem 1.4.13 holds for an almost-complex manifold as well. By Theorem 1.4.15, the existence of an almost-complex structure is a cohomological question.

Exercises 1.4.16. (a) Prove that S^4, $(S^2 \times S^2)\#(S^2 \times S^2)$ and $\mathbb{CP}^2\#\mathbb{CP}^2$ do not admit any almost-complex structure.

(b)* More generally, prove (using Theorem 1.4.15) that a simply connected, smooth, closed 4-manifold X admits an almost-complex structure iff $b_2^+(X)$ is odd.

The definitions $c_2[X] = \chi(X)$ and $c_1^2[X] = 3\sigma(X) + 2\chi(X)$ extend the notions of c_2 and c_1^2 to all closed, oriented 4-manifolds (even to manifolds without an almost-complex structure). The next formula has fundamental importance in the study of the smooth structures of 4-manifolds. (For generalizations see Theorems 2.4.8 and 11.4.7; for applications see, e.g., Theorem 2.1.6.) Let S be a complex surface (so a real 4-dimensional manifold) with $i\colon C \hookrightarrow S$ a smooth (nonsingular), connected complex curve in it.

Theorem 1.4.17. (Adjunction Formula) *Denoting the genus of C by $g(C)$ and the self-intersection by $[C]^2$, we have $2g(C)-2 = [C]^2 - c_1(S)[C]$ (where $c_1(S)[C]$ means $\langle c_1(S), [C]\rangle \in \mathbb{Z}$).*

Proof. Restrict the tangent bundle of S to C and apply characteristic class computations: $TS|C = TC \oplus \nu C$, where $\nu C \to C$ is the normal bundle of C in S; hence $c_1(S)[C] = c_1(TS|C)[C] = c_1(TC)[C] + c_1(\nu C)[C] = e(TC)[C] + e(\nu C)[C] = \chi(C) + e(\nu C)[C] = 2 - 2g(C) + e(\nu C)[C]$. The solution of Exercise 1.4.11(b) now gives $c_1(S)[C] = 2 - 2g(C) + [C]^2$, which proves the adjunction formula. □

A similar argument shows:

Proposition 1.4.18. *For a given oriented 4-manifold X and $\alpha \in H_2(X;\mathbb{Z})$ we have that $\langle w_2(X), \alpha\rangle \equiv Q_X(\alpha,\alpha) \pmod 2$.*

Proof. Represent $\alpha \in H_2(X;\mathbb{Z})$ by an embedded orientable surface $\Sigma \subset X$. Then $\langle w_2(X), \alpha\rangle = \langle w_2(TX|\Sigma), [\Sigma]\rangle = w_2(T\Sigma)[\Sigma] + w_2(\nu\Sigma)[\Sigma] + (w_1(T\Sigma) \cup w_1(\nu\Sigma))[\Sigma] \equiv e(T\Sigma)[\Sigma] + e(\nu\Sigma)[\Sigma] \equiv e(\nu\Sigma)[\Sigma] = Q_X([\Sigma],[\Sigma]) = \alpha^2 \pmod 2$. Note that $w_1(T\Sigma) = 0$ and $e(T\Sigma)[\Sigma] = \chi(\Sigma) \equiv 0 \pmod 2$, since Σ is orientable. (In the expression $\langle w_2(X), \alpha\rangle$ we took the mod 2 reduction of the integral homology class α.) □

Assuming that X is orientable, the same relation holds for homology elements with $\mathbb{Z}_2$-coefficients: For $a \in H_2(X;\mathbb{Z}_2)$ one has $\langle w_2(X), a\rangle = Q_X(a,a)$. This equation is called the *Wu formula* (cf. Exercise 5.7.3). Note also that Proposition 1.4.18 holds even if X has boundary or is noncompact.

Exercise 1.4.19. Prove the Wu formula using Remark 1.2.4. (For a nonorientable $\mathbb{R}^n$-bundle F, $w_n(F)$ can be defined like the Euler class in Claim 1.4.7, using $\mathbb{Z}_2$-coefficients, and $\langle w_n(X), [X]\rangle \equiv \chi(X) \pmod 2$ for X closed.)

Proposition 1.4.18 shows that if there is no 2-torsion in $H^2(X;\mathbb{Z})$ (for example, if X is simply connected), then $w_2(X)$ vanishing is equivalent to Q_X being even. This is not true for all 4-manifolds, since in the presence of 2-torsion the mod 2 reduction $H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z}_2)$ is not onto. There exists a manifold X (e.g., the Enriques surface, cf. Section 3.4) with nontrivial $w_2(X)$, $Q_X \cong (-E_8) \oplus H$ and $\pi_1(X) \cong \mathbb{Z}_2$. (In fact, $w_2(X)$ lifts to a class of order 2 in $H^2(X;\mathbb{Z})$ and pairs nontrivially with a class in $H_2(X;\mathbb{Z}_2)$ with no integer lift.) Note also that since $c_1(X,J) \equiv w_2(X) \pmod 2$, Proposition 1.4.18 implies that the first Chern class of an almost-complex structure is a characteristic element, as we needed when proving Theorem 1.4.13.

To demonstrate how useful characteristic classes are, we describe the classification of $U(2)$, $SU(2)$, $SO(3)$ and $SO(4)$-bundles over a 4-manifold X. (Note that the case $SO(2) \cong U(1)$ was already discussed at the beginning of this section.)

Theorem 1.4.20. **(a)** *Two $U(2)$-bundles E_1 and E_2 on X are isomorphic iff $c_1(E_1) = c_1(E_2)$ and $c_2(E_1) = c_2(E_2)$. Moreover, for every pair $(c_1, c_2) \in H^2(X;\mathbb{Z}) \times H^4(X;\mathbb{Z})$ there is a $U(2)$-bundle E with $c_1 = c_1(E)$ and $c_2 = c_2(E)$. Furthermore, a $U(2)$-bundle E can be reduced to an $SU(2)$-bundle iff $c_1(E) = 0$. Consequently, two $SU(2)$-bundles E_1 and E_2 are isomorphic iff $c_2(E_1) = c_2(E_2)$.*

(b) *Two $SO(4)$-bundles F_1 and F_2 are isomorphic iff $w_2(F_1) = w_2(F_2)$, $p_1(F_1) = p_1(F_2)$ and $e(F_1) = e(F_2)$.*

(c) *Two $SO(3)$-bundles $F_1, F_2 \to X$ are isomorphic iff $w_2(F_1) = w_2(F_2)$ and $p_1(F_1) = p_1(F_2)$. Moreover $p_1(F_1) \equiv \mathcal{P}(w_2(F_1)) \pmod 4$, and for every pair $(p_1, w_2) \in H^4(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}_2)$ with $p_1 \equiv \mathcal{P}(w_2) \pmod 4$ there is an $SO(3)$-bundle $F \to X$ such that $p_1 = p_1(F)$ and $w_2 = w_2(F)$.* □

In the above theorem, the map $\mathcal{P}\colon H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_4)$ denotes the *Pontrjagin square*. (If the cohomology class $c \in H^2(X;\mathbb{Z})$ is an integral lift of w_2 (so $c \equiv w_2 \pmod 2$), then $c^2 \equiv \mathcal{P}(w_2) \pmod 4$. We do not give the definition of $\mathcal{P}$ in full generality.)

Exercises 1.4.21. **(a)** Prove that if $c, c' \in H^2(X;\mathbb{Z})$ are lifts of a fixed element $w_2 \in H^2(X;\mathbb{Z}_2)$, then $c^2 \equiv (c')^2 \pmod 4$.

(b) Prove Theorem 1.4.20(a). (*Hint*: Take a CW decomposition of X and apply obstruction theory. For Theorem 1.4.20(b) and (c) see [**DW**].)

(c)* Using Theorem 1.4.20, prove Theorem 1.4.15.

(d) Show that for an $SO(n)$-bundle F we have $w_{2i}^2(F) \equiv p_i(F) \pmod 2$. (*Hint*: Recall that $p_i(F) = (-1)^i c_{2i}(F \otimes_{\mathbb{R}} \mathbb{C}) \equiv w_{4i}(F \otimes_{\mathbb{R}} \mathbb{C}) \pmod 2$; by the Whitney product formula determine $w_{4i}(F \otimes_{\mathbb{R}} \mathbb{C})$ in terms of the $w_j(F)$'s.)

We close this subsection by quoting the *Lefschetz Hyperplane Theorem* used in various places in the text. (The proof appears in [**M2**], see also Exercise 11.2.3(b).)

Theorem 1.4.22. (Lefschetz Hyperplane Theorem) *Let X be a compact, complex n-dimensional submanifold of $\mathbb{CP}^N$. If H is a hyperplane in $\mathbb{CP}^N$, then the homomorphisms $\pi_i(X \cap H) \to \pi_i(X)$ and $H_i(X \cap H) \to H_i(X)$ are isomorphisms for $i < n-1$ and surjections for $i = n-1$.* □

1.4.2. Spin structures. We will now sketch the theory of spin structures and their relation to $w_2(X)$. For an obstruction-theoretic approach of the same notions see Section 5.6.

For a given real n-plane bundle $F \to X$ one can always reduce the structure group $GL(n;\mathbb{R})$ to $O(n)$ by introducing a Riemannian metric on F. The Lie group $O(n)$ is not connected, however, and the possibility of a further reduction of the structure group to $SO(n)$ (a connected component of $O(n)$) depends on a characteristic class of F. If such a reduction exists, $F \to X$ is an *orientable* bundle, and the choice of a reduction is an *orientation* for F (cf. Remark 1.1.3; note that by definition $GL^+(n;\mathbb{R}) \cap O(n) = SO(n)$). Since $w_1(\det F) = w_1(F)$ and the line bundle $\det F$ is trivial iff the structure group of F can be reduced to $SO(n)$, we have

Lemma 1.4.23. *The bundle $F \to X$ is orientable iff $w_1(F) \in H^1(X;\mathbb{Z}_2)$ vanishes; the orientations are parametrized by $H^0(X;\mathbb{Z}_2)$ (which is isomorphic to $\mathbb{Z}_2$ if X is connected).* □

If F is the tangent bundle TX, the orientability of F means that X is an orientable manifold (cf. Section 1.1). Although $SO(n)$ is connected, it is not a simply connected group; for $n \geq 3$ we have $\pi_1(SO(n)) \cong \mathbb{Z}_2$. (We always reduce to the case $n \geq 3$ by summing F with a trivial bundle if necessary.) The universal (double) cover of $SO(n)$ is the *spin group* $Spin(n)$. Let $F \to X$ be an oriented Riemannian (i.e., $SO(n)$-) bundle; the corresponding principal frame bundle will be denoted by $P_{SO(n)} \to X$.

Definition 1.4.24. The bundle $F \to X$ is *spinnable* if $P_{SO(n)} \to X$ can be covered by a principal $Spin(n)$-bundle $P_{Spin(n)} \to X$ such that the double covering $P_{Spin(n)} \to P_{SO(n)}$ is the universal cover $\rho\colon Spin(n) \to SO(n)$ fiberwise, i.e., $P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$. (Hence a spin structure comprises a principal $Spin(n)$-bundle $P_{Spin(n)} \to X$ together with an identification $c\colon P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$.) Fixing such a cover of $P_{SO(n)}$ — a *spin* *structure* — realizes F as a *spin bundle*. A spin structure on $F = TX$ turns X into a *spin manifold*.

From spectral sequences we get an exact sequence

$$0 \to H^1(X;\mathbb{Z}_2) \to H^1(P_{SO(n)};\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n);\mathbb{Z}_2) \xrightarrow{\delta} H^2(X;\mathbb{Z}_2),$$

where $H^1(SO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\delta(1)$ equals $w_2(F)$ in $H^2(X;\mathbb{Z}_2)$ (cf. Remark 5.6.9(b)). The exercise below shows that the double covers of $P_{SO(n)}$ are in 1-1 correspondence with elements of $H^1(P_{SO(n)};\mathbb{Z}_2)$. By the assumption that fiberwise the cover needs to be $Spin(n) \to SO(n)$, the spin structures are in 1-1 correspondence with elements of $(i^*)^{-1}(1) \subset H^1(P_{SO(n)};\mathbb{Z}_2)$. This means that a required cover exists iff $1 \in \mathrm{Im}\ i^*$, hence (by exactness) iff $\delta(1) = w_2(F) = 0$.

Proposition 1.4.25. *The $SO(n)$-bundle $\pi\colon F \to X$ is spinnable iff its second Stiefel-Whitney class $w_2(F) \in H^2(X;\mathbb{Z}_2)$ vanishes. If so, then the different spin structures are parametrized by $(i^*)^{-1}(1) \cong \ker\ i^*$, which is isomorphic to $H^1(X;\mathbb{Z}_2)$. (The identification of $H^1(X;\mathbb{Z}_2)$ with the set of spin structures is not canonical; it becomes canonical only after choosing a "base spin structure" corresponding to $0 \in H^1(X;\mathbb{Z}_2)$.)* □

Exercises 1.4.26. (a)* Show that the double covers of a manifold X are in 1-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$.

(b) Using obstruction theory, show that if G is a connected Lie group and X is a CW complex, then any principal G-bundle P_G is trivial over the 1-skeleton X_1. If G is simply connected, show that $P_G|X_2$ is trivial. (See Section 5.6 for related discussions.) What can we say about $P_G|X_3$ if $\pi_1(G) = 1$? (*Hint*: Use the fact that for any Lie group, $\pi_2(G) = 0$.)

We say that the oriented manifold X is spinnable if $w_2(X) = 0$. Note that when X is simply connected (so $H^1(X;\mathbb{Z}_2) = 0$), the spin structure is unique.

Remarks 1.4.27. (a) By slight (and standard) abuse of terminology we will refer to a manifold with $w_2(X) = 0$ as a *spin* manifold, although technically this implies the choice of a particular spin structure.

(b) For a 3-manifold X^3 we have $w_1^2(X) = w_2(X)$ [**MS**], so an orientable 3-manifold always has a spin structure. (In fact, if X^3 is orientable, then its tangent bundle is trivial, since $\pi_2(SO(3)) = 0$.)

(c) A simply connected (not necessarily closed) 4-manifold X is spin iff Q_X is even (since both statements are equivalent to $w_2(X)$ vanishing). If $H_1(X;\mathbb{Z})$ has 2-torsion, this equivalence no longer holds. (See also Corollary 5.7.6.)

Rohlin's Theorem (Theorem 1.2.29) was stated only for simply connected 4-manifolds. It can be generalized to arbitrary (closed, smooth) 4-manifolds with the following theorem — replacing the assumption about Q_X with the assumption that X is spin. (See [**K2**] or [**LaM**] for a proof.)

Theorem 1.4.28. (Rohlin, [**R2**]) *If X is a smooth, closed, spin 4-manifold, then $\sigma(X) \equiv 0 \pmod{16}$.* □

In the rest of this section we will give various bundle constructions and define operators one can associate to a spin structure. (For further details see [**LaM**], [**Mr2**].) The importance of these constructions in dimension four becomes clear once we generalize the notion of spin structures to *spin^c structures* and list the spectacular results based on that theory (cf. Section 2.4). The group $Spin(n)$ can be constructed as a subgroup of a 2^n-dimensional real algebra, the *Clifford algebra Cl_n*. By definition $Cl_n = T(\mathbb{R}^n)/I(\mathbb{R}^n)$, where $T(\mathbb{R}^n)$ is the tensor algebra $\bigoplus_k (\mathbb{R}^n)^{\otimes k}$ and $I(\mathbb{R}^n)$ is the ideal generated by elements of the form $v \otimes v + \langle v, v\rangle 1 \in T(\mathbb{R}^n)$, $v \in \mathbb{R}^n$. We denote the complexification of Cl_n by $\mathbb{C}l_n$. The following algebraic statement can be found, e.g., in [**LaM**].

Proposition 1.4.29. *If $n = 2k+1$ is odd, then the complex Clifford algebra $\mathbb{C}l_n$ is isomorphic to the direct sum of two isomorphic matrix algebras: $\mathbb{C}l_n \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$ (where $M_m(\mathbb{C}) = \{m \times m$ complex matrices$\}$). In this way we get two complex 2^k-dimensional representations of $Spin(n) \subset \mathbb{C}l_n$; these are irreducible and isomorphic. (We will denote them by S_n.) If $n = 2k$ is even, then $\mathbb{C}l_n \cong M_{2^k}(\mathbb{C})$; the corresponding complex 2^k-dimensional representation S_n of $Spin(n) \subset \mathbb{C}l_n$ splits into two (nonisomorphic) irreducible representations, hence $S_n = S_n^+ \oplus S_n^-$ as a $Spin(n)$-module.* □

Exercise 1.4.30. Check that for the real Clifford algebras we have $Cl_1 \cong \mathbb{C}, Cl_2 \cong \mathbb{H}, Cl_3 \cong M_2(\mathbb{C})$ and $Cl_4 \cong M_2(\mathbb{H}) = \{2\times 2$ quaternionic matrices$\}$. Determine $\mathbb{C}l_n$ for $n \leq 4$.

Assume now that X is a spin manifold with a fixed spin structure $P_{Spin(n)} \to X$ and $c\colon P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$. Using the above complex representation S_n of $Spin(n)$, one can associate the vector bundle $S \to X$ to $P_{Spin(n)} \to X$. Sections of $S \to X$ are the *spinors* over X. If $\dim X$ is even, then the bundle S splits as $S = S^+ \oplus S^-$ (corresponding to the decomposition $S_n = S_n^+ \oplus S_n^-$); sections of S^+ (S^-) are the *positive (negative) spinors*, respectively. The bundle associated to $P_{Spin(n)} \to X$ by the $Spin(n)$-representation $\mathbb{C}l_n$ is the *Clifford bundle* $\mathbb{C}l(X)$ of the spin structure. The action of $\mathbb{C}l_n$ on S_n induces an action of the Clifford bundle $\mathbb{C}l(X)$ on the spinor bundle $S \to X$; this action is called the *Clifford multiplication*. Recall that for defining the principal bundle $P_{SO(n)} \to X$ of $TX \to X$ we fixed a metric g and an orientation on X (or equivalently, reduced the structure group of TX from $GL(n;\mathbb{R})$ to $SO(n)$). There is a canonical object associated to the metric g on X, which is the *Levi-Civita connection* $\nabla_g\colon \Gamma(X;TX) \to \Gamma(X;TX \otimes T^*X)$. (As usual, $\Gamma(X;F)$ denotes the vector space of C^∞-sections of the vector bundle $F \to X$.) This connection can be pulled back to the $Spin(n)$-bundle $P_{Spin(n)} \to X$, defining a covariant differentiation $\nabla\colon \Gamma(X;S) \to \Gamma(X;S \otimes T^*X)$ on the associated bundle $S \to X$.

Remark 1.4.31. It is a standard fact of differential geometry (cf. [**DK**]) that a covariant differentiation on the associated bundle F determines a $Lie(G)$-valued 1-form on the principal G-bundle P_G corresponding to F, and vice versa. (Here $Lie(G)$ denotes the Lie algebra of the Lie group G.) Hence the Levi-Civita connection ∇_g determines a $Lie(SO(n))$-valued 1-form on $P_{SO(n)}$, which can be pulled back to $P_{Spin(n)}$. (Note that the corresponding Lie algebras satisfy $Lie(SO(n)) = Lie(Spin(n))$.) In the following ∇ denotes the associated covariant differentiation on the associated bundle $S \to X$.

Since T^*X is a subbundle of the Clifford bundle $\mathbb{C}l(X)$, T^*X acts on the spinor bundle S. Hence one can define a map (the Clifford multiplication) $C\colon \Gamma(X;S \otimes T^*X) \to \Gamma(X;S)$.

Definition 1.4.32. For a given Riemannian manifold X with a fixed spin structure $P_{Spin(n)} \to X$, the composition

$$\partial\!\!\!/ = C \circ \nabla\colon \Gamma(X;S) \to \Gamma(X;S)$$

is called the *Dirac operator* of the spin manifold X. If $\dim X$ is even (so $S = S^+ \oplus S^-$), the Dirac operator $\partial\!\!\!/\colon \Gamma(X;S^\pm) \to \Gamma(X;S^\mp)$ interchanges spinors of opposite sign.

Finally, we examine the above constructions on 4-manifolds. Note that $Spin(3) \cong SU(2) \cong \{\text{unit quaternions}\} = Sp(1) = S^3$. (For $q \in Sp(1)$ associate the map $q\colon \mathbb{H} \to \mathbb{H}$ given by $x \mapsto qxq^{-1}$ (with quaternionic multiplication); this determines an action of $Sp(1)$ on the imaginary quaternions Im $\mathbb{H}$, giving the double cover $Sp(1) = SU(2) \to SO(3) \approx \mathbb{RP}^3$.) Similarly, we have that $Spin(4) = SU(2)\times SU(2)$: For a pair $(q_+, q_-) \in SU(2)\times SU(2)$, take the linear transformation $\mathbb{H} \to \mathbb{H}$ defined by $x \mapsto q_+xq_-^{-1}$, and get the desired universal (double) cover $SU(2) \times SU(2) \to SO(4)$.

In dimension 4 there is an alternative (and obviously equivalent) way of defining spin structures. Let V be a 4-dimensional oriented Euclidean vector space — so the symmetry group of V is isomorphic to $SO(4)$. We define a *spin structure* for V as a pair of 1-dimensional quaternionic vector spaces V^+, V^- with hermitian metrics and a fixed isomorphism $\gamma\colon V \to \mathrm{Hom}_{\mathbb{H}}(V^+, V^-)$ compatible with the metrics.

Exercise 1.4.33. Prove that the symmetry group of a spin structure (V^+, V^-, γ) is isomorphic to $SU(2) \times SU(2) \cong Spin(4)$.

Applying the above definition fiberwise, we get an alternative definition of spin structures over a 4-manifold: A *spin structure* for the 4-dimensional (oriented) Riemannian manifold X^4 is a pair of $SU(2)$-bundles $S^\pm \to X$ and an isomorphism $\gamma \colon TX \to \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$ compatible with the metrics. Once the $SU(2)$-bundles $S^\pm$ are given, the principal bundle $P_{Spin(4)} \to X$ can be easily constructed. (Put the cocycle structures of S^+ and S^- together to map into $SU(2) \times SU(2) \cong Spin(4)$ and get $P_{Spin(4)} \to X$; the isomorphism $c \colon P_{Spin(4)} \times_\rho SO(4) \cong SO(4)$ can be derived from γ.) This shows that the triple $(S^\pm, \gamma)$ determines a spin structure on X in the previous sense. On the other hand, we have already seen how to derive $S^\pm$ and γ (which corresponds to the Clifford multiplication) from $P_{Spin(4)}$, so the two definitions of spin structures over a 4-dimensional manifold X are obviously equivalent.

Remark 1.4.34. It is known that $n = 4$ is the unique dimension in which $Lie(SO(n))$ splits as a Lie algebra. The splitting of $Lie(SO(4))$ as $Lie(SO(3)) \oplus Lie(SO(3))$ will be exploited in the definition of the 4-manifold invariants discussed in Section 2.4.

Chapter 2

Surfaces in 4-manifolds

In light of gauge-theoretic results (see Section 2.4 and especially Theorem 2.4.8), understanding the *genus function* G should lead us to a better understanding of the smooth structure of 4-manifolds. The genus function G is defined on $H_2(X;\mathbb{Z})$ as follows: For $\alpha \in H_2(X;\mathbb{Z})$, consider

$$G(\alpha) = \min\{\text{genus}(\Sigma) \mid \Sigma \subset X \text{ represents } \alpha, \text{ i.e., } [\Sigma] = \alpha\},$$

where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that by the above definition $G(-\alpha) = G(\alpha)$, $G(0) = 0$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X;\mathbb{Z})$. The first section of this chapter is devoted to the description of G for $\mathbb{CP}^2$. Later on we will show certain techniques (removing singularities and the blow-up process) which give partial information about G for other manifolds as well. We close this chapter with a brief introduction to Seiberg-Witten theory and its use in understanding G.

2.1. Surfaces in $\mathbb{CP}^2$

In this section we will determine the constraints on the genera of smoothly embedded surfaces in $\mathbb{CP}^2$ representing a given element of $H_2(\mathbb{CP}^2;\mathbb{Z})$; recall (Exercise 1.3.1(e)) that $H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$, and the complex projective line $H = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0\}$ defines a generator $h = [H] \in H_2(\mathbb{CP}^2;\mathbb{Z})$, hence $G(h) = 0$.

Exercise 2.1.1. Show that $D = \{[x : y : z] \in \mathbb{CP}^2 \mid x^d + y^d + z^d = 0\}$ is a smooth, connected submanifold of $\mathbb{CP}^2$ representing $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$. (*Hint*: Apply the Implicit Function Theorem to prove that D is smooth and prove that it intersects H transversally in exactly d points with positive sign, so $Q_{\mathbb{CP}^2}([D],[H]) = d$. Each component intersects some complex line positively; conclude connectedness from this fact.)

Since D is a connected complex curve in $\mathbb{CP}^2$, the adjunction formula shows that $2g(D) - 2 = [D]^2 - c_1(\mathbb{CP}^2)[D] = d^2 - 3d$, thus

$$g(D) = \frac{1}{2}(d^2 - 3d + 2) = \frac{1}{2}(d-1)(d-2).$$

This proves that $G(dh) \leq \frac{1}{2}(d-1)(d-2)$; as we will see later on, $G(dh)$ actually equals $\frac{1}{2}(d-1)(d-2)$. In the above example we chose a particular representative for $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$; as it was shown in Claim 1.3.11, the particular choice of the homogeneous polynomial plays no important role.

In the following, we will present a method for constructing smooth surfaces representing various homology classes, by starting with polynomials defining *singular* subsets of $\mathbb{CP}^2$ and then "resolving" the singular points. This method will then be generalized to arbitrary ambient 4-manifolds X, and in this way we can explore the behavior of the genus function G. Assume that the (closed) smooth, oriented surfaces Σ_1 and Σ_2 intersect each other transversally in $P \in \mathbb{CP}^2$. Although $\Sigma_1 \cup \Sigma_2$ is not a smooth surface (at P it fails to be a manifold), it still defines a homology class, which is equal to $[\Sigma_1] + [\Sigma_2] \in H_2(\mathbb{CP}^2;\mathbb{Z})$. In a 4-ball neighborhood D of the intersection point P, the union $\Sigma_1 \cup \Sigma_2$ is modeled (up to reversing the ambient orientation) on $F = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1z_2 = 0,\ |z_1|^2 + |z_2|^2 \leq 1\}$: two 2-dimensional disks intersecting each other in one point in the 4-ball. We cut out the pair (D,F) and replace it with (D,R), where $R \subset D$ is obtained by perturbing

$$R' = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1z_2 = \varepsilon,\ |z_1|^2 + |z_2|^2 \leq 1\} \quad (0 < |\varepsilon| \ll 1)$$

to achieve that $\partial F = \partial R \subset \partial D$. R' is simply the graph of $z_2 = \frac{\varepsilon}{z_1}$, hence it is easy to see that topologically it is an annulus (and hence so is R). By replacing (D,F) with (D,R) we eliminate the singular point P, but we do not change either the ambient manifold $\mathbb{CP}^2$ (since we cut out D and then glue it back) or the homology class of $\Sigma_1 \cup \Sigma_2$ (since the subsets F and R are homologous in $(D,\partial D)$).

Remark 2.1.2. Note that $\partial F \subset \partial D = S^3$ is the *Hopf link* (see Figure 2.1 and Section 4.6) and R is a Seifert surface on the Hopf link (Example 6.2.7). While we could use any other Seifert surface to eliminate P, the surface R is the optimal choice because it is the unique minimal genus Seifert surface for the Hopf link. (For more about links and their Seifert surfaces see Section 4.5.) By doing the process described above, one *"removes the singular point P"*. Because we worked locally around P, this method is valid for every 4-manifold X and pair of transversally intersecting surfaces Σ_1 and Σ_2 in it, as well as for transverse self-intersections.

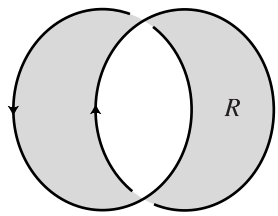


Figure 2.1. Seifert surface on the Hopf link.

Exercise 2.1.3. Represent $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$ by the union of d lines in $\mathbb{CP}^2$ in general position. By repeating the above process, remove all the singular points of this union to get a smooth representative of dh. What is the genus of the resulting surface?

The polynomial defining the d distinct lines in the above exercise is a product of d linear polynomials, and so does not satisfy the hypothesis of the Implicit Function Theorem everywhere. By perturbing this polynomial to a generic (homogeneous) degree-d polynomial p, one gets a smooth representative of $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$ (as the zero set of p) — in this way one changes the surface defined by the polynomial everywhere, but obtains a complex submanifold. The desingularization process described above changes $\Sigma_1 \cup \Sigma_2$ only in a small neighborhood of P (so it is a local process), but we lose the property that the resulting manifold is a complex submanifold. However, the two processes coincide in a small neighborhood of P, and the resulting smooth surfaces are related by a small smooth isotopy of $\mathbb{CP}^2$. (This connection with algebraic geometry serves as a further reason to choose R as opposed to other Seifert surfaces on the Hopf link.)

Both arguments above (the algebro-geometric and the cut-and-paste) gave that $G(2h) = 0$ and that $3h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ can be represented by a torus. Can we represent this latter homology class with surfaces of genus different from 1? An easy argument shows that (smoothly) we can always add to the genus: take a torus in a small 4-disk $D^4 \subset \mathbb{CP}^2$ disjoint from the surface at hand and tube the two surfaces together. It clearly does not change the homology class (the torus represents the 0 class, since it is a boundary in D^4), but obviously it adds 1 to the genus. This is the reason why we are interested in the *minimum* possible value of genera representing a given homology class. (See the definition of G.) Since we have found a torus representative of $3h$ showing $G(3h) \leq 1$, we can now determine $G(3h)$ by examining its representability by an embedded sphere.

Proposition 2.1.4. *The class* $3h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ *cannot be represented by a smoothly embedded sphere.*

The proof of this statement is an interesting application of Rohlin's Theorem 1.2.29 and the technique we will discuss in the next section; hence the proof will be given at the end of the next section. Proposition 2.1.4 suggests that the representatives given by the equations $x^d + y^d + z^d = 0$ should have minimal genera for the classes $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$ ($d = 1, 2, 3, \dots$). This conjecture (frequently attributed to Thom) was recently proved using Seiberg-Witten theory:

Theorem 2.1.5. ([**KM1**], [**MSzT**]) *A smooth surface representing* $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$ ($d \in \mathbb{N}$) *has genus at least* $\frac{1}{2}(d-1)(d-2)$ *(the genus of the smooth complex curve representing the given homology class). Thus, among the smooth surfaces representing* dh*, the complex submanifolds have minimal genus.* □

Using Theorem 2.1.5 we can determine the value of the function G on every element in $H_2(\mathbb{CP}^2;\mathbb{Z})$: if $d \in \mathbb{Z}$ is nonzero, then $G(dh) = \frac{1}{2}(|d|-1)(|d|-2)$. (Obviously $G(0) = 0$.) Theorem 2.1.5 has been generalized to all simply connected complex surfaces:

Theorem 2.1.6. (The generalized Thom Theorem, [**OSz**]) *Suppose that* $C \subset S$ *is a connected, smooth, complex curve in a closed, simply connected complex surface* S*. If a connected, smooth (real 2-dimensional) surface* $C' \subset S$ *represents the same homology class as* C*, then* $g(C') \geq g(C)$*, i.e., the minimal genus is attained by the complex representative.* □

Remarks 2.1.7. (a) Theorem 2.1.6 can be derived as an application of the generalized adjunction formula (Theorem 2.4.8) and the nonvanishing result Theorem 2.4.7. Since both 2.4.7 and 2.4.8 hold for symplectic 4-manifolds (regardless of fundamental groups) as well, Theorem 2.1.6 can be extended verbatim to 2-dimensional symplectic submanifolds of symplectic 4-manifolds. (For more about symplectic manifolds see Chapter 10.) The generalization also shows, in particular, that the requirement in Theorem 2.1.6 that S be simply connected can be weakend to requiring that the first Betti number $b_1(S)$ be even. This is because for complex surfaces the evenness of $b_1(S)$ is equivalent to the Kähler condition (cf. Theorem 10.1.4), and this latter implies that the manifold is symplectic.

(b) Theorems 2.1.5 and 2.1.6 are false for topological embeddings. Lee and Wilczynski [**LW**] showed that $dh \in H_2(\mathbb{CP}^2;\mathbb{Z})$ can be represented by a topological embedding of a surface F of genus $\frac{1}{4}d^2 - 1$ (rounded down) that has a normal bundle (i.e., the embedding extends to an embedding of a complex line bundle over F). Furthermore, $\pi_1(\mathbb{CP}^2 - F)$ is abelian, and F minimizes the genus of such representatives of dh. They also gave a topological minimal genus theorem for arbitrary homology classes in simply connected topological 4-manifolds.

Using Theorem 2.1.6 and other arguments involving the existence of orientation-reversing diffeomorphisms, one can determine G for $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (see [**Rb2**]):

Theorem 2.1.8. (a) *Assume that* $\alpha = [S^2 \times \{\text{pt.}\}]$ *and* $\beta = [\{\text{pt.}\} \times S^2]$ *are the obvious basis elements of* $H_2(S^2 \times S^2; \mathbb{Z})$. *If* $ab \neq 0$, *then* $G(a\alpha + b\beta) = (|a| - 1)(|b| - 1)$. *Obviously* $G(a\alpha) = G(b\beta) = 0$.

(b) *For the homology class* $a_1 h_1 + a_2 h_2 \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z})$ *with* $|a_1| > |a_2|$ *we have* $G(a_1 h_1 + a_2 h_2) = \frac{1}{2}(|a_1| - 1)(|a_1| - 2) - \frac{1}{2}|a_2|(|a_2| - 1)$. *If* $|a_2| > |a_1|$, *reverse the roles of* a_1 *and* a_2. *For* $|a_1| = |a_2|$ *we have* $G(a_1 h_1 + a_2 h_2) = 0$. □

Exercise 2.1.9. Prove that $G(a\alpha) = G(b\beta) = 0$ in Theorem 2.1.8(a). For $ab \neq 0$ find a smooth representative for $a\alpha + b\beta$ in $S^2 \times S^2$ with genus $(|a| - 1)(|b| - 1)$. (*Hint*: Take $|a|$ disjoint copies of $S^2 \times \{\text{pt.}\}$, $|b|$ disjoint copies of $\{\text{pt.}\} \times S^2$ and resolve the singularities.) For Theorem 2.1.8(b) see Exercise 2.3.6(e).

The examples above are very special 4-manifolds ($\mathbb{CP}^2$, $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$). This is the reason we have so much information about the genus function G in these cases. For more results of this kind, see [**LL**] and [**Lw2**]. Later on we will show other examples of 4-manifolds for which G is at least partially known; to do so, however, we have to discuss certain techniques presented in the following sections.

2.2. The blow-up process

In the previous section we considered a method for constructing smooth submanifolds from ones having (mild) singularities; we did not change the ambient manifold X or the homology class of the surface. In the following, we will discuss another method for resolving singular points — now, however, we will change both X and the homology class of the surface.

Take $\tau = \{(l, p) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid p \in l\} = \{([u : v], (x, y)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid xv = yu\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$. The space τ can be projected to the first or to the second factor of $\mathbb{CP}^1 \times \mathbb{C}^2$. It is easy to prove that the projection $\pi_1 \colon \tau \to \mathbb{CP}^1$ gives a complex line bundle structure to τ. (Trivialize τ over $\mathbb{CP}^1 - [0 : 1]$ and over $\mathbb{CP}^1 - [1 : 0]$.) This fibration $\pi_1 \colon \tau \to \mathbb{CP}^1$ is called the *tautological bundle* over $\mathbb{CP}^1$ (and can be generalized with a similar formula to any $\mathbb{CP}^n$ or $\mathbb{RP}^n$).

Claim 2.2.1. *The Chern number* $\langle c_1(\tau), [\mathbb{CP}^1] \rangle$ *of the complex line bundle* $\pi_1 \colon \tau \to \mathbb{CP}^1$ *is equal to* -1.

Proof. If we delete the image of the zero section σ_0 from τ, we get $\tau - \sigma_0 \cong \mathbb{C}^2 - \{0\}$. Now take the dual bundle $\tau^* = \text{Hom}(\tau, \mathbb{C})$; a section $\alpha \in \Gamma(\tau^*)$ of this is by definition a (fiberwise linear) map $\alpha \colon \tau \to \mathbb{C}$. Restricting α

to $\tau - \sigma_0$, we get a continuous map $\mathbb{C}^2 \to \mathbb{C}$, and conversely, a smooth map $\mathbb{C}^2 \to \mathbb{C}$ that is linear along lines in $\mathbb{C}^2$ extends to a section of τ^*. Now the map $(z_1, z_2) \mapsto z_1$ (for example) gives a section with a single zero. (There is a single line — $\{z_1 = 0\}$ — on which the above map is the zero homomorphism.) The intersection number of the zero section with the image of this section is $+1$ (cf. Remark 1.2.6(b)); consequently $c_1(\tau^*) = 1$, hence $c_1(\tau) = -c_1(\tau^*) = -1$. □

Remark 2.2.2. There is an alternative way to prove Claim 2.2.1, by adopting the definition of the first Chern class given by Theorem 1.4.3. Note that $\mathbb{CP}^1$ admits a CW-decomposition with one 0-, one 1- and two 2-cells; we can think of the 1-skeleton as the equator while the two 2-cells are the southern and the northern hemispheres S and N of $\mathbb{CP}^1$. To determine $c_1(\tau)$ (which can be given by a cochain c, cf. Claim 1.4.2), we compare the trivializations of τ over the southern and northern hemispheres.

Exercise 2.2.3. Prove that if u (resp. v) denotes the complex coordinate on the southern (northern) hemisphere of $\mathbb{CP}^1$, and y (resp. x) is the fiber coordinate in a trivialization, then over the equator we have $v = \frac{1}{u}$ and $x = yu$. (For an answer, see the text before Example 7.2.3.)

The generator of $H_2(\mathbb{CP}^1; \mathbb{Z})$ can be represented by the cycle $N - S$. By the above exercise the trivializations give the transition function $v \mapsto \frac{1}{v}$ on the common boundary of the 2-cells, which is a map of degree -1. Consequently $[c](N - S) = -1$, which also proves the claim.

Exercise 2.2.4. * Find a smooth section of τ intersecting the 0-section transversely in a single point with negative sign. (Note that this verifies our sign convention, and in fact gives a third proof that $\langle c_1(\tau), [\mathbb{CP}^1] \rangle = -1$.)

Proposition 2.2.5. *For a fixed point $P \in \overline{\mathbb{CP}^2}$ the manifold $\overline{\mathbb{CP}^2} - \{P\}$ is diffeomorphic to τ.*

Proof. The 4-manifold $\overline{\mathbb{CP}^2} - \{P\}$ admits a bundle structure over $\mathbb{CP}^1$: Once we choose a line L_0 ($\approx \mathbb{CP}^1$) in $\overline{\mathbb{CP}^2}$ missing P, the map $\pi(Q) = L_{PQ} \cap L_0$, where L_{PQ} is the line passing through P and Q, gives a complex line bundle structure $\pi \colon \overline{\mathbb{CP}^2} - \{P\} \to L_0 \approx \mathbb{CP}^1$. By taking a generic section (e.g., another line not going through P) we see that $\langle c_1(\overline{\mathbb{CP}^2} - \{P\}), [\mathbb{CP}^1] \rangle = -1$. Thus, τ and $\overline{\mathbb{CP}^2} - \{P\}$ are isomorphic as complex line bundles, implying, in particular, that they are diffeomorphic. □

This diffeomorphism preserves the orientations of the 4-manifolds, and we can choose it to preserve the orientations of the fibers. It follows that if Z denotes the image of the zero section in τ, then the restriction of the above diffeomorphism to $Z \to L_0$ *reverses* the given orientations, since Z

intersects (holomorphic) fibers positively, while L_0 in $\overline{\mathbb{CP}^2} - \{P\}$ intersects fibers negatively. The projection $\pi_2 \colon \tau \to \mathbb{C}^2$ to the second factor is no longer a bundle map, but it has a very interesting property: for a point $p \in \mathbb{C}^2$ the inverse image $\pi_2^{-1}(p)$ is a single point if $p \neq 0$, and $\pi_2^{-1}(0) = \mathbb{CP}^1$. This can be verified easily, since $\pi_2^{-1}(p)$ consists of all lines in $\mathbb{C}^2$ (= points of $\mathbb{CP}^1$) going through p and the origin, which is a unique line if $p \neq 0$ and the space of all lines through 0 if $p = 0$. Consequently, the map π_2 is a biholomorphism between $\tau - \pi_2^{-1}(0)$ and $\mathbb{C}^2 - \{0\}$. If L_1, L_2 are complex lines in $\mathbb{C}^2$ intersecting each other in the origin, then although their preimages $\pi_2^{-1}(L_i)$ are not manifolds — since $\pi_2^{-1}(L_i) = \pi_2^{-1}(L_i - \{0\}) \cup \pi_2^{-1}(0)$ — the closures $\widetilde{L}_i = c\ell(\pi_2^{-1}(L_i - \{0\}))$ are complex lines in τ. Note that $\widetilde{L}_1$ and $\widetilde{L}_2$ are disjoint, since in τ not only the points of the lines L_i, but also their directions, are recorded (in the $\mathbb{CP}^1$-factor). We call $\pi_2^{-1}(L_i) \subset \tau$ the *total transform* and $\widetilde{L}_i \subset \tau$ the *proper transform* of $L_i \subset \mathbb{C}^2$. Let S be a complex surface with $P \in S$ and fix a neighborhood $U \subset S$ of P which is biholomorphic to an open subset V of $\mathbb{C}^2$ (with P mapped to $0 \in \mathbb{C}^2$). Removing U and replacing it with $\pi_2^{-1}(V) \subset \tau$, we get a new complex manifold S'. (Recall that $U - \{P\}$ and $\pi_2^{-1}(V) - \pi_2^{-1}(0)$ are biholomorphic.)

Definition 2.2.6. The surface S' is called the *blow-up* of S at P.

Extending π_2 to S' in the obvious way, one obtains a map $\pi \colon S' \to S$ which has similar properties to π_2: the map π is a biholomorphism between $S' - \pi^{-1}(P)$ and $S - \{P\}$, and $\pi^{-1}(P)$ is biholomorphic to $\mathbb{CP}^1$. The subset $\pi^{-1}(P)$ is called the *exceptional curve* (or *exceptional sphere*). Informally, when we blow up S at P, we replace the point P with the space of all lines going through P, which is a copy of $\mathbb{CP}^1$. As in the case of $\tau \to \mathbb{C}^2$, if C_1, C_2 are complex curves in S intersecting each other transversally (and only) in P, then the closures $\widetilde{C}_i = c\ell(\pi^{-1}(C_i - \{P\}))$ will be disjoint in S'. More generally, by blowing up one can reduce the number of intersection points of transversally intersecting complex curves. Since S' is constructed from S by deleting the point P and gluing back $\tau = \overline{\mathbb{CP}^2} - \{\text{pt.}\}$, S' is diffeomorphic to $S \# \overline{\mathbb{CP}^2}$. This observation motivates the following definition — the extension of blowing up to the smooth category.

Definition 2.2.7. For a smooth, oriented manifold X, the connected sum $X' = X \# \overline{\mathbb{CP}^2}$ is called the *blow-up* of X. The sphere $\overline{\mathbb{CP}^1}$ in the $\overline{\mathbb{CP}^2}$ summand is called the *exceptional sphere*, and its homology class $[\overline{\mathbb{CP}^1}]$ is usually denoted by $e = [\overline{\mathbb{CP}^1}] \in H_2(X'; \mathbb{Z}) = H_2(X; \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})$; note that $Q_{X'}(e, e) = -1$. Extending the map $\pi_2 \colon \overline{\mathbb{CP}^2} - \{\text{pt.}\} \to \mathbb{C}^2 \approx \mathbb{R}^4$ in the obvious way, we obtain a map $\pi \colon X' \to X$ with the previous properties: π is a diffeomorphism between $X' - \overline{\mathbb{CP}^1}$ and $X - \{P\}$, while $\pi^{-1}(P) = \overline{\mathbb{CP}^1}$ is the exceptional sphere.

The choice of the sign of e is somewhat arbitrary in the smooth setting, since complex conjugation gives an orientation-preserving self-diffeomorphism of $\mathbb{CP}^2$ that reverses signs of second homology classes. We have chosen the sign of e to agree with that of $\pi_2^{-1}(0) = \overline{L}_0$ in the holomorphic case. Since $X' = X\#\overline{\mathbb{CP}^2}$, we have that $b_2(X') = b_2(X) + 1$ and $\chi(X') = \chi(X) + 1$ ($c_2(X') = c_2(X) + 1$). Moreover $\sigma(X') = \sigma(X) - 1$, consequently $c_1^2(X') = c_1^2(X) - 1$.

Exercise 2.2.8. For a complex surface S, prove that the first Chern class satisfies $c_1(S') = c_1(S) - PD(e)$. (*Hint*: Use the adjunction formula 1.4.17.)

Definition 2.2.9. Take a smooth surface Σ in the smooth 4-manifold X and blow up a point $P \in \Sigma$; denote the projection by $\pi\colon X' \to X$. The inverse image $\Sigma' = \pi^{-1}(\Sigma) \subset X'$ is called the *total transform* of Σ; the closure $\widetilde{\Sigma} = c\ell(\pi^{-1}(\Sigma - \{P\}))$ is the *proper transform* of Σ.

Note that since $P \in \Sigma$, the exceptional sphere is part of the total transform; consequently Σ' is not a smooth submanifold (even if Σ is smooth), since $\Sigma' = \widetilde{\Sigma} \cup \{\text{exceptional sphere}\}$. In fact, Σ' has a normal crossing, a singularity at the point $\widetilde{\Sigma} \cap \pi^{-1}(P)$ modeled on $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$. The surface $\widetilde{\Sigma}$ has the same genus as Σ, and by the above description (since $[\Sigma] = [\Sigma'] \in H_2(X;\mathbb{Z})$, cf. Lemma 7.1.4) $[\widetilde{\Sigma}] = [\Sigma] - e \in H_2(X';\mathbb{Z})$. For this last statement we can argue in a different way: when we blow up P, we replace its 4-ball neighborhood D by $\tau = \overline{\mathbb{CP}^2} - \{\text{pt.}\}$, and when we take the proper transform of Σ, we replace a small neighborhood of P in Σ with a fiber of $\tau \to \mathbb{CP}^1$, which represents the same homology class in $\overline{\mathbb{CP}^2}$ as minus the exceptional sphere. (Note that when P is not in Σ, the blow-up process does not affect Σ. Note also that throughout the above arguments we assumed that Σ is a smooth submanifold of the 4-manifold X.)

Assume now that we have two surfaces $\Sigma_1, \Sigma_2 \subset X$ intersecting each other transversally at P, and that the sign of this intersection is $+1$. A more "differential topological" description of the blow-up process and the proper transforms can be given in the following way. By blowing up P — as we already saw earlier — we just replace a 4-ball neighborhood D of P with $\tau = \overline{\mathbb{CP}^2} - \{\text{a 4-ball neighborhood } D' \text{ of a point } Q\}$. The proper transforms of Σ_1 and Σ_2 can easily be seen as follows: We can choose two lines L_1, L_2 going through Q in $\overline{\mathbb{CP}^2}$ in such a way that the pairs $(D, D \cap (\Sigma_1 \cup \Sigma_2))$ and $(D', D' \cap (L_1 \cup L_2))$ correspond via a diffeomorphism f that reverses the orientations of D, Σ_1 and Σ_2. This is possible because we are reversing an odd number of orientations to identify the positive double point $\Sigma_1 \cap \Sigma_2$ with the negative one $L_1 \cap L_2$. If we use the restriction of f to glue $X - \text{int } D$ and $\overline{\mathbb{CP}^2} - \text{int } D'$ along their boundaries, the proper transforms $\widetilde{\Sigma}_i$ are equal to $\Sigma_i \# L_i$ (with the orientations corresponding correctly). Since $L_1 - \{Q\}$ and

$L_2 - \{Q\}$ are disjoint, the intersection point $P \in \Sigma_1 \cap \Sigma_2$ has disappeared from the intersection $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$. Note that in the case of complex curves in a complex surface we did not hypothesize that the intersection is positive; the reason is that the transverse intersection of two complex curves in a complex surface is always positive (cf. Remark 1.2.6(b)).

We have more freedom if we perform the blow-up in the smooth category. Assume that Σ_1 and Σ_2 intersect each other transversally (and only) in P, but the sign of the intersection is negative now. By taking $\mathbb{CP}^2$ instead of $\overline{\mathbb{CP}^2}$ we reverse the sign of the intersection point of L_1 and L_2 in D' (where D' is a neighborhood of $Q \in \mathbb{CP}^2$), hence $(D', D' \cap (L_1 \cup L_2))$ again becomes orientation-reversing diffeomorphic to $(D, D \cap (\Sigma_1 \cup \Sigma_2))$. In $X \# \mathbb{CP}^2$ the proper transforms $\widetilde{\Sigma}_i$ will be disjoint. As before, $[\widetilde{\Sigma}_i] = [\Sigma_i] - e$, where $e \in H_2(\mathbb{CP}^2; \mathbb{Z})$ is the generator $[\overline{\mathbb{CP}^1}]$; since $e^2 = 1$, we have $[\widetilde{\Sigma}_1] \cdot [\widetilde{\Sigma}_2] = [\Sigma_1] \cdot [\Sigma_2] + 1 = 0$ as required. There is no complex analogue of this process: $S \# \mathbb{CP}^2$ has no complex structure, since if S is complex, then $1 - b_1(S \# \mathbb{CP}^2) + b_2^+(S \# \mathbb{CP}^2) = 2 - b_1(S) + b_2^+(S)$ has the wrong parity, cf. Theorem 1.4.13. The above process is not well-suited for gauge theoretic purposes either; we have described it here because it will be used in Part 2. There is, however, another way to get around the problem of having a negative intersection point $P \in \Sigma_1 \cap \Sigma_2$: take the connected sum of Σ_1 and Σ_2 with L_1 and $\overline{L}_2$ in $\overline{\mathbb{CP}^2}$. One can easily see that $(D', D' \cap (L_1 \cup \overline{L}_2))$ will be orientation-reversing diffeomorphic to $(D, D \cap (\Sigma_1 \cup \Sigma_2))$ as required, hence the operation can be carried out. In computing the homology classes and self-intersections we have $[\widetilde{\Sigma}_1] = [\Sigma_1] + [L_1] = [\Sigma_1] - e$, but $[\widetilde{\Sigma}_2] = [\Sigma_2] - [L_2] = [\Sigma_2] + e$. Note that in this case $[\widetilde{\Sigma}_1] \cdot [\widetilde{\Sigma}_2] = [\Sigma_1] \cdot [\Sigma_2] - e^2 = [\Sigma_1] \cdot [\Sigma_2] + 1$, so we have raised the intersection number (and P has disappeared from $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$). To complete our description, we mention the counterpart of this last operation: If $P \in \Sigma_1 \cap \Sigma_2$ is a transverse intersection point with positive sign (so the regular blow-up would work perfectly), we might use $\mathbb{CP}^2$ with L_1 and $\overline{L}_2$ inside to resolve P. Needless to say, this last operation is purely topological and has no algebraic geometric counterpart. These versions of the blow-up process will be used in resolving double points of immersed spheres. Until now, $P \in \Sigma$ has always been a smooth point of Σ. (Actually, for the sake of simplicity we assumed that Σ was a smooth submanifold.) The blow-up process is even more useful in treating singularities of a surface Σ; this topic will be discussed in the next section.

The following algebraic geometric result (see, e.g., [**BPV**]) gives rise to the notion of *blow-down* (the reverse process).

Proposition 2.2.10. *Suppose that the smooth complex surface S contains a rational curve C (a complex submanifold biholomorphic to $\mathbb{CP}^1$) with $[C]^2 =$*

$[C] \cdot [C] = -1$. *Then there is a smooth complex surface T such that S is biholomorphic to the blow-up T' of T and C is the exceptional curve. We say that C can be* contracted *in S.* □

A corresponding statement holds for smooth 4-manifolds:

Proposition 2.2.11. *If the 4-manifold X contains a sphere Σ_- with $[\Sigma_-]^2 = -1$, then $X = Y \# \overline{\mathbb{CP}^2}$ for some 4-manifold Y. Similarly, if X contains a sphere Σ_+ with $[\Sigma_+]^2 = +1$, then $X = Y \# \mathbb{CP}^2$. The copy of $\overline{\mathbb{CP}^2} - \{\text{pt.}\}$ ($\mathbb{CP}^2 - \{\text{pt.}\}$ resp.) can be chosen to be a tubular neighborhood of Σ_- (Σ_+ resp.).*

Proof. A tubular neighborhood of Σ_- in X is diffeomorphic to the normal bundle $\nu\Sigma_- \to \Sigma_-$, which is an oriented 2-plane bundle over Σ_-. Since $e(\nu\Sigma_-)[\Sigma_-] = [\Sigma_-]^2 = -1$, the total space of $\nu\Sigma_-$ is obviously diffeomorphic to $\overline{\mathbb{CP}^2} - \{\text{pt.}\}$, consequently $\partial(X - \nu\Sigma_-) = S^3$. By taking the union $Y = (X - \nu\Sigma_-) \cup_{S^3} D^4$, we obtain our first assertion. The case of $\Sigma_+ \subset X$ with $[\Sigma_+]^2 = 1$ follows a similar pattern; note that the conclusion now has no holomorphic counterpart. □

In algebraic geometry, a complex surface is called *minimal* if it does not contain any rational -1-curve, so it is not the blow-up of another complex surface. Since the blow-down operation reduces the second Betti number b_2 of the manifold, it can be repeated only finitely many times. Starting from S and blowing down the rational -1-curves, we get a new surface S_{min}, called a *minimal model* of S. The minimal model is not always unique, since for some S the order of blowing down is important. For example, blowing up $\mathbb{CP}^2$ twice and blowing up $\mathbb{CP}^1 \times \mathbb{CP}^1$ once can produce the same complex surface S (as we will prove later), so S has more than one minimal model.

Exercises 2.2.12. (a) Prove that $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ are minimal.

(b)* Show that if d is even, then the hypersurface S_d (of Section 1.3) is minimal. (Actually, for any $d \geq 4$, the corresponding surface S_d of Section 1.3 is minimal. The proof of this statement is more complicated if d is odd.)

Now we return to Proposition 2.1.4, and prove it.

Proof of Proposition 2.1.4. Assume that $3h \in H_2(\mathbb{CP}^2; \mathbb{Z})$ can be represented by an embedded sphere $\Sigma \subset \mathbb{CP}^2$. Then $[\Sigma]^2 = (3h)^2 = 9$, so blowing up $\Sigma \subset \mathbb{CP}^2$ eight times gives a sphere $\widetilde{\Sigma} \subset \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ (the proper transform) with $[\widetilde{\Sigma}]^2 = +1$. If e_i denotes the homology class of the exceptional sphere in the $i^{\underline{th}}$ blow-up ($i = 1, \dots, 8$), then $[\widetilde{\Sigma}] = 3h - \sum_{i=1}^{8} e_i$ in $H_2(\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}; \mathbb{Z})$. Applying Proposition 2.2.11 to $\widetilde{\Sigma}$, we write $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ in the form $Y \# \mathbb{CP}^2$, where Y is a smooth, simply connected, closed manifold, and $\{e_1 - e_2,\ e_2 - e_3, \dots, e_7 - e_8, e_6 + e_7 + e_8 - h\}$ gives a basis of

$H_2(Y;\mathbb{Z})$. To verify the latter assertion, one only needs to check that all the above elements are orthogonal to $[\widetilde{\Sigma}] = 3h - \sum_1^8 e_i$ and that the matrix of Q_Y in these elements has determinant ± 1 (cf. Corollary 1.2.13). The intersection matrix of Q_Y in this basis is exactly the $(-E_8)$-matrix, so Y is an even, simply connected (hence spin) manifold with signature $\sigma(Y) = -8$. Such a smooth manifold, however, does not exist (by Rohlin's Theorem 1.2.29), so our original assumption about the representability of $3h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ was false. This proves Proposition 2.1.4. □

A similar argument gives the following theorem; the proof is left to the reader as an exercise. (See Exercises 5.7.7(b) and 5.7.17(d) for more general versions.)

Theorem 2.2.13. ([**KeM1**]) *Assume that $\Sigma \subset X$ is an embedded sphere in the simply connected, closed, smooth 4-manifold X, and that $[\Sigma]$ in $H_2(X;\mathbb{Z})$ is characteristic, i.e., for all $\alpha \in H_2(X;\mathbb{Z})$ we have $Q_X(\alpha,\alpha) \equiv Q_X(\alpha,[\Sigma]) \pmod 2$. Then $[\Sigma]^2 \equiv \sigma(X) \pmod{16}$.* □

(*Hint*: Prove that the blow-up of a characteristic element is characteristic, and that the complement of a neighborhood of a characteristic sphere has an even intersection form, then apply Theorem 1.2.29; cf. also Exercise 5.7.7(b).)

2.3. Desingularization of curves

After quoting two theorems to demonstrate the usefulness of the blow-up process, we will perform the blow-up of a singular curve in detail; more examples will be given in Section 7.2. We begin with the two (algebraic geometric) theorems demonstrating that one can use blow-ups to turn an arbitrary (singular) complex curve into a smooth one. (For the proofs see [**BPV**] or [**La1**].)

Theorem 2.3.1. *Let S be a nonsingular complex surface and $C \subset S$ a (possibly singular) complex curve. Then there is a complex surface T and a map $\pi\colon T \to S$ such that π is a composition of several blow-ups and the proper transform $\widetilde{C} \subset T$ is a smooth complex curve.* □

Theorem 2.3.2. *Let S and C be as above. Then there is a complex surface T' (which can be chosen to be a further blow-up of T of Theorem 2.3.1) and a map $\pi\colon T' \to S$ such that π is a composition of several blow-ups and the total transform $C' \subset T'$ has only normal crossings (like the origin in $V = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1^n z_2^m = 0\}$) as singularities.* □

The following two simple examples will be worked out in detail:

$$C_1 = \{[x:y:z] \in \mathbb{CP}^2 \mid zy^2 = x^3 + zx^2\} \subset \mathbb{CP}^2,$$

$$C_2 = \{[x : y : z] \in \mathbb{CP}^2 \mid zy^2 = x^3\} \subset \mathbb{CP}^2.$$

First we will determine the topology of C_1 and C_2, and then we will describe the proper and total transforms of C_1 in the blow-up (and ask the reader to do the same for C_2).

Proposition 2.3.3. *The curves C_1 and C_2 are smooth except at the point $P = [0 : 0 : 1] \in \mathbb{CP}^2$. The first curve C_1 is homeomorphic to a sphere with two points identified, while C_2 is homeomorphic to S^2 and has one singular point.*

Proof. The first statement is an obvious consequence of the Implicit Function Theorem. Parametrize C_1 in the following way: Take the space of projective lines in $\mathbb{CP}^2$ passing through $P = [0 : 0 : 1]$; this space can be parametrized by $[t_0 : t_1] \in \mathbb{CP}^1$ as $L_{[t_0:t_1]} = \{[x : y : z] \in \mathbb{CP}^2 \mid t_0 x = t_1 y\}$. A line $L_{[t_0:t_1]}$ intersects C_1 in exactly one more point besides P — except for the lines $L_{[1:1]}$ and $L_{[1:-1]}$, which intersect C_1 only in P. Hence, the map $\mathbb{CP}^1 \mapsto C_1$ sending $[t_0 : t_1]$ to the other intersection point of $L_{[t_0:t_1]}$ with C_1 (and $[1 : \pm 1]$ to P) is a 1-1 map except that both $[1 : \pm 1]$ map to P, so C_1 is homeomorphic to $\mathbb{CP}^1$ with $[1 : 1]$ and $[1 : -1]$ identified. We use the same idea for parametrizing C_2. Starting with the same set of lines going through P, one can easily prove that $C_2 \cap L_{[t_0:t_1]} = \{P, Q_{[t_0,t_1]}\}$, except in the case of $[0 : 1]$ when we have $C_2 \cap L_{[0:1]} = \{P\}$. Hence, the map $[t_0 : t_1] \mapsto Q_{[t_0:t_1]}$ (with $Q_{[0:1]} = P$) gives the desired homeomorphism between $\mathbb{CP}^1$ and C_2. □

Remark 2.3.4. The above point $P \in C_2$ is a singular point, whose neighborhood can be modeled (up to homeomorphism) on a cone over the right-handed trefoil knot in S^3. (See Figure 2.2.)

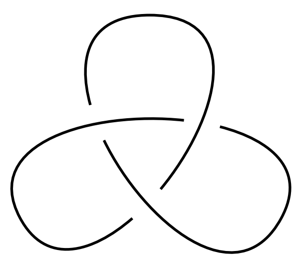


Figure 2.2. Right-handed trefoil knot.

Next we will blow up $\mathbb{CP}^2$ at $P \in C_1$ and analyze its total and proper transforms. We will work on the chart $U_z = \{[x : y : 1] \in \mathbb{CP}^2 \mid x, y \in \mathbb{C}\}$ containing P; in this chart C_1 is described by the equation $y^2 = x^3 + x^2$. When performing the blow-up, we replace the chart $U_z \approx \mathbb{C}^2$ with $\tau = \{([u : v], (x, y)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid xv = yu\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$. Consequently, the total transform of C_1 in $\tau \subset (\mathbb{CP}^2)' = \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is given by the two equations $y^2 = x^3 + x^2$ and $xv = yu$. Now $\mathbb{CP}^1 \times \mathbb{C}^2$ can be covered with two charts: $A_1 = \{(u, 1, x, y)\} \approx \mathbb{C}^3$ and $A_2 = \{(1, v, x, y)\} \approx \mathbb{C}^3$; for $u \neq 0$ the point $(u, 1, x, y) \in A_1$ corresponds to $(1, \frac{1}{u}, x, y) \in A_2$. The subset $\tau \subset \mathbb{CP}^1 \times \mathbb{C}^2$ is given by $x = yu$ in A_1 and by $xv = y$ in A_2, so in A_1 the total transform of C_1 has the form $\{y^2 = x^3 + x^2,\ x = yu\}$, which is equivalent to $\{y^2(u^3y + u^2 - 1) = 0,\ x = yu\}$. This curve is the union of two curves:

$$\{y^2 = 0, \quad x = yu\} \quad \text{and} \quad \{u^3y + u^2 - 1 = 0, \quad x = yu\}.$$

The first curve in this union consists of the set $\{(u, 0, 0) \mid u \in \mathbb{C}\}$, which is the intersection of the exceptional curve with A_1. (Note that since the exponent of y is 2, the curve has multiplicity two. We will discuss multiplicities more thoroughly in Section 7.1.) The curve $\{u^3y + u^2 - 1 = 0, \quad x = yu\}$ is the intersection of the proper transform $\widetilde{C}_1$ of C_1 with A_1; the Implicit Function Theorem shows that it is a smooth curve. Similarly, on A_2 the equations describing the total transform are $\{y^2 = x^3 + x^2,\ xv = y\}$, and these are equivalent to $\{x^2(v^2 - x - 1) = 0,\ xv = y\}$. Again, this is the union of

$$\{x^2 = 0, \quad xv = y\} \quad \text{and} \quad \{v^2 - x - 1 = 0, \quad xv = y\};$$

the first curve $\{(v, 0, 0)\}$ is the intersection of the exceptional curve with A_2, while the other curve $\{v^2 - x - 1 = 0, \quad xv = y\}$ is the intersection of the proper transform $\widetilde{C}_1$ with A_2. Of course, on $A_1 \cap A_2$ the two equations define the same curve. (Replace v with $\frac{1}{u}$.) According to the Inverse Function Theorem, $\widetilde{C}_1$ is smooth and intersects the exceptional curve transversally in the two points $\{([1 : \pm 1], 0, 0)\} = \{([\pm 1 : 1], 0, 0)\} \in \tau$. Consequently, a single blow-up has given the configuration guaranteed by Theorem 2.3.1.

Note that P is not a smooth point of the curve C_1, consequently the homology class of the proper transform in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is no longer the difference of the original curve and the exceptional curve. Instead, we have $[\widetilde{C_1}] = [C_1] - 2e$, as can be seen in the following way: Taking the total transform does not change the homology class (cf. Lemma 7.1.4). In our case, we saw that the total transform contained the exceptional curve with multiplicity 2, so when we took the proper transform (threw the exceptional curve away), we were left with $[C_1] - 2e$. (In Chapter 7 the delicate notion of the multiplicity of a complex curve will be discussed in detail.) There is an alternative interpretation of the term $-2e$ appearing in the above formula: Two branches of the curve C_1 meet in the double point P, and each branch will be connected to a copy of a projective line in $\overline{\mathbb{CP}^2}$. Since P is a *positive* double point, the signs of the homology classes of the two projective lines are the same for each branch. Note that if we blow up a *negative* double point, the homology class of the proper transform will be the same as the homology class of the original (immersed) surface, since we add $[L_1]$ and $[\overline{L}_2]$ to $[\Sigma]$, and these two terms cancel each other. In particular, $[\widetilde{\Sigma}]^2 = [\Sigma]^2$ in this case. Recall that in discussing the smooth interpretation of the blow-up process we introduced an additional operation (without an algebraic geometric analogue), namely taking the connected sum with $\mathbb{CP}^2$ and using L_1, L_2 or $L_1, \overline{L}_2$ to resolve an intersection point (the choice depending on the sign of the intersection point). Using this latter method in the case of a negative self-intersection of an immersed surface Σ, we will change the homology class to $[\widetilde{\Sigma}] = [\Sigma] - 2e$ with $[\widetilde{\Sigma}]^2 = [\Sigma]^2 + 4$; for a positive double point the homology class (and so the self-intersection) of Σ will be unchanged under this process. Taking the connected sum with $\mathbb{CP}^2$, however, not only destroys any complex structure on X but also annihilates the Seiberg-Witten invariants of it (cf. Section 2.4). For this reason, we usually prefer the ordinary blow-up as opposed to the connected sum with $\mathbb{CP}^2$. Summing up, we have:

Proposition 2.3.5. *Blowing up a positive transverse double point of Σ using $\overline{\mathbb{CP}^2}$ reduces $[\Sigma]^2$ by 4; blowing up a negative double point using $\mathbb{CP}^2$ increases $[\Sigma]^2$ by 4. For the other two choices of sign, the homology class $[\Sigma]$ and its self-intersection are preserved. Only the first of these four operations has a holomorphic interpretation.* □

For dimension reasons, a generic C^∞-map $\alpha\colon \Sigma \to X$ of a surface Σ into a 4-manifold X is an immersion [**GP**] with only transverse double points as singularities. Hence, by blowing up the double points of a generic map, we can turn it into an embedding, and the homology class of the resulting submanifold $\widetilde{\Sigma}$ can easily be computed using the above principles.

Exercises 2.3.6. **(a)** Let X be a simply connected 4-manifold. Prove that for n large enough, the second homology of the n-fold blow-up $X\# n\overline{\mathbb{CP}^2}$ admits a basis such that all basis elements can be represented by embedded spheres (cf. Remark 1.2.4).

(b) Go through the above computation for the curve $C_2 = \{zy^2 = x^3\}$. Prove that the proper transform $\widetilde{C}_2$ is a smooth curve in $(\mathbb{CP}^2)' = \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ and it is tangent to the exceptional curve. Thus the smoothing of the singular curve C_2 (guaranteed by Theorem 2.3.1) can be achieved by a single blow-up, while the configuration guaranteed by Theorem 2.3.2 needs more blow-ups. (For more complicated examples, see Section 7.2.)

(c)* Prove that $3h \in H_2(\mathbb{CP}^2;\mathbb{Z}) \subset H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2};\mathbb{Z})$ cannot be represented by an embedded sphere in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. (*Hint*: Imitate the proof of Proposition 2.1.4 and conclude with Theorem 1.2.30.)

(d)* Prove that if $3h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is represented by an immersed sphere $f\colon S^2 \to \mathbb{CP}^2$, then this immersion has positive double point. (*Hint*: After blowing up all the double points, follow the proof of Proposition 2.1.4 and conclude with Theorem 1.2.30.)

(e) Realize the minimum genus given by Theorem 2.1.8(b) for each homology class in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. (*Hint*: How many smooth curves can be separated by a single blow-up?)

Remark 2.3.7. Note that by blowing up C_1 at $P = [0:0:1]$ we can represent the proper transform $\widetilde{C}_1$ by an embedded sphere — we have just separated the images of $[1:1]$ and $[1:-1]$ under the map $\mathbb{CP}^1 \to C_1$ of Proposition 2.3.3. This means that $[\widetilde{C}_1] = [C_1] - 2e = 3h - 2e \in H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2};\mathbb{Z})$ can be represented by a sphere although $3h$ cannot (cf. Exercise 2.3.6(c)). Hence $G(3h) = 1$, $G(2e) = 0$ and $G(3h-2e) = 0$, showing that the genus function G is far from being linear.

2.4. Appendix: Introduction to gauge theory

We close this chapter by introducing the *Seiberg-Witten invariants* of a smooth, closed, oriented, simply connected 4-manifold X. The definition involves notions such as spin^c structures, Dirac operators and various bundle constructions — these are outlined at the end of this section. (For a complete treatment see [**Sa**], [**KKM**], [**Mr1**] or [**Mr2**], for example.)

Let X be a smooth, closed, oriented, simply connected 4-manifold with $b_2^+(X) > 1$ and odd. Let $\mathcal{C}_X = \{K \in H^2(X;\mathbb{Z}) \mid K \equiv w_2(X) \pmod 2\}$ be the set of characteristic elements; recall that for $K \in \mathcal{C}_X$ and $\alpha \in H_2(X;\mathbb{Z})$ this means $\langle K, \alpha\rangle \equiv \alpha^2 = Q_X(\alpha,\alpha) \pmod 2$. For a given metric g on X, $K \in \mathcal{C}_X$ and perturbation $\delta \in \Omega^+(X)$ the moduli space $\mathcal{M}_K^\delta(g)$ of solutions of the (perturbed) monopole equation can be defined. For a generic metric and perturbation this moduli space is a closed, orientable manifold of dimension $\dim \mathcal{M}_K^\delta(g) = \frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X)))$, cf. Theorem 2.4.24. The space $\mathcal{M}_K^\delta(g)$ is a subset of an infinite dimensional space $\mathcal{B}_K^*$ which is homotopy equivalent to $\mathbb{CP}^\infty$. (We postpone the definition of the monopole equations, $\mathcal{M}_K^\delta(g)$ and $\mathcal{B}_K^*$ until the end of this section.) The above homotopy equivalence means $H^*(\mathcal{B}_K^*;\mathbb{Z})$ is a polynomial ring $\cong \mathbb{Z}[\mu]$, where $\mu \in H^2(\mathcal{B}_K^*;\mathbb{Z})$; note that since $K \in \mathcal{C}_X$ and $b_2^+(X)$ is odd, $\dim \mathcal{M}_K^\delta(g) = 2m$ is even. Since $\mathcal{M}_K^\delta(g)$ is a closed and oriented manifold (cf. Remark 2.4.4(c)), it defines a homology class $[\mathcal{M}_K^\delta(g)] \in H_{2m}(\mathcal{B}_K^*;\mathbb{Z})$.

Exercise 2.4.1. * Verify that if X is simply connected, $K \in \mathcal{C}_X$ and $b_2^+(X)$ is odd, then $\dim \mathcal{M}_K^\delta(g)$ is even. For even $b_2^+(X)$ show that $\dim \mathcal{M}_K^\delta(g)$ is odd.

Definition 2.4.2. The *Seiberg-Witten invariant* $SW_X\colon \mathcal{C}_X \to \mathbb{Z}$ of the simply connected, smooth 4-manifold X (with $b_2^+(X) > 1$ and odd) on $K \in \mathcal{C}_X$

is defined as $SW_X(K) = \langle \mu^m, [\mathcal{M}_K^\delta(g)]\rangle$, where $\dim \mathcal{M}_K^\delta(g) = 2m$. If $\dim \mathcal{M}_K^\delta(g) < 0$ then $SW_X(K) = 0$ by definition.

Theorem 2.4.3. *The Seiberg-Witten function $SW_X\colon \mathcal{C}_X \to \mathbb{Z}$ is a diffeomorphism invariant of the smooth 4-manifold X, i.e., SW_X does not depend on the chosen metric g or perturbation δ. For an orientation-preserving diffeomorphism $f\colon X \to X'$ we have $SW_{X'}(K) = \pm SW_X(f^*K)$.* □

Remarks 2.4.4. **(a)** One could use the same definition if $b_2^+(X) > 1$ was even, but since in that case $[\mathcal{M}_K^\delta(g)] \in H_{2m+1}(\mathcal{B}_K^*; \mathbb{Z}) \cong 0$, the Seiberg-Witten function would contain no information about X. (For the case $b_2^+(X) = 0$, see the end of this section.)

(b) The case $b_2^+(X) = 1$ is not covered by our definition; in this case the value of SW_X *depends* on the chosen metric g and perturbation δ. This dependence is well-understood; we skip this case only for sake of brevity, cf. Exercise 2.4.26(c).

(c) The ambiguity of the sign in Theorem 2.4.3 comes from the fact that the orientation of the moduli space $\mathcal{M}_K^\delta(g)$ depends on an additional choice (an orientation for $H^0(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$, that is, a *homology orientation* for X), and depending on whether f^* preserves or reverses this latter orientation, we have a plus or a minus sign in the formula of Theorem 2.4.3.

(d) As a consequence of a certain symmetry in the equations defining SW_X, we have that $SW_X(-K) = (-1)^\varepsilon SW_X(K)$, where $\varepsilon = \frac{1}{2}(1 + b_2^+(X))$ in the simply connected case.

(e) For the sake of simplicity we assumed that $\pi_1(X) = 1$. This assumption can be easily relaxed — in the general case we meet only a few technical rather than essential difficulties.

Definition 2.4.5. The cohomology class $K \in \mathcal{C}_X \subset H^2(X;\mathbb{Z})$ is a *Seiberg-Witten basic class* of X if $SW_X(K) \neq 0$. The set of basic classes of X will be denoted by $\mathcal{B}as_X \subset H^2(X;\mathbb{Z})$. Note that, as a consequence of Remark 2.4.4(d), $K \in \mathcal{B}as_X$ iff $-K \in \mathcal{B}as_X$. The simply connected 4-manifold X is of *simple type* if each basic class K satisfies the equation $K^2 = c_1^2(X) = 3\sigma(X) + 2\chi(X)$, hence the moduli spaces $\mathcal{M}_K^\delta(g)$ giving nonzero invariants have dimension zero. (There is no known example of a simply connected 4-manifold with $b_2^+ > 1$ which is *not* of simple type; the existence of a symplectic structure, for example, implies that the manifold at hand has simple type.)

Before turning our attention to outlining the gauge-theoretic background needed for defining SW_X, we give the most important theorems concerning Seiberg-Witten invariants and Seiberg-Witten basic classes. No proofs of these statements will be given in the present volume; we will use these results to analyze the smooth topology of the 4-manifolds discussed in later chapters. (Regarding the proofs, see the short notes towards the end of this section.) First we quote vanishing and nonvanishing results concerning SW_X.

Theorem 2.4.6. (Vanishing theorems) *Suppose that X is a smooth, closed, oriented, simply connected 4-manifold with $b_2^+(X) > 1$ and odd.*

1. *If $X = X_1 \# X_2$ and $b_2^+(X_i) > 0$ $(i = 1, 2)$, then $SW_X \equiv 0$.*
2. *If X admits a metric with positive scalar curvature, then $SW_X \equiv 0$.*
3. *If $\Sigma \subset X$ is an embedded sphere with $[\Sigma]^2 \geq 0$ and $0 \neq [\Sigma]$ in $H_2(X; \mathbb{Z})$, then $SW_X \equiv 0$.* □

Theorem 2.4.7. (Nonvanishing theorems)

1. *If S is a simply connected complex surface (hence $b_2^+(S)$ is odd) and $b_2^+(S) > 1$, then $SW_S(\pm c_1(S)) \neq 0$.*
2. (Taubes [**T2**]) *More generally, if (X, ω) is a simply connected symplectic manifold and $b_2^+(X) > 1$, then $SW_X(\pm c_1(X, \omega)) = \pm 1$.* □

Recall that a 2-form ω is a *symplectic form* on X if it is nondegenerate ($\omega \wedge \omega > 0$) and $d\omega = 0$. Every symplectic manifold admits an almost-complex structure, hence simply connected symplectic manifolds have odd b_2^+. The space of almost-complex structures tamed by ω is nonempty and connected, consequently $c_1(X, \omega)$ can be defined for a symplectic manifold (X, ω) as $c_1(X, J)$ for any J tamed by ω. (For the definition of tame almost-complex structures and more about symplectic manifolds see Chapter 10.) Since every simply connected complex surface is Kähler, hence symplectic, 2.4.7(2) generalizes 2.4.7(1). In light of Definition 2.4.5 one can interpret Theorem 2.4.7 as saying that the classes $\pm c_1(S)$ are basic classes of the complex surface S (and the classes $\pm c_1(X, \omega)$ are Seiberg-Witten basic classes of the symplectic 4-manifold (X, ω)). The following important relation between basic classes and the smooth topology of the 4-manifold X was first proved by Kronheimer and Mrowka [**KM1**] in the case $[\Sigma]^2 \geq 0$. The case of negative self-intersections (with the assumption that X is of simple type) was proved by Ozsváth and Szabó [**OSz**].

Theorem 2.4.8. (Generalized adjunction formula) *Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $[\Sigma]^2 \geq 0$ (and $[\Sigma] \neq 0$). Then for every Seiberg-Witten basic class $K \in \mathcal{B}as_X$ we have $2g(\Sigma) - 2 \geq [\Sigma]^2 + |K([\Sigma])|$. If X is of simple type and $g(\Sigma) > 0$, then the same inequality holds for $\Sigma \subset X$ with arbitrary square $[\Sigma]^2$.* □

The following theorem describes the connection between Seiberg-Witten basic classes and the blow-up process.

Theorem 2.4.9. (The blow-up formula) *Let X be a simply connected 4-manifold of simple type with $\mathcal{B}as_X = \{K_i \mid i = 1, \ldots, s\}$. If $X' = X \# \overline{\mathbb{CP}^2}$ is the blow-up of X and $E \in H^2(X'; \mathbb{Z})$ denotes the Poincaré dual of the homology class $e \in H_2(X'; \mathbb{Z})$ of the exceptional sphere, then the set of basic classes of X' equals $\{K_i \pm E \mid i = 1, \ldots, s\}$.* □

The blow-up formula has been proved for every 4-manifold (without the assumption that X is a manifold of simple type); the general case is similar to Theorem 2.4.9, but somewhat more complicated to formulate [**FS1**]. In particular, it has also been shown that X is of simple type iff the blown up manifold $X \# \overline{\mathbb{CP}^2}$ is. Theorem 2.4.9 can be generalized to an arbitrary connected sum $X \# N$ with $b_2^+(N) = 0$; again, we formulate this generalization only for X of simple type.

Theorem 2.4.10. *Assume that the simply connected 4-manifold X' decomposes as $X' = X \# N$, where X is of simple type. If $b_2^+(N) = 0$, whence $H^2(N; \mathbb{Z})$ has an orthogonal basis $\{E_i \in H^2(N; \mathbb{Z}) \mid i = 1, 2, \ldots, b_2(N)\}$ with $E_i^2 = -1$, then $\mathcal{B}as_{X'} = \{K_i \pm E_1 \pm \ldots \pm E_{b_2(N)} \mid K_i \in \mathcal{B}as_X\}$.* □

Remark 2.4.11. In accordance with the definition of the Seiberg-Witten invariants, we assumed that the 4-manifolds in the above theorems are simply connected. These results can be easily extended to the general case — once again, the difficulties are technical rather than essential in nature. The assumption $b_2^+(X) > 1$ is, however, more fundamental. Note, for example, that $\mathbb{CP}^2$ admits both a complex structure and a metric with positive scalar curvature — so Theorems 2.4.6(2) and 2.4.7(1) would conflict without the assumption on b_2^+.

Exercises 2.4.12. (a) Combining Theorem 2.4.8, the adjunction formula (1.4.17) and Theorem 2.4.7, prove Theorem 2.1.6 — at least for simply connected complex surfaces with $b_2^+(S) > 1$.

(b)* Using the generalized adjunction formula, prove that a manifold has only finitely many basic classes. (*Hint*: Apply Theorem 2.4.8 to a basis $\{\alpha_1, \ldots, \alpha_n\}$ of $H^2(X; \mathbb{Z})$.)

(c) Using the blow-up formula (Theorem 2.4.9) and (1) of Theorem 2.4.6, prove (3) of Theorem 2.4.6 for X of simple type. (*Hint*: If $[\Sigma]^2 = n > 0$, then blow it up $n-1$ times and split off a copy of $\mathbb{CP}^2$ from the resulting manifold; if $[\Sigma]^2 = 0$, blow up X once, represent the classes $n[\Sigma] + e$ by spheres of square -1 and then conclude the existence of infinitely many basic classes if $SW_X \neq 0$.) Now prove Theorem 2.4.6(3) without the simple type assumption using the generalized adjunction formula of Theorem 2.4.8.

By analyzing the genera of embedded surfaces and applying Theorem 2.4.8, one gets restrictions on the set of basic classes; for example, by choosing appropriate representatives of certain homology classes in $H_2(S_4; \mathbb{Z})$ and then applying Theorems 2.4.7 and 2.4.8, one can show that $0 \in H^2(S_4;\mathbb{Z})$ is the unique basic class of the $K3$-surface S_4 (cf. Section 3.1). On the other hand, through the generalized adjunction formula of Theorem 2.4.8, knowledge of the set of basic classes gives information about the genus function G introduced at the beginning of this chapter.

Exercise 2.4.13. * Show that $X = S_4\#\overline{\mathbb{CP}^2}$ and $Y = \#3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ are homeomorphic but nondiffeomorphic 4-manifolds. (As we will see in Section 9.3, the existence of such pairs leads us to a proof of the existence of exotic $\mathbb{R}^4$'s.)

The remainder of this section consists of a discussion of spin^c structures and the definition of the moduli space $\mathcal{M}_K^\delta(g)$. (This outline is intended to be short; for a more detailed discussion see [**A5**], [**KKM**], [**Mr1**], [**Mr2**] or [**Sa**]).

2.4.1. Spin^c structures. First we discuss spin^c structures on 4-manifolds. Recall that $U(2) = \{2 \times 2$ complex unitary matrices$\}$, while by definition $SU(2) = \{A \in U(2) \mid \det(A) = 1\}$. This latter Lie group is isomorphic to the group of unit quaternions, so $SU(2)$ is diffeomorphic to S^3. We also know that $Spin(4) = SU(2) \times SU(2)$, hence $SO(4)$ is isomorphic to $SU(2) \times SU(2)/\{\pm(I,I)\}$, while $U(2) = S^1 \times SU(2)/\{\pm(1,I)\}$ (cf. text following Definition 1.4.32). The *spin^c group* $Spin^c(4)$ is defined as $Spin^c(4) = \{(A,B) \in U(2) \times U(2) \mid \det(A) = \det(B)\}$.

Exercise 2.4.14. * Prove that $Spin^c(4)$ is isomorphic to the Lie group $S^1 \times SU(2) \times SU(2)/\{\pm(1,I,I)\}$. (Using similar ideas, one can define the 3-dimensional spin^c group $Spin^c(3)$ as $S^1 \times SU(2)/\{\pm(1,I)\}$; note that this group is isomorphic to $U(2)$.)

Recall that for an oriented Riemannian 4-manifold X, a spin structure means a double cover $P_{Spin(4)} \to P_{SO(4)}$ of the oriented orthonormal frame bundle $P_{SO(4)} \to X$ by a principal $Spin(4)$-bundle $P_{Spin(4)} \to X$. Since there is a natural S^1-fibration $Spin^c(4) \xrightarrow{\rho} SO(4)$, the above analogy results in the following definition.

Definition 2.4.15. A *spin^c structure* $\mathcal{L}$ for X is specified by fixing a principal $Spin^c(4)$-bundle $P_{Spin^c(4)} \to X$ and a bundle map $P_{Spin^c(4)} \to P_{SO(4)}$ which is $\rho\colon Spin^c(4) \to SO(4)$ fiberwise, i.e., $P_{Spin^c(4)} \times_\rho SO(4) = P_{SO(4)}$. Hence a spin^c structure is given by fixing a principal $Spin^c(4)$-bundle $P_{Spin^c(4)} \to X$ together with an identification $c\colon P_{Spin^c(4)} \times_\rho SO(4) \cong P_{SO(4)}$.

By associating $\det(A)$ to the pair $(A,B) \in Spin^c(4)$, we get a homomorphism $\alpha\colon Spin^c(4) \to S^1$; using α a line bundle $L = P_{Spin^c(4)} \times_\alpha \mathbb{C}$ can be associated to the spin^c structure $\mathcal{L}$. The resulting complex line bundle L is called the *determinant line bundle* of the given spin^c structure.

Proposition 2.4.16. *Suppose that $\mathcal{L}$ is a given spin^c structure with determinant line bundle L. Then the first Chern class $c_1(L) \in H^2(X;\mathbb{Z})$ of L satisfies $c_1(L) \equiv w_2(X) \pmod 2$, hence it is a characteristic element. For every characteristic element $K \in \mathcal{C}_X$ there is a spin^c structure with determinant line bundle L satisfying $c_1(L) = K$. Every oriented (possibly noncompact) 4-manifold admits a spin^c structure. If X is simply connected, the determinant line bundle determines the spin^c structure, and the set of spin^c structures $\mathcal{S}^c(X)$ is in 1-1 correspondence (via $c_1(L)$) with the set $\mathcal{C}_X = \{K \in H^2(X;\mathbb{Z}) \mid K \equiv w_2(X) \pmod 2\}$ of characteristic elements.*

Proof. Let P_{S^1} denote the principal S^1-bundle corresponding to L. By Exercise 2.4.14 there is an exact sequence

$$1 \to \mathbb{Z}_2 \to Spin^c(4) \to S^1 \times SO(4) \to 1,$$

hence $Spin^c(4)$ is a double cover of $S^1 \times SO(4)$. Consequently, there exists a spin^c structure with determinant line bundle L iff the principal bundle $P_{S^1 \times SO(4)}$ (obtained by combining the cocycle structures of P_{S^1} and $P_{SO(4)}$) admits a double cover which is $Spin^c(4) \to S^1 \times SO(4)$ fiberwise.

The group $S^1 \times SO(4)$ admits three different nontrivial double covers. (This follows from the fact that $H^1(S^1 \times SO(4);\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, cf. also Exercise 1.4.26.) It is easy to see that these double covers are $S^1 \times Spin(4)$, $S^1 \times SO(4)$ (where $S^1 \to S^1$ is a double cover) and $Spin^c(4)$. Note that since $S^1 \cong SO(2)$, the group $S^1 \times SO(4)$ can be embedded in $SO(6)$.

Exercise 2.4.17. * Prove that the double cover $\varphi\colon G \to SO(2) \times SO(4)$ extends to a double cover of $SO(6)$ iff $\operatorname{Im}\varphi_* \subset \pi_1(S^1) \oplus \pi_1(SO(4)) = \langle x\rangle \oplus \langle a\rangle$ is generated by the element (x, a). (For $G = S^1 \times Spin(4)$ the group $\operatorname{Im}\varphi_*$ equals $\langle (x,0)\rangle$; while for $G = S^1 \times SO(4)$ we have $\operatorname{Im}\varphi_* = \langle (0,a),(2x,0)\rangle$. If $G = Spin^c(4)$, then we have $\operatorname{Im}\varphi_* = \langle (x,a)\rangle$; consequently $Spin^c(4) \to SO(2) \times SO(4)$ is the unique double cover which can be extended to a double cover of $SO(6)$.)

By the above exercise there exists a spin^c structure with determinant bundle L iff $P_{S^1 \times SO(4)}$ is spinnable, hence iff $0 = w_2(P_{S^1 \times SO(4)}) = w_2(L) + w_2(X)$; since $w_2(L)$ is the mod 2 reduction of $c_1(L)$, this observation proves Proposition 2.4.16. Note that if X is simply connected, the spin structure on $P_{S^1 \times SO(4)}$ (i.e., the spin^c structure with determinant line bundle L) is unique; the same holds if $H^1(X;\mathbb{Z}_2) = 0$ (i.e., when there is no 2-torsion in $H^2(X;\mathbb{Z})$). The existence of a spin^c structure on an arbitrary 4-manifold X now follows from the fact that $\mathcal{C}_X \neq \emptyset$, which is a consequence of Proposition 5.7.4 (cf. also Remark 5.7.5). □

Note that for a fixed element $c \in \mathcal{C}_X$ the map $a \mapsto c + 2a$ ($a \in H^2(X;\mathbb{Z})$) gives a mapping from $H^2(X;\mathbb{Z})$ to $\mathcal{C}_X$. If $H^2(X;\mathbb{Z})$ has no 2-torsion (e.g., if X is simply connected), the above map is obviously a bijection. Note, however, that this map is not canonical; it depends on the choice of $c \in \mathcal{C}_X$. If $H^2(X;\mathbb{Z})$ has 2-torsion, neither of the above maps (i.e., the one $\mathcal{S}^c(X) \to \mathcal{C}_X$ associating the first Chern class of the determinant line bundle to a spin^c structure, and the above map $H^2(X;\mathbb{Z}) \to \mathcal{C}_X$) will remain isomorphisms — both maps are surjections but not injections in general. In Section 10.4 we will define a new map $H^2(X;\mathbb{Z}) \to \mathcal{S}^c(X)$ which will be an isomorphism even in the presence of 2-torsion in $H^2(X;\mathbb{Z})$. In this section, however, we restrict ourselves to the case when X is simply connected, hence we can identify the set of spin^c structures $\mathcal{S}^c(X)$ with $\mathcal{C}_X$ via $c_1(L)$ as described in Proposition 2.4.16.

By taking the two projections $\mu^\pm\colon Spin^c(4) \to U(2)$ (determined by $Spin^c(4) \subset U(2) \times U(2)$), we define the *positive (negative) spinor bundles* $W^\pm$ as the associated $U(2)$-bundles $W^\pm = P_{Spin^c(4)} \times_{\mu^\pm} \mathbb{C}^2$. The sections of W^+ (resp. W^-) are the *positive (negative) spinors*. The definition of $Spin^c(4)$ (as a subgroup of $U(2) \times U(2)$) implies that $\det W^+$ and $\det W^-$ are both isomorphic to the determinant line bundle of the spin^c structure. Next we exhibit the above representations $\mu^\pm$ using the presentation of $Spin^c(4)$ as $S^1 \times SU(2) \times SU(2)/\{\pm(1,I,I)\}$. An element $a = [\lambda, q_1, q_2] \in S^1 \times SU(2) \times SU(2)/\mathbb{Z}_2 = Spin^c(4)$ is given by a unit complex number λ and two unit quaternions q_1, q_2. If $h \in \mathbb{H}$ is a quaternion, then the representation $\rho_0([\lambda, q_1, q_2])(h) = q_1 h \overline{q}_2$ (quaternionic multiplication) results in $P_{Spin^c(4)} \times_{\rho_0} \mathbb{H} \cong TX$; the map ρ_+ defined as $\rho_+([\lambda, q_1, q_2])(h) = q_1 h \lambda$ gives $P_{Spin^c(4)} \times_{\rho_+} \mathbb{H} \cong W^+$; finally $\rho_-([\lambda, q_1, q_2])(h) = q_2 h \lambda$ yields $P_{Spin^c(4)} \times_{\rho_-} \mathbb{H} \cong W^-$. (These statements can be checked using the isomorphism described in Exercise 2.4.14.) One can easily see from this representation-theoretic description that $TX \otimes \mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$. Using this isomorphism we define the *Clifford multiplication* C as the obvious map $C\colon \Gamma(X; W^+ \otimes T^*X) \to \Gamma(X; W^-)$.

Remark 2.4.18. As in Section 1.4.2, there is an alternative way to define spin^c structures: Assume that V is a 4-dimensional Euclidean vector space. A *spin^c structure* for V is given once we fix a pair of Hermitian 2-dimensional complex vector spaces W^+ and W^-, an isomorphism of the corresponding determinant lines $\Lambda^2_{\mathbb{C}} W^+ \cong \Lambda^2_{\mathbb{C}} W^-$ and an isomorphism $\gamma\colon V \otimes \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$ satisfying $\gamma(v)^*\gamma(v) = -|v|^2 \mathrm{id}_{W^+}$. Note that the symmetry group of a spin^c structure is isomorphic to $Spin^c(4)$. Globally, if X is a Riemannian manifold, then a pair of $U(2)$-bundles $W^\pm \to X$ with identified determinant line bundles $\Lambda^2_{\mathbb{C}} W^+ \cong \Lambda^2_{\mathbb{C}} W^-$ and an isomorphism $\gamma\colon T_{\mathbb{C}} X \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$ (with $\gamma(v)^*\gamma(v) = -|v|^2 \mathrm{id}_{W^+}$) is by definition a *spin^c structure* for X; this spin^c structure is frequently denoted by the triple $(W^{\pm}, \gamma)$. As in the spin case, the proof of the equivalence of the two definitions is an easy exercise.

Let $L \to X$ denote the determinant line bundle of the spin^c structure $\mathcal{L}$ corresponding to $K \in \mathcal{C}_X$. (Recall that we have assumed $\pi_1(X) = 1$, hence the map $\mathcal{S}^c(X) \xrightarrow{c_1} \mathcal{C}_X$ is an isomorphism.) Furthermore, let $\mathcal{A}_L$ denote the space of $U(1)$-connections on L. By choosing $A \in \mathcal{A}_L$ and coupling it with the Levi-Civita connection on X we get a covariant differentiation $\nabla_A \colon \Gamma(X; W^+) \to \Gamma(X; W^+ \otimes T^*X)$.

Remark 2.4.19. Recall that a connection on $P_{Spin^c(4)} \to X$ is a $Spin^c(4)$-equivariant Lie algebra-valued 1-form giving a fixed isomorphism along the tangent of each fiber. Since $Lie(Spin^c(4)) = Lie(SO(4)) \oplus Lie(S^1)$, the pull-back of the Levi-Civita connection defined on $P_{SO(4)}$ will not provide a connection on $P_{Spin^c(4)}$. On the other hand, the pull-back of a connection on $P_{SO(4)\times S^1}$ does define a connection on $P_{Spin^c(4)}$. By fixing $A \in \mathcal{A}_L$ (and the Levi-Civita connection on $P_{SO(4)}$) we specify a connection on $P_{SO(4)\times S^1}$, hence — by pulling it back — on $P_{Spin^c(4)}$; the associated covariant differentiation on W^+ is denoted by $\nabla_A \colon \Gamma(X; W^+) \to \Gamma(X; W^+ \otimes T^*X)$.

The composition of the Clifford multiplication C and ∇_A gives an operator

$$\partial\!\!\!/_A = C \circ \nabla_A \colon \Gamma(X; W^+) \to \Gamma(X; W^-),$$

which is called the *coupled (*or *twisted) Dirac operator* of the spin^c structure K coupled to the connection $A \in \mathcal{A}_L$.

2.4.2. The Seiberg-Witten moduli space. We close this section with a short description of the Seiberg-Witten equations and Seiberg-Witten moduli spaces, followed by an indication of the proofs of the most important results concerning Seiberg-Witten invariants.

Recall that on a 4-dimensional, oriented, Riemannian manifold X, the *Hodge $*_g$-operator* $*_g \colon \Omega^2(X) \to \Omega^2(X)$ (given by the metric g) is defined on a basis $\{e_i \wedge e_j\}$ as $*_g(e_i \wedge e_j) = e_k \wedge e_l$ whenever (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$. (By convention, $\Lambda^p(X)$ is the bundle of exterior p-forms, $\Omega^p(M) = \Gamma(X; \Lambda^p(X))$, and $\{e_1, \dots, e_4\}$ is a positively oriented local orthonormal frame of T^*X.) If the metric g is obvious from the context, we will drop it from the notation and denote the Hodge star-operator by $*$.

Definition 2.4.20. The vector space $\Omega^+(X)$ of *self-dual* 2-forms is defined as $\Omega^+(X) = \{\omega \in \Omega^2(X) \mid *\omega = \omega\}$. Similarly, $\Omega^-(X) = \{\omega \in \Omega^2(X) \mid *\omega = -\omega\}$ is the vector space of *anti-self-dual* (ASD) 2-forms. For $\omega \in \Omega^2(X)$ set $\omega^+ = \frac{1}{2}(\omega + *\omega)$ and $\omega^- = \frac{1}{2}(\omega - *\omega)$; note that $\omega = \omega^+ + \omega^-$, and since $*^2 = \mathrm{id}_{\Omega^2(X)}$, we have $\omega^{\pm} \in \Omega^{\pm}(X)$. If it is necessary, we will use the notation $\Omega^{\pm}_g(X)$ to indicate that we are considering 2-forms self-dual (or anti-self-dual) with respect to the metric g.

Exercise 2.4.21. * Prove that the dimension of the space $H^+(X;\mathbb{R})$ of self-dual closed 2-forms (considered as a subspace of $H^2(X;\mathbb{R})$) is equal to $b_2^+(X)$.

One can identify the space of 2-forms $\Lambda^2(\mathbb{R}^4)$ on $\mathbb{R}^4$ with the Lie algebra $Lie(SO(4)) = Lie(SU(2)\times SU(2))$ (or more generally $\Lambda^2(\mathbb{R}^n)$ with $Lie(SO(n))$) by the following definition: If $A \in Lie(SO(n))$ — i.e., A is a matrix satisfying $A + A^T = 0$ — then for $\alpha, \beta \in \mathbb{R}^n$ the formula $w_A(\alpha,\beta) = \langle A\alpha, \beta\rangle$ gives a 2-form w_A. Realizing that $\dim Lie(SO(n)) = \dim \Lambda^2(\mathbb{R}^n)$, one can show that the above map $w\colon Lie(SO(n)) \to \Lambda^2(\mathbb{R}^n)$ is an isomorphism. Since $Lie(SO(3)) = Lie(SU(2)) \cong \text{Im}\ \mathbb{H}$, we have a splitting of $Lie(SO(4))$ as $\text{Im}\ \mathbb{H} \oplus \text{Im}\ \mathbb{H}$. (As we mentioned earlier, the splitting of the Lie algebra $Lie(SO(n))$ is unique to dimension $n = 4$.) On the other hand, by Definition 2.4.20, $\Lambda^2(\mathbb{R}^4)$ splits as $\Lambda^+ \oplus \Lambda^-$. It is a fundamental fact that these two splittings coincide via w, hence the bundles $\Lambda^\pm(X)$ can be given by the following representations of $Spin^c(4)$ on $\text{Im}\ \mathbb{H}$: If $h \in \text{Im}\ \mathbb{H}$, then $r_+([\lambda, q_1, q_2])(h) = q_1 h \overline{q}_1$ gives $P_{Spin^c(4)} \times_{r_+} \text{Im}\ \mathbb{H} \cong \Lambda^+(X)$, and $r_-([\lambda, q_1, q_2])(h) = q_2 h \overline{q}_2$ gives $P_{Spin^c(4)} \times_{r_-} \text{Im}\ \mathbb{H} \cong \Lambda^-(X)$.

For a $U(1)$-connection $A \in \mathcal{A}_L$, the curvature F_A is a Lie algebra-valued 2-form, and since $Lie(U(1)) \cong i\mathbb{R}$, the 2-form F_A is in $i\Omega^2(X)$. The splitting $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$ can be extended to imaginary-valued 2-forms as well; F_A^+ will denote the self-dual part of the curvature F_A.

If $Spin^c(4)$ acts on $\mathbb{H}$ via ρ_+ and on $\text{Im}\ \mathbb{H}$ via r_+, then the map $\sigma\colon \mathbb{H} \to \text{Im}\ \mathbb{H}$ defined as $h \mapsto -hi\overline{h}$ is $Spin^c(4)$-equivariant. As such, it induces a map $\sigma\colon \Gamma(W^+) \to \Omega^+(X)$ between the sections of the associated bundles, hence for a positive spinor $\psi \in \Gamma(W^+)$ the expression $i\sigma(\psi)$ defines an imaginary valued self-dual 2-form. The group of maps $\mathcal{G} = \text{Map}(X, S^1)$ (the *gauge group*) acts on $\Gamma(X; W^+)$ as $\psi \mapsto -g\cdot\psi$ (where $g \in \mathcal{G}$, $\psi \in \Gamma(X; W^+)$ and g acts by pointwise multiplication), and on $\mathcal{A}_L$ as $A \mapsto A + 2dg$ (where $A \in \mathcal{A}_L$). (In fact, $\mathcal{G}$ can be identified as the group of bundle automorphisms of $P_{Spin^c(4)} \to X$ inducing the trivial action on the frame bundle $P_{SO(4)} \to X$. The factor of two in the above action is a manifestation of the fact that the map $\alpha\colon Spin^c(4) \to S^1$ (defined after Definition 2.4.15) is the double cover map when restricted to the central $S^1 \subset Spin^c(4)$.) Consequently $\mathcal{G}$ acts on the product $\mathcal{A}_L \times \Gamma(X; W^+)$; the quotient $\mathcal{B}_K = \mathcal{A}_L \times \Gamma(X; W^+)/\mathcal{G}$ is called the *configuration space*, while $\mathcal{B}_K^* = \{[A,\psi] \in \mathcal{B}_K \mid \psi \text{ is not identically } 0\}$.

Remarks 2.4.22. (a) It is not hard to see that the pairs (A,ψ) with ψ not identically 0 form the subspace of $\mathcal{A}_L \times \Gamma(X; W^+)$ on which the gauge group $\mathcal{G} = \text{Map}(X, S^1)$ acts freely. The constant functions $X \to S^1$ obviously fix all elements of the form $(A, 0) \in \mathcal{B}_K$; by the above description of the $\mathcal{G}$-action, it is easy to see that the stabilizer $Stab_{(A,0)}$ is equal to the subgroup

$\{f \in \mathcal{G} \mid f \text{ is constant}\}$. Consequently, the *based gauge group* $\mathcal{G}_0$ given as $\{g \in \mathcal{G} \mid g(x_0) = 1 \text{ for a fixed } x_0 \in X\}$ acts freely on $\mathcal{A}_L \times \Gamma(X; W^+)$.

(b) For analytic reasons, it is more convenient to consider certain *Sobolev completions* of the space $\mathcal{A}_L \times \Gamma(X; W^+)$ and the gauge group $\mathcal{G}$. Using these completed spaces one can prove that the quotient $\mathcal{B}_K$ of the completion of $\mathcal{A}_L \times \Gamma(X; W^+)$ by the (appropriate) completion of $\mathcal{G}$ is a (singular) Banach manifold, and the usual theorems of analysis (e.g., the Implicit Function Theorem) generalize to this infinite dimensional setting. For more about Sobolev completions and the necessary analytic background see [**KKM**] or [**Sa**].

After this quick preparation, we are ready to describe the *Seiberg-Witten* (or *monopole*) equations. For $A \in \mathcal{A}_L$ and $\psi \in \Gamma(X; W^+)$, these equations are given by

$$\partial\!\!\!/_A \psi = 0 \quad \text{and} \quad F_A^+ = i\sigma(\psi).$$

The *moduli space* $\mathcal{M}_K(g)$ is defined as the set of $[A, \psi] \in \mathcal{B}_K$ satisfying these equations. As the definition suggests, the monopole equations are *gauge invariant*, that is, if $(A, \psi) \in \mathcal{A}_L \times \Gamma(X; W^+)$ satisfies the above equations and $g \in \mathcal{G}$, then $g^*(A, \psi) \in \mathcal{A}_L \times \Gamma(X; W^+)$ will satisfy the equations as well. In general, it is not clear whether $\mathcal{M}_K(g) \subset \mathcal{B}_K$ is a smooth manifold, but if we take a generic perturbation $\delta \in \Omega_g^+(X)$, the solution set $\mathcal{M}_K^\delta(g)$ of the perturbed equations

$$\partial\!\!\!/_A \psi = 0 \quad \text{and} \quad F_A^+ + i\delta = i\sigma(\psi)$$

gives rise to a smooth manifold.

Remark 2.4.23. Singularities of the moduli space $\mathcal{M}_K^\delta(g)$ can arise from two sources: The equations defining the moduli space may not cut it out transversally from $\mathcal{B}_K$ (i.e., the hypotheses of the Implicit Function Theorem may not be satisfied) or the gauge group $\mathcal{G}$ may not act freely. Consider the map $\mathcal{SW}\colon \mathcal{A}_L \times \Gamma(X; W^+) \times \Omega^+(X) \to \Gamma(X; W^-) \times \Omega^+(X)$ given by

$$(A, \psi, \delta) \mapsto (\partial\!\!\!/_A \psi, F_A^+ + i\delta - i\sigma(\psi)).$$

Direct computation shows that the linearization of $\mathcal{SW}$ at a solution is onto. Consequently, $\mathcal{SW}^{-1}(0)$ is a smooth (infinite dimensional) manifold, so a Sard-type argument implies that for almost all fixed $\delta \in \Omega^+(X)$ the subset $\{(A, \psi, \delta) \mid (A, \psi) \text{ solves the perturbed equation with the fixed } \delta\}$ is a smooth (infinite dimensional) manifold. Hence a generic choice of g and δ eliminates singularities of the first type encountered. On the other hand (as we have already seen), $\mathcal{G}$ does not act freely in general. At this point we must use our assumption that $b_2^+(X) > 0$. It turns out that in the space of perturbations $\mathcal{P}ert(X) = \{(g, \delta) \in Met(X) \times \Omega_g^+(X) \mid g \text{ is a metric on } X \text{ and } \delta \in \Omega_g^+(X)\}$ there is a codimension-$b_2^+(X)$ subset for which the corresponding moduli space will not lie in $\mathcal{B}_K^*$. If (g, δ) is from that subspace, then the Seiberg-Witten equation will admit solutions of the form $(A, 0)$ (called *reducible* solutions), cf. Exercise 2.4.26(a). In that case the gauge group $\mathcal{G}$ does not act freely on the space of solutions (cf. Remark 2.4.22(a)), hence the quotient $\mathcal{M}_K^\delta(g)$ will have singularities corresponding to the reducible solutions. On the other hand, if $b_2^+(X) > 0$ then for a generic pair (g, δ) the space of solutions $\mathcal{M}_K^\delta(g)$ is, in fact, a smooth submanifold of $\mathcal{B}_K^*$. Since the based gauge group $\mathcal{G}_0$ acts freely on $\mathcal{A}_L \times \Gamma(X; W^+)$, the space of $\mathcal{G}_0$-equivalence classes of solutions of the Seiberg-Witten equations — the so-called *based moduli space* $(\mathcal{M}_K^\delta(g))^0$ — is a smooth manifold of dimension $\dim \mathcal{M}_K^\delta(g) + 1$ for arbitrary b_2^+. (Again, we assume generic choices of g and δ.) It admits a $\mathcal{G}/\mathcal{G}_0 \cong S^1$ action with fixed points corresponding to reducible solutions. We will take advantage of the above based setting and the S^1-action later (cf. Exercise 2.4.26(d)).

The next theorem summarizes the most important properties of the moduli space $\mathcal{M}_K^\delta(g)$; these properties ensure that Definition 2.4.2 gives a diffeomorphism invariant of X.

Theorem 2.4.24. *Let X be a simply connected, oriented, closed 4-manifold with $b_2^+(X)$ odd; fix the spin^c structure corresponding to $K \in \mathcal{C}_X$. For a generic metric g and perturbation $\delta \in \Omega_g^+(X)$ the moduli space $\mathcal{M}_K^\delta(g)$ is a smooth, closed submanifold of $\mathcal{B}_K^*$ of dimension*

$$d = \frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X))).$$

Furthermore, a homology orientation of X (that is, an orientation of the vector space $H^0(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$) determines an orientation for $\mathcal{M}_K^\delta(g)$. If $b_2^+(X) > 1$, the homology class $[\mathcal{M}_K^\delta(g)] \in H_d(\mathcal{B}_K^*;\mathbb{Z})$ is independent of the choice of g and δ, hence the map $SW_X \colon \mathcal{C}_X \to \mathbb{Z}$ given in Definition 2.4.2 is a smooth invariant of X.

Proof (sketch). As Remark 2.4.23 outlines, for generic choices of g and δ the moduli space $\mathcal{M}_K^\delta \subset \mathcal{B}_K^*$ is a smooth manifold. Its dimension can be computed using the Atiyah-Singer Index Theorem, which shows that $\dim \mathcal{M}_K(g) = \frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X)))$. The proof of orientability of $\mathcal{M}_K^\delta(g)$ proceeds in the following way: The determinant line bundle of the elliptic operator $\mathcal{SW}_\delta$ (with a fixed perturbation δ) over the moduli space can be identified with the top power $\Lambda^{\max} T\mathcal{M}_K^\delta(g)$ of its tangent bundle. Since the determinant line bundle is trivial over $\mathcal{B}_K^*$, orientability follows. Note that by restricting trivializations of this bundle from the connected space $\mathcal{B}_K^*$ to $\mathcal{M}_K^\delta(g)$ we get two preferred orientations of the moduli space — even if it consists of more than one component. By analyzing the determinant line bundle it can be shown that a trivialization depends on the orientation of $H^0(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$ (and of $H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$ in the nonsimply connected case).

The above-listed properties of $\mathcal{M}_K^\delta(g)$ (i.e., smoothness and orientability) follow from general analytic properties of the operators involved in the defining equations. The compactness of the moduli space, however, is special to Seiberg-Witten theory — for example, the moduli spaces encountered in Donaldson theory fail to be compact, although they possess all other properties discussed above. The compactness of $\mathcal{M}_K^\delta(g)$ originates from the existence of a bound on solutions $\psi \in \Gamma(X; W^+)$ depending only on the geometry of X: If (A, ψ) solves the Seiberg-Witten equations, then either $\psi \equiv 0$ or

$$|\psi(x)|^2 \leq s_{(X,g)} = \max\{-s(x) \mid x \in X\},$$

where $s(x)$ denotes the scalar curvature of (X, g) at x. Since this bound provides a bound for A as well, standard analysis provides the compactness of $\mathcal{M}_K^\delta(g)$. (See also Exercise 2.4.26(b).)

As we saw in Remark 2.4.23, for $b_2^+(X) > 0$ almost every pair (g, δ) gives rise to a smooth moduli space — we can choose (g, δ) outside of a codimension-$b_2^+(X)$ subspace. On the other hand, SW_X will be a diffeomorphism invariant only in the case $b_2^+(X) > 1$. The general way to prove independence from g and δ proceeds as follows: Choose perturbations (g_i, δ_i) $(i = 0, 1)$ and try to prove that the corresponding moduli spaces are homologous in $\mathcal{B}_K^*$. The two perturbations can be joined by an arc $\gamma_t = (g_t, \delta_t)$ in the space $\mathcal{P}ert(X)$, and the parametrized moduli space $\bigcup_{t\in[0,1]} \mathcal{M}_K^{\delta_t}(g_t)$ gives the desired homology — if γ_t does not meet the codimension-$b_2^+(X)$ subspace giving reducible solutions. For $b_2^+(X) > 1$ it is not hard to find such a path; if $b_2^+(X) = 1$, however, there will in general be finitely many points $t_1, \ldots, t_n \in [0, 1]$ with the property that for (g_{t_i}, δ_{t_i}) the Seiberg-Witten equations admit reducible solutions. Hence, for $b_2^+(X) = 1$ the homology class $[\mathcal{M}_K^\delta(g)]$ does depend on (g, δ). (To produce smooth invariants for manifolds with $b_2^+(X) = 1$ and obtain results about the smooth structure of these manifolds, one must understand the relation between $[\mathcal{M}_K^{\delta_t}(g_t)]$ and $[\mathcal{M}_K^{\delta_{t'}}(g_{t'})]$ for $t < t_i < t'$. The resulting formulae are usually called *wall-crossing formulae*, cf. also Exercise 2.4.26(c).) □

Remark 2.4.25. The definition of a spin^c structure and the choice of a homology orientation of X require a metric g on the 4-manifold X. In the above argument, however, we used a 1-parameter family of moduli spaces corresponding to various metrics. It can be shown [**Mr2**] that a spin^c structure and homology orientation fixed for one metric g canonically determine such structures for all other metrics on X.

Exercises 2.4.26. **(a)** Show that the (unperturbed) Seiberg-Witten equations admit reducible solutions iff the harmonic representative of $c_1(\det \mathcal{L})$ is ASD. Conclude that if $b_2^+(X) > 0$ and $c_1(\det \mathcal{L}) \neq 0$, then for generic metric g and small perturbation $\delta \in \Omega^+(X)$ the moduli space $\mathcal{M}_K^\delta(g)$ does not contain reducibles.

(b)* Applying the Weitzenböck formula

$$\partial\!\!\!/_A \partial\!\!\!/_A \psi = \nabla_A^* \nabla_A \psi + \frac{1}{4} s\psi + \frac{1}{2} F_A \cdot \psi,$$

show that if ψ solves the Seiberg-Witten equations and ψ is not identically zero, then $|\psi|^2 \leq s_{(X,g)}$. (The curvature term F_A acts on ψ by Clifford multiplication.)

(c) Suppose that $\pi_1(X) = 1$ and $b_2^+(X) = 1$. For a generic metric g, take the 2-form ω_g of unit length generating $\mathcal{H}^+(X;\mathbb{R}) \cong \mathbb{R}$ and positive with respect to a preassigned homology orientation. (Recall from Exercise 2.4.21 that the dimension of $\mathcal{H}^+(X;\mathbb{R})$ is one for X with $b_2^+(X) = 1$.) Show that for a given spin^c structure $\mathcal{L}$ the Seiberg-Witten equations admit reducible solutions iff the number $c_g = c_1(\det \mathcal{L}) \cup [\omega_g]$ is zero. Prove that in that case the reducible solution is unique (up to gauge equivalence). Suppose that g_1, g_2 are generic metrics and $\delta_1, \delta_2 \in \Omega^+(X)$ are sufficiently small perturbations. Conclude from the above discussion that if $c_{g_1} \cdot c_{g_2} > 0$, then the Seiberg-Witten invariants of X on $\mathcal{L}$ using moduli spaces corresponding to (g_1, δ_1) and (g_2, δ_2) are equal. Furthermore, show that if $c_{g_1} \cdot c_{g_2} < 0$, then the values of the Seiberg-Witten invariants using $\mathcal{M}_K^{\delta_1}(g_1)$ and $\mathcal{M}_K^{\delta_2}(g_2)$ differ by 1. (As usual, K denotes the first Chern class of the determinant bundle of the spin^c structure $\mathcal{L}$.)

(d) Suppose that $\pi_1(X) = 1$ and $b_2^+(X) = 0$. Show that for a spin^c structure $\mathcal{L}$ the moduli space contains a unique reducible solution. Show furthermore that a neighborhood of the (singular) point corresponding to this reducible solution can be modeled on the cone over the projective space $\mathbb{CP}^{\frac{d-1}{2}}$, where d denotes the dimension of the moduli space corresponding to the spin^c structure $\mathcal{L}$. (*Hint*: The uniqueness of the reducible solution — up to gauge equivalence — follows from standard Chern-Weil theory. Considering the S^1-action given by $\mathcal{G}/\mathcal{G}_0$ on the based moduli space (cf. Remark 2.4.23) completes the solution.)

The above exercise gives a fairly detailed description of the moduli space $\mathcal{M}_K(g)$ for a manifold with $b_2^+(X) = 0$. By deleting an open neighborhood of the singular point of the moduli space we get a manifold $V \subset \mathcal{B}_K^*$ with boundary diffeomorphic to $\mathbb{CP}^{\frac{d-1}{2}}$. This shows, in particular, that $[\partial V] \in H_{d-1}(\mathcal{B}_K^*;\mathbb{Z})$ is nullhomologous. On the other hand, the generator $\mu \in H^2(\mathcal{B}_K^*;\mathbb{Z})$ can be identified as the first Chern class of the S^1-fibration provided by the base point fibration $(\mathcal{M}_K^\delta(g))^0 \to \mathcal{M}_K^\delta(g)$. Since this fibration is the tautological S^1-bundle over $\partial V \approx \mathbb{CP}^{\frac{d-1}{2}}$, we conclude that $\langle \mu^{\frac{d-1}{2}}, \partial V\rangle = \pm 1$. This shows, however, that $[\partial V]$ is nonzero in $H_{d-1}(\mathcal{B}_K^*;\mathbb{Z})$ — implying a contradiction with the above observation for all characteristic elements K with $d(K) - 1 = \frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X))) - 1 \geq 0$, i.e., $d(K) > -1$. (Recall that the parity of $d(K)$ is determined by X, cf. Exercise 2.4.1.) This contradiction shows that if X is a smooth (simply connected) 4-manifold with $b_2^+(X) = 0$, then no characteristic element K can exist for which $\frac{1}{4}(K^2 - (3\sigma + 2\chi)) > -1$. Since $b_2^+ = 0$, we have that $\frac{1}{4}(K^2 - (3\sigma + 2\chi)) = \frac{1}{4}(K^2 + b_2(X)) - 1$, which is greater than -1 iff $K^2 + b_2(X) > 0$. Hence we get a contradiction once we find a characteristic element $K \in \mathcal{C}_X$ having square less than $b_2(X)$ in absolute value.

Exercises 2.4.27. (a) Show that there exists no smooth simply connected 4-manifold X with $Q_X = 2(-E_8)$. (*Hint*: Since $2(-E_8)$ is even, we have that 0 is characteristic, so the contradiction described above implies the solution.) In fact, this result — coupled with Freedman's Theorem 1.2.27 — already implies the existence of (large) exotic $\mathbb{R}^4$'s, cf. Section 9.4.

(b) Prove that if Q is a nontrivial, negative definite, even intersection form, then for $n \geq 0$ the form $Q \oplus n\langle -1\rangle$ cannot be realized as the intersection form of a smooth, simply connected 4-manifold X. (*Hint*: Take the characteristic element $K = \sum_{i=1}^n e_i$, where the vectors e_i are linearly independent with square -1. For this K we have $|K^2| = n < \mathrm{rk}(Q) + n = \mathrm{rk}(Q \oplus n\langle -1\rangle)$, leading to a contradiction.)

The above argument, coupled with the purely algebraic Theorem 2.4.28 due to Elkies, quickly lead us to a proof of Donaldson's Theorem 1.2.30.

Theorem 2.4.28. ([**Elk**]) *The form $n\langle -1\rangle$ can be characterized as being the only negative definite rank n unimodular form with the property that* $\min\{-Q(K,K) \mid K$ *is characteristic*$\}$ *equals n. For all other negative definite forms of rank n,* $\min\{-Q(K,K) \mid K$ *is characteristic*$\} < n$. □

Corollary 2.4.29. (Donaldson) *If X is a simply connected smooth 4-manifold with negative definite intersection form (i.e., $b_2^+(X) = 0$), then $Q_X \cong n\langle -1\rangle$.* □

Remark 2.4.30. The above theorem holds, in fact, for closed 4-manifolds with arbitrary fundamental groups [**KKM**], [**Sa**]. The original proof of Corollary 2.4.29 (which is the same as Theorem 1.2.30) used an analysis of ASD $SU(2)$-connections on the principal $SU(2)$-bundle $P \to X$ with $\langle c_2(P), [X]\rangle = 1$ [**D1**]. (Recall that an $SU(2)$-bundle over a 4-manifold X is determined by its second Chern class $c_2(P)$.) The analysis required for the original proof, however, is much more delicate than that outlined above.

Finally, we give a sketch of the proof of Theorem 2.4.6(1). (A similar outline of the proof of Theorem 2.4.7 will be given in Section 10.4.) We restrict ourselves to indicating the proof that under the given hypotheses, $SW_X(K) = 0$ for spin^c structures with $d(K) = \frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X))) = 0$. The basic idea of the proof is that if X decomposes as $X_1 \# X_2$ with $b_2^+(X_i) > 0$, then for an appropriate metric the moduli space $\mathcal{M}_K(g)$ is empty. Decompose X as $(X_1 - D^4) \cup_{S^3} S^3 \times [0,t] \cup_{S^3} (X_2 - D^4)$, equip $X_i - D^4$ with fixed metrics h_i and define the metric g_t on X as the given h_i's on $X_i - D^4$ and the natural product metric on $S^3 \times [0,t]$. (Assume that each h_i is chosen in such a way that this construction provides a smooth metric on X.) In this way the sequence g_t introduces a long "neck" between $X_1 - D^4$ and $X_2 - D^4$. It can be shown that for t large enough, $\mathcal{M}_{K,X}(g_t)$ is diffeomorphic to the direct product $\mathcal{M}_{K_1,X_1}(\tilde{h}_1) \times \mathcal{M}_{K_2,X_2}(\tilde{h}_2)$ for suitable metrics $\tilde{h}_i$ on X_i. (Here $K_i = K|X_i \in H^2(X_i;\mathbb{Z})$ is the restriction of the spin^c structure on X_i.) The dimension formula, however, tells us that $0 = \dim \mathcal{M}_{K,X}(g_t) = \dim \mathcal{M}_{K_1,X_1}(\tilde{h}_1) + \dim \mathcal{M}_{K_2,X_2}(\tilde{h}_2) + 1$, implying that one of the smooth manifolds $\mathcal{M}_{K_i,X_i}(\tilde{h}_i)$ is negative dimensional, hence empty. (The assumption $b_2^+(X_i) > 0$ is used in the argument that both moduli spaces $\mathcal{M}_{K_i,X_i}(\tilde{h}_i)$ are smooth manifolds of the expected dimension.) This argument shows that for t large enough $\mathcal{M}_{K,X}(g_t) = \emptyset$, completing the outline of the proof of Theorem 2.4.6(1). Note that, for example, we conclude that $SW_{\#n\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}} \equiv 0$ for $n \geq 2$ and odd, cf. Exercise 2.4.13. Similar ideas prove Theorems 2.4.9 and 2.4.10 as well; in these cases $b_2^+(X_2) = 0$, hence we encounter reducible solutions on X_2. By studying reducibles along the lines of Exercise 2.4.26(d), one can obtain the required formulae. Similar reasoning gives the proof of Theorem 2.4.8. After we decompose X as $(X - \nu\Sigma) \cup_T \nu\Sigma$, however, extra complications arise from the fact that the 3-manifold T, along which X is pulled apart, is a circle bundle over Σ with Chern number $[\Sigma]^2$ — rather than S^3. In addition, the description of reducibles over $\nu\Sigma$ (in the case of negative self-intersection $[\Sigma]^2$) is more complicated than in the case of the blow-up formulae.

Chapter 3

Complex surfaces

Having introduced the theory of 4-manifolds, it is time to present more examples. This chapter is devoted to the description of certain families of complex surfaces. We will always pay special attention to the genus function G introduced in Chapter 2. We begin with a detailed study of elliptic surfaces; these examples of complex surfaces form a class wide enough to show the special properties of 4-manifolds — for example, the existence of infinitely many exotic smooth structures. On the other hand, we understand elliptic surfaces fairly well, so these surfaces serve as good examples for the theory of Kirby calculus presented in the further chapters of the present volume. For that reason we will return to the discussion of elliptic surfaces in Chapter 7 from a more algebraic and in Chapter 8 from a more topological point of view. Most of the theorems stated in this chapter will be proved in Chapters 7 and 8. We finish this chapter with an outline of the classification of complex surfaces; more examples of complex surfaces (e.g., surfaces of general type) will be given in Chapter 7. We will examine complex surfaces from the differential topological point of view, hence (unless otherwise stated) we will regard diffeomorphic complex surfaces as the same.

3.1. $E(1)$ and fiber sum

Definition 3.1.1. A complex surface S is an *elliptic surface* if there is a holomorphic map $\pi\colon S \to C$ to a complex curve C such that for generic $t \in C$ the inverse image $\pi^{-1}(t)$ is a smooth *elliptic curve* — that is, topologically a real 2-dimensional torus. The map π is called a (*holomorphic*) *elliptic fibration*. A smooth map $\pi\colon X \to C$ (X a closed, oriented 4-manifold) will be called a (C^∞-) *elliptic fibration* if each (possibly singular) fiber $\pi^{-1}(t)$ "looks like" a fiber in a holomorphic elliptic fibration, i.e., it has a neighborhood U and an orientation-preserving diffeomorphism $\varphi\colon U \to \varphi(U) \subset S$ into an elliptic surface, with φ commuting with the maps π.

Before giving examples of elliptic surfaces, we construct a $\mathbb{CP}^1$-fibration over $\mathbb{CP}^1$ for sake of motivation. Take all complex projective lines in $\mathbb{CP}^2$ going through the point $P = [0 : 0 : 1] \in \mathbb{CP}^2$. By associating $[t_0 : t_1] \in \mathbb{CP}^1$ to the line $\{[x : y : z] \in \mathbb{CP}^2 \mid t_0 x = t_1 y\}$, we parametrize the set of such lines by $\mathbb{CP}^1$. It is easy to see that this family of lines gives a one-sheet cover of $\mathbb{CP}^2 - \{P\}$, i.e., for each point $Q \in \mathbb{CP}^2 - \{P\}$ there is a unique element of the above family going through Q. Moreover, all these lines intersect each other transversally in P. A map $f\colon \mathbb{CP}^2 - \{P\} \to \mathbb{CP}^1$ can be defined in the following way: For $Q \in \mathbb{CP}^2 - \{P\}$ associate the parameter of the unique line of the above family going through Q. This map cannot be extended to $\mathbb{CP}^2$, but by blowing up $\mathbb{CP}^2$ at P we replace P by the set of all lines going through it, so f can obviously be extended to $(\mathbb{CP}^2)' = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, resulting in a $\mathbb{CP}^1$-fibration of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ over $\mathbb{CP}^1$.

Exercise 3.1.2. * Prove that this fibration is not the trivial S^2-bundle over S^2. (The complex surfaces admitting $\mathbb{CP}^1$-fibrations over $\mathbb{CP}^1$ are also called *Hirzebruch surfaces*; these complex surfaces will be discussed in Section 3.4. More general S^2-fibrations will be discussed in detail later on.)

The total space of the above bundle $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ sometimes will also be denoted by $S^2 \tilde{\times} S^2$. Note that the exceptional sphere intersects each fiber of $S^2 \tilde{\times} S^2 \to \mathbb{CP}^1$ transversally in one point, so it is a section of the fibration. (In our subsequent discussions we will not distinguish between a section — a map — and its image, which is a submanifold.)

Next we generalize the above construction. As we saw, the polynomials defining the lines passing through $P \in \mathbb{CP}^2$ are linear combinations of two linear polynomials. (In our case, these two linear polynomials were $p_0 = x$ and $p_1 = y$.) Generalizing this point of view, take two generic homogeneous *quadratic* polynomials p_0 and p_1 in the variables x, y, z. By a general position argument, $V_{p_0} = \{[x : y : z] \in \mathbb{CP}^2 \mid p_0(x, y, z) = 0\}$ and $V_{p_1} = \{[x : y : z] \in \mathbb{CP}^2 \mid p_1(x, y, z) = 0\}$ (the curves in $\mathbb{CP}^2$ corresponding to the polynomials) intersect each other in 4 points P_1, P_2, P_3 and P_4. Take the family $\mathcal{Q} = \{t_0 p_0 + t_1 p_1 \mid [t_0 : t_1] \in \mathbb{CP}^1\}$ of quadratic polynomials. The curves corresponding to the polynomials of this family obviously give a one-sheet cover of $\mathbb{CP}^2 - \{P_1, \dots, P_4\}$, and any two curves in $\mathcal{Q}$ intersect each other transversally in $\{P_i \mid i = 1, \dots, 4\}$. Thus, as before, a map $f\colon \mathbb{CP}^2 - \{P_1, \dots, P_4\} \to \mathbb{CP}^1$ can be defined, and although f cannot be extended to $\mathbb{CP}^2$, blowing up each P_i defines an extension to a map $\tilde{f}\colon \mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. Note, however, that this map is *not* a bundle map, since although the *generic* fiber is $\mathbb{CP}^1$ (because a generic quadric curve in $\mathbb{CP}^2$ is a copy of $\mathbb{CP}^1$), there are singular fibers as well — namely quadric curves which are unions of two lines. (Since there are no singular linear subspaces of $\mathbb{CP}^2$, the first construction (resulting in $S^2\tilde{\times}S^2 \to \mathbb{CP}^1$) gave a fiber bundle.) By a little abuse of notation we still call $\tilde{f}\colon \mathbb{CP}^2\#4\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ a (singular) fibration, although $\tilde{f}\colon \mathbb{CP}^2\#4\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ is not a fiber bundle, since different fibers may not be diffeomorphic. This kind of fibration — called a *Lefschetz fibration* — will be discussed in Chapter 8 in detail. Note that each exceptional sphere intersects each fiber $\tilde{f}^{-1}(t)$ transversally in a unique point, so the exceptional spheres of the blow-ups are sections of $\tilde{f}\colon \mathbb{CP}^2\#4\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$.

Exercise 3.1.3. Prove that the same fibration can be defined by choosing 4 points in $\mathbb{CP}^2$ in general position and taking all quadric curves passing through these points. What are the singular fibers and how many of them can be found in such a fibration? What goes wrong if the points are not in general position, for example, if all four are on the same (projective) line? Can we generalize this approach to curves of higher degree? (See also Exercise 8.1.8(b).)

Now taking two generic cubics p_0 and p_1 (intersecting each other in $P_1, \dots, P_9$) and constructing the corresponding *pencil* of curves $\{t_0p_0 + t_1p_1 \mid [t_0 : t_1] \in \mathbb{CP}^1\}$, we can define the map $f\colon \mathbb{CP}^2 - \{P_1, \dots, P_9\} \to \mathbb{CP}^1$ in the same way: For $Q \in \mathbb{CP}^2 - \{P_1, \dots, P_9\}$ take the unique cubic $p_{[t_0:t_1]} = t_0p_0 + t_1p_1$ which passes through Q, and then define $f(Q) = [t_0 : t_1] \in \mathbb{CP}^1$. (In the notation we sometimes confuse the curve in $\mathbb{CP}^2$ with the homogeneous polynomial defining it.) By blowing up $\mathbb{CP}^2$ at $P_1, \dots, P_9$, we extend f to a fibration $\pi\colon \mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ whose fibers are cubic curves, hence the generic fiber is a smooth elliptic curve (i.e., a torus). Consequently, this process provides a holomorphic elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. Depending on the choice of the cubic polynomials p_0 and p_1, we will have different types of singular fibers. Note that if $\pi\colon X \to \mathbb{CP}^1$ is a fibration such that *all* fibers are tori, then $\chi(X) = 0$; this argument shows that $\pi\colon \mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ must have fibers not diffeomorphic to the torus T^2. The above procedure easily generalizes to curves with degree higher than 3: Take p_0, p_1 generic smooth curves in $\mathbb{CP}^2$ of degree d. The family $\{t_0p_0 + t_1p_1 \mid [t_0 : t_1] \in \mathbb{CP}^1\}$ of degree-d curves gives a (one-sheet) cover of $\mathbb{CP}^2 - p_0 \cap p_1$. Blowing up the d^2 points of the intersection $\{p_0 \cap p_1\}$, we get

Lemma 3.1.4. *The manifold $\mathbb{CP}^2\#d^2\overline{\mathbb{CP}^2}$ admits a (singular) fibration $\mathbb{CP}^2\#d^2\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$, where the generic fiber is a complex curve of genus $\frac{1}{2}(d-1)(d-2)$.* □

We will return to this kind of fibration in Chapter 8; now we turn back to the case of elliptic (degree-3) curves. The next proposition describes the singular fibers of certain elliptic fibrations of $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. For a proof, see Exercise 8.1.2(b).

Proposition 3.1.5. *For a generic choice of the degree-3 curves p_0 and p_1, the singular curves in the family $\mathcal{Q} = \{t_0p_0 + t_1p_1 \mid [t_0 : t_1] \in \mathbb{CP}^1\}$ are ambiently isotopic to $C_1 = \{[x : y : z] \in \mathbb{CP}^2 \mid zy^2 = x^3 + zx^2\} \subset \mathbb{CP}^2$.* □

The fibers in $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ coming from blowing up curves ambiently isotopic to C_1 are called *fishtail fibers*. A fiber originating from a curve ambiently isotopic to $C_2 = \{[x : y : z] \in \mathbb{CP}^2 \mid zy^2 = x^3\} \subset \mathbb{CP}^2$ is called a *cusp fiber*. Note that the polynomials p_0 and p_1 can be chosen so that at least one singular curve in the family $\mathcal{Q}$ will be ambiently isotopic to C_2: Take, for example, $p_0 = x^3 - zy^2$ and $p_1 = x^3 + y^3 + z^3$. A fishtail fiber has Euler characteristic 1, hence the fact that $\chi(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}) = 12$ implies that a generic fibration has 12 singular fibers. As we will see in Chapter 7, $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ admits elliptic fibrations with only cusp fibers as singular fibers. Since a cusp fiber has Euler characteristic 2, there are 6 such fibers in those fibrations. For a complete list of singular fibers in elliptic fibrations, see [**HKK**] or [**BPV**]. In the next section we will show other constructions of elliptic surfaces having singular fibers different from the above examples.

If we think of $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ as being equipped with an elliptic fibration, we denote it by $E(1)$. The invariants of $E(1)$ are easy to compute:

Lemma 3.1.6. *For the 4-manifold $E(1)$ we have $\pi_1(E(1)) = 1$, $\chi(E(1)) = 12$, $\sigma(E(1)) = -8$, $b_2(E(1)) = 10$, $b_2^+(E(1)) = 1$, $c_1^2(E(1)) = 0$, and finally, $c_2(E(1)) = 12$. Since $E(1)$ is a complex surface, $c_1(E(1))$ is defined and equals the Poincaré dual of $3h - \sum_1^9 e_i \in H_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}; \mathbb{Z})$.* □

Here we use the convention that h denotes the canonical generator originating from $H_2(\mathbb{CP}^2; \mathbb{Z})$ in $H_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}; \mathbb{Z}) = H_2(\mathbb{CP}^2; \mathbb{Z}) \oplus 9H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})$, while e_i is the homology class of the exceptional sphere of the $i^{\underline{th}}$ blow-up (generating the $i^{\underline{th}}$ $H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})$-factor). Since the fiber of the elliptic fibration is exactly the blow-up of a cubic curve, its homology class is equal to $3h - \sum_1^9 e_i$. As before, every exceptional sphere of the blow-up will be a section of $\pi\colon E(1) \to \mathbb{CP}^1$. Thus $E(1)$ has 9 disjoint sections, whose corresponding homology elements are the classes $e_i \in H_2(E(1); \mathbb{Z})$ $(i = 1, \dots, 9)$. A more convenient basis for $H_2(E(1); \mathbb{Z})$ can be given by changing $\langle h, e_1, \dots, e_9\rangle$ to $\langle f = 3h - \sum_1^9 e_i, e_9, e_1 - e_2, e_2 - e_3, \dots, e_7 - e_8, -h + e_6 + e_7 + e_8\rangle$. In the first basis, $Q_{E(1)}$ explicitly splits as $\langle 1\rangle \oplus 9\langle -1\rangle$, while in the second basis $Q_{E(1)}$ is given by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right] \oplus (-E_8)$. Assume that $E(1) \to \mathbb{CP}^1$ contains a cusp fiber and let $N(1)$ denote the (closure of the regular neighborhood of the union of a cusp fiber and the $9^{\underline{th}}$ exceptional sphere. Then $N(1)$ is simply connected, $H_2(N(1);\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and the intersection form $Q_{N(1)} \cong \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ is unimodular, hence (by Corollary 5.3.12, cf. also Remark 1.2.11) the boundary $\partial N(1)$ is a homology sphere; that is, $H_*(\partial N(1);\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$. The manifold $N(1)$ is called the *nucleus* of $E(1)$. Since the remaining basis elements $\{e_1 - e_2, \dots, e_7 - e_8, e_6 + e_7 + e_8 - h\}$ are orthogonal to $H_2(N(1);\mathbb{Z})$, these homology classes can be represented in the complement $\Phi(1) = E(1) - \text{int } N(1)$; the intersection matrix of $H_2(\Phi(1);\mathbb{Z})$ in this basis is $-E_8$. Note that $\Phi(1) = E(1) - \text{int } N(1)$ is not a closed manifold; it has a (homology sphere) boundary. Thus Rohlin's Theorem 1.2.29 does not apply in this case, and the intersection form is allowed to be $-E_8$ (which is impossible for a smooth, *closed* 4-manifold). Actually, *every* unimodular form occurs as the intersection form of a 4-manifold with (homology sphere) boundary — we will give an easy proof of this statement later on (see Exercise 5.3.13(e)). The intersection form $Q_{\Phi(1)}$ being $-E_8$ can be explained by the fact that $\Phi(1)$ is the manifold we get by plumbing according to the diagram corresponding to the $-E_8$-matrix. (For the definition of plumbing see Example 4.6.2; for a proof of the above statement see Corollary 7.3.23 and Exercise 8.3.4(c).)

Exercises 3.1.7. (a) Show that $e_i - e_{i+1}$ $(i = 1, \dots, 7)$ and $e_6 + e_7 + e_8 - h$ can be represented by embedded spheres in $\Phi(1)$.

(b) Prove that a homology element $a \in H_2(\Phi(1);\mathbb{Z})$ with $a^2 = -2$ has one of the following forms: $e_i - e_j$ $(i \neq j)$, $\pm(h - e_i - e_j - e_k)$ $(i, j, k$ different indices), $\pm(2h - e_{i_1} - \dots - e_{i_6})$ (where $i_1, \dots, i_6$ are six different indices), or $\pm(3h - 2e_{i_1} - e_{i_2} - \dots - e_{i_8})$ (where $i_1, \dots, i_8$ are eight different indices). (*Hint*: From the fact that an element $a = \alpha \cdot h + \sum_1^9 \beta_i e_i$ is in $H_2(\Phi(1);\mathbb{Z})$, one gets two equations ($a \cdot e_9 = 0$ and $a \cdot f = 0$), giving constraints for the integers $\alpha, \beta_1, \dots, \beta_9$. The condition $a^2 = -2$ and the Cauchy inequality $(\sum_1^n k_i)^2 \leq n \sum_1^n k_i^2$ imply that $|\alpha| \leq 4$, and then a case-by-case argument concludes the solution.)

(c) Prove that every homology element with square -2 in $H_2(\Phi(1);\mathbb{Z})$ can be represented by a sphere in $\Phi(1)$. (*Hint*: The first 3 cases can be answered by tubing the spheres representing h, $2h$ and e_i together — a good choice of the points where the blow-ups are performed insures that the resulting surface is actually in $\Phi(1)$. For the homology elements of the last type, one can represent $3h$ by the curve C_1 of Section 2.3, and proceed further.)

Using the fiber sum operation, many elliptic surfaces can be constructed from the single example $E(1)$. Assume that C^∞-elliptic fibrations $\pi_i \colon S_i \to C_i$ $(i = 1, 2)$ are given. The *fiber sum* $S_1 \#_f S_2$ is defined as follows: Take $t_i \in C_i$ $(i = 1, 2)$ such that the fibers $F_i = \pi_i^{-1}(t_i)$ are generic, and take regular neighborhoods of these fibers in S_i; topologically we specify a copy of $D^2 \times T^2$ in each S_i. Using a fiber-preserving, orientation-reversing diffeomorphism φ between the boundary 3-tori of the manifolds $S_i - \nu F_i$, we can glue these manifolds together and construct a new manifold $S_1 \#_f S_2$, which will admit a C^∞-elliptic fibration $\pi\colon S_1 \#_f S_2 \to C_1 \# C_2$. The diffeomorphism type of the resulting 4-manifold might depend on the choice of the diffeomorphism φ. If one of the fibrations contains a cusp fiber, however, there is no such dependence on φ (see Lemma 8.3.6), and $S_1 \#_f S_2$ is a well-defined 4-manifold. The elliptic surface $E(n)$ is defined as the n-fold fiber sum of copies of $E(1)$; in particular, $E(n) = E(n-1) \#_f E(1)$.

Remark 3.1.8. Since the fibration $E(1) \to \mathbb{CP}^1$ can be chosen to contain a cusp fiber, the above construction gives a well-defined 4-manifold. It is not clear from the definition of fiber summing that the resulting manifold will have a *complex* structure as well. The cyclic n-fold branched covering construction can be applied to $E(1)$ along 2 regular fibers to construct a complex elliptic surface which is diffeomorphic to the 4-manifold $E(n)$ defined above (see Section 7.3). In this way we can talk about $E(n)$ as a *complex surface*. Note that the complex structure on $E(n)$ is not unique — it depends on the choices we make in the construction. We always think of $E(n)$ as a smooth 4-manifold, and equip it with an arbitrary (compatible) complex structure when we need it. It is also worth mentioning that the restriction that the fibration $S_1 \to C_1$ contains a cusp fiber is not very serious. In fact, if an elliptic surface S has $\chi(S) \neq 0$, then it admits an elliptic fibration with a cusp fiber (cf. Theorem 8.3.12). We will not need this fact in our later arguments.

Next we determine the relevant invariants of $E(n)$, beginning with $E(2)$. Suppose that F is a regular fiber in $E(1)$, a neighborhood of which will be denoted by νF. Observe that $E(1) - \nu F$ is simply connected: The normal circle to the fiber can be contracted along the remaining hemisphere of any section. Using the Seifert-Van Kampen theorem, this implies that $E(2) = (E(1) - \nu F) \cup_{T^3} (E(1) - \nu F)$ is simply connected. Well-known properties of the Euler characteristic χ imply that $\chi(E(2)) = 2\chi(E(1) - \nu F) - \chi(T^3) = 2\chi(E(1) - \nu F)$ and $\chi(E(1) - \nu F) = \chi(E(1)) - \chi(\nu F) + \chi(T^3) = 12$, so the Euler characteristic of $E(2)$ is 24. This means that $b_2(E(2)) = 22$, and since $\pi_1(E(2)) = 1$, we have $H_2(E(2);\mathbb{Z}) \cong \mathbb{Z}^{22}$. In the following we describe a convenient basis for this free abelian group. Choose the regular fibers along which the fiber sum is performed to be in the two copies of the nucleus $N(1) \subset E(1)$. Thus the two copies of $\Phi(1) \subset E(1)$ are in $E(2)$, providing 16 spheres of square -2 (corresponding to $\{e_1 - e_2, \ldots, e_7 - e_8, e_6 + e_7 + e_8 - h\}$), which realize two $-E_8$'s in the intersection matrix. The $9^{\underline{th}}$ section of each $E(1)$ intersects the boundary of $E(1) - \nu F$ in a circle, and we may assume that the gluing map φ identifies these circles. Then a section $\sigma\colon \mathbb{CP}^1 \to E(2)$

is constructed by sewing the two sections of the $E(1)$'s together. Since $\sigma^2 = -2$, the homology classes σ and f of the section and the fiber give two more homology elements (orthogonal to the previous 16), for which the intersection matrix has the form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right]$. To find the missing 4 homology elements for a basis of $H_2(E(2);\mathbb{Z})$, we have to go back to $E(1) - \nu F$ for a moment. Note that $H_2(\partial(E(1) - \nu F);\mathbb{Z}) = H_2(T^3;\mathbb{Z})$ is spanned by 3 copies of the 2-torus T^2; one is the fiber of $E(1) \to \mathbb{CP}^1$, and the two others are homologous to 0 in $E(1)$ but give nonzero homology elements in $E(1) - \nu F$. We choose the gluing map $\varphi\colon \partial(E(1) - \nu F) \to \partial(E(1) - \nu F)$ in such a way that these tori are identified, so 2 new homology elements (represented by tori) have been detected.

Exercise 3.1.9. Prove that these tori have self-intersection 0. As we will see in a moment, these tori are homologically nontrivial. Applying Seiberg-Witten theory, prove that the homology classes of these tori cannot be represented by spheres in $E(2)$. (*Hint*: Use the fact that $E(2)$ admits a complex structure and $b_2^+(E(2)) > 1$.)

Each of the above-mentioned 2-dimensional tori in $T^3 = \partial(E(1) - \nu F)$ has a dual circle in T^3 ("the third circle") intersecting it transversally at a point, and since $E(1) - \nu F$ is simply connected, these two circles can be contracted in $E(1) - \nu F$. By contracting each circle on both sides, we can construct two new closed surfaces (hence homology elements) in $E(2)$; these surfaces will pair nontrivially with the tori of Exercise 3.1.9 in $Q_{E(2)}$ (proving that those tori are not homologous to zero).

Lemma 3.1.10. *The dual circles in $\partial(E(1) - \nu F)$ bound embedded disks in $E(1) - \nu F$. These disks can be chosen to be in $N(1)$ and disjoint from the section and from each other. The spheres in $E(2)$ defined by the above disks have self-intersection -2.* □

(For the proof of Lemma 3.1.10 see Section 8.2.) The above spheres of square -2 intersect only the tori to which the corresponding circles were dual. Hence we have found 22 homology elements (19 spheres and 3 tori), for which the intersection matrix is $2(-E_8) \oplus 3\left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right]$. Note that this matrix has determinant -1, implying that the homology classes listed above form a basis of $H_2(E(2);\mathbb{Z})$ (cf. Corollary 1.2.13). A similar argument provides a basis for $H_2(E(n);\mathbb{Z})$ with intersection matrix $n(-E_8) \oplus 2(n-1)\left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & -n \end{smallmatrix}\right]$. Recall that $E(n) = E(1)\#_f E(n-1)$, hence the assertions $\pi_1(E(n)) = 1$ and $H_2(E(n);\mathbb{Z}) \cong \mathbb{Z}^{12n-2}$ can be proved by induction. The elements of square 0 in the above basis are tori (and cannot be represented by spheres); all other elements in this basis are spheres. The sphere with square $-n$ is constructed by sewing sections of $E(1)$ together, hence it gives rise to a section of $E(n) \to \mathbb{CP}^1$ (cf. also Exercise 3.1.12(a)). In this way we can find

9 disjoint sections in $E(n)$. By Exercises 1.2.17(a) and (b) we know that $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -n \end{smallmatrix}\right]$ represents H or $\langle 1\rangle \oplus \langle -1\rangle$ depending on the parity of n; thus the intersection form of $E(n)$ is equivalent to $n(-E_8) \oplus (2n-1)H$ if n is even and to $(2n-1)\langle 1\rangle \oplus (10n-1)\langle -1\rangle$ if n is odd. (Recall the classification of indefinite forms.) We conclude

Proposition 3.1.11. *The invariants of the elliptic surface $E(n)$ are given as follows: $\chi(E(n)) = 12n$ and $\pi_1(E(n)) = 1$, hence $b_2(E(n)) = 12n-2$. For the signature we have $\sigma(E(n)) = -8n$; hence $b_2^+(E(n)) = 2n-1$ and $c_1^2(E(n)) = 0$. Moreover, $E(n)$ is spin iff n is even.* □

Since $E(n)$ admits a complex structure (cf. Remark 3.1.8), it has a first Chern class as well. We just remark here that $c_1(E(n))$ is the Poincaré dual of $(2-n)f$ (where f is the homology class of the fiber); in particular, $c_1(E(2)) = 0$, meaning that $E(2)$ is a $K3$-surface. (By the classification of complex surfaces, $E(2)$ is in fact diffeomorphic to the $K3$-surface S_4 of Section 1.3.) Taking a regular neighborhood of a cusp fiber and a section in $E(n)$, one can define the nucleus $N(n) \subset E(n)$. The complement $\Phi(n) = E(n) - \operatorname{int} N(n)$ is a simply connected manifold with boundary. $\partial\Phi(n)$ is a homology sphere and the intersection form of $\Phi(n)$ is equivalent to $n(-E_8) \oplus 2(n-1)H$, hence $\Phi(n)$ is spin for all n. We just note here that $\Phi(n)$ is a well-known manifold, called the *Milnor fiber* associated to the polynomial $f(x,y,z) = x^2 + y^3 + z^{6n-1}$, and $\partial\Phi(n)$ is the Seifert-fibered homology sphere $\Sigma(2,3,6n-1)$. (For more details, see Sections 6.3, 7.3, 8.3 or [**G9**].)

Exercises 3.1.12. **(a)*** Prove that if $C \subset E(n)$ is a rational complex curve (so $C \approx \mathbb{CP}^1$) with $[C]^2 = -n \neq -2$, then C is a section of the elliptic fibration $E(n) \to \mathbb{CP}^1$. (*Hint*: Use the adjunction formula 1.4.17 and the fact that $c_1(E(n)) = PD((2-n)f)$.)

(b)* Show that $N(n+2)$ embeds in $E(n)$. Find an embedding of $N(n+1)$ in the blow-up $E(n)\#\overline{\mathbb{CP}^2}$.

(c) Identify 3 disjoint copies of $N(2)$ in $E(2)$. Similarly, show that $2(n-1)$ disjoint copies of $N(2)$ can be embedded in $E(n)$.

(d) Show that the sum of the fiber and a section in $E(3)$ can be represented by a torus of square -1. (*Hint*: Resolve the singularity of the union of a torus and a sphere representing a fiber and a section as discussed in Section 2.1.)

(e) Take the fiber sum of $E(n)$ with the trivial elliptic fibration $\Sigma_g \times T^2$, where Σ_g is the Riemann surface of genus g. Determine the characteristic numbers of the resulting manifold $E(n,g) \to \Sigma_g$.

Next we will determine the Seiberg-Witten basic classes of $E(n)$ ($n \geq 2$). Let $K \in H^2(E(n);\mathbb{Z})$ be a Seiberg-Witten basic class; we will find the constraints for K provided by the generalized adjunction formula Theorem 2.4.8.

Lemma 3.1.13. *Given a copy of $\Phi(1)$ embedded in a 4-manifold X which has simple type, any Seiberg-Witten basic class $K \in H^2(X;\mathbb{Z})$ of X will vanish on $H_2(\Phi(1);\mathbb{Z}) \subset H_2(X;\mathbb{Z})$.*

Proof. The generalized adjunction formula shows that for any sphere S in $\Phi(1) \subset X$ having square -2 we have $|K([S])| \leq 2$. (Add a trivial torus to the sphere to obtain $g(S') > 0$ and apply Theorem 2.4.8.) An elementary combinatorial argument shows that on $(H_2(\Phi(1);\mathbb{Z}), Q_{\Phi(1)}) \cong (\mathbb{Z}^8, -E_8)$ only the identically 0 function has the property that it is linear and takes one of the values $0, \pm 2$ on each element of square -2. Exercise 3.1.7(c) now proves the lemma. □

Exercise 3.1.14. Prove that if a linear function $\varphi\colon (\mathbb{Z}^8, -E_8) \to \mathbb{Z}$ takes the values $0, \pm 2$ on each element of square -2, then φ is identically 0. (For the solution, see [**S1**].)

If $t, s \in H_2(E(n);\mathbb{Z})$ are the classes induced by a torus and sphere as above generating a $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right]$ summand, then the generalized adjunction formula shows that K vanishes on t and $t+s$ (since each can be represented by a torus with self-intersection 0), hence it vanishes on the subgroup they generate. Since the intersection matrix of $E(n)$ is $n(-E_8) \oplus 2(n-1)\left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & -n \end{smallmatrix}\right]$, K vanishes on the subspace $n(-E_8) \oplus 2(n-1)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. (Note that we have actually proved that if $\Phi(n)$ is a submanifold of a 4-manifold X of simple type and $K \in H^2(X;\mathbb{Z})$ is a Seiberg-Witten basic class of X, then K vanishes on $H_2(\Phi(n);\mathbb{Z}) \subset H_2(X;\mathbb{Z})$.) To finish the computation of the Seiberg-Witten basic classes of $E(n)$, we have to determine the values $K(f)$ and $K(\sigma)$. (Recall that f and σ are the homology classes of the fiber and a section respectively.) Again, the generalized adjunction formula shows that for any basic class K we have $K(f) = 0$ and so $K(nf+\sigma) = K(\sigma)$. The class $nf + \sigma$ can be represented by a surface of genus n — take n disjoint copies of the fiber, one copy of the section and resolve the singular points. Since $(nf+\sigma)^2 = n$, the generalized adjunction formula shows that $|K(\sigma)| \leq n-2$. Since $K(\sigma) \equiv \sigma^2 = -n \pmod 2$, the possibilities for K are the elements of the set

$$\{PD(k \cdot f) \in H^2(E(n);\mathbb{Z}) \mid k \equiv n \pmod 2,\ |k| \leq n-2\}.$$

Delicate gauge theoretic arguments [**FS2**] show that all these classes are actually SW basic classes.

Corollary 3.1.15. *The set of basic classes of $E(n)$ ($n \geq 2$) is equal to* $\{PD(k \cdot f) \in H^2(E(n);\mathbb{Z}) \mid k \equiv n \pmod 2,\ |k| \leq n-2\}$. □

Exercises 3.1.16. (a)* Show that the result of Exercise 3.1.12(c) is optimal, i.e., we cannot embed $2n-1$ disjoint copies of $N(2)$ in $E(n)$ when $n > 2$. Show that $E(2)$ does not contain 4 disjoint copies of $N(2)$.

(b)* Show that the homology class of the -1-torus in $E(3)$ found in Exercise 3.1.12(d) cannot be represented by an embedded sphere.

The values of the Seiberg-Witten function on the above classes have been determined: $SW_{E(n)}(PD(k\cdot f)) = \pm\binom{n-2}{|k|}$ [**FM3**]. Note that for the $K3$-surface $E(2)$ the class $0 \in H^2(E(2);\mathbb{Z})$ is the only basic class. From the fact that $SW_{E(n)} \neq 0$, the genus function G can be computed on many elements of $H_2(E(n);\mathbb{Z})$ — including the given basis. This, however, does not imply the knowledge of $G\colon H_2(E(n);\mathbb{Z}) \to \mathbb{Z}$ in general. We have seen that G is not linear, and in general there is no known way of computing $G(\alpha+\beta)$ in terms of $G(\alpha)$ and $G(\beta)$.

Finally we show that $S_3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid \sum z_i^3 = 0\} \subset \mathbb{CP}^3$ of Section 1.3 is, in fact, diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$. Note that cubic curves in $\mathbb{CP}^2$ can be parametrized by points of the projective space $\mathbb{CP}^9$. (A homogeneous cubic polynomial in 3 variables has 10 coefficients, hence determines a point of $\mathbb{CP}^9$; conversely a point in $\mathbb{CP}^9$ determines a polynomial up to a constant factor, cf. Claim 1.3.11.)

Lemma 3.1.17. *Fix six different points $\{P_1,\dots,P_6\} \in \mathbb{CP}^2$ in general position. (By general position we mean here that the points P_i are not on a quadric curve and no three of them are collinear.) The set of cubics passing through these points forms a subspace Z of $\mathbb{CP}^9$ isomorphic to $\mathbb{CP}^3$. If we add a point Q to $\{P_1,\dots,P_6\}$, the set of cubics passing through $P_1,\dots,P_6,Q$ defines a hyperplane $H_Q \subset Z$.* □

Note that if $Q \in \{P_1,\dots,P_6\}$, then $H_Q = Z$. For Q not belonging to the set $\{P_1,\dots,P_6\}$, the hyperplane H_Q in $Z \approx \mathbb{CP}^3$ determines a point in the dual projective space $(\mathbb{CP}^3)^* \approx \mathbb{CP}^3$, hence the above construction gives a map $f\colon \mathbb{CP}^2 - \{P_1,\dots,P_6\} \to \mathbb{CP}^3$ by $Q \mapsto H_Q \in Z^* \approx \mathbb{CP}^3$. The map f cannot be extended to $\mathbb{CP}^2$, but by blowing up the latter at the points $\{P_1,\dots,P_6\}$, we can extend the map to $\tilde f\colon \mathbb{CP}^2\#6\overline{\mathbb{CP}^2} \to \mathbb{CP}^3$. (A point Q' of the exceptional sphere over P_i corresponds to a direction through P_i, so we can interpret $\tilde f(Q')$ as the set of cubics passing through $\{P_1,\dots,P_6\}$ and having a prescribed tangent at P_i.) One can easily see that $\tilde f$ is injective, and since we know the Euler characteristics of the complex surfaces in $\mathbb{CP}^3$ (cf. Section 1.3), we have that $S_3 \approx \mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.

Exercises 3.1.18. **(a)** By choosing $\{P_1,\dots,P_6\} \subset \mathbb{CP}^2$ explicitly (i.e., fixing homogeneous coordinates for each P_i), go through the above construction and prove that $\operatorname{Im} \tilde f = S_3$.

(b) Prove that $\operatorname{Im} \tilde f$ is, in fact, a cubic hypersurface by showing that the intersection of it with a hyperplane is a cubic curve.

3.2. Other constructions of elliptic fibrations

In this section we will present other methods for constructing the same elliptic surfaces $E(n)$. In Section 7.3 we will prove that the newly constructed manifolds are diffeomorphic to the ones defined above. For more about elliptic surfaces see also Section 8.3.

First we will show that the $K3$-surface $S_4 \subset \mathbb{CP}^3$ admits an elliptic fibration. We have already mentioned that all $K3$-surfaces are diffeomorphic, and since both S_4 and $E(2)$ are $K3$-surfaces, S_4 admits a *smooth* elliptic fibration. In the following, we will explicitly describe a *holomorphic* elliptic fibration for S_4 (see also [**HKK**]). Recall that S_4 has been defined as $\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$. We change this setup slightly (cf. Claim 1.3.11), and redefine S_4 by the more convenient equation

$$S_4 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^4 - z_1^4 + z_2^4 - z_3^4 = 0\}.$$

(The following constructions would work perfectly for the original equation as well; we merely change the signs for the sake of simplicity.) Let $L_1 = \{z_0 = z_1,\ z_2 = z_3\}$ and $L_2 = \{z_0 = -z_1,\ z_2 = -z_3\}$ be skew projective lines in $\mathbb{CP}^3$. Recall that skew means that there is no hyperplane $\mathbb{CP}^2 \subset \mathbb{CP}^3$ containing both L_1 and L_2; note that $L_1, L_2 \subset S_4$. A map $\pi\colon S_4 \to L_2 \approx \mathbb{CP}^1$ will be constructed, and by explicitly determining the fibers we will show that π is an elliptic fibration. For $P \in S_4$ we define $\pi(P)$ in the following way:

- if $P \in S_4$ is not in L_1, then take the unique hyperplane H_P ($\approx \mathbb{CP}^2$) spanned by P and L_1, and define $\pi(P) = H_P \cap L_2$;
- if $P \in L_1 \subset S_4$, then let H_P be the tangent plane of S_4 at P, and define $\pi(P) = H_P \cap L_2$.

To prove that $\pi\colon S_4 \to L_2$ is an elliptic fibration, we need to check that for generic $Q \in L_2$ the inverse image $\pi^{-1}(Q) \subset S_4$ is a smooth elliptic curve. By the (complicated) definition of π, if H_Q denotes the hyperplane spanned by Q and L_1, then $\pi^{-1}(Q) = ((H_Q \cap S_4) - L_1) \cup \{P \in L_1 \mid$ the tangent plane of S_4 at P intersects L_2 in $Q\}$. A generic point $Q \in L_2$ has coordinates $[y : -y : x : -x]$, and the corresponding equation of H_Q is $x(z_0 - z_1) = y(z_2 - z_3)$. Intersecting this plane with S_4, we get that $H_Q \cap S_4$ equals

$$\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^4 - z_1^4 + z_2^4 - z_3^4 = 0,\ x(z_0 - z_1) = y(z_2 - z_3)\}.$$

Note that $L_1 \subset H_Q \cap S_4$ for every Q, since both H_Q and S_4 contain L_1 by definition. A little computation (factorization of $z_0^4 - z_1^4$ and $z_2^4 - z_3^4$) shows that the intersection $H_Q \cap S_4$ is equal to the union of $\{z_0 = z_1$ and $z_2 = z_3\}$ with $\{y(z_0^3 + z_0^2 z_1 + z_0 z_1^2 + z_1^3) + x(z_2^3 + z_2^2 z_3 + z_2 z_3^2 + z_3^3) = 0$ and $x(z_0 - z_1) = y(z_2 - z_3)\}$. The first component of this union is L_1, hence to determine

$\pi^{-1}(Q)$ we only have to drop the points of L_1 from the set $\{x(z_0 - z_1) = y(z_2 - z_3)$ and $y(z_0^3 + z_0^2 z_1 + z_0 z_1^2 + z_1^3) + x(z_2^3 + z_2^2 z_3 + z_2 z_3^2 + z_3^3) = 0\}$, and then add those points $P \in L_1$ for which the tangent intersects L_2 in Q. For $P = [u : u : v : v] \in L_1$, the tangent plane of S_4 is given by the equation $u^3(z_0 - z_1) + v^3(z_2 - z_3) = 0$. This plane intersects L_2 in $Q = [y : -y : x : -x]$ iff $u^3 y + v^3 x = 0$, meaning that $P \in L_1 \subset S_4$ actually satisfies

$$y(z_0^3 + z_0^2 z_1 + z_0 z_1^2 + z_1^3) + x(z_2^3 + z_2^2 z_3 + z_2 z_3^2 + z_3^3) = 0.$$

Consequently, we have identified $\pi^{-1}([y : -y : x : -x])$ with

$$\{y(z_0^3+z_0^2 z_1+z_0 z_1^2+z_1^3)+x(z_2^3+z_2^2 z_3+z_2 z_3^2+z_3^3) = 0,\ x(z_0-z_1) = y(z_2-z_3)\},$$

which is a cubic curve in the hyperplane $H = \{x(z_0 - z_1) = y(z_2 - z_3)\}$, that is, it is an elliptic curve. For generic $Q \in L_2$ this inverse image is smooth. Note also that L_2 is a section of π.

Exercise 3.2.1. Using the same method, prove that the complex surface $S_2 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^2 - z_1^2 + z_2^2 - z_3^2 = 0\}$ admits a $\mathbb{CP}^1$-fibration over $\mathbb{CP}^1$. Note that (since S_2 has even intersection form) this implies that S_2 is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Another description of $E(2)$ can be given in the following way. (This time the new description can be generalized to $E(n)$ for all $n \geq 2$.) It is easy to see that if we divide $\mathbb{C}^2$ by the $\mathbb{Z}^4$-action $(z_1, z_2) \mapsto (z_1 + n_1 + n_2 i, z_2 + n_3 + n_4 i)$ $(n_j \in \mathbb{Z},\ \ j = 1, \ldots, 4)$, we get the 4-dimensional real torus T^4 as a quotient. Dividing it further by the $\mathbb{Z}_2$-action $(z_1, z_2) \mapsto (-z_1, -z_2)$, we get a new (singular) manifold $\widetilde{X}$. The singularities of $\widetilde{X}$ correspond to the fixed points of the $\mathbb{Z}_2$-action on T^4, since — in contrast to the $\mathbb{Z}^4$-action above — the action of $\mathbb{Z}_2$ is *not* free; for example, $0 \in \mathbb{C}^2$ is a fixed point.

Exercises 3.2.2. **(a)** Prove that the induced $\mathbb{Z}_2$-action on $T^4 = \mathbb{C}^2/\mathbb{Z}^4$ has exactly 16 fixed points $\{p_1, \ldots, p_{16}\}$.

(b) Show that a neighborhood of the image of each fixed point p_i in $\widetilde{X}$ is a cone over the 3-dimensional real projective space $\mathbb{RP}^3$.

(c) Define a $\mathbb{Z}_2$-action on T^2 in the same manner as on T^4 (cf. also the hyperelliptic action defined before Exercises 3.2.5). Prove that $T^2/\mathbb{Z}_2$ is homeomorphic to S^2 and has 4 “corner” points (corresponding to the fixed points of the $\mathbb{Z}_2$-action), where the induced metric is singular but over which the complex (and smooth) structure naturally extends. (*Hint*: Compute $\chi(T^2/\mathbb{Z}_2)$.) The quotient $T^2/\mathbb{Z}_2$ is usually called the *pillowcase*, see Figure 3.1.

There is a canonical way to resolve complex singularities (cf. Section 7.2); this process is particularly simple in the above case. Recall that the unit disk bundle of the cotangent bundle of the sphere S^2 is a smooth

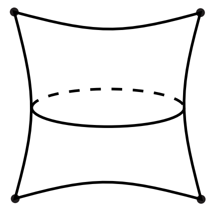


Figure 3.1. The pillowcase $T^2/\mathbb{Z}_2$.

manifold $W = \{v \in T^*S^2 \mid ||v|| \leq 1\}$ with $\partial W = \mathbb{RP}^3$. Now delete the cone-like neighborhoods of the singular points in $\widetilde{X}$ and get a manifold X_1 with $\partial X_1 = \bigcup_1^{16} \mathbb{RP}^3$. Gluing 16 copies of W to X_1, one gets a smooth manifold $X = X_1 \cup_{16\mathbb{RP}^3} \bigcup_1^{16} W$.

Remark 3.2.3. For the sake of convenience, we write down the desingularization process in this special case explicitly. Blow up T^4 at the fixed points $\{p_1, \dots, p_{16}\}$ and extend the $\mathbb{Z}_2$-action to the resulting surface $T^4 \# 16\overline{\mathbb{CP}^2}$ — it extends trivially to the exceptional curves. The quotient $X = T^4 \# 16\overline{\mathbb{CP}^2}/\mathbb{Z}_2$ obviously admits a complex structure. It is a smooth manifold, since the quotient of $\tau = \overline{\mathbb{CP}^2} - [1 : 0 : 0]$ by the $\mathbb{Z}_2$-action $[x : y : z] \to [-x : y : z]$ is (despite the presence of the fixed points $\{[0 : y : z]\}$) a manifold diffeomorphic to $\tau \otimes \tau = \tau^2 \approx T^*S^2$. (Recall the definition of the tautological bundle $\tau \to \mathbb{CP}^1$ from Section 2.2.) In the language of branched covers, we have just shown that $T^4 \# 16\overline{\mathbb{CP}^2}$ is the double branched cover of X branched along 16 spheres (the images of the exceptional curves in $T^4 \# 16\overline{\mathbb{CP}^2}$), cf. Section 7.1. Resolution of more general singularities will be discussed in Chapter 7.

Exercises 3.2.4. **(a)** Prove that $\pi_1(X) = 1$. (*Hint*: First determine the fundamental group of X_1 using the fact that $T^4 - \{p_1, \dots, p_{16}\}$ is the double cover of X_1, and then apply the Seifert-Van Kampen theorem).

(b) Compute the Euler characteristic $\chi(X)$.

(c) Show that $c_1(X) = 0$ (cf. [**HKK**], page 30).

Note that by Exercise 3.2.4(a) and (c) the complex surface X is a $K3$-surface, so (by the algebro-geometric statement we quoted in Section 1.3) it is diffeomorphic to S_4 and $E(2)$. The projection $\mathrm{pr}_1 \colon T^4 = T^2 \times T^2 \to T^2$ to the first factor is an elliptic fibration, and it descends to an elliptic fibration $\tilde{\pi} \colon \widetilde{X} \to T^2/\mathbb{Z}_2 \approx S^2$. The fibers of $\tilde{\pi} \colon \widetilde{X} \to T^2/\mathbb{Z}_2$ are tori except over the corner points, where the fibers are pillowcases themselves. Resolving the 16 singular points does not change the generic fiber, so we have defined an elliptic fibration of X over $T^2/\mathbb{Z}_2 \approx S^2$. Note that the first factor of $T^4 = T^2 \times T^2$ gives a section of $\pi \colon X \to \mathbb{CP}^1$ — this section is a copy of the pillowcase $T^2/\mathbb{Z}_2$ in X.

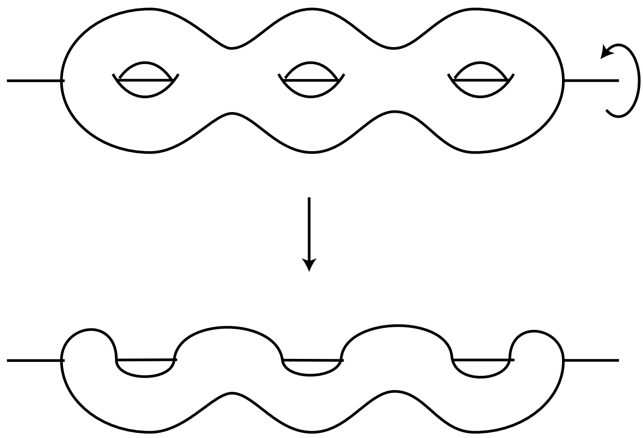


Figure 3.2. Hyperelliptic action on Σ_3.

Now take Σ_n to be a surface of genus n (e.g., $\Sigma_0 = S^2$ and $\Sigma_1 = T^2$). Imagine $\Sigma_n \subset \mathbb{R}^3$ such that the y-axis intersects it in $2n+2$ points and Σ_n is invariant under the 180° rotation around the y-axis. (See Figure 3.2.) This rotation defines a $\mathbb{Z}_2$-action $\sigma_n\colon \Sigma_n \to \Sigma_n$ with $2n+2$ fixed points — frequently called the *hyperelliptic action*. Taking the $\mathbb{Z}_2$-action $\sigma_n \times \sigma_1$ on $\Sigma_n \times T^2$ and resolving the $4(2n+2)$ singular points of $(\Sigma_n \times T^2)/\mathbb{Z}_2$ as before, we get a smooth manifold $X(n+1)$. The ideas of Exercise 3.2.4 extend to solve the following

Exercises 3.2.5. (a) Prove that $\pi_1(X(n)) = 1$ and that $\chi(X(n)) = 12n$.

(b) Show that Σ_n/σ_n is homeomorphic to S^2 with $2n+2$ "corner points".

(c) Extend the projection $\mathrm{pr}_1\colon \Sigma_n \times T^2 \to \Sigma_n$ to the quotient and then to the resolution. Show that the resulting map $\pi\colon X(n+1) \to \mathbb{CP}^1$ is an elliptic fibration and that it admits a section.

(d) Prove that $X(n)$ decomposes as the fiber sum of n copies of $X(1)$.

(e) Determine the number of singular fibers in $\pi\colon X(n) \to T^2/\mathbb{Z}_2$ and compute the Euler characteristic of a singular fiber. Describe the topology of the singular fibers of this fibration.

Remarks 3.2.6. (a) By taking the product of the surfaces Σ_n and Σ_m, resolving the singularities of the quotient $\Sigma_n \times \Sigma_m/(\sigma_n \times \sigma_m)$ and extending $\mathrm{pr}_1\colon \Sigma_n \times \Sigma_m \to \Sigma_n$ to the resulting manifold $X(n+1, m+1)$, we get a singular fibration $X(n+1, m+1) \to \Sigma_n/\sigma_n \approx S^2$. In this case, the fiber has genus m. Similar fibrations will be discussed in Chapter 8.

(b) Since the product $T^4 = S^1 \times S^1 \times S^1 \times S^1$ admits six different projections to factors T^2, the $K3$-surface $X(2) = X(2,2)$ has six different (C^∞) elliptic fibrations. The other manifolds $X(n)$ (for $n > 2$) have no such freedom

— the above construction gives only one way to find an elliptic fibration on these manifolds. Note that if we project $\Sigma_n \times T^2$ to the second factor, we get a Σ_n-fibration $X(n+1) \to T^2/\mathbb{Z}_2 \approx S^2$. More generally, the identification $X(n,m) = X(m,n)$ provides two different fibrations on the same complex manifold $X(n,m)$; both fibrations admit sections, cf. Exercise 7.3.16(d). In Section 7.3 we will prove that $X(2,n) = X(n)$ is, in fact, diffeomorphic to the elliptic surface $E(n)$.

(c) The map $\Sigma_n \to \Sigma_n/\sigma_n \approx S^2$ is a manifestation of the fact that a surface Σ_n of genus n can be given as a double branched cover of $\mathbb{CP}^1$ branched in $2n+2$ points (in the "corner points" of Σ_n/σ_n). The actions σ_n on Σ_n and σ_1 on T^2 induce a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action $\langle \sigma_n, \sigma_1 \rangle$ on $\Sigma_n \times T^2$. Thus $\mathbb{Z}_2$ acts on the (singular) manifold $(\Sigma_n \times T^2)/\mathbb{Z}_2$.

Exercise 3.2.7. Determine the topology of the quotient $(\Sigma_n \times T^2)/\langle \sigma_n, \sigma_1 \rangle$, and find the fixed point set. (*Answer*: See Lemma 7.3.4.)

Finally, we give one more description of $E(n)$: Take a generic bihomogeneous polynomial P_n of bidegree $(n,3)$ in the variables $(x,y;z_0,z_1,z_2)$ (that is, P_n is homogeneous of degree n in the variables (x,y) and of degree 3 in (z_0,z_1,z_2)). While a homogeneous polynomial in the variables $(z_0,\dots,z_n)$ defines a subset of $\mathbb{CP}^n$, a bihomogeneous polynomial in the variables $(x_0,\dots,x_n;z_0,\dots,z_m)$ gives rise to a subset of $\mathbb{CP}^n \times \mathbb{CP}^m$ (cf. the last paragraph of Section 1.3). Define $V(n)$ as $\{p \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid P_n(p) = 0\}$; projecting $V(n)$ to the first factor of $\mathbb{CP}^1 \times \mathbb{CP}^2$, we obtain a map whose fibers are cubic curves in $\mathbb{CP}^2$, hence $V(n)$ admits an elliptic fibration.

Exercise 3.2.8. Show that the bihomogeneous polynomial P_n can be chosen in such a way that the above elliptic fibration admits a section. (*Hint*: Consider P_n of the form $x^n p_0(z_0,z_1,z_2) + y^n p_1(z_0,z_1,z_2)$ for generic cubic polynomials p_0, p_1, and show that each component of the curve given as $\{([x:y],[z_0:z_1:z_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid p_0(z_0,z_1,z_2) = p_1(z_0,z_1,z_2) = 0\} \subset V_n$ is a section of the elliptic fibration. How many sections of $V(n) \to \mathbb{CP}^1$ do we find in this way?)

Note that since $\mathbb{CP}^1 \times \mathbb{CP}^2$ is a complex submanifold of $\mathbb{CP}^5$ (by the same method that we used to embed $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{CP}^3$ as a quadric surface, cf. S_2' in Section 1.3), the surface $V(n)$ is a complex projective manifold (cf. Definition 1.3.7). A deep result (due to Kas [**Ks**] for holomorphic elliptic surfaces, and Moishezon [**Msh**] in general) shows that for fixed n all the above 4-manifolds $E(n)$, $X(n)$ and $V(n)$ are diffeomorphic. These diffeomorphisms will be discussed in Section 7.3, cf. also Theorem 8.3.12.

Theorem 3.2.9. *Two minimal, simply connected, elliptic surfaces with sections are diffeomorphic iff their Euler characteristics are equal.* □

Exercises 3.2.10. (a) Prove that $V(1) \approx E(1)$ using the construction of $E(1)$ ($= \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$) discussed at the beginning of Section 3.1. (For the solution, see Lemma 7.3.10.)

(b)* Show that although $V(n)$ is a complex projective manifold, it is not a complete intersection unless $n = 2$. (*Hint*: Use Exercise 1.3.13 and the fact that $V(n) \approx E(n)$.)

3.3. Logarithmic transformation

There is one more important construction regarding elliptic surfaces besides the fiber sum operation — this is the logarithmic transformation (cf. also Sections 8.3 and 8.5). We begin the discussion by outlining the definition in the complex category. (For more details see [**GH**].)

For a given elliptic surface $\pi\colon S \to C$ choose $t \in C$ and a neighborhood Δ of t isomorphic to $\{z \in \mathbb{C} \mid |z| < 1\}$ such that there is no singular fiber in $\pi^{-1}(\Delta)$. Fix a holomorphic section $\alpha\colon \Delta \to \pi^{-1}(\Delta) \subset S$ of the fibration $\pi\colon \pi^{-1}(\Delta) \to \Delta$. Note that for $z \in \Delta$ the elliptic curve $\pi^{-1}(z)$ has a unique holomorphic (abelian) group structure, once we declare the origin to be $\alpha(z) \in \pi^{-1}(z)$ ($z \in \Delta$). Taking the elements of order p in each fiber (for p any fixed positive integer), we get a cover of Δ. Choosing one branch of that cover gives a section $\beta\colon \Delta \to \pi^{-1}(\Delta)$; by definition $p \cdot \beta(z) = \alpha(z)$ for all $z \in \Delta$. Pull back $\pi^{-1}(\Delta) \to \Delta$ by the map $\phi\colon \Delta \to \Delta$ given as $\phi(z) = z^p$; in other words, take $\Psi = \{(w, r) \in \Delta \times \pi^{-1}(\Delta) \mid \pi(r) = w^p\} \subset \Delta \times \pi^{-1}(\Delta)$. The projection $\Psi \to \Delta$ is an elliptic fibration whose fiber over each point $e^{\frac{2\pi i}{p}n} w \in \Delta$ is canonically identified with $\pi^{-1}(w^p)$. Now if we divide Ψ by the (non-free) $\mathbb{Z}_p$-action $(w, r) \to (e^{\frac{2\pi i}{p}} w, r)$, we get back $\pi^{-1}(\Delta) \to \Delta$. On the other hand, if $\mathbb{Z}_p$ acts freely as $\varphi(w, r) = (e^{\frac{2\pi i}{p}} w, r + \beta(w^p))$, then the quotient $\Psi_1 = \Psi/\mathbb{Z}_p$ will have a different fibration.

Exercise 3.3.1. Prove that Ψ_1 is diffeomorphic to $T^2 \times D^2 \approx \pi^{-1}(\Delta)$, but it has a different elliptic fibration. (*Hint*: For $x = [(w, r)] \in \Psi_1$ take $\psi(x) = w^p \in \Delta$ and determine the fibers of ψ.) Visualize the fiber over $0 \in \Delta$. Draw an analogous S^1-fibration of $S^1 \times D^2$.

The free $\mathbb{Z}_p$-action identifies p different fibers over each $w \neq 0$, so when w approaches 0 in Δ, the corresponding fibers $\psi^{-1}(w)$ give a p-fold cover of $\psi^{-1}(0)$. We see from the construction that $\psi^{-1}(\Delta - \{0\})$ and $\pi^{-1}(\Delta - \{0\})$ are isomorphic fibrations: the map $\lambda(w, r) = (w^p, r - (\frac{p}{2\pi i} \log w)\beta(w^p))$ gives the isomorphism. Hence by cutting $\pi^{-1}(\Delta)$ out of S and gluing in Ψ_1 via this isomorphism we get a new elliptic surface S_p. Since a regular fiber is a p-fold cover of the *multiple fiber* $\psi^{-1}(0)$, the homology class of a regular fiber is given by $f = p[\psi^{-1}(0)] \in H_2(S_p; \mathbb{Z})$. The resulting surface S_p will not depend significantly on the choices made (namely $t \in \Delta$, the trivialization α

and the section β). More precisely, if $\chi(S) \neq 0$ (hence we may assume that S has a cusp fiber), then different choices of Δ, α and β give diffeomorphic (actually deformation equivalent, see Definition 3.4.1) surfaces. The procedure described above is called the *logarithmic transformation of multiplicity* p along the fiber $\pi^{-1}(t)$.

Next we discuss the C^∞ (smooth) version of the operation (cf. also Section 8.3). Since we simply removed $T^2 \times D^2$ ($\approx \pi^{-1}(\Delta)$) and glued it back in (as Ψ_1) with a different fibration, we can reformulate the definition of logarithmic transformation in the following way: Consider the elliptic surface $\pi\colon S \to C$ and fix a generic fiber $\pi^{-1}(t) = F$ ($t \in C$). We denote a closed tubular neighborhood of the fiber F in S by νF; this is obviously diffeomorphic to $T^2 \times D^2$. Deleting int νF from S and regluing $T^2 \times D^2$ via a diffeomorphism $\varphi\colon T^2 \times S^1 \to \partial(S - \text{int } \nu F) = \partial \nu F$, we get a new manifold S_φ. Note that $\partial\nu F$ inherits a trivial T^2-fibration over S^1.

Definition 3.3.2. For a diffeomorphism $\varphi\colon T^2 \times S^1 \to \partial\nu F$ as above, let p denote the absolute value of the degree of the map $\pi \circ \varphi\colon \{\text{pt.}\} \times S^1 \to \pi(\partial\nu F) = S^1$. Then gluing by φ as above is called a *(generalized) logarithmic transformation of multiplicity p.*

Theorem 3.3.3. ([**G9**], see also Theorem 8.3.5) *Assume that the elliptic fibration $\pi\colon S \to C$ contains a cusp fiber. If φ, φ' determine logarithmic transformations with the same multiplicity p, then S_φ and $S_{\varphi'}$ are diffeomorphic (and will be denoted by S_p).* □

Our first (algebraic geometric) construction of logarithmic transformation gave a specific identification φ for each $p \geq 1$. Note that both processes can be performed simultaneously on k fibers: Starting with k regular fibers $F_1, \dots, F_k$, nonnegative integers $p_1, \dots, p_k$, and diffeomorphisms $\varphi_i\colon T^2 \times S^1 \to \partial\nu F_i$ with multiplicity p_i, a new manifold $S_{p_1,\dots,p_k}$ can be constructed. As the notation suggests, the diffeomorphism type of the resulting manifold will depend only on the multiplicities p_i of the maps φ_i. (Again, we assume that S contains a cusp fiber.) The smooth version of the logarithmic transformation makes sense even for multiplicity $p = 0$; this construction, however, has no complex analog and destroys the fibration. Note that a logarithmic transformation with $p = 1$ is trivial. Since the nuclei $N(n) \subset E(n)$ contain regular fibers in addition to the cusp fiber, logarithmic transformations can be performed within $N(n)$, resulting in $N(n)_{p_1,\dots,p_k} \subset E(n)_{p_1,\dots,p_k}$.

Lemma 3.3.4. ([**G9**]) *The inclusion $N(n)_{p_1,\dots,p_k} \subset E(n)_{p_1,\dots,p_k}$ induces an isomorphism on the fundamental groups. The nucleus $N(n)_{p_1,\dots,p_k}$ ($p_1, \dots, p_k \neq 1$) is simply connected iff $k \leq 1$ or $k = 2$ and* $\gcd(p_1, p_2) = 1$. *Assuming that* $\gcd(p, q) = 1$ *and $n \in \mathbb{N}$, the complex surface $E(n)_{p,q}$ is spin iff $N(n)_{p,q}$ is spin, and this occurs iff n is even and pq is odd. For* *the invariants of* $E(n)_{p,q}$ *we have* $b_2(E(n)_{p,q}) = b_2(E(n)) = 12n - 2$ *and* $\sigma(E(n)_{p,q}) = \sigma(E(n)) = -8n$. □

Hence, using fiber sum and logarithmic transformation repeatedly, a family $\mathcal{F} = \{E(n)_{p,q} \mid n, p, q \in \mathbb{N},\ \gcd(p,q) = 1\}$ of simply connected elliptic surfaces can be constructed from the basic example $E(1)$. By Lemma 3.3.4 and Theorem 1.2.27, two elements $E(n)_{p,q}$ and $E(n')_{p',q'}$ of $\mathcal{F}$ are homeomorphic iff $n = n'$ and $(n-1)(pq - p'q')$ is even. Thus for odd n, all $E(n)_{p,q}$ with $\gcd(p,q) = 1$ are homeomorphic, but if n is even, there are two homeomorphism types: those with odd pq (when $E(n)_{p,q}$ is spin) and those with even pq (when $E(n)_{p,q}$ is nonspin).

Remark 3.3.5. Remember that by performing a logarithmic transformation with multiplicity $p > 1$ on the elliptic surface S we create a new homology class f_p (the class of the multiple fiber over the chosen point t) with the property that $f = pf_p$ (where f is the homology class of a regular fiber). Consequently — since f is no longer a primitive class — there is no section for $S_p \to C$ even if $S \to C$ has a section. (A section intersects a generic fiber in one point, but in $H_2(S_p;\mathbb{Z})$ the class f is divisible by p.)

By introducing the notion of *rational blow-down*, Fintushel and Stern gave a simple way to determine the Seiberg-Witten basic classes of $E(n)_{p,q}$ $(n \geq 2)$. (See Section 8.5.)

Theorem 3.3.6. ([**FS2**]) *Assume that* $\gcd(p,q) = 1$ *and let* $f_{p,q}$ *denote the primitive class* $\frac{f}{pq} \in H_2(E(n)_{p,q};\mathbb{Z})$. *The Seiberg-Witten basic classes of* $E(n)_{p,q}$ $(n \geq 2)$ *comprise the set*

$$\mathcal{B} = \mathcal{B}as_{E(n)_{p,q}} = \{PD(k \cdot f_{p,q})) \mid k \equiv npq - p - q \pmod 2,\ |k| \leq npq - p - q\}.$$

□

Note that $E(n)_{p,q} = N(n)_{p,q} \cup_\partial \Phi(n)$ (where $\Phi(n)$ is the Milnor fiber of $f(x,y,z) = x^2 + y^3 + z^{6n-1}$), and — as we saw in Section 3.1 — any Seiberg-Witten basic class K is orthogonal to $H_2(\Phi(n);\mathbb{Z}) \subset H_2(E(n)_{p,q};\mathbb{Z})$. We also have $K(f_{p,q}) = 0$, since $f_{p,q}$ can be represented by a torus with square 0. By finding a suitable representative for the other basis element of $H_2(N(n)_{p,q};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, we can find an additional restriction on the set of basic classes, as shown for $E(n)$ earlier. (Details are left to the reader.) Given an appropriate representative, the generalized adjunction formula (Theorem 2.4.8) shows that any basic class must be an element of the set $\mathcal{B}$ given in Theorem 3.3.6. To show that all elements of $\mathcal{B}$ are, in fact, basic classes (i.e., $SW_{E(n)_{p,q}}(h) \neq 0$ for all $h \in \mathcal{B}$), we need delicate gauge theoretic arguments, cf. Section 8.5. Since the Seiberg-Witten basic classes are diffeomorphism invariants of the smooth 4-manifolds $E(n)_{p,q}$

($n \geq 2$), the following corollary can be derived from Theorem 3.3.6 (cf. Theorem 8.3.12).

Corollary 3.3.7. (see also [**MM**], [**SSz**], [**Ls1**]) *Assume that* $n \geq 2$. *Then* $E(n)_{p,q}$ *is diffeomorphic to* $E(n)_{p',q'}$ *iff* $\{p,q\} = \{p',q'\}$ *(as unordered pairs).* □

Thus, for a fixed odd n all manifolds $E(n)_{p,q}$ are homeomorphic, but they are pairwise nondiffeomorphic. (For even n, the parity of pq determines the homeomorphism type.) Consequently, the topological manifold $E(n)$ admits infinitely many smooth structures. Applying the blow-up formula (Theorem 2.4.9), one can prove the same for $E(n)\#k\overline{\mathbb{CP}^2}$; note that by computing the intersection form of $E(n)\#k\overline{\mathbb{CP}^2}$ we obtain a proof of Theorem 1.2.32.

Seiberg-Witten theory becomes more complicated for the manifolds $E(1)_{p,q}$ since $b_2^+(E(1)_{p,q}) = 1$, cf. Remark 2.4.4. The surfaces $E(1)_{p,q}$ with $\gcd(p,q) = 1$ and $p, q \geq 2$ are called *Dolgachev surfaces*. The diffeomorphism classification in this case is:

Theorem 3.3.8. ([**Fr**], see also Theorem 8.3.11) $E(1)_{1,p}$ *and* $E(1)_{1,q}$ *are diffeomorphic for all* p *and* q. *If* $p, q, p', q' > 1$ *then* $E(1)_{p,q}$ *is diffeomorphic to* $E(1)_{p',q'}$ *iff* $\{p,q\} = \{p',q'\}$; *these are never diffeomorphic to* $E(1)$. □

Remarks 3.3.9. (a) Throughout the last few paragraphs, we focused only on the simply connected case, i.e., when $k \leq 2$ and $\gcd(p_1,p_2) = 1$. If $k = 2$ and $\gcd(p_1,p_2) = p$, then $\pi_1(E(n)_{p_1,p_2}) \cong \mathbb{Z}_p$, and similar statements (to Theorems 3.3.6 through 3.3.8) hold for these elliptic surfaces with finite fundamental group. If $k \geq 3$, then the classification is much easier: The diffeomorphism type of $E(n)_{p_1,\ldots,p_k}$ is determined by n and the fundamental group $\pi_1(E(n)_{p_1,\ldots,p_k})$ (cf. Theorem 8.3.12).

(b) In Lemma 8.3.10 we will see that all diffeomorphisms of the boundary $\partial\Phi(n)$ of the Milnor fiber extend to $\Phi(n)$, proving that $E(n)_{p_1,\ldots,p_k}$ is uniquely determined from its nucleus by gluing $\Phi(n)$ to its boundary. (See also [**G9**].)

(c) We mainly focused on the logarithmic transformations with nonzero multiplicity — these are the ones which have a holomorphic interpretation. In [**G9**] it is shown by Kirby calculus that a logarithmic transformation with multiplicity 0 transforms the elliptic surface into a trivial connected sum: $E(n)_0 \approx (2n-1)\mathbb{CP}^2\#(10n-1)\overline{\mathbb{CP}^2}$ (cf. Exercise 8.3.16(d)).

3.4. Classification of complex surfaces

In this final section we describe results concerning the Enriques-Kodaira classification of compact complex surfaces. We assume that the reader is familiar with the basic notions of algebraic geometry (see, e.g., [**GH**]).

Definition 3.4.1. Two complex surfaces S_1, S_2 are *deformation equivalent* if there are connected complex spaces $\mathcal{T}$ and $\mathcal{C}$, a surjective, proper holomorphic map $\pi\colon \mathcal{T} \to \mathcal{C}$ with smooth fibers and points $t_1, t_2 \in \mathcal{C}$ such that S_i is biholomorphic to $\pi^{-1}(t_i)$ $(i = 1, 2)$.

Two complex surfaces are considered to be the same (from the classification point of view we adopt now) if they are deformation equivalent. It is not hard to see that if S_1 and S_2 are deformation equivalent, then S_1 is diffeomorphic to S_2 [**FM1**]. In the following we will describe the deformation equivalence classes of compact complex surfaces. In order to discuss the classification, we first define the Kodaira dimension of a complex surface. The *canonical bundle* K_S is by definition the determinant line bundle of the cotangent bundle T^*S (in particular, $c_1(K_S) = -c_1(S)$). The space of holomorphic sections of the tensor power $K_S^{\otimes n}$ is a finite-dimensional (complex) vector space, whose dimension is denoted by $P_n(S)$.

Definition 3.4.2. The *Kodaira dimension* $\kappa(S)$ of the complex surface S is defined in the following way:

- $\kappa(S) = -\infty$ if $P_n(S) = 0$ for all n.
- $\kappa(S) = 0$ if some $P_n(S)$ is nonzero and $\{P_n(S)\}$ is a bounded sequence.
- $\kappa(S) = 1$ if $\{P_n(S)\}$ is unbounded but $\{P_n(S)/n\}$ is bounded.
- $\kappa(S) = 2$ if $\{P_n(S)/n\}$ is unbounded.

Remarks 3.4.3. (a) A (partially defined) map $\phi_L\colon S^* \to \mathbb{CP}^n$ is defined by a holomorphic line bundle $L \to S$ in the following way: Fix a basis $f_0, \ldots, f_n$ for the vector space $H^0(L)$ of holomorphic sections of L and associate $[f_0(x) : \ldots : f_n(x)] \in \mathbb{CP}^n$ to $x \in S^* = \{p \in S \mid \text{there is an } i \text{ with } f_i(p) \neq 0\}$. (Note that although $f_i(x)$ is not a well-defined element of $\mathbb{C}$, the ratio $[f_0(x) : \ldots : f_n(x)] \in \mathbb{CP}^n$ makes sense if $x \in S^*$.) The maps $\phi_n = \phi_{K_S^{\otimes n}}$ induced by the powers of the canonical line bundle are called the *pluricanonical maps*. Now $\kappa(S)$ can be given as the maximum over all n of $\dim \phi_n(S_n^*)$, and $\kappa(S) = -\infty$ if $S_n^* = \emptyset$ for all n [**BPV**], [**Bea**]. Note that the condition $\kappa(S) = 0$ implies that each $P_n(S) \in \{0, 1\}$: If $P_m(S) \geq 2$ for some m, then there are linearly independent sections $f_1, f_2 \in H^0(K_S^{\otimes m})$, hence $K_S^{\otimes nm}$ admits linearly independent sections $f_1^{\otimes i} f_2^{\otimes (n-i)}$ $(i = 0, \ldots, n)$; consequently $P_{nm}(S) \geq n + 1$, i.e., $P_n(S)$ is unbounded.

(b) A similar definition of the Kodaira dimension for complex curves gives the following: $\kappa(\mathbb{CP}^1) = -\infty$ (since the tensor powers $K_{\mathbb{CP}^1}^{\otimes n}$ do not admit holomorphic sections). If C is a complex curve of genus 1 (i.e., a 2-dimensional torus), then K_C is (holomorphically) trivial, hence the space of holomorphic sections of $K_C^{\otimes n}$ is 1-dimensional; this implies $\kappa(C) = 0$.

For a complex curve of genus ≥ 2 we find that $P_n(C) = \dim H^0(K_C^{\otimes n})$ is unbounded, hence $\kappa(C) = 1$.

It is easy to verify that $\kappa(\mathbb{CP}^2) = -\infty$ and $\kappa(\mathbb{CP}^1 \times \mathbb{CP}^1) = -\infty$; since the canonical bundle of the $K3$-surface $E(2)$ is trivial, each $P_n(E(2)) = 1$, hence $\kappa(E(2)) = 0$. A standard algebraic geometric argument shows that the blow-up process does not change the Kodaira dimension, so if S' is a blow-up of S, then $\kappa(S') = \kappa(S)$. Recall that a surface S is called *minimal* if there is no rational -1-curve in S (which could be blown down). Every complex surface S admits a minimal model S_{min}, but — as we have already mentioned — the minimal model might not be unique, cf. the explanation before Exercise 2.2.12. (It turns out, however, that if $\kappa(S) \geq 0$, then S has a unique minimal model [**BPV**].) In the following we will focus on minimal surfaces.

Definition 3.4.4. The *holomorphic Euler characteristic* $\chi_h(S)$ of a surface S is given by $\chi_h(S) = \frac{1}{12}(c_1^2[S] + c_2[S])$. (Note that by Theorem 1.4.13, $\chi_h(S) = \frac{\sigma(S)+\chi(S)}{4} = \frac{1}{2}(1 - b_1(S) + b_2^+(X)) \in \mathbb{Z}$.)

Exercise 3.4.5. Suppose that C_1 and C_2 are complex curves. Determine the Kodaira dimension and holomorphic Euler characteristic of $C_1 \times C_2$. (*Hint*: Suppose that $g(C_1) \leq g(C_2)$. Consult Remark 3.4.3(b) and conclude that $\kappa(C_1 \times C_2)$ is $-\infty$ if $g(C_1) = 0$, it is 0 or 1 if C_1 is an elliptic curve (i.e., it is of genus 1) and $\kappa(C_1 \times C_2) = 2$ if $g(C_1), g(C_2) \geq 2$.) The above computation can be extended to determine the Kodaira dimension $\kappa(X(n,m))$ of the complex surfaces $X(n,m)$ introduced in Section 3.2. Assuming $n \leq m$ we get that $\kappa(X(n,m)) = -\infty$ if $n = 1$, $\kappa(X(n,m)) = 0$ or 1 for $n = 2$ and $\kappa(X(n,m)) = 2$ if $n \geq 3$.

A surface S is *geometrically ruled* if there is a holomorphic map $\pi \colon S \to C$ to a complex curve C such that $\pi^{-1}(p) \approx \mathbb{CP}^1$ for every $p \in C$. Appealing to Remark 3.4.3(b), it is easy to see that if S is geometrically ruled, then $\kappa(S) = -\infty$. As the following theorem shows, the converse of the above statement essentially holds for simply connected surfaces. (The general case will be given in Theorem 3.4.29.)

Theorem 3.4.6. *If S is a simply connected, minimal surface with $\kappa(S) = -\infty$, then S is either geometrically ruled or biholomorphic to $\mathbb{CP}^2$.* □

Hence, in order to understand surfaces with $\kappa = -\infty$, we only need to discuss geometrically ruled surfaces.

Example 3.4.7. Consider the holomorphic line bundle $L_n \to \mathbb{CP}^1$ with $c_1(L_n) = n$ ($n \geq 0$). Let $\mathbb{F}_n$ denote the (fiberwise) projectivization of the $\mathbb{C}^2$-bundle $L_n \oplus \underline{\mathbb{C}} \to \mathbb{CP}^1$. The complex surface $\mathbb{F}_n$ obviously inherits a map

$\mathbb{F}_n \to \mathbb{CP}^1$ which turns it into a geometrically ruled surface. The surfaces $\mathbb{F}_n$ $(n \geq 0)$ are called *Hirzebruch surfaces*; in particular, $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. (For the discussion of ruled surfaces over curves of positive genus, see the text after Theorem 3.4.29.)

Theorem 3.4.8. *If S is a simply connected, minimal, geometrically ruled surface, then S is biholomorphic to a Hirzebruch surface $\mathbb{F}_n$. The surface $\mathbb{F}_n$ is biholomorphic to $\mathbb{F}_m$ iff $n = m$. $\mathbb{F}_n$ is deformation equivalent to $\mathbb{F}_m$ iff $n \equiv m \pmod 2$. We have $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{F}_1 = (\mathbb{CP}^2)' = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (cf. the explanation before Exercise 3.1.2); moreover, $\mathbb{F}_n$ is minimal iff $n \neq 1$ (cf. Exercise 7.1.10(c)).* □

The above description asserts that $\mathbb{F}_1 = (\mathbb{CP}^2)' = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is nonminimal; consequently minimal and nonminimal surfaces can be deformation equivalent — for example $\mathbb{F}_3$ and $\mathbb{F}_1$. By Theorems 3.4.6 and 3.4.8, the simply connected minimal surfaces with Kodaira dimension $-\infty$ comprise three deformation equivalence classes, represented by $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. (The discussion of the nonsimply connected case is postponed until the end of the section.) Note that although $\mathbb{CP}^2$ is not ruled, it can be blown up to admit a ruling, cf. Section 3.1. Hence Theorem 3.4.6 can be rephrased as follows: A simply connected complex surface has Kodaira dimension $-\infty$ iff (after possibly blowing up) it admits a (possibly singular) $\mathbb{CP}^1$-fibration. Next we turn to the cases $\kappa(S) = 0$ and 1.

Theorem 3.4.9. *If S is a simply connected, minimal surface with $\kappa(S) = 0$, then S is a $K3$-surface. All $K3$-surfaces are deformation equivalent, hence diffeomorphic.* □

We have already given several constructions of $K3$-surfaces (S_4 in Section 1.3, $E(2)$ in Section 3.1, $X(2)$ and $V(2)$ in Section 3.2). We will see more about these complex surfaces in Chapters 7 and 8.

Theorem 3.4.10. *If S is a minimal complex surface with $\kappa(S) = 1$, then S is an elliptic surface.* □

Remark 3.4.11. Note that $\kappa(S) = 1$ implies (by Remark 3.4.3(a)) that for appropriate n the pluricanonical map ϕ_n maps S to a complex curve C. By determining the genus of the generic fiber, one can complete the above reasoning to show that a complex surface with $\kappa(S) = 1$ admits an elliptic fibration [**GH**]. Note that the converse is not true: the $K3$-surface $E(2)$ and the rational surface $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ both admit elliptic fibrations, although the Kodaira dimensions of these surfaces are 0 and $-\infty$, respectively.

Recall that the family $\mathcal{F} = \{E(n)_{p,q} \mid n, p, q \in \mathbb{N},\ p \leq q,\ \gcd(p,q) = 1\}$ of simply connected elliptic surfaces was defined in Section 3.3. It turns out that the family $\mathcal{F}$ contains a representative from each deformation equivalence class of simply connected elliptic surfaces. More precisely,

Theorem 3.4.12. *If S is a minimal, simply connected elliptic surface, then there is a surface $E(n)_{p,q} \in \mathcal{F}$ such that S and $E(n)_{p,q}$ are deformation equivalent. Moreover, $E(n)_{p,q}$ and $E(n')_{p',q'}$ are deformation equivalent iff*

- *$n = n'$ and $\{p, q\} = \{p', q'\}$ or*
- *$n = n' = 1$ and $\min(p, q) = \min(p', q') = 1$, i.e., $E(1)_{1,q}$ is deformation equivalent to $E(1)_{1,q'}$ for all q, q'.* □

Coupling the above result with Corollary 3.3.7 and Theorem 3.3.8, we have

Theorem 3.4.13. *Two simply connected complex surfaces S_1 and S_2 with $\kappa(S_i) \leq 1$ are deformation equivalent iff S_1 is diffeomorphic to S_2.* □

(Again, the case of elliptic surfaces with nontrivial fundamental group will be considered later, see Definition 3.4.26 and the subsequent text.)

Finally, we turn our attention to the $\kappa = 2$ case.

Definition 3.4.14. If $\kappa(S) = 2$, then S is a *surface of general type*.

Examples 3.4.15. By Exercise 3.4.5, the product $C_1 \times C_2$ of two complex curves of genus ≥ 2 is a surface of general type. Similarly, the surfaces $X(n, m)$ $(n, m \geq 3)$ are surfaces of general type — as we will see, $\pi_1(X(n, m)) = 1$ (cf. Exercise 7.4.16); hence we have found examples of simply connected complex surfaces of general type.

Exercise 3.4.16. Prove that the complex surface $C_1 \times C_2$ is minimal when $g(C_i) \geq 1$ $(i = 1, 2)$. (*Hint*: Determine $\pi_2(C_1 \times C_2)$ and conclude that $C_1 \times C_2$ does not contain a homologically essential sphere.) A somewhat more complicated argument shows that $X(n, m)$ is minimal when $n, m \geq 2$. For the case $n = 1$ see Exercise 7.3.8(b).

Not much is known about the classification of minimal surfaces of general type. Here we give a few relevant results and examples; more examples will be given in Chapter 7. The following theorem (due to Gieseker) shows that for a fixed topological type there are only finitely many (deformation equivalence classes of) surfaces of general type — in contrast to what we saw for elliptic surfaces.

Theorem 3.4.17. *For fixed nonnegative integers b_1 and b_2, there are only finitely many deformation equivalence classes of surfaces of general type with the given b_1, b_2 as first and second Betti numbers.* □

Remark 3.4.18. It can be shown that the condition $\kappa(S) = 2$ implies that $P_n(S) = \frac{n(n-1)}{2} c_1^2(S) + \chi_h(S)$, where $c_1^2(S)$ and $\chi_h(S)$ depend only on $\chi(S)$ and $\sigma(S)$ [**BPV**]. By the equivalent formulation of $\kappa(S)$ (given in Remark 3.4.3(a)) we know that if S is a surface of general type, then for some n the image of the pluricanonical map ϕ_n is 2-dimensional. By a result of Bombieri [**Bm**], in fact, the 5-canonical map ϕ_5 essentially embeds each complex surface S of general type in $\mathbb{CP}^{P_5(S)-1}$ with degree d, with d only depending on $\chi(S)$ and $\sigma(S)$. By applying a result of Gieseker, it can be proved that for n and d fixed the projective space $\mathbb{CP}^n$ contains only finitely many deformation types of complex surfaces of degree d. Since $P_5(S)$ depends only on the topology of S, the above argument leads to the proof of Theorem 3.4.17, and also shows that a surface S of general type is projective, implying that it is Kähler, hence, for example, $b_1(S)$ is even. By the procedure given in Section 8.1, a surface of general type — as a submanifold of some $\mathbb{CP}^n$ — can be equipped with a Lefschetz pencil, hence (after blowing up) with a Lefschetz fibration. The above classification results and Theorem 3.4.19 now show that the genus of such a fibration is necessarily at least 2.

The question of determining the simply connected topological manifolds (in particular, intersection forms) corresponding to minimal surfaces of general type is called (after Persson [**Pe**]) the *geography question* for minimal surfaces of general type. Recall that the homeomorphism type of a simply connected smooth 4-manifold X is determined by three data: its Euler characteristic $\chi(X)$, its signature $\sigma(X)$ and the parity of its intersection form. The geography literature mainly discusses the possible values of χ and σ of minimal surfaces of general type, so we only return to the parity question briefly (cf. Theorem 7.4.18). For historical reasons, geography deals with the characteristic numbers $\chi_h(S) = \frac{1}{4}(\sigma(S) + \chi(S))$ and $c_1^2(S) = 3\sigma(S) + 2\chi(S)$ rather than the equivalent invariants $\chi(S)$ and $\sigma(S)$. Thus we would like to determine the set of pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ corresponding to minimal surfaces S of general type with $\chi_h(S) = a$ and $c_1^2(S) = b$.

Theorem 3.4.19. ([**BPV**]) *Suppose that S is a minimal surface of general type. Then $c_1^2(S) > 0$, $c_2(S) > 0$ and $2\chi_h(S) - 6 \leq c_1^2(S) \leq 9\chi_h(S)$. (See Figure 3.3.)* □

(The above inequalities are frequently called the *Noether* and *Bogomolov-Miyaoka-Yau inequalities* [**BPV**].) As we will see in Section 7.4, almost all pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $b > 0$ and $2a - 6 \leq b \leq 9a$ actually correspond to surfaces of general type (via $\chi_h = a$ and $c_1^2 = b$). If we restrict ourselves to simply connected surfaces, the answer becomes less satisfactory and more complicated — for the precise statement see Section 7.4. (See also [**Pe**].) The above results only deal with $\chi_h(S)$ and $c_1^2(S)$; from these the parity of Q_S cannot always be recovered. By Rohlin's Theorem 1.2.29, if S is spin then $\sigma(Q_S)$ is divisible by 16; thus Rohlin's Theorem allows an even form

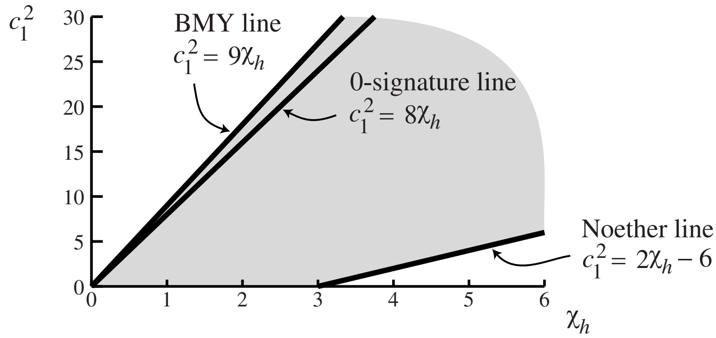


Figure 3.3. Geography of minimal surfaces of general type.

Q_S iff 8 divides $c_1^2(S)$ and $\frac{c_1^2(S)}{8} \equiv \chi_h(S) \pmod 2$. In these cases there are two intersection forms having equal χ_h and c_1^2; in all other cases (χ_h, c_1^2) determines Q_S. In [**PPX**] a discussion about the geography of *spin* surfaces can be found (cf. also Theorem 7.4.18). We will return to the geography problem for surfaces of general type in Section 7.4.

Summarizing the classification results we have quoted so far, we have

Theorem 3.4.20. *Any minimal, simply connected, complex surface is deformation equivalent to one of the surfaces* $\mathbb{CP}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, *a member of the family* $\mathcal{F} = \{E(n)_{p,q} \mid n, p, q \in \mathbb{N},\ \gcd(p,q) = 1\}$ *or a surface of general type.* (*Note that* $\mathcal{F}$ *contains the K3-surface as* $E(2)$.) □

One might hope for a natural generalization of Theorem 3.4.13 showing a very close connection between the smooth and the holomorphic structures of complex surfaces — this question is, however, still open (cf. Corollary 7.3.28 and the subsequent text):

Conjecture 3.4.21. *Two simply connected complex surfaces* S_1 *and* S_2 *are deformation equivalent iff* S_1 *is diffeomorphic to* S_2.[1]

Before outlining the Enriques-Kodaira classification of complex surfaces (without the simple connectivity assumption), we describe the Seiberg-Witten invariants of surfaces of general type.

Theorem 3.4.22. ([**Wi**], [**Mr1**]) *If* K *is a Seiberg-Witten basic class of a minimal surface* S *of general type with* $b_2^+(S) > 1$, *then* $K = \pm c_1(S)$. *Moreover,* $SW_S(\pm c_1(S)) = \pm 1$. □

[1] A recent preprint of Manetti [**Man**] claims that the conjecture is false under slightly weaker hypotheses. He asserts that for each k there exists a smooth 4-manifold X_k with $b_1(X_k) = 0$ supporting at least k deformation inequivalent complex structures.

Note that by Theorem 2.4.7, the classes $\pm c_1(S)$ are always basic classes of a Kähler surface. The above theorem states that for a minimal surface of general type these are the only basic classes. By analyzing the Seiberg-Witten function for simply connected complex surfaces (remembering that the case $b^+ = 1$ needs special attention), one can prove

Theorem 3.4.23. ([**FM2**]) *The cohomology classes $\pm c_1(S)$ of a minimal, simply connected complex surface S are diffeomorphism invariants; more precisely, if $f\colon S \to S'$ is an orientation preserving diffeomorphism between minimal, simply connected complex surfaces, then $f^*(c_1(S')) = \pm c_1(S)$.* □

More examples of simply connected surfaces of general type are provided by the complete intersections described in Section 1.3. We have already seen that $S(2) \approx \mathbb{CP}^1 \times \mathbb{CP}^1$, $S(3) \approx \mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$, $S(2,2) \approx \mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$ and $S(4)$, $S(2,3)$ and $S(2,2,2)$ are $K3$-surfaces. For all other multidegrees we have

Theorem 3.4.24. *The complete intersection surface $S(d_1, \dots, d_{n-2})$ is a minimal, simply connected surface of general type, except in the above cases (i.e., when $(d_1, \dots, d_{n-2})$ is (2), (3), (4), $(2,2)$, $(2,3)$ or $(2,2,2)$).* □

Ebeling [**Eb**] observed that the complete intersection surfaces with multidegrees (3,3,6,7,7,10) and (2,2,3,3,3,3,3,5,9) are homeomorphic but nondiffeomorphic 4-manifolds. This statement can be proved by computing the characteristic numbers of the surfaces at hand (see Exercise 1.3.13); applying Freedman's Theorem 1.2.27 then provides the desired homeomorphism. By Theorem 3.4.23 the first Chern class of a minimal complex surface is a diffeomorphism invariant, but in the above two examples $c_1(S)$ has different divisibilities in $H^2(S;\mathbb{Z})$ (see Exercise 1.3.13(e)), consequently these complete intersections are nondiffeomorphic.

We close this section with a quick overview of the classification of complex surfaces in general (without the simple connectivity assumption). The definitions of the Kodaira dimension $\kappa(S)$ and of surfaces of general type were given regardless of the fundamental group; the same can be said about Theorems 3.4.17, 3.4.19, 3.4.22, 3.4.23 and Remark 3.4.18. In the rest of this section we will reconsider complex surfaces with $\kappa \leq 1$ of arbitrary fundamental group — we will say more about surfaces of general type in Section 7.4. If the Kodaira dimension κ is 0 or 1, we have the following result.

Theorem 3.4.25. *If $\kappa(S) = 0$ or 1, then the minimal complex surface S is deformation equivalent to a complex surface which admits a (holomorphic) elliptic fibration (in the sense of Definition 3.1.1).* □

Definition 3.4.26. A surface S with $\kappa(S) = 1$ is called *properly elliptic* (cf. also Remark 3.4.11).

The diffeomorphism type of an elliptic surface S with $\chi(S) > 0$ and $|\pi_1(S)| = \infty$ is determined by its fundamental group (cf. Theorem 8.3.12). The case of finite fundamental group is much more interesting; results about the smooth structures of elliptic surfaces with finite fundamental group were given in Section 3.3. Every elliptic surface with $\chi > 0$ can be constructed (up to diffeomorphism) from the basic examples $E(1)$ and $T^2 \times T^2$ using the fiber sum and the logarithmic transformation operations. For more about (properly) elliptic surfaces see [**FM1**]. Surfaces with $\kappa(S) = 0$ — besides the $K3$-surfaces — are the primary and secondary Kodaira surfaces, hyperelliptic surfaces, complex tori and Enriques surfaces. (For a detailed description of each of these classes see [**BPV**].) A complex surface S is called a *hyperelliptic* (resp. *primary Kodaira*) *surface* if it has $b_1(S) = 2$ (resp. $b_1(S) = 3$) and admits a locally trivial T^2-fibration over T^2. There are seven deformation equivalence classes of hyperelliptic surfaces [**GH**], [**Bea**]; all these surfaces are projective. A surface is called a *secondary Kodaira surface* if it admits a finite cover which is a primary Kodaira surface. A secondary Kodaira surface admits a T^2-fibration over $\mathbb{CP}^1$ and has first Betti number equal to 1. A *complex torus* is a quotient of $\mathbb{C}^2$ by a lattice isomorphic to $\mathbb{Z}^4$, hence it is diffeomorphic to T^4. A complex surface S is an *Enriques surface* if the double cover of S is a $K3$-surface.

Lemma 3.4.27. (see [**BPV**]) *All Enriques surfaces are deformation equivalent, hence diffeomorphic. The intersection form of an Enriques surface is even. All Enriques surfaces are projective.* □

Obviously an Enriques surface S has $\pi_1(S) \cong \mathbb{Z}_2$. Since the double cover of S is a $K3$-surface, we have $\chi(S) = 12$, $\sigma(S) = -8$, hence $Q_S = (-E_8) \oplus H$. Note that Rohlin's Theorem 1.4.28 implies that an Enriques surface is not spin. In fact, an Enriques surface is diffeomorphic to $E(1)_{2,2}$ of Section 3.3. Note that the above classification results imply Theorem 3.4.25. By Theorem 10.1.4 we also see that surfaces with $\kappa(S) = 0$ are Kähler except for the primary and secondary Kodaira surfaces.

Example 3.4.28. Suppose that p_i, q_i $(i = 1, 2, 3)$ are quadratic polynomials in 3 variables and the submanifold $S = \{[z_0 : \ldots : z_5] \in \mathbb{CP}^5 \mid p_i(z_0, z_1, z_2) + q_i(z_3, z_4, z_5) = 0,\ i = 1, 2, 3\} \subset \mathbb{CP}^5$ is smooth. (Recall that $S = S(2, 2, 2)$ is a $K3$-surface.) It is easy to see that if p_i, q_i satisfy $\{[x : y : z] \in \mathbb{CP}^2 \mid p_i(x, y, z) = 0,\ i = 1, 2, 3\} = \emptyset$ and $\{[x : y : z] \in \mathbb{CP}^2 \mid q_i(x, y, z) = 0,\ i = 1, 2, 3\} = \emptyset$, then the intersection $S \cap \{[z_0 : \ldots : z_5] \in \mathbb{CP}^5 \mid z_0 = z_1 = z_2 = 0 \text{ or } z_3 = z_4 = z_5 = 0\}$ is empty. (Generic choices of p_i, q_i satisfy these properties.) Now the $\mathbb{Z}_2$-action $[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [z_0 : z_1 : z_2 : -z_3 : -z_4 : -z_5]$ (given on $\mathbb{CP}^5$) induces a fixed-point free involution on S, hence the quotient $S/\mathbb{Z}_2$ is an Enriques surface.

Finally, we list the surfaces with $\kappa = -\infty$.

Theorem 3.4.29. *A minimal complex surface S with $\kappa(S) = -\infty$ is either $\mathbb{CP}^2$, geometrically ruled or of Class VII.* □

The generalization of Theorem 3.4.8 shows that a geometrically ruled surface S is biholomorphic to the projectivization $\mathbb{P}(E)$ of a $\mathbb{C}^2$-bundle $E \to C$ (where C is called the *base curve* of the ruling), and it is deformation equivalent to one of the form $\mathbb{P}(L \oplus \underline{\mathbb{C}})$, where $L \to C$ is a holomorphic line bundle. Moreover, $\mathbb{P}(L_1 \oplus \underline{\mathbb{C}})$ and $\mathbb{P}(L_2 \oplus \underline{\mathbb{C}})$ are deformation equivalent iff $c_1(L_1) \equiv c_1(L_2) \pmod 2$. For a geometrically ruled surface S we have $H_2(S;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$; Q_S is even iff the corresponding Chern number $c_1(L)[C]$ is even. Hence the deformation type of a ruled surface S is determined by the parity of its intersection form Q_S and by its first Betti number $b_1(S)$; this last integer is equal to $2g(C)$. An easy computation shows that the signature $\sigma(S)$ of a geometrically ruled surface is 0, and $\chi(S) = 2\chi(C)$. Consequently we have $c_1^2(S) = 4\chi(C) = 8 - 8g(C)$ and $\chi_h(S) = 1 - g(C)$. The genus function G (introduced in Section 2.1) has been determined for geometrically ruled surfaces over base curves of arbitrary genus [**LL**]. Extending the notation introduced at the beginning of Section 3.1, we can represent the two deformation equivalence classes of geometrically ruled surfaces over the curve C by the 4-manifolds $C \times \mathbb{CP}^1$ and $C \tilde{\times} \mathbb{CP}^1$ — the former is spin, and the latter is a nonspin 4-manifold. Along the lines of the observation following Theorem 3.4.8, we get that S has $\kappa(S) = -\infty$ iff it can be blown up to admit a (possibly singular) $\mathbb{CP}^1$-fibration or it is of Class VII.

Exercise 3.4.30. Show that $(C \times \mathbb{CP}^1)\#\overline{\mathbb{CP}^2}$ and $(C \tilde{\times} \mathbb{CP}^1)\#\overline{\mathbb{CP}^2}$ are diffeomorphic. (*Hint*: Take a section of the trivial $\mathbb{CP}^1$-bundle $C \times \mathbb{CP}^1 \to C$ and blow up the intersection of a fiber with that section. Show that after blowing down the proper transform of the fiber we get $C \tilde{\times} \mathbb{CP}^1$.)

A *surface of Class VII* is by definition a complex surface S with $\kappa(S) = -\infty$ and $b_1(S) = 1$. An example of a surface of Class VII can be constructed by taking the quotient of $\mathbb{C}^2 - \{0\}$ by the $\mathbb{Z}$-action generated by $(z_1, z_2) \mapsto (\frac{1}{2}z_1, \frac{1}{2}z_2)$. A surface S with $\mathbb{C}^2 - \{0\}$ as its universal cover is called a *Hopf surface*; these surfaces are of Class VII. The above example of a Hopf surface is, in fact, diffeomorphic to $S^1 \times S^3$, giving a complex structure on this latter manifold; note that $b_2(S^1 \times S^3) = 0$. Any surface S of Class VII has $b_2^+(S) = 0$, i.e., its intersection form is negative definite. There are surfaces of Class VII other than Hopf surfaces (cf. [**BPV**]); the complete classification of these surfaces, however, is still lacking. We only note that a surface of Class VII has $\chi_h = 0$, $c_1^2 \leq 0$ and $c_2 \geq 0$.

In summary, a minimal complex surface S is either

(1) $\mathbb{CP}^2$,

(2) geometrically ruled,

(3) of Class VII,

(4) deformation equivalent to a surface with an elliptic fibration, or

(5) a surface of general type.

As we remarked earlier, surfaces in classes (1) and (2) admit $\mathbb{CP}^1$-fibrations (after possibly blowing up); surfaces in (4) can be equipped with (possibly singular) torus fibrations. Finally, surfaces of general type (i.e., in class (5)) admit Lefschetz fibrations of genus ≥ 2 — after possibly blowing up. For the characteristic numbers of complex surfaces the following can be proved.

Theorem 3.4.31. *If S is not a ruled surface, then we have $c_2(S) \geq 0$ and $\chi_h(S) \geq 0$. (Recall that for a minimal ruled surface, $c_2(S) = 4 - 4g(C)$ and $\chi_h(S) = 1 - g(C)$.) If S is minimal and $\kappa(S) \geq 0$, then $c_1^2(S) \geq 0$.* □

We close this overview with a statement which will be useful later on:

Theorem 3.4.32. *A complex surface S is deformation equivalent to a projective surface iff $b_1(S)$ is even. If $b_1(S)$ is odd, the complex surface S is either elliptic or of Class VII.* □

The books [**FM1**], [**BPV**] and [**Bea**] give systematic and complete descriptions of complex surfaces — here we have just highlighted the aspects which might be most interesting from the topological point of view we have adopted in this volume.

Part 2

Kirby Calculus

The aim of Kirby calculus is to understand smooth 4-manifolds by decomposing them into simple pieces (diffeomorphic to balls after smoothing corners) and analyzing the resulting gluing maps. The analogous procedure in arbitrary dimensions is called handlebody theory and is equivalent to Morse theory. The main theorems of handlebody theory are proved in careful detail in Milnor's *Lectures on the h-Cobordism Theorem* [**M4**] (from the Morse theory viewpoint) and in Rourke and Sanderson [**RS**] (in the PL-category, which is equivalent to the smooth category in dimensions ≤ 4), so we will sometimes refer to these references for detailed proofs. Throughout Part 2 we assume (except where otherwise stated) that all manifolds and maps are smooth. Local diffeomorphisms between oriented manifolds are assumed to preserve orientation unless otherwise stated. Recall that Figure 0.1 approximately indicates the logical dependence of the sections of Part 2.

Chapter 4

Handlebodies and Kirby diagrams

4.1. Handles

Definition 4.1.1. For $0 \le k \le n$, an *n-dimensional k-handle* h is a copy of $D^k \times D^{n-k}$, attached to the boundary of an n-manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$.

There is a canonical way to smooth corners (cf. Remark 1.3.3), so we will interpret $X \cup_\varphi h$ as a smooth n-manifold. Note that there is a deformation retraction of $X \cup_\varphi h$ onto $X \cup_{\varphi|\partial D^k \times 0} D^k \times 0$, so up to homotopy, attaching a k-handle is the same as attaching a k-cell. (It is often useful to think of a k-handle as being a k-cell "thickened up" to be n-dimensional.) As in Figure 4.1, we will call $D^k \times 0$ the *core* of the handle, $0 \times D^{n-k}$ the *cocore*, φ the *attaching map*, $\partial D^k \times D^{n-k}$ (or its image $\varphi(\partial D^k \times D^{n-k})$) the *attaching region*, $\partial D^k \times 0$ (or its image) the *attaching sphere* and $0 \times \partial D^{n-k}$ the *belt sphere*. (The attaching and belt spheres are sometimes called *descending* and *ascending spheres*, respectively.) The number k is called the *index* of the handle.

Exercise 4.1.2. Figure 4.1 shows a 2-dimensional 1-handle. Draw pictures of the other handles with $0 \le k \le n \le 3$. For $n = 4$, a handle has boundary $S^3 = \mathbb{R}^3 \cup \{\infty\}$. What do the attaching region and attaching and belt spheres look like in $\mathbb{R}^3$ for each k?

Since we are mainly interested in the diffeomorphism type of $X \cup_\varphi h$, it suffices to specify φ up to isotopy. More precisely, an isotopy between φ and φ' specifies (up to ambient isotopy) a diffeomorphism $X \cup_\varphi h \approx X \cup_{\varphi'} h$. (By the Isotopy Extension Theorem (see Definition 1.1.5), we can extend the isotopy of φ to an ambient isotopy $\Phi\colon I \times \partial X \to \partial X$. The

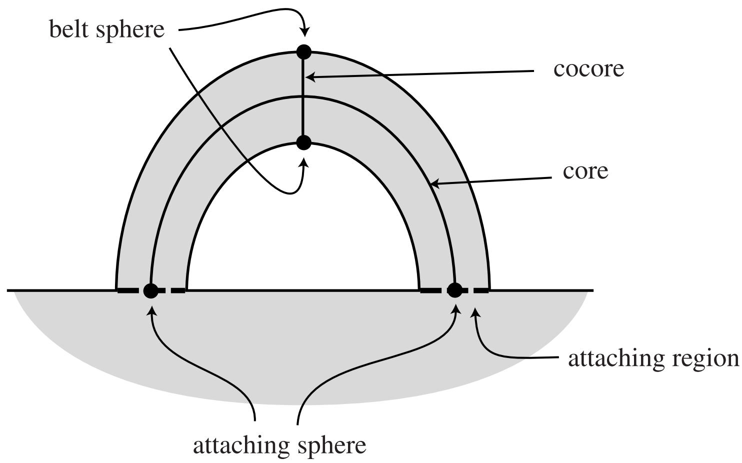


Figure 4.1. Anatomy of a handle.

required diffeomorphism can easily be constructed from the diffeomorphism $\mathrm{id}_I \times \Phi\colon I \times \partial X \to I \times \partial X$ by identifying $I \times \partial X$ with a neighborhood of ∂X in X, cf. [**RS**].) By the Tubular Neighborhood Theorem [**GP**], an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$ can be constructed from an embedding $\varphi_0\colon \partial D^k \times 0 \to \partial X$ together with an identification f of the normal bundle of $\operatorname{Im}\varphi_0$ with $\partial D^k \times \mathbb{R}^{n-k}$, and this data determines φ up to isotopy. (We require $\varphi|\partial D^k \times 0 = \varphi_0$ and $d\varphi|0 \times T_0 D^{n-k} = f^{-1}$.) Thus, $X \cup_\varphi h$ is specified by two pieces of data:

1. an embedding $\varphi_0\colon S^{k-1} \to \partial X$ (a *knot* in ∂X) with trivial normal bundle, and
2. a (*normal*) *framing* f of $\varphi_0(S^{k-1})$, or identification of the normal bundle $\nu\varphi_0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$.

Note that it makes sense to talk about an isotopy (smooth family parametrized by $[0,1]$) of framed embeddings (φ_0, f), and an isotopy from (φ_0, f) to (φ_0', f') determines (up to isotopy) a diffeomorphism between the resulting manifolds $X \cup_\varphi h$ and $X \cup_{\varphi'} h$. In particular, the diffeomorphism type of $X \cup_\varphi h$ depends only on the isotopy class of (φ_0, f).

Example 4.1.3. For $2(l+1) \leq m$, two homotopic embeddings $N^l \hookrightarrow M^m$ of an l-manifold into an m-manifold will always be isotopic. This is because a generic homotopy $F\colon I \times N \to M$ will be an embedding of $I \times N$ unless $2(l+1) = m$, in which case F will be an immersion with isolated double points. Each double point will correspond to a pair of points in $I \times N$ with distinct I-coordinates, so F will be an isotopy whenever $2(l+1) \leq m$. We conclude that for $2k \leq n-1$ the diffeomorphism type of $X^n \cup k$-handles is determined by X, the number of handles and the framing and homotopy class of φ_0 in ∂X for each. In particular, $D^n \cup k$-handles is determined by the number and framings of the k-handles ($2k \leq n-1$). In contrast, we will see that for handles of larger index (e.g. 2-handles in 4-manifolds) the isotopy class of φ_0 can contain much more information than just its homotopy class.

To understand framings on a sphere S^{k-1} in ∂X^n with trivial normal bundle (up to isotopy fixing S^{k-1}), we pick one framing f_0. When we compare any framing f with f_0, we obtain an element of $GL(n-k)$ at each point of S^{k-1}. By composing φ with a self-diffeomorphism of the second factor of $D^k \times D^{n-k}$, we can arrange for this element to be the identity I at a preassigned base point $p \in S^{k-1}$. Thus, we obtain an element of $\pi_{k-1}(GL(n-k)) \cong \pi_{k-1}(O(n-k))$, where we have suppressed the base point I from the notation. (The Gram-Schmidt procedure gives a deformation retraction from $GL(m)$ onto $O(m)$.) It is easy to check that this procedure determines a bijection from isotopy classes of framings of S^{k-1} in ∂X^n (fixed at p) onto $\pi_{k-1}(O(n-k))$, which is a known abelian group for small n. Note, however, that the bijection depends on our initial choice of framing f_0. In general, the set of framings will not have a canonical group structure, but rather a canonical group *action* by $\pi_{k-1}(O(n-k))$ that is free and transitive. We will see that under some circumstances there is a canonical choice of f_0, and then the set of framings will be canonically identified with $\pi_{k-1}(O(n-k))$.

Examples 4.1.4. (a) A 0-handle is attached to ∂X along $\partial D^0 \times D^n = \emptyset$, so attaching a 0-handle to an n-manifold is the same as taking the disjoint union with D^n. In particular, 0-handles are the only handles we can attach to the empty set.

(b) The attaching sphere of a 1-handle is $\partial D^1 \times 0$, which is a pair of points. If ∂X is connected and nonempty, then there is a unique isotopy class of embeddings $\varphi\colon \partial D^1 \times 0 \to \partial X$ (up to interchanging the points if $n = 2$ with ∂X noncompact). Since $\pi_0(O(n-1)) \cong \mathbb{Z}_2$ for $n \geq 2$, there are exactly two framings on $\varphi(\partial D^1 \times 0)$ (fixed at one point p). Thus, for each $n \geq 2$ there are exactly two n-manifolds that can be obtained by attaching a 1-handle to a 0-handle (or any other orientable manifold with connected boundary), distinguished by whether they are orientable. See Figure 4.2 for the $n = 2$ case, the annulus and Möbius band. Similarly, for $n \geq 3$ there is a unique orientable manifold obtained by attaching ℓ 1-handles to D^n, and this is diffeomorphic to the boundary sum $\natural \ell S^1 \times D^{n-1}$. (To construct the diffeomorphism, begin with the $n = 2$ case, then cross with D^{n-2}.) Note that taking a boundary sum is a special case of attaching a 1-handle.

(c) For $(n-1)$-handles with $n \neq 2$ and n-handles in general, there is a unique framing (since $\pi_{n-2}(O(1)) = \pi_{n-1}(O(0)) = 0$ except for $\pi_0(O(1)) \cong \mathbb{Z}_2$.) An n-handle is the same as D^n attached along $\partial D^n = S^{n-1}$. Thus, an n-handle

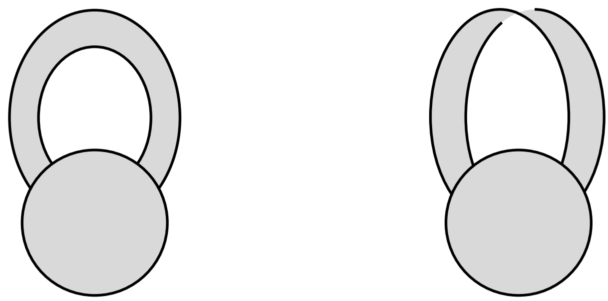


Figure 4.2. 1-handles attached to D^2.

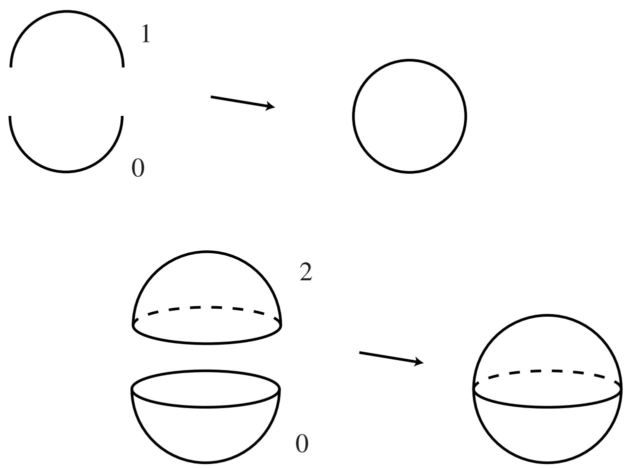


Figure 4.3. $S^n = 0\text{-handle} \cup_{\mathrm{id}} n\text{-handle}$, $n = 1, 2$.

can only be attached to a manifold X with a boundary component diffeomorphic to S^{n-1}. For $n \leq 4$, it is known that any self-diffeomorphism of S^{n-1} is isotopic to either the identity or a reflection, so there is a unique way to attach an n-handle to an S^{n-1} boundary component. In many higher dimensions, however, there are exotic self-diffeomorphisms of S^{n-1}. Attaching an n-handle to a 0-handle by the identity map yields S^n (Figure 4.3), but for most $n \geq 7$, exotic diffeomorphisms of S^{n-1} yield *exotic spheres* that are homeomorphic to S^n but not diffeomorphic to it ([**M1**], cf. Remark 9.2.10). In dimensions $n \geq 5$, detaching and reattaching an n-handle is equivalent to taking the connected sum with a possibly exotic n-sphere, so n-handle attaching is unique in dimensions ≤ 6.

(d) For $k < n$, there is a unique isotopy class of "unknotted" embeddings $S^{k-1} \to S^{n-1}$, characterized by the condition that they extend to embeddings $D^k \to S^{n-1}$. (It is not hard to show that there is a unique smooth embedding $\varphi\colon D^k \to S^{n-1}$ up to isotopy (and reflection if $k = n - 1$) by approximating φ by $d\varphi_0$.) Clearly, $\partial D^k \times 0 \subset \partial(D^k \times D^{n-k}) = S^{n-1}$ is such an unknotted sphere. If we identify a 0-handle D^n with $D^k \times D^{n-k}$ and attach a k-handle to it by an unknotted embedding of S^{k-1}, then by isotoping the framed attaching sphere we may assume that the attaching map is a diffeomorphism $\varphi\colon \partial D^k \times D^{n-k} \to \partial D^k \times D^{n-k}$ that projects to $\mathrm{id}_{\partial D^k}$ in the first factor and is an element of $O(n-k)$ on each copy of $\{\mathrm{pt.}\} \times D^{n-k}$. The resulting manifold X has a canonical projection $\pi\colon X \to S^k$ induced by projection onto the first factor in each handle, and each point preimage is a copy of D^{n-k}. Thus, if we attach a k-handle to a 0-handle by an unknotted embedding of the attaching sphere, we obtain a D^{n-k}*-bundle over* S^k (with structure group $O(n-k)$). Not surprisingly, such bundles $\pi : X \to S^k$ are classified by the framings of the k-handles, that is, they correspond bijectively to $\pi_{k-1}(O(n-k))$. (To see that any D^{n-k}-bundle over S^k is realized as $D^n \cup_\varphi h$, note that its restriction to each hemisphere must be trivial.) This time, the correspondence is canonical, since we have a canonical framing f_0 corresponding to the identity $\mathrm{id}_{\partial D^k \times D^{n-k}}$, which determines the trivial bundle $S^k \times D^{n-k}$.

(e) In addition to the special cases of bundles which appeared in previous examples ($k = 0, 1, n-1, n$), we consider the important case $k = 2$ — that is, D^{n-2}-bundles over S^2, obtained by attaching a 2-handle to D^n along an unknot. We have $\pi_1(O(n-2)) \cong \mathbb{Z}$ for $n = 4$ and $\mathbb{Z}_2$ for $n > 4$. Thus D^2-bundles X^4 over S^2 correspond bijectively to integers $e(X)$. It is not hard to verify that the intersection form on $H_2(X) \cong \mathbb{Z}$ is $\langle e(X)\rangle$. (Compute the self-intersection of the 0-*section*, the generating sphere $D^2 \times 0 \cup_\varphi D^2 \times 0$.) Thus, the oriented 4-manifolds X corresponding to different integers are all distinct (cf. Exercise 1.2.7(d)). The integer $e(X)$ is called the *Euler number* (or *Chern number*) of X (or more correctly, of the D^2-bundle $X \to S^2$). It is easy to see that X arises as the unit disk bundle of a complex (Hermitian) line bundle L over S^2, whose Chern number is $\langle c_1(L), [S^2]\rangle = e(X)$ (cf. Section 1.4). For $\ell \geq 3$, there are exactly two D^ℓ-bundles over S^2. One is the trivial bundle $S^2 \times D^\ell$, and the other is denoted $S^2 \tilde{\times} D^\ell$. We have already encountered the closed 4-manifold $S^2 \tilde{\times} S^2 = \partial(S^2 \tilde{\times} D^3)$ in connection with Hirzebruch surfaces (Example 3.4.7). We will examine it from a topological perspective in Exercise 4.2.6(b).

Remark 4.1.5. It is not hard to visualize $\pi_1(O(m))$. The natural inclusion $O(\ell) \to O(m)$, $2 \leq \ell < m$, preserves the generator g of π_1, which is rotation in the x_1-x_2 plane through an angle which increases continuously from 0 to 2π. To see that g^2 is the identity in $\pi_1(O(3))$ (hence, in $\pi_1(O(m))$, $m \geq 3$), think of g as a family of rotations about the z-axis. Now homotope g in $O(3)$ by continuously rotating $\mathbb{R}^3$ so that the z-axis is gradually turned over and restored to its original position upside down. Then g has been transformed into a rotation about the z-axis in the opposite direction, proving $g = g^{-1}$ in $\pi_1(O(3))$.

Exercise 4.1.6. Prove that for $2k \leq n - 1$ any n-manifold of the form $D^n \cup k$-handles is diffeomorphic to a boundary sum of D^{n-k}-bundles over S^k. (*Hint*: Example 4.1.3.)

4.2. Handle decompositions

Definition 4.2.1. Let X be a compact n-manifold with boundary ∂X decomposed as a disjoint union $\partial_+ X \coprod \partial_- X$ of two compact submanifolds (either of which may be empty). If X is oriented, orient $\partial_\pm X$ so that $\partial X = \partial_+ X \coprod \overline{\partial_- X}$ in the boundary orientation. A *handle decomposition* of X (relative to $\partial_- X$) is an identification of X with a manifold obtained from $I \times \partial_- X$ by attaching handles, such that $\partial_- X$ corresponds to $\{0\} \times \partial_- X$ in the obvious way. A manifold X with a given handle decomposition is called a *relative handlebody* built on $\partial_- X$, or if $\partial_- X = \emptyset$ it is called a *handlebody*.

Note that in the oriented case, the inclusion $I \times \partial_- X \to X$ preserves orientation. (For standard orientation conventions see [**GP**], for example.) Also note that since attaching a k-handle is the same up to homotopy as attaching a k-cell, a handle decomposition of $(X, \partial_- X)$ determines a relative cell complex on $\partial_- X$ with the same homotopy type. Beware that some 3-manifold topologists reserve the term "handlebody" for $D^3 \cup 1$-handles.

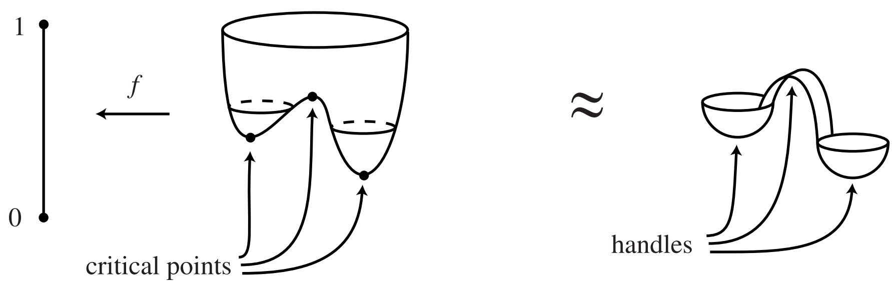


Figure 4.4. A handle decomposition from a Morse function.

Every smooth, compact manifold pair $(X, \partial_- X)$ as above admits a handle decomposition, by Morse Theory [**M2**], [**M4**]. The basic idea is that any smooth function $f\colon X \to [0,1]$ with $f^{-1}(0) = \partial_- X$ and $f^{-1}(1) = \partial_+ X$ can be perturbed into a special form called a *Morse function* (with no critical points on ∂X). By definition, the critical points of a Morse function are locally modeled on quadratic critical points, $-\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$, and these critical points will correspond to handles (with index equal to the

Morse index k of the critical point). See Figure 4.4. Similarly, a noncompact manifold with compact boundary will admit a proper Morse function $f\colon X \to [0,\infty)$ with $f^{-1}(0) = \partial X$, providing a theory of handle decompositions of noncompact manifolds. In contrast with the compact case, we may have infinitely many handles in the noncompact setting. This may lead to technical complications, since we can only fit finitely many attaching regions into a given compact boundary. Dropping the smoothness condition, we find that any PL-manifold pair admits a PL handle decomposition constructed from a triangulation [**RS**]. By difficult theorems of Moise [**Mo**] ($n = 3$), Kirby and Siebenmann [**KS**] ($n \geq 6$), Freedman and Quinn [**FQ**] ($n = 5$), a topological manifold pair $(X, \partial_- X)$ with $\dim X = n \neq 4$ always admits a topological handle decomposition (with attaching maps that are homeomorphic embeddings). However, if $n = 4$, then $(X, \partial_- X)$ admits a topological handle decomposition if and only if X is smoothable, since the attaching maps can always be smoothed by an isotopy. (Any homeomorphic embedding of smooth 3-manifolds is uniquely smoothable [**Mo**], so a handle decomposition of a topological 4-manifold determines a smooth structure.) For example, Freedman's closed 4-manifold with intersection form E_8 admits no handle decomposition (Theorems 1.2.27 and 1.2.29).

Example 4.2.2. The torus T^2 admits a handle decomposition with a 0-handle, two 1-handles and a 2-handle (Figure 4.5). The projective plane admits a decomposition with a 0-handle, one 1-handle and a 2-handle. (The first two handles form a Möbius band as in Figure 4.2.) Compare with the standard cell decompositions of T^2 and $\mathbb{RP}^2$.

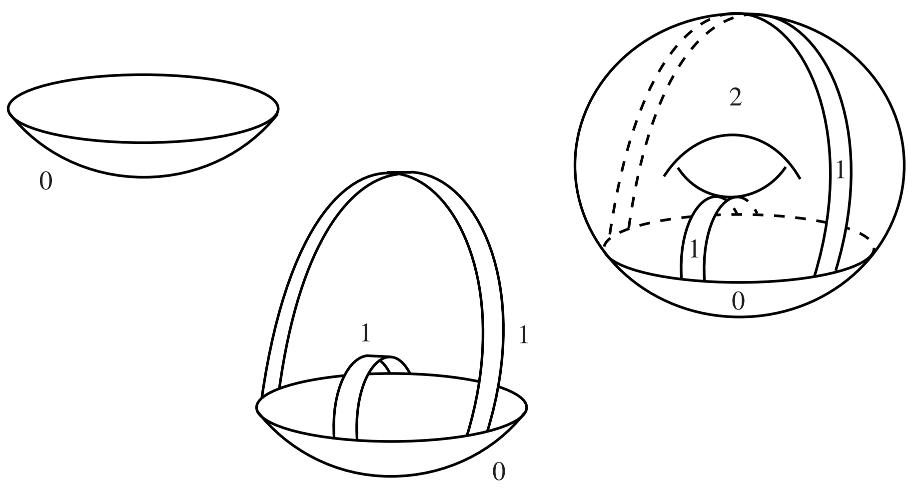


Figure 4.5. Handle decomposition of the torus.

Exercise 4.2.3. Find a handle decomposition for any closed surface. Find a decomposition for S^2 with four handles.

Example 4.2.4. The manifolds $\mathbb{RP}^n$ and $\mathbb{CP}^n$ (Section 1.3) each have a handle decomposition with $n+1$ handles. There is one handle of each index from 0 through n (for $\mathbb{RP}^n$) or each even index from 0 through $2n$ (for $\mathbb{CP}^n$). We construct such a decomposition for $\mathbb{CP}^n$; substituting $\mathbb{R}$ for $\mathbb{C}$ throughout gives the decomposition for $\mathbb{RP}^n$. Recall that $\mathbb{CP}^n$ is covered by $n+1$ local parametrizations $\psi_i \colon \mathbb{C}^n \to \mathbb{CP}^n$, $i = 0, \dots, n$, where $\psi_i(z_1, \dots, z_n) = [z_1 : \dots : z_i : 1 : z_{i+1} : \dots : z_n]$. Let D be the closed unit disk in $\mathbb{C}$, and let $B_i = \psi_i(D \times \cdots \times D) \subset \mathbb{CP}^n$. Each point $p \in \mathbb{CP}^n$ has homogeneous coordinates $[z_0 : \dots : z_n]$, which we normalize so that $\max_i |z_i| = 1$. Then $p \in B_i$ if and only if $|z_i| = 1$, and $p \in \operatorname{int} B_i$ if and only if $|z_j| < 1$ for all $j \neq i$. It follows immediately that the balls B_i cover $\mathbb{CP}^n$, and that they only intersect along their boundaries. Since B_k intersects $\bigcup_{i<k} B_i$ precisely on $\psi_k(\partial(D \times \cdots \times D) \times (D \times \cdots \times D))$ (with k copies of D in the first product), we can interpret B_k as a $2k$-handle attached to $\bigcup_{i<k} B_i$, exhibiting the required handle decomposition. Note that by symmetry under coordinate permutations, we can attach the balls B_i in any order, but the indices of the resulting handles will always increase as we add them.

Exercise 4.2.5. Explicitly compute the attaching maps and draw the handle decompositions for $\mathbb{RP}^1$, $\mathbb{CP}^1$ and $\mathbb{RP}^2$ (and $\mathbb{RP}^3$ for a more challenging example). What does the permutation symmetry look like in these pictures?

We conclude the example by determining the attaching map of the 2-handle in $\mathbb{CP}^2$. A point p in $B_0 \cap B_1$ can be written in two ways: $p = \psi_0(w_1, w_2) = [1 : w_1 : w_2]$ and $p = \psi_1(z_1, z_2) = [z_1 : 1 : z_2]$. Comparing homogeneous coordinates, we find that $w_1 = z_1^{-1}$ and $w_2 = z_1^{-1} z_2$, so $\varphi(z_1, z_2) = (z_1^{-1}, z_1^{-1} z_2)$ defines the attaching map $\varphi\colon \partial D \times D \to \partial D \times D$. This map preserves the fibration by disks $z \times D$, but as we travel once around ∂D ($z_1 = e^{2\pi i t}$, $0 \le t \le 1$), the identification of fibers ($z_2 \mapsto e^{-2\pi i t} z_2$) rotates once, realizing a generator of $\pi_1(O(2)) \cong \mathbb{Z}$. Thus, by Example 4.1.4(d), $B_0 \cup B_1$ is the D^2-bundle over S^2 with Euler class 1. To check the sign, observe that the holomorphic curves given by $w_2 = 1$ in B_0 and $z_1 = z_2$ in B_1 fit together to form a holomorphic section of the bundle (and a $\mathbb{CP}^1 \subset \mathbb{CP}^2$) whose intersection number with the 0-section is $+1$. (Compare with our proof that $Q_{\mathbb{CP}^2} = \langle 1 \rangle$ in Section 1.3.) This bundle is called the (positive) *Hopf disk bundle*. Compare with the Möbius band (a D^1-bundle over S^1) in $\mathbb{RP}^2$ (Exercise 4.2.5). We conclude that $\mathbb{CP}^2$ is obtained from a Hopf disk bundle by gluing a 4-ball (4-handle) onto its boundary. In particular, the boundary of the Hopf bundle must be diffeomorphic to S^3. Thus, we have exhibited S^3 as a circle bundle over S^2, the famous *Hopf fibration* of S^3.

Exercises 4.2.6. (a) For the D^2-bundle over S^2 with Euler number n, find an embedding into a *holomorphic line bundle* over S^2, i.e., a complex line bundle L over $S^2 = \mathbb{CP}^1$ whose transition functions are holomorphic, so that the projection $L \to \mathbb{CP}^1$ is a holomorphic map of complex manifolds. Show that L admits a holomorphic section (not identically zero) iff $n \geq 0$.

(b) Show that $S^2 \tilde{\times} S^2$, the twisted S^2-bundle over S^2, is diffeomorphic to the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. (*Hint*: Find an embedding $\mathbb{CP}^2 - \operatorname{int} D^4 \hookrightarrow S^2 \tilde{\times} S^2$.) Compare with the Klein bottle $S^1 \tilde{\times} S^1 = \mathbb{RP}^2 \# \mathbb{RP}^2$. (For a topological solution, see Example 4.6.3. For an algebro-geometric one, see the beginning of Section 3.1)

In our examples so far, we have observed certain symmetries, and the handles have been attached in order of increasing index. We will see that these are general phenomena.

Proposition 4.2.7. *Any handle decomposition of a compact pair $(X, \partial_- X)$ can be modified (by isotoping attaching maps) so that the handles are attached in order of increasing index. Handles of the same index can be attached in any order (or simultaneously).*

Proof. Suppose we attach a k-handle h to an n-manifold Y, followed by an ℓ-handle h', with $\ell \leq k$. Then the belt sphere of h has dimension $n-k-1$, and the attaching sphere of h' has dimension $\ell - 1$. Since $\dim \partial Y = n-1$ is strictly larger than the sum of the dimensions of these spheres, we can perturb them to be disjoint. It is now routine to construct an isotopy of $Y \cup_\varphi h$ that pushes the attaching sphere of h' off of h (for example, by integrating a vector field directed radially away from the cocore of h). □

We will always assume that handle decompositions are ordered in this manner, by increasing index, and we will use the notation X_k to denote the union of $I \times \partial_- X$ with all handles of index $\leq k$. Note that the proposition is not true for noncompact manifolds X with $\partial_- X$ compact, since there may be infinitely many handles of a given index, and we cannot attach them all along a compact boundary before proceeding to higher index handles. Sometimes we can attach infinitely many handles in finitely many stages, however, by (for example) replacing a 0-handle D^n by the half-space $(-\infty, 0] \times \mathbb{R}^{n-1}$. Proposition 4.2.7 generalizes to such a collection of handles.

As for symmetry, any handle decomposition on a compact pair $(X, \partial_- X)$ determines a *dual* handle decomposition on $(X, \overline{\partial_+ X})$ as follows. (We set $\overline{\partial_+ X} = \partial_+ X$ in the unoriented case.) First, we glue a collar $I \times \overline{\partial_+ X}$ to X (setting $\{1\} \times \partial_+ X$ equal to $\partial_+ X \subset X$), and remove the collar $I \times \partial_- X$ on which the handlebody is built. Then we notice that each k-handle $D^k \times D^{n-k}$ can be interpreted as an $(n-k)$-handle glued to the part of X *above* it (reversing the roles of core and cocore). Intuitively, we turn the handlebody "upside down." In terms of Morse theory, we are replacing a Morse function f by $1 - f$. Note that this inversion preserves the property of handles being attached in order of increasing index. As an example, Figure 4.6 shows $(S^1 \times D^1, \emptyset)$ described as 0-handle $\cup$ 1-handle, and its dual decomposition as $I \times S^1 \times \partial D^1 \cup$ 1-handle $\cup$ 2-handle.

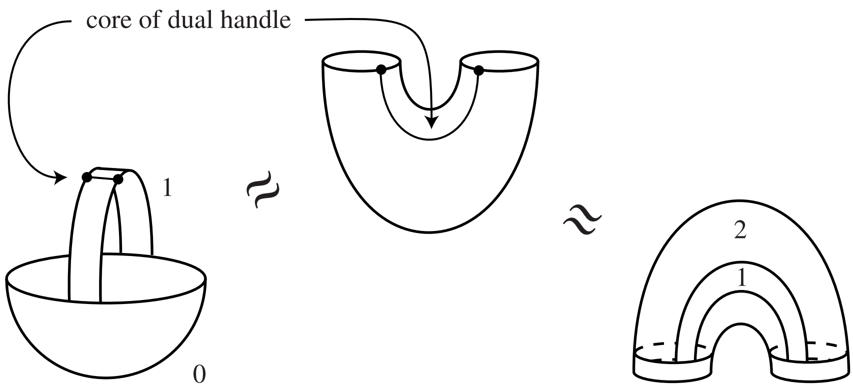


Figure 4.6. Dualizing a handlebody.

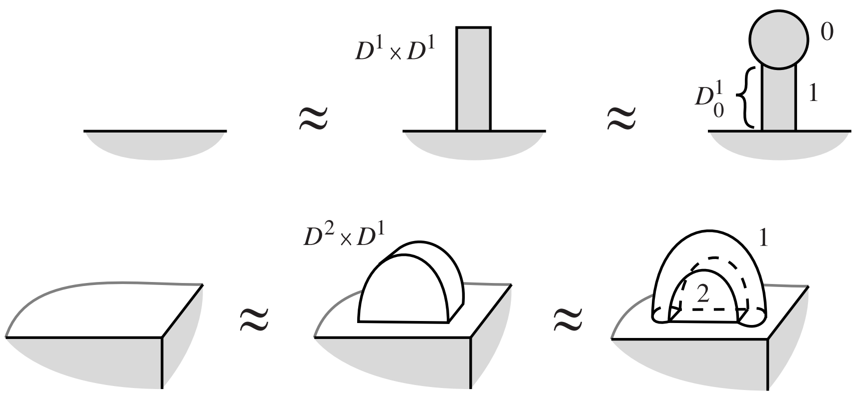


Figure 4.7. Handle pair creation.

Exercise 4.2.8. Construct dual decompositions for the handlebodies in Example 4.2.2 and Exercise 4.2.3. (It may help to draw the cores and cocores of the 1-handles.)

As we saw in Exercise 4.2.3, some handle decompositions have more handles than necessary. We can always create a "cancelling" pair of handles with indices $k-1$ and k, $1 \le k \le n$. (See Figure 4.7.) To see this, write ∂D^k as $S^{k-1} = D_+^{k-1} \cup_\partial D_-^{k-1}$. If we form the boundary sum of X with $D^n = D^k \times D^{n-k}$ by gluing along $D_-^{k-1} \times D^{n-k}$, the diffeomorphism type of X is not changed. Now we slice a neighborhood of D_+^{k-1} off of D^k, $D^k = D_0^k \cup_{D_+^{k-1}} D_+^{k-1} \times D^1$. Then $D_+^{k-1} \times D^1 \times D^{n-k}$ is the $(k-1)$-handle attached to ∂X, and $D_0^k \times D^{n-k}$ is the cancelling k-handle. This procedure is reversible:

Proposition 4.2.9. *A $(k-1)$-handle h_{k-1} and a k-handle h_k $(1 \leq k \leq n)$ can be cancelled, provided that the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.*

Proof (sketch). If we try to push h_k off of h_{k-1} as in the proof of Proposition 4.2.7, we will end up with the attaching sphere of h_k intersecting h_{k-1} in $D^{k-1} \times p \subset D^{k-1} \times \partial D^{n-k+1}$, and we can reverse the previous construction. That is, we can explicitly identify $h_{k-1} \cup h_k$ with the above standard model, exhibiting it as a ball boundary-summed onto the manifold. Specifically, we identify $h_{k-1} = D^{k-1} \times D^{n-k+1}$ with $D_+^{k-1} \times D^1 \times D^{n-k}$ by diffeomorphisms $D^{k-1} \approx D_+^{k-1}$, $D^{n-k+1} \approx D^1 \times D^{n-k}$ with $p \in \partial D^{n-k+1}$ mapping to $(-1, 0) \in D^1 \times D^{n-k}$, and we identify $h_k = D^k \times D^{n-k}$ with $D_0^k \times D^{n-k}$ in the obvious way, so as to obtain a well-defined diffeomorphism $h_{k-1} \cup h_k \to D^k \times D^{n-k}$. See [**M4**] Theorem 5.4 for a careful proof from the Morse theory viewpoint, and [**RS**] 6.4 for the PL version. □

There is one more important move, which we will discuss in detail in Section 5.1. (See also [**M4**] Theorem 7.6 and [**RS**] 6.7.)

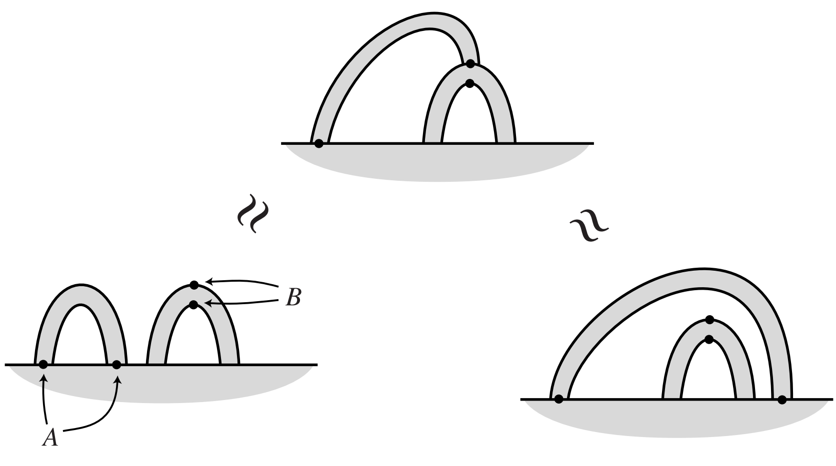


Figure 4.8. Handle slide.

Definition 4.2.10. Given two k-handles h_1 and h_2 $(0 < k < n)$ attached to ∂X, a *handle slide* of h_1 over h_2 is given by the following procedure. Isotope the attaching sphere A of h_1 in $\partial(X \cup h_2)$, pushing it through the belt sphere B of h_2 (Figure 4.8). At the intermediate stage, the spheres will intersect in one point p (with $T_pA \oplus T_pB$ of codimension 1 in $T_p\partial(X \cup h_2)$). If we apply Proposition 4.2.7 to this intermediate stage, we will have a choice of directions for pushing A off of B. One direction gives the original picture, and the other gives the result of the handle slide.

Exercise 4.2.11. Show that the Klein bottle is diffeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$ by sliding handles. (See Figure 5.1 for the answer.)

It is a fundamental fact that handle sliding and cancellation form a complete set of moves for handlebodies.

Theorem 4.2.12. ([**Ce**]) *Given any two relative handle decompositions (ordered by increasing index) for a compact pair $(X, \partial_- X)$, it is possible to get from one to the other by a sequence of handle slides, creating/annihilating cancelling handle pairs and isotopies within levels.* □

The basic idea of the proof is that the handlebodies are given by Morse functions $f_0, f_1 \colon X \to [0,1]$, and these are clearly homotopic, so it suffices to analyze a generic homotopy f_t from f_0 to f_1 (rel ∂X). In the space of all smooth functions $X \to [0,1]$, Morse functions generating suitably ordered handlebodies fill the complement of a codimension-1 subset, so there will only be finitely many values of t for which f_t must be analyzed carefully. At these values, we may find an annihilating pair of critical points (Figure 4.9), which corresponds to a handle cancellation. At the other anomalous values of t, the gradient flow of f_t has a trajectory running between two critical points of the same index. This translates into handle theory as the intermediate stage of a handle slide, at which $A \cap B = \{p\}$ in Definition 4.2.10 (cf. Figure 4.8).

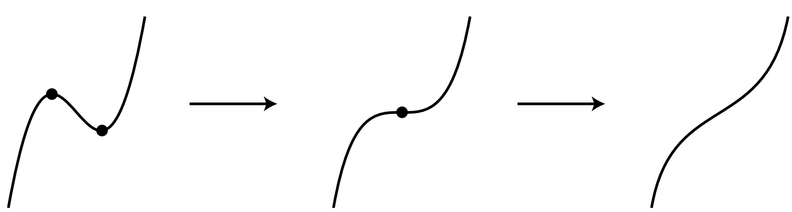


Figure 4.9. Cancelling pair of critical points (handles).

Proposition 4.2.13. *If X^n is compact and connected, then $(X, \partial_- X)$ admits a handle decomposition with exactly one 0-handle (if $\partial_- X = \emptyset$) or no 0-handles (if $\partial_- X \neq \emptyset$). We can also assume that there is exactly one n-handle (if $\partial_+ X = \emptyset$) or no n-handles (if $\partial_+ X \neq \emptyset$).*

Proof. If $\partial_- X = \emptyset$, then any handle decomposition of $(X, \partial_- X)$ has at least one 0-handle (which is the only type of handle we can attach to $\emptyset$). If there is more than one 0-handle and X is connected, then two 0-handles must be connected by a 1-handle (since handles of index ≥ 2 have connected attaching spheres and, hence, cannot connect a disconnected manifold). Clearly, the 1-handle cancels one 0-handle. Continuing by induction, we eliminate all but one 0-handle. If $\partial_- X \neq \emptyset$, a similar argument eliminates all 0-handles. The rest of the proposition follows by applying the same argument to the dual decomposition. □

Exercise 4.2.14. Classify compact, connected manifolds of dimension ≤ 2.

Having discussed the basic topology of handle decompositions, we now turn to their algebraic topology. As we have already observed, a handle decomposition of $(X, \partial_- X)$ determines a relative cell complex with the same homotopy type. Thus, the computation of homotopy and homology groups of a handlebody is essentially the same as for a CW-complex. For reference, we sketch the computations of π_1, H_* and H^* in the language of handle decompositions, and refer the reader to any algebraic topology text for more details in the language of CW-complexes. (See also [**M4**], [**RS**].) A handle decomposition of $(X, \emptyset)$ with a unique 0-handle determines a presentation of $\pi_1(X)$: Each 1-handle (with an orientation chosen on its core D^1) determines a generator, and the attaching circle of each 2-handle, attached by an arc to the base point, gives a relation. Handles of higher index do not affect π_1, since their attaching spheres are simply connected. For the homology of $(X, \partial_- X)$, start with any handle decomposition (with handles ordered by increasing index) and define the group of relative k-chains to be $C_k(X, \partial_- X) = H_k(X_k, X_{k-1}; \mathbb{Z})$; this is freely generated by the k-handles (once we have oriented their core disks). The boundary operator $\partial_* \colon C_k(X, \partial_- X) \to C_{k-1}(X, \partial_- X)$ is defined by the long exact sequence of the triple (X_k, X_{k-1}, X_{k-2}), but for X oriented, it can also be described directly by the formula $\partial_* h = \sum (B_i \cdot A) h_i$. (Here A is the attaching sphere of the k-handle h, B_i is the belt sphere of the $(k-1)$-handle h_i, and $B_i \cdot A$ denotes their intersection number in $\partial_+ X_{k-1}$, cf. Remark 1.2.6(a).) Standard theory shows that $\partial_*^2 = 0$, so we have $\operatorname{Im} \partial_* \subset \ker \partial_* \subset C_k(X, \partial_- X)$, and $H_k(X, \partial_- X; \mathbb{Z})$ is given by $\ker \partial_* / \operatorname{Im} \partial_*$. In particular, $H_k(X; \mathbb{Z}) = H_k(X, \emptyset; \mathbb{Z})$. As usual, we obtain the cochain complex $C^k(X, \partial_- X; G)$ of $(X, \partial_- X)$ (where G is any abelian group) by dualizing the chain complex, resulting in $H^k(X, \partial_- X; G)$; the groups $H_k(X, \partial_- X; G)$ are obtained by tensoring the chain complex with G (or equivalently, using $C_k(X, \partial_- X; G) = H_k(X_k, X_{k-1}; G)$). If $Y \subset X$ is a compact, codimension-0 submanifold, we can use excision to interpret $H_k(X, Y)$ as $H_k(X - \operatorname{int} Y, \partial Y)$ (and similarly for cohomology), so it suffices to use a handle decomposition of $(X - \operatorname{int} Y, \partial Y)$ to compute $H_k(X, Y)$ and $H^k(X, Y)$. For cohomology, an equivalent formulation is to take Y to be a subhandlebody of X and define

$C^k(X,Y;G) \subset C^k(X;G) = C^k(X,\emptyset;G)$ to be the subcomplex of cochains vanishing on k-handles in Y; the resulting cohomology is $H^k(X,Y;G)$.

Finally, we consider the effect of handle moves on the homology of an oriented manifold $(X, \partial_- X)$. (See [**M4**], [**RS**] for more details.) If h is a k-handle cancelling a $(k-1)$-handle h_i, then $B_i \cdot A = \pm 1$, so $\partial_* h = \pm h_i + \sum_{j \neq i} a_j h_j$. Cancelling the pair eliminates the generators h, h_i from their respective chain groups via the relations $h = 0$, $h_i = \mp \sum_{j\neq i} a_j h_j$. It is easily checked directly that the homology of $(X, \partial_- X)$ is unchanged. Beware that an *algebraically* cancelling pair, i.e., one with $B_i \cdot A = \pm 1$, need not actually cancel (*geometrically*). Much deep topology has been devoted to this issue — see Section 9.2. If we slide a k-handle h over another k-handle h', the algebraic result is a change of the canonical basis for $C_k(X, \partial_- X)$. Specifically, we replace the basis element h by $h \pm h'$, with the sign depending on the direction we push from the intermediate stage described in Definition 4.2.10 (relative to the orientation induced on $T_p\partial(X \cup h')/(T_pA \oplus T_pB) \cong \mathbb{R}$ by the given ones on the three tangent spaces). It is easily verified that for $0 < k < n-1$ and $\partial_+ X_{k-1}$ connected, any basis change of the form $h \mapsto h \pm h'$ can be realized by a handle slide. (See Section 5.1 for the 4-dimensional case.) Together with reordering handles and the moves $h \mapsto -h$ (by reversing the orientation on the core disk), these moves generate all changes of basis on $C_k(X, \partial_- X)$. Our remaining operation, dualizing a handlebody, also has an elegant interpretation: It gives a simple proof of Poincaré duality for compact, oriented n-manifolds, $H^k(X, \partial_- X) \cong H_{n-k}(X, \overline{\partial_+ X})$. (Identify $C^k(X, \partial_- X)$ with $C_{n-k}(X, \overline{\partial_+ X})$ by the obvious correspondence of canonical bases, then show that the boundary operators correspond up to sign.)

Exercise 4.2.15. Determine the effect of handle moves on the presentation of $\pi_1(X)$ given above.

4.3. Dimension three — Heegaard splittings

Having dispensed with lower dimensions (Exercise 4.2.14), we now consider handle decompositions of compact, connected 3-manifolds X with $\partial_- X = \emptyset$. By Proposition 4.2.13, we can assume that X has a handle decomposition with a unique 0-handle and 3-handle (or no 3-handle if $\partial_+ X \neq \emptyset$). Recall that X_1 denotes the union of the 0-handle and 1-handles. By Example 4.1.4(b), if X is orientable then $X_1 \approx \natural g\, S^1 \times D^2$, where g is the number of 1-handles (see Figure 4.10 for $g = 3$), and the nonorientable case is not much harder to understand. According to Example 4.1.4(c), the remaining handle attaching will not depend on choices of framing. Thus, the handlebody will be completely specified if we draw the attaching circles of the 2-handles in ∂X_1, as we have done in Figure 4.11. (If $\partial_+ X = \emptyset$, then the boundary of

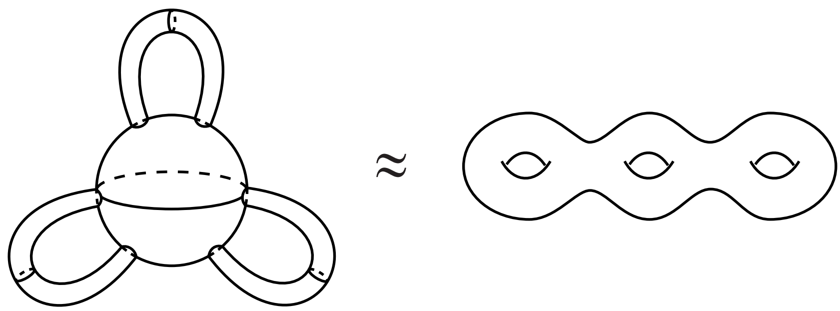


Figure 4.10. X_1 for a 3-manifold.

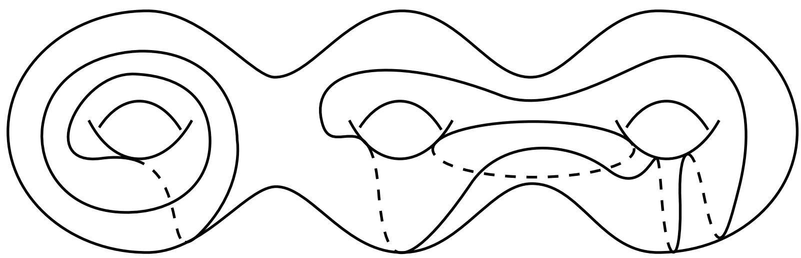


Figure 4.11. Heegaard diagram.

$X_2 = X_1 \cup 2$-handles must be S^2 so that we can attach the unique 3-handle, and this gluing is uniquely determined.) When $\partial_+ X = \emptyset$, such a diagram is called a *Heegaard diagram*. For an alternate way of understanding this construction, observe that by turning $X - \mathrm{int}\ X_1$ ($= 2, 3$-handles) upside down (dualizing), we can write it as 0-handle $\cup$ 1-handles $\approx \natural \ell\, S^1 \times D^2$ (assuming X is closed and oriented). Since X_1 and $X - \mathrm{int}\ X_1$ have the same boundary, we must have $\ell = g$, and so we have written X as $H \cup_\psi H$ for $H = \natural g S^1 \times D^2$ and some diffeomorphism $\psi\colon \partial H \to \partial H$, a *Heegaard splitting* of genus g. Thus, handle decompositions of closed 3-manifolds are essentially the same as Heegaard splittings, and the resulting diagrams are prescriptions for gluing together two copies of a handlebody H. (That is, the circles in ∂H determine the images under ψ of the belt circles of H, and that specifies $X = H \cup_\psi H$ up to diffeomorphism.)

If it becomes too cumbersome to draw collections of circles in ∂X_1 as in Figure 4.11, we can represent the same information by a diagram in $\mathbb{R}^2$. We simply note that the 0-handle D^3 has boundary $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and draw the attaching regions in $\mathbb{R}^2$. The attaching region of each 1-handle is a pair of disks $D^2 \coprod D^2$, and attaching the 1-handle is equivalent to gluing together the disks in ∂D^3. The resulting manifold X_1 will be orientable if and only if the gluing map *reverses* orientation, so without loss of generality we can take it to be the obvious reflection $(x, y) \mapsto (-x, y)$ (Figure 4.12) when

X_1 is orientable. We add 2-handles as before, by drawing their attaching circles in ∂X_1, and then attaching the 3-handle (when $\partial_+ X = \emptyset$) is uniquely determined. Circles need not appear as circles in $\mathbb{R}^2$, however, but rather as collections of arcs with endpoints on the boundaries of the attaching regions of the 1-handles. They will become circles when we identify these boundaries by the given reflection to get ∂X_1. For example, Figure 4.11 becomes the planar diagram shown in Figure 4.13. Figure 4.14 shows the (nondiffeomorphic) *lens spaces* $L(5,1)$ and $L(5,2)$, each made with a single handle of each index 0,1,2,3.

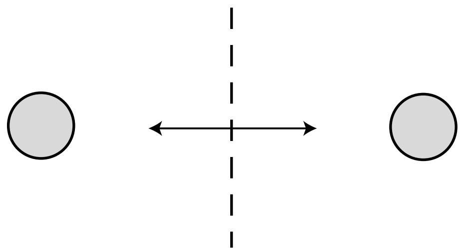


Figure 4.12. Gluing map to attach a 1-handle to D^3.

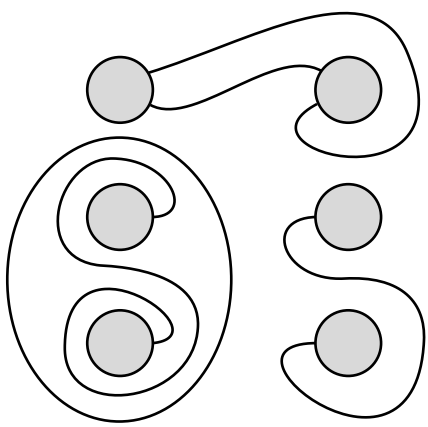


Figure 4.13. Heegaard diagram.

Exercises 4.3.1. (a)* Draw diagrams for $S^1 \times S^2$ and $I \times T^2$. Draw one for the 3-*torus* $T^3 = S^1 \times S^1 \times S^1$. (*Hint*: T^3 is obtained from a cube by gluing opposite faces by translations. Its diagram is most symmetrical if one 1-handle is attached at $\infty \in S^2 = \mathbb{R}^2 \cup \{\infty\}$.) Draw a diagram for $\mathbb{RP}^3$. (See Exercise 4.2.5.)

(b)* What does a handle cancellation look like in a Heegaard diagram? A 2-handle slide? Identify the familiar manifold shown in Figures 4.11 and 4.13.

Figure 4.14. Heegaard diagrams.

(c)* For the lens spaces shown in Figure 4.14, compute π_1, H_* and H^*. For each $n \geq 0$, construct a 3-manifold X with $\pi_1(X) \cong \mathbb{Z}_n$.

4.4. Dimension four — Kirby diagrams

A *Kirby diagram* is a description of a 4-dimensional (relative) handlebody by a diagram in $\mathbb{R}^3$. For now, we consider the case of a compact 4-manifold X with $\partial_- X = \emptyset$. (For the relative version $\partial_- X \neq \emptyset$, see Section 5.5.) The procedure is analogous to that of the previous section. We assume that there is a unique 0-handle D^4, which has boundary $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and we draw the attaching regions of the remaining handles in $\mathbb{R}^3$. The attaching region of each 1-handle is $D^3 \coprod D^3$, which we draw as a pair of round balls (Figure 4.15).

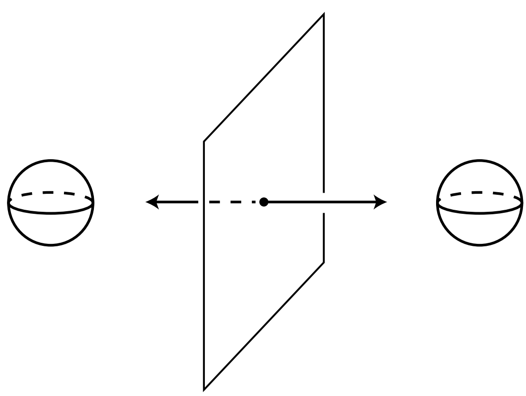


Figure 4.15. Gluing map to attach a 1-handle to D^4.

Assuming that X is orientable, the union X_1 of 0- and 1-handles will be $\natural n\, S^1 \times D^3$, where n is the number of 1-handles (and $\natural 0\, S^1 \times D^3 = D^4$), and we can form it by gluing $2n$ 3-balls in pairs by reflecting through the planes perpendicularly bisecting the segments joining their centers (Figure 4.15). (For the nonorientable case, one also needs to establish a convention for orientation-preserving gluings. See, for example, [**A2**].) We add 2-handles along circles in ∂X_1 as before, but in $\mathbb{R}^3$ circles can be knotted and linked (see Figure 4.16), and we must also deal with framings. For 3- and 4-handles, there are no framings (by Example 4.1.4(c)). In the case $\partial_+ X = \emptyset$, we can assume that there is a unique 4-handle, and by duality the union of 3- and 4-handles will be diffeomorphic to $\natural m\, S^1 \times D^3$, where m is the number of 3-handles. In particular, for $\partial_+ X = \emptyset$, the union X_2 of all 0-, 1- and 2-handles must have $\partial X_2 = \partial(\natural m\, S^1 \times D^3) = \# m\, S^1 \times S^2$. (We present techniques for analyzing ∂X_2 in general in Section 5.3.) Any self-diffeomorphism of $\# mS^1 \times S^2$ extends over $\natural m\, S^1 \times D^3$ [**LP**], so if $\partial X_2 = \# m\, S^1 \times S^2$ there is a unique manifold with $\partial_+ X = \emptyset$ that can be obtained by attaching 3- and 4-handles to X_2. Thus, if $\partial_+ X = \emptyset$ we do not need to keep track of the 3- and 4-handles attached to X_2. Similarly, if $\partial_+ X$ is nonempty but connected and X is simply connected, then X is determined by X_2 and the number of 3-handles [**Tr**]. We conclude that the complexity of a 4-dimensional handlebody is mainly due to the 2-handles.

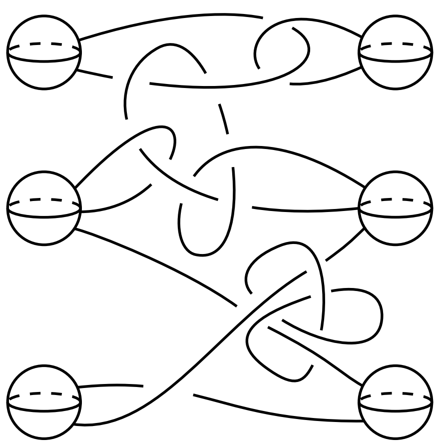


Figure 4.16. Kirby diagram.

Remark 4.4.1. Laudenbach and Poénaru [**LP**] use basic 3-manifold theory to show that any self-diffeomorphism of $\# mS^1 \times S^2$ can be written as a composite of handle slides of $\natural mS^1 \times D^3 = D^4 \cup$1-handles (together with isotopies in ∂D^4 permuting the attaching balls, and reflection in the orientation-reversing case). This actually implies that any two closed (oriented) handlebodies $X_2 \cup$ 3-handles $\cup$ 4-handle (X_2 fixed) are related by a sequence of 3-handle slides. For the basic idea of Trace's argument [**Tr**], assume (for the sake of simplicity) that X, X' are each obtained from Y by attaching a single 3-handle, with disjoint attaching spheres $S, S' \subset \partial_+ Y$, respectively. If $\partial_+ X$ and $\partial_+ X'$ are connected, then one can construct a knot $K \subset \partial_+ Y$ intersecting S, S' each transversely once. If X (hence Y) is simply connected, then K bounds an immersed disk $D \subset Y$, which we may assume is embedded (after using parallel copies of S to remove double points). If we interpret D as the cocore of a 2-handle h attached to $Y - \nu D$, then either of the 3-handles will cancel h, so $X \approx Y - \nu D \approx X'$. Note that for $\partial X'$ disconnected, we have the easy counterexample $Y = S^2 \times D^2$, $S = S^2 \times \{\text{pt.}\}$ (so $X \approx D^4$), S' unknotted (so $X' \approx S^2 \times D^2 - \text{int } D^4$). Similarly, $Y = S^2 \times D^2 \natural S^1 \times D^3$, $S = S^2 \times \{\text{pt.}\}$, $S' = \{\text{pt.}\} \times \partial D^3$ ($X \approx S^1 \times D^3$, $X' \approx S^2 \times D^2 \# S^1 \times S^3$) is a counterexample with $\partial X, \partial X'$ connected, $\pi_1(X) \cong \mathbb{Z}$.

It remains to deal with framings for 2-handles. Recall (cf. Example 4.1.4(d)) that these are classified by $\pi_1(O(2)) \cong \mathbb{Z}$, although the bijection is not determined until we decide which framing should correspond to 0. If K is a knot (i.e., embedding of S^1) in an oriented 3-manifold M, we can specify a framing on K by choosing a single nowhere-zero transverse vector field v to K (Figure 4.17). Then the normal orientation on K induced by the orientations of S^1 and M determines a unique way (up to isotopy) to extend v to a correctly oriented basis for each normal fiber. Although this framing depends on our choices of orientation, the ambiguity does not affect the result of handle attaching (as we implicitly proved in Section 4.1 when we arranged our map $S^{k-1} \to O(n-k)$ to be based at $I \in SO(n-k) \subset O(n-k)$). Thus, the pair (K, v) uniquely specifies the result of attaching a 2-handle to a 4-manifold X with $\partial_+ X = M$, and it is common to abuse terminology by referring to v itself as a framing. An equivalent (and more practical) way to specify the attachment is to construct a knot K' *parallel* to K by pushing K in the direction of v (Figure 4.18). (Intuitively, we connect the arrowheads.) If $\varphi\colon S^1 \times D^2 \to M^3$ is the attaching map, then $K = \varphi | S^1 \times 0$, and we can interpret K' as $\varphi | S^1 \times p$ for some nonzero $p \in D^2$.

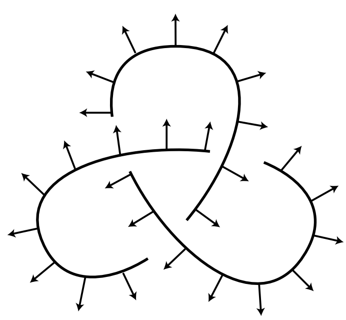


Figure 4.17. Framing specified by a vector field.

Once we arbitrarily choose one framing to correspond to $0 \in \pi_1(O(2)) \cong \mathbb{Z}$, we immediately see how to get the framing corresponding to any other integer n — we simply add n twists as in Figures 4.18 and 4.19 (with $n = 3$ and a nonstandard choice of 0-framing in Figure 4.18). To fix the sign convention, we endow S^3 with the standard orientation as ∂D^4 (Definition 1.1.2),

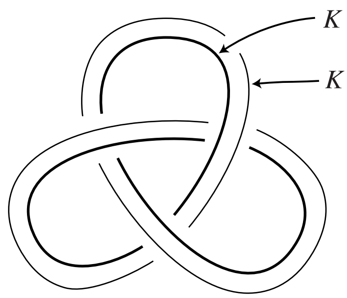


Figure 4.18. Framing specified by a parallel knot K'.

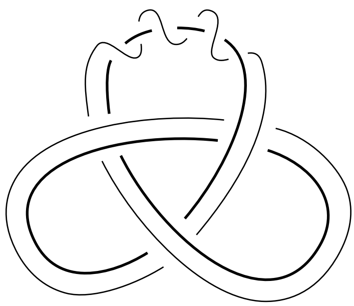


Figure 4.19. Another framing.

which corresponds to the standard orientation on $\mathbb{R}^3 = T_{(1,0,0,0)}S^3 \subset \mathbb{R}^4$, given by a right-handed basis in our pictures. Now we choose the sign so that a positive integer n corresponds to n right-handed twists (Figure 4.19), and a negative n corresponds to $|n|$ left twists. Note that while the sign depends on the orientation of S^3 (or M^3 in general), it does *not* depend on the orientation of S^1. (*Proof*: Unscrew a nut from a bolt, then flip it over and screw it back on.)

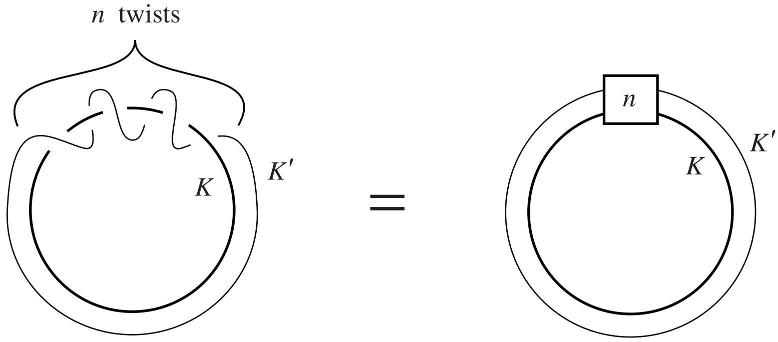


Figure 4.20. Kirby diagram of a D^2-bundle over S^2 (with K' indicating framing).

Example 4.4.2. We saw in Example 4.1.4(d) that D^2-bundles X over S^2 are given by attaching a 2-handle to an unknotted circle K in ∂D^4. This is shown in Figure 4.20 (using the above convention when n is negative). We introduce the convention of using a box labelled "n" to represent n full (360°) twists in a collection of strands. We can explicitly see a 2-sphere S in int X (the 0-section) generating its homology: Take the obvious disk bounded by the circle K, imagine its interior being pushed into D^4, and then glue on the core of the 2-handle. (This is a standard procedure for constructing surfaces in Kirby diagrams; we generalize it to arbitrary surfaces in Section 6.2.) We can also compute the self-intersection of S: To construct a sphere S' isotopic to S but transverse to it, we start with a disk $D^2 \times p$ parallel to the core of the 2-handle. This intersects S^3 in the parallel curve $K' = S^1 \times p$ of Figure 4.20. Extend the disk by an annulus that dives straight down into D^4 (i.e., identify a collar of ∂D^4 with $I \times S^3$ and take the annulus to be $I \times K'$). At some level $t \times S^3$ we see the rest of S as a disk spanning K and intersecting K' transversely in $|n|$ points. Below this level, we span K' by a disk to complete S'. (Compare with the 2-dimensional case, Figure 4.21.) The spheres S and S' are isotopic and intersect transversely in $|n|$ points of the same sign, which equals the sign of n (provided that S' inherits its orientation from S in the obvious way), as can be computed directly via the previous paragraph and Definition 1.1.2. Thus, the intersection pairing of X is given by $\langle n \rangle$, and so n is the Euler number of X. In particular, $S^2 \times D^2$ is the case $n = 0$ (and we can visualize the product structure directly), and $\mathbb{CP}^2$ is obtained by adding a 4-handle to the $n = 1$ case. Similarly, $\overline{\mathbb{CP}^2}$ corresponds to $n = -1$.

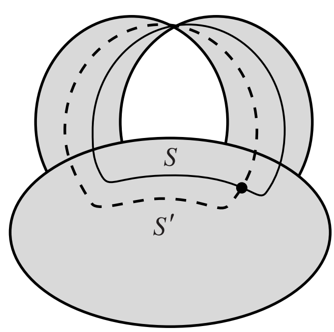


Figure 4.21. Computing a self-intersection number.

Exercise 4.4.3. Check directly in Figure 4.20 that the intersection number $S \cdot S'$ is positive for $n > 0$.

To simplify notation, we will replace the double-strand notation of Figure 4.20 with the notation of Figure 4.22. However, some simple examples such as Figure 4.23 show that this is a more subtle operation than one might think. Which framing should be called the 0-framing, particularly if K is knotted? To determine this, we must introduce the concept of a *linking number*.

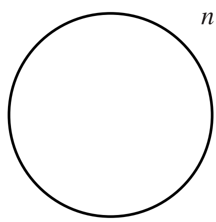


Figure 4.22. D^2-bundle over S^2 with Euler number n.

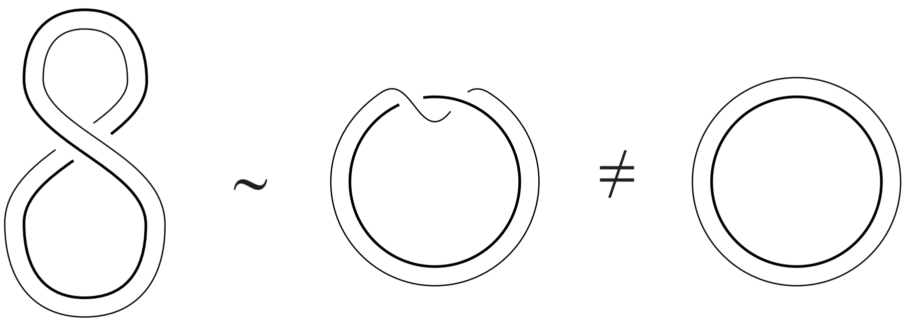


Figure 4.23. A subtlety of framings.

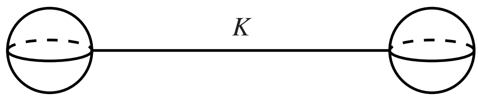


Figure 4.24. A knot in $S^1 \times S^2$.

Exercise 4.4.4. * For the knot K in $S^1 \times S^2$ shown in Figure 4.24, find an isotopy (fixing neighborhoods of the balls) that returns K to itself but changes any framing on K by two twists. How does this relate to $\pi_1(O(3))$? How does it relate to Philippine dancing?

4.5. Linking numbers and framings

An *m-component link* in a 3-manifold M is an embedding of $\coprod_{i=1}^m S^1$ into M, or equivalently, a collection of m disjoint knots. We are mainly interested in links up to isotopy. A *link diagram* of a link L in $\mathbb{R}^3 = S^3 - \{\infty\}$ is a generic projection of L into $\mathbb{R}^2$ (to an immersion of $\coprod_{i=1}^m S^1$ which is 1–1 except at transverse double points), together with information specifying which strand crosses underneath at each double point (denoted as in Figure 4.25). Clearly, the diagram uniquely determines L up to isotopy. If L is oriented (i.e., an orientation is fixed on $\coprod_{i=1}^m S^1$), then each crossing in the diagram will look like exactly one of the pictures in Figure 4.25 (up to isotopy in $\mathbb{R}^2$), allowing us to assign a sign to each crossing. You should check that for self-crossings of a single component of L, the sign does not depend on the orientation of L. Two diagrams represent isotopic links if and only if they are equivalent via a sequence of isotopies of $\mathbb{R}^2$ and the *Reidemeister moves* shown in Figure 4.26 [**AB**]. (The mirror image of the last move should also be allowed. Sufficiency of these moves can be proved by studying generic projections into $\mathbb{R}^2$ of isotopies $I \times (\coprod S^1) \to \mathbb{R}^3$, in the same spirit as the proof of Theorem 4.2.12.)

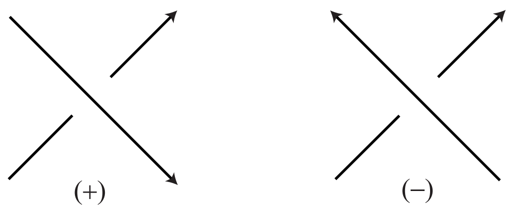


Figure 4.25. Signs of crossings in a link diagram.

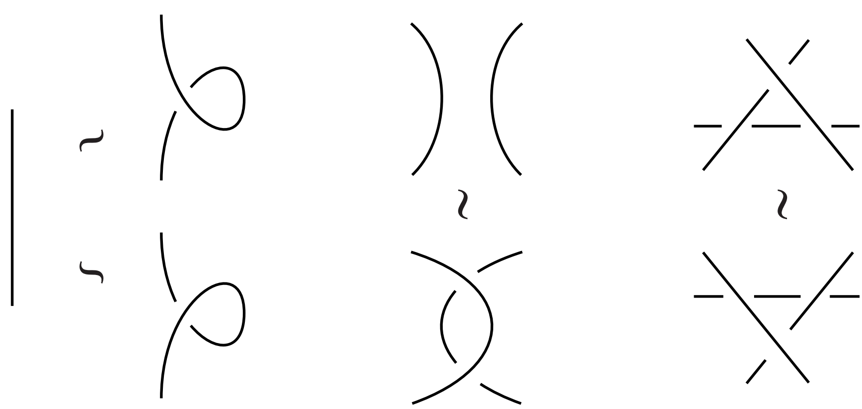


Figure 4.26. The Reidemeister moves.

If K_1 and K_2 are components of an oriented link in S^3 (i.e., they are disjoint, oriented knots), we wish to define their *linking number* $\ell k(K_1, K_2)$. We will give three equivalent ways of doing this. First, note that since $H_2(S^3, S^3 - K_1; \mathbb{Z}) \cong H_2(\nu K_1, \partial \nu K_1; \mathbb{Z}) \cong \mathbb{Z}$ (where $\nu K_1 \approx S^1 \times D^2$ denotes a tubular neighborhood of K_1), the long exact homology sequence shows that $H_1(S^3 - K_1; \mathbb{Z}) \cong \mathbb{Z}$, generated by a *meridian* μ of K_1 (i.e., a circle μ in $S^3 - K_1$ bounding a disk D in S^3 intersecting K_1 exactly once, transversely). We fix the orientation of μ by the right-hand rule as in Figure 4.27 (so that the intersection point of D with K_1 has positive sign).

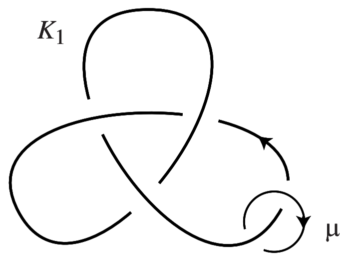


Figure 4.27. A canonically oriented meridian.

Definition 4.5.1. For K_1, K_2 and μ as above, K_2 represents the class $n[\mu] \in H_1(S^3 - K_1; \mathbb{Z})$ for a unique $n \in \mathbb{Z}$. We define the *linking number* $\ell k(K_1, K_2)$ to be n.

Although this definition can actually be applied to any pair of disjoint, nullhomologous knots in an oriented 3-manifold by replacing $H_1(S^3 - K_1; \mathbb{Z})$ with $\ker(H_1(M - K_1) \to H_1(M)) \cong \mathbb{Z}$, we will use linking numbers primarily in the boundary S^3 of a 0-handle. Note that $\ell k(K_1, K_2)$ reverses sign if we reverse the orientation on any one of K_1 (hence μ), K_2 or S^3.

Proposition 4.5.2. *For K_1 and K_2 (as above) given by a link diagram, $\ell k(K_1, K_2)$ equals the signed number of times that K_2 crosses underneath K_1. (That is, it is the number of positive undercrossings minus the number of negative undercrossings.)*

Proof. First note that if we change K_2 into a new knot K_2' by changing each crossing of K_2 underneath K_1 to an overcrossing, then K_2' will lie entirely over K_1, so that we can pass a plane between them in $\mathbb{R}^3$. Clearly, $\ell k(K_1, K_2') = 0$. Now we transform K_2' back into K_2. It is easy to see that each crossing change adds the sign of the crossing to the linking number. □

Corollary 4.5.3. *The linking number is symmetric, that is, $\ell k(K_1, K_2) = \ell k(K_2, K_1)$.*

Proof. If we stand on the other side of $\mathbb{R}^2$, we will see every undercrossing of K_1 as an undercrossing of K_2. It is easily checked that the signs are preserved. □

Exercise 4.5.4. Prove directly that the signed number of undercrossings is isotopy invariant by checking its invariance under Reidemeister moves.

For the third characterization of $\ell k(K_1, K_2)$, note that any knot K_1 in S^3 (or nullhomologous knot in an orientable 3-manifold) bounds a *Seifert surface* F, an embedded compact surface (orientable unless otherwise specified) with $\partial F = K_1$. This can be constructed directly from a link diagram using Seifert's algorithm (Exercise 4.5.12(a)), or algebraically as follows: Since $H^1(S^3 - K_1; \mathbb{Z}) \cong \mathbb{Z}$ and this cohomology group is classified by $K(\mathbb{Z}, 1) = S^1$, there is a map $\varphi\colon S^3 - K_1 \to S^1$ inducing a cohomology isomorphism, and the preimage of a regular value will be the required surface. Let $F \cdot K_2$ denote the intersection number of these two manifolds, where F is oriented so that its oriented boundary is K_1. (See Remark 1.2.6(a) or [**GP**].)

Proposition 4.5.5. *For K_1, K_2 and F as above, $\ell k(K_1, K_2) = F \cdot K_2$.*

Proof. Produce K_2' disjoint from F as in the proof of Proposition 4.5.2, then isotope it back to K_2 in S^3 as before. The only new complication is that the isotopy may produce intersections with F away from K_2, but it is easy to see that such intersections appear and disappear in pairs with opposite sign as in Figure 4.28, so they have no effect on the final answer. □

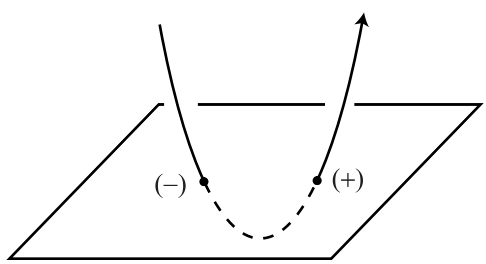


Figure 4.28. Cancelling pair of intersections.

Remark 4.5.6. The fact that new intersections appear in cancelling pairs is the main idea of the proof that the intersection pairing is well-defined on homology classes (Remark 1.2.6(a)). In this case, we are implicitly using the pairing between $H_1(S^3 - \text{int } \nu K; \mathbb{Z})$ and $H_2(S^3 - \text{int } \nu K, \partial\nu K; \mathbb{Z})$, which shows that F can be taken to be any smooth *cycle* representing the given relative class, rather than an embedded surface.

Definition 4.5.7. For a framed knot (K, v) in the boundary of a 0-handle, we define the *framing coefficient* to be the integer $\ell k(K, K')$, where K' is a parallel copy of K determined by v as in Section 4.4, and the orientations of K and K' are chosen to be parallel.

Note that this is independent of the orientation of K, since reversing K also reverses K'. Now we have the promised canonical correspondence between integers and ways of attaching a 2-handle to D^4 along K. Clearly, it is consistent with our previous discussion. (Adding a right twist increases the framing coefficient by 1, and the coefficient in Figure 4.20 is n.) Henceforth, we will use integers to denote framings on knots in ∂D^4, as in Figure 4.22. (Note that the integer is always computed in ∂D^4. If we obtain a different copy of S^3 by adding other 2-handles and looking at the resulting boundary, the framing coefficient computed in this other S^3 may be different.) For knots that run over 1-handles, one needs additional care, as Exercise 4.4.4 demonstrates. We will return to this problem in Section 5.4.

To compute the coefficients of all framings on a knot K, it suffices to do the computation for any one framing on K; any other framing will differ by n twists, where n is the difference of framing coefficients. A convenient choice is the *blackboard framing*, determined by a knot diagram by requiring v to lie in $\mathbb{R}^2$ as in Figures 4.17 and 4.18. (Beware that this is not isotopy invariant — see Figure 4.23.) Another useful choice is the 0-framing. Propositions 4.5.2 and 4.5.5 immediately imply the following:

Proposition 4.5.8. *The framing coefficient of the blackboard framing on a knot K (given by a diagram) equals the* writhe $w(K)$*, the signed number of self-crossings of K. The 0-framing is obtained from the outward normal to any orientable Seifert surface.* □

Exercise 4.5.9. * What is the framing coefficient in Figure 4.19? What is the 0-framing? Compare with an orientable Seifert surface. (In this diagram, you can find such a surface by 2-coloring (like a checkerboard) the regions in $\mathbb{R}^2$ separated by the knot. The pieces will be connected by bands with 180° twists. One color produces an orientable surface; the other is nonorientable. Compare with Exercise 4.5.12(a).) What framing is induced by the nonorientable (Möbius band) Seifert surface?

Many interesting 4-manifolds arise as 2-*handlebodies*, i.e., handlebodies made by attaching 2-handles to D^4. In particular, many closed manifolds such as $\mathbb{CP}^2$ can be described as 2-handlebody ∪ 4-handle. (As we will see, the hypersurfaces S_d of Section 1.3 have this property (Corollary 6.3.19), as do elliptic surfaces over S^2 with as most one multiple fiber (Corollary 8.3.17) and various other examples. It is not known whether all smooth, closed, simply connected 4-manifolds have such handle decompositions.) A 2-handlebody is described by a framed link L in ∂D^4 (or $\mathbb{R}^3$). If we orient L and choose an order $K_1, \ldots, K_m$ for the components of L, then we may encode the linking number data from K as a matrix.

Definition 4.5.10. Let L be an ordered, oriented framed link in S^3 (or $\mathbb{R}^3$), with components $K_1, \ldots, K_m$. The *linking matrix* of L is the symmetric

$m \times m$ matrix $[a_{ij}]$, where $a_{ij} = \ell k(K_i, K_j)$ for $i \neq j$ and a_{ii} is the framing coefficient of K_i.

It is sometimes convenient to interpret the linking matrix as a bilinear pairing ℓk on formal linear combinations of components of L; thus, $\ell k(K_i, K_i)$ is the framing of K_i. Do not confuse this linking pairing with the linking form of a 3-manifold defined in Exercise 4.5.12(c) below.

The basic algebraic invariants of a 2-handlebody X are easy to determine. Since attaching a k-handle is the same up to homotopy as attaching a k-cell, X has the homotopy type of a wedge of m 2-spheres (where m is the number of 2-handles). Thus, X is simply connected and $H_2(X) \cong \mathbb{Z}^m$. Attaching a 4-handle (if possible) merely creates the group $H_4(X \cup 4\text{-handle}) \cong \mathbb{Z}$. If we orient the link L determining the handlebody, we obtain a canonical basis for $H_2(X)$. We can find surfaces in int X representing this basis as follows: Each component K_i of L has a Seifert surface F_i in ∂D^4. We push int F_i into int D^4 (as we did in Example 4.4.2), then add the core of the 2-handle to obtain a closed surface $\hat{F}_i$. We orient F_i so that $\partial F_i = K_i$ in the boundary orientation, and extend the orientation over $\hat{F}_i$. Now the classes $\alpha_i = [\hat{F}_i] \in H_2(X)$, $i = 1, \dots, m$, form the canonical basis. We use this basis to describe the intersection form Q_X by a matrix.

Proposition 4.5.11. *Let X be a connected handlebody without* 1- *or* 3-*handles, described by an ordered, oriented, framed link L. Then the matrix of Q_X with respect to the canonical ordered basis $\alpha_1, \dots, \alpha_m$ (described above) is given by the linking matrix of L. In short, the intersection form of a 2-handlebody equals its linking pairing.*

Proof. We use the method of Example 4.4.2. Fix $i \neq j$ and assume that $\hat{F}_j$ is deeper in D^4 than $\hat{F}_i$, i.e., $\hat{F}_i \cap D^4$ is contained in some collar $I \times S^3$ of ∂D^4 on which $\hat{F}_j$ is vertical ($I \times K_j$). Then $\alpha_i \cdot \alpha_j = \hat{F}_i \cdot \hat{F}_j = F_i \cdot K_j = \ell k(K_i, K_j)$. (Check the signs.) Similarly, we compute α_i^2 by constructing a copy $\hat{F}_i'$ of $\hat{F}_i$, beginning with a disk $D^2 \times p$ parallel to the core. This intersects ∂D^4 in a parallel copy K_i' of K_i given by the framing, so as in Example 4.4.2 we have $\alpha_i^2 = \hat{F}_i \cdot \hat{F}_i' = F_i \cdot K_i' = \ell k(K_i, K_i')$, which is the framing coefficient of K_i. □

The same method can be used to compute Q_X for a handlebody with 1- or 3-handles, although $H_2(X)$/torsion need not have a canonical basis. For example, if there are 3-handles but no 1-handles, one can apply the above proposition before attaching the 3-handles, and the latter will simply mod out a certain subspace on which the pairing vanishes.

Exercises 4.5.12. (a)* (Seifert's algorithm) Given a diagram for an oriented link L in S^3, construct an oriented Seifert surface with oriented boundary L as follows: Change the diagram in a small neighborhood of each crossing to obtain an embedded collection of circles in $\mathbb{R}^2$, with orientations induced by L. This collection bounds an obvious collection of oriented disks in $\mathbb{R}^3$. Now recover L by adding twisted bands.

(b)* Let X be an arbitrary 4-manifold (not necessarily compact or orientable). Give a geometric proof that every homology class in $H_2(X;\mathbb{Z})$ is represented by an embedded, closed, oriented surface F. Prove the same for $H_2(X;\mathbb{Z}_2)$ with F not necessarily orientable. (*Hint*: As at the end of Section 4.2, we can realize any 2-homology class by a 2-cycle $z \in \ker \partial_* \subset C_2(X) = H_2(X_2, X_1)$, or by a disjoint union of core disks $D^2 \times \{\text{pt.}\}$ whose boundaries form a link (oriented in one case) in ∂X_1. Show that by attaching bands, you can obtain a link in ∂D^4.) Note that a similar (easier) argument works when $\dim X > 4$, since any link $\coprod S^1 \to S^n$ is trivial for $n \geq 4$ (by Example 4.1.3).

(c)* In an oriented 3-manifold M, let K_1 and K_2 be disjoint, oriented, rationally nullhomologous knots (i.e., $[K_i] = 0 \in H_1(M;\mathbb{Q})$). Define a linking number $\ell k_{\mathbb{Q}}(K_1, K_2) \in \mathbb{Q}$. Show that the same answer is obtained by generalizing Definition 4.5.1 or Proposition 4.5.5. For K rationally nullhomologous in M, give a procedure for assigning rational numbers to framings on K that generalizes framing coefficients in S^3. For a fixed K, describe the image of this map in $\mathbb{Q}$. Now define a $\mathbb{Q}/\mathbb{Z}$-valued, symmetric bilinear form on the torsion subgroup of $H_1(M;\mathbb{Z})$. This is called the *linking form* of M. (Also see Exercise 5.3.13(g).) For M closed, prove that the linking form is nonsingular, that is, every nonzero element pairs nontrivially with something. (*Hint*: For this last part, construct K_2 representing a suitable class in $H_1(M - K_1;\mathbb{Z})/\text{torsion}$ by using the dual space $H^1(M - K_1;\mathbb{Z})$. To understand the latter, recall that $PD[K_1] \in H^2(M;\mathbb{Z})$ can be defined to be the image of one generator of $H^2(M, M - K_1;\mathbb{Z})$, then consider the long exact sequence.) Nonsingular, $\mathbb{Q}/\mathbb{Z}$-valued symmetric forms on finite abelian groups have been classified [**W3**], [**KK**], and any such form is realized by a closed 3-manifold [**KK**].

(d)* Given an embedding $M^3 \hookrightarrow S^4$ (M closed), prove that the torsion subgroup of $H_1(M;\mathbb{Z})$ splits as a direct sum of two subgroups G_1, G_2 such that the linking form vanishes on each G_i. (*Hint*: The splitting comes from the Mayer-Vietoris sequence. Now use the fact that Q_{S^4} is trivial.) Compute the linking form of $\mathbb{RP}^3$ and prove that $\mathbb{RP}^3$ does not embed in S^4.

4.6. Examples

So far, we have Kirby diagrams describing the disk bundle over S^2 with Euler number n (Figure 4.22), $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ (Figure 4.22 with $n = \pm 1$ and a 4-handle attached), $\natural n\, S^1 \times D^3$ ($D^4 \cup n$ 1-handles), and of course, S^4

(the empty diagram with a 4-handle attached). The next obvious candidate is $S^2 \times S^2$. We begin with the obvious handle decomposition for S^2 as a pair of disks, $S^2 = D_- \cup_\partial D_+$. Then the product $S^2 \times S^2$ decomposes as $(D_- \times D_-) \cup (D_- \times D_+) \cup (D_+ \times D_-) \cup (D_+ \times D_+)$. We can interpret this as a handle decomposition with 0-handle $D_- \times D_-$, 4-handle $D_+ \times D_+$, and a pair of 2-handles in between. (This is a general construction for products of handlebodies — a product of a k-handle and an ℓ-handle is a $(k+\ell)$-handle. Compare with $T^2 = S^1 \times S^1$, Example 4.2.2 and $T^3 = S^1 \times S^1 \times S^1$, Exercise 4.3.1(a).) Now $(D_- \times D_-) \cup (D_- \times D_+) = D_- \times S^2$, with the obvious handle decomposition (Example 4.4.2), so it is given by a 0-framed unknot. Similarly, $(D_- \times D_-) \cup (D_+ \times D_-) = S^2 \times D_-$ is a 0-framed unknot. Since 4-handles are trivial, the only question is how the unknots are linked. By Proposition 4.5.11, we should expect them to have linking number 1, but there are many such links of unknots. (See Figure 4.29.)

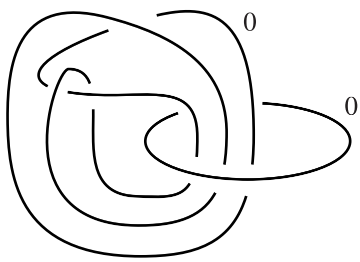


Figure 4.29. Handlebody whose boundary is a nontrivial homology sphere.

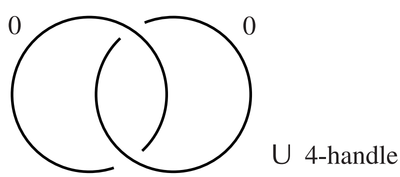


Figure 4.30. $S^2 \times S^2$.

To specify the link precisely, we observe that the attaching circles of the 2-handles are given by $S^1 \times 0$ and $0 \times S^1$ in $S^3 = \partial(D_- \times D_-)$. Now $0 \times S^1$ is clearly isotopic (disjointly from $S^1 \times 0$) to $p \times S^1$ for some $p \in \partial D_-$. The latter circle bounds the embedded disk $D = p \times D_- \subset \partial(D_- \times D_-)$. Clearly, D intersects $S^1 \times 0$ transversely in S^3, in the unique point $p \times 0$. Thus, the attaching circle ∂D is (by definition) a meridian of $S^1 \times 0$. (It is routine to check that any two disks in S^3 that each intersect a knot K transversely once are ambiently isotopic, fixing K setwise, so a meridian is uniquely determined up to isotopy.) The link $S^1 \times 0 \cup 0 \times S^1$, shown in

Figure 4.30, is called a *Hopf link*. We now have our picture of $S^2 \times S^2$. The linking matrix is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ as required, and we can explicitly see a transverse wedge of 2-spheres with product neighborhoods (by the method of proof of Proposition 4.5.11) — compare with Figure 4.29, which is not $S^2 \times S^2$. (The two obvious spheres intersect in three points.) Figure 4.31 shows another picture of $S^2 \times S^2$, in which one circle goes through ∞, and we see the torus $\partial D_- \times \partial D_-$ separating the solid tori $\partial D_- \times D_-$ and $D_- \times \partial D_-$.

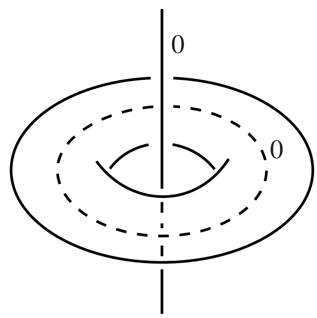


Figure 4.31. Genus-1 Heegaard splitting of ∂(0-handle) associated to the canonical handle decomposition of $S^2 \times S^2$.

Exercise 4.6.1. * Draw $S^3 \times S^1$ and $S^2 \tilde{\times} S^2$.

A diagram of a boundary sum of two handlebodies is obtained by drawing both handle decompositions in the same picture, separated by a plane. A connected sum $X \# Y$ with $\partial X = \emptyset$, $\partial Y \neq \emptyset$ is the same as $X^* \natural Y$, where $X = X^* \cup$ 4-handle. Similarly, a connected sum of closed 4-manifolds is obtained by removing both 4-handles, boundary summing and attaching a single 4-handle. Now we can draw the manifolds $\# n S^2 \times S^2$ and $\# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ $(n, m \geq 0)$ — enough simply connected, closed, smooth 4-manifolds to realize all possible intersection forms except for even forms with nonzero signature. For examples in the remaining case, we will exhibit pictures of elliptic surfaces in Section 8.3 (e.g., Figure 8.15).

Example 4.6.2. – Plumbings. Let F be a closed, possibly disconnected or nonorientable surface, and let $\pi\colon X \to F$ be a D^2-bundle over F. If D_1 and D_2 are disjoint disks in F, then each $\pi^{-1}(D_i)$ is a trivial bundle $D_i \times D^2$. We *plumb* X at D_1 and D_2 by identifying $D_1 \times D^2$ with $D_2 \times D^2$, using a map that preserves the product structures but interchanges the factors. (As usual, we smooth corners.) See Figure 4.32 for the corresponding construction with half as many dimensions. This gluing introduces a transverse self-intersection in F. A *plumbing* is a manifold obtained by finitely many applications of this procedure. Such a manifold is a regular neighborhood of (hence, deformation retracts onto) the immersed surface coming from F. Conversely, for any immersed closed surface with only transverse double point singularities in a 4-manifold, a regular neighborhood will be a plumbing. For any plumbing P, we can form a graph (or *plumbing diagram*) with a vertex for each component of F, and an edge for each plumbing that was performed. If F and X are oriented, then we can assign a sign to each edge (the sign of the corresponding intersection) and a pair of numbers to each vertex (the genus and Euler number of the corresponding component), and this data will determine P. Conversely, any such decorated finite graph will determine a plumbing.

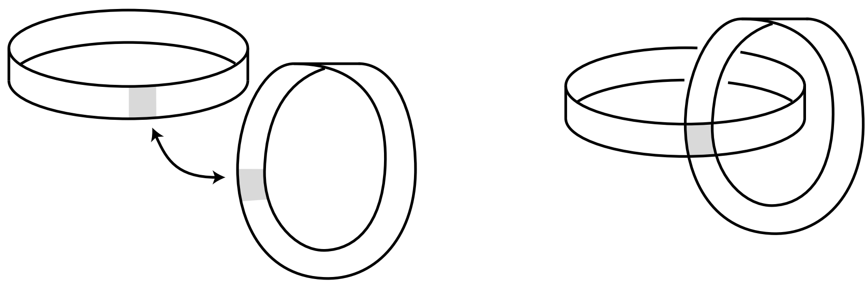


Figure 4.32. Plumbing.

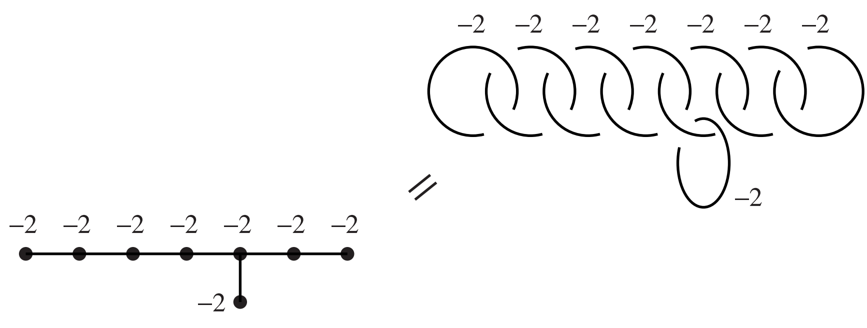


Figure 4.33. E_8-plumbing (negative).

We will now show how to draw a Kirby diagram for any plumbing of spheres for which the graph is a tree. (For more general plumbings, see Section 6.1.) The diagram generalizes that of the plumbing of two 0-framed spheres, Figure 4.30 without the 4-handle. The generalization will have an unknot for each vertex of the tree, and these will be linked in the simplest possible way such that any two vertices joined by an edge form a Hopf link. (See Figure 4.33 for the (negative) E_8-*plumbing*, whose intersection form is $-E_8$.) Each framing will be the Euler number of the corresponding D^2-bundle. Each of the plumbed spheres will be visible in the picture as the core of the corresponding 2-handle union the (pushed in) spanning disk of the unknot, and the cocore of the 2-handle will be a fiber of the D^2-bundle. (By the method of Proposition 4.5.11, we can verify directly that these spheres intersect as desired, cf. Figures 4.29 and 4.30.) To prove that this picture is correct, we use induction. We have already verified it for the case of a single vertex. For the general case, if we add a vertex and edge (without introducing π_1 in the graph), this corresponds to plumbing the new sphere bundle E onto a sphere represented by a certain 2-handle h in the diagram — that is, we identify one hemisphere of the 0-section of E with the cocore of h. It is now easy to see (as we did for $S^2 \times S^2$) that this plumbing corresponds to attaching a 2-handle along the belt circle of h, and this belt circle is isotopic to a meridian of the attaching circle of h. The proof is completed by observing that the new framing must be the Euler number $e(E)$ by Proposition 4.5.11.

Example 4.6.3. – Doubles. For a compact n-manifold X, we define the *double* of X to be $DX = \partial(I \times X) = X \cup_{\mathrm{id}_{\partial X}} \overline{X}$. Similarly, for a compact pair $(X, \partial_- X)$, the *relative* double $D(X, \partial_- X)$ is $(X \cup_{\mathrm{id}_{\partial_+ X}} \overline{X}, \partial_- X)$. (Then $\partial_- D(X, \partial_- X) = \partial_- X \subset X$ and $\partial_+ D(X, \partial_- X) = \partial_- X \subset \overline{X}$.) For example, if $X = X^* \cup n$-handle is closed, then $DX^* = X \# \overline{X}$, and the double of any D^2-bundle over S^2 is $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, depending on its Euler number modulo 2. (This solves Exercise 4.2.6(b).) If $(X, \partial_- X)$ is given a handle decomposition, then its double automatically inherits a handle decomposition. (Use the given decomposition on X, and turn it upside down (dualize) on $\overline{X}$.) If the original decomposition is ordered by increasing index, with no index exceeding $\frac{n}{2}$, then the resulting decomposition will also be ordered by increasing index.

Now suppose X is a 4-dimensional handlebody without 3- or 4-handles. We construct a diagram of DX. (See Example 5.5.4 for the relative case.) Clearly, $DX = X \cup$ handles, where each 2-handle in X generates a new 2-handle, each 1-handle generates a 3-handle and the 0-handle generates a unique 4-handle. It suffices to understand the new 2-handles. Since DX is formed by gluing using $\mathrm{id}_{\partial X}$, and each dual 2-handle h' is a copy of some 2-handle h of X with the roles of core and cocore interchanged, h' is attached to ∂X along the belt circle of h, and the attaching map is essentially $\mathrm{id}_{\partial D^2 \times D^2}$ (up to interchanging the factors of $\partial D^2 \times D^2$). Thus, the core of h' and cocore of h fit together to create a sphere with trivial normal bundle. As in the previous example, it follows immediately that h' is attached along a 0-framed meridian of h. To summarize, we transform the handlebody for X into one for DX by adding a 0-framed meridian to each link component, then attaching 3- and 4-handles to create a closed manifold. For example, the double of the disk bundle over S^2 with Euler number n is shown in Figure 4.34. This must be diffeomorphic to $S^2 \times S^2$ for n even and to

$S^2 \tilde{\times} S^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ for n odd. We will exhibit such diffeomorphisms in Section 5.1, and also show that doubles of 4-dimensional 2-handlebodies always decompose as $\# m\ S^2 \times S^2$ or $\# m\ S^2 \tilde{\times} S^2$. It is also possible to double handlebodies with 3-handles, but more work is required to reorder the handles by increasing index as in Proposition 4.2.7.

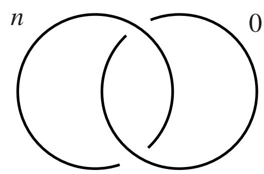


Figure 4.34. S^2-bundle over S^2.

Exercises 4.6.4. **(a)*** What is $D(\natural m\, S^k \times D^{n-k})$? (It has a simple description.) What can you say about the double of a boundary sum in general?

(b)* Prove Theorem 1.2.33, that any finitely presented group G can be realized as the fundamental group of a closed, oriented 4-manifold (or n-manifold for any fixed $n \geq 4$). Given a presentation for G, how do you construct a Kirby diagram representing a closed 4-manifold with fundamental group G? We will deduce from this construction (Exercise 5.1.10(c)) that there can be no algorithm for classifying closed 4-manifolds with arbitrary fundamental groups. For the unclassifiability of manifolds of fixed dimension $n \geq 4$, see Exercise 5.2.2(c).

Example 4.6.5. – Bundles over surfaces. We now consider bundles $\pi\colon X \to F$, where F is a closed, connected but not necessarily orientable surface, X is oriented, and the fibers are D^2 or S^2. Since the inclusion $O(2) \hookrightarrow \mathrm{Diff}(D^2)$ is a homotopy equivalence, any D^2-bundle can be assumed to have structure group $O(2)$ as in Example 4.1.4(d) (i.e., it is the unit disk bundle in a vector bundle with fiber $\mathbb{R}^2$). Similarly, the homotopy equivalence $O(3) \hookrightarrow \mathrm{Diff}(S^2)$ implies that any S^2-bundle is the unit sphere bundle in some $\mathbb{R}^3$-vector bundle. (For these homotopy equivalences, see e.g. Theorem 3.10.11 of [**Th2**].) In particular, any S^2-bundle over a surface F will be the double of a D^2-bundle (since the $\mathbb{R}^3$-bundle will have a nonzero section by transversality, splitting it as an $\mathbb{R}^2$-bundle summed with $\mathbb{R}$).

First, we assume that X is a D^2-bundle. Fix a handle decomposition of F with a unique 0- and 2-handle, and hence, $m = 2 - \chi(F)$ 1-handles. (As usual, χ denotes the Euler characteristic.) We obtain a handle decomposition of X whose k-handles are the preimages under π of k-handles in F. Since X is oriented, the union of 0- and 1-handles is determined by m. (In fact, the bundle $\pi\colon X \to F$ is determined over $F - \{\text{pt.}\}$, since the orientability of X implies that the bundle is twisted over a 1-handle $(\pi_0(O(2)) \cong \mathbb{Z}_2)$

if and only if the 1-handle in F is twisted; $w_1(\pi) = w_1(F)$ in the language of Section 1.4.) Now it only remains to determine the framed attaching circle of the 2-handle in $\partial(\natural m\, S^1 \times D^3)$. We will compute this explicitly in examples below. As we saw for $F = S^2$ (Example 4.4.2), the framing will be determined by an integer invariant called the *Euler number* $e(X)$ (which also determines the bundle structure since it is already specified over $F - \{\text{pt.}\}$). If F is orientable, then $e(X)$ is defined (as in Example 4.1.4(d) or Section 1.4) to be the entry in the 1×1 intersection matrix of X. If F is nonorientable, then $H_2(X; \mathbb{Z}) = 0$, but one can interpret $e(X)$ as before using homology with twisted coefficients. More geometrically, we can define $e(X)$ to be the self-intersection number of the 0-section F_0. (To define this, let F_0' be a surface transverse to F_0 and isotopic to it. Then the intersections $p \in F_0 \cap F_0'$ have well-defined signs, since an orientation defined near p on F_0 defines one on F_0', and reversing both orientations preserves the sign of p.) Alternatively, $e(X) = \frac{1}{2} e(\widetilde{X})$, where $\widetilde{X}$ is the cover of X corresponding to the orientable double cover of F.

The case of S^2- (D^3-) bundles is simpler. As before, such a bundle $\pi\colon X \to F$ is determined over $F - \{\text{pt.}\}$ (given that X is oriented). However, the last framing lies in $\pi_1(O(3)) \cong \mathbb{Z}_2$, so for fixed F there are only two S^2-bundles $\pi\colon X \to F$ with X oriented. (These bundles are distinguished by the fact that exactly one X is spin, or by their intersection pairings, using $\mathbb{Z}_2$-coefficients if F is nonorientable.) If F is oriented, these bundles will be $X = F \times S^2$ and $F \tilde{\times} S^2$. In general, the two bundles will arise as doubles of D^2-bundles X' over F, and the resulting bundle $X = DX'$ will be determined by $e(X')$ modulo 2. (When $e(X')$ is even, so is the $\mathbb{Z}_2$-intersection pairing of X, and assuming F is orientable, $X = F \times S^2$.)

Now we draw D^2-bundles X over T^2 explicitly, keeping track of the 0-section T_0, beginning with the standard handle decomposition of T^2, Figure 4.35. The 0-handle of T_0 lies in the 0-handle of X, and is visible in Figure 4.36(a) as a disk whose interior has been pushed into int D^4. We attach the 1-handles of (X, T_0) *pairwise*, i.e., attach the 1-handles of X so that they contain the 1-handles of T_0. We obtain Figure 4.36(b), where each 1-handle is given by a pair of diametrically opposite balls, and the gluing identifies arcs on the boundary of the 0-handle of T_0 as required (cf. Figure 4.35). Now T_0 appears in Figure 4.36(b) as a punctured torus T_0^* in $S^1 \times D^3 \natural S^1 \times D^3$, bounded by the given circle. Since the attaching circle of the 2-handle must be this same circle ∂T_0^*, we are done once we specify the framing. Although we have not yet defined framing coefficients for circles running over 1-handles, we can do so in this case by taking 0 to represent the blackboard framing in the given diagram, or equivalently, defining the coefficient n to be the intersection number of a parallel curve with the obvious punctured-torus Seifert surface. (In fact, the latter definition is isotopy-invariant; see Section 5.4.) We immediately see (as in the proof of Proposition 4.5.11) that $n = e(X)$. Note that T_0 is clearly visible as T_0^* union the core of the 2-handle, and the cocore of the 2-handle represents a fiber of $\pi\colon X \to T^2$.

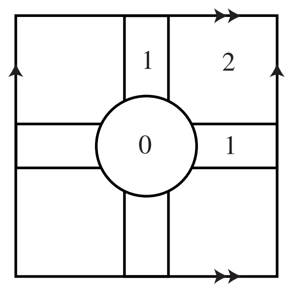


Figure 4.35. Handle decomposition of T^2.

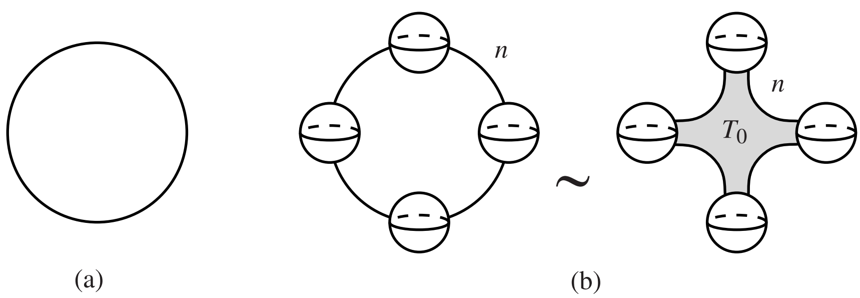


Figure 4.36. D^2-bundle over T^2 with Euler number n.

Exercises 4.6.6. **(a)*** Visualize the fibration of $T^2 \times D^2$ (Figure 4.36 with $n = 0$) by tori $T^2 \times \{\text{pt.}\}$.
(b)* Draw pictures of arbitrary D^2- and S^2-bundles over orientable surfaces, and prove your answer correct. Do the same for a plumbing of two D^2-bundles over T^2.

Finally, we consider disk bundles X over $\mathbb{RP}^2$. We proceed as before, beginning with the standard decomposition of $\mathbb{RP}^2$ in Figure 4.37. Since the 0-handle of the 0-section F_0 must be glued to itself with a half-twist, we obtain Figure 4.38, with the framing on the 2-handle still to be determined. The 0-section F_0 is visible as a Möbius band F_0^* in $S^1 \times D^3$, together with the core of the 2-handle. It is more delicate to deal with framings now, since the attaching circle K is not nullhomologous in $S^1 \times S^2$. Our previous argument still shows that the framing coefficient on K should equal $e(X)$, *provided* that we define coefficients by declaring that the outward normal v

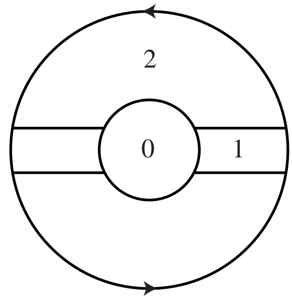


Figure 4.37. Handle decomposition of $\mathbb{RP}^2$.

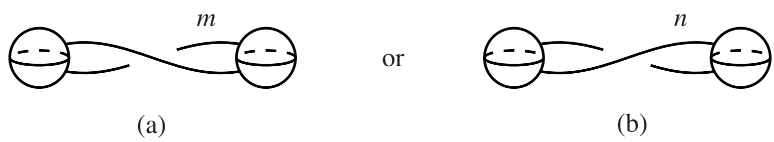


Figure 4.38. D^2-bundle over $\mathbb{RP}^2$ with Euler number $m - 2 = n + 2$.

to the Möbius band should determine the 0-framing. (The argument works even though the normal bundle νF_0^* is nontrivial, since it still has a trivial summand. Specifically, we form a parallel copy of F_0^* bounded by a curve K' determined by v, by dropping K' below the level of F_0^* in D^4, and then filling in the Möbius band.) Defining v to be 0 is not a natural convention, however — even in S^3, nonorientable Seifert surfaces may not determine the 0-framing. (See Exercise 4.5.9.) A better convention in Figure 4.38 (see Section 5.4) is to generalize Proposition 4.5.8, defining the coefficient of the blackboard framing to equal the writhe $w(K)$, $+1$ and -1 in Figure 4.38(a) and (b), respectively. (Beware that the isotopy from (a) to (b) (by flipping one strand around a ball) does not preserve the 0-framing!) Now we can measure any framing by counting undercrossings, and the coefficient of v is $+2$ and -2 in (a) and (b), respectively. Thus, Figure 4.38 represents a given X if $m = e(X) + 2$ and $n = e(X) - 2$.

Exercises 4.6.7. (a) Verify directly that for $m, n = e \pm 2$ as above (e fixed), the manifolds of (a) and (b) of Figure 4.38 are diffeomorphic. (Isotope one picture to the other, using the double-strand notation (Figures 4.18–4.20) for framings.) Generalize to a picture with any odd number of half-twists.

(b) Draw a picture for any oriented 4-manifold arising as a D^2- or S^2-bundle over a nonorientable surface.

Example 4.6.8. – Products with 3-manifolds. We give procedures for constructing diagrams for $I \times M^3$ and $S^1 \times M^3$. (The latter generalizes to M^3-bundles over S^1. For a different approach, see Exercise 6.2.5(b) or

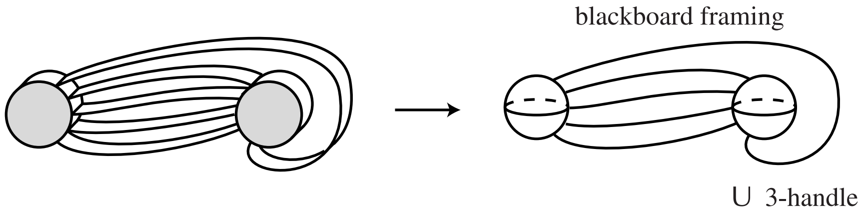


Figure 4.39. $I \times L(5,1)$.

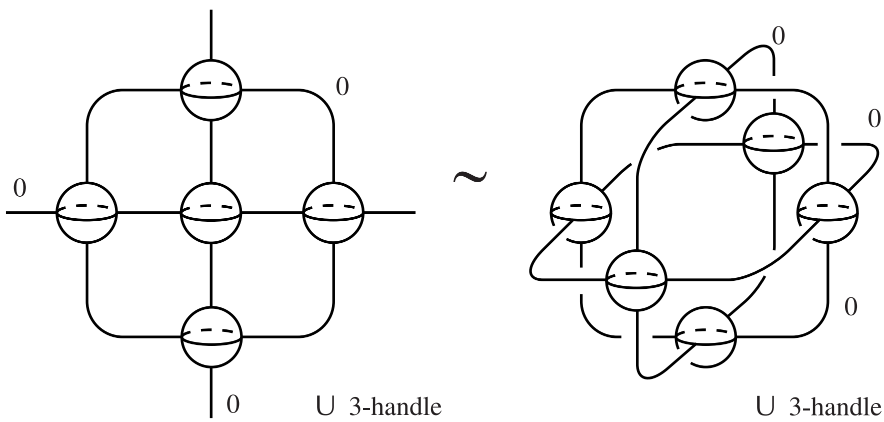


Figure 4.40. $I \times T^3$.

[**A6**].) We saw in Section 4.3 that any handle decomposition of a 3-manifold M can be given by a diagram in $\mathbb{R}^2$. To obtain the corresponding handle decomposition of $I \times M$, simply cross the diagram with I to get a Kirby picture in $I \times \mathbb{R}^2 \subset \mathbb{R}^3$. For example, Figure 4.39 shows $I \times L(5,1)$ (cf. Figure 4.14), and Figure 4.40 shows $I \times T^3$ (cf. Exercise 4.3.1(a)). (In the latter figure, one 1-handle is added at 0 and ∞, with the spheres identified by a radial contraction.) The 2-handles will always be framed by the blackboard framing, coming from the unique framing of the corresponding attaching circle in $\mathbb{R}^2$. To see this construction more clearly, recall that $\mathbb{R}^2 \cup \{\infty\} = \partial D^3$, so $I \times \mathbb{R}^2$ represents $I \times \partial D^3 \subset I \times D^3 = D^4$. The remaining boundary of D^4, $\{0,1\} \times D^3 \subset I \times D^3$, is given by the two balls D_i (oppositely oriented) obtained by one-point compactifying the components of $\mathbb{R}^3 - (\text{int } I \times \mathbb{R}^2)$. If M is closed, then $\mathbb{R}^2$ is filled by attaching regions (including the 3-handle boundary), and $I \times \mathbb{R}^2$ will also be filled, i.e., the part not in $I \times$ (diagram) is part of the attaching region of the 3-handle. Thus, $\partial(I \times M)$ is seen as the two balls D_i, together with the parts of the handles lying over $\{0,1\} \times \mathbb{R}^2$. These parts comprise 3-dimensional handles added to the balls D_i by the original diagram, so we see directly that the boundary of

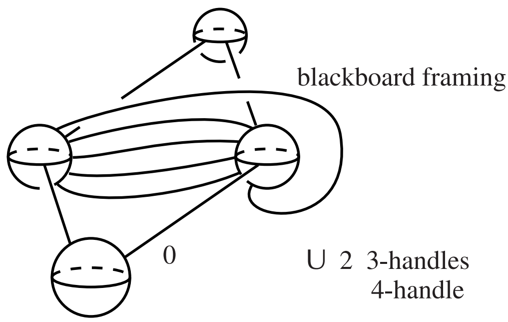


Figure 4.41. $S^1 \times L(5,1)$.

our 4-dimensional handlebody is $M \coprod \overline{M}$. (We will discuss boundaries more generally in Section 5.2.)

We now turn to $S^1 \times M$. Since $S^1 = $ 0-handle $\cup$ 1-handle, $S^1 \times M$ will inherit a handle decomposition, with each k-handle of M generating a k- and a $(k+1)$-handle. We have just described the subhandlebody coming from the 0-handle of S^1, so we only need to determine how a k-handle h of M generates a $(k+1)$-handle h' of $S^1 \times M$. If h has core C, then h' has core $(S^1 \times C) - (\text{int } I) \times C$ and attaching sphere $\{0,1\} \times C \cup (S^1 - \text{int } I) \times \partial C$, and h' has the effect of gluing $h \times \{0\}$ to $h \times \{1\}$. The 0-handle generates a 1-handle whose attaching region is $D_0 \cup D_1$. For convenience, we will shrink these balls to standard size. (See Figure 4.41.) Then one ball will lie in each boundary component of $I \times M$ if M is closed. Each 1-handle of M generates a 2-handle, whose attaching circle is made from $\{0,1\} \times C$ (where C is the core of the 1-handle) by connecting $\{0\} \times \partial C$ to $\{1\} \times \partial C$ with product arcs in the new 1-handle. In fact, the union of the 0-handle, old and new 1-handle and the 2-handle is a copy of $T^2 \times D^2$, generated from 0-handle $\cup$ 1-handle $\approx S^1 \times D^2$ in M^3. Thus, the framing on the 2-handle is the one determined by the punctured torus Seifert surface. Since the remaining handles of $S^1 \times M$ have index ≥ 3, the construction is now complete. We obtain Figures 4.41 and 4.42. (Again, one 1-handle in Figure 4.42 is attached at ∞.)

Exercises 4.6.9. (a) Find four ways of identifying Figure 4.42 as $S^1 \times T^3$. Find six copies of T^2 in the figure. Find three plumbings of pairs of copies of $T^2 \times D^2$.

(b) Verify that the graph in S^3 given by Figure 4.42 represents the intersection of S^3 with the four coordinate axes and six coordinate 2-planes in $\mathbb{R}^4$. Use this for a different proof that the diagram represents T^4. (*Hint*: See Exercise 4.3.1(a).) Where are the attaching spheres of the 3-handles?

(c) Use the method of Example 4.6.8 to verify the correctness of our previous pictures of $F \times D^2$, F oriented (Exercise 4.6.6).

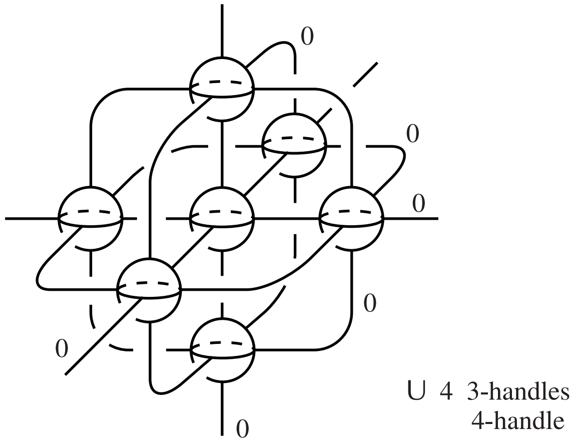


Figure 4.42. T^4.

(d)* Draw a handle picture of $I \times \mathbb{RP}^3$. (See Exercise 4.3.1(a).) Explain the similarity with Figure 4.38. What is the corresponding value of $e(X)$ given by Figure 4.38? Why?

Chapter 5

Kirby calculus

In Section 4.2, we introduced a complete set of moves for handlebodies, namely handle pair creation/cancellation and handle sliding, which (together with isotopies) are sufficient for getting between any two relative handle presentations of a given pair $(X, \partial_- X)$ (Theorem 4.2.12). We begin this chapter with a section describing these moves in the context of Kirby diagrams, as well as the operations of blowing up and down that we introduced in Section 2.2. These *Kirby moves* are the basic tools of Kirby calculus. In the next two sections, we study the boundaries of handlebodies, leading to surgery and related constructions. These constructions allow us to study 3-manifolds with the techniques of Kirby calculus. They also facilitate our introduction of new notation for 1-handles (Section 5.4), with which we can eliminate the ambiguity of framing coefficients that we encountered in the presence of 1-handles in Chapter 4. In Section 5.5, we use surgery on 3-manifolds to develop Kirby diagrams for relative handlebodies $(X^4, \partial_- X^4)$. Finally, we return to spin structures, studying them in arbitrary dimensions from the perspective of handlebodies (obstruction theory) in Section 5.6, and then specializing to dimensions 3 and 4 with Kirby calculus in Section 5.7.

5.1. Handle moves

We begin with handle sliding (Definition 4.2.10). Given a pair of handles h_1 and h_2 of the same index k attached to a manifold Y, we isotope the attaching sphere for h_1 in $\partial(Y \cup h_2)$, sliding it along a disk $D^k \times \{\text{pt.}\} \subset \partial h_2$ (where $h_2 = D^k \times D^{n-k}$), and returning it to ∂Y. For example, Figure 5.1 shows a 1-handle slide on a 2-manifold, which proves (after we attach a 2-handle) that the Klein bottle is diffeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$ (Exercise 4.2.11). Note that the handle slide changed the attaching sphere of h_1 (unlinked it from that of

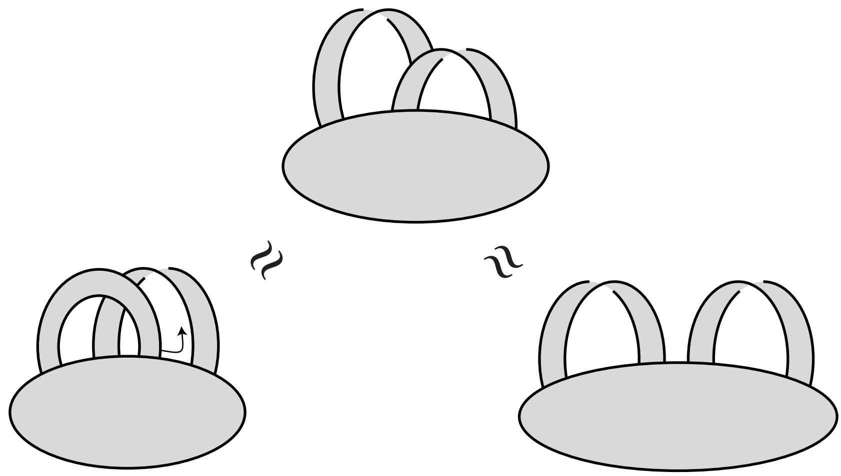


Figure 5.1. Handle slide — $S^1 \tilde{\times} S^1 \approx \mathbb{RP}^2 \# \mathbb{RP}^2$.

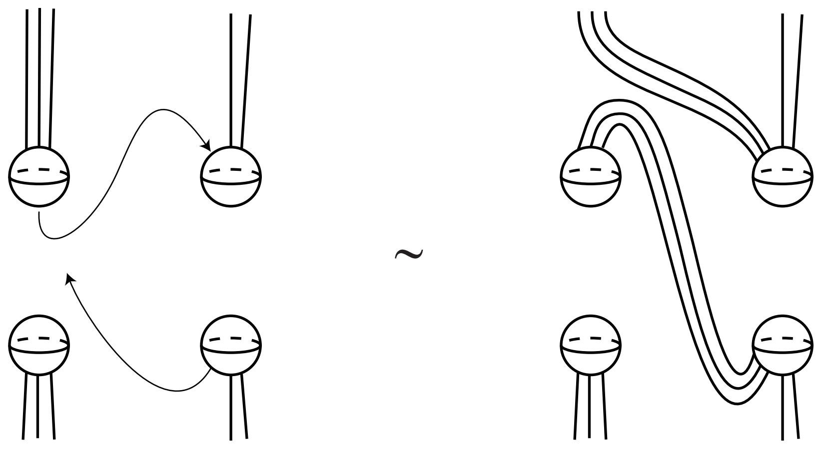


Figure 5.2. 1-handle slide.

h_2), and also changed the framing (which in this case is a well-defined element of $\pi_0(O(1)) \cong \mathbb{Z}_2$), since it no longer respects the orientation of D^2. It is also easy to draw 1-handle slides in Kirby diagrams. One simply takes one attaching ball of h_1 and pushes it through the 1-handle h_2 as in Figure 5.2 (where the attaching balls of each 1-handle are aligned vertically). One can keep track of framings in the obvious way using the double-strand notation. (See Section 5.4 for defining and keeping track of framing coefficients in this setting.)

Exercise 5.1.1. Figure 5.3 shows a D^2-bundle over the Klein bottle. Prove (by Kirby calculus) that it is a D^2-bundle over $\mathbb{RP}^2 \# \mathbb{RP}^2$. How do the Euler

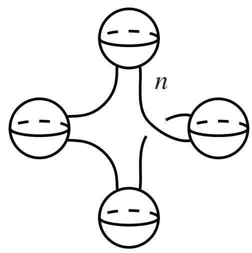


Figure 5.3. D^2-bundle over Klein bottle with Euler number n.

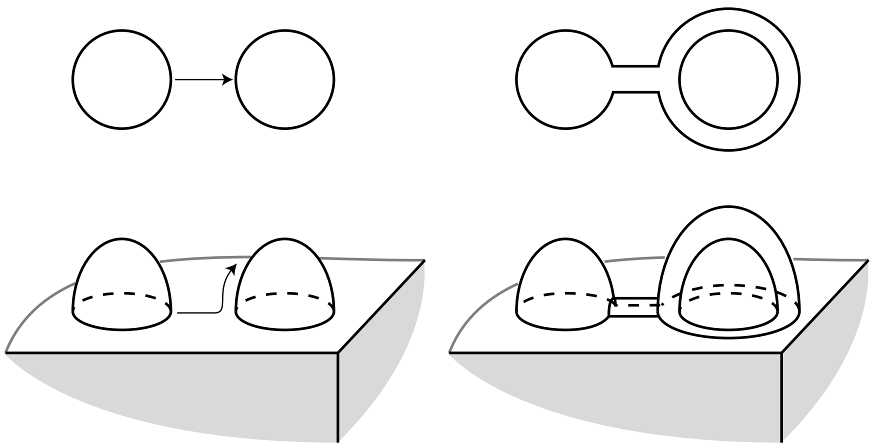


Figure 5.4. 2-handle slide.

numbers correspond? (The coefficient of the blackboard framing is taken to be $w(K) = -1$.)

Sliding 2-handles requires a bit more work. Figure 5.4 shows how the core disks move in the 3-dimensional case. In the 4-dimensional version, 2-handles h_1 and h_2 will be attached along framed knots K_1 and K_2. A parallel curve K_2' determining the framing on K_2 will bound a disk $D^2 \times \{\text{pt.}\} \subset \partial(Y \cup h_2)$; in fact, the framing determines arbitrarily many such parallel curves bounding disjoint disks. We slide h_1 by isotoping K_1 over one such disk. In practice, this means we form a *band-sum* of K_1 and K_2', i.e., we form the connected sum along some band as in Figure 5.5. Since we may precede the slide by any isotopy, we are allowed to use any band disjoint from the rest of the link, and the choice will affect the resulting link. If K_1 and K_2 are oriented, we call the move a *handle addition* if the sum respects the orientations of K_1 and K_2', and a *handle subtraction* otherwise.

We still need to determine the new framing of h_1. The most elementary and general way to do this is with the double-strand notation. One simply

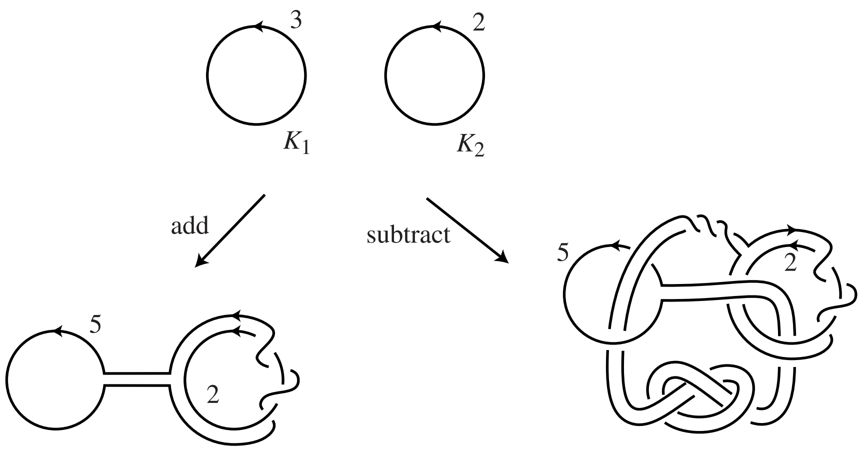


Figure 5.5. 2-handle slides.

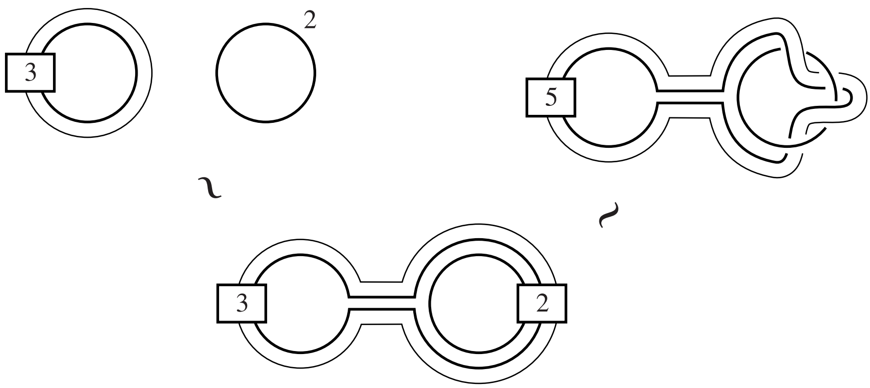


Figure 5.6. Change of framing under a handle slide.

isotopes both strands over parallel disks in ∂h_2 by making two parallel band-sums as in Figure 5.6 (which justifies the addition in the previous figure). If our 2-handles are attached to D^4, however, it is easier to use framing coefficients. Recall that if we orient the framed link L representing a 2-handlebody X, then we obtain a canonical basis $\alpha_1, \dots, \alpha_m$ for $H_2(X)$, and the intersection form is given with respect to this basis by the linking matrix of L (Proposition 4.5.11). Now observe that if we slide h_i over h_j, then we change the basis for $H_2(X)$ by replacing α_i by $\alpha_i' = \alpha_i \pm \alpha_j$, adding or subtracting depending on whether we add or subtract handles (Figure 5.7). (Thus, changing the band by a half-twist reverses the sign, and the formula is otherwise independent of the choice of band.) Now the new framing coefficient will be given by

$$(*) \qquad (\alpha_i \pm \alpha_j)^2 = \alpha_i^2 + \alpha_j^2 \pm 2\alpha_i \cdot \alpha_j = n_i + n_j \pm 2\ell k(K_i, K_j),$$

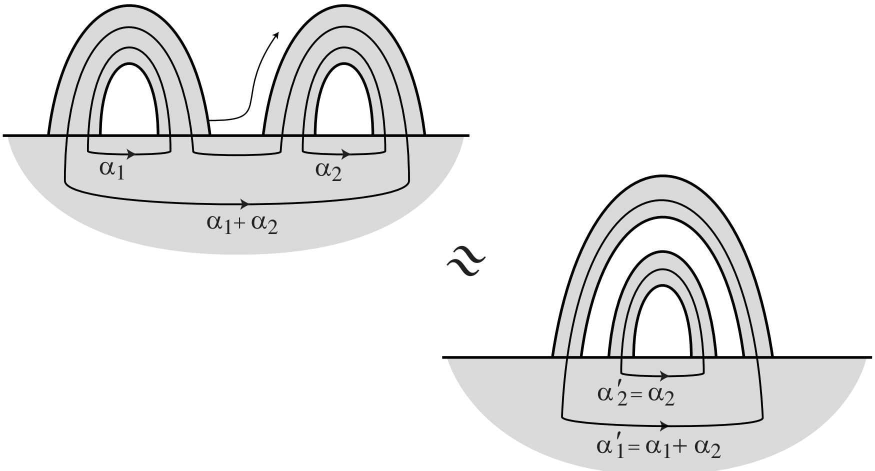


Figure 5.7. Change of basis induced by handle addition.

where n_k is the framing coefficient of K_k, and the sign is $(+)$ for handle addition and $(-)$ for subtraction. Note that this is independent of our choice of orientation of L (as the double strand notation implies), since changing the orientation of either K_i or K_j will change both the linking number and the sign appearing in the formula. This formula can also be applied in the presence of 1-handles; see Section 5.4.

Exercises 5.1.2. (a) Check Formula $(*)$ by using the double-strand notation. How does the linking number come in?

(b)* Show that any handle slide can be followed by another slide that reverses it. Describe the reversing slide explicitly, and check that the framing coefficients transform as required.

(c)* Give an algorithm for simultaneously sliding many strands of one attaching curve over another one and computing the resulting framing, as in Figure 5.8. (Check the figure.) In particular, what happens to the framing when the signed number of strands is 0?

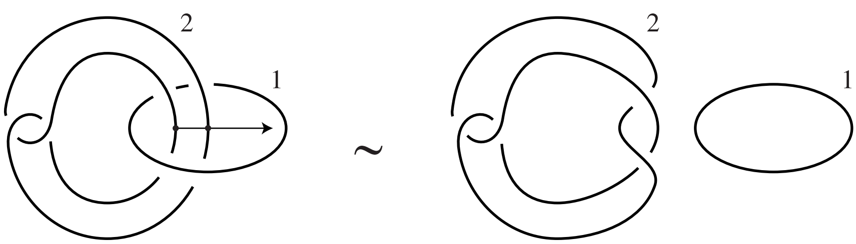


Figure 5.8. Multiple handle slide.

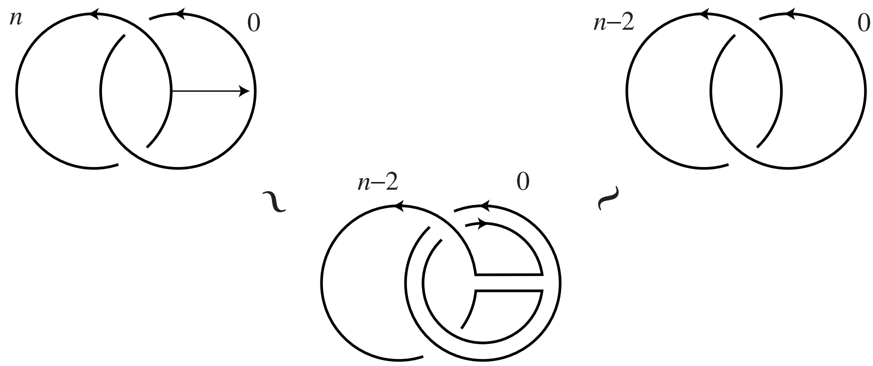


Figure 5.9. Diffeomorphism of S^2-bundles over S^2.

Examples 5.1.3. (a) Consider an S^2-bundle over S^2 realized as the double of the disk bundle with Euler number n, Figure 5.9 with a 4-handle added. If we subtract handles as shown, we will recover a Hopf link, but n will be reduced by 2. This gives a direct proof that the diffeomorphism types of these manifolds only depend on n modulo 2. Note that instead of applying Formula $(*)$, we can keep track of the entire linking matrix: We start with $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then the change of basis subtracts the second row from the first, followed by the same operation on columns, and we obtain $\left[\begin{smallmatrix} n-2 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. This can be a useful technique when the algebra guides the topology.

(b) The slide indicated in Figure 5.10 shows that $S^2 \tilde{\times} S^2$ is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. This slide is suggested by diagonalizing the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Compare with $S^1 \tilde{\times} S^1$, Figure 5.1.

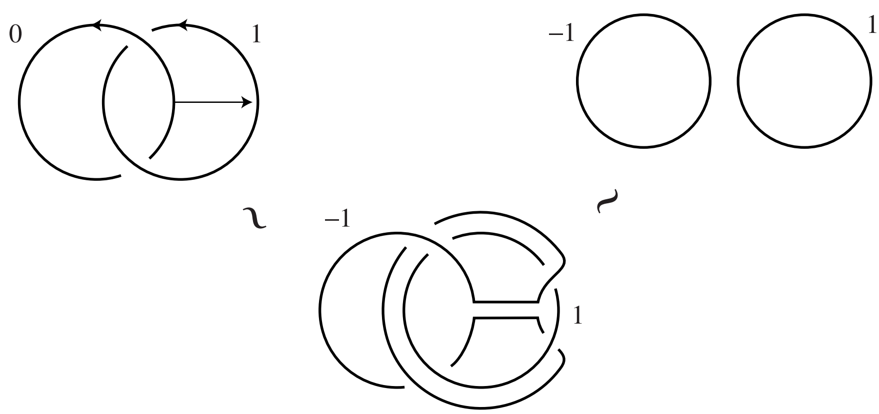


Figure 5.10. $S^2 \tilde{\times} S^2$ is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Proposition 5.1.4. *Let X^4 be a handlebody given by a Kirby diagram. Suppose that K_1 and K_2 are attaching circles in the diagram such that K_1 lies*

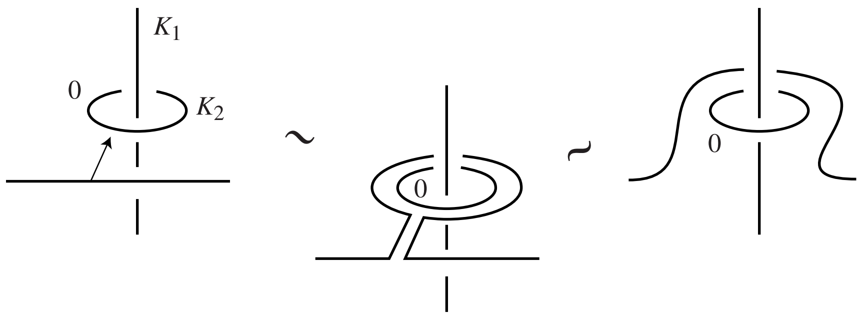


Figure 5.11. Changing a crossing by a handle slide.

entirely in ∂D^4 and K_2 is a 0-framed meridian of K_1 (i.e., K_2 bounds a disk in ∂D^4 intersecting the link in a single transverse intersection with K_1). Then $X = Y \# S$, where Y is obtained from X by erasing K_1 and K_2, and S equals $S^2 \times S^2$ if the framing coefficient n of K_1 is even and $S^2 \tilde{\times} S^2$ otherwise.

Proof. By Figure 5.11, we can change any undercrossing of K_1 to an overcrossing. In the case of a self-crossing, n will change by 2, and otherwise all framings will be unchanged. We can use this procedure to bring K_1 and K_2 entirely in front of the rest of the picture, and then to unknot K_1. Then we can isotope K_1 and K_2 away from the rest of the diagram, where they form an S^2-bundle summand as in Figure 5.9. □

Corollary 5.1.5. *Let X^4 be a handlebody without 1-handles and with an odd intersection form Q_X. Then $X \# S^2 \times S^2$ and $X \# S^2 \tilde{\times} S^2$ are diffeomorphic.*

Proof. Since X has an odd intersection form, its Kirby diagram has a component K with odd framing. Sum X with $S^2 \times S^2$ by adding a 0-framed Hopf link to the diagram, and slide one component of this over K, so that its framing becomes odd. Now apply Proposition 5.1.4. □

This corollary is a special case of a theorem of Wall; see Proposition 5.2.4.

Corollary 5.1.6. *Let X^4 be a 2-handlebody with m 2-handles. Then the double DX is diffeomorphic to $\# m\, S^2 \times S^2$ if Q_X is even, and to $\# m\, \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ otherwise. In particular, if Y is a closed 4-manifold built without 1- or 3-handles, then $Y \# \overline{Y}$ admits such a connected sum splitting.* □

Again, examples of such manifolds Y include S_d and $E(n)_p$, and it is not known if all closed, simply connected 4-manifolds have such handle decompositions. For more on connected sum splittings, see Theorem 9.1.15 and the associated text (cf. also Exercise 5.1.10(b)).

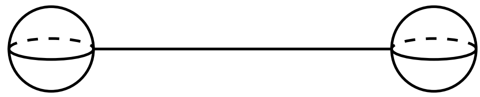


Figure 5.12. Cancelling handle pair.

Exercises 5.1.7. (a) Prove Corollary 5.1.6.

(b)* Prove that Proposition 5.1.4 is still true if K_1 is allowed to run over 1-handles, provided that it is nullhomotopic in the boundary of the union X_1 of 0- and 1-handles. Is it still true if K_1 is only required to be nullhomologous?

Next, we consider handle cancellation. Recall (Proposition 4.2.9 and Figure 4.7) that a $(k-1)$-handle and a k-handle can be cancelled if the attaching sphere of the latter intersects the belt sphere of the former transversely in a unique point (regardless of framings). For $k = 2$, this is shown in Figure 5.12. The cancellation consists of erasing both handles. Note that since there is a unique isotopy class of framed embeddings of an interval in any 3-manifold, there is essentially a unique way to draw a cancelling 1-handle/2-handle pair. That is, we can unknot the attaching circle and slide it off of any other 1-handles by an isotopy in ∂X_1. The only complication occurs when there are other 2-handles running over the 1-handle (Figure 5.13, with the framing indicated by double-strand notation). If this occurs, we can reduce to the previous case by sliding the extra handles over the 2-handle that we wish to cancel, removing them from the 1-handle (and then untangling and erasing the cancelling pair as before).

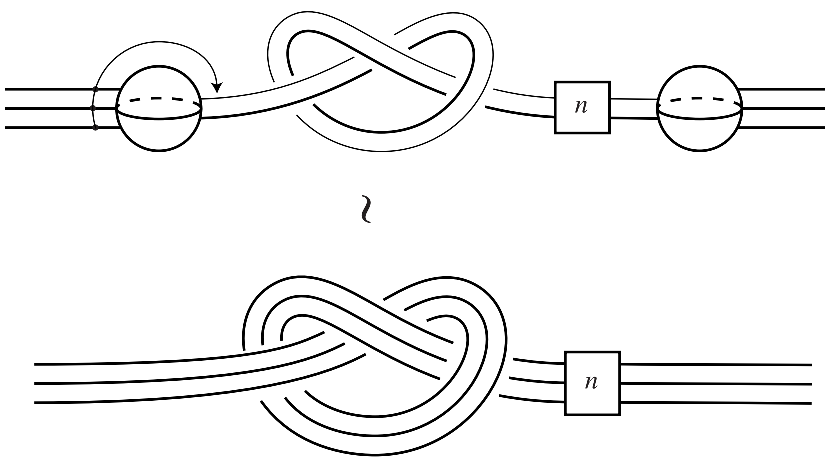


Figure 5.13. Cancelling handle pair.

Exercises 5.1.8. (a) Verify the equivalence in Figure 5.13. (Remember that handle slides can be performed simultaneously, Exercise 5.1.2(c).)

(b) Show that if we attach two 2-handles to $T^2 \times D^2$ as in Figure 5.14 (with framing coefficients given relative to the blackboard framings), we get a 2-handlebody on a 0-framed (right-handed) trefoil knot (Figure 4.27). (To compute the framings easily, see Exercise 5.1.2(c).)

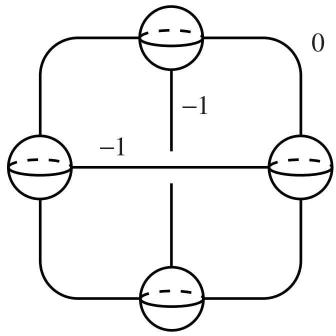


Figure 5.14. $T^2 \times D^2 \cup$ two 2-handles.

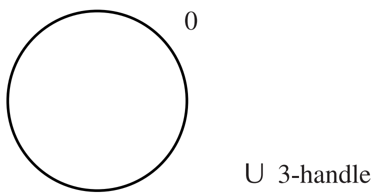


Figure 5.15. Cancelling handle pair.

A model for a cancelling 2-handle/3-handle pair is shown in Figure 5.15. The 3-handle is attached to the obvious S^2 in $\partial(D^4 \cup 2\text{-handle}) = \partial(S^2 \times D^2)$, which intersects the belt sphere of the 2-handle (a meridian, i.e., $\{\text{pt.}\} \times S^1$) in a unique point, as required. We cancel by erasing both handles. In fact, any cancelling 2-3 pair can be made to look like Figure 5.15 by suitably sliding the 2-handle. To see this, suppose that X is an arbitrary handlebody with a cancelling 2-3 pair. Then we can assume that the attaching sphere of the 3-handle intersects the cancelling 2-handle h in $D^2 \times \{\text{pt.}\} \subset D^2 \times \partial D^2$. The complementary disk D of the attaching sphere will be embedded in ∂Y, where Y is the union of D^4 with the 1,2-handles other than h, with ∂D sharing a tubular neighborhood $\nu\partial D = \nu K$ with the attaching circle K of h (cf. the 1-2 pair given by Figure 5.12). We can arrange D to lie in ∂D^4 by precomposing the embedding $D \hookrightarrow \partial Y$ with an isotopy radially shrinking D, and extending to an ambient isotopy in ∂Y, dragging along $K \subset \nu\partial D$. This isotopy in ∂Y will appear in the diagram as a sequence of handle slides by K. After the slides, D will appear in ∂D^4 as a spanning disk to K as in

Figure 5.15 (with no other 1,2-handles intersecting the picture), where the framing must be 0 since ∂D bounds $D^2 \times \{\text{pt.}\} \subset \partial h$.

There is also a converse that allows us to cancel 0-framed unknots. Suppose that X is an orientable handlebody with $\partial_+ X = \emptyset$, and that we have located a 2-handle h attached to a 0-framed unknot in ∂D^4 that is isolated from the rest of the diagram. If $Y = D^4$ union the 1,2-handles other than h, then $\partial(Y \cup h)$ is diffeomorphic to $\# m\ S^1 \times S^2$ (since $\partial_+ X = \emptyset$). Clearly, h determines an $S^1 \times S^2$-summand of $\partial(Y \cup h)$. By uniqueness of prime connected-sum decompositions of oriented 3-manifolds [**He**], it follows that $\partial Y = \#(m-1)\ S^1 \times S^2$. Thus, we can make a closed manifold Z from $Y \cup h$ by first cancelling h and then adding other 3,4-handles. By uniqueness of 3,4-handle addition Z is diffeomorphic to X, and there is a 3-handle in X that cancels h (after 3-handle slides). Thus, when $\partial_+ X = \emptyset$, any time we find an isolated 0-framed unknot in the diagram we can cancel it against a 3-handle. If $\partial_+ X$ is nonempty but connected and X is simply connected (with at least one 3-handle), we obtain a similar result from [**Tr**]. Summarizing, we have the following:

Proposition 5.1.9. *Let X be an oriented handlebody with $\partial_+ X = \emptyset$. Then a 3-handle can be cancelled (after sliding 2- and 3-handles) if and only if it is possible to slide 2-handles to obtain a 0-framed unknot isolated from the rest of the diagram. If so, then we cancel by erasing the unknot and 3-handle. Similarly, if $\partial_+ X$ is connected and X is simply connected, then erasing an isolated 0-framed unknot and 3-handle will preserve the diffeomorphism type of the manifold X.* □

In practice, it can be quite difficult to apply this proposition, since it gives no clue as to how to slide the 2-handles.

Exercises 5.1.10. **(a)*** Identify the familiar closed manifold shown in Figure 5.16. Check your answer by computing the intersection form.

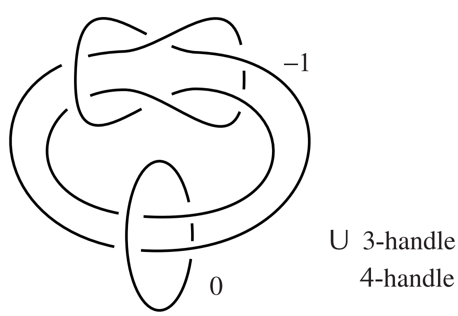


Figure 5.16. Identify this 4-manifold.

(b)* Let X be a handlebody of the form $D^4 \cup (\ell$ 1-handles$) \cup (m$ 2-handles$)$. Prove that if X is simply connected, then $DX \# \ell S^2 \times S^2$ is diffeomorphic to $\# mS^2 \times S^2$ or $\# m\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$. (*Hint*: For each 1-handle h in X, there is a knot K in ∂X such that attaching a 2-handle to X along K cancels h. What does the π_1-condition imply about K?)

(c)* Prove Markov's theorem that there exists no algorithm that can be applied to arbitrary pairs of closed, orientable 4-manifolds to determine whether or not they are diffeomorphic. Use the fact ([**Ad**], [**Ra**], see also [**Mi**]) that there is no algorithm that can be applied to arbitrary finite group presentations to determine whether they present the trivial group. (*Hint*: Apply (b) above to Exercise 4.6.4(b).)

(d)* Let P be a presentation of the trivial group with the same number of relators as generators. Then we can manipulate P by *Andrews-Curtis* moves, namely inversion and permutation of generators and of relators, conjugation of relators by generators, multiplying one generator (resp. relator) by another one, and adding or deleting a generator g together with a relator equal to g. (Recall that relators are elements of the free group on the generators, normally generating the kernel of the epimorphism to the desired group. They are determined up to Andrews-Curtis moves by a complete set of relations, by writing the relations in the form relator $= 1$.) It is easy to see that Andrews-Curtis moves do not change the (trivial) group presented by P. Suppose that P is *Andrews-Curtis trivial*, i.e., it can be reduced to the empty presentation by Andrews-Curtis moves. Prove that the closed 4-manifold constructed using P in Exercise 4.6.4(b) (the double of a handlebody realizing P) is diffeomorphic to S^4.

Remark 5.1.11. The *Andrews-Curtis Conjecture*, that any P as above (presenting the trivial group with the same number of generators as relators) should be Andrews-Curtis trivial, is still unresolved, but there are many likely counterexamples. For example, for presentations of the form $\langle x, y \mid y = w^{-1}xw,\ x^{n+1} = y^n\rangle$, where w is any word in $x^{\pm 1}$ and $y^{\pm 1}$, even the simple cases $w = yx$, $n \geq 3$, are not known to be Andrews-Curtis trivial. (Check for yourself that the cases $n = 0, 1$ *are* AC-trivial, as are the cases $w = x^k y^l$ for all n. The case $w = yx$, $n = 2$ is also trivial by [**Ge**].) However, these presentations always give the trivial group, since the element x^{n+1} (being equal to y^n) commutes with both generators, so $y^{n+1} = (w^{-1}xw)^{n+1} = w^{-1}x^{n+1}w = x^{n+1} = y^n$. The manifolds DX_P associated to conjecturally AC-nontrivial presentations P by Exercise 4.6.4(b) are candidates for exotic 4-spheres. (They are homeomorphic to S^4 by Freedman's Theorem 1.2.27 since they are simply connected with Euler characteristic 2, but are not generally known to be diffeomorphic to S^4.) Note, however, that even when P is not known to be Andrews-Curtis trivial, it is sometimes possible to trivialize the handle decomposition of X_P (if $\partial X_P = S^3$) by introducing a cancelling 2-handle/3-handle pair (introducing a new relator with no corresponding generator). This is done for $w = yx$ (for example) in [**G8**].

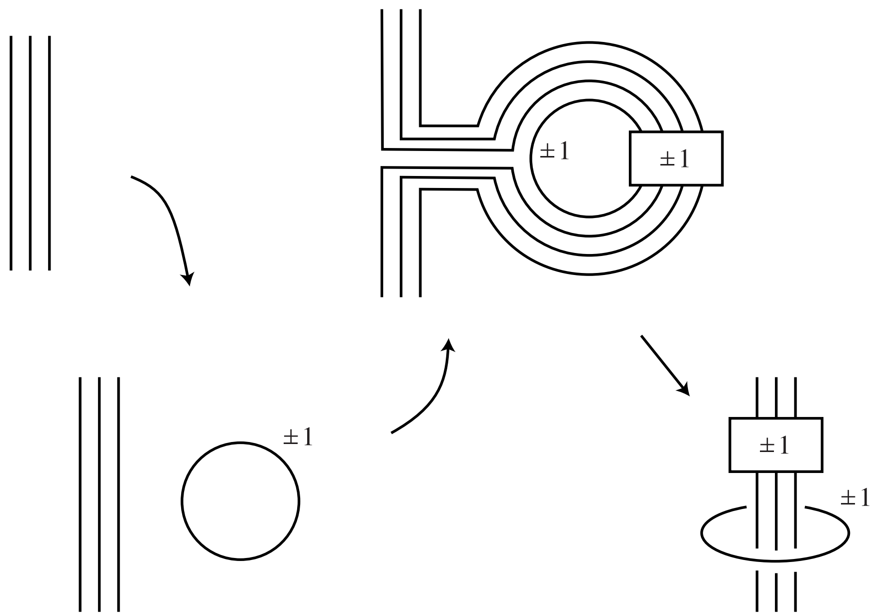


Figure 5.17. Blowing up.

Finally, we consider the effect of blowing up on Kirby diagrams. Recall (Section 2.2) that this consists of taking a connected sum with $\overline{\mathbb{CP}^2}$, or sometimes with $\mathbb{CP}^2$ in the smooth setting. In its simplest form, the blow-up operation consists of adding a ± 1-framed unknot to a diagram (without linking it). To obtain the general case, we then slide handles over the new unknot as in Figure 5.17. In this form, the operation consists of choosing some strands of the attaching circles, putting a full ± 1 twist in the bunch, and then drawing a ± 1-framed unknot around the twist. (We choose the signs consistently throughout, with the top sign corresponding to $\mathbb{CP}^2$ and the bottom one to $\overline{\mathbb{CP}^2}$.) If an attaching circle has k strands in the bunch (counted with sign), its framing will increase by $\pm k^2$ (by Exercise 5.1.2(c)). The change in linking numbers of the strands corresponds to the change in intersection number when we blew up an intersection point of two surfaces in Section 2.2. The reverse operation, blowing down, is shown in Figure 5.18. We start with any ± 1-framed unknot K in our diagram, and remove it after applying a ∓ 1 twist to all curves running through it, adding $\mp(\ell k(K, K_i))^2$ to the framing of each component K_i (cf. Figure 5.8). This procedure shows that any ± 1-framed unknot in a link diagram (possibly linking other components) represents a $\mathbb{CP}^2$ ($\overline{\mathbb{CP}^2}$) summand, and shows how to obtain the complementary summand (cf. Proposition 2.2.11).

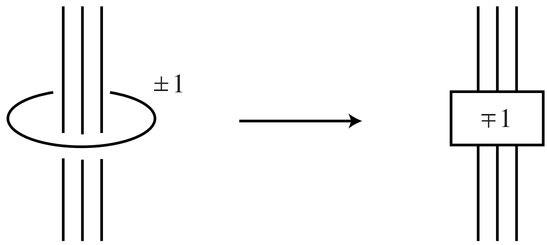


Figure 5.18. Blowing down.

The blow-up operation is useful in settings where we are allowed to change our 4-manifold by sums with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$. For example, nontriviality of the Seiberg-Witten invariants is preserved under sum with $\overline{\mathbb{CP}^2}$ (Theorem 2.4.9). Blowing down is useful for simplifying connected sums with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$. For example, many simply connected complex surfaces S have the property that $S \# \mathbb{CP}^2 \approx \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ (Theorem 9.1.15), as one can sometimes show by blowing up and down. (See Exercise 8.3.4(d) for the example $S = E(n)$.) Another important application is when we are mainly interested in the 3-manifold bounding the handlebody, which does not change under blowing up and down (since the connected sum occurs in the interior of the 4-manifold). We will examine this application more closely in Section 5.3, where we will see that blowing up and down provide a complete set of moves reducing the theory of closed, oriented 3-manifolds to that of link diagrams. Some useful moves with blow-ups include reversing crossings in a link diagram (Figure 5.19), undoing clasps (Figure 5.20) and changing the sign of a clasp (Figure 5.21, which is taken from [**Kp**]). (The framing changes in Figures 5.19 and 5.21 are given for the case of self-crossings of a knot, oriented as shown.) Compare Figure 5.19 with Proposition 2.3.5. Note that Figure 5.20, followed by blowing down the rightmost circle, shows again that $S^2 \times S^2 \# \overline{\mathbb{CP}^2} \approx S^2 \tilde{\times} S^2 \# \overline{\mathbb{CP}^2} \approx \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (cf. Exercise 4.2.6(b)), with the last diffeomorphism obtained by blowing down the left diagram when $n = 1$. In fact, this is a translation into Kirby calculus of the corresponding algebro-geometric proof for $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$ (which are classified as complex surfaces by the integer $|n|$ (cf. Theorem 3.4.8) but are all equivalent to $\mathbb{CP}^2$ up to holomorphically blowing up and down). Note that the effect of the blow-up was to make the two spheres disjoint (changing their self-intersection).

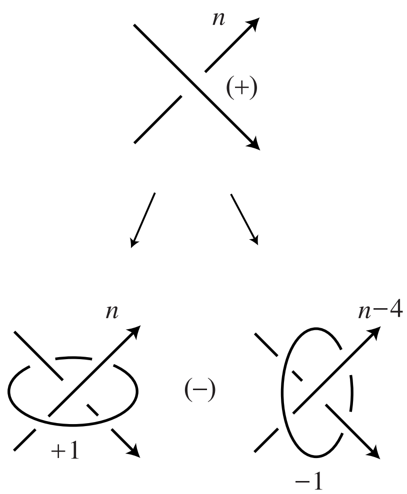


Figure 5.19. Blowing up to reverse a crossing.

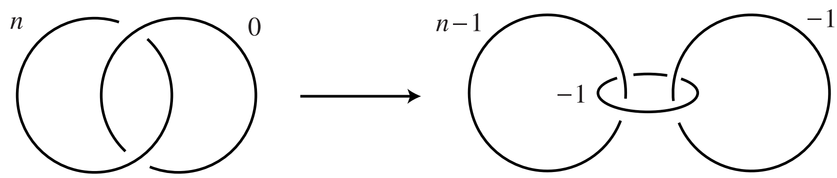


Figure 5.20. Blowing up to undo a clasp.

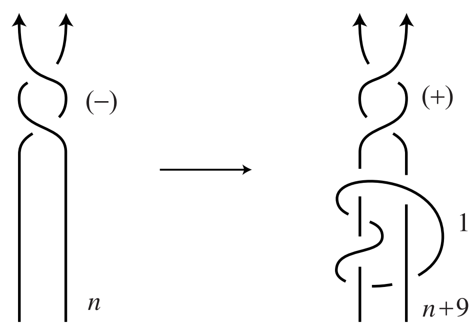


Figure 5.21. Blowing up to reverse a clasp.

Exercises 5.1.12. (a)* Let P denote the (negative) E_8-plumbing, Figure 4.33. (This is actually the same as the Milnor fiber $\Phi(1)$ that we encountered in Section 3.1. There are copies of P embedded in elliptic surfaces $E(n)_{p_1,\dots,p_k}$ and many other complex surfaces, cf. Chapters 7 and 8.) Let Q be the 2-handlebody on the left-handed *trefoil knot* shown in Figure 5.22. Prove that $P \# \mathbb{CP}^2 \approx Q \# \mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$. (*Hint*: Blow up along a meridian

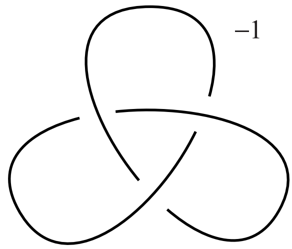


Figure 5.22. 4-manifold bounded by the Poincaré homology sphere.

at the end of the long arm of the E_8-plumbing. This creates a -1 that you can blow down.) What happens if you change the length of the long arm of the E_8-graph (leaving all coefficients -2)? The 3-manifold $\partial P = \partial Q$ is called the *Poincaré homology sphere*. See [**KSc**] for other descriptions of this ubiquitous manifold.

(b)* Let L and L' be framed links in $\mathbb{R}^3$, and suppose L' is obtained from L by a handle slide. Prove that L' can also be obtained from L by a sequence of blow-ups and blow-downs. (*Hint*: First do the case of sliding over a +1-framed unknot as in Figure 5.23. Then obtain the general case by Figure 5.19 and blowing up meridians.) This is due to Fenn and Rourke [**FR**]. Note that the assertion is false if we replace $\mathbb{R}^3$ by a more general 3-manifold — for example, if we are sliding over a homologically nontrivial curve.

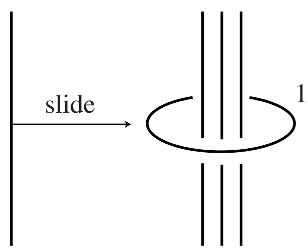


Figure 5.23

5.2. Surgery

To understand boundaries of handlebodies, we consider *surgery theory*. (For more applications of this theory in high dimensions, see e.g., [**Br**].) When we add a handle to a manifold, what happens to its boundary? The interior of the attaching region disappears into the interior of the new manifold, and it is replaced by the rest of the boundary of the handle. This motivates the following definition. For uniformity of notation, we define the -1-sphere S^{-1} to be $\partial D^0 = \emptyset$.

Definition 5.2.1. Let $\varphi\colon S^k \to M^n$ $(-1 \leq k \leq n)$ be an embedding of a k-sphere in an n-manifold, with a (normal) framing f on $\varphi(S^k)$ (which we assume lies in int M). Then the pair (φ, f) determines an embedding $\hat{\varphi}\colon S^k \times D^{n-k} \to M$ (uniquely up to isotopy), and *surgery* on (φ, f) is the procedure of removing $\hat{\varphi}(S^k \times \mathrm{int}\ D^{n-k})$ and replacing it by $D^{k+1} \times S^{n-k-1}$, with gluing map $\hat{\varphi}|S^k \times S^{n-k-1}$.

The smooth manifold obtained by surgery on (φ, f) is uniquely determined up to diffeomorphism by the isotopy class of (φ, f) (and an isotopy of (φ, f) determines a diffeomorphism up to isotopy). For $k \leq 3$, any self-diffeomorphism of S^k is isotopic either to the identity or a reflection, so in this case it suffices to specify the image of φ, and we talk about surgery on a sphere in M with framing f.

There are several relations between surgery and attaching handles. As we have seen, attaching a handle to $(X, \partial_- X)$ has the effect of surgery on $\partial_+ X$, and conversely any surgery on a closed manifold M is realized as $\partial_+(I \times M \cup h)$ where h is attached by (φ, f). In particular, for closed, unoriented manifolds M_1 and M_2, we can transform M_1 to M_2 by a sequence of surgeries if and only if there is a compact, unoriented manifold whose boundary is $M_1 \cup M_2$. Similarly, closed, oriented manifolds M_1 and M_2 are *oriented cobordant*, i.e., there is a compact, *oriented* manifold with boundary $\overline{M}_1 \cup M_2$ (cf. Chapter 9), if and only if M_1 can be transformed to M_2 by surgeries such that each surgery on a 0-sphere preserves orientations (Exercise 5.2.2(d)). A different relation with attaching handles is that surgery on $\varphi\colon S^k \to M^n$ can be interpreted as attaching a $(k+1)$-handle and an n-handle to the complement of $\hat{\varphi}(S^k \times \mathrm{int}\ D^{n-k})$ in M^n (by turning the obvious handle decomposition of $D^{k+1} \times S^{n-k-1}$ upside down). The $(k+1)$-handle attaches to $\hat{\varphi}(S^k \times \{\mathrm{pt.}\})$ with the product framing induced by the embedding $\hat{\varphi}$.

Note that surgery on M produces a manifold with a canonical embedding of $D^{k+1} \times S^{n-k-1}$. If we surger on this framed S^{n-k-1}, we recover M. This corresponds to turning the relative handlebody $I \times M \cup h$ upside down. We call this procedure *reversing the surgery*.

Exercises 5.2.2. (a) Draw surgeries and their reversals for all choices of $k, n \leq 2$ (cf. Exercise 4.1.2). What is surgery on S^{-1}? (*Answer*: Disjoint union with S^n.) Try to visualize the $n = 3$ case. Describe connected summing as surgery.

(b)* Prove that for any smooth (proper) embedding $N^k \hookrightarrow M^n$ of manifolds of dimensions n and k, the inclusion map $i\colon M - N \hookrightarrow M$ induces an isomorphism on π_1 for $k \leq n-3$ and an epimorphism for $k = n-2$. In the latter case, show that $\ker i_*$ is generated by meridians of N (attached somehow to the base point). (*Hint*: Any continuous map between smooth manifolds can be approximated by a smooth map homotopic to it. Now use transversality.)

(c)* Fix $n \geq 4$. Use surgery to prove that any finitely presented group G is the fundamental group of a closed, oriented n-manifold. Why is this construction the same as in Exercise 4.6.4(b)? Rework Exercise 5.1.10(c) in this language ($n \geq 4$). (See Example 4.1.3.) Prove that any n-manifold constructed by this procedure from an Andrews-Curtis trivial presentation will be diffeomorphic to S^n (Exercise 5.1.10(d)).

(d) Prove the above statements relating cobordism (oriented or not) and equivalence up to surgery.

Proposition 5.2.3. *Let M be an n-manifold (not necessarily compact) with $n \geq 4$, and let C be a nullhomotopic circle embedded in M. Then any surgery on C produces a manifold diffeomorphic to $M \# S$, where S is one of the two S^{n-2}-bundles over S^2 (with structure group $O(n-1)$).*

Proof. Write $M = M \# S^n$, and let $C_0 \subset M \# S^n$ be the circle $\partial D^2 \times 0 \subset \partial(D^2 \times D^{n-1}) = S^n$. Since C is nullhomotopic, it is homotopic to C_0. But homotopic embedded circles in an n-manifold with $n \geq 4$ are isotopic (Example 4.1.3). Thus, we can assume $C = C_0$, and surgery on C produces $M \# S$, where S is obtained by surgering S^n on C_0. But $S = \partial(D^{n+1} \cup h)$, where h is a 2-handle attached to C_0 in $S^n = \partial D^{n+1}$, and $D^{n+1} \cup h$ is a D^{n-1}-bundle over S^2 (Example 4.1.4(d)). □

When M is spin, the two manifolds $M \# S^2 \times S^{n-2}$ and $M \# S^2 \tilde{\times} S^{n-2}$ are different (Exercise 5.6.8(b)), since the latter one has no spin structure. (In fact, $\langle w_2, \alpha \rangle \neq 0$, where α is the homology class of the section of $S^2 \tilde{\times} S^{n-2}$ determined by the north pole of each fiber. See Section 1.4 or 5.6 for further discussion of spin structures and w_2. A simply connected 4-manifold M is spin if and only if Q_M is even.) When M is not spin, however, these manifolds may be the same, by the following (which is essentially due to Wall [**W2**]).

Proposition 5.2.4. *Let M be a simply connected n-manifold with $n \geq 4$. If M is not spin, then $M \# S^2 \times S^{n-2}$ is diffeomorphic to $M \# S^2 \tilde{\times} S^{n-2}$.*

Proof. Since M is simply connected, we have $H_2(M; \mathbb{Z}_2) \cong H_2(M; \mathbb{Z}) \otimes \mathbb{Z}_2$ (Universal Coefficient Theorem) and $H_2(M; \mathbb{Z}) \cong \pi_2(M)$ (Hurewicz Theorem), so every element of $H_2(M; \mathbb{Z}_2)$ is represented by an immersed sphere (which we can assume is embedded if $n > 4$). Since $w_2(M) \neq 0$, it has nonzero value on some 2-sphere Σ, whose normal bundle is twisted (and has odd Euler number if $n = 4$, cf. Exercise 6.1.1(a)). Now $M \# S^2 \times S^{n-2}$ is obtained from M by surgery on a circle C bounding some 2-disk $D \subset M$, with framing determined by the unique normal framing of D, and $M \# S^2 \tilde{\times} S^{n-2}$

is obtained by surgery on C with the other framing. If we take D to be the north polar cap of Σ, then isotoping C over Σ to the south polar cap (avoiding collisions at double points when $n = 4$ as in Example 4.1.3) will interchange the framings. $\square$

Example 5.2.5. Setting $n = 4$, we see (yet again) that $S^2 \times S^2 \# \overline{\mathbb{CP}^2} \approx S^2 \tilde{\times} S^2 \# \overline{\mathbb{CP}^2} \approx \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$. We see that we can replace $\overline{\mathbb{CP}^2}$ by any simply connected, nonspin manifold. (We proved this for manifolds without 1-handles in Corollary 5.1.5.) A corollary of Proposition 5.2.3 and Exercise 5.2.2(b) is that if $F \subset M^4$ is any embedded sphere with a trivial normal bundle in a 4-manifold, and if a meridian of F is nullhomotopic in $M - F$, then we can split M as $N \# S^2 \times S^2$ or $N \# S^2 \tilde{\times} S^2$, with F a fiber of the bundle. (Surger out F, then reverse the surgery.) Compare with K_2 in Proposition 5.1.4 and Exercise 5.1.7(b). Note that the condition on the meridian of F is satisfied if and only if there is an immersed sphere in M intersecting F transversely in a single point. (In particular, $[F] \in H_2(M;\mathbb{Z})/\text{torsion}$ must be nonzero.)

Exercises 5.2.6. (a) By reducing dimensions in the proof of Proposition 5.2.4, construct an analogous argument to show that for any surface F, $F \# S^1 \times S^1 \approx F \# S^1 \tilde{\times} S^1$ if and only if F is nonorientable.

(b)* For an arbitrary orientable n-manifold M with $n \geq 4$, prove the equivalence of the following statements:

(i) $M \# S^2 \times S^{n-2} \approx M \# S^2 \tilde{\times} S^{n-2}$.

(ii) The universal cover of M is nonspin.

(iii) $w_2(M)$ has nonzero value on some immersed 2-sphere in M.

One more variation of these ideas is the *Gluck construction* [**Gk**]. Given a 2-sphere S with a trivial normal bundle in a 4-manifold M, we can obtain a new manifold M' by surgering out S, switching the framing on the resulting framed circle, and then surgering again. (Equivalently, we cut out $S^2 \times D^2$ and reglue it by the self-diffeomorphism of $S^2 \times S^1$ that rotates each 2-sphere $S^2 \times \{\theta\}$ through the angle θ.) Applying this to a knotted 2-sphere in S^4, we obtain a manifold homeomorphic to a 4-sphere (Exercise 5.2.7(a)). Many such examples are potentially exotic 4-spheres. (For some families of knots these are standard ([**Gk**], [**Go**], [**Me**], [**P**], Exercise 6.2.11(b)), but the general case is still open.) For drawing Kirby diagrams of such manifolds, see Exercises 5.4.3(d), 6.2.4(e) and 6.2.12(d).

Exercises 5.2.7. (a) Suppose that M is simply connected. Prove that M' is simply connected. If M is also closed and S is nullhomologous, prove that M' has the same intersection form (hence, homeomorphism type by

Freedman's Theorem 1.2.27) as M. Find a counterexample to this if S is nontrivial in homology.

(b)* For any M and S, prove that M and M' become diffeomorphic after connected summing with any simply connected, nonspin manifold. (Now summing with $\overline{\mathbb{CP}^2}$ shows that under the hypotheses of the previous exercise, M and M' cannot be distinguished by their Seiberg-Witten invariants; cf. Theorem 2.4.9.)

5.3. Dehn surgery

On 3-manifolds, surgery on circles has a natural generalization, called *Dehn surgery* or *rational surgery* (or sometimes just *surgery*, a term that we will avoid in this context). (See Rolfsen [**Ro**] for further reading.)

Definition 5.3.1. Let K be a knot in an oriented 3-manifold, with a closed tubular neighborhood $\nu K \approx S^1 \times D^2$. A *Dehn surgery* on K is the operation of removing int νK and gluing in $S^1 \times D^2$ by any diffeomorphism φ of the boundary tori.

The self-diffeomorphisms of a torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ are given (up to isotopy) by $GL(2,\mathbb{Z})$. However, since a solid torus $S^1 \times D^2$ is built from its boundary with a 2-handle and a 3-handle, it suffices to keep track of the attaching circle of the 2-handle, $\varphi(\{\text{pt.}\} \times \partial D^2)$. This circle is determined by its homology class α in $H_1(\partial \nu K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, which can be any primitive element. To specify this class, we orient K and let μ be a right-handed meridian (Section 4.5, Figure 4.27). We let $\lambda \in H_1(\partial \nu K; \mathbb{Z})$ be a *longitude* given by some parallel copy of K. If $M = S^3$, we define λ using the 0-framing. (For the general case, we may have to choose λ arbitrarily.) Then (μ, λ) is an oriented basis for $H_1(\partial \nu K; \mathbb{Z})$, and $\alpha = p\mu + q\lambda$ for unique relatively prime integers p and q. Reversing the orientation of K or α reverses the signs of both p and q but doesn't affect the diffeomorphism type obtained by the Dehn surgery, so we lose no information by taking the quotient p/q, and we call this a Dehn surgery with *coefficient* (or *slope*) $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. For a general 3-manifold M, this depends on our choice of longitude λ, but for links in S^3 a diffeomorphism type is uniquely determined by specifying a surgery coefficient in $\mathbb{Q} \cup \{\infty\}$ for each link component. Note that a Dehn surgery with coefficient ∞ is trivial. For a standard surgery on S^3 (arising as the boundary of a 2-handlebody as in Section 5.2), α is the parallel copy of K determined by the framing, which can be any class with $q = \pm 1$. Thus, standard surgeries correspond to Dehn surgeries with integer coefficients, and the surgery coefficient equals the framing coefficient. Now we can specify 3-manifolds by rational surgeries on links in S^3, and without ambiguity think of them as boundaries of 2-handlebodies whenever

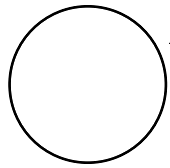


Figure 5.24. Lens space $L(p, q)$.

all coefficients are integral. (In the latter case, we frequently use the term *integral surgery*, to avoid confusion with more general rational surgeries.)

Example 5.3.2. The *lens space* $L(p, q)$ is defined to be $-\frac{p}{q}$-surgery on the unknot (Figure 5.24). Since the complement of a tubular neighborhood of the unknot is a solid torus $D^2 \times S^1 \subset \partial(D^2 \times D^2)$, lens spaces are precisely those closed, oriented 3-manifolds that have genus-1 Heegaard splittings (cf. Figure 4.14). Each lens space $L(p, 1)$ arises as the boundary of the disk bundle over S^2 with Euler number $-p$. In particular, $L(0, 1) = S^2 \times S^1$ and $L(\pm 1, 1) = S^3 = L(1, 0)$.

Exercises 5.3.3. (a) Prove that $L(\pm 2, 1) = \mathbb{RP}^3$ (cf. Exercise 4.3.1(a)).

(b) For $p, q > 0$, let M be the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}_p$-action generated by $(z, w) \mapsto (e^{2\pi i/p} z, e^{2\pi i q/p} w)$. Prove that M is diffeomorphic to $L(p, q)$ as an oriented manifold. (This explains our sign convention. Note that the flow $(e^{it} z, e^{iqt} w)$ is right-handed on S^3, in the sense that its trajectories are circles with positive linking numbers.) (*Hint*: Split S^3 into solid tori along $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ (for S^1 of radius $\frac{1}{\sqrt{2}}$). Now $\mathbb{Z}_p$ acts on each solid torus. Find fundamental domains and obtain a splitting of M into solid tori. Which one corresponds to the unknot complement?)

(c)* Prove that for $p \geq 2$, the lens space $L(p, q)$ does not embed in S^4. (*Hint*: Exercise 4.5.12(d).)

(d)* We will see later (Exercise 5.4.3(c)) that T^3 is obtained by 0-surgery on the *Borromean rings*, Figure 5.25. Find a torus in T^3 as described by the figure that is disjoint from two of the solid tori attached during the surgery and intersects the third in a disk. (*Hint*: Ambiently perform surgery on S^2.) By pushing your torus through the 3-manifold, visualize its structure as $S^1 \times T^2$. What does the 3-fold symmetry represent? Find two tori intersecting transversely in a (homologically nontrivial) circle and three tori intersecting in a point.

It is natural to ask which 3-manifolds can be obtained by rational or integral surgery on links in S^3. Clearly, such manifolds are always closed and oriented. The converse follows from the following theorem of Rohlin.

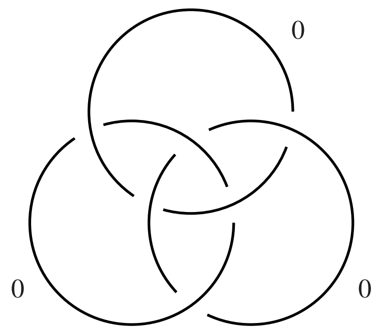


Figure 5.25. 3-torus T^3.

Theorem 5.3.4. ([**R1**]) *Any closed, oriented* 3*-manifold* M *bounds a compact, oriented* 4*-manifold* X*.* □

Corollary 5.3.5. *Any closed, oriented, connected* 3*-manifold* M *is realized by integral surgery on a link* L *in* S^3*.*

(We can also assume the framing coefficients are even; see Theorem 5.7.14.)

Proof. Use the theorem to write M as ∂X, and decompose X as a handlebody. Then the union of 0- and 1-handles is $\natural n S^1 \times D^3$. By surgery on circles in X, we can replace this by $\natural n S^2 \times D^2$, so without loss of generality we can assume (after changing X) that X has no 1-handles. Similarly, we can eliminate 3-handles by turning the handlebody upside down and surgering $I \times M \natural m S^1 \times D^3$. Now M bounds a 2-handlebody and hence the corollary follows. □

The basic idea of Rohlin's proof is to immerse M in S^5, then surger M to eliminate curves of self-intersection, resulting in a new manifold M' embedded in S^5 which is oriented cobordant to M. Now M' bounds a 4-dimensional "Seifert surface" in S^5 (by the same argument as preceding Proposition 4.5.5) and we are done. An alternate approach, due to Lickorish [**L1**], is to prove Corollary 5.3.5 directly by 3-dimensional techniques, then obtain Theorem 5.3.4 as a corollary. For this proof, we begin with a Heegaard splitting of M. There is a standard Heegaard splitting of S^3 with the same genus, and so we can get from S^3 to M by cutting along a surface F and regluing by some orientation-preserving diffeomorphism of F. As was first shown by Dehn [**De**], any such diffeomorphism is a composition of *Dehn twists* (cutting F along a circle and regluing it after a 360° twist). It is not hard to realize these Dehn twists by ± 1-surgeries in S^3. (See Exercise 8.2.4.)

Since we now know that any closed, oriented 3-manifold can be realized by integral surgery on a link in S^3, we would like to reduce the theory of such 3-manifolds to that of framed links in S^3 — that is, we wish to find a set of moves that suffices for getting between any two surgery descriptions of the same 3-manifold. In fact, we have already found such moves — namely, blowing up and down (Section 5.1), as the following theorem of Kirby [**K1**] (augmented by Fenn and Rourke [**FR**]) shows.

Theorem 5.3.6. *Let L and L' be two framed links in S^3 describing (orientation-preserving) diffeomorphic 3-manifolds (by integral surgery). Then L can be transformed into L' by blowing up and down (and isotopy). In fact, any preassigned orientation-preserving diffeomorphism can be realized in this manner.*

Kirby [**K1**] proved the analogous theorem with handle slides also allowed, and Fenn and Rourke [**FR**] eliminated the handle slides as in Exercise 5.1.12(b). The latter paper also proves a corresponding theorem for nonorientable 3-manifolds, where S^3 is replaced by the twisted S^2-bundle over S^1, handle sliding is retained and an additional move is introduced. Kirby's proof, which we sketch below, generalizes to the case where S^3 is replaced by any fixed compact 3-manifold (with boundary) [**Rob**]. In that case, one needs blow-ups, handle slides, the slam-dunk move described below and (in the nonorientable case only) a fourth move. For a different approach to Kirby's Theorem using Heegaard splittings, see [**Lu**] or [**MP**]. Kirby's Theorem has turned out to be particularly useful in the discovery in recent years of many new 3-manifold invariants. One can define such invariants by means of integral surgery diagrams, and then one only needs to prove invariance under blowing up and down. For introductions to the various types of new invariants, see e.g., [**KMe1**], [**L2**], [**O**], [**Tu**]. For additional reading, see e.g., [**BHMV**], [**CM**], [**KMeZ**], [**LMO**].

To interpret the last assertion of the theorem, which is implicit in Kirby's proof but not explicit in previous literature, recall from Section 4.1 that an isotopy of a framed attaching circle (hence a handle slide) determines a diffeomorphism between the corresponding handlebodies (up to isotopy). Blowing up an isolated ± 1-framed circle in a surgery diagram only changes the 3-manifold by replacing a 3-ball with another 3-ball (exhibited as ± 1-surgery on the first ball), so the resulting 3-manifold is canonically diffeomorphic to the original one. (Any self-diffeomorphism of ∂B^3 extends uniquely over B^3.) We will exhibit this diffeomorphism explicitly in Exercise 5.3.8(a) below. The same applies to adding a 0-framed Hopf link (which can be achieved by blowing up and down as in Figure 5.20). By our construction of arbitrary blow-ups (Figure 5.17), it should now be clear that these canonically induce diffeomorphisms of the corresponding 3-manifolds.

Proof of Theorem 5.3.6 (sketch). The given framed links L, L' determine 2-handlebodies X, X' with a specified diffeomorphism $\psi\colon \partial X \to \partial X'$. After blowing up, we may assume that the closed 4-manifold $\overline{X} \cup_{\mathrm{id}_{\partial X}} I \times \partial\overline{X} \cup_{\psi} X'$

has signature 0, so by a theorem of Thom (Theorem 9.1.6) it bounds a compact, oriented 5-manifold W. Let $f\colon W \to I$ be a Morse function with $f^{-1}(0) = X$, $f^{-1}(1) = X'$ and $f|I \times \partial\overline{X}$ given by projection to the first factor. The last condition guarantees that Morse theory still works as it did in Chapter 4 (when we assumed $\partial\overline{X} = \emptyset$), so we obtain a handle decomposition of (W, X). As before, we can cancel 0- and 5-handles, and as in the proof of Corollary 5.3.5 we can surger W to eliminate 1- and 4-handles. Now $W = I \times X \cup$ 2-handles $\cup$ 3-handles, and by Proposition 5.2.3, $\partial_+ W_2$ is obtained from X by summing with copies of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. By turning W upside down, we see that $\partial_+ W_2$ is also obtained similarly from $X' = \partial_+ W$. Thus, after blowing up or adding 0-framed Hopf links to L, L', we obtain a diffeomorphism between the corresponding 4-manifolds (which we still denote by X, X') extending $\psi\colon \partial X \to \partial X'$. (This is a standard argument going back to Wall, cf. Theorem 9.1.12.) Now we apply Cerf theory as in Theorem 4.2.12. We identify the 2-handlebodies X and X' by the above diffeomorphism and compare the given handle structures. If we could get between these just by sliding 2-handles, we would be done. However, we must expect the creation of cancelling handle pairs in general. By a long computation using Cerf's machinery, one can eliminate 0- and 4-handles and arrange the 1- and 3-handles to be trivial in the sense that we create them in our original handlebody X, after which they remain unchanged until we reach our final handlebody X'. Now we can surger out these 1- and 3-handles (which adds 0-framed Hopf links to L and L' after we apply Figure 5.11), and the resulting links will be equivalent by handle slides as required. □

Exercises 5.3.7. (a)* Prove that for each integer k there is an oriented 3-manifold M that can be realized both by (k^2+1)-surgery on a knot K_+ and by $-(k^2+1)$-surgery on a knot K_-. (*Hint*: Figure 5.26 can be used to solve the $k = 0$ case.)

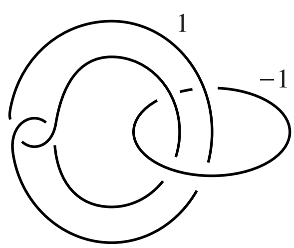


Figure 5.26

(b)* Let X be a closed, simply connected 4-manifold. Use Kirby's Theorem 5.3.6 to deduce that for sufficiently large m, $X\#m\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ is diffeomorphic to $\#(b_2^+(X)+m)\mathbb{CP}^2\#(b_2^-(X)+m)\overline{\mathbb{CP}^2}$. (*Hint*: In the case without 1- or 3-handles, $\partial X_2 \approx S^3$.) This also follows directly from Wall's Theorem 9.1.12; see Corollary 9.1.14.

Although it suffices to use integral surgeries with blow-ups and blow-downs to represent 3-manifold theory, it is convenient to enlarge our family of moves and allow rational coefficients. We will begin by generalizing the blow-up operation to a move called a *Rolfsen twist* [**Ro**]. We will see that these twists (together with inserting and deleting link components with coefficient ∞) suffice for getting between any two rational surgery diagrams of a given 3-manifold. To perform a Rolfsen twist, we need an unknotted component K in our surgery diagram. Then $S^3 - \operatorname{int} \nu K$ is a solid torus $S^1 \times D^2$, and we perform the Rolfsen twist by putting a Dehn twist φ in this solid torus. (That is, we map $S^1 \times D^2 \to S^1 \times D^2$ by $\varphi(e^{i\theta}, z) = (e^{i\theta}, e^{i\theta} z)$, and then perform an isotopy so that the twist is supported near a single disk $D = \{\text{pt.}\} \times D^2$.) By iterating and/or inverting this map, we can add n twists for any $n \in \mathbb{Z}$, as shown in Figure 5.27. If K initially has coefficient $r = \frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$, it is easily checked that the resulting coefficient should be $\frac{p}{q+np} = (\frac{1}{r} + n)^{-1}$. The rest of the link will be unchanged except near D, where it picks up n full (360°) twists as shown. For each component K_i intersecting D, the surgery coefficient changes from r_i to $r_i + n(\ell k(K_i, K))^2$. (To see this, draw each twist as in Figure 5.28, and note that the writhe of K_i increases by $\Delta w_i = n(\ell k(K_i, K))^2$. Thus, the new longitude and meridian of K_i are given by $\mu_i' = \varphi_* \mu_i$, $\lambda_i' = \varphi_* \lambda_i - \Delta w_i \mu_i'$.)

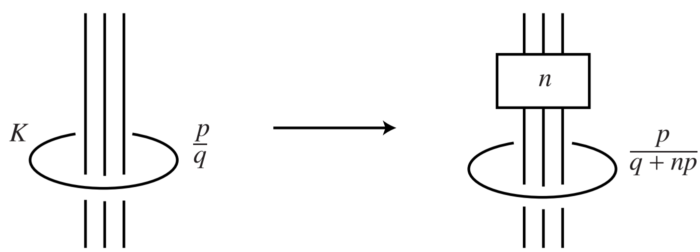


Figure 5.27. Rolfsen twist.

Exercises 5.3.8. (a)* Show by Rolfsen twists that the 3-manifolds in Figure 5.29 are diffeomorphic to S^3 (as given by the empty link). (Recall that we showed this abstractly while proving that we can add a 4-handle to obtain $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ and $S^2 \tilde{\times} S^2$, respectively.) Now show that blowing up and down are special cases of Rolfsen twists.

(b)* Prove that the lens spaces $L(p,q)$ and $L(p, q+np)$ are diffeomorphic for any integer n. (See Example 5.3.2.) Thus, when considering lens spaces we can assume that either $0 < q < p$ or $(p,q) = (1,0)$ or $(0,1)$. (The only other relations among lens spaces (up to orientation-preserving diffeomorphism)

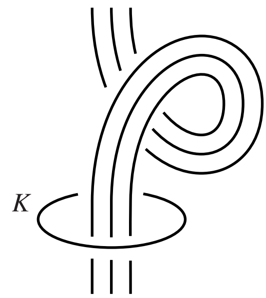


Figure 5.28

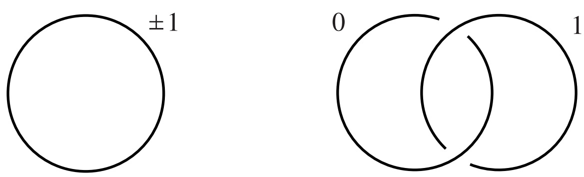


Figure 5.29. Surgery diagrams of S^3.

are $L(p,-q) = \overline{L(p,q)}$ and $L(p,q) = L(p,q')$ for $qq' \equiv 1 \pmod p$, the latter obtained by interchanging the solid tori, cf. also Exercise 5.3.9(b).)

Another useful move is the *slam-dunk*, a classical operation whose recent dynamic name is due to T. Cochran. This move can be derived from Rolfsen twists (assuming our link L is in S^3, Exercise 5.3.9(c)), but it is most easily seen directly. Suppose that one component K_1 of our link is a meridian of another component K_2, and that the coefficients of these are $r \in \mathbb{Q} \cup \{\infty\}$ and $n \in \mathbb{Z}$, respectively (Figure 5.30). Let M be the manifold obtained from S^3 by surgery on just K_2, and let $T \subset M$ be the solid torus glued in during the surgery. Then K_1 is a knot in M that we can pull into T through the boundary torus. Since the surgery coefficient n of K_2 was integral, K_1 will intersect the disk $\{\text{pt.}\} \times D^2$ in T exactly once, and so it will be isotopic to $S^1 \times \{\text{pt.}\}$ in T. (It cannot be knotted, since it is isotopic to a circle in ∂T.)

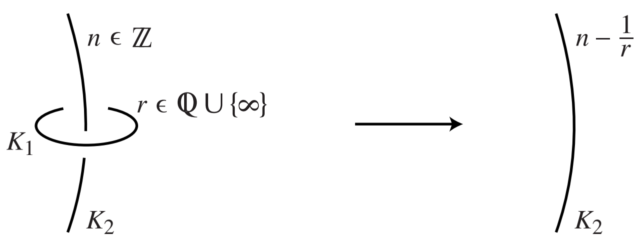


Figure 5.30. Slam-dunk.

Then T will be a tubular neighborhood of K_1, so the required surgery on K_1 will be the same as cutting out T and regluing it a second time. Thus, we obtain a new surgery diagram with K_1 deleted and a new coefficient on K_2 (and the remaining surgeries unchanged). To compute the coefficient, note that in the $n = 0$ case, the first surgery was obtained from the trivial gluing of T by a 90° rotation of $H_1(\partial T)$ ($\mu \mapsto \lambda$, $\lambda \mapsto -\mu$). This changes the slope of the surgery on K_1 from r to its orthogonal slope $-\frac{1}{r}$. For the general case, we add n twists to get $n - \frac{1}{r}$.

Exercises 5.3.9. (a)* Show by a slam-dunk that surgery on a Hopf link with coefficients 0 and n gives S^3. (Try both choices for K_1.) More generally, let X be a 4-dimensional handlebody without 3- or 4-handles. We showed (Example 4.6.3) how to construct a handlebody for the double DX. Check that the union of 0-, 1- and 2-handles of DX has boundary $\# m\, S^1 \times S^2$, where m is the number of 1-handles of X.

(b)* Let X be a plumbing of spheres whose graph is linear. (See Figure 5.31.) Prove that ∂X is a lens space $L(p, q)$. Which one is it? (The expression you get is called a *continued fraction expansion* of $-\frac{p}{q}$.) Note that you get two different answers, depending on where you begin. This corresponds to the diffeomorphism $L(p, q) \approx L(p, q')$ if $qq' \equiv 1 \bmod p$. Prove that every lens space has a description as in Figure 5.31, i.e., lens spaces are precisely the class of 3-manifolds occurring as boundaries of plumbings of spheres on linear graphs. Now find an algorithm for turning any rational surgery diagram into an integral surgery diagram for the same manifold.

(c)* Suppose that K_2 (in our above notation for the slam-dunk) is unknotted. (See Figure 5.32.) Show that the slam-dunk is the same as a sequence of Rolfsen twists. (*Hint*: Reduce to the case $n = 1$, then "blow down." Where does K_1 end up? Now remove the extra twist.) Now prove that *any* slam-dunk (on a link in S^3) is a sequence of Rolfsen twists. (See Exercise 5.1.12(b).)

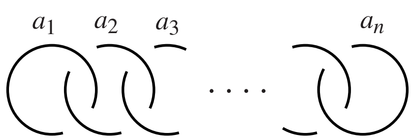


Figure 5.31. Lens space as boundary of a linear plumbing.

Combining these exercises with Theorem 5.3.6, we obtain the following:

Proposition 5.3.10. *Let L and L' be links with rational coefficients in S^3. If the resulting 3-manifolds obtained by Dehn surgery are (orientation-preserving) diffeomorphic, then L can be transformed into L' by a sequence of*

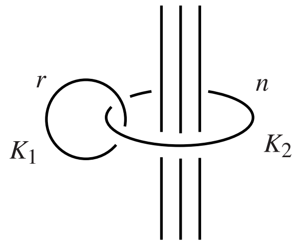


Figure 5.32

Rolfsen twists (together with isotopies and inserting and deleting components with coefficient ∞), and any preassigned orientation-preserving diffeomorphism can be realized in this manner.

Proof. Turn L and L' into integral surgery diagrams by inverse slam-dunks, or equivalently by Rolfsen twists (Exercises 5.3.9(b) and (c)), then apply Theorem 5.3.6 and Exercise 5.3.8(a). To realize a preassigned diffeomorphism ψ, note that our discussion of Rolfsen twists produced a canonical diffeomorphism between the manifolds in Figure 5.27. Thus, ψ determines a diffeomorphism between the integral surgery diagrams constructed from L and L', and this is realized by blowing up and down by Theorem 5.3.6. □

Finally, we consider the homology of a 3-manifold M obtained by rational surgery on an m-component oriented link L. As in Section 4.5, the long exact homology sequence of $(S^3, S^3 - L)$ shows that $H_1(S^3 - L; \mathbb{Z})$ is free abelian of rank m, and the right-handed meridians μ_i to the components K_i of L form a canonical basis. Since M is obtained from $S^3 - \operatorname{int} \nu L$ by adding a 2-handle and a 3-handle for each component K_i, $H_1(M; \mathbb{Z})$ will be a quotient of $H_1(S^3 - L; \mathbb{Z})$ by m relators, one for each 2-handle. If K_i has surgery coefficient $\frac{p_i}{q_i}$ and 0-framed longitude λ_i, then the corresponding relation is $p_i\mu_i + q_i\lambda_i = 0$. But λ_i is the boundary of a Seifert surface F_i in S^3 for K_i. In $S^3 - L$, F_i will be punctured by the other components K_j, resulting in a surface with additional boundary components representing $\pm\mu_j$, so F_i determines a relation $\lambda_i = \sum_{j\neq i} \ell k(K_i, K_j)\mu_j$. We conclude:

Proposition 5.3.11. *For M given by Dehn surgery as above, $H_1(M; \mathbb{Z})$ is generated by the meridians μ_i $(i = 1, \dots, m)$ with a complete set of relations given by $p_i\mu_i + q_i \sum_{j\neq i} \ell k(K_i, K_j)\mu_j = 0$, $i = 1, \dots, m$.* □

Corollary 5.3.12. *Let X^4 be a 2-handlebody. Then any matrix for the intersection form of X is also a presentation matrix for $H_1(\partial X; \mathbb{Z})$. (That is, the matrix determines a homomorphism $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}^m$ with $\mathbb{Z}^m / \operatorname{Im} \varphi \cong H_1(\partial X; \mathbb{Z})$.) In particular, $H_1(\partial X; \mathbb{Z})$ is finite if and only if $\det Q_X \neq 0$, and if so, $|H_1(\partial X; \mathbb{Z})| = |\det Q_X|$.*

Proof. Since the surgery coefficients for ∂X are integral, we can set each $q_i = 1$ in Proposition 5.3.11. The matrix for Q_X with respect to the standard ordered basis (determined by orienting and ordering L) is the linking matrix (Proposition 4.5.11), which obviously gives the required presentation. Clearly, a change of basis in $H_2(X)$ corresponds to a change of presentation of $H_1(\partial X)$. □

The corollary is true for any compact 4-manifold X with $H_1(X;\mathbb{Z}) = 0$ (cf. Exercise 5.3.13(f)). In particular, this hypothesis implies that ∂X is a (possibly empty) disjoint union of homology spheres if and only if Q_X is unimodular. (If $H_1(X) \neq 0$, unimodularity still follows from ∂X being a union of homology spheres by Remark 1.2.11, but $S^1 \times D^3$ is a counterexample to the converse.) See Section 5.4 for computing $H_1(\partial X)$ for a general handlebody X^4. A statement analogous to Proposition 5.3.11 can be proved for $\pi_1(M)$, using the Wirtinger presentation of $\pi_1(S^3 - L)$. (See [**Ro**].) The latter group is generated by meridians (Exercise 5.2.2(b)), but now there will be more than m of them. One obtains a generator for each segment of L bounded by a pair of consecutive undercrossings in the diagram, and each crossing contributes a relator. As before, we obtain $\pi_1(M)$ using m additional relators.

Exercises 5.3.13. **(a)** Compute $H_1(M)$ for rational surgery on any knot. Visualize the relation for an unknot (when $M = L(p,q)$) or an integral surgery on the trefoil knot (Figure 5.22).

(b) Compute $H_1(M)$ for rational surgery on a Hopf link, and visualize the relations when one coefficient is 0 or 1. For integral surgery with one coefficient 0, visualize why any loop in M is nullhomotopic.

(c) Compute $H_1(M)$ for integral surgery on a 2-component link with linking number 2. (It may not be cyclic.)

(d) Check that the boundary of the E_8-plumbing (Figure 4.33) is a homology sphere. Compare with (a) and Exercise 5.1.12(a).

(e)* Freedman [**F**], [**FQ**] proved that any homology 3-sphere bounds a compact, contractible topological 4-manifold. Deduce that any unimodular symmetric form Q on $\mathbb{Z}^n$ can be realized as the intersection form of a closed, simply connected topological 4-manifold. This is a main ingredient of Freedman's Classification Theorem 1.2.27. Find a homology sphere that doesn't bound a smooth, contractible 4-manifold.

(f)* Let M be the boundary of a compact, oriented 4-manifold X with $H_1(X;\mathbb{Z}) = 0$. Prove that any matrix for Q_X presents $H_1(M;\mathbb{Z})$. Interpret your proof geometrically in the case when X is a 2-handlebody. (*Hint*: Exercise 1.2.10 — Note that the hypothesis implies $H_2(X;\mathbb{Z})$ and $H_2(X,M;\mathbb{Z})$ have no torsion.)

(g)* Let M^3 be a rational homology 3-sphere (i.e., $H_1(M;\mathbb{Q}) = 0$) given by integral surgery on an oriented, framed link L in S^3 with linking matrix A. Show that the linking form on M (Exercise 4.5.12(c)) is given with respect to the generating set $\{\mu_1, \dots, \mu_m\}$ by the matrix $(-A)^{-1}$ (reduced mod 1). Show that the same holds for Dehn surgery, where A is interpreted to have the rational surgery coefficients on its diagonal. Now for any compact, oriented 4-manifold X with $H_1(X;\mathbb{Z}) = 0$ and $M = \partial X$ a rational homology sphere, show that if A represents Q_X then $(-A)^{-1}$ represents the linking form on $H_1(M;\mathbb{Z})$. (*Hint*: Lift the form from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}$ by choosing the obvious circles representing the classes μ_i. How does the relator $p_i\mu_i + q_i \sum_{j\neq i} \ell k(K_i, K_j)\mu_j$ pair with μ_k? For the case of arbitrary X, see the previous exercise.)

5.4. 1-handles revisited

As we saw in dealing with disk bundles over $\mathbb{RP}^2$ (Example 4.6.5), our notation for 1-handles causes technical problems related to framings of 2-handles. We will now introduce new notation for (orientation preserving) 1-handles that resolves these problems and also makes our handle pictures with 1-handles compatible with our discussion of surgery on links in S^3. This notation was developed by S. Akbulut [**A1**], [**AK1**].

The main observation is that the handlebody 0-handle $\cup$ m 1-handles $\approx \natural m\, S^1 \times D^3$ has the same boundary as $\natural m\, S^2 \times D^2$, which is obtained by adding 0-framed 2-handles to an m-component *unlink* (i.e., the boundary of an embedding in S^3 of m disjoint disks). In fact, the latter 4-manifold contains a canonical collection of m (uniquely) framed 2-spheres $S^2 \times 0 \subset S^2 \times D^2$, which are obtained from the disks in S^3 by pushing their interiors into int D^4 and adding the cores of the 2-handles, and surgery on these framed spheres gives back $\natural m\, S^1 \times D^3$. We will denote such a surgery by erasing the framing coefficient of the unknot and putting a dot on it as in Figure 5.33. (The symbol "∂" in the figure indicates that we have a diffeomorphism between the boundaries of the pictured 4-manifolds.) Thus, an m-component unlink with a dot on each component is the same as m 1-handles.

Exercise 5.4.1. * Visualize the S^1-family of 2-spheres $S^2 \times \{\text{pt.}\} \subset S^2 \times S^1$ in each picture of Figure 5.33, and compare the pictures.

For an alternate description of this construction, recall that a 1-handle can be cancelled by attaching an appropriate 2-handle (Proposition 4.2.9), $X \cup$ 1-handle $\cup$ 2-handle $\approx X$. Thus, adding a 1-handle to X is the same as removing the cancelling 2-handle. It is easy to check that the cocore of the 2-handle corresponds to an unknotted 2-disk in X (obtained from a 2-disk in

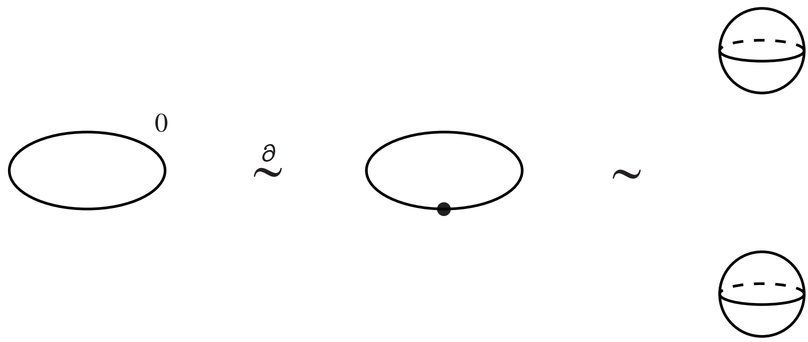


Figure 5.33. Surgering a 2-handle to a 1-handle.

∂X by pushing the interior into int X). Thus, adding a 1-handle is the same as pushing the interior of a disk D into int X and then removing a tubular neighborhood of D. The disk D is visible in Figure 5.33 spanning the dotted circle. Compare with the 3-dimensional case, Figure 5.34, where we first dig a ditch underneath the 1-handle, then slide the attaching region into the ditch. The bridge (the 1-handle) becomes level with ∂X, and the underpass (the region under the 1-handle) becomes a tunnel (a deleted νD^1).

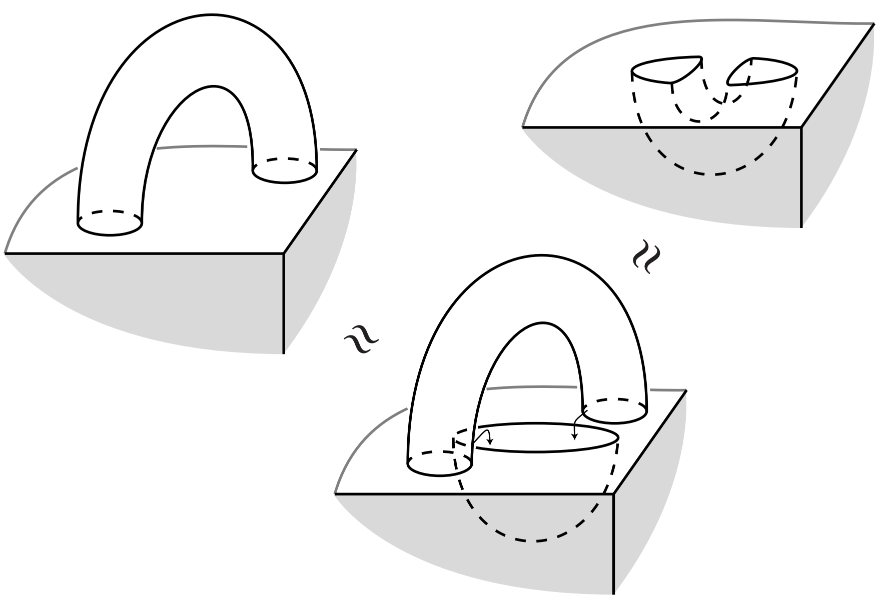


Figure 5.34. 3-dimensional 1-handle.

Exercises 5.4.2. (a)* Examine a pair of curves in Figure 5.34, where one runs over the 1-handle and the other runs underneath it (between the attaching disks in ∂X). Where do they go when we push down the 1-handle?

Then do the same in Figure 5.33, with a curve running over the 1-handle and a surface separating the two balls. Now compare with Exercise 5.4.1.

(b)* For each picture in Figure 5.33, find the torus inducing the genus-1 Heegaard splitting of $S^1 \times S^2$. How do these correspond? How do the solid tori in the Heegaard splitting correspond under the diffeomorphism?

It is sometimes helpful to imagine a dotted circle as being obtained by squeezing together the two balls, so that they become flat and close together like pancakes. Then we can simultaneously visualize curves running over the 1-handle and surfaces running underneath (between the pancakes); we merely need to remember that these are disjoint from each other. (See the solution to Exercise 5.4.2(a).) Yet another viewpoint is to imagine D in a collar $I \times \partial X$ of ∂X, with the I-coordinate represented by time. Then as we descend into X, D will appear as a dotted circle that persists until a particular time when it bounds a disk and disappears. This way, we can see the 4-manifold $I \times \partial X - \nu D \approx I \times \partial X \cup$ 1-handle in its entirety.

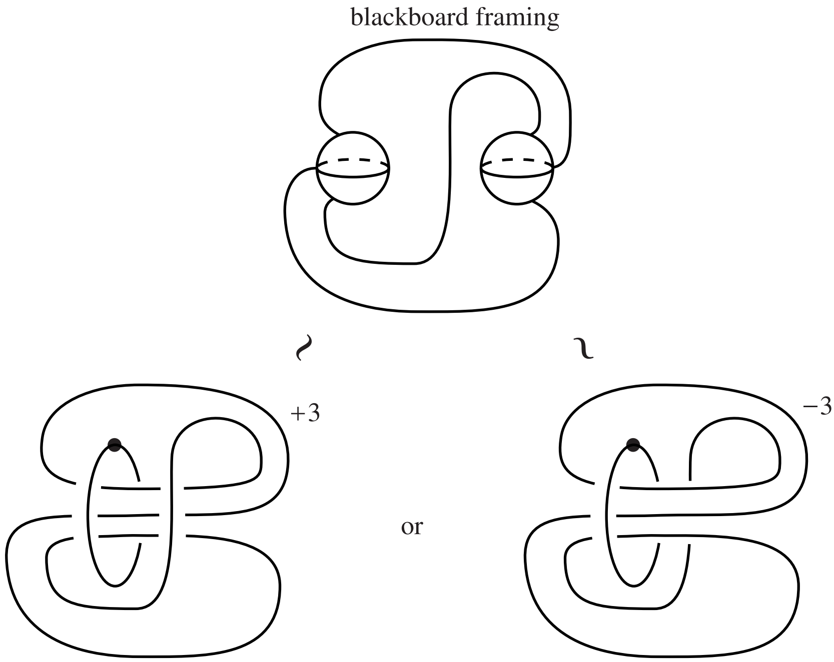


Figure 5.35. Changing notation for 1-handles.

Now if X^4 is an arbitrary handlebody in our old notation, we can isotope the attaching circles so that they avoid the regions between the attaching balls of the 1-handles. Then we can push the balls together and switch to dotted circle notation. We obtain a picture in which every attaching circle is given by a knot in S^3, so there is a canonical way to define framing coefficients, and these transform in the usual way under 2-handle slides. (See Figure 5.35. Note that the blackboard framing is preserved when we push the balls together.) Thus, we have removed the previous ambiguity in framing coefficients. In fact, the previous difficulties with framings arose when we slid a 2-handle h under a 1-handle — that is, through the region between the attaching balls. In our new notation, this appears as in Figure 5.36. Note the formal similarity with sliding over a 0-framed 2-handle. In fact, the latter operation differs from sliding under a 1-handle only by a surgery in the interior of the 4-manifold to which h is attached. In particular, the coefficient of h must transform the same way — it changes by twice the linking number of the attaching circle with the dotted circle. Because of this similarity, this move is frequently referred to as sliding "over" a 1-handle, when in fact we are sliding under it.

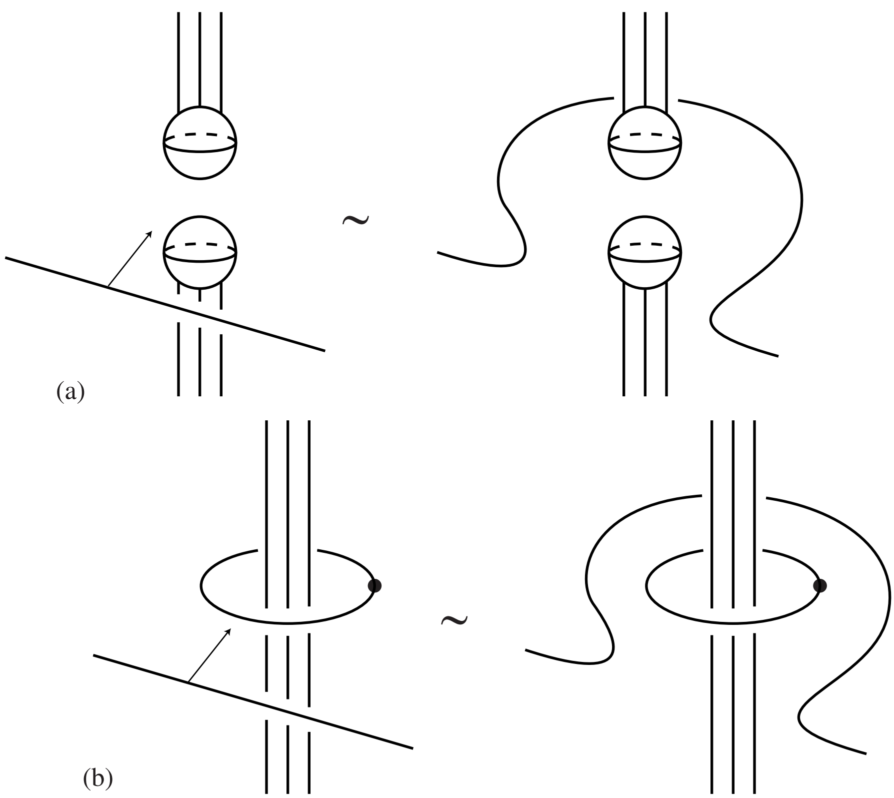


Figure 5.36. Sliding a 2-handle under a 1-handle.

Exercises 5.4.3. (a) Check the equivalence of the two bottom pictures of Figure 5.35 directly.

(b)* Check directly that if a framed knot K slides under a 1-handle given by a dotted circle K_0, then its framing coefficient changes by $2\ell k(K, K_0)$. (What happens to $w(K)$?)

(c) Draw an arbitrary D^2-bundle X over T^2 (Figure 4.36(b)) in dotted circle notation. Where does the 0-section T_0 go? Compare your answer with Figure 5.25 and Exercise 5.3.3(d). (*Answer*: Figure 6.1.) Do the same for bundles over $\mathbb{RP}^2$ (Figure 4.38). (*Answer*: Figure 6.2.) Using the new notation, verify that (a) and (b) of Figure 4.38 are diffeomorphic for $m = e + 2$, $n = e - 2$ (e fixed) as in Example 4.6.5 (cf. Exercise 4.6.7(a)).

(d)* Given a Kirby diagram for a 4-manifold X, suppose that one attaching circle K is a 0-framed unknot in the boundary of the 0-handle. Then K determines an embedded 2-sphere S in X with a trivial normal bundle. How can a diagram be constructed for the manifold obtained from X by the Gluck construction on S (Exercises 5.2.7 and the preceding text)?

The above observations suggest an alternate approach to dealing with framings. We return to the old notation, but draw reference arcs (with dashed lines or in a different color) to indicate how each pair of balls should be joined to obtain dotted circle notation (Figure 5.37). Now framing coefficients are well-defined (via dotted circle notation), and we are free to perform isotopies. The only catch is that whenever an attaching circle isotopes through a reference arc, we must recalculate its framing as above. Note that for a nullhomologous knot in $\partial(D^4 \cup 1\text{-handles})$, framings are well-defined without reference arcs. In practice, the reference arcs are frequently suppressed from the notation when this can be done without confusion; cf. Figure 5.14. (Beware that if a 2-handle runs over two 1-handles whose reference arcs cross, its framing coefficient will depend on which arc crosses over the other, unless its attaching circle has intersection number 0 with one belt sphere.)

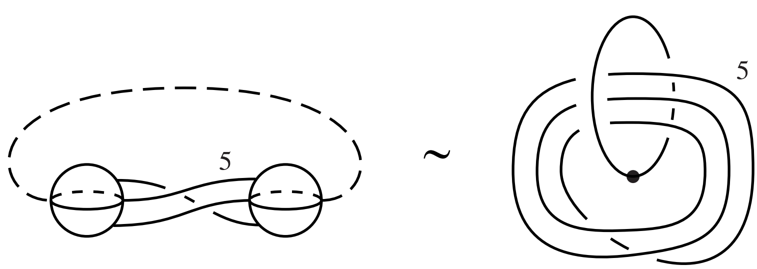


Figure 5.37. Specifying a framing using a reference arc.

Another advantage of the dotted circle notation for (oriented) handlebodies X is that we have a general formula for $H_1(\partial X)$ — If there are no

3-handles, we simply surger out all 1-handles in X (i.e., replace dots with 0 coefficients) to obtain a 2-handlebody Y with $\partial Y = \partial X$, and then apply Corollary 5.3.12. If there are 3-handles, then each 3-handle surgers out an S^2 from the 2-handlebody ∂Y, so it either disconnects ∂Y (undoing a connected sum) or removes an $S^1 \times S^2$ summand (deleting a $\mathbb{Z}$-summand from $H_1(\partial Y)$).

Exercise 5.4.4. * Compute $H_1(M)$ for an arbitrary circle bundle $M \to F$ over a closed surface (M orientable), using your answers to Exercises 5.4.3(c), 4.6.6(b) and 4.6.7(b). (See also Figure 6.4.)

To complete our discussion of Kirby calculus on handlebodies, we must see what Kirby moves look like in the new notation. The discussion of 2-handle/3-handle cancellations is unchanged (provided we allow slides under 1-handles in Proposition 5.1.9). A cancelling 1-handle/2-handle pair is shown in Figure 5.38. Note that the attaching circle of the 2-handle intersects the spanning disk of the dotted circle in a unique point as required.

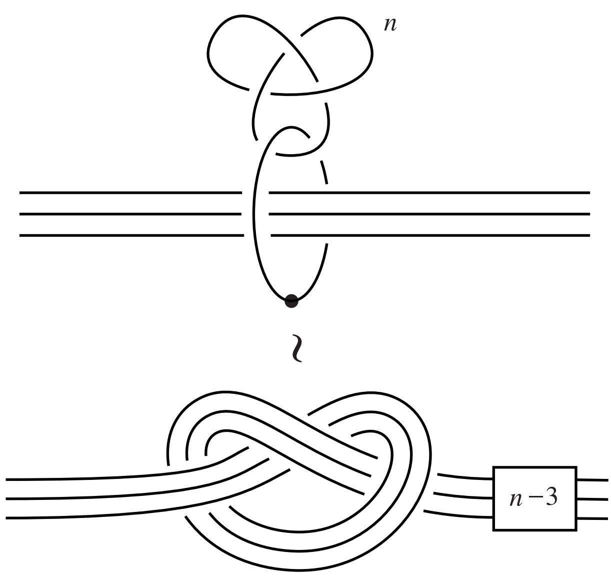


Figure 5.38. Handle cancellation.

As before, other 2-handles must be slid off of the 1-handle (which we can easily do simultaneously), and then we erase the 1-2 pair. If we think of the 2-handle h as being attached before we remove the disk D spanning the cancelling dotted circle, then we can identify D with the cocore of h (since the dotted circle is a meridian of the attaching circle of h), making it clear why deleting νD is the same as removing h. Alternatively, if h is attached to a 0-framed meridian of the dotted circle, then we can identify h with νD. (We can always reduce to this case by slides under the 1-handle and Figure 5.42, discussed below.) A 2-handle slide proceeds as before (and slides under 1-handles are formally similar). A 1-handle slide is shown in Figure 5.39. Note the formal similarity to sliding 0-framed 2-handles, except that the sliding 2-handle corresponds to the 1-handle that does *not* slide. We must be careful, however, to make sure that the union of dotted circles remains an unlink; Figure 5.40 shows a 2-handle slide with no analogue for 1-handles in the present notation (although we will return to this in Section 6.2). We can ensure that our handle slide is valid by imagining a plane separating the two 1-handles (and disjoint from any other dotted circles), and then choosing the band so that its core only intersects the plane once. The picture will then be isotopic to Figure 5.39. (Shrink one half-space to a small ball and pull on the band to eliminate any knotting of it in the other half-space, then repeat with the half-spaces interchanged.) Thus, we can slide dotted circles over each other as if they were 0-framed 2-handles, provided that the band satisfies the above property. Changing the manifold X by blowing up or down proceeds as in Section 5.1; we need only make sure that the ± 1-framed unknot is unlinked from the dotted circles.

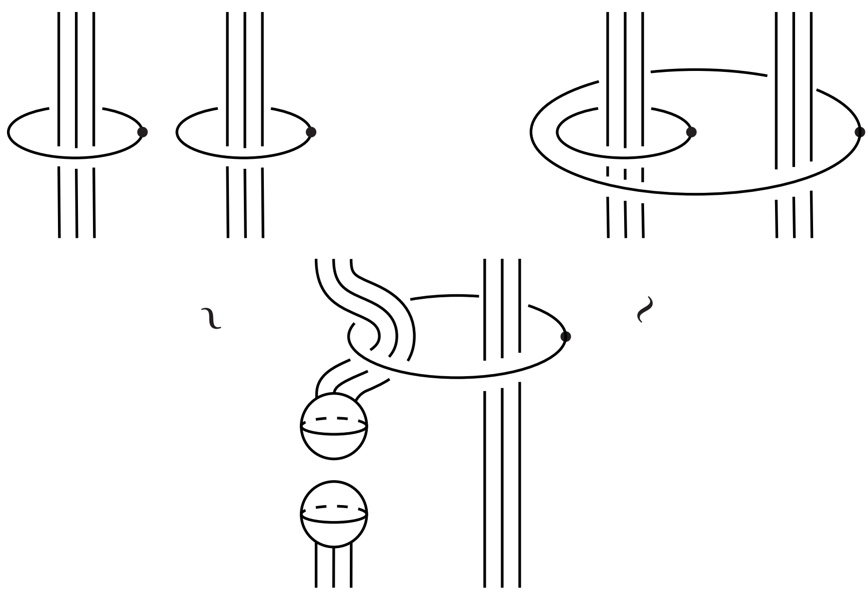


Figure 5.39. 1-handle slide.

Exercises 5.4.5. **(a)*** Identify the familiar closed manifold shown in Figure 5.41.

(b)* Repeat Exercise 5.1.1 using dotted circle notation.

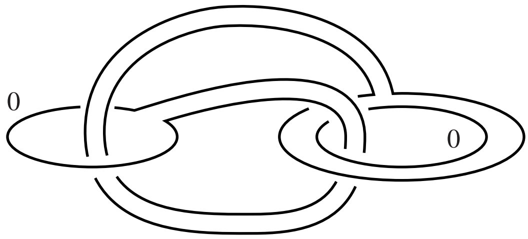


Figure 5.40. 2-handle slide producing a nontrivial link.

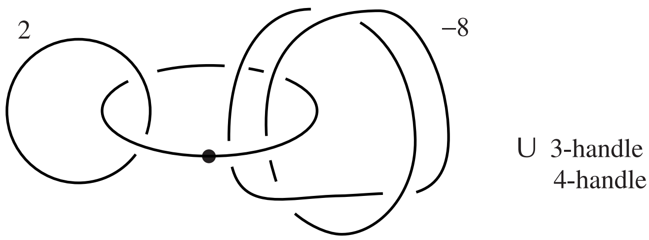


Figure 5.41. Identify this manifold.

One more useful trick is shown in Figure 5.42. It is obtained by rotating one attaching ball 360° as shown. A different derivation using standard Kirby moves in dotted circle notation is given in Figure 5.43; see also Exercise 5.5.2.

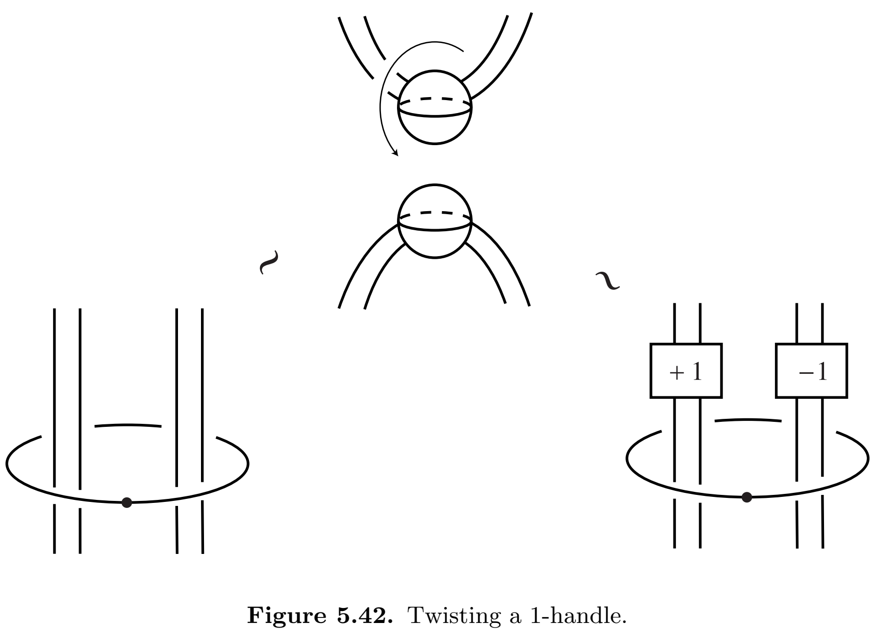


Figure 5.42. Twisting a 1-handle.

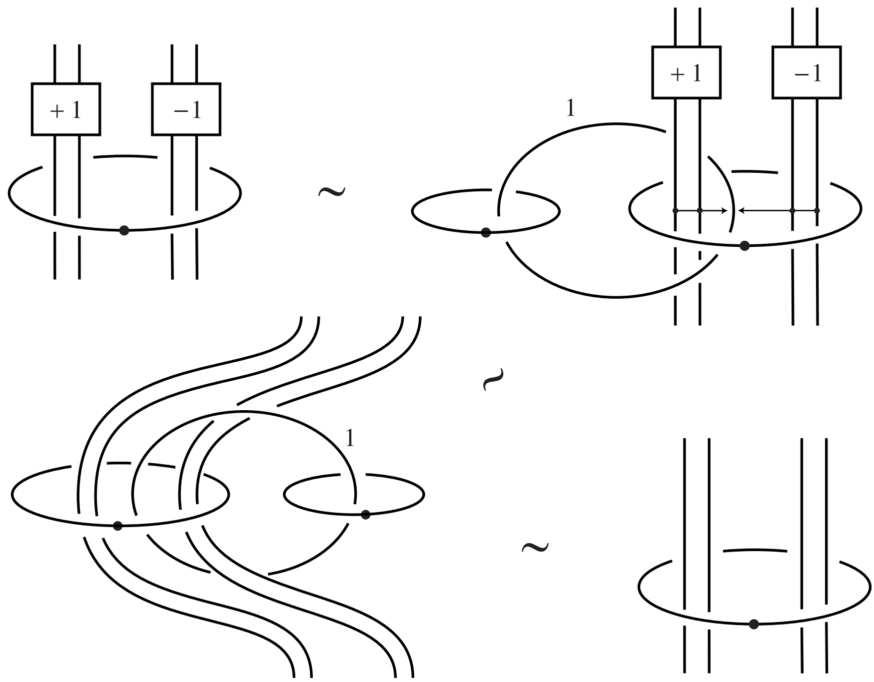


Figure 5.43. Derivation of previous figure using Kirby moves.

5.5. Relative Kirby calculus

As we saw in Section 4.2, handle decompositions are defined in greatest generality on manifold *pairs* $(X, \partial_- X)$. So far, we have only drawn Kirby diagrams for the case $\partial_- X = \emptyset$. We now generalize to the case of an arbitrary compact pair $(X^4, \partial_- X)$ with $\partial_- X$ oriented. (Recall that if X is oriented then $\partial X = \partial_+ X \coprod \overline{\partial_- X}$ in the boundary orientation.) We continue to assume X is connected (or work with each component separately). Thus, without losing generality, we can assume that there are no 0-handles if $\partial_- X \neq \emptyset$, and (by the following remark) that $\partial_- X$ is connected.

Remark 5.5.1. If $\partial_- X$ is disconnected, there is a family of 1-handles in X connecting the components of $I \times \partial_- X$ (cf. Proposition 4.2.13), and deleting these 1-handles from X (along with $I \times$ (attaching balls)) creates a new boundary $\partial_-^{\#} X$ that is the connected sum of the original components of $\partial_- X$. We can build the rest of the handlebody on the connected manifold $I \times \partial_-^{\#} X$ by the techniques of this section, and then recover X by attaching dual 3-handles to $\{0\} \times \partial_-^{\#} X$ along the 2-spheres on which we summed.

To describe $(X, \partial_- X)$ with $\partial_- X \neq \emptyset$, we must begin by describing the 3-manifold $\partial_- X$. We can always do this by a rational (or integral) surgery diagram in $\mathbb{R}^3$ as in Section 5.3. Then we interpret the resulting diagram as

$\{1\} \times \partial_- X \subset I \times \partial_- X$, and use our previous notation for attaching the handles of X, obtaining a Kirby diagram superimposed on a surgery diagram. To distinguish the two diagrams, we put brackets around the surgery coefficients of $\partial_- X$, as in $\langle r \rangle$. (Using a different color is also helpful.) We call the resulting diagram a (*relative*) *Kirby diagram* for $(X, \partial_- X)$. Note that $\partial_+ X$ is obtained by doing surgery on the entire diagram, $\partial_- X$ is obtained by just doing surgery on the components with coefficients in brackets, and $\partial X = \partial_+ X \coprod \{0\} \times \overline{\partial_- X}$. We always define framing coefficients with respect to the copy of $\mathbb{R}^3$ in which the diagram is drawn. (Even if $\partial_- X = S^3$, framings defined with respect to $\partial_- X$ may differ from those defined by $\mathbb{R}^3$ — consider a meridian to a $\langle 1 \rangle$-framed unknot.)

Now that we have described $(X, \partial_- X)$ by a link with coefficients attached and two kinds of components (or a suitable generalization thereof to allow 1-handles), we wish to understand the allowable moves of such diagrams. First, we can clearly perform Kirby and Rolfsen moves on the $\langle r \rangle$-framed components representing $\partial_- X$, avoiding the spanning disks of dotted circles (which are necessarily disjoint from the components representing $\partial_- X$) and interpreting the 2-handles of X as a framed link in $\partial_- X$, dragging it along in the obvious way (changing coefficients as usual during Rolfsen twists, etc.). Next, we can isotope the 2-handle attaching circles in $\partial_- X$, in particular, sliding handles over $\langle n \rangle$-framed link components. Finally, we can perform Kirby moves on the handles of X. Of course, we are not allowed to slide an $\langle r \rangle$-framed component over an n-framed component, which also restricts blow-downs of ± 1-framed unknots. If we use dotted circles to represent 1-handles in X, these will be unlinked from the surgery diagram of $\partial_- X$, and we can slide surgery curves under them provided that we use suitable bands to avoid linking (cf. Figure 5.40) as we did for 1-handle slides. (Check this by switching to ball notation for 1-handles.) To see that our previous theory of Kirby moves works without change on the handles of X, simply turn the rational surgery diagram into an integral surgery diagram (by inverse slam-dunks as in Exercise 5.3.9(b)). Then we have implicitly written $\partial_- X$ as the boundary of a 2-handlebody, so we have reduced to the case $\partial_- X = \emptyset$. In particular, the framing formulas are still valid.

Exercise 5.5.2. * Derive the move in Figures 5.42 and 5.43 by removing int D^4 and performing Kirby moves on the new $\partial_- X = S^3$.

Theorem 5.5.3. *Given relative Kirby diagrams for the (compact) pairs $(X_i, \partial_- X_i)$ ($i = 1, 2$, $\partial_- X_i$ nonempty, connected and oriented), any diffeomorphism ψ from $(X_1, \partial_- X_1)$ to $(X_2, \partial_- X_2)$ that preserves orientation on $\partial_- X_1$ can be realized by isotopy in $\mathbb{R}^3$ and a sequence of Rolfsen twists and ∞-insertions/deletions in the Dehn surgery diagram for $\partial_- X_1$ (or by blow-ups/blow-downs if all coefficients are integral), together with handle* *pair creations/cancellations and handle slides (and switching 1-handles from dotted circle to ball notation).*

Proof. We can make the Dehn surgery diagrams of $\partial_- X_i$ agree (realizing $\psi|\partial_- X_1$) by Rolfsen twists (Proposition 5.3.10) or blowing up and down in the integral case (Theorem 5.3.6). We can assume all coefficients are integral (Exercises 5.3.9(b) and (c)). Now we can identify $\partial_- X_2$ with $\partial_- X_1$ and $(X_2, \partial_- X_2)$ with $(X_1, \partial_- X_1)$ by ψ to obtain two relative handle decompositions of the same manifold. By Theorem 4.2.12 we can then make the decompositions agree by pair creations/cancellations, slides and isotopy in $\partial_- X_1$. The latter includes slides over $\langle n \rangle$-framed knots, but Exercise 5.1.12(b) reduces such slides to $\langle \pm 1 \rangle$-blow-ups. If there are 1-handles in dotted circle notation, isotopy in $\partial_- X_1$ may also require slides under these 1-handles. If we switch to ball notation, these moves become isotopies in $\mathbb{R}^3$. □

Example 5.5.4. – Relative doubles. Recall from Example 4.6.3 that the relative double $D(X, \partial_- X)$ of $(X, \partial_- X)$ is $(X \cup_{\mathrm{id}_{\partial_+ X}} \overline{X}, \partial_- X)$, and a handle decomposition of $(X, \partial_- X)$ that is ordered by increasing index and has no handles of index $> \frac{1}{2} \dim X$ induces a decomposition of $D(X, \partial_- X)$ that is also ordered by index. In that example, we saw how to construct a Kirby diagram for this decomposition in the case $\dim X = 4$, $\partial_- X = \emptyset$. The construction generalizes immediately to the relative case, starting with a relative Kirby diagram for $(X, \partial_- X)$ without 3- or 4-handles. Each 2-handle h induces a dual 2-handle in $(\overline{X}, \partial_+ X)$ that attaches to a 0-framed meridian of the attaching circle of h. Each 1-handle induces a 3-handle, and if $\partial_- X \neq \emptyset$ there are (by assumption) no 0- or 4-handles.

Example 5.5.5. – Turning handlebodies upside down. As we saw in Section 4.2, any handle decomposition of $(X, \partial_- X)$ induces a dual decomposition of $(X, \overline{\partial_+ X})$. We will see how to obtain this dual decomposition, beginning with the case without 3- or 4-handles in $(X, \partial_- X)$. Given such a decomposition of $(X, \partial_- X)$, we have already obtained the required decomposition of $(\overline{X}, \partial_+ X)$ inside $D(X, \partial_- X)$. To see it by itself, we merely need to remove $(X, \partial_- X)$, i.e., interpret the picture of the latter inside $D(X, \partial_- X)$ as the 3-manifold $\partial_+ X$. Thus, we begin with the diagram of $(X, \partial_- X)$, using dotted circle notation for 1-handles, surger these to 0-framed 2-handles, and put brackets on all coefficients to obtain $\partial_+ X$. To build $(\overline{X}, \partial_+ X)$, we simply add a 0-framed meridian to each of the original 2-handles of $(X, \partial_- X)$, then add a 3-handle for each 1-handle of $(X, \partial_- X)$, and a 4-handle if $\partial_- X = \emptyset$. If we wish to obtain $(X, \overline{\partial_+ X})$ with the original orientation on X, we must then take the mirror image of the diagram and reverse the signs of all coefficients. (Sometimes it is easier to fix the orientation on $\partial_+ X$ and allow the orientation on X to reverse when we dualize.) In practice, we can usually simplify $\partial_+ X$ significantly by Kirby or Rolfsen moves.

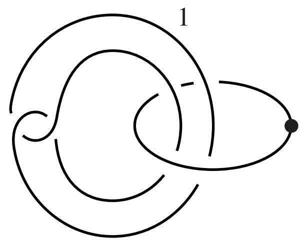


Figure 5.44

Exercise 5.5.6. * Turn the handlebody X of Figure 5.44 upside down.

To turn a more general relative handlebody $(X, \partial_- X)$ upside down, we lift off the 3- and 4-handles to get a new relative handlebody $(X_2, \partial_- X)$, and then apply the above procedure, obtaining a handlebody on $\partial_+ X_2$. If $\partial_+ X = \emptyset$, then the 3- and 4-handles dualize to $\natural m\, S^1 \times D^3$ $(m \geq 0)$, and there is a unique way to glue this into the manifold. Thus, the surgery diagram we obtain for $\partial_+ X_2$ can be transformed into a $\langle 0 \rangle$-framed unlink ($\# m\, S^1 \times S^2$), and we can mount it on top of $\natural m\, S^1 \times D^3$ simply by turning the components into dotted circles, obtaining the required dual handle decomposition of $(\overline{X}, \partial_+ X)$ (up to 1-handle slides). A similar procedure works if $\partial_+ X$ is nonempty but connected and X is simply connected. In that case $\partial_+ X_2 \approx \partial_+ X \# m S^1 \times S^2$, and the choice of identification of it with the corresponding boundary $\partial_+(I \times \partial_+ X \cup m1\text{-handles})$ does not matter by [**Tr**].

Exercises 5.5.7. (a)* Embed $T^2 \times D^2$ (Exercise 5.4.3(c)) in S^4 by attaching cancelling handles to it to obtain D^4. Draw a Kirby diagram of the complement X of $T^2 \times \text{int } D^2$, with $\partial_- X = \emptyset$. Repeat for bundles over $\mathbb{RP}^2$. (Which two bundles work?) Compare with Section 6.2.

(b)* Show directly that a pair of consecutive applications of the algorithm for dualizing results in the original manifold pair $(X, \partial_- X)$.

(c)* Let h_1 and h_2 be 2-handles in a relative Kirby diagram, and let h_1^* and h_2^* be the corresponding dual 2-handles in the dual handle decomposition. Verify that sliding h_1 over h_2 corresponds to sliding h_2^* over h_1^* in the dual diagram, and if we orient the handles (so that the cores of h_i and h_i^* have intersection number $+1$) then adding handles corresponds to subtracting dual handles. (*Hint*: What does the slide look like in the dual diagram? What must the dual handles look like?)

Example 5.5.8. – Gluing along a common boundary. Suppose we are given relative handlebodies $(X, \partial_- X)$ and $(Y, \partial_- Y)$ and a diffeomorphism $\varphi \colon \partial_+ X \to \partial_- Y$. Then we can obtain a handle decomposition of the pair $(X \cup_\varphi Y, \partial_- X)$, simply by pulling the attaching data back from $\partial_- Y$ to $\partial_+ X$ via φ. In the context of Kirby calculus, we are typically given φ in the following form: $\partial_+ X$ is exhibited by taking the diagram for X, converting 1-handles to dotted circles and then to $\langle 0\rangle$-framed unknots, and then putting brackets on all remaining coefficients. We are then given a sequence of Kirby or Rolfsen moves making the picture identical to that of $\partial_- Y$, followed by an explicit identification of the pictures. (Beware that a different choice of identification may result in a different diffeomorphism φ and a different 4-manifold $X \cup_\varphi Y$.) To obtain $X \cup_\varphi Y$, we pull the handle diagram for Y in $\partial_- Y$ back to $\partial_+ X$, and reverse the above construction (removing brackets and replacing dotted circles) to convert $\partial_+ X$ back into X. A common situation is where we have handlebodies X and Y with connected $\partial X \approx \partial Y$ and wish to form a closed manifold $X \cup_\partial \overline{Y}$. In this case, we turn one handlebody upside down and then apply the previous procedure using the given diffeomorphism.

Exercises 5.5.9. (a)* Split $\partial(T^2 \times D^2)$ as $T^2 \times \partial D^2 = (S^1 \times S^1) \times S^1$. What familiar manifold is obtained from two copies of $T^2 \times D^2$ by gluing by a cyclic permutation φ of the three S^1 factors? What manifold is obtained by embedding $T^2 \times D^2$ in S^4 with complement X as in Exercise 5.5.7(a), then cutting it out and regluing it by φ? What is $X \cup_\varphi X$?

(b)* Let P denote the (negative) E_8-plumbing (Figure 4.33) and let Q denote $D^4 \cup$ 2-handle glued along a -1-framed left trefoil knot (Figure 5.22). We showed in Exercise 5.1.12(a) that $\partial P \approx \partial Q$. Prove that $P \cup_\partial \overline{Q} \approx \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. (There is only one such manifold, since any self-diffeomorphism of the Poincaré homology sphere ∂P is isotopic to the identity [**BO**].)

Remarks 5.5.10. (a) There are times when it is useful to have a general solution to a gluing problem as above. For example, suppose we wish to understand in general how to cut a copy of Z out of a manifold and replace it by X via a fixed diffeomorphism $\varphi\colon \partial X \to \partial Z$. Then we will begin with a family $\{Y_i\}$ of manifolds containing Z, which are given as handlebodies built on ∂Z, and we wish to simultaneously pull all sets of gluing data back to ∂X via φ. This can be done using a Heegaard splitting of ∂Z. (See Exercise 6.2.2 for locating such a splitting.) If $H \approx \natural g S^1 \times D^2$ is one of the resulting 1-handlebodies in ∂Z, we can assume that all attaching data for 1- and 2-handles of manifolds Y_i lie in H, so it suffices to pull H back to ∂X via φ. One way to do this is to identify the 0-handle of H with a copy of D^3 at ∞ in the diagram, and leave this fixed during the Kirby moves. Now the 1-handles of H can be represented as arcs with endpoints attached to the fixed ∂D^3, with framing coefficients that are well-defined because there is essentially a unique arc in ∂D^3 connecting each pair of endpoints. The Kirby moves will send this family of framed arcs to a similar family in ∂X, and the correspondence between these framed arcs will determine how the handles of each Y_i pull back to ∂X. (This procedure reflects the fact that a diffeomorphism between closed 3-manifolds is determined up to isotopy by its restriction to a 1-handlebody H as above.) As an example, we will use this procedure for analyzing logarithmic transformations in Section 8.3.

(b) Given a relative Kirby diagram for $(X, \partial_- X)$, there is also a procedure for turning over a single boundary component, which in turn allows us to glue components of ∂X in pairs. For example, we can construct a handlebody for $(X, \emptyset)$ (with the entire boundary appearing as $\partial_+ X$). First, we find a Heegaard decomposition for $\partial_- X$. (We can always do this by the method of Exercise 6.2.2.) Then we construct $(I \times \partial_- X, \emptyset)$ as a handlebody as in Example 4.6.8, where we explicitly see the Heegaard decomposition of the 3-manifold $\{1\} \times \partial_- X$. Finally, we add handles to $\{1\} \times \partial_- X$ as prescribed by the original diagram of $(X, \partial_- X)$. Similarly, we can turn over a single component of $\partial_- X$ or glue two components of ∂X by a diffeomorphism (as we did to create $S^1 \times M$ in Example 4.6.8). For a different approach, see Exercise 6.2.5(b) or [**A6**].

5.6. Spin structures

We now describe w_2 and spin structures from an elementary viewpoint, for the purpose of analyzing their meaning for Kirby diagrams in the next section. Some of this discussion is parallel to that of Section 1.4, but the two discussions are essentially independent. The division of material reflects the differing viewpoints of high- and low-dimensional topology. In the present section, our technique for classifying spin structures will be a special case of *obstruction theory* [**St**], [**Wh**], a general approach for understanding fiber bundles and structures on them. This theory works for bundles over arbitrary CW-complexes, but for our purposes it is convenient to take the base to be a handlebody X (ordered by increasing index). Recall that at the end of Section 4.2 we described the homology and cohomology of X using its handle structure, via the chain complex $C_k(X) = H_k(X_k, X_{k-1})$. This description allows us to conveniently define obstructions such as w_2 in the cohomology group $H^*(X)$.

Let E be a real vector bundle over X, with oriented fibers of dimension m. After summing with a trivial bundle if necessary, we can assume $m \geq 3$. (It is easily verified that for $m \geq 3$, summing with an additional trivial bundle will have no effect on our subsequent discussion of spin structures, since the inclusion $SO(m) \to SO(m + r)$ induces isomorphisms on π_1 and π_2.) By Milnor [**M3**] (or Remark 5.6.9(a) below) we can define a *spin structure* on E to be a (positively oriented) trivialization of $E|X_2$ (up to fiber homotopy). To construct a spin structure, we begin with a trivialization τ of $E|X_1$. (Such trivializations exist because E is oriented; cf. Exercise 5.6.2(b).) Now we wish to extend τ over each 2-handle h. But $E|h$

is canonically trivial, so τ over the attaching region determines an element of $\pi_1(SO(m)) \cong \mathbb{Z}_2$ (cf. Example 4.1.4(d)), and τ extends over h if and only if this element vanishes. Applying this to each 2-handle of X_2, we obtain an element of $\mathbb{Z}_2$ assigned to each 2-handle, that is, a cochain $c(\tau) \in C^2(X;\mathbb{Z}_2)$, and τ extends over X_2 if and only if $c(\tau) = 0$. Of course, we may have made a bad choice for τ. For a different τ', we can assume (by homotopy) that $\tau|X_0 = \tau'|X_0$. Then the difference between τ and τ' will be an (arbitrary) element of $\pi_1(SO(m))$ assigned to each 1-handle, or equivalently, a cochain $d(\tau,\tau') \in C^1(X;\mathbb{Z}_2)$. For each 2-handle h, the extendability over h of τ and τ' will be equivalent if and only if $0 = \langle d(\tau,\tau'), \partial_* h\rangle = \langle \delta d(\tau,\tau'), h\rangle$. Thus, $c(\tau') - c(\tau) = \delta d(\tau,\tau')$, so changing τ corresponds to changing $c(\tau)$ by arbitrary coboundaries. Now for any 3-handle h_3, $E|h_3$ is trivial, so for any τ we must have $0 = \langle c(\tau), \partial_* h_3\rangle = \langle \delta c(\tau), h_3\rangle$, hence $\delta c(\tau) = 0$, i.e., $c(\tau)$ is a cocycle. Since $c(\tau)$ is independent of τ up to coboundaries, we obtain a class $w_2(E) = [c(\tau)] \in H^2(X;\mathbb{Z}_2)$, the *second Stiefel-Whitney class* of E. We conclude (cf. Proposition 1.4.25):

Proposition 5.6.1. *An oriented vector bundle E over X admits a spin structure if and only if $w_2(E) = 0$ in $H^2(X;\mathbb{Z}_2)$.* □

Since $\pi_2(SO(m)) = 0$, any trivialization over X_2 extends over X_3. Thus, if $\dim X \leq 3$ or if $\dim X = 4$ and X has no closed components, we conclude that $E \to X$ admits a spin structure if and only if it is trivial. For a spin bundle over a closed, connected, oriented 4-manifold X, $E|(X - \{\text{pt.}\})$ is trivial, and the remaining obstruction to trivializing E is in $H^4(X;\pi_3(SO(m))) \cong \pi_3(SO(m))$. This is $\mathbb{Z}\oplus\mathbb{Z}$ for $m = 4$, and $\mathbb{Z}$ for other $m \geq 3$. The resulting integer obstructions are the *Pontrjagin class* $p_1(E)$ and (when $m = 4$) the *Euler class* $e(E)$ in $H^4(X;\mathbb{Z})$ (cf. Theorem 1.4.20). The vanishing of $\pi_2(SO(m))$ also implies that if τ extends from X_1 to X_2, then the extension is unique. Thus we can define a spin structure to be a trivialization of $E|X_1$ that extends over X_2. (It can be shown [**G15**] that spin^c structures have an analogous characterization, as complex structures on $E|X_2$ that extend over X_3, provided we arrange the fibers of E to have even dimension ≥ 4.)

Exercises 5.6.2. (a)* Suppose that E is an oriented vector bundle over $X = Y \cup$ handles, where $E|Y$ is given a spin structure s. Define a relative class $w_2(E,s) \in H^2(X,Y;\mathbb{Z}_2)$ that vanishes if and only if s extends to a spin structure on E.

(b) Let E be an arbitrary (unoriented) vector bundle over X (not necessarily connected). We can define a *(fiber) orientation* of E to be a trivialization of $E|X_0$ that extends over X_1 (up to fiber homotopy). Show that this is the same as a continuous choice of orientation of all fibers. Define the *first*

Stiefel-Whitney class $w_1(E) \in H^1(X;\mathbb{Z}_2)$ so that E is fiber-orientable if and only if $w_1(E)=0$ (cf. Lemma 1.4.23).

(c) For a complex vector bundle $E \to X$, the *first Chern class* $c_1(E) \in H^2(X;\mathbb{Z})$ is the obstruction to a complex trivialization of $E|X_2$. Construct this as above. (The determinant function $\det\colon U(m) \to S^1$ induces an isomorphism on π_1.) Show that $c_1(E)|_2 = w_2(E)$, i.e., $c_1(E)$ maps to $w_2(E)$ under the coefficient homomorphism $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2)$. Generalize to the relative case as in (a). (Compare with Theorem 1.4.3 and Proposition 1.4.9.)

(d) For a real, oriented 2-plane bundle E, define the *Euler class* $e(E) \in H^2(X;\mathbb{Z})$ so that E is trivial if and only if $e(E)=0$. How does e relate to w_2? To c_1? (Compare with Proposition 1.4.9.) How does this compare with the usual definition of $e(E) \in H^m(X;\mathbb{Z})$ (m = fiber dimension of E) as the Poincaré dual to the zero set of a generic section (Claim 1.4.7)?

The other Stiefel-Whitney classes $w_i(E) \in H^i(X;\mathbb{Z}_2)$ and Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ (cf. Proposition 1.4.5) can be defined using the obstructions to finding a trivial subbundle of codimension $i-1$ over $E|X_i$ (resp. $E|X_{2i}$) [**MS**]. If E is nonorientable, a subtlely of the theory arises that we have managed to avoid — the coefficient groups of the cohomology may be twisted.

Now suppose that E admits a spin structure s. We wish to classify all spin structures s' on E. The structures s and s' determine trivializations τ and τ' of $E|X_1$ up to homotopy. As before, we can homotope τ to agree with τ' on $E|X_0$, and then define $d(\tau,\tau') \in C^1(X;\mathbb{Z}_2)$. Since τ and τ' both extend over X_2, we have $\delta d(\tau,\tau') = c(\tau') - c(\tau) = 0 - 0 = 0$. Thus, $d(\tau,\tau')$ is a cocycle. However, it depends on our choice of homotopy to make $\tau = \tau'$ over X_0. If we change the homotopy, it changes τ' by a full twist (the generator of $\pi_1(SO(m))$) over a collection of 0-handles comprising a 0-cochain b, and so $d(\tau,\tau')$ will change on any 1-handle h for which $0 \neq \langle b, \partial_* h\rangle = \langle \delta b, h\rangle$. Thus, $d(\tau,\tau')$ changes by the (arbitrary) coboundary δb, and so we obtain a well-defined *difference class* $\Delta(s,s') = [d(\tau,\tau')] \in H^1(X;\mathbb{Z}_2)$. Clearly, $\Delta(s,s')=0$ if and only if $s=s'$. Furthermore, $\Delta(s,s')+\Delta(s',s'') = \Delta(s,s'')$. In the other direction, given s, any cocycle $d \in C^1(X;\mathbb{Z}_2)$ changes $\tau = s|X_1$ to τ' with $c(\tau') = c(\tau) + \delta d = 0$, so τ' extends uniquely to a spin structure s' with $\Delta(s,s') = [d]$. We immediately obtain (cf. Proposition 1.4.25):

Proposition 5.6.3. *If $E \to X$ is an oriented vector bundle with $w_2(E)=0$, then the group $H^1(X;\mathbb{Z}_2)$ acts freely and transitively on the set $\mathcal{S}(E)$ of spin structures on E. Equivalently, choosing a base point $s \in \mathcal{S}(E)$ identifies $\mathcal{S}(E)$ with $H^1(X;\mathbb{Z}_2)$ by $s' \mapsto \Delta(s,s')$.* □

Exercises 5.6.4. (a)* For $E \to X = Y \cup$ handles as in Exercise 5.6.2(a), classify spin structures that restrict to a fixed s on $E|Y$.

(b)* Show that for spin structures s and s' on an oriented bundle E over a compact n-manifold X, we have

$$[I] \times \Delta(s, s') = w_2(E^*, s^*) \in H^2(I \times X, \partial I \times X; \mathbb{Z}_2),$$

where $[I] \in H^1(I, \partial I; \mathbb{Z}_2)$ is the relative fundamental class of $I = [0, 1]$, E^* is E pulled back over $I \times X$, and s^* equals s on $E|\{0\} \times X$ and s' on $E|\{1\} \times X$. Equivalently, the Poincaré duals of $\Delta(s, s')$ in $H_{n-1}(X, \partial X; \mathbb{Z}_2)$ and $w_2(E^*, s^*)$ in $H_{n-1}(I \times X, I \times \partial X; \mathbb{Z}_2)$ correspond under the obvious isomorphism. (By the relative version of Proposition 5.6.7 below, you can choose the handle decomposition conveniently.)

(c) Classify fiber orientations on a bundle $E \to X$ with X not necessarily connected. What group acts? (*Answer*: Lemma 1.4.23.)

(d)* For a complex vector bundle $E \to X$, classify the complex trivializations of $E|X_2$ up to homotopy. (Note that $\pi_2(U(m)) = 0$.) How do these relate to spin structures? What happens in the relative case?

Proposition 5.6.5. *Let $\pi\colon E \to X$ and $\pi'\colon E' \to X'$ be oriented vector bundles with a smooth function $f\colon X' \to X$ that is covered by a bundle map $F\colon E' \to E$ (i.e., for each $x \in X'$, F maps $(\pi')^{-1}(x)$ isomorphically to $\pi^{-1}(f(x))$). Then $w_2(E') = f^*w_2(E)$, and any spin structure s on E induces a spin structure f^*s on E' (depending on F). Furthermore, $\Delta(f^*s_0, f^*s_1) = f^*\Delta(s_0, s_1)$. (Compare with Proposition 1.4.5.)*

Proof. After a homotopy, we can assume that $f(X_i') \subset X_i$ for each i. Then $f^*\colon H^k(X; G) \to H^k(X'; G)$ is obtained as usual by dualizing the map $f_\#\colon C_k(X') \to C_k(X)$ that sends each handle h' to $\sum a_i h_i$, where a_i is the intersection number of the core of h' with the cocore of h_i. Clearly, any trivialization τ of $E|X_1$ determines a trivialization $f^*\tau$ of $E'|X_1'$, and the obstruction cochain $c(f^*\tau)$ will be $f^\# c(\tau)$. Thus, $w_2(E') = f^*w_2(E)$. Similarly, spin structures and difference classes pull back as required. The only difficulty is that the construction of f^*s depends on a choice of homotopy of f, so we must check that for a homotopy f_t (with f_0 and f_1 preserving handles as above) we will have $f_1^*s = f_0^*s$. But we may assume that for all t, $f_t(X_1') \subset X_2$. Since s trivializes $E|X_2$, we obtain the required homotopy from $f_0^*\tau$ to $f_1^*\tau$. □

Exercise 5.6.6. Prove analogous propositions for w_1 and c_1.

Proposition 5.6.7. *For an oriented bundle $E \to X$, it follows that spin structures, w_2 and Δ are independent of the choice of handle decomposition of X.*

Proof. Apply the previous proposition to the identity map. □

Our main interest is in tangent bundles. As in Section 1.4, we define $w_i(X)$ to be $w_i(TX)$ and refer to spin structures on TX as spin structures on X. A spin structure s on X restricts to a spin structure $s|\partial X$ on ∂X, and $(\partial X, s|\partial X)$ is called the *spin boundary* of the spin manifold (X, s). Proposition 5.6.5 shows that for any local diffeomorphism $f\colon Y \to X$, we have $w_2(Y) = f^*w_2(X)$, and any spin structure s on X pulls back to a structure f^*s on Y. (Set $F = df$.)

Exercises 5.6.8. (a)* Prove that for $n \geq 2$, the manifold $S^1 \times D^n$ admits an orientation-preserving self-diffeomorphism that is trivial in homology (and also in $H_*(S^1 \times \partial D^n)$ if $n \geq 3$) but not isotopic to the identity map.

(b)* Prove that the two S^n-bundles over S^2 (with structure group $O(n+1)$, $n \geq 2$) are nondiffeomorphic.

Remarks 5.6.9. (a) To verify that the present notion of spin structure agrees with that of Section 1.4, let $P_{Spin(n)} \to X$ be a spin structure in the previous sense associated to a bundle $E \to X$. Since $\pi_i(Spin(n)) = 0$ for $i = 0, 1, 2$ and $n \geq 3$, obstruction theory shows that $P_{Spin(n)}|X_2$ has a unique section (up to homotopy). Pushing this section down to $P_{SO(n)}|X_2$ determines a trivialization of $E|X_2$. We have now defined a map from the spin structures of Section 1.4 to those of the present section. By tracing through the definitions, one can verify that this map is equivariant under the $H^1(X; \mathbb{Z}_2)$-action, so it must be a bijection. (The possibility of empty domain and nonempty range is ruled out by the next remark.)

(b) We now have three versions of the characteristic class w_2: Subsection 1.4.1, $\delta(1)$ as in Subsection 1.4.2 following Definition 1.4.24, and the present section. One can prove that these are the same by using the universal bundle $ESO(n) \to BSO(n)$ [**MS**]. For any bundle $E \to X$ there is a bundle map $E \to ESO(n)$. Since all three versions of w_2 are natural with respect to bundle maps as in Proposition 5.6.5, it now suffices to check that they agree for the bundle $ESO(n)$. But since $H^2(BSO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$ $(n \geq 3)$ and none of the versions of w_2 is identically zero, $w_2(ESO(n))$ is the unique nonzero element in each case.

5.7. Spin structures in Kirby diagrams

Let $X_1 = D^4 \cup$ 1-handles be given by a Kirby diagram in dotted circle notation. Then X_1 is given as a submanifold of D^4, so it inherits a canonical spin structure s_0 from D^4. Thus, the set $\mathcal{S}(X_1)$ of all spin structures on X_1 is canonically identified with $H^1(X_1; \mathbb{Z}_2) \cong \mathbb{Z}_2^m$ (where m is the number of 1-handles) by the correspondence $s \mapsto \Delta(s_0, s)$, once we have drawn X_1 in this notation (Proposition 5.6.3). (Note, however, that the orientation-preserving diffeomorphisms of X_1 act transitively on $\mathcal{S}(X_1)$ (cf. Exercise 5.6.8(a)), so the correspondence depends on our choice of representation of X_1 by a Kirby diagram.) The canonical spin structure on X_1 has the following characterization, which also allows us to define it in the other notation for 1-handles once we have specified a suitable convention for framing coefficients.

Proposition 5.7.1. *The canonical spin structure s_0 on an oriented 1-handlebody X_1 in dotted circle notation is the unique spin structure that extends over an arbitrary 2-handle h attached to X_1 if and only if the framing coefficient n of h is even. Any other structure s extends over h if and only if $n \equiv \langle \Delta(s_0, s), K \rangle \pmod 2$, where K is the attaching circle of h.*

Proof. First we try to extend s_0 over h. Since s_0 extends over D^4, it suffices to assume that h is attached to D^4 (with its unique spin structure). Now K bounds an immersed disk in D^4, which we glue to the core of h to obtain a sphere S immersed in $D^4 \cup h$. The immersion extends to a local diffeomorphism of some disk bundle Y over S, and extending s_0 over h is equivalent to putting a spin structure on the 2-handlebody Y. Since $S^2 \times D^2$ admits a spin structure (for example, it embeds in $\mathbb{R}^4$), it is easy to see that Y admits a spin structure if and only if $e(Y)$ is even. (Each twist in the bundle Y twists $TY|S^2$ by the generator of $\pi_1(SO(4))$.) However, $n = [S]^2 \equiv e(Y) \pmod 2$ (since each self-intersection of S contributes an even number to $[S]^2$, cf. Exercise 6.1.1(a)). Thus, s_0 extends over h if and only if n is even. The generalization to an arbitrary s on X_1 follows immediately from the definition of $\Delta(s_0, s)$, and now s_0 is uniquely characterized, by the classification of spin structures. □

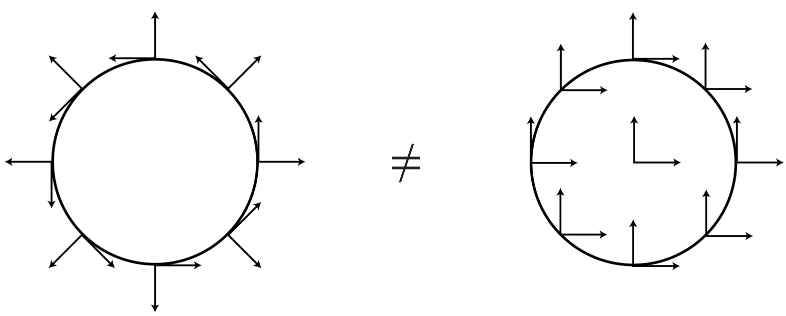


Figure 5.45. The two spin structures on ∂D^2.

Beware that the spin structure that extends over h is the unique product framing of $h = D^2 \times D^2$, *not* the one induced by the tangent vector field to K and the given normal framing, as Figure 5.45 indicates. (The outward normal to D^2 in the first picture of $D^2 \times \{0\}$ matches up with the inward normal to X_1, and the normal framing of $D^2 \times \{0\} \subset D^2 \times D^2$ matches the normal framing of K.)

Corollary 5.7.2. *For an oriented handlebody X given by a Kirby diagram in dotted circle notation, $w_2(X) \in H^2(X;\mathbb{Z}_2)$ is represented by the cocycle $c \in C^2(X;\mathbb{Z}_2)$ whose value on each 2-handle is its framing coefficient modulo* 2. □

Exercise 5.7.3. * Prove the *Wu formula*, that for any oriented 4-manifold X (not necessarily compact) and $x \in H_2(X;\mathbb{Z}_2)$ we have $\langle w_2(X), x\rangle = x^2$ (using the $\mathbb{Z}_2$-intersection pairing). Note that this formula completely determines $w_2(X)$, since $H^2(X;\mathbb{Z}_2)$ is the dual vector space of $H_2(X;\mathbb{Z}_2)$. (*Hint*: Use Exercise 4.5.12(b) to represent x by a (possibly nonorientable) surface, then apply the method of the previous proof.) Compare with Proposition 1.4.18.

Proposition 5.7.4. (Hirzebruch and Hopf [**HH**]). *For any compact, oriented 4-manifold X, $w_2(X)$ is the* mod 2 *reduction of a class in $H^2(X;\mathbb{Z})$. Thus, the Poincaré dual of $w_2(X)$ is represented by an oriented surface $(W, \partial W) \subset (X, \partial X)$.*

Proof. Let $T_i \subset H_i(X;\mathbb{Z})$ and $T^i \subset H^i(X;\mathbb{Z})$ be the torsion subgroups. By the Universal Coefficient Theorems, we have $H^3(X;\mathbb{Z}) \cong \mathrm{Hom}(H_3(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_2(X;\mathbb{Z}),\mathbb{Z})$, implying that $T^3 \cong T_2$. Similarly, $H^2(X;\mathbb{Z}_2) \cong H^2(X;\mathbb{Z})\otimes\mathbb{Z}_2 \oplus T^3 * \mathbb{Z}_2$ and $H_2(X;\mathbb{Z}_2) \cong H_2(X;\mathbb{Z})\otimes\mathbb{Z}_2 \oplus T_1 * \mathbb{Z}_2$ (where the star denotes torsion product). The natural pairing between $H^2(X;\mathbb{Z}_2)$ and $H_2(X;\mathbb{Z}_2)$ is nondegenerate, but the first term of $H^2(X;\mathbb{Z}_2)$ pairs trivially with $T_2 \otimes \mathbb{Z}_2$ because the pairing on these subspaces lifts to the integers (killing all torsion elements). Thus, the pairing between $T^3 * \mathbb{Z}_2$ and $T_2 \otimes \mathbb{Z}_2$ is nondegenerate (since these spaces have the same dimension). But the $\mathbb{Z}_2$-intersection pairing on $H_2(X;\mathbb{Z}) \otimes \mathbb{Z}_2$ also vanishes on $T_2 \otimes \mathbb{Z}_2$ (since it lifts to $\mathbb{Z}$), so the Wu formula (Exercise 5.7.3) shows that $w_2(X)$ pairs trivially with $T_2 \otimes \mathbb{Z}_2$, hence lies in $H^2(X;\mathbb{Z}) \otimes \mathbb{Z}_2$. The last statement of the proposition follows from the fact that any class in $H_2(X, \partial X;\mathbb{Z})$ is represented by an embedded surface with boundary, which follows by the method of either Proposition 1.2.3 or Exercise 4.5.12(b) (applied to a relative handle structure on $(X, \partial X)$). □

Remark 5.7.5. The proposition is true even for noncompact 4-manifolds (where the surface may be noncompact but properly embedded), and it implies that all oriented 4-manifolds admit spin^c structures (cf. Proposition 2.4.16). For completeness, we sketch a proof by Teichner and Vogt that applies to any oriented 4-manifold X. The map $\mathbb{Z} \to \mathbb{Z}_2$ induces a commutative diagram of short exact sequences:

$$\begin{array}{ccccc} \mathrm{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z}_2) & \longrightarrow & H^2(X;\mathbb{Z}_2) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}_2) \end{array}$$

The left vertical map is an isomorphism by homological algebra ($\mathrm{Ext}^2_{\mathbb{Z}} = 0$). Thus, to lift $w_2(X)$ by the middle map, it suffices to lift its image $w \in \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}_2)$ by the right map. Now each $y \in H_2(X;\mathbb{Z})$ defines a map $H_2(X;\mathbb{Z}) \to \mathbb{Z}$ by $x \mapsto x \cdot y$. Let T be the kernel of the direct product $H_2(X;\mathbb{Z}) \to \Pi_y\mathbb{Z}$ of these maps. (Thus, $T = T_2$ when X is closed.) By the Wu formula, the map $w\colon H_2(X;\mathbb{Z}) \to \mathbb{Z}_2$ is given by $w(x) = x^2$, so w descends to a map $w'\colon H_2(X;\mathbb{Z})/T \to \mathbb{Z}_2$. But the new domain is a countable subgroup of the free product $\Pi_y\mathbb{Z}$, so it is free abelian (e.g. Theorem 19.2 of [**Fs**]). Thus, w' lifts to an integer-valued homomorphism, as required.

If $w_2(X^4) \neq 0$ then connected surfaces W dual to w_2 are characterized by the fact that $X - W$ admits spin structures, but these cannot be extended across a normal disk to W. (For X a connected 4-manifold, it follows that $T(X - W)$ is trivial.) If X is closed, then the Wu formula shows that any integral homology class dual to w_2 projects (into $H_2(X,\mathbb{Z})/T_2$) to a characteristic element of Q_X (cf. Proposition 1.4.18). For more general 4-manifolds, we have (cf. Remark 1.4.27(c)):

Corollary 5.7.6. *For any oriented 4-manifold X (not necessarily compact), if X is spin then Q_X is even. The converse holds if $H_1(X;\mathbb{Z})$ has no 2-torsion.*

Proof. Apply the Wu formula and the observation that if $T_1 * \mathbb{Z}_2 = 0$ then every element of $H_2(X;\mathbb{Z}_2)$ lifts to $H_2(X;\mathbb{Z})$. □

We have already seen a counterexample to the converse of the corollary in the presence of 2-torsion, namely the Enriques surface $E(1)_{2,2}$ (Lemma 3.4.27). For a simpler example, we have the following exercise.

Exercises 5.7.7. (a)* Consider the two oriented manifolds X that are S^2-bundles over $\mathbb{RP}^2$ (Exercise 4.6.7(b)). These are drawn in Figure 5.46, and only depend on $n \bmod 2$. (Why?) Compute $H_2(X;\mathbb{Z})$ and the intersection pairings over $\mathbb{Z}$ and $\mathbb{Z}_2$, and determine $w_2(X)$ in each case. For $n = 1$, find a closed, orientable surface dual to $w_2(X)$. Note that the corresponding integral homology class is not divisible by 2 — why does this not contradict the above discussion about Q_X (which is even) and characteristic elements? Draw the attaching sphere of the 3-handle and compute $\partial_*\colon C_3(X) \to C_2(X)$ explicitly. (*Hint*: For the attaching sphere, simplify ∂X_2 by changing the

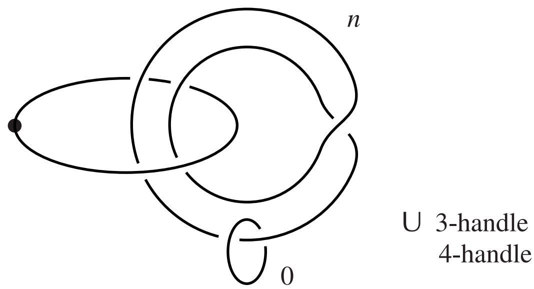


Figure 5.46. S^2-bundle over $\mathbb{RP}^2$.

meridian to a dotted circle and cancelling it. Then find the attaching sphere and pull it back to the original picture.)

(b)* Let X be a closed, oriented 4-manifold with an embedded sphere S dual to $w_2(X)$. Prove that $[S]^2 \equiv \sigma(X) \pmod{16}$. (*Hint*: See Theorem 2.2.13 and Rohlin's Theorem 1.4.28.)

Now let X be a handlebody in dotted circle notation, and suppose that $w_2(X) = 0$. By Corollary 5.7.2, the cocycle $c \in C^2(X;\mathbb{Z}_2)$ given by the framings mod 2 must be a coboundary (since $[c] = w_2(X) = 0$). That is, there is a class $\Delta \in H^1(X_1;\mathbb{Z}_2) \cong C^1(X;\mathbb{Z}_2)$ with $\delta\Delta = c$. Then Δ determines a spin structure s on X_1 with $\Delta(s_0, s) = \Delta$, and it follows that s extends (uniquely) over X. The structure s is determined from Δ by twisting s_0 on each 1-handle h_1 for which $\langle \Delta, h_1\rangle \neq 0$. We can realize these twists by self-diffeomorphisms of the 1-handles as in Figures 5.42 and 5.43, so that s is the canonical spin structure of the new picture and all 2-handles have even framing coefficients. Suitable classes Δ for this procedure can be recognized by the condition that $\delta\Delta = c$, or equivalently, that the framing coefficient of each 2-handle h_2 must be congruent mod 2 to $\langle \Delta, \partial_* h_2\rangle$ (which is the mod 2 number of times that h_2 runs over 1-handles on which we have twisted). In general, Δ is not unique, since we can change it by any class with vanishing coboundary, or equivalently, any class in $H^1(X;\mathbb{Z}_2)$. Thus, we recover the classification of spin structures given in the previous section. Summarizing, we have

Proposition 5.7.8. *Any spin structure on a compact 4-manifold X can be realized as the canonical spin structure of a Kirby diagram in dotted circle notation with all framing coefficients even.* □

Exercise 5.7.9. * Describe all spin structures on oriented manifolds X diffeomorphic to S^2-bundles over $\mathbb{RP}^2$, using Kirby diagrams with the canonical framing on X_1, and verify that the orientation-preserving diffeomorphisms of X act transitively on $\mathcal{S}(X)$. (See Exercise 5.7.7(a).)

Next, we analyze spin structures on 3-manifolds. Any oriented 3-manifold M has a trivial tangent bundle (Remark 1.4.27(b)), and there is an obvious correspondence between the spin structures on M and on $I \times M$. If we pick a collection of circles K_i in M representing a basis of $H_1(M;\mathbb{Z}_2)$, then we can specify a spin structure as before, by choosing (mod 2) a framing coefficient n_i for each K_i and requiring that the structure should extend over a 2-handle attached to K_i with framing given by n_i. For a less cumbersome approach for closed 3-manifolds, however, we follow Kaplan [**Kp**], writing M as the boundary of a 2-handlebody X on a framed link L in S^3 and defining a correspondence between spin structures and certain sublinks of L. By Exercise 5.6.2(a), any spin structure s on $M = \partial X$ determines a relative obstruction to extending s over X, $w_2(X, s) \in H^2(X, M;\mathbb{Z}_2) \cong H_2(X;\mathbb{Z}_2)$ (where the isomorphism comes from Poincaré duality), and this maps to $w_2(X) \in H^2(X;\mathbb{Z}_2) \cong H_2(X, M;\mathbb{Z}_2)$. Now classes in $H_2(X;\mathbb{Z}_2)$ correspond bijectively to sublinks L' of L (where the sublink L' corresponds to the sum of all handles attached to L'). Recall the correspondence between the intersection form Q_X and the linking pairing of L (Proposition 4.5.11), which also holds with $\mathbb{Z}_2$-coefficients.

Definition 5.7.10. Let $L = \{K_1, \dots, K_m\}$ be a framed link in S^3. A *characteristic sublink* $L' \subset L$ is a sublink such that for each K_i in L, its framing $\ell k(K_i, K_i)$ is congruent mod 2 to $\ell k(K_i, L')$.

Proposition 5.7.11. *The map $s \mapsto w_2(X, s) \in H_2(X;\mathbb{Z}_2)$ determines a bijection from $\mathcal{S}(M)$ to the set of characteristic sublinks of L.*

Proof. By the Wu formula (Exercise 5.7.3), a sublink is characteristic if and only if it corresponds to a class in $H_2(X;\mathbb{Z}_2)$ mapping to $w_2(X)$ in $H_2(X, M;\mathbb{Z}_2)$ (up to Poincaré duality). Thus, a spin structure s on M determines a characteristic sublink via $w_2(X, s)$. To check surjectivity, observe that a characteristic sublink determines a class in $H_2(X;\mathbb{Z}_2)$ that is represented by a closed surface W dual to $w_2(X)$. Then $X - W$ admits a spin structure whose restriction to M corresponds to the given sublink. For injectivity, let s' be a different spin structure on M. Starting with s and W as before, we change s to s' on ∂X by adding a collar $I \times M$ to X (with s' on the outside boundary of $I \times M$ and s inside as before). By Exercise 5.6.4(b) applied to $I \times M$, this will change the dual of $w_2(X, s)$ in $H_2(X;\mathbb{Z}_2)$ to that of $w_2(X, s')$ by adding the dual of $\Delta(s, s')$. Since inclusion induces an injection $H_2(M;\mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2)$ (as we see by turning the 2-handlebody upside down), we conclude that the characteristic sublink changes if $s' \neq s$, so the correspondence is injective. □

Now we can specify spin structures by their characteristic sublinks L'. The spin structure corresponding to a given L' is characterized by the fact that it extends over a new 2-handle h attached to a knot K with framing n if and only if $n \equiv \ell k(K, L') \pmod 2$. (This is because when the structure extends, the resulting structure on $\partial(X \cup h)$ still has characteristic sublink L'.)

Exercises 5.7.12. (a)* For disk bundles over $\mathbb{RP}^2$, analyze the map $\mathcal{S}(\partial X) \to H_2(X; \mathbb{Z}_2)$ given by $s \mapsto w_2(X, s)$. Compare with the case of 2-handlebodies.

(b)* Use characteristic sublinks to classify spin structures on the 3-manifold given in Figure 5.47, and describe these in terms of extending over new handles.

(c) For a relative handlebody $(X, \partial_- X)$ with a fixed spin structure s_0 on $\partial_- X$, analyze $w_2(X, s_0)$ and spin structures on X extending s_0 the way we did for handlebodies with $\partial_- X = \emptyset$ in Corollary 5.7.2 and the text following Exercises 5.7.7.

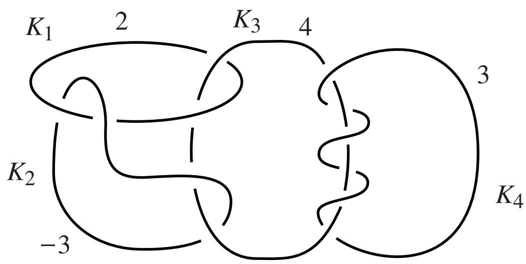


Figure 5.47

Having expressed spin structures on 3-manifolds in terms of characteristic sublinks of their surgery diagrams, we wish to understand the effect of Kirby moves. Given a spin structure s on $M = \partial X$ as above, suppose we slide a 2-handle h_1 over h_2. The resulting 4-manifold is canonically diffeomorphic to X, so its boundary inherits a spin structure from s, and if we identify the new manifolds and spin structure with (X, M, s) in the obvious way, the relative class $w_2(X, s)$ is preserved. The new canonical basis for $H_2(X; \mathbb{Z}_2)$ is given by $h_i' = h_i$, $i \neq 1$, and $h_1' = h_1 + h_2$. If $w_2(X, s) = \sum \varepsilon_i h_i$ ($\varepsilon_i \in \mathbb{Z}_2$), then in terms of the new basis it is given by $-\varepsilon_1 h_2' + \sum \varepsilon_i h_i'$. Thus, if we draw characteristic sublinks in a different color from the other link components, the only color change induced by sliding h_1 over h_2 is that K_2 changes color if and only if K_1 is in the characteristic sublink. (This can also be seen by the method of Exercise 5.5.7(c).) To understand blowing up and down, note that the unique characteristic sublink of an isolated ± 1-framed unknot is the unknot itself. The general case easily follows by sliding handles (Figure 5.17): If we blow up an unknot K, then the new characteristic sublink includes K if and only if $\ell k(K, L_0')$ is even (where L_0' is the original characteristic sublink), and the original colors of L are preserved. To blow down, note that this condition on K is guaranteed by Definition 5.7.10.

Exercise 5.7.13. * We can obtain $\mathbb{RP}^3$ as either $+2$- or -2-surgery on the unknot. Realize the diffeomorphism between these by Kirby moves and determine how the spin structures correspond. Check your answer by keeping track of the framing on a meridian.

Theorem 5.7.14. *Any closed, connected, spin 3-manifold (M, s) is the spin boundary of a spin 2-handlebody X.*

Of course, a 2-handlebody admits a spin structure if and only if all framings are even. Since $H^1(X; \mathbb{Z}_2) = 0$, the spin structure on X is unique, and so s is uniquely determined by the Kirby diagram of X by restricting the spin structure to ∂X (or equivalently, taking the characteristic sublink to be empty). The theorem can be proved in the manner of Corollary 5.3.5, using the fact that every spin 3-manifold is a spin boundary, which was known to Milnor [**M3**]. (See also [**K2**].) The following proof, relying only on Theorem 5.3.4 (that an *oriented* 3-manifold is an *oriented* boundary), is Kaplan's application of characteristic sublinks [**Kp**].

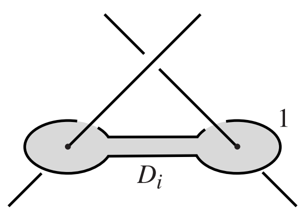


Figure 5.48

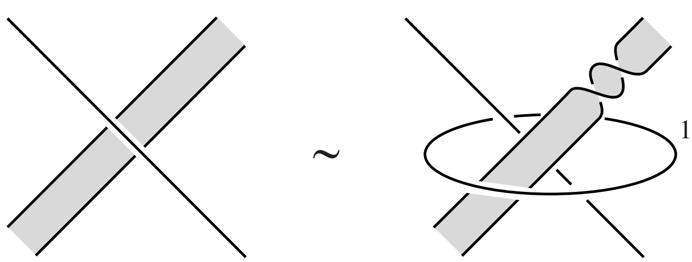


Figure 5.49

Proof. Represent (M, s) by integral surgery on a link L (Corollary 5.3.5) with a characteristic sublink L'. We assume $L' \neq \emptyset$. (Otherwise, we are done.) By sliding one component of L' over the others, we arrange for L' to be a knot K_0. Blowing up as in Figure 5.19, we may unknot K_0 by changing suitable crossings, at the expense of adding a collection of unknots $K_1, \dots, K_r$ to the characteristic sublink. Each of the new circles K_i ($i = 1, \dots, r$) bounds a canonical disk D_i that intersects K_0 in two points. We can visualize each disk D_i as a band-sum of a pair of meridianal disks of K_0 (Figure 5.48), but if we draw K_0 as round, the bands will appear tangled with each other and K_0. By blowing up as in Figure 5.49, we can reverse a crossing of any band with any strand of $K_0, \dots, K_r$ without adding components to the characteristic sublink (since the blown-up curve links an odd number of strands of $\coprod K_i$). Now we can assume the characteristic sublink has the simple form of Figure 5.50 (where $n_1, \dots, n_r \in \frac{1}{2}\mathbb{Z}$). Blowing up as in Figure 5.21 allows us to change the sign of any clasp without otherwise changing the characteristic sublink, so we can change each $n_i \in \frac{1}{2}\mathbb{Z}$ arbitrarily as in Figure 5.51. Now we can assume that the characteristic sublink is an unlink. After blowing up meridians, we can assume that all framings on the sublink are ± 1. Now we can blow down the sublink, so that the new characteristic sublink is empty and the spin structure s extends over the 4-manifold. □

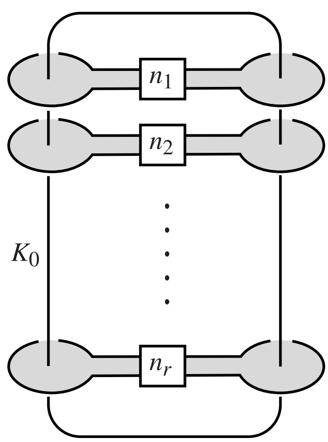


Figure 5.50

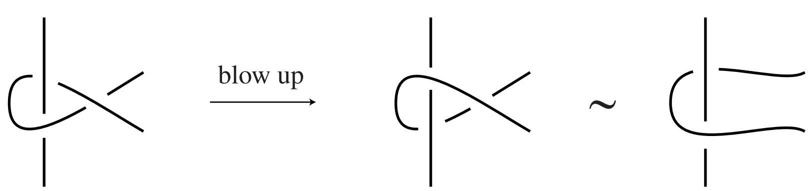


Figure 5.51

Exercises 5.7.15. (a)* For the lens spaces $L(p,1)$, $p > 0$, realize all spin structures as spin boundaries. (If you are careful, you can get plumbings on linear graphs.) Repeat for 0-surgery on the trefoil knot.

(b)* Prove that every closed, oriented 3-manifold embeds smoothly in $\#mS^2 \times S^2$ for some m.

Definition 5.7.16. For a closed, spin 3-manifold (M,s), the *Rohlin invariant* (or *mu invariant*) $\mu(M,s) \in \mathbb{Z}_{16}$ is the signature $\sigma(X)$ reduced modulo 16, where X is any (smooth) compact, spin 4-manifold with spin boundary (M,s).

To see that this invariant is well-defined, first observe that for any oriented manifold X there is a canonical correspondence between spin structures on X and on $\overline{X}$ (by reversing the sign of the first vector in the trivialization). Now if X and Y are two compact spin 4-manifolds with spin boundary (M,s), then the closed manifold $X \cup_M \overline{Y}$ inherits a spin structure. By Rohlin's Theorem 1.4.28, $\sigma(X \cup_M \overline{Y}) \equiv 0 \pmod{16}$. But by Novikov additivity of the signature (see Remark 9.1.7), we have $\sigma(X \cup_M \overline{Y}) = \sigma(X) + \sigma(\overline{Y}) = \sigma(X) - \sigma(Y)$, and so $\sigma(X) \equiv \sigma(Y) \pmod{16}$.

Note that $\mu(\overline{M},s) = -\mu(M,s)$, and that $\mu(M_1 \# M_2, s_1 \# s_2) = \mu(M_1,s_1) + \mu(M_2,s_2)$. (We define $s_1 \# s_2$ in the obvious way by interpreting the sum as attaching a 4-dimensional 1-handle and extending the trivialization over the 1-handle.) If M is a $\mathbb{Z}_2$-homology sphere (i.e., $H_1(M;\mathbb{Z}_2) = 0$), then it admits a unique spin structure s, and $\mu(M) = \mu(M,s)$ is an invariant of the oriented 3-manifold. If M is an integral homology sphere (that is, $H_1(M;\mathbb{Z}) = 0$), then any compact spin manifold X with boundary M has an even, unimodular intersection pairing (Remark 1.2.11; cf. also Corollary 5.3.12 and the subsequent discussion), so (by Lemma 1.2.20) $\sigma(X) \equiv 0 \pmod 8$, and the Rohlin invariant descends to an invariant $\frac{1}{8}\mu(M) \in \mathbb{Z}_2$ of the unoriented manifold.

Exercises 5.7.17. (a)* Compute the Rohlin invariant of every spin structure on a lens space $L(p,1)$. For even $p \not\equiv 0 \pmod{16}$, show that $L(p,1)$ admits no orientation-preserving self-diffeomorphism that interchanges the two spin structures. What can you say about orientation-reversing diffeomorphisms? What happens when $p = 2$?

(b)* Prove that the Poincaré homology sphere Σ (-1-surgery on the left trefoil (Figure 5.22), see Exercise 5.1.12(a)) is not the boundary of any contractible (or even acyclic) smooth 4-manifold. (Many homology spheres do bound contractible 4-manifolds. For example, surger out a 2-sphere in Figure 4.29.) Prove that Σ does not embed smoothly in $\mathbb{R}^4$. By Freedman, all homology spheres bound contractible topological 4-manifolds (cf. Exercise 5.3.13(e)). (Gluing such a contractible manifold with boundary Σ

onto the 4-manifold Q of Figure 5.22 produces Freedman's unsmoothable manifold homotopy equivalent to $\overline{\mathbb{CP}^2}$.) Using Freedman, prove that $I \times \Sigma$ embeds topologically in $\mathbb{R}^4$.

(c)* Compute the Rohlin invariants of the 8 spin structures on T^3. Which one is exceptional? (*Hint*: You can keep track of the signature in Kaplan's algorithm by counting blow-ups.)

(d)* Let X be a compact, oriented 4-manifold with a spin structure s on ∂X. Suppose that $S \subset X$ is an embedded sphere dual to $w_2(X, s)$. What can you say about $[S]^2$? (*Hint*: Exercise 5.7.7(b).)

It is natural to ask how the previous discussion of spin structures and characteristic sublinks extends to rational surgery diagrams. To answer this, we observe that if M is obtained by rational surgery on a link L in S^3 with coefficients $\frac{p_i}{q_i}$, then a spin structure on M is determined by its restriction to the link complement $S^3 - \nu L$.

Claim 5.7.18. *If a new manifold M' is obtained from M by changing each p_i and q_i by even integers, then a spin structure on $S^3 - \nu L$ will extend over M' if and only if it extends over M.*

The claim gives us a canonical isomorphism $\mathcal{S}(M) \cong \mathcal{S}(M')$. Now we can replace each even q_i by 0 and each odd q_i by 1, reducing to the previous case.

Definition 5.7.19. Let L be a link in S^3 with rational surgery coefficients $\frac{p_i}{q_i}$. Let L'' be the sublink of components with q_i odd. A *characteristic sublink* L' of L is a sublink of L'' such that for each K_i in L'', $p_i \equiv \ell k^*(K_i, L') \pmod 2$, where ℓk^* is the linking pairing obtained by setting each framing $\ell k^*(K_j, K_j) = p_j$.

As before, we obtain a canonical bijection $\mathcal{S}(M) \cong \{\text{characteristic sublinks}\}$, and the spin structure corresponding to a given L' is characterized as extending over a 2-handle attached to $I \times M$ along a knot K with framing n if and only if $n \equiv \ell k(K, L') \pmod 2$.

Proof of Claim 5.7.18. Identify each boundary component of $S^3 - \nu L$ with $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The group of orientation-preserving diffeomorphisms of this (up to isotopy) is given by $SL(2, \mathbb{Z})$, and the integers p_i and q_i correspond to the first column of the matrix of a gluing diffeomorphism. It is easily checked that our hypothesis on the change in p_i and q_i guarantees that after a column operation (on the second column) corresponding to a self-diffeomorphism of the solid torus, the matrices of the two gluing maps will agree modulo 2. Each matrix acts on $H^1(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ through its mod 2 reduction in $SL(2, \mathbb{Z}_2)$, so it clearly suffices to show that any matrix acting trivially on $H^1(T^2; \mathbb{Z}_2)$ corresponds to a diffeomorphism of T^2 fixing all spin structures. (The corresponding statement for general manifolds is false by Exercise 5.6.8(a).) But there is a canonical trivialization τ of the tangent bundle of T^2 descending from the standard basis of $\mathbb{R}^2$. For any $A \in SL(2,\mathbb{Z})$, the trivialization $A_*\tau$ induced by the diffeomorphism is also obtained from τ by applying A to the tangent space at each point separately. Thus, $A_*\tau$ is homotopic to τ via a path in $GL(2,\mathbb{R})$ connecting A to the identity, so τ determines a spin structure that is fixed by all diffeomorphisms of T^2. Since any other spin structure differs from this by an element of $H^1(T^2;\mathbb{Z}_2)$ (and difference classes pull back under diffeomorphisms by Proposition 5.6.5), we are done. □

Exercise 5.7.20. Define a canonical spin structure on T^3. Which one is it? (See Exercise 5.7.17(c).) Verify (using difference classes) that the self-diffeomorphisms of T^3 act transitively on the remaining spin structures; cf. the solution of Exercise 5.7.17(c). (The group of orientation-preserving diffeomorphisms of T^3 up to isotopy is $SL(3;\mathbb{Z})$.)

To analyze the effects of Rolfsen moves on characteristic sublinks, it is convenient to keep track of a spin structure by adding a 2-handle h_i to $I \times M$ along a meridian μ_i of each link component K_i in such a way that the spin structure extends over the 2-handles. Thus, the framing on μ_i is odd if and only if K_i is in the characteristic sublink L' (which we again imagine drawn in a different color). It is now clear that if we transform the diagram by a Rolfsen twist or slam-dunk, then the colors will be preserved except for a possible change on the unknot that is being twisted or slam-dunked. A Rolfsen twist as in Figure 5.27 will change the meridian μ of the unknot K on which we twist by adding n twists to its framing and wrapping it n times around the other strands passing through K. We can return it to its original position (as a meridian of K) by an isotopy in the spin 4-manifold $I \times M \cup \bigcup h_i$, sliding it over the new handles h_i when necessary (and then unlinking it from the attaching circles μ_i with nonzero framing by passing μ through μ_i). The net change in framing $(\mathrm{mod}\, 2)$ is $n(1 + \ell k(K, L^*))$, where L^* is obtained from L' by deleting K if it is present. Thus, K changes color if and only if this number is odd. For a slam-dunk as in Figure 5.30, K_1 disappears and (as we have already seen) the other colors are unchanged. To understand the inverse move, simply note that the color of K_1 is determined by the colors of the other link components, by Definition 5.7.19 applied to K_1 and K_2. For example, if the coefficient of K_1 is $\frac{p}{q}$ with p odd, then K_1 is in L' if and only if K_2 is not in L' and q is odd. (Compare with blowing up.)

To complete the discussion, we observe that Rohlin invariants are not determined by the mod 2 residues of p_i and q_i. (Consider lens spaces $L(p, 1)$.) However, we can still compute Rohlin invariants by obtaining an integral surgery diagram (Exercise 5.3.9(b)) and applying Kaplan's method (Theorem 5.7.14).

Exercises 5.7.21. (a) Analyze the behavior of characteristic sublinks under Rolfsen moves by reducing to the case where all coefficients are 0, 1 or ∞ as in Claim 5.7.18 and applying the integral theory. Verify that your answers are equivalent to the ones given above.

(b)* Compute the Rohlin invariant of any spin structure on an arbitrary lens space $L(p,q)$ in terms of a continued fraction expansion of $-\frac{p}{q}$. (*Hint*: There is such an expansion with at most one odd entry.) For a related discussion of spin structures on T^2-bundles over S^1, see [**KMe2**] (Section 5 and Appendix).

Chapter 6

More examples

We now turn to more advanced examples of Kirby diagrams. We begin by discussing plumbings in full generality (in the context of oriented 4-manifolds). A related example is given by Casson handles, which are infinite handlebodies that are fundamental in Freedman's work (as we will see in Chapter 9). As an application, we draw a Kirby diagram of an exotic $\mathbb{R}^4$, a smooth manifold homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to it. In Section 6.2, we discuss how to draw arbitrary surfaces in 4-manifolds. For example, we draw arbitrary smooth complex curves in $\mathbb{CP}^2$. We also give an algorithm for constructing Kirby diagrams of the complements of surfaces in 4-manifolds. The third section discusses Kirby diagrams of covers and branched covers. This construction is particularly useful in that many complex surfaces arise as branched covers, as we will see in Chapter 7. Additional examples will be given in Part 3, for example elliptic surfaces in Chapter 8.

6.1. Plumbings and related constructions

First, we summarize what we already know about plumbings, beginning with the case with a trivial graph, i.e., disk bundles over closed, connected surfaces. We considered arbitrary D^2-bundles $\pi\colon X \to F$ (X oriented) in Example 4.6.5, and saw that they are classified by F and the Euler number $e(X)$. The case $F = T^2$ was given by Figure 4.36 with $n = e(X)$. By changing to dotted circle notation (Exercise 5.4.3(c)), we obtain the Borromean rings, Figure 6.1. (Note that while dotted circles representing 1-handles are required to form an unlink, it is frequently convenient to draw this unlink in a plane projection with crossings. In this case, they form the *Bing double* of

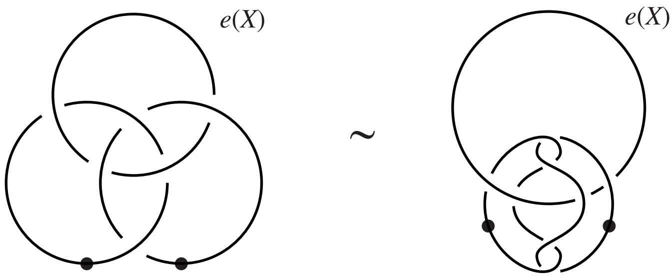


Figure 6.1. D^2-bundle X over T^2.

a meridian of the attaching circle, and we should think of them as representing a trivial pair of disks in D^4 that have been bent like taco shells around each other and then deleted from D^4.) Similarly, we drew bundles over $\mathbb{RP}^2$ (Figure 4.38), which we can redraw as in Figure 6.2. By the same method of building handlebodies pairwise, we saw how to draw any D^2-bundle over a surface (Exercises 4.6.6(b) and 4.6.7(b)). This is most easily described using the *connected sum* operation for oriented knots. The connected sum $K_1 \# K_2$ of two oriented knots is formed by drawing them in the same picture, separated by a plane, and then connecting them by a surgery following a band (respecting preassigned orientations) that only passes through the plane once (Figure 6.3). Since the connected sum of two surfaces is obtained by fusing together (boundary summing) their 0-handles and replacing their 2-handles by a single 2-handle, it should be clear from Example 4.6.5 that a bundle over a connected sum is obtained from bundles over the summands by connected summing their attaching circles (Figure 6.4). The framing will be $e(X) + 2w$, where w is the sum of the writhes of the dotted circles if we draw them as on the right side of Figure 6.2.

We considered more general plumbings X previously in Section 4.6.2. We saw that such oriented manifolds are classified by their decorated graphs, with each vertex corresponding to a disk bundle over a connected surface as above (with the surface oriented if possible) and each edge corresponding to a plumbing, with an associated sign if the corresponding surfaces are oriented. We drew explicit pictures of plumbings of spheres on trees (Figure 4.33), and saw how to generalize the technique to arbitrary plumbings on trees (Exercise 4.6.6(b)). The main idea is that when each plumbing is performed, a new disk bundle over a surface F is introduced, and the 0-handle of F is identified with a cocore of some 2-handle h. Since this cocore is isotopic to an unknotted disk in D^4 bounded by a meridian of h, the plumbing will be accomplished by applying the previous paragraph to this meridian. Thus, for an arbitrary plumbing on a tree, we draw a disk bundle as before for each vertex, linking their 2-handles in the simplest possible way for each edge as in

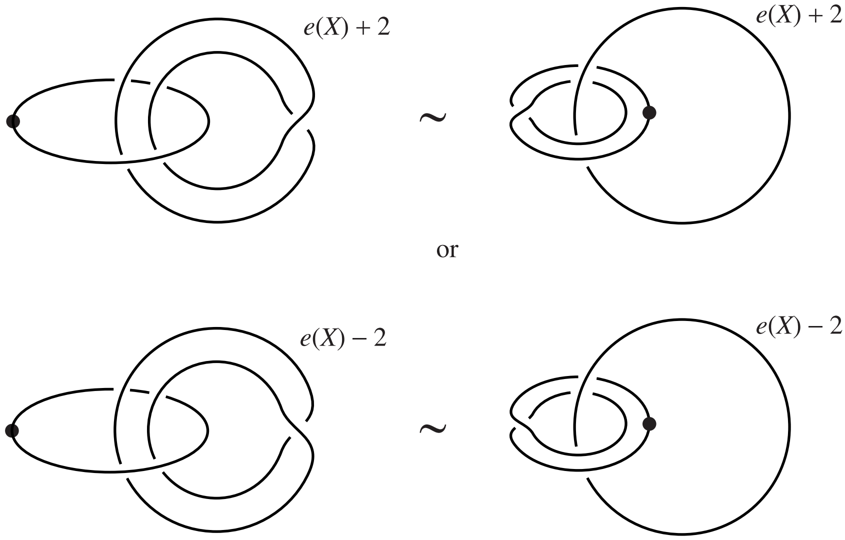


Figure 6.2. D^2-bundle X over $\mathbb{RP}^2$.

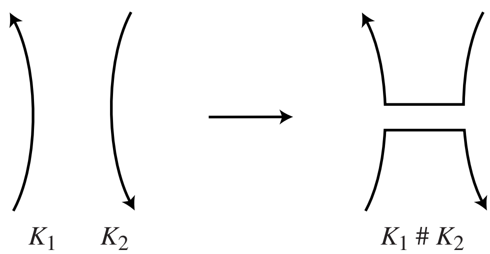


Figure 6.3. Connected sum of knots.

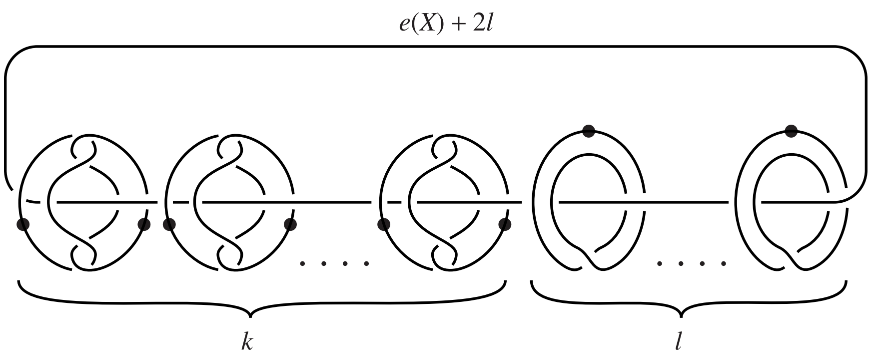


Figure 6.4. D^2-bundle X over $\# kT^2 \# \ell\mathbb{RP}^2$.

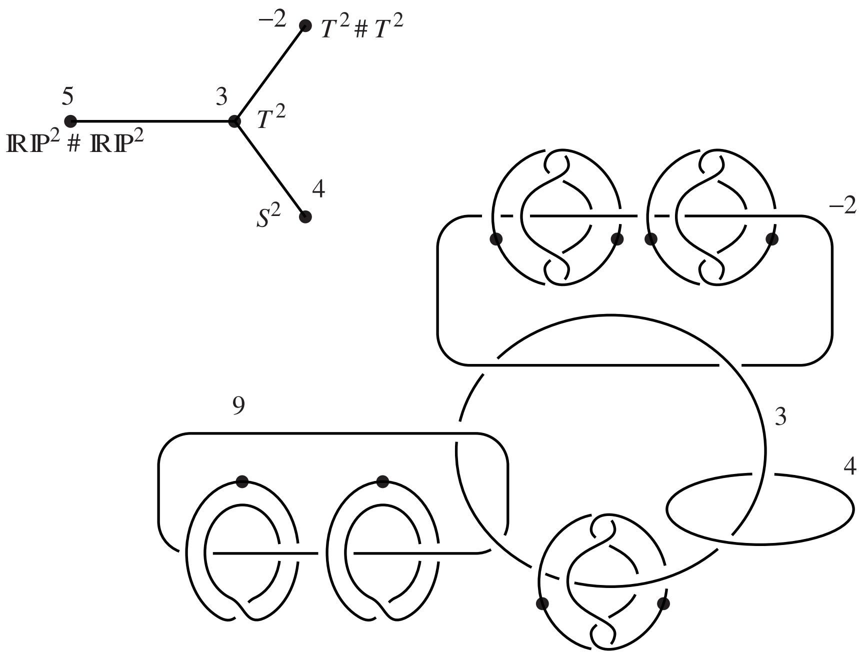


Figure 6.5. A typical plumbing on a tree.

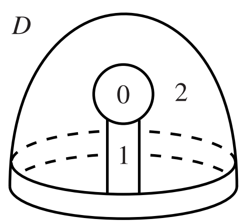


Figure 6.6. Handle decomposition of $(D^2, \partial D^2)$.

Figure 6.5. As part of the induction hypothesis, we see that each 2-handle cocore is a fiber of a bundle. We can also see each 0-section by starting with a spanning disk for the corresponding attaching circle and modifying it to avoid the dotted circles, using a Möbius band for each $\mathbb{RP}^2$-summand (Figure 6.2) and a surgery for each T^2-summand (Exercises 5.3.3(d) and 5.4.3(c)).

It remains to consider plumbings on nonsimply connected graphs (including self-plumbings). These are obtained from plumbings on trees by plumbing the 2-handles. In general, if h and h' are 2-handles attached to a 4-manifold X (possibly with $h = h'$), we can plumb them as follows: Starting with the relative handle decomposition of the core $(D, \partial D)$ of h with a single handle, introduce a cancelling 0- and 1-handle (Figure 6.6). We

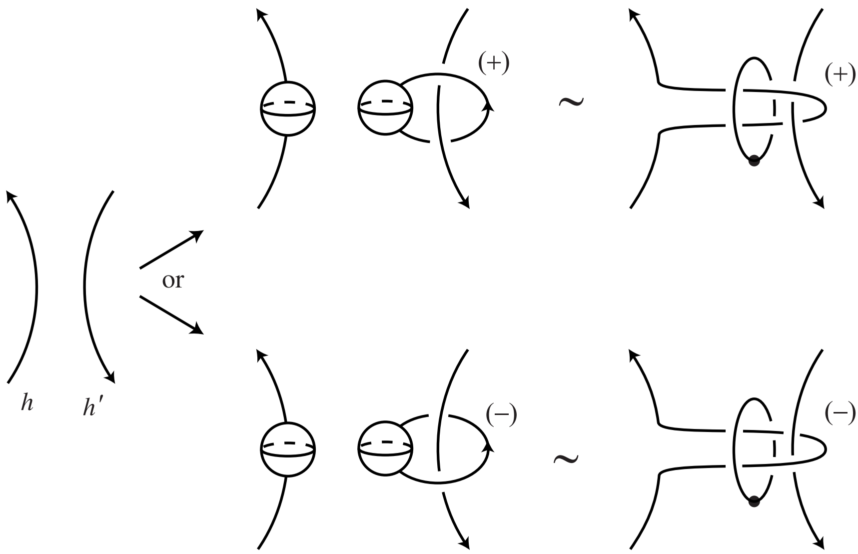


Figure 6.7. Plumbing 2-handles of a connected handlebody.

identify the 0-handle with a cocore of h' (realized as a disk in X bounded by a meridian of the attaching circle of h'). Then we attach h by attaching the 1-handle and 2-handle (pairwise, as we did to draw disk bundles in Example 4.6.5). See Figure 6.7. In dotted circle notation, we create a clasp just as in the previous case, but now it also runs through a dotted circle. Note that we can choose the sign of the clasp as prescribed by the graph. (For plumbings on trees, we can change signs arbitrarily by changing orientations of the surfaces, but in the general case, signs may affect the resulting diffeomorphism type.) Clearly, for $h \neq h'$ this plumbing operation does not change the relation between framing and Euler number (since each attaching circle by itself was only changed by an isotopy), so we have a complete picture of plumbings without self-plumbings of components of the surface. See Figure 6.8. Note that although we have choices in the bands we use to

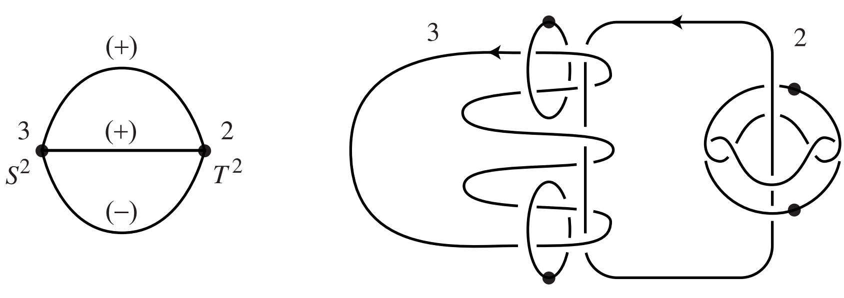


Figure 6.8. Plumbing on a nonsimply connected graph.

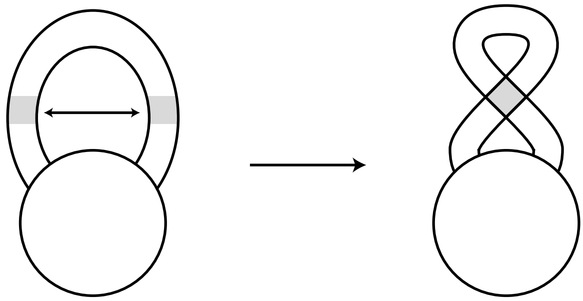


Figure 6.9. Self-plumbing in dimension 2.

create the clasps, these do not affect the resulting diffeomorphism type, as we see from the old notation for the 1-handles. (In dotted circle notation, we can change the band by sliding curves under the 1-handle.) Similarly, we can switch one of the dotted circles of Figure 6.8 onto the center clasp by a 1-handle slide.

Finally, we consider self-plumbings, $h = h'$. (See Figure 6.9.) Since the segments pictured in Figure 6.7 are arbitrary on the attaching circle K, we can take them to be contiguous as in Figure 6.10(a) and (b). Then we can straighten K again by an isotopy (Figure 6.10(c)), twisting the dotted circle into a *Whitehead double* of a meridian of K. When we analyze the framing, we encounter a new complication. For an immersion of a closed, oriented surface F in a 4-manifold, we still have a well-defined normal bundle νF (defined by pulling back to the domain F), but its Euler number no longer gives $F \cdot F$. In fact, when F is generically immersed, $F \cdot F - e(\nu F)$ equals twice the signed number self(F) of self-intersections (transverse double points) of F, as can be seen by forming a transverse copy F' of F and examining $F \cap F'$ at the self-intersections of F (Exercise 6.1.1(a)). Now if we interpret h as a 2-handle attached to D^4 (ignoring dotted circles), it determines (up to sign) a homology class represented by an oriented surface F made from the core union a Seifert surface. If we add self-plumbings to h by drawing dotted Whitehead curves as in Figure 6.10(c) without changing the framing on h, then the framing coefficient will be preserved, and hence, so will the self-intersection number $[F] \cdot [F]$. Thus, $e(\nu F)$ will drop by twice the signed number of clasps (= 2 self(F)). Equivalently, the framing on h determines a (different) framing in Figure 6.10(b) if we create the clasps by a regular homotopy of K, and this choice of framing will preserve $e(\nu F)$, but now the framing coefficient (and hence, self-intersection number) will increase by 2 self(F) (as we see directly by the change in writhe of K). Either way, we can relate the framing coefficient for a self-plumbing to the Euler number (and for an oriented surface, to the self-intersection number $F \cdot F$ — this equals the framing coefficient). For example, Figure 6.11 represents

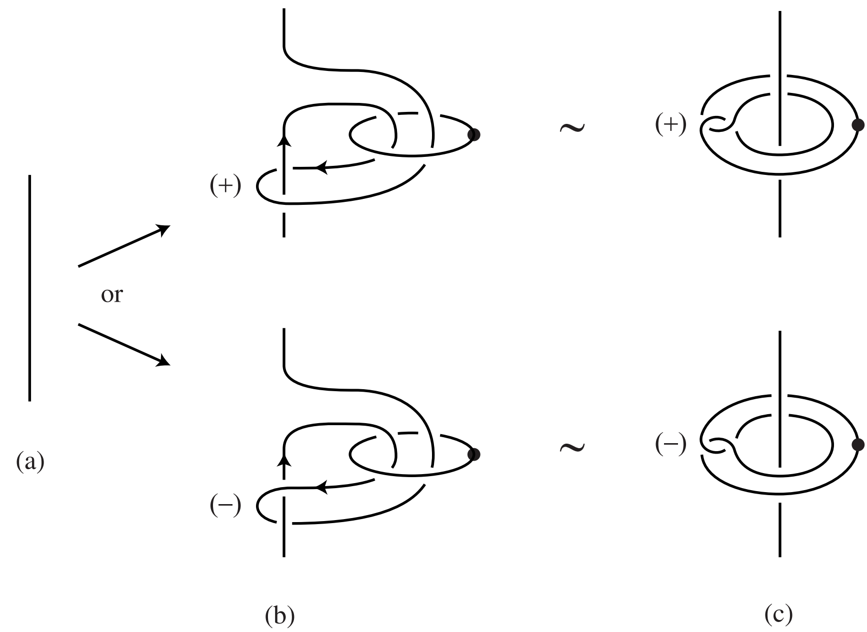


Figure 6.10. A self-plumbing.

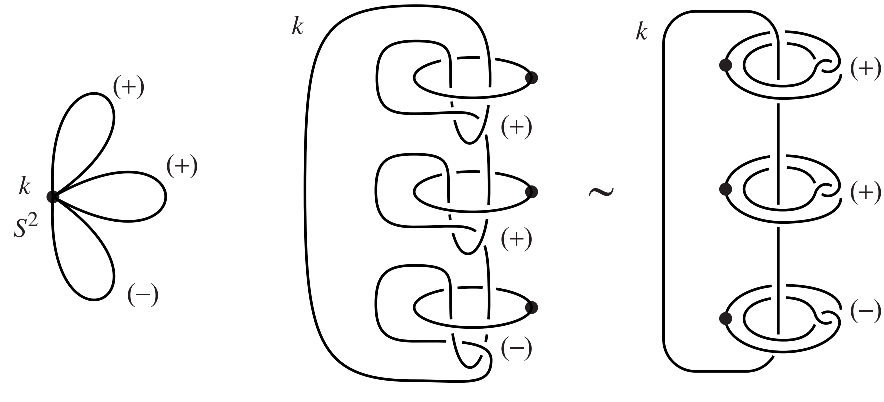


Figure 6.11. Self-plumbed D^2-bundle over S^2 ($D^4 \cup$ kinky handle), $F \cdot F = k = e(\nu F) + 2$.

a sphere with three self-plumbings (with the indicated signs), $F \cdot F = k$ and $e(\nu F) = k - 2$.

Exercises 6.1.1. (a)* Prove that for a generic immersion $F^2 \to X^4$ (both oriented), $F \cdot F = e(\nu F) + 2\ \mathrm{self}(F)$.

(b) Draw a generic homotopy $\varphi_t \colon S^1 \to \mathbb{R}^3$, $t \in I = [0,1]$, such that $\varphi_0 = \varphi_1 = \mathrm{id}_{S^1}$, $\varphi_{\frac{1}{2}}$ is a figure-eight (i.e., two strands intersect) and for each $t \neq \frac{1}{2}$, φ_t is an embedding. Consider the immersed surface F comprising the image of $(\mathrm{id}_I, \varphi)\colon I \times S^1 \to I \times \mathbb{R}^3$, with a unique transverse double point. Draw a normal framing of F, using double-strand notation in each picture $\{t\} \times \mathbb{R}^3$. Compare the framings at $t = 0, 1$. How does this relate to part (a)?

Remark 6.1.2. Bing and Whitehead doubling are actually operations that can be performed on any framed circle K in an oriented 3-manifold. One simply uses the framing and orientation to identify a tubular neighborhood of K with one of the solid tori in Figure 6.12, obtaining a new knot or link. (There are two ways to do this, depending on the orientation of K, but the answers are the same because of the symmetry, rotating 180° about the z-axis in Figure 6.12.) For $K \subset S^3$, one usually uses the 0-framing; the case of framing n is called the *n-twisted Bing (positive/negative Whitehead) double* of K. For example, the (untwisted) positive Whitehead double of the right trefoil knot is shown in Figure 6.13.

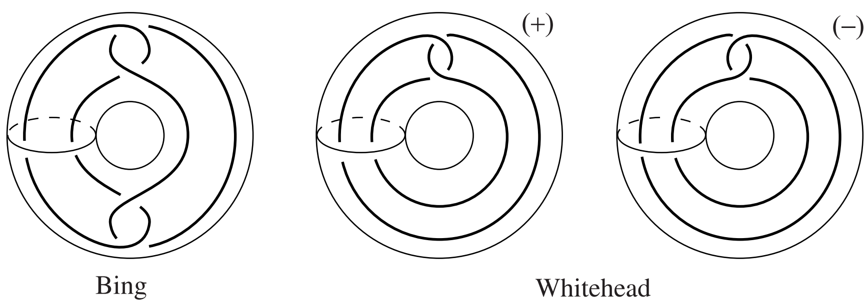


Figure 6.12. Bing and Whitehead doubling.

Figure 6.13. Whitehead double of the trefoil.

Example 6.1.3. The case of 2-handles with self-plumbings, or *kinky handles* [**C**], is of independent interest because of its importance in Freedman's work on topological 4-manifolds [**F**], [**FQ**]. The natural convention for framings is that attaching a kinky handle to a 4-manifold X should look the same as attaching an ordinary handle on the level of intersection forms. That is, we twist the product framing of $\partial D^2 \times 0 \subset \partial D^2 \times D^2$ by twice the signed number of self-plumbings, and match this new framing with the given one on the attaching circle in ∂X. Thus, Figure 6.11 shows a kinky handle attached to a k-framed unknot. (It may be helpful to imagine the dotted circles isotoped into the 2-handle, so that a kinky handle is obtained from a 2-handle by removing some disks.) Observe that if we attach 2-handles to 0-framed meridians of the dotted circles, we obtain a standard 2-handle attached to the same framed curve K. If we instead attach kinky handles to these 0-framed meridians, we obtain a *2-stage Casson tower* T_2 attached to K. Repeating the process with the new kinky handles yields a 3-stage tower T_3, and so on. Taking the infinite union of $T_2 \subset T_3 \subset T_4 \subset \dots$, one obtains a *Casson handle* (after removing all boundary except the attaching region of the original kinky handle T_1). The simplest Casson handle (with only one self-plumbing at each stage and all signs positive) is shown in Figure 6.14, attached to a k-framed unknot. (To interpret this infinite picture, we replace D^4 by the noncompact manifold $(-\infty, 0] \times \mathbb{R}^3$.) More general Casson handles have branching due to kinky handles with more than one self-plumbing, and self-plumbings of both signs may be present, Figure 6.15.

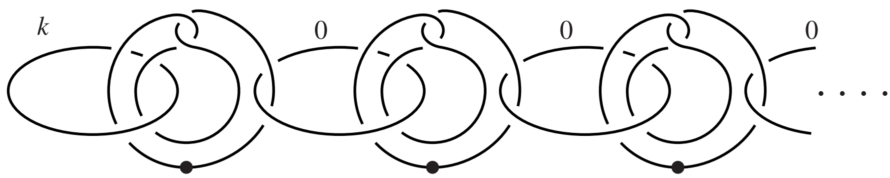


Figure 6.14. Simplest Casson handle attached to D^4 along a k-framed unknot.

Casson handles are important because of their relevance to a key lemma from high-dimensional topology. (See Section 9.2.) The main theorems of high-dimensional topology depend on a trick of Whitney (Theorem 9.2.7) which involves embedding a 2-handle $(D^2 \times D^{n-2}, \partial D^2 \times D^{n-2})$ in $(X^n, \partial_- X^n)$ with a specified framed attaching circle in $\partial_- X^n$. For $n \geq 5$ this is straightforward, but when $n = 4$ it usually fails. (Roughly, this is because a generic map $(D^2, \partial D^2) \to (X^n, \partial_- X^n)$ is an embedding for $n \geq 5$ but only an immersion for $n = 4$.) Casson showed that under the required hypotheses in dimension 4, we can at least find an embedded Casson handle. Freedman's key lemma is that any Casson handle is homeomorphic to an open 2-handle $D^2 \times \text{int } D^2$, such that the given framing on its attaching circle maps to the product framing. This ultimately allows the high-dimensional machinery to run for *topological* 4-manifolds, resulting in the Classification Theorem 1.2.27 (for example). Casson handles cannot always be *diffeomorphic* to $D^2 \times \text{int } D^2$, since the Classification Theorem is known to fail for

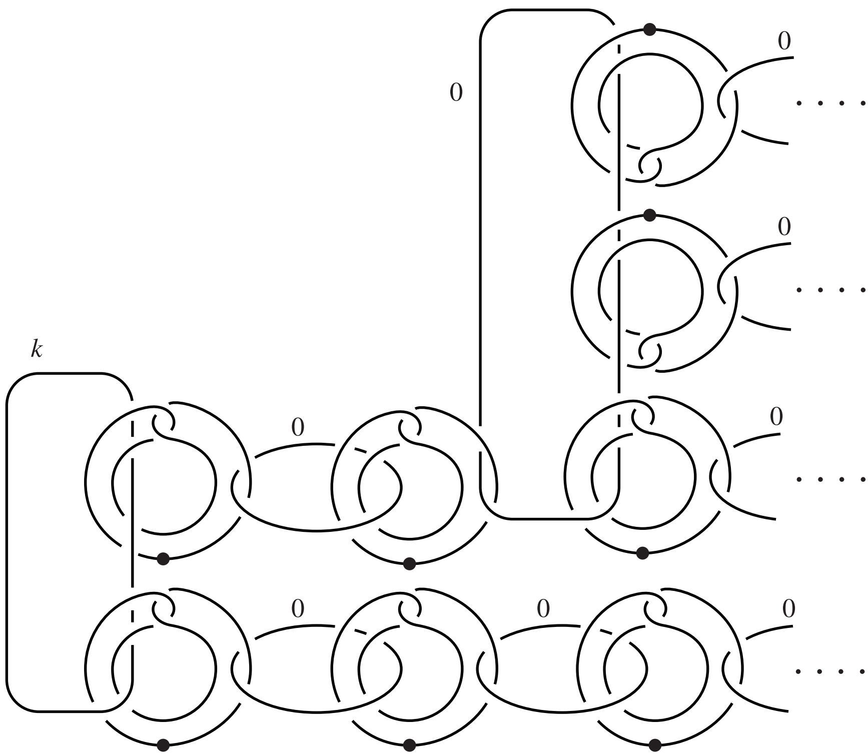


Figure 6.15. Casson handle attached to D^4 along a k-framed unknot.

smooth 4-manifolds (e.g., Donaldson's Theorem 1.2.30). For example, for $k = 0$ Figure 6.14 contains no smoothly embedded sphere generating its homology (by the text following Theorem 11.2.6), so the pictured Casson handle contains no smoothly embedded disk bounded by its attaching circle. There are uncountably many diffeomorphism types of Casson handles [**G2**], [**G6**] (see also Exercise 9.4.13(b)), but it is still not known if they are all distinct, or if any one is diffeomorphic to $D^2 \times \text{int } D^2$. The interior of any Casson handle is diffeomorphic to $\mathbb{R}^4$ by Exercise 9.4.1(c), but a slight modification of one yields the *exotic* $\mathbb{R}^4$ shown as the interior R of Figure 6.16 (taken from [**BG**]), a manifold homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to it. (See Theorem 9.3.8.) More recent versions of Freedman's argument [**FQ**] involve more complicated towers with some kinky handles replaced by manifolds of the form $F \times D^2$, for F an orientable surface with $\partial F \approx S^1$. This replacement corresponds to replacing Whitehead doubles by Bing doubles.

Exercises 6.1.4. (a)* Let $X_{n,k}$ be the manifold $D^4 \cup n$-stage tower obtained by cutting off Figure 6.14 after the $n^{\underline{th}}$ stage. Show directly that adding a 0-framed meridian to the top stage dotted circle produces a disk

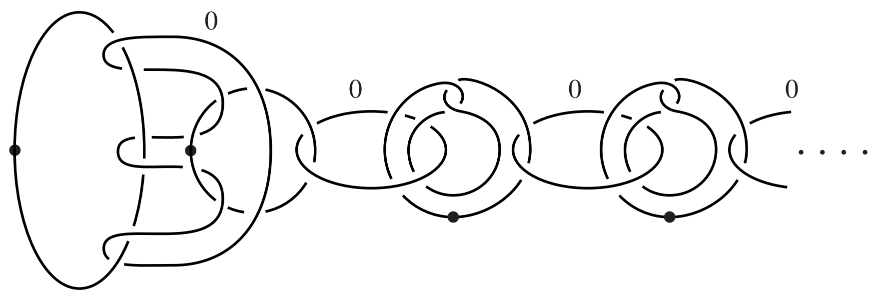


Figure 6.16. Exotic $\mathbb{R}^4$.

bundle. Now draw a diagram for $X_{n,k}$ with only one 1-handle and one 2-handle. Describe this in the language of Remark 6.1.2. How does the diagram change if the kinky handles have more self-plumbings? Self-plumbings of both signs?

(b) Prove that if the Casson handle in Figure 6.16 is replaced by a standard 2-handle, the manifold becomes diffeomorphic to D^4. Thus, by Freedman's lemma, R is homeomorphic to $\mathbb{R}^4$. For a proof that R is not diffeomorphic to $\mathbb{R}^4$, see Theorem 9.3.8.

(c)* Prove directly that R is contractible. (It suffices to show that $\pi_1(R)$ and $H_*(R)$ are trivial.) Prove that R is *simply connected at infinity*, i.e., any compact subset C of R is contained in a compact D with connected complement such that the inclusion $R - D \to R - C$ induces the trivial map on π_1. By Freedman [**F**], [**FQ**], any 4-manifold (without boundary) that is both contractible and simply connected at infinity is homeomorphic to $\mathbb{R}^4$. (*Hint*: Let Y_n be the compact manifold obtained from Figure 6.16 by cutting off the Casson handle after the $n^{\underline{th}}$ stage. What is the minimum number of components in a surgery diagram for ∂Y_n? Now choose D so that $\pi_1(R - D)$ is trivial.)

6.2. Embedded surfaces and their complements

In this section we study submanifolds and their complements. After a brief discussion of ambient handle decompositions of submanifolds $(Y, \partial Y) \subset (D^n, \partial D^n)$ in arbitrary dimensions, we focus on cases of interest to low-dimensional topologists. We warm up with an exercise on link complements in S^3 and Heegaard decompositions of Dehn surgeries, then turn to the problem of drawing embedded (and then immersed) surfaces $(F, \partial F) \subset (D^4, \partial D^4)$, as well as their complements. Finally, we discuss the general problem of drawing surfaces in arbitrary 4-manifolds, e.g., complex curves in $\mathbb{CP}^2$, and understanding handle moves in such diagrams. In each case, we study first the submanifolds themselves, and then handle decompositions of their complements.

We begin by considering an embedding $(Y^m, \partial Y^m) \hookrightarrow (D^n, \partial D^n)$ of a compact pair in a ball, with $m < n$ and Y intersecting ∂D^n transversely. We identify D^n with $I \times D^{n-1}$ in such a way that $\partial Y \subset \{1\} \times D^{n-1}$. As in Section 4.2, we can perturb the first coordinate t of the embedding so that it becomes a Morse function on Y (with $\partial_+ Y = \partial Y$). Then t induces a handle decomposition of Y with a handle for each critical point. On any interval $J \subset I$ without critical values of $t|Y$, the embedding of Y is vertical up to isotopy — that is, a routine flow argument produces a product structure on the pair $(D^n, Y^m) \cap (J \times D^{n-1})$ as $J \times (D^{n-1}, (t|Y)^{-1}(\text{pt.}))$. Each index-$k$ critical point of $t|Y$ corresponds to a k-handle h of Y, and (up to smoothing corners) we can flatten h so that it lies in a single level $t = b$. (See Figure 6.17.) As in Proposition 4.2.7, we can ambiently (i.e., in $I \times D^{n-1}$) order the handles of Y by increasing index, with the caveat that in the codimension-1 case ($m = n - 1$), we may not be able to interchange handles of the same index. (To see why this reordering is possible, imagine attaching a second handle h' at level $t = c$ in Figure 6.17. The dimension hypotheses and transversality allow us to assume that the core of h' and cocore of h have disjoint projections to D^{n-1}. It is now routine to construct an isotopy of Y sliding h' down to the level $t = a$.)

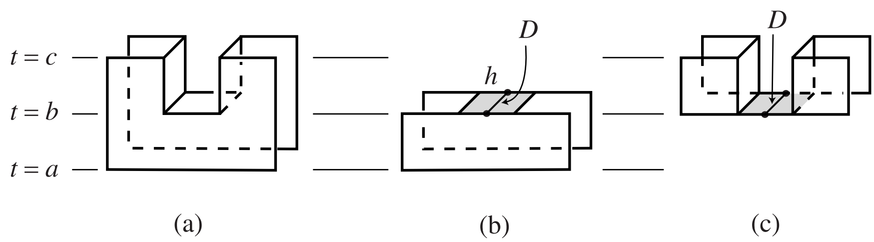


Figure 6.17. Local model of ambiently attaching a handle h with core D.

We now wish to describe a handlebody structure on the complement X of an open tubular neighborhood $\nu Y \subset I \times D^{n-1}$. For $t \in (0, 1]$, let $Y_t = Y \cap ([0,t] \times D^{n-1})$, $X_t = X \cap ([0,t] \times D^{n-1})$ and $\partial_+ X_t = X \cap (\{t\} \times D^{n-1})$. (Note that we have modified our previous convention for ∂_+ in that $\partial_+ X_t$ has nonempty boundary.) As t increases, the topology of X_t only changes when we pass a critical value of $t|Y$. For each k-handle of Y (index-k critical point of $t|Y$), X_t changes by the attachment of a handle, so X inherits a handle decomposition from Y. This is not hard to visualize by imagining νY sitting in a basin $I \times D^{n-1}$ that is being filled with water (representing X_t), but for the sake of completeness we check the details carefully:

Proposition 6.2.1. *Let $[a,d] \subset I$ $(a \neq 0)$ be an interval containing a unique critical value $b \in (a,d)$ of $t|Y$ corresponding to a unique k-handle h of Y. Then $X_d \approx X_a \cup (k+n-m-1)$-handle.*

Proof. For convenience, pick $c \in (b,d)$. Then the local description of $Y \subset D^n$ is $([a,b] \times \partial Y_a) \cup h \cup ([b,c] \times \partial Y_c)$ (Figure 6.17). In the case of codimension 1 (i.e., $m = n-1$) we can assume the open k-handle in νY obtained by thickening h has the form $(a,c) \times h$, so we can write νY locally as $([a,c) \times (\nu\partial Y_a)) \cup ((a,c) \times h) \cup ((a,c] \times (\nu\partial Y_c))$. Although the levels $\partial_+ X_t$ have different topologies for $t = a,b,c$, the manifolds X_t are essentially the same for $a \leq t < c$. (Removing $h \cup \nu\partial h$ from $\partial_+ X_a$ to create $\partial_+ X_b$ merely puts a depression in ∂X_b, and can be achieved by a suitable flow in $I \times D^{n-1}$.) To create $\partial_+ X_c$, however, we must add material back in. Specifically, we must add a neighborhood of the core D of h, extended to the edge of $\nu\partial Y_a$, which is a k-handle attached to $\partial_+ X_b$. Thus, X_d is obtained from X_a as required, by attaching a k-handle with core parallel to D. Before passing to the case of arbitrary codimension, it is useful to consider the dual viewpoint: $\partial_+ X_b$ is obtained from $\partial_+ X_c$ by removing $h \cup \nu\partial h$, which we can interpret as the open $(m-k)$-handle h^* dual to h and attached to $\nu\partial Y_c$ (Figure 6.17(c)). To reverse the procedure and recover $\partial_+ X_c$, we attach a k-handle to $\partial_+ X_b$ along the cocore D of h^*. Now the case of arbitrary codimension $n-m$ is similar. The neighborhood νY is given by the same local picture crossed with D^{n-m-1}. (See Figure 6.18 for $m=1$, $n=3$.) The only difference is that the $(m-k)$-handle h^* has cocore $D \times D^{n-m-1}$, so $\partial_+ X_c$ and X_d are obtained from $\partial_+ X_b$ and X_a by attaching a $(k+n-m-1)$-handle with core $D \times D^{n-m-1}$. □

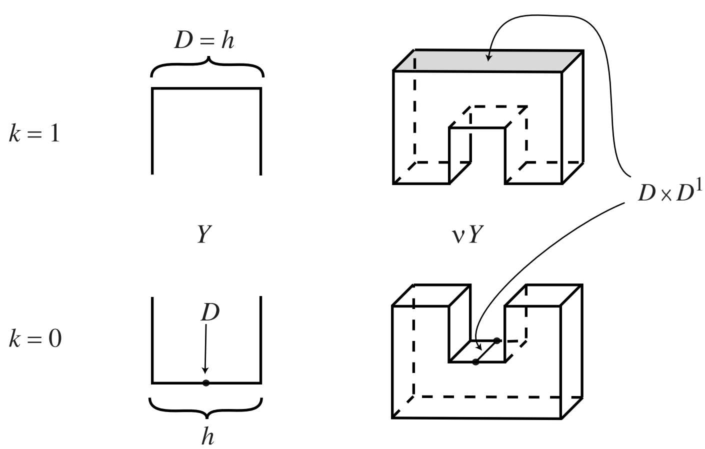


Figure 6.18. Local models of $\nu Y^1 \subset I \times D^2$.

For our present purposes, the most interesting case is when the codimension $n - m$ is 2. (The codimension-1 case is also useful — for example, when studying the *Schoenflies problem*, whether every smooth embedding $S^3 \hookrightarrow S^4$ is unknotted, cf. [**Sch**], [**G8**].) In codimension 2, each k-handle of Y determines a $(k+1)$-handle of X (Figure 6.18). When $k = 1$, the attaching circle $\partial(D \times D^1)$ of the 2-handle of X consists of two copies $D \times \partial D^1$ of the core D of the 1-handle, attached together at the attaching region of the 1-handle (along $\partial D \times D^1$).

Exercise 6.2.2. * Let K denote the (right-handed) trefoil knot in S^3 (Figure 4.27). Draw a handle diagram representing $S^3 - \nu K$. Now draw a Heegaard diagram for the closed 3-manifold obtained by $\frac{p}{q}$-surgery on K. Check your answers by computing H_1. Generalize to describe a procedure for drawing a Heegaard diagram (handle diagram with a unique 3-handle) for any manifold given by Dehn surgery on a link in S^3.

Now we restrict to the case where Y is a surface $F \subset D^4$, so $\partial F \subset S^3$ is a link. An important special case is when F is a disk or disjoint union of disks. Another important case is when F is embedded without 2-handles.

Definition 6.2.3. If $(D, \partial D) \subset (D^4, S^3)$ is an embedded disk, then it is called a *slice disk* for the knot $\partial D \subset S^3$, which is called a *slice knot*. More generally, if $(F, \partial F) \subset (D^4, S^3)$ is a disjoint union of disks, then ∂F is called a (*smoothly*) *slice link*. Similarly, a *topologically slice link* is the boundary of a topologically embedded union of *flat* disks, where a topologically embedded surface F is *flat* if the embedding extends to a homeomorphic embedding of $F \times D^2$. A smooth surface $(F, \partial F) \subset (D^4, S^3)$ is called a *ribbon surface* (or *ribbon disk* if $F \approx D^2$) if it can be isotoped (with $\partial F \subset S^3$) so that the height function $D^4 = I \times D^3 \to I$ induces a handle decomposition without 2-handles (and $\partial F \subset \{1\} \times D^3$). Boundaries of ribbon disks are called *ribbon knots*, and similarly, *ribbon links* bound ribbon surfaces which are disjoint unions of disks.

Note that if we drop the condition of flatness, then any knot $K \subset S^3$ bounds an embedded topological (in fact, PL) disk in D^4, as seen by coning the pair (S^3, K) to obtain an embedding $D^2 \hookrightarrow D^4$ that is "locally knotted" at the cone point. Any knot K also bounds a ribbon surface in D^4, obtained by pushing a Seifert surface for K into D^4, but we cannot obtain a ribbon disk in this manner unless K is unknotted. It is an interesting but difficult problem to understand which knots and links are ribbon or (smoothly or topologically) slice. Clearly, ribbon implies smoothly slice, which implies topologically slice. It is not known if smoothly slice knots (or links) are always ribbon, and it is known via gauge theory and Freedman's work that many topologically slice knots are not smoothly slice. For example, any (untwisted) Whitehead double of a knot (or more generally, any knot with Alexander polynomial 1) is topologically slice [**FQ**], but the positive double of the right trefoil, Figure 6.13 (for example), is not smoothly slice. For this and other examples, see Exercises 6.2.10(b) and 11.4.11(e) or [**CG**], [**Ru1**], [**BG**]. The trefoil itself is not even topologically slice. By Freedman [**F**] (and the fact that any topologically embedded 2-handle contains a Casson handle [**Q**]), each component of a flat surface F can be assumed (after a topological isotopy) to be smooth except at a single point p. Thus, in principle, a connected, flat surface in D^4 can be described by a (usually infinite) handle decomposition in $D^4-p \approx (-\infty, 0]\times S^3$, but no explicit description is known of (e.g.) a topological slice disk for a nonribbon knot. See [**K4**] for more open problems.

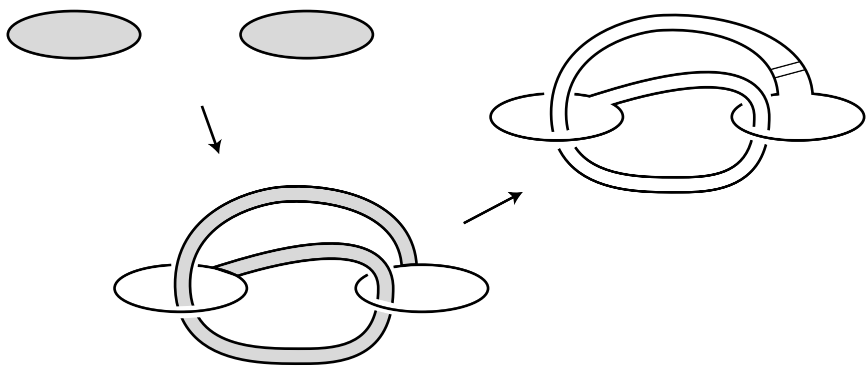


Figure 6.19. Ribbon disk.

We can describe a smoothly embedded surface F in $D^4 = I \times D^3$ by a *level picture*, interpreting the first coordinate as time and drawing successive D^3 levels. (See Figure 6.19.) First, we see the 0-handles of F, appearing as a collection of disks in D^3. (This picture is uniquely determined up to isotopy by the number of 0-handles.) Immediately thereafter, we will see F as an unlink (the boundaries of the disks). Then the 1-handles will appear as a collection of bands $D^1 \times D^1$ attached to the unlink along $\partial D^1 \times D^1$. At later times, F will appear as the link L generated by the band-sums. Any 2-handles will subsequently appear as disks spanning unknotted components of L, and the remaining components will form $\partial F \subset S^3$. For example, Figure 6.19 (without the fine band in the last picture) shows a ribbon disk with a knotted boundary. Alternatively, we can draw F from the top down (reversing time). We begin with ∂F, together with an unknot for each 2-handle, then make *ribbon moves* — that is, we do band sums along the cocores of the 1-handles to turn ∂F into an unlink — see the ribbon move indicated by the band in the last picture of Figure 6.19. (Note that an analogous but infinite procedure describes properly embedded (smooth) surfaces in $(-\infty, 0] \times D^3$, and hence, connected flat topological surfaces.) Given such a level picture (in either direction), we can determine the diffeomorphism type of each component of F by its Euler characteristic, number of boundary components, and whether it is orientable. For example, a knot is ribbon if and only if for some n there is a way to make n ribbon moves to obtain an $(n+1)$-component unlink. (This statement does not involve orientations because no nonorientable, connected surface with boundary S^1 has Euler characteristic 1.)

Now we draw a Kirby diagram for the complement $X = D^4 - \nu F$. The case when F has only 0-handles was discussed in Section 5.4; we obtain a dotted circle (a 1-handle) for each 0-handle of F. (Compare with Figures 5.33 and 5.34, the latter of which shows the complement of a 1-dimensional 0-handle in D^3.) In the general case, each 1-handle of F requires us to add a 2-handle to X as in Proposition 6.2.1. If we attach our 1-handles to F at level $t = b$ (in our previous notation), we have a picture of $X_a \approx X_b$ as a collection of dotted circles, and the 1-handles of F are visible as a collection of bands connecting the dotted circles (as in Figure 6.19). To obtain the 2-handles of X, we first remove a neighborhood of each band from X_a, drilling tunnels in $\partial_+ X_a$ to create $\partial_+ X_b$. Then each 2-handle h will be attached to the boundary of $\partial_+ X_b$ along two parallel copies of the core of the 1-handle (one in front of the band and one behind it) which connect in the boundary of $\partial_+ X_b$ as in Figure 6.20 (where we have identified X_b with X_a). The framing will be 0, as we can verify by drawing a parallel copy of the core of h. To complete the picture of X, we add a 3-handle for each 2-handle of F (if any). The attaching sphere will bound a neighborhood of the corresponding disk of F in S^3. (That is, we put a roof over the local maximum of F.)

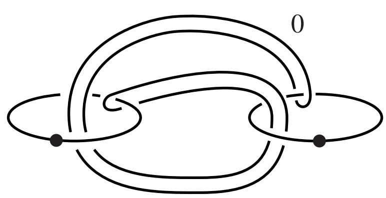


Figure 6.20. Ribbon disk complement.

Exercises 6.2.4. (a)* In Figure 6.20, locate the torus $\partial\partial_+ X_c$, i.e., the boundary torus of a tubular neighborhood of the ribbon knot seen in Figure 6.19.

(b)* Let $F \subset S^3$ be an unknotted punctured torus. Push int F into int D^4 and then draw $D^4 - \nu F$. Compare your answer with Exercise 5.5.7(a).

(c)* Let $F \subset S^3$ be an unknotted Möbius band with a single half-twist. Push int F into int D^4 and prove that $D^4 - \nu F$ is a disk bundle over $\mathbb{RP}^2$.

Conclude that S^4 is obtained by gluing two disk bundles over $\mathbb{RP}^2$ along their boundaries. What are their Euler numbers? Compare with Exercise 5.5.7(a) (and Example 6.3.17). Draw both surfaces in the same level picture of $S^4 = I \times D^3 \cup$ 4-handle.

(d)* For any oriented knot $K \subset S^3$, the connected sum $K \# \overline{K} \subset S^3$ is ribbon, where $\overline{K}$ is obtained from K by reversing orientations on both K and S^3. To see this, remove a small ball from S^3 at a point on K to obtain a knotted arc $K_0 \subset D^3$, and consider the disk $I \times K_0 \subset I \times D^3 \approx D^4$. Find a procedure for drawing the ribbon disk (with boundary isotoped into $\{1\} \times D^3$) explicitly as above. (*Hint*: What happens to K_0 if you begin shrinking D^3?) Now let K be the trefoil knot and draw a Kirby diagram (with $\partial_- X = \emptyset$) for the above ribbon disk complement associated to $K \# \overline{K}$.

(e)* Let $S \subset S^4$ be an embedded 2-sphere, and let $X = S^4 - \nu S$ be given by a handlebody via Proposition 6.2.1. Prove that the homotopy 4-sphere obtained by the Gluck construction on S (Exercise 5.2.7) is constructed from X by "blowing down" one dotted circle (as if it were a $+1$- (or -1-) framed 2-handle) and adding a 4-handle. (*Hint*: Start with a relative handle decomposition of $(S^2 \times D^2, S^2 \times S^1)$.)

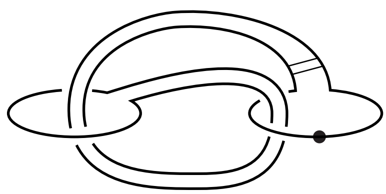


Figure 6.21. Ribbon disk complement.

The dotted circle notation for complements of unknotted disks (1-handles) can now be generalized to complements of ribbon disks as in [**AK2**]. Since a given link may bound different collections of ribbon disks, we must keep track of the ribbon moves defining the desired ribbon disks. The notation is shown in Figure 6.21, which is equivalent to Figure 6.20. (Sometimes ribbon moves are denoted by single arcs, with the blackboard framing being understood, but this carries the danger of not being isotopy invariant. Other times the ribbons are entirely missing from the picture, which is justified if there is a canonical ribbon disk being used, for example, in the case of $K \# \overline{K}$ described above, drawn with the symmetry exhibited.) The new notation has the advantage that we can perform "illegal" 1-handle slides (cf. Figure 5.40). We simply think of the ribbon disks as representing 2-handles attached to ∂D^4 from the inside, and slide these as usual. (Of course, the linking matrix of the dotted circles will be identically 0.) The resulting new disk will be obtained as an ambient boundary sum of one disk with a parallel

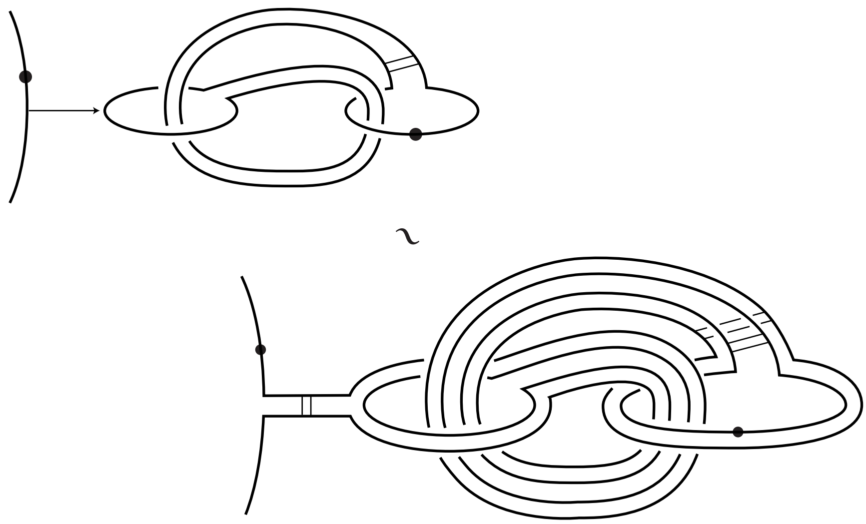


Figure 6.22. Sliding ribbon disks.

copy of the other, and the connecting band will be a 1-handle (ribbon move) as in Figure 6.22. This move can also be derived from our previous Kirby moves as in Figure 6.23. (Each diagram in the figure is obtained from the previous one by a 1- or 2-handle slide as indicated. In the first two diagrams we have also added an obvious cancelling 1-2 pair.)

Exercises 6.2.5. (a) The exotic $\mathbb{R}^4$ in Figure 6.16 is obtained from a ribbon disk complement by adding a Casson handle and removing the boundary. To see this, erase the Casson handle from the figure, leaving a handlebody X given by two dotted circles and a 2-handle. If we also erase the dotted circle C that has linking number zero with the 2-handle, the remaining 1-2 pair will cancel to leave D^4. Thus, the disk spanning C will be a slice disk in D^4 whose complement is X. Draw the slice disk in D^4 (given by the empty link), and verify that it is ribbon (and a single ribbon move suffices). (*Hint*: Ribbon moves represent 1-handles in int D^4, so we are free to push them through attaching circles in ∂D^4. It helps to work on a blackboard, isotoping by drawing the new position of a strand and then erasing it from its old position. You should get the ribbon disk for the $(-3,-3,3)$ pretzel knot shown in Figure 6.24.)

(b)* Give a procedure for using dotted ribbon link notation to draw $I \times M$ when M is a knot complement in S^3. What if M is Dehn surgery on a knot? On a link? Compare with Example 4.6.8. What about $S^1 \times M$? See also [**A6**].

We can also draw the complement $X = D^4 - \nu F$ of a generically immersed surface F in D^4. The self-intersections of F will be transverse double

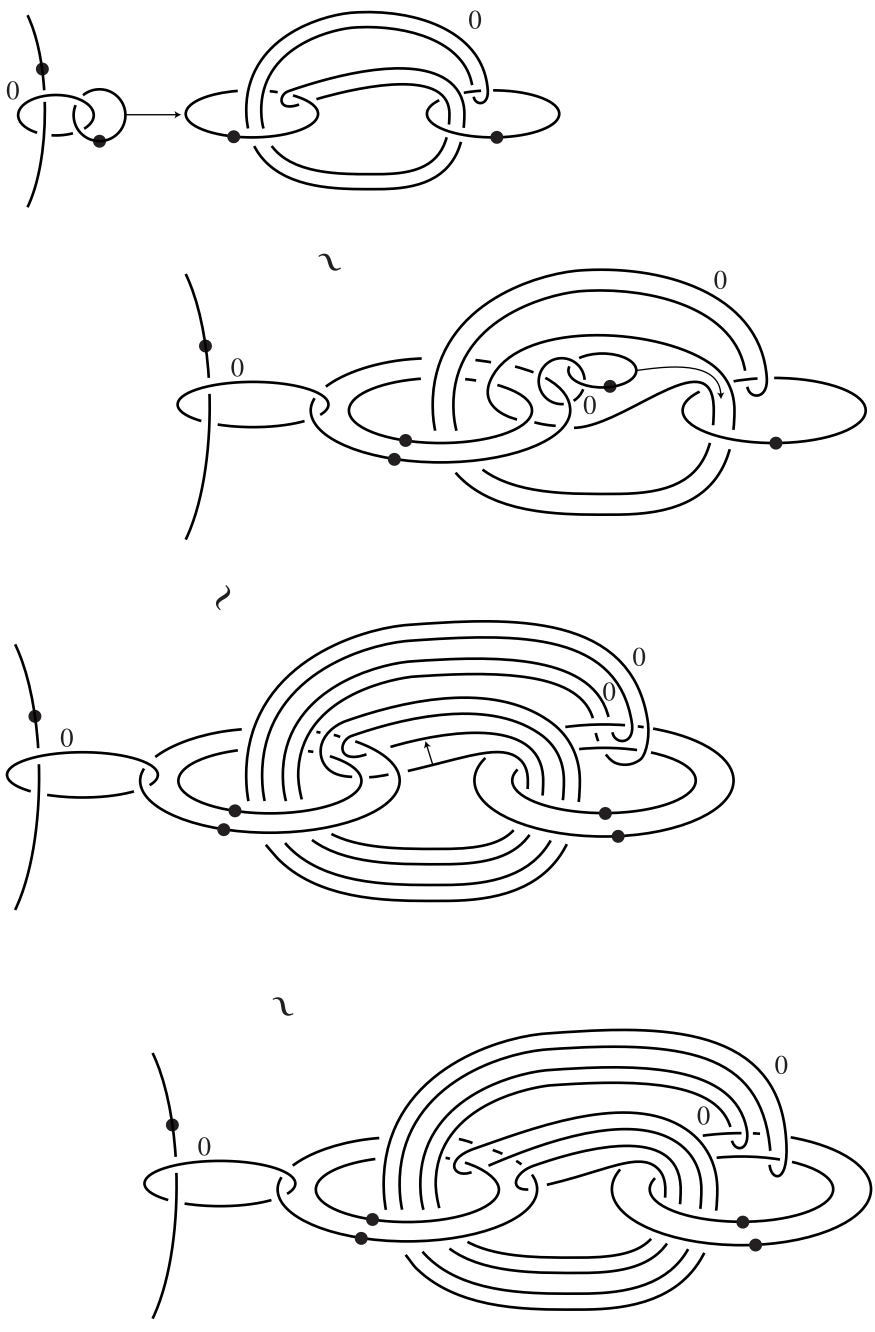


Figure 6.23. Derivation of previous figure using Kirby moves.

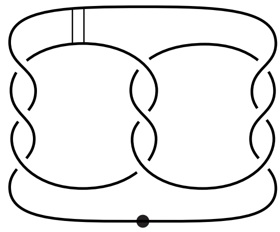


Figure 6.24. A ribbon disk complement.

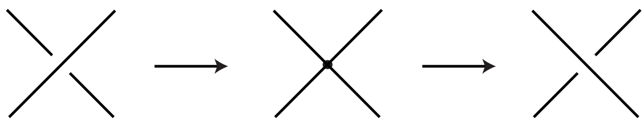


Figure 6.25. Transverse double point of immersed surface in D^4.

points, which will appear in our pictures of $I \times D^3$ as in Figure 6.25. For F oriented, a positive double point will correspond to a crossing changing from negative to positive as t increases (or positive to negative as t decreases), as we can easily verify by examining the obvious spanning disks for a Hopf link (cf. Proposition 4.5.11). Near the double point, a tubular neighborhood νF appears as in Figure 6.26. Thus, to build the complement X, we start with X_a (in the notation of Proposition 6.2.1), drill a tunnel between the tubes in $\partial_+ X_a$ to create $\partial_+ X_b$ (leaving $X_b \approx X_a$), then fill in a different tunnel to create $\partial_+ X_c$. This last move is attaching a 2-handle at $t = c$ in the figure, along a 0-framed meridian K to the tunnel. Pulling this curve back to $t = a$, we obtain the curve in $\partial_+ X_a$ shown in Figure 6.27. Now we see how to represent a self-intersection of F in our handle picture of X. We can assume (by isotopy or use of dotted ribbon link notation) that the crossing occurs between sheets of F that are given by dotted curves. Then we add a 2-handle to reverse the crossing of ∂F as in Figure 6.27. An example is given in Figure 6.28.

Exercises 6.2.6. (a)* Locate the torus $\partial\partial_+ X_c$ in Figures 6.27 and 6.28. (It should be nullhomologous in ∂X.)

(b) Let $F \subset D^4$ be the obvious disk with unknotted boundary, a unique self-intersection and no 1- or 2-handles (cf. Exercise 6.1.1(b)). Prove that the complement $D^4 - \nu F$ is obtained as a self-plumbing of a disk bundle over S^2. Conclude that S^4 is obtained from a pair of self-plumbings by gluing along their boundaries (cf. Example 8.4.7). How do the signs of intersection compare? What are the Euler numbers? Draw both surfaces in the same level picture.

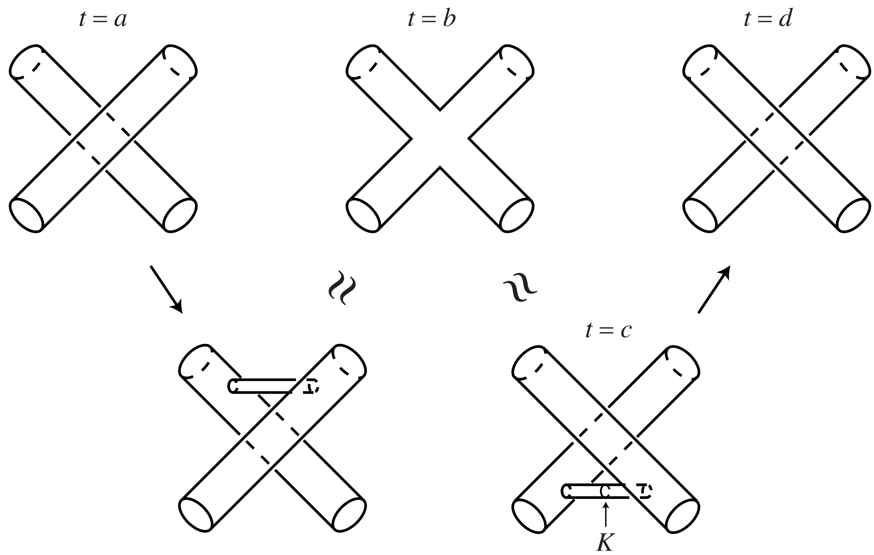


Figure 6.26. Neighborhood of surface at transverse double point.

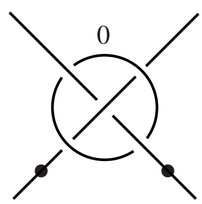


Figure 6.27. Complement of surface at transverse double point.

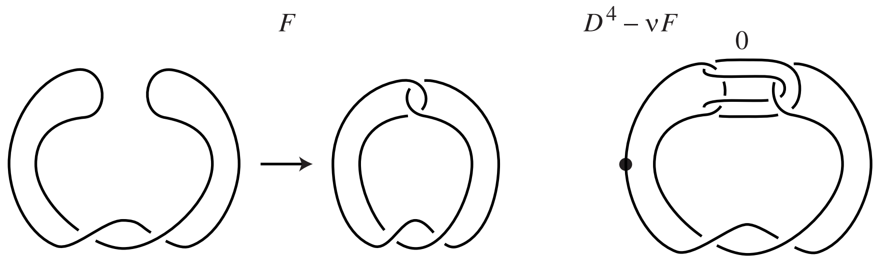


Figure 6.28. Immersed ribbon disk and complement.

Finally, we consider immersed surfaces $(F, \partial F)$ in general compact 4-manifolds $(Y, \partial Y)$. Given a handle decomposition of Y (or $(Y, \partial_- Y)$), we can assume (by pushing F off of handles as in the proofs of Propositions 4.2.7 and 4.2.9) that F lies in the subhandlebody Y_2 consisting of 0-, 1-, and

2-handles (and $I \times \partial_- Y$ in the relative case), and that F intersects each 2-handle in a collection of disks $D_i = D^2 \times \{p_i\}$ parallel to the core. To draw F, it now suffices to remove the 2-handles of Y_2 and draw the remaining surface $F' = F - \bigcup \text{int } D_i \subset Y_1$. The disks D_i will be bounded by a link in $\partial_+ Y_1$ consisting of parallel copies (determined by the framings) of the attaching circles of the 2-handles. If $Y_1 = D^4$, we have now reduced to our previous situation. The case where $Y_1 = D^4 \cup$ 1-handles is no harder, since the dotted circle notation provides a simple embedding $Y_1 \subset D^4$. (The dotted circles will appear as disks at, say, $t = \frac{1}{2}$, and as circles thereafter.) Alternatively, we can deal with 1-handles pairwise as in Example 4.6.5. The case with $\partial_- Y \neq \emptyset$ (and possibly $\partial F \cap \partial_- Y \neq \emptyset$) is analogous. We represent the first coordinate of $I \times \partial_- Y$ as time, which we can assume is a Morse function on F', and proceed as before (representing 1-handles of $(Y_1, \partial_- Y)$ as complements of trivial disks). The only complication is that the links and handles of F will appear in a surgery description of $\partial_- Y$, so we must allow them to slide over surgery curves. (It may also be useful to manipulate the surgery description by Kirby or Rolfsen moves.)

For an alternate viewpoint, we can assume that F is disjoint from the cores of the 2-handles of Y_2 and think of the disks D_i as 2-handles attached to $\partial F'$ in $I \times \partial_+ Y_2$. More generally, we can write $F = F_+ \cup_\partial F_-$, with $F_+ \subset I \times \partial_+ Y_2$ and $F_- \subset Y_1$. For example, we can choose F_+ to project to an embedding in $\partial_+ Y_2$. This is particularly convenient if F_+ is a canonical Seifert surface for ∂F_+ in $\partial_+ Y_2$ (for example, a disk in S^3 or a fiber for a fibered knot as in the next example), since in this case we do not need to keep careful track of F_+. This latter method is used by Akbulut and Kirby [**AK2**] to study holomorphic submanifolds of $\mathbb{CP}^2$.

Example 6.2.7. – Holomorphic curves in $\mathbb{CP}^2$. We will now show how to draw a nonsingular complex curve of any degree d in $\mathbb{CP}^2$, recovering the picture of [**AK2**] by a different method. We begin by considering an (m, n)-*torus link*, which is the oriented link $T_{m,n}$ in S^3 consisting of $\gcd(m, n)$ oriented circles in the boundary of a tubular neighborhood of the unknot, representing the class $m\mu + n\lambda$ in $H_1(T^2)$. Thus, we have $T_{m,1}$ = the unknot, $T_{m,0}$ = an m-component unlink, $T_{2,3}$ = the right-handed trefoil knot and $T_{n,m} = T_{m,n} = \overline{T}_{-m,n}$. The link $T_{m,m}$ is called an *m-component (right-handed) Hopf link*. (Note that any two components of $T_{m,m}$ form the usual right-handed Hopf link $T_{2,2}$.) There is a canonical Seifert surface $F_{m,n}$ for $T_{m,n}$ (with $\partial F_{m,n} = T_{m,n}$ as an oriented manifold), consisting of n disks connected by $m(n-1)$ bands as in Figure 6.29. The complement $S^3 - T_{m,n}$ is a fiber bundle over S^1 with fiber $F_{m,n}$, implying (by standard 3-manifold theory) that $F_{m,n} = F_{n,m}$ is the unique oriented Seifert surface for $T_{m,n}$ with minimal genus. According to [**M5**], the locus $x^m + y^n = \varepsilon$ in $\mathbb{C}^2$ intersects S^3

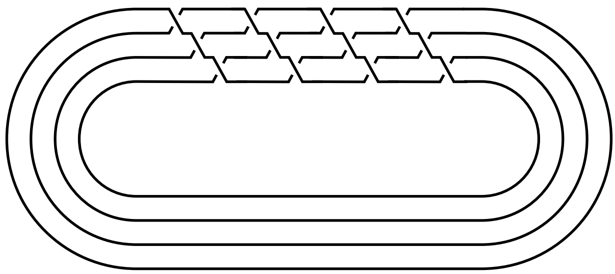


Figure 6.29. Torus link Seifert surface $F_{m,n}$: n disks, m columns of bands ($m = n = 4$).

in $T_{m,n}$, and D^4 in $F_{m,n}$ (with its interior pushed into int D^4). Our current discussion, however, is independent of these assertions.

Recall (Exercise 2.1.3 and subsequent text) that we can obtain a singular degree-d curve in $\mathbb{CP}^2$ as the union of d generic complex lines. A generic perturbation of the polynomial will yield a nonsingular curve, and any two nonsingular curves are smoothly isotopic (Claim 1.3.11). Near each singular point, the perturbation is given as in Section 2.1 — that is, we remove a pair of intersecting disks and replace them by an annulus. We now see that the annulus is the Seifert surface $F_{2,2}$ of the Hopf link in ∂D^4 (cf. Remark 2.1.2). In level pictures, this surgery corresponds to replacing Figure 6.25 by 6.30.

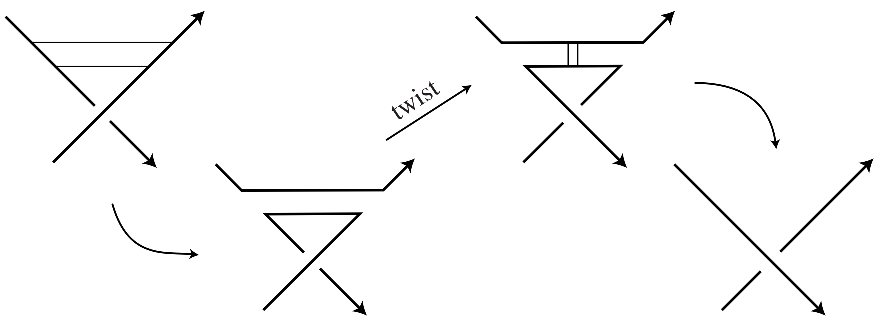


Figure 6.30. Resolved transverse double point.

Exercises 6.2.8. **(a)** Check this last assertion carefully. You should be able to see the 4-ball D in which the surgery occurs in Figure 6.25, and verify that the 360° twist of a horizontal segment in Figure 6.30 corresponds to the full twist in the annulus $F_{2,2}$ in ∂D.

(b) Check that both bands can be added at the same level as shown in Figure 6.31. Compare with $F_{2,2}$.

Now we draw a smooth degree-d curve F in $\mathbb{CP}^2$, beginning with a singular curve F^* consisting of d complex lines. F^* intersects the 2-handle

Figure 6.31. Resolved transverse double point.

in d parallel copies of the core, so in ∂D^4 it is visible as a (d, d)-torus link in the boundary of the attaching region of the 2-handle. Letting t decrease, we descend into $I \times D^3$, and see the components of the link pull through each other to form an unlink, which bounds d 0-handles. Each pair of circles has a unique crossing, corresponding to the unique intersection of each pair of lines, and we can assume the crossings all lie in a single level, where they form the configuration shown (for $d = 5$) in Figure 6.32. (For example, if the handle decomposition of $\mathbb{CP}^2$ is given by Example 4.2.4, we can assume that in the affine coordinates ψ_0 on the 0-handle B_0, the function t is constant on a 2-disk D at 0 in $\mathbb{R}^2 \subset \mathbb{C}^2$ and has a unique critical point on the line $\mathbb{C} \times \{0\}$ — e.g., we can take t to be a suitable modification of the function $\|z_1 - \frac{i}{2}\|^2 + \operatorname{Im} z_2$. Now specify the d lines of F^* by equations with real coefficients, such that over $\mathbb{R}$ they form Figure 6.32 in the disk $D \subset \mathbb{R}^2$. If these complex lines are sufficiently close to $\mathbb{C} \times \{0\}$, t will have a unique critical point on each, so the level pictures will have the required description.)

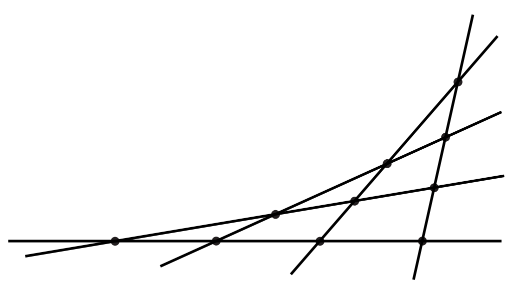


Figure 6.32. Configuration of lines in $\mathbb{R}^2$.

We obtain F by resolving the intersections (of F^*) as in Figure 6.31. Since a small perturbation of the function t allows us to perform the crossings in any order as t increases, we can draw F as in Figure 6.33 (where the crossings occur from bottom to top as t increases). This can clearly be isotoped (in the t direction) so that all handles of $F \cap I \times D^3$ occur at the same level, and then F is a Seifert surface for the link $T_{d,d}$. It is not hard to slide handles in Figure 6.33 to exhibit F as the canonical surface $F_{d,d}$ as in Figure 6.29. (The bands in Figure 6.33 are arranged in columns of increasing length, alternately ascending and descending (from left to right). Proceeding from left to right, slide each band but the first in each ascending column leftward and down as far as possible.) Thus, the degree-d surface F in $\mathbb{CP}^2$ is the union of d parallel copies of the core of the 2-handle with the canonical Seifert surface $F_{d,d}$ in ∂D^4.

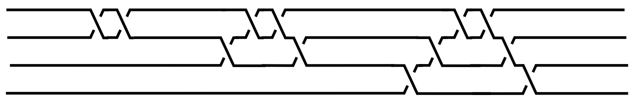


Figure 6.33. Surface obtained by resolving the complexification of Figure 6.32.

To obtain a slightly different picture, we begin by assuming that the disks of F in the 2-handle lie in ∂Y_2, where $Y_2 = D^4 \cup$ 2-handle. Thus, we have a decomposition $F = F_+ \cup F_-$ as above, with $F_- = F_{d,d}$ (pushed into D^4) and F_+ a union of d disks in $\partial Y_2 \approx S^3$. To see F_+ explicitly, observe that ∂Y_2 is given by +1-surgery on the unknot K with $\partial \nu K$ the torus defining $T_{d,d}$. Note that K is isotopic in $S^3 - T_{d,d}$ to the core K' of the complementary solid torus $S^3 - \nu K$, since the union of either circle with $T_{d,d}$ is $T_{d+1,d+1}$. Blowing down K' turns $T_{d,d}$ into the unlink $T_{0,d} \subset \partial Y_2$, which bounds the unique collection of disks F_+. Now following [**AK2**], we observe that any collection of 1-handles of F_- can be lifted into ∂Y_2, so that they disappear from F_- and attach dually to F_+. By blowing down K', one can still see F_+ explicitly and check that it is embedded in a single level $\partial_+ Y_2$. Thus, one obtains many ways of splitting F as $F_+ \cup F_-$ with each of $F_\pm$ a Seifert surface in a copy of S^3. In particular, one obtains various decompositions with $F_\pm$ being canonical Seifert surfaces of torus links. In the case where we remove one column of 1-handles from F_- to obtain $F_{d-1,d}$, F_+ will be a disk $F_{-1,d}$, so we infer:

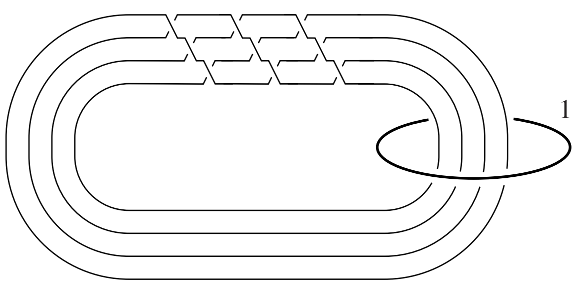


Figure 6.34. Quartic curve in $\mathbb{CP}^2$.

Proposition 6.2.9. ([**AK2**]) *A nonsingular degree-d holomorphic curve in $\mathbb{CP}^2$ is given by the canonical Seifert surface $F_{d-1,d}$ (pushed into D^4) for the $(d-1,d)$-torus knot in the boundary of the attaching region of the 2-handle of $\mathbb{CP}^2$ (and linking the attaching circle d times), together with a disk in $\partial(D^4 \cup$ 2-handle). (See Figure 6.34 for the case $d = 4$.)* □

We conclude Example 6.2.7 with the following exercises. See also Exercises 6.2.12(b) and (c) for drawing complements of holomorphic curves in the projective plane $\mathbb{CP}^2$.

Exercises 6.2.10. (a) Draw F_+ explicitly in the case where $F_- = F_{d-1,d}$, and verify that it is a disk embedded in $\partial_+ Y_2$. What happens if we remove k columns of 1-handles from F_-?

(b) Use Theorem 2.1.5 to show that $F_{m,n}$ realizes the minimum genus of all oriented surfaces in D^4 with boundary $T_{m,n}$. This argument can be generalized to a much larger class of surfaces [**Ru1**], including the punctured-torus Seifert surface of Figure 6.13 (inherited from Figure 6.12), which proves that the positive Whitehead double of the right trefoil is not smoothly slice (cf. Exercise 11.4.11(e)).

(c) There is a canonical embedding $\mathbb{RP}^n \subset \mathbb{CP}^n$ inherited from the inclusion $\mathbb{R} \subset \mathbb{C}$. (See Example 4.2.4.) Draw $\mathbb{RP}^2 \subset \mathbb{CP}^2$.

We have already used various moves for manipulating pictures of pairs (Y, F). In general, we can perform handle moves in both F and Y and isotopies of F in Y. Only a few of these moves require additional comment. A 2-handle/3-handle cancellation in Y appears as in Section 5.1, with the additional observation that disks of F running over the 2-handle will become visible in ∂D^4 in the obvious way. (This is a special case of sliding F over a 3-handle.) To draw 1-2 cancellations in Y, reduce by handle sliding (see below) to the case of a Hopf link with a 0-framing and a dot. (Adjust the framing by Figure 5.42 if necessary.) Then cancel the pair by pushing the 2-handle down so that it refills the disk complement represented by the dotted circle. The two circles will disappear as before (regardless of the presence of F) with disks of F in the 2-handle dropping into ∂D^4 as before. If we slide 2-handles in Y, say h_1 over h_2, then disks of F in h_1 will be dragged across h_2 (cf. Figure 5.7). That is, each will be connected by a ribbon move to a new disk in h_2, as in Figure 6.35. (Note that after we make the ribbon moves, $F \cap \partial D^4$ will appear as it did before the slide, so $F \cap D^4$ is unchanged below the level of the new ribbon moves.) Sliding h_1 under a 1-handle (dotted circle) is formally similar, except that the new disks will be 0-handles parallel to the deleted disk bounded by the dotted circle. (As t *increases*, these disks band into the original F to yield parallel copies of the new attaching circle.) We also note that F can be slid over 2-handles. In particular, if two disks of F in a 2-handle are connected by a trivial ribbon move as in Figure 6.36, then we can eliminate them by pushing the ribbon over the 2-handle as shown. (To verify this, imagine the isotopy occurring in a copy of $I \times D^2$ in the 2-handle, with $\partial I \times D^2$ given by the disks and $I \times \partial D^2$ by the two bands in the figure.) Similarly, if F has no closed components (and $F \cap \partial_- Y = \emptyset$), then we can slide it completely off of the 2-handles by

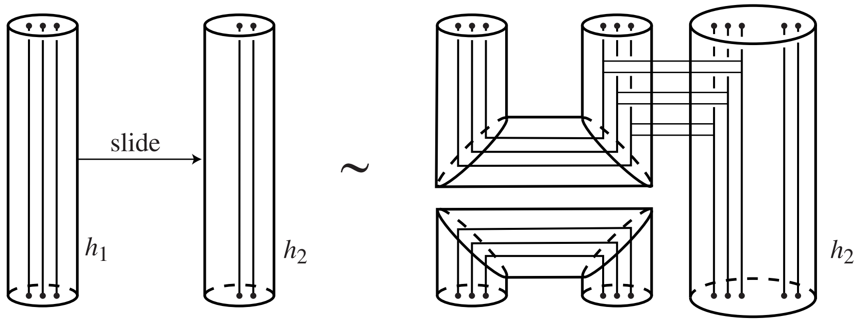


Figure 6.35. Handle slide dragging embedded surfaces.

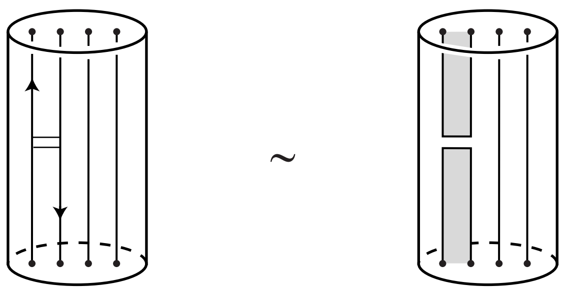


Figure 6.36. Sliding a surface off of a 2-handle.

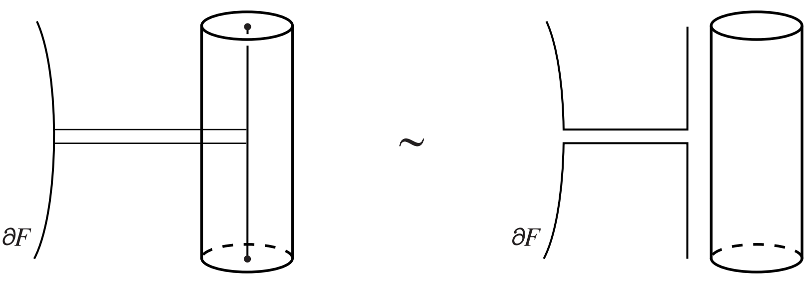


Figure 6.37. Sliding a surface with boundary off of a 2-handle.

sliding ∂F over the 2-handles along ribbon moves (Figure 6.37), provided that none of the required ribbons runs below a 2-handle of F.

Exercises 6.2.11. (a) Show that the reverse of the handle slide in Figure 6.35 can be realized as a handle slide followed by moves as in Figure 6.36 (cf. Exercise 5.1.2(b)).

(b)* Let $F \subset D^4$ be a ribbon disk, and let $S \subset S^4$ be the sphere obtained by doubling the pair (D^4, F). Prove that the manifold obtained by the Gluck construction on S is diffeomorphic to S^4. (*Hint*: First slide 1-handles in

F to simplify its abstract handle structure. Then double and apply Exercise 6.2.4(e).)

It remains to draw the complement $X = Y - \nu F$ of an arbitrary immersed surface $(F, \partial F) \to (Y, \partial Y)$. We have already done this if $Y = D^4$, and the relative version $Y = I \times \partial_- Y$ (assuming that $F \cap (\{0\} \times \partial_- Y) = \emptyset$) is no harder. Adding 1-handles to D^4 (or $I \times \partial_- Y$) also causes no complication; with dotted circle notation a 1-handle is equivalent to an unknotted disk added disjointly to F. It now suffices to understand the effect of 2-handles of Y. If F intersects a 2-handle h in k disks, then h will appear as a hollowed-out 2-handle in X. That is, the second factor of $h = D^2 \times D^2$ will be punctured k times. Since a k-punctured disk is given by $D^2 \cup k$ 1-handles, h will contribute a 2-handle (essentially h again) and k 3-handles to X. (Compare with 2-handles of $F \subset I \times D^3$.) As we have just observed, the 3-handles are frequently unnecessary, since we may often remove F from the 2-handles by sliding ∂F as in Figure 6.37. Similarly, if Y is closed, then we may eliminate the 4-handle by pushing a disk of F into it and observing that $Y - \nu F = Y^* - \nu F^*$, where $F = F^* \cup$ 2-handle and $Y = Y^* \cup$ 4-handle.

Exercises 6.2.12. **(a)** Suppose that F is contained in Y with $K = \partial F$ a knot in ∂Y. Let $(\hat{Y}, \hat{F})$ be obtained from (Y, F) by adding a handle pair (2-handle, core) along K. Prove that $\hat{Y} - \nu\hat{F}$ is obtained from $Y - \nu F$ by adding a 2-handle h_2 and 3-handle h_3, where h_2 is attached along a parallel copy of K. Prove that the attaching sphere of h_3 is obtained from the torus $\partial\nu K$ by surgery along the core of h_2.

(b) Let $F_d \subset \mathbb{CP}^2$ be a smooth holomorphic curve of degree d. Prove that $\mathbb{CP}^2 - \nu F_2$ is a disk bundle over $\mathbb{RP}^2$. What is its Euler number? Draw both surfaces in the same picture. Compare with Exercise 6.2.10(c). (If $F_2 = \{[x : y : z] \in \mathbb{CP}^2 \mid x^2 + y^2 + z^2 = 0\}$, then we can assume the $\mathbb{RP}^2$ is canonically embedded. Note that $F_2 \cap \mathbb{RP}^2 = \emptyset$ in this case. Compare with Example 6.3.17.)

(c)* For F_d as in (b), draw $X_d = \mathbb{CP}^2 - \nu F_d$. What is $\pi_1(X_d)$? Find a picture of X_d with the minimal number of 1-handles.

(d) Let Y^4 be a closed handlebody, $S \subset Y$ be an embedded 2-sphere and $X = Y - \nu S$ be the complement, with a handle description as above. Let C_+ and C_- be dotted circles corresponding to 0-handles of S in $I \times D^3$. Prove that the manifold Y' obtained from Y by the Gluck construction on S is obtained from X by "blowing down" C_+ and C_- (as if $C_\pm$ was a ± 1-framed 2-handle) and adding a 3- and 4-handle. (*Hint*: The Gluck twist on $S^2 \times S^1$ has two invariant circles $\{\text{pt.}\} \times S^1$. Write $(S^2 \times D^2, S^2 \times S^1)$ as a relative handlebody with two 2-handles attached along invariant circles, cf. Exercise 6.2.4(e).) If Y has no 1-handles and S has at most two 0-handles, conclude that Y' has a handle decomposition without 1-handles (hence, by dualizing, a decomposition without 3-handles). This is due to Melvin [**Me**] by a different argument.

6.3. Branched covers

Branched (or *ramified*) coverings have fundamental importance in both algebraic geometry and topology. In Chapter 7, we will examine (possibly singular) branched coverings in the context of complex surfaces; for now we consider how to draw nonsingular branched covers of smooth 4-manifolds. Our discussion is partly based on the paper [**AK2**] of Akbulut and Kirby, and we refer the reader there for additional reading. For the present, we use the following definition.

Definition 6.3.1. A (nonsingular) *d-fold branched covering* is a smooth, proper map $f\colon X^n \to Y^n$ with critical set $B \subset Y$ called the *branch locus*, such that $f|X - f^{-1}(B)\colon X - f^{-1}(B) \to Y - B$ is a covering map of degree d, and for each $p \in f^{-1}(B)$ there are local coordinate charts $U, V \to \mathbb{C} \times \mathbb{R}^{n-2}_{+}$ about $p, f(p)$ on which f is given by $(z, x) \mapsto (z^m, x)$ for some positive integer m called the *branching index* of f at p.

Clearly, the critical points of f form a codimension-2 submanifold C of X, and $f|C$ is an immersion whose image is the branch locus $B \subset Y$. The branching index is constant and ≥ 2 on each component of C (and 1 on $f^{-1}(B) - C$), and the sum of the indices of all points in $f^{-1}(p)$ is the degree of f. Given (Y, B) with $\dim Y \leq 4$ (or B embedded), the branched covering $f\colon X \to Y$ (X connected) is completely determined by the index-d subgroup $\pi_1(X - f^{-1}(B)) \subset \pi_1(Y - B)$ (since $X - f^{-1}(B)$ is determined, and the circle bundle $\partial\nu B$ determines a circle bundle structure on $f^{-1}(\partial\nu B)$, which can be uniquely filled in by a D^2-bundle). In Chapter 7, we will consider singular branched coverings where B and X may have singularities and the above local description of f near C only applies to generic points of C. For the present, we only consider the nonsingular case. We focus on *cyclic* branched covers, where $f|X - f^{-1}(B)$ is a (regular) cyclic covering, so it is determined by an epimorphism $\pi_1(Y - B) \to \mathbb{Z}_d$, and $Y = X/\mathbb{Z}_d$. We sometimes also assume (after orienting Y and B if $d > 2$) that each (positively oriented) meridian of B maps to the same generator of $\mathbb{Z}_d$. For a given (Y, B) with $H_1(Y) = 0$, there is at most one such branched cover, called the *canonical* d-fold cyclic branched cover of (Y, B). For such a branched cover, f maps C diffeomorphically onto B, and $Y = X/\mathbb{Z}_d$ with $\mathbb{Z}_d$ acting freely away from the fixed set C.

Example 6.3.2. Let $(F, \partial F)$ be a connected, compact, orientable surface embedded in a 4-manifold $(Y, \partial Y)$ with $H_1(Y; \mathbb{Z}_d) = 0$, and suppose $[F] = 0 \in H_2(Y, \partial Y; \mathbb{Z}_d)$. (Consider any $(F, \partial F) \subset (D^4, S^3)$, for example.) Then

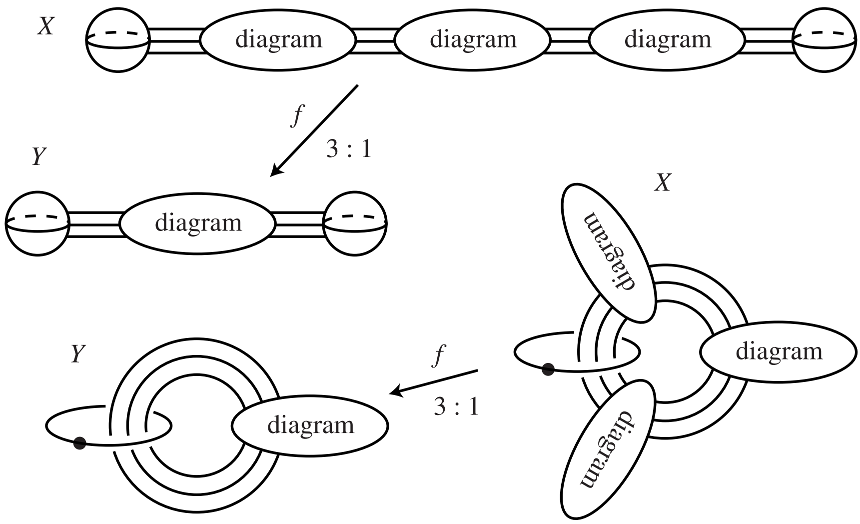


Figure 6.38. 3-fold cover.

$H_1(Y - F; \mathbb{Z}_d) \cong \mathbb{Z}_d$. (It is cyclic by the long exact homology sequence; if it had order $< d$ we could construct a class in $H_2(Y; \mathbb{Z}_d)$ with nontrivial $\mathbb{Z}_d$-intersection number with $[F]$.) Thus, Y has a unique d-fold cyclic cover branched over F. What is this cover when the surface F is an unknotted disk in $Y = D^4$? When $d = 2$, the same reasoning gives a unique branched cover when F is nonorientable.

To understand how Kirby diagrams transform under branched coverings, we first consider the case of empty branch locus, i.e., ordinary finite coverings. Given such a covering $f \colon X^4 \to Y^4$, we fix a Kirby diagram for Y and construct a corresponding diagram for X. The simplest case is when f is cyclic with a single 1-handle h of Y mapping to the generator $1 \in \mathbb{Z}_d$ and any other 1-handles mapping to $0 \in \mathbb{Z}_d$. In this case, we identify $D^4 \cup h$ with $S^1 \times D^3$, and this is d-fold covered in the obvious way by $S^1 \times D^3 = D^4 \cup$ 1-handle in X. Each of the remaining handles of Y lifts to d handles of X. Figure 6.38 shows how the diagram for Y lifts to X (in the case $d = 3$); the covering map $f \colon X \to Y$ and the $\mathbb{Z}_d$-action on X are easy to see. The only subtlely arises in computing framing coefficients. To do this, we revert to double-strand notation, then transform the diagram as in Figure 6.38. We see that the blackboard framing on each knot lifts to the blackboard framing on each of the d corresponding knots in the diagram for X, even though the corresponding writhes may differ. Changing the framing in Y by 1 twist changes each of the d corresponding framings in X by 1 twist, so we have determined how to lift any framing.

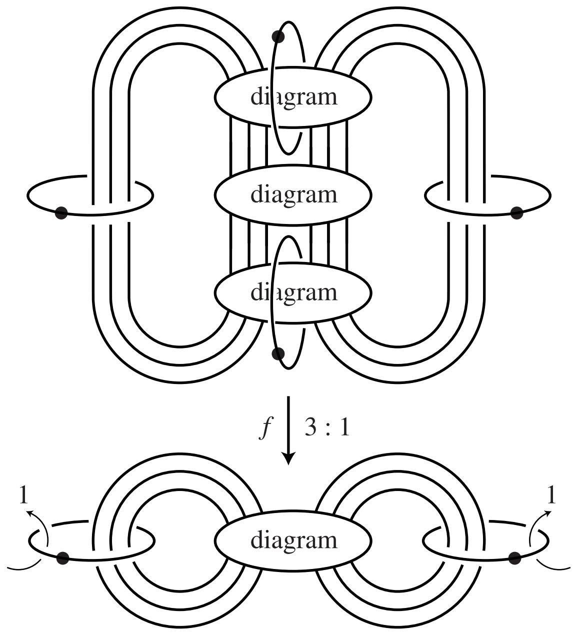


Figure 6.39. 3-fold cover.

Exercises 6.3.3. **(a)*** Let Y be a D^2-bundle over $\mathbb{RP}^2$, drawn as in Figure 6.2. Construct the corresponding diagram for the double cover X of Y, and verify that it represents the D^2-bundle over S^2 with $e(X) = 2e(Y)$ as required.

(b)* Draw a disk bundle Y over S^2 with a single positive self-plumbing and intersection pairing $\langle n \rangle$. Draw the double cover. Interpret the resulting diagram as a plumbing and explain the Euler numbers. Repeat for the d-fold cover.

Now consider a cyclic cover $X \to Y$ where two 1-handles h_1, h_2 of Y map to the same generator 1 of $\mathbb{Z}_d$ and any remaining 1-handles map to 0. Write $D^4 \cup h_1 \cup h_2$ as $S^1 \times D^3 \natural S^1 \times D^3$. Each copy of $S^1 \times D^3$ lifts to an $S^1 \times D^3$ in X as before, but now the 1-handle joining the copies (forming the boundary sum) will lift to d 1-handles. Thus, we obtain $S^1 \times D^3 \natural S^1 \times D^3 \cup (d-1)$ 1-handles in X. Again, the remaining handles of Y each lift to d handles in the obvious way (Figure 6.39 when $d = 3$), and blackboard framings lift as before. The same basic strategy can be used to draw any cover of Y: First, understand the cover on the 1-handles of Y, then lift the diagram in the obvious way.

Exercise 6.3.4. Let Y be a D^2-bundle over S^2 with two positive self-plumbings. Draw a diagram of the cyclic cover of Y for which the two

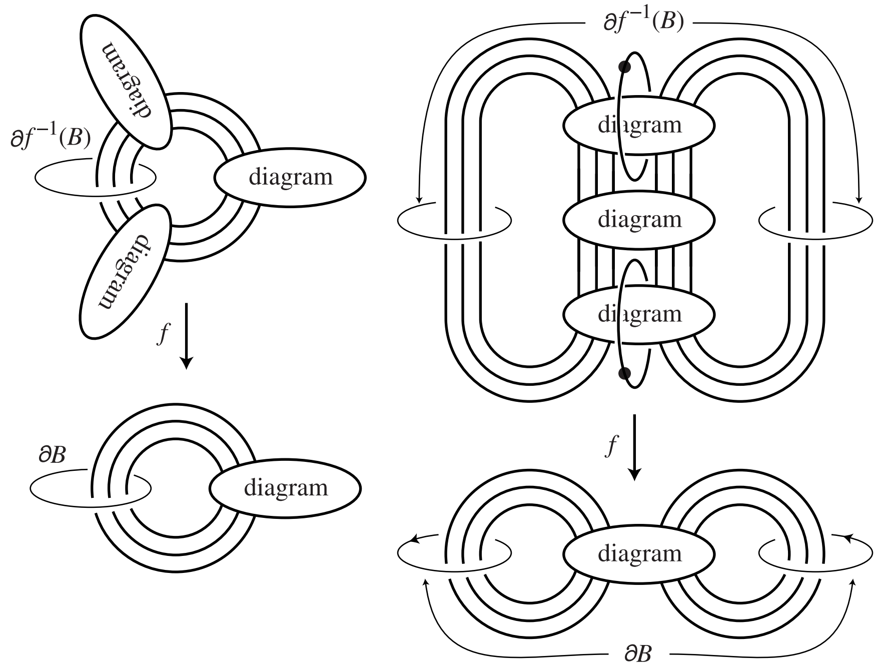


Figure 6.40. 3-fold branched covers.

1-handles map to (a) 1 in $\mathbb{Z}_5$, (b) 1 and 2 in $\mathbb{Z}_5$, respectively (c) 1 and 2 in $\mathbb{Z}_4$. Draw an irregular 3-fold cover of Y.

Next, we consider a branched covering $f\colon X \to Y$ with branch locus B an unknotted collection of disks in the 0-handle of Y. Then $D^4 - \nu B \approx D^4 \cup$ 1-handles, and we produce a diagram of $Y - \nu B$ from one of Y by drawing dotted circles along ∂B. We can now form the cover of $Y - \nu B$ as before, and X is obtained from this by filling in $f^{-1}(B)$, that is, erasing the dotted circles in X coming from B in Y. For example, Figure 6.40 shows the canonical 3-fold cyclic branched covers of two diagrams, with B given by 1 and 2 disks, respectively (cf. Figures 6.38 and 6.39). Note that the canonical d-fold cover of D^4 branched along a trivial collection of k disks is given by $D^4 \cup (k-1)(d-1)$ 1-handles, where the 1-handles arise as extra lifts of the connecting 1-handles in $\natural\, k\, S^1 \times D^3$, as above. Compare with Figure 3.2 in Section 3.2.

Exercises 6.3.5. (a)* Let Y be a D^2-bundle over S^2. Use Kirby calculus to show that the d-fold cyclic cover of Y branched along a pair of fibers is a disk bundle over S^2 with Euler number $de(Y)$.

(b) For Y as in (a), let B be the pair of disks in the 0-handle indicated in Figure 6.41. Draw the branched covers of Y corresponding to the covers described in Exercise 6.3.4. (Note that in the last 2 cases, B lifts to more than 2 disks.)

(c)* To draw a cyclic branched cover of a knot $K \subset S^3$, simply unknot K by blowing up as in Figure 5.19 with linking number 0, then take the corresponding branched cover along the unknot. (The zero linking numbers insure that the corresponding branched cover of the 4-manifold exists.) Prove that the 2-fold branched cover of S^3 along the right trefoil knot is the lens space $L(3, 2)$.

(d)* We can draw branched covers of ribbon disks in D^4 by first drawing their complements as in Section 6.2. Draw the 2-fold cover of D^4 branched along the disk shown in Figure 6.19. Repeat using the obvious ribbon disk for the square knot (which is $K\#\overline{K}$ for K the trefoil knot, Exercise 6.2.4(d) and Figure 12.35). In the latter case, you should get $I \times (L(3, 2) - \text{int } D^3)$. Why?

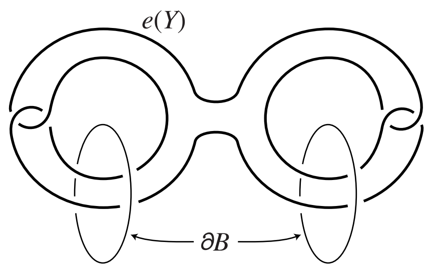


Figure 6.41. Pair of disks B in D^2-bundle Y over S^2.

In principle, we can now draw an arbitrary branched cover by first drawing $Y - B$ as in Section 6.2 and its corresponding cover, then filling in $f^{-1}(B)$ by adding a $(k+2)$-handle for each k-handle of the surface $f^{-1}(B)$. Each such $(k+2)$-handle cancels a $(k+1)$-handle in the cover, since the corresponding handle in Y cancels a $(k+1)$-handle. (The intersection condition of Proposition 4.2.9 lifts.) The method works particularly well for canonical cyclic branched covers of $Y = D^4 \cup$ handles where B is obtained from an embedded surface $F = \bigcup 0, 1$-handles in ∂D^4 by pushing int F into int D^4. In this case, the isotopy from F to B determines a map $I \times F \to D^4$ that is an embedding away from ∂F. As we saw in Section 6.2, $D^4 - \nu(I \times F) \approx D^4$, and $D^4 - \nu B$ is obtained from this by adding handles to fill in $\nu(I \times F) - \nu B$, with a $(k+1)$-handle for each k-handle of B. Filling in B adds a $(k+2)$-handle for each k-handle of B, cancelling the previous handles so that we recover D^4. When we pass to the branched cover X with F orientable, we obtain d maps $I \times F \to X$ with images intersecting in $f^{-1}(B)$ (and for F nonorientable we have $d = 2$ and a similar local picture over each handle). Each 1-handle of B contributes d 2-handles to $X - f^{-1}(B)$ (one for each copy

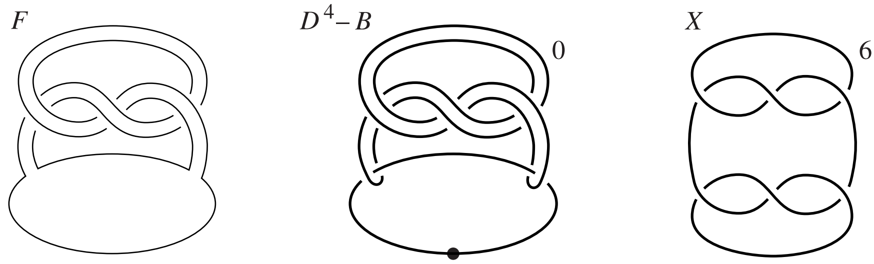


Figure 6.42. 2-fold branched cover.

of $I \times F$), and filling in B (or equivalently, deleting one copy of $\nu(I \times F)$) cancels one of these handles as before. Thus, the induced handle structure on X has $d-1$ 2-handles for each 1-handle of B. Note that removing $\nu(I \times F)$ from X breaks the symmetry of the picture, so that the $\mathbb{Z}_d$-action is no longer visible. To show how this construction works in practice, we give some examples and exercises.

Example 6.3.6. Consider the knotted annulus $F \subset S^3$ shown in Figure 6.42. By Section 6.2, the corresponding manifold $D^4 - B$ is the handlebody shown. The double cover $X - f^{-1}(B)$ has two 2-handles, but one of these disappears when we fill in $f^{-1}(B)$. The resulting branched cover X is obtained from D^4 by adding a 2-handle along a granny knot K with the blackboard framing $w(K) = 6$. An easier way to visualize the construction is to first take the double cover of D^4 branched along the 0-handle of F (pushed into int D^4). We obtain D^4 again, but F has become an annulus made from K with the blackboard framing, and this new annulus determines the 2-handle of X.

Exercise 6.3.7. Justify this last description by constructing X from two copies of $Y - I \times F$, gluing one pair of corresponding faces $I \times F$ by attaching suitable handles. How does this description generalize to d-fold cyclic covers? (Compare with [**AK2**].)

Example 6.3.8. The previous example generalizes to any double cover of $Y = D^4 \cup$ handles branched over $B \subset D^4$ obtained from $F \subset \partial D^4$ as above. We first draw the picture so that F appears as a trivial disk for each component of F, with a band attached for each 1-handle. In general, the bands will be twisted, knotted and linked (Figures 6.42 and 6.43). The attaching circles of any 2-handles of Y may wind around and through F as in Figure 6.43 (but beware that the double branched cover will not exist unless each attaching circle has even linking number with ∂F). If F has k components, then $f^{-1}(D^4) \subset X$ will have $k-1$ 1-handles, as we observed in the case of F without 1-handles. In Figure 6.43 we isotoped the dotted circle

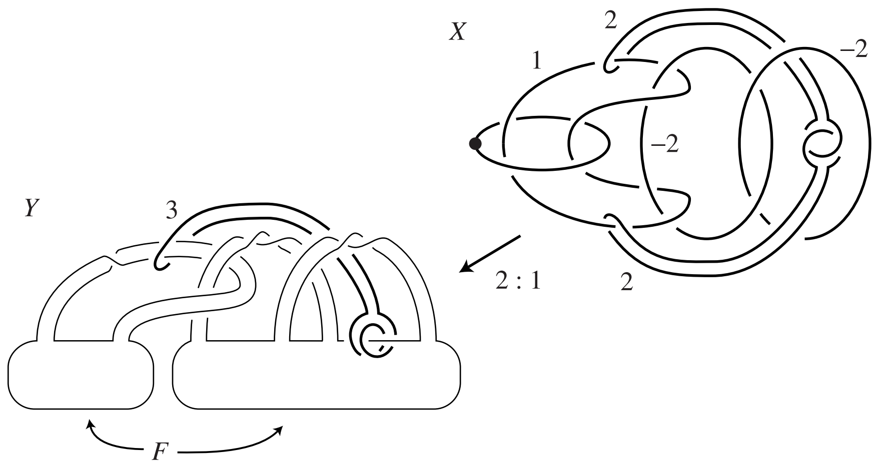


Figure 6.43. 2-fold branched cover.

into a position illustrating a special feature of 2-fold branched covers: The $\mathbb{Z}_2$-action is still visible in the diagram, as 180° rotation about the y-axis.

Exercises 6.3.9. (a) In Figure 6.43, verify directly that rotation about the y-axis induces a $\mathbb{Z}_2$-action on X whose quotient is Y, and that the fixed-point set maps to $B \subset Y$. (*Hint*: The $\mathbb{Z}_2$-action extends over the 2-handles of $f^{-1}(D^4)$ as reflection on each factor of $D^2 \times D^2$. Verify that the images of these 2-handles are 4-balls in Y attached to D^4 along D^3, so they can be absorbed into D^4. What happens to the fixed-point sets in these 2-handles?)

(b)* What is the double cover of D^4 branched along an unknotted annulus or Möbius band in S^3 (pushed into D^4) with k half-twists? Verify your answer without Kirby diagrams using (a).

(c)* What is the double cover of D^4 branched along the punctured-torus Seifert surface of the right trefoil knot? Compare with Exercise 6.3.5(c).

(d)* Draw the double cover of D^4 branched along the punctured-torus Seifert surface of the ($\pm$) Whitehead double $D_\pm K$ of any knot K. What about twisted doubles? (*Hint*: What happens if you knot one band of the punctured torus visible in Figure 6.43?)

(e)* Prove that any plumbing of spheres whose graph is a tree can be realized as a double cover of D^4 branched along a surface pushed in from ∂D^4.

Example 6.3.10. – Milnor Fibers. For $p, q, r \in \mathbb{N}$ and $\varepsilon \in \mathbb{C} - \{0\}$, the manifold $M(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = \varepsilon\}$ is called the *Milnor fiber* of the singularity $x^p + y^q + z^r = 0$. It is independent of the order of (p,q,r) and the choice of ε (as seen by suitably rescaling the coordinates) and it has a canonical compactification as the interior of a smooth manifold with boundary, namely $M_c(p,q,r) = M(p,q,r) \cap D^6$, where $D^6 \subset \mathbb{C}^3$ is

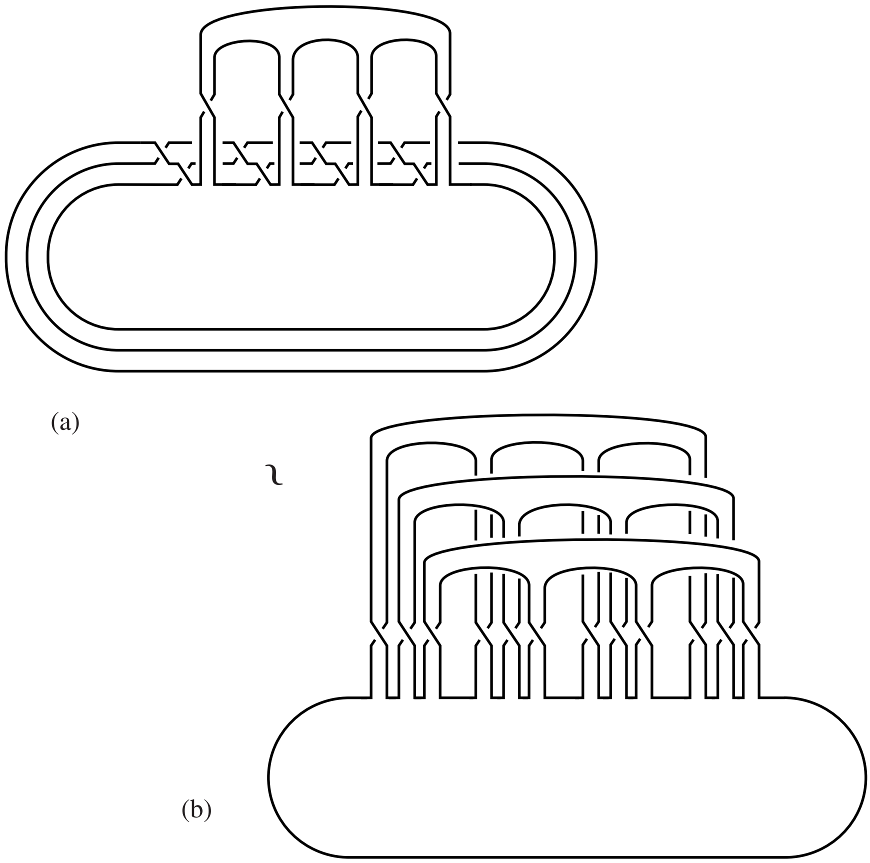


Figure 6.44. Torus link Seifert surface $F_{q,r}$: $r-1$ gates with $q-1$ arches ($q=r=4$).

any round ball with sufficiently large radius. Clearly, $M(1,q,r) \approx \mathbb{C}^2$ and $M_c(1,q,r) \approx D^4$. In Section 7.3 (following Exercise 7.3.17) we will show that $M_c(p,q,r)$ is the p-fold cover of $D^4 \subset \mathbb{C}^2$ branched along the surface $B = \{(y,z) \mid y^q + z^r = \varepsilon\}$. According to [**M5**], B is obtained by pushing in the canonical Seifert surface $F_{q,r}$ of the torus link $T_{q,r}$ (Example 6.2.7), so we can apply the above theory to draw $M_c(2,q,r)$. Begin with the standard picture of $F_{q,r}$ (Figure 6.29 in the case of $F_{4,4}$). By flipping up the back disk and sliding its connecting 1-handles down to the front disk, we obtain Figure 6.44(a). Repeating the procedure for each subsequent disk, we realize $F_{q,r}$ as Figure 6.44(b). (In general, the picture has $r-1$ gates with $q-1$ archways in each gate.)

Exercise 6.3.11. Prove that $M_c(2,q,r)$ is given by Figure 6.45, with $q-1$ rings each containing $r-1$ circles. (*Hint*: First separate the archways by isotoping the top of each gate. It may help to start with special cases.)

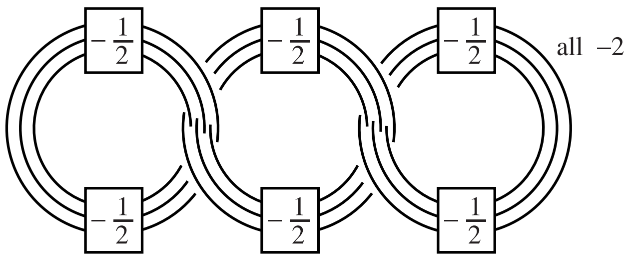


Figure 6.45. Milnor fiber $M_c(2,q,r)$: $q-1$ rings of $r-1$ circles ($q=r=4$).

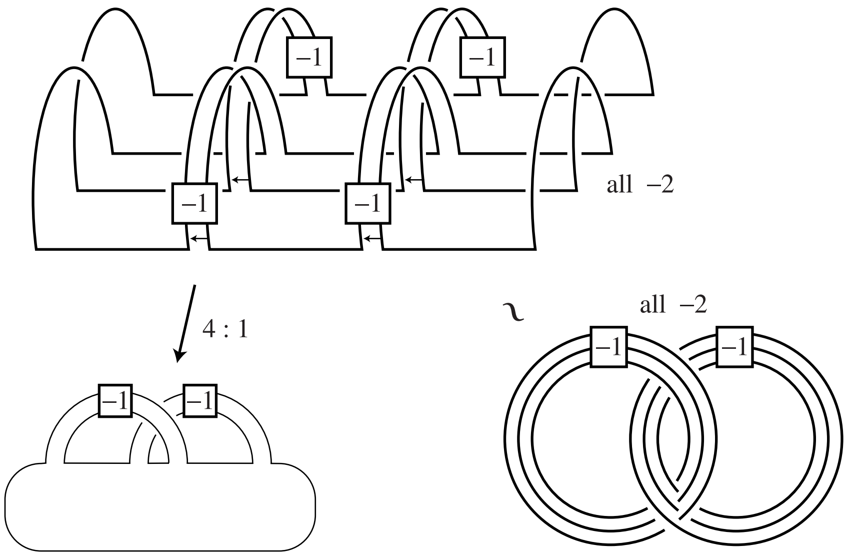


Figure 6.46. 4-fold branched cover.

Note that $M_c(2,q,r) \subset M_c(2,q',r')$ whenever $q \leq q'$, $r \leq r'$. Verify that $M_c(2,2,r) \approx M_c(2,r,2)$.

Example 6.3.12. – *d***-fold covers.** Recall that each 1-handle of B generates $d-1$ 2-handles in the canonical d-fold cyclic cover X, since the $d^{\underline{th}}$ 2-handle is cancelled by a 3-handle when we fill in $f^{-1}(B)$. An example is given in Figure 6.46. The diagram becomes easier to draw if we slide each of the $d-1$ 2-handles over the previous one, proceeding from left to right in the figure, and then fold the 2-handles down as in the last diagram. Figure 6.47 shows other examples obtained by this procedure. (Check the details yourself.)

Exercises 6.3.13. (a)* Draw the r-fold cover $M_c(r,q,2)$ of D^4 branched along the surface $F_{q,2}$ (pushed into int D^4). Verify that you get $M_c(2,q,r)$ as drawn in Figure 6.45.

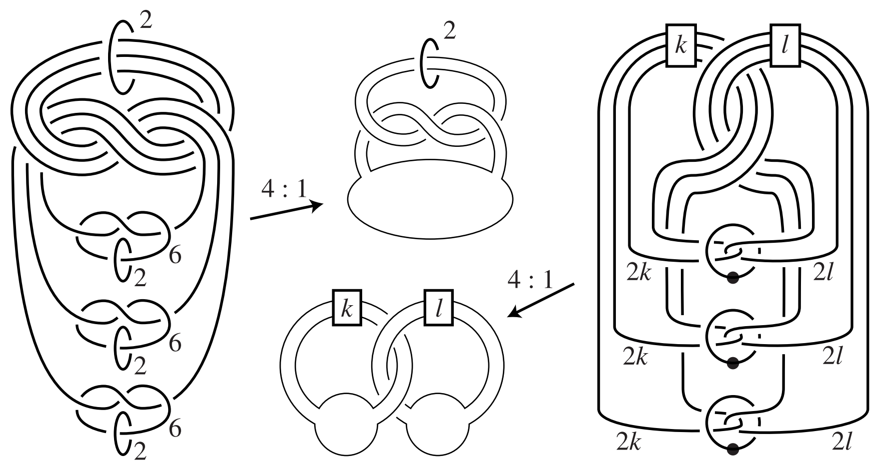


Figure 6.47. 4-fold branched covers.

(b) Show that $M_c(p,q,r)$ embeds in $M_c(p',q',r')$ if $p \leq p'$, $q \leq q'$, $r \leq r'$.

Example 6.3.14. – Closed manifolds. If Y is closed, then so is B, and if $B \neq \emptyset$ we can draw (Y,B) so that B intersects the 4-handle h of Y in an unknotted disk D. Since any cover of h branched along D will again be a 4-ball (or disjoint union of balls), we can draw a cover of Y branched along B by first drawing the corresponding cover of $Y - \operatorname{int} h$ branched along $B - \operatorname{int} D$, then adding 4-handles.

Remark 6.3.15. Suppose that $f\colon X \to Y$ is a cyclic branched covering with $\partial X \approx S^3$ and $\partial B \neq \emptyset$. Then we can infer that $\partial Y \approx S^3$, and we can extend f over 4-handles attached to X and Y as in Example 6.3.14 to obtain a branched covering of closed manifolds. This follows from the solution of the Smith Conjecture [**MB**], which states that any smooth, orientation-preserving, finite cyclic group action on S^3 with nonempty fixed-point set is equivalent to the standard action $(z,w) \mapsto (z, e^{2\pi i/d}w)$ on $S^3 \subset \mathbb{C}^2$.

Exercises 6.3.16. (a)* What is the 2-fold cover of S^4 branched along the standard embedding of T^2? of $\mathbb{RP}^2$? How does a 2-fold cover of a 4-manifold Y branched along B change if we tube B into a small standard T^2 or $\mathbb{RP}^2$ in Y? What about a canonical d-fold cyclic cover?

(b)* What is the double cover of $\mathbb{CP}^2$ branched along a quadric curve? (See Proposition 6.2.9.)

Example 6.3.17. The above exercises (a) and (b) fit together nicely. We can identify $S^2 \times S^2$ with the double of the disk bundle over S^2 with Euler number 2; the 0-sections of the resulting two glued disk bundles have normal Euler numbers 2 and -2, and can be identified with the diagonal and antidiagonal (graph of the antipodal map), respectively. The involution $(x, y) \mapsto (y, x)$ on $S^2 \times S^2$ has the diagonal as its fixed-point set, and it acts as the antipodal map on the antidiagonal. As in (b) above, its quotient is $\mathbb{CP}^2$. The diagonal descends to the quadric curve $\sum z_i^2 = 0$ in $\mathbb{CP}^2$ (a sphere with normal Euler number 4), and the antidiagonal descends to $\mathbb{RP}^2$. (The latter has normal Euler number -1. Explain the behavior of these Euler numbers under the branched covering map.) We recover the decomposition of $\mathbb{CP}^2$ as two disk bundles, Exercise 6.2.12(b). Now complex conjugation on $\mathbb{CP}^2$ has fixed-point set $\mathbb{RP}^2$, and it acts without fixed points on the above quadric curve. As in (a), the quotient is S^4, and we have recovered the decomposition of S^4 as two disk bundles over $\mathbb{RP}^2$ with normal Euler numbers ± 2, Exercise 6.2.4(c).

Exercise 6.3.18. * Consider the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on $S^2 \times S^2$ generated by the above involution and a simultaneous reflection of both S^2-factors. Draw $S^2 \times S^2$ so that the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action is visible. What are the quotients by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and its three $\mathbb{Z}_2$-subgroups?

Exercise 6.3.16(b) generalizes in a different way. As we will see in the next chapter, many interesting complex surfaces arise as branched covers of $\mathbb{CP}^2$ or $S^2 \times S^2$. Given a nonsingular complex curve F_d of degree d in $\mathbb{CP}^2$, for example, we can take the k-fold cyclic branched cover $X_{k,d}$ whenever d is divisible by k. (For Kirby diagrams of some analogous covers of $S^2 \times S^2$, see Section 8.4.) By Proposition 6.2.9, we can exhibit $F_d \subset \mathbb{CP}^2$ as a pushed-in Seifert surface $F_{d-1,d}$ union a disk in the 4-handle, so by Example 6.3.14 it is routine (but tedious for large d) to draw $X_{k,d}$. Since a branched cover of D^4 along $F_{d-1,d}$ is a Milnor fiber, $X_{k,d} = M_c(k, d-1, d) \cup k$ 2-handles $\cup$ 4-handle. In fact, many complex surfaces are obtained from Milnor fibers by attaching a few additional handles — we will show this for elliptic surfaces in Corollary 7.3.23 and again in Section 8.3. We will also see (Exercise 7.1.6) that $X_{d,d}$ is the degree-d hypersurface S_d in $\mathbb{CP}^3$. Thus, Exercise 6.3.16(b) again shows that $S_2 \approx S^2 \times S^2$. For diagrams of S_3, S_4 and S_5, see [**AK2**]. We have the following corollary (which was originally proven in the first 2 cases by Harer [**H1**] using different methods, and generalizes to complete intersection surfaces in $\mathbb{CP}^N$ by work of Mandelbaum [**Ma2**]):

Corollary 6.3.19. *The Milnor fibers $M_c(p, q, r)$, hypersurfaces S_d in $\mathbb{CP}^3$ and branched covers $X_{k,d}$ of smooth holomorphic curves in $\mathbb{CP}^2$ all admit handle decompositions without 1- or 3-handles.* □

Exercise 6.3.20. Draw the manifold $X_{2,8}$. (This is an example of a *Horikawa surface*, which we will encounter again in the next two chapters.) Draw S_3 and prove directly that it is diffeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$.

Part 3

Applications

Having introduced the basics of 4-manifold theory and Kirby calculus in the first two parts, we now give an overview of related topics and problems. In Chapter 7, a very effective way for constructing complex surfaces — the branched cover construction — is discussed. Chapter 8 is more topological in nature, discussing Lefschetz fibrations and providing Kirby diagrams for them. In particular, elliptic surfaces are reconsidered. In Chapter 9, cobordisms and h-cobordisms between 4-manifolds are analysed. We sketch the proof of the famous h-Cobordism theorem, and from its failure for smooth 4-manifolds we deduce the existence of exotic 4-dimensional Euclidean spaces. We study such "exotic $\mathbb{R}^4$'s" and related examples in depth. Finally, Chapters 10 and 11 discuss the topological behavior of 4-manifolds admitting symplectic and Stein structures.

Chapter 7

Branched covers and resolutions

Many examples of complex surfaces (and hence real 4-dimensional manifolds) are given as branched covers of familiar surfaces along (possibly singular) complex curves. This chapter is devoted to the study of some simple branched covers. We begin by reviewing parts of the material discussed in Section 6.3 from a more algebraic point of view. After showing further examples of branched covers, in Section 7.2 we will investigate how to resolve singularities in branched covers and describe the topology of the desingularized objects. For the sake of simplicity we will mainly focus on double branched covers. We will also show that the various constructions of elliptic surfaces given in Chapter 3 produce diffeomorphic manifolds. (We will consider elliptic surfaces from another point of view in Chapter 8.) Throughout this chapter we will adopt a more algebraic attitude in discussing branched covers than in Section 6.3. We will provide ways of constructing many examples of 4-manifolds with only partial information about their topology (e.g., characteristic numbers, fundamental groups) — as opposed to Section 6.3, where detailed topological descriptions of certain branched covers were given.

7.1. Definitions and examples

Recall from Definition 6.3.1 that a smooth, proper map $f\colon X \to Y$ is a *d-fold branched covering* if away from the critical set $B \subset Y$ the restriction $f|X - f^{-1}(B)\colon X - f^{-1}(B) \to Y - B$ is a covering map of degree d, and for $p \in f^{-1}(B)$ the map f is $(z, x) \mapsto (z^m, x)$ in appropriate coordinate charts around p and $f(p)$ (cf. Definition 6.3.1). The set $B \subset Y$ is called the *branch locus* of the branched cover $f: X \to Y$. In the following, we will restrict ourselves to *cyclic* branched covers, i.e., when the index-d subgroup $\pi_1(X - f^{-1}(B)) \subset \pi_1(Y - B)$ describing the above covering is determined by a surjection $\pi_1(Y - B) \to \mathbb{Z}_d$. In the forthcoming discussions we will mainly work in the holomorphic category. Suppose that Y is a smooth complex surface and $B \subset Y$ is given as the zero set $\sigma_B^{-1}(0)$ of a holomorphic section σ_B of a holomorphic line bundle $L_B \to Y$. If $c_1(L_B) \in H^2(Y; \mathbb{Z})$ is divisible by d, then a choice $[A] \in H_2(Y; \mathbb{Z})$ satisfying $PD(c_1(L_B)) = d[A]$ determines a cyclic d-fold branched cover of Y branched along B in the following way.

Construction 7.1.1. Since $PD(c_1(L_B)) = d[A]$, the line bundle $L_A \to Y$ corresponding to $[A] \in H_2(Y; \mathbb{Z})$ (via the condition $PD(c_1(L_A)) = [A]$) satisfies $L_A^{\otimes d} = L_B$. Hence the definition

$$X = \{l_p \in L_A \mid l_p^{\otimes d} = \sigma_B(p)\}$$

specifies a subset of the total space of L_A.

Lemma 7.1.2. *Restricting the projection map $\pi: L_A \to Y$ to the above defined subset $X = \{l_p \in L_A \mid l_p^{\otimes d} = \sigma_B(p)\}$, we get a d-fold (cyclic) branched cover $\phi: X \to Y$ branched along B.* □

Throughout this chapter we will investigate properties of branched covers given by Construction 7.1.1. In the above construction we did not assume that B is a smooth submanifold of the complex surface Y; as we will see, singularities of B might introduce singularities on X. In Section 7.2 we will describe a method for resolving these kinds of singularities — at least for double branched covers. Note that in Construction 7.1.1 the complex curve B is specified by the section $\sigma_B \in \Gamma(L_B)$. Based on this definition of B, an integer $m_p(B)$ — the *multiplicity* of B at $p \in B$ — can be defined as the order of vanishing of σ_B at $p \in B$. That is, $m_p(B)$ is the greatest integer such that all partial derivatives $\frac{\partial^k \sigma_B}{\partial x_{i_1} \dots \partial x_{i_k}}(p)$ vanish (after we locally project σ_B into the fiber) for $k \leq m_p(B) - 1$. Recall that B is not necessarily a submanifold of Y; the set of smooth points of B will be denoted by B^*. Using algebraic geometric arguments [**GH**], it can be shown that (since B is of complex dimension 1) $B - B^*$ consists of finitely many points.

Definition 7.1.3. The closure of a connected component of the topological space B^* in B is called a *component* of the complex curve B.

(Note that in the above sense a curve B might have more than one component, even if B is connected as a topological space. Since we will be using results of algebraic geometry, we will use the term “component” throughout this chapter in the sense it is given in Definition 7.1.3.) An easy argument

[**GH**] shows that if $p \in B^*$, then $m_p(B)$ can be identified with the degree of the map $f_p\colon S^1_p \to \mathbb{C} - \{0\}$, where $S^1_p \subset Y$ is a small normal circle of B with center p and $f_p = \sigma_B | S^1_p$. This description of the multiplicity also shows that $m_p(B)$ is constant along components of B^*; hence we can talk about the multiplicity of a component of B. We say that B has multiplicity k if each component of it has multiplicity k. To preserve the equality $PD[B] = c_1(L_B)$, we must count the homology class represented by each component of B with the appropriate multiplicity; for example, if $B \subset Y$ is a smooth submanifold with multiplicity d, then the homology class $[B]$ associated to B equals $d\beta \in H_2(Y;\mathbb{Z})$, where $\beta \in H_2(Y;\mathbb{Z})$ is the fundamental class of the submanifold of Y defined by B.

Recall that for the blown-up manifold Y', the second homology $H_2(Y';\mathbb{Z})$ canonically splits as $H_2(Y;\mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z})$, hence we obtain an embedding $H_2(Y;\mathbb{Z}) \subset H_2(Y';\mathbb{Z})$. Using this embedding, we may think of $[B]$ as an element of $H_2(Y';\mathbb{Z})$. As before, B' and $\widetilde{B}$ denote the total and proper transforms of the curve $B \subset Y$. The exceptional curve of the blow-up will be denoted by E, and we assume that the multiplicity of E in B' is m.

Lemma 7.1.4. *Under the above circumstances,* $[B'] = [\widetilde{B}] + m[E]$*; consequently* $[B] = [B']$ *in* $H_2(Y';\mathbb{Z})$.

Proof. Recall that $\widetilde{B}$ is defined as the closure of $B' - E$. Since E has multiplicity m, we get $[\widetilde{B}] = [B'] - m[E]$ in homology, which proves the first assertion. Since the multiplicity of the point $p \in Y$ that we blew up is equal to m, the intersection of $\widetilde{B}$ (counted with multiplicity) with the exceptional curve E is m. (We can make the intersection transversal by using a small C^∞-perturbation of the complex curves.) Consequently $[B'] \cdot [E] = 0$; since B and B' coincide in $Y - \{p\}$, the equality $[B] = [B']$ follows. □

To avoid codimension-1 singularities in X, we assume from now on that the curve B used in Construction 7.1.1 has multiplicity one; for such B the section σ_B provides no information other than the subset $\sigma_B^{-1}(0)$. (We always assume that the complex surface Y is smooth.) In the desingularization algorithm described in Section 7.2, however, we will frequently meet curves for which the above distinction between the curve (as an algebraic geometric object defined by the zero set of a section) and the submanifold (as a topological subspace of the ambient 4-manifold) becomes crucial. Note also that in Construction 7.1.1 we had to specify the complex curve B (and not just its homology class $[B]$); on the other hand, A was specified only up to homology. If $H_2(Y;\mathbb{Z})$ is torsion free, then $[A]$ is determined by $[B]$, so in that particular case Y and the complex curve B are the only necessary data. A local description of Construction 7.1.1 is given as follows. Suppose that $\varphi_\alpha\colon U_\alpha \to \mathbb{C}^2$ is a chart for Y, and the image of $U_\alpha \cap B$ in $\mathbb{C}^2$ is given by the equation $f(x,y)=0$. Then $\phi^{-1}(U_\alpha)\subset X$ is given in $\mathbb{C}^3$ by the equation $\{(x,y,z)\in\mathbb{C}^3 \mid z^d - f(x,y)=0\}$.

Lemma 7.1.5. *If B is a smooth curve (of multiplicity one) in the smooth complex surface Y, then the branched cover X (branched along B) given by Construction 7.1.1 is smooth.*

Proof. We will use the above local description of X. By the Implicit Function Theorem, if $(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})\neq(0,0,0)$ along $\{g(x,y,z)=0\}$, then $\{(x,y,z)\in\mathbb{C}^3\mid g(x,y,z)=z^d-f(x,y)=0\}$ is smooth. $\frac{\partial g}{\partial z}=dz^{d-1}$ equals 0 iff $z=0$, hence $f(x,y)=0$. At these points, $(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})=(0,0,0)$ iff $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})=(0,0)$. Since $B=B^*$ and the multiplicity of B is one, we have that $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})\neq(0,0)$ along $\{f(x,y)=0\}$, which proves the lemma. □

Examples of d-fold branched covers for various d can be found by examining complex curves in the projective plane $\mathbb{CP}^2$. Take $F_d\subset\mathbb{CP}^2$ defined by $\{x_0^d+x_1^d+x_2^d=0\}$. (Recall that the genus of F_d is $\frac{1}{2}(d-1)(d-2)$.) The projection of F_d from $[0:0:1]\in\mathbb{CP}^2$ to $H=\{x_2=0\}$ gives a d-fold branched cover $F_d\to H\approx\mathbb{CP}^1$ branched at the points $[1:\zeta:0]$ where $\zeta^d=-1$. (Note that for complex 1-dimensional manifolds the branch locus B is a finite set of points.) This phenomenon can be seen from another point of view as well: If $B=\{b_1,\ldots,b_d\}\subset\mathbb{CP}^1$ is the branch locus, then the above d-fold branched cover of $\mathbb{CP}^1$ branched along B can be thought of (by the description of Construction 7.1.1) as a subset of the total space of the line bundle L_A, where $A=\{b_1\}$. We have already seen that $L_A\approx\mathbb{CP}^2-\{[0:0:1]\}$, so the d-fold branched cover branched along B is a curve (of degree d) in $\mathbb{CP}^2$.

Exercise 7.1.6. Extend the above idea to surfaces and determine the d-fold branched cover of $\mathbb{CP}^2$ provided by Construction 7.1.1 when B is a smooth curve of degree d in $\mathbb{CP}^2$. (*Hint*: Take $B=\{x_0^d+x_1^d+x_2^d=0\}\subset\mathbb{CP}^2$, identify L_A with $\mathbb{CP}^3-\{[0:0:0:1]\}$ and describe the d-fold branched cover by an equation. Cf. also Exercise 6.3.16(b) and the text after Exercise 6.3.18.)

In the following we will mainly concentrate on double branched covers of (closed) complex surfaces. We would like to determine the topology (characteristic numbers, intersection form, fundamental group) of the double branched cover X in terms of Y and the branch locus (or *branch curve*) B. Some of these questions are hard to answer; for example, we will not give a general formula for $\pi_1(X)$, only a sufficient condition for the simple connectivity of X. (See Lemma 7.4.15 and [**Pe**].) A double branched cover $\phi\colon X\to Y$ provides an obvious $\mathbb{Z}_2$-action on X (mapping the points with equal ϕ-image into each other), and conversely, a $\mathbb{Z}_2$-action without isolated fixed points gives a double branched cover. The image of the fixed point set of this action corresponds to the branch curve. Note that every double cover is cyclic. We begin the description of the characteristic numbers of X with the case of a smooth branch curve B — the singular case will be handled in the next section. By Lemma 7.1.5, the smoothness of B implies that X is a smooth manifold. (The smoothness of Y is always assumed.) The characteristic numbers of X are given in terms of Y and the branch curve B in the following way:

Lemma 7.1.7. *Suppose that the double branched cover $\phi\colon X \to Y$ is determined by the (smooth) complex curve $B \subset Y$ and the homology class $[A] \in H_2(Y;\mathbb{Z})$, using Construction 7.1.1. Then we have*

$$c_2(X) = 2(c_2(Y) - \chi(B)) + \chi(B) = 2c_2(Y) - \chi(B) \quad \textit{and}$$

$$\sigma(X) = 2\sigma(Y) - \frac{1}{2}[B]^2 = 2\sigma(Y) - 2[A]^2.$$

Consequently $c_1^2(X) = 2(c_1(Y) - \frac{1}{2}[B])^2 = 2(c_1(Y) - [A])^2$.

Proof. Since $c_2(X) = \chi(X)$, the expression for $c_2(X)$ can easily be derived using well-known properties of the Euler characteristic. For the formula giving $\sigma(X)$, see [**Hi2**]. Since $c_1^2(X) = 3\sigma(X) + 2\chi(X)$ and $c_1(Y)[B] = \chi(B) + [B]^2$ (by the adjunction formula, Theorem 1.4.17), the expression for $c_1^2(X)$ follows. Note that $2(c_1(Y) - \frac{1}{2}[B])^2 = 2(c_1^2(Y) - \chi(B)) - \frac{3}{2}[B]^2$. (Throughout the above formulae we have identified the homology classes $[A]$ and $[B]$ with their Poincaré duals.) □

Remarks 7.1.8. **(a)** In fact, the first Chern class $c_1(X)$ of X is equal to $\phi^*(c_1(Y) - [A])$. It is easy to prove that for any $a \in H^2(Y;\mathbb{Z})$ the square of $\phi^*(a) \in H^2(X;\mathbb{Z})$ equals $\langle \phi^*(a)^2, [X]\rangle = 2\langle a^2, [Y]\rangle$. (This can be seen geometrically in homology by taking Σ_1, Σ_2 two transversally intersecting representatives of $PD(a)$ in such a way that $\Sigma_1 \cap \Sigma_2 \cap B = \emptyset$. Then $\langle \phi^*(a)^2, [X]\rangle = \#(\phi^{-1}(\Sigma_1) \cap \phi^{-1}(\Sigma_2))$, which is $2\#(\Sigma_1 \cap \Sigma_2)$.) This reasoning recomputes $c_1^2(X)$ for the double branched cover X from a slightly different point of view. Note that since B is a smooth complex submanifold, its Euler characteristic $\chi(B)$ is determined by the homology class $[B]$ (through the adjunction formula). If, in addition, B is connected, then $\chi(B)$ can be replaced by $2 - 2g(B)$, where $g(B)$ is the genus of B.

(b) If $\phi\colon X \to Y$ is a d-fold branched cover defined by the complex surface Y and the smooth curve $B \subset Y$ using Construction 7.1.1, then the formulae given by Lemma 7.1.7 generalize as $c_2(X) = dc_2(Y) - (d-1)\chi(B)$, $c_1^2(X) = d(c_1(Y) - \frac{d-1}{d}[B])^2 = d(c_1(Y) - (d-1)[A])^2$. Moreover, it can be shown that $c_1(X) = \phi^*(c_1(Y) - (d-1)[A])$.

Next we give a few examples of double branched covers. Again, we start with complex 1-dimensional examples. For every (oriented, real) 2-dimensional surface Σ_g there is a map $\phi\colon \Sigma_g \to \mathbb{CP}^1$ which is a double branched cover. To see this, we only need a suitable $\mathbb{Z}_2$-action on Σ_g — take the hyperelliptic $\mathbb{Z}_2$-action σ_g defined in Section 3.2 (180° rotation around the y-axis in Figure 3.2). It is easy to see that $\Sigma_g/\sigma_g \approx \mathbb{CP}^1$, and since σ_g has $2g+2$ fixed points, the branch locus B is $\{b_1, \dots, b_{2g+2}\} \subset \mathbb{CP}^1$. Note that, in particular, there is a double branched cover $\varphi_2 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ branched at two points, say $[1:0]$ and $[0:1] \in \mathbb{CP}^1$; this map, in fact, can be given as $z \mapsto z^2$. (We have already met this double branched cover as $F_2 \to H \approx \mathbb{CP}^1$, cf. the text after Lemma 7.1.5.) One can also think of the above map as the quotient by the involution $z \mapsto -z$ $(z \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1)$. In the same fashion, the map $z \mapsto z^d$ gives a d-fold cyclic cover $\varphi_d \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ — we will make use of this d-fold cyclic cover in our later discussions, cf. Theorem 7.3.12.

Let $L \to \Sigma$ be a complex line bundle over the (real) 2-dimensional surface Σ. The fiberwise map $z \mapsto -z$ defines a $\mathbb{Z}_2$-action on L, hence a double branched cover $L \to L \otimes L$ (given fiberwise by the map $z \mapsto z^2$, cf. φ_2 above) with the zero section of $L \otimes L$ as branch locus. Recall that the Hirzebruch surface $\mathbb{F}_n$ is defined as the projectivization of the $\mathbb{C}^2$-bundle $L_n \oplus \underline{\mathbb{C}} \to \mathbb{CP}^1$, where $L_n \to \mathbb{CP}^1$ is the complex line bundle with $c_1(L_n) = n$ (cf. Example 3.4.7); it is a geometrically ruled surface over $\mathbb{CP}^1$. It admits a zero section S_n with square $[S_n]^2 = n$ and a section S_∞ at infinity formed by the "ideal" points ∞ of the fibers $\mathbb{CP}^1$. Since S_∞ is disjoint from the zero section S_n and intersects every fiber of $\mathbb{F}_n$ in a unique point, the homology class of S_∞ is $[S_n] - n[F_n]$ (where F_n is the fiber of the ruling of $\mathbb{F}_n$), hence $[S_\infty]^2 = -n$. Extending the $\mathbb{Z}_2$-action defined on $L_n \to \mathbb{CP}^1$ to $\mathbb{F}_n$ (mapping $z \in \mathbb{CP}^1$ to $-z$ fiberwise), we get a $\mathbb{Z}_2$-action τ on $\mathbb{F}_n$ with $S_n \cup S_\infty$ as the fixed point set. It is easy to see that $\mathbb{F}_n/\tau$ is a geometrically ruled surface. By computing the self-intersection of the zero set one gets that $\mathbb{F}_n/\tau \approx \mathbb{F}_{2n}$; hence we have found a double branched cover $\pi \colon \mathbb{F}_n \to \mathbb{F}_{2n}$ branched along $S_{2n} \cup S_\infty$ — the compact analogue of $L_n \to L_n \otimes L_n$. The same argument shows that the geometrically ruled surface $\mathbb{G}_{n,g}$ over the base curve Σ_g of genus g can be expressed as the double branched cover of $\mathbb{G}_{2n,g}$ branched along the zero and infinity sections.

Definition 7.1.9. In the following, a section of $\mathbb{F}_n \to \mathbb{CP}^1$ originating from a holomorphic section of the line bundle $L_n \to \mathbb{CP}^1$ will be called an *affine section*, to distinguish it, for example, from the *infinity section* S_∞ of $\mathbb{F}_n$. Note that the zero section of the line bundle L_n gives rise to an affine section S_n; in fact, each affine section represents the same homology class $[S_n]$. In the literature the zero section is frequently denoted by S_0; since the following discussions involve various Hirzebruch surfaces appearing in the same argument, we prefer to record the self-intersection n in the index of the zero section.

Exercises 7.1.10. **(a)** For the above double branched cover $\pi\colon \mathbb{F}_n \to \mathbb{F}_{2n}$, prove that $\pi^*(a[S_{2n}]+b[F_{2n}]) = 2a[S_n]+b[F_n] \in H^2(\mathbb{F}_n;\mathbb{Z})$. (Here F_n (F_{2n}) denotes the fiber and S_n (S_{2n} resp.) an affine section of the corresponding Hirzebruch surfaces. We have identified the second homology with the second cohomology via Poincaré duality.)

(b)* Show that a smooth complex curve C with $[C] = [S_n] + \beta[F_n]$ in $H_2(\mathbb{F}_n;\mathbb{Z})$ ($\beta \in \mathbb{Z}$) is a section of the ruled surface $\mathbb{F}_n \to \mathbb{CP}^1$. (*Hint*: Follow the proof of Exercise 3.1.12(a), cf. Exercise 7.4.1(a) for $c_1(\mathbb{F}_n)$.) Hence a curve C is an affine section iff $[C] = [S_n]$.

(c)* Show that $\mathbb{F}_n$ is minimal iff $n \neq 1$. (Recall that $\mathbb{F}_1$ is the blow-up of the complex projective plane $\mathbb{CP}^2$.)

Before the next example of a double branched cover, we discuss a definition — for future generalization, it is formulated in the smooth setting.

Definition 7.1.11. Assume that $F_i \subset X_i$ are (real) 2-dimensional surfaces in the smooth 4-manifolds X_i, the surfaces F_i have equal genus and $[F_i]^2 = 0$ ($i = 1, 2$). We identify tubular neighborhoods νF_i of F_i with $F_i \times D^2$, and fix a diffeomorphism $f\colon F_1 \to F_2$. The *generalized fiber sum* $X_1 \#_F X_2$ of (X_1, F_1) and (X_2, F_2) is defined as $(X_1 - \nu F_1) \cup_\varphi (X_2 - \nu F_2)$, where φ is $f \times$ (complex conjugation) on the boundary $\partial(X_i - \nu F_i) = F_i \times S^1$.

The above operation generalizes the fiber sum operation we defined for elliptic surfaces in Section 3.1. Note that in Definition 7.1.11 we did not assume that X admits any fibration — in that sense the name of the operation is slightly misleading. The diffeomorphism type of $X_1 \#_F X_2$ might depend on the identifications chosen in the definition — for the sake of brevity we do not record those dependencies. Further generalization of the above construction will be given in Section 10.2.

Assume that the 4-manifold X contains a surface F with $[F]^2 = 0$, and take $B = F \times \{1\} \cup F \times \{-1\} \subset F \times D^2 \subset X$. The $\mathbb{Z}_2$-action on $X \#_F X$ (given by interchanging the two copies of $X - \nu F$ together with $\varphi = \mathrm{id}_F \times$ (complex conjugation) on $\partial \nu F = F \times S^1$) shows that $X \#_F X$ is the double branched cover of X along B — the image of the fixed point set of the above $\mathbb{Z}_2$-action. In particular, the elliptic surface $E(2n)$ can be given as the double branched cover of $E(n)$ branched along a pair of generic fibers. In the same way, we get a double branched cover $\mathbb{F}_{2n} \to \mathbb{F}_n$ branched along a pair of fibers, cf. also Exercise 6.3.5(a).

Exercise 7.1.12. Determine the double branched cover of the ruled surface $\mathbb{G}_{n,g}$ (over base curve Σ_g with genus g) branched along a pair of fibers.

7.2. Resolution of singularities

In our applications we often meet examples of branched covers involving branch loci with singular points — in these cases the corresponding (cyclic) cover X might admit singular points (cf. Lemma 7.1.5). In the following we will describe an algorithm which produces a smooth manifold X' out of X. In this section we will mainly follow [**HKK**] and give a detailed description only in the case of double covers; for the general case, see [**La1**] or [**Nm**]. Various different approaches to resolving singularities are avaliable in the literature — in the following we will emphasize the topological aspects of the theory.

Definition 7.2.1. The complex surface X' is a *resolution* (or *desingularization*) of the singular complex surface X, if X' is a smooth complex surface with a holomorphic map $\pi\colon X' \to X$ such that π is a biholomorphism away from the singular points $Sing(X)$, that is, the restriction $\pi\colon X' - \pi^{-1}(Sing(X)) \to X - Sing(X)$ is a biholomorphism.

By a famous theorem of Hironaka [**Hrn**] such a resolution X' always exists. X' is obviously not unique — for example, a blow-up of X' is a resolution as well. (There is, however, a unique *minimal* resolution of X [**BPV**].) In the following, we would like to describe a construction for X' when X is defined as a double branched cover of a smooth complex surface Y along a (singular) complex curve B. In our examples, singularities are always isolated, hence we can resolve them independently. Assume that X is singular at $P \in X$; by the description of X' we mean the description of the configuration of curves in $\pi^{-1}(P) \subset X'$. Since P is an isolated singularity, we can use the local description of the double branched cover X as $\{(x, y, z) \in \mathbb{C}^3 \mid z^2 = f(x, y)\}$, where the singular point is at the origin $(0, 0, 0)$.

Remark 7.2.2. We have always assumed (and will assume) that each component of the branch curve B has multiplicity one. If this assumption does not hold, the surface X (given by Construction 7.1.1) might be singular along a codimension 1 subspace, and in this latter case X is not even *normal*. (A singular complex manifold is normal if its singularities have codimension ≥ 2 and all holomorphic maps defined on the smooth part can be extended. There is a standard way, called *normalization*, for turning a singular complex manifold into a normal one; later we will briefly return to this point.) Normal surfaces admit only *isolated* singularities; any isolated singularity of a complex surface locally can be thought of as a branched cover of $\mathbb{C}^2$ branched along a singular curve [**La1**]. Hence, by finding a way for dealing with (isolated) singularities originating from branched covers, one can desingularize any normal surface.

The resolution of an isolated singularity of the above type involves two steps; we begin by discussing the first. We will resolve the branch curve $B = \{f(x, y) = 0\}$ by repeated blow-ups of the $\mathbb{C}^2$-chart of Y containing the singular point of B. As we have already seen (Theorem 2.3.2), after finitely many blow-ups we end up with a curve (the total transform B') having only normal crossing singularities. (A normal crossing singularity is one which can be modeled on $\{(x, y) \in \mathbb{C}^2 \mid x^l y^m = 0\}$. Note that even if B had multiplicity one, the total transform might contain components of various multiplicities.) Hence there is an integer $n \in \mathbb{N}$ and a map $\rho\colon \mathbb{C}^2\#n\overline{\mathbb{CP}^2} \to \mathbb{C}^2$ such that $\rho^{-1}(B)$ is a curve which has only normal crossings as singularities. We would like to record $\rho^{-1}(0)$ in a diagram and also denote where $\widetilde{B}$ (the proper transform of the branch locus B) intersects it — examples will be shown for that procedure shortly. Thus, as the result of the first step we expect a configuration of exceptional curves and the proper transform $\widetilde{B}$; note that the exceptional curves are all rational ($\approx \mathbb{CP}^1$). The only important topological data of these rational curves are their self-intersections; this number is -1 when the rational curve first appears in the process, and it drops by 1 every time we blow up a point of the curve at hand. At the end of this procedure we get a configuration of n rational curves — with a negative integer attached to each — and the proper transform $\widetilde{B}$ in $\mathbb{C}^2\#n\overline{\mathbb{CP}^2}$; these data completely describe $\rho^{-1}(B)$ as a subspace of $\mathbb{C}^2\#n\overline{\mathbb{CP}^2}$. Since the total transform $B' = \rho^{-1}(B)$ is defined as the zero set of the composition $\varphi = f \circ \rho$, the mutiplicities of the components of B' are defined. Note that $\varphi^{-1}(0) = \rho^{-1}(B)$, so the order of vanishing of φ along a component of $\rho^{-1}(0) \subset \rho^{-1}(B)$ is a positive integer; this integer will not change under further blow-ups. Thus, as a result of the first step we get a configuration of n rational curves and the proper transform $\widetilde{B}$ such that

1. all curves are smooth and intersect each other in normal crossings, with no triple intersections;
2. the $i^{\underline{th}}$ rational curve is decorated by a negative integer e_i (its self-intersection) and by a positive integer m_i (its multiplicity).

From the topological point of view the multiplicity m_i can be interpreted as the coefficient of the homology class $[E_i]$ of the $i^{\underline{th}}$ exceptional curve in the difference $[B'] - [\widetilde{B}] \in H_2(D^4\#n\overline{\mathbb{CP}^2}, \partial D^4; \mathbb{Z})$. (Recall that B' is the total transform, while $\widetilde{B}$ is the proper transform of the curve B, which is now viewed as a subset of $D^4 \subset \mathbb{C}^2$). The above interpretation of the multiplicities can be verified by assuming first that we performed a unique blow-up; by Lemma 7.1.4 we get $[B'] = [\widetilde{B}] + m[E]$. The general expression now follows by induction and shows that $[B'] = [\widetilde{B}] + \sum_j m_j[E_j]$.

The above configuration will be visualized by a diagram in which straight intervals symbolize the rational curves corresponding to the exceptional spheres (decorated by the two integers e_i and m_i), and the intervals intersect each other iff the corresponding curves do so. Fine curves indicate the proper transform $\widetilde{B}$ intersecting the configuration. Note that there are no integers attached to these curves — their multiplicities are (by assumption) equal to 1.

Next we present a way to carry out this program in practice. Recall that the blow-up of $(0,0) \in \mathbb{C}^2$ is $\tau = \{([u:v],x,y) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid xv = yu\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$. The manifold $\mathbb{CP}^1 \times \mathbb{C}^2$ is covered by the charts $V_1 = \{([u:1],x,y)\}$ and $V_2 = \{([1:v],x,y)\}$, so τ is covered by the charts $U_1 = \{([u:1],x,y) \mid x = yu\} = \{u, yu, y\} \approx \mathbb{C}^2$ and $U_2 = \{([1:v],x,y) \mid xv = y\} = \{v, x, xv\} \approx \mathbb{C}^2$. If (u,y) are the coordinates on U_1 and (v,x) on U_2, then the gluing map between the two charts can be seen from the above description: $x = yu$ and $v = \frac{1}{u}$ (where $u \neq 0$) or $y = xv$ and $u = \frac{1}{v}$ (where $v \neq 0$). The projection $\rho\colon \tau \to \mathbb{C}^2$ is simply the map $(u,y) \to (yu,y)$ on $U_1(u,y)$ and $(v,x) \to (x,xv)$ on $U_2(v,x)$. The exceptional sphere E is $\rho^{-1}(0)$, which means $\{yu = 0, y = 0\}$ on U_1 and $\{x = 0, xv = 0\}$ on U_2. This shows that E is $\{([u:1],0,0)\}$ in U_1 and $\{([1:v],0,0)\}$ in U_2, hence can be given by $\{y = 0\}$ and $\{x = 0\}$ respectively. Consequently, the blow-up of $B = \{f(x,y) = 0\} \subset \mathbb{C}^2$ can be described in the following way (where we change letters for convenience): Take coordinate charts (s,t), (s',t') with the identification $s' = \frac{1}{s}$, $t' = st$ $(s \neq 0)$ and maps $(s,t) \to (st,t)$, $(s',t') \to (t', s't')$ determining the projection ρ. By replacing x with st and y with t in the equation $f(x,y) = 0$ we get an equation giving $\rho^{-1}(B)$ in U_1; similarly the replacement of x with t' and y with $s't'$ gives $\rho^{-1}(B)$ in U_2. If we factor out t, the rest of the expression determines the proper transform $\widetilde{B}$ in U_1, and the power of t is the multiplicity of the exceptional sphere. Similarly, factoring out t', we get $\widetilde{B}$ in U_2, and (as the power of t') the same multiplicity of the exceptional curve. Now the Implicit Function Theorem helps us determine whether $\widetilde{B}$ is smooth or not; if not, then we have to repeat the above process. Even if all curves are smooth, we might need additional blow-ups to avoid nontransversal and triple intersections. By Theorem 2.3.2, after finitely many steps this process terminates and gives us the desired decorated configuration of curves. We illustrate the above process with some examples.

Examples 7.2.3. (a) $f(x,y) = x^2 + y^3$

By blowing up $\mathbb{C}^2$ at $(0,0)$, we get the equations $(st)^2 + t^3 = t^2(s^2 + t) = 0$ on U_1 and $(t')^2 + (s't')^3 = (t')^2(1 + (s')^3 t') = 0$ on U_2. Since the multiplicity of t (and t') is 2, the exceptional sphere E_1 has $m_1 = 2$ (and, of course $e_1 = -1$). The Implicit Function Theorem shows that the proper transform ($s^2 + t = 0$ on U_1 and $1 + (s')^3 t' = 0$ on U_2) is smooth. The exceptional sphere does not intersect $\widetilde{B}$ on U_2, and it is tangent to $\widetilde{B}$ on U_1, so we have

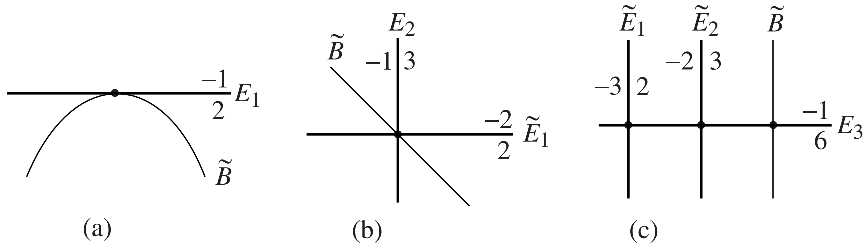


Figure 7.1. Blow-up of a cusp.

to apply additional blow-ups to get normal crossings — see Figure 7.1(a). Blow up $(0,0) \in U_1$; call the new coordinates (u,v), (u',v') with the usual transformation rules ($u' = \frac{1}{u}$, $v' = uv$). Now the inverse image of the curve $\{t^2(s^2+t) = 0\} \subset U_1$ in $U_1\#\overline{\mathbb{CP}^2}$ is (with (s,t) replaced by (uv,v) on $U_{1,1}$ and by $(v', u'v')$ on $U_{1,2}$):

$$v^2((uv)^2 + v) = v^3(u^2v + 1) = 0 \quad \text{and}$$

$$(u'v')^2((v')^2 + u'v') = (u')^2(v')^3(v' + u') = 0.$$

The proper transform $\widetilde{E}_1$ of the first exceptional sphere has self-intersection $e_1 = -2$ ($m_1 = 2$); the new exceptional sphere has $e_2 = -1$ and $m_2 = 3$ (by definition, the exponent of v or v'). On $U_{1,1}$ the proper transform $\widetilde{B}$ does not intersect E_2 (or $\widetilde{E}_1$); on $U_{1,2}$, however, all three curves pass through the origin (Figure 7.1(b)). To have normal crossing, we have to perform one more blow-up of $U_{1,2}$. The new coordinates will be denoted by (a,b), (a',b') (with u' replaced by ab, v' by b on $U_{1,2,1}$ and u' replaced by b', v' by $a'b'$ on $U_{1,2,2}$); these substitutions give the total transform on the new coordinate charts $U_{1,2,1}$ and $U_{1,2,2}$ ($U_{1,2,1} \cup U_{1,2,2} = U_{1,2}\#\overline{\mathbb{CP}^2}$). The curves are given as $(ab)^2b^3(ab+b) = a^2b^6(a+1) = 0$ on $U_{1,2,1}$ and $(b')^2(a'b')^3(a'b'+b') = (b')^6(a')^3(a'+1) = 0$ on $U_{1,2,2}$. Now we have three exceptional spheres: $\widetilde{E}_1$, $\widetilde{E}_2$ and E_3 with $e_1 = -3$, $m_1 = 2$; $e_2 = -2$, $m_2 = 3$; and $e_3 = -1$, $m_3 = 6$. It is easy to see that these three spheres and $\widetilde{B}$ form the required configuration. (See Figure 7.1(c).)

(b) $f(x,y) = x^3 + y^6$

We will follow the same recipe (and describe the steps only briefly): the new coordinates (s,t) and (s',t') transform B into $(st)^3 + t^6 = t^3(s^3 + t^3) = 0$ and $(t')^3 + (s't')^6 = (t')^3(1 + (s')^6(t')^3) = 0$, hence $e_1 = -1$ and $m_1 = 3$. Blowing up U_1 again (now (u,v), (u',v') are the new coordinates on the charts covering $U_1\#\overline{\mathbb{CP}^2}$) we have: $v^3((uv)^3 + v^3) = v^6(u^3+1) = 0$ and $(u'v')^3((v')^3 + (u'v')^3) = (u')^3(v')^6(1 + (u')^3) = 0$. This gives E_2 with $e_2 = -1$ and $m_2 = 6$; we have smooth curves with normal crossings, hence the process has terminated (with $e_1 = -2$ and $m_1 = 3$); see Figure 7.2.

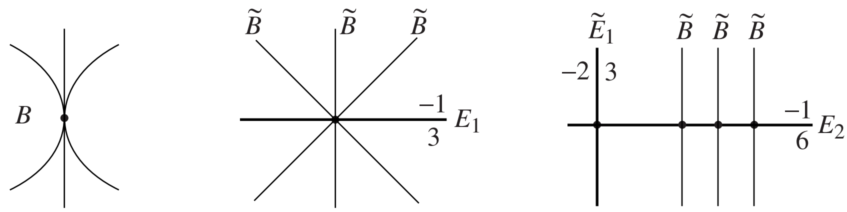


Figure 7.2. Blow-up of an infinitely close triple point.

Exercises 7.2.4. Go through the same computation for

(a) $f(x, y) = x^4 + y^8$,

(b)* and more generally, for $f(x, y) = x^n + y^{2n}$;

(c)* $f(x, y) = (x^2 + y^3)(x^3 + y^2)$;

(d) $f(x, y) = x^3 + y^{3(n+1)}$. (See [**HKK**] and [**La1**] for additional examples.)

Note that in some cases both charts U_1 and U_2 contain singularities of the total transform (cf. Exercise 7.2.4(c)). One can also describe the configuration of curves in $\rho^{-1}(0)$ by its *dual graph*. In that graph each vertex stands for a rational curve, and an edge means that the corresponding curves intersect each other. The vertices are decorated by the above described two integers e_i and m_i; an arrow points out from a vertex if a component of $\widetilde{B}$ intersects the corresponding rational curve. (Compare with plumbing diagrams, Example 4.6.2.) Note that the dual graph is always a tree. In our description we will mainly use the diagrams introduced in the text before Example 7.2.3, where rational curves are symbolized by intervals.

Exercises 7.2.5. Determine the diagram and the dual graph of the curves in $\rho^{-1}(0)$ corresponding to $f(x, y) =$

(a)* $x^2 + y^k$ $(k \geq 2)$;

(b) $x^2y + y^{k+1}$ $(k \geq 3)$;

(c)* $x^3 + y^4$;

(d) $x^3 + xy^3$;

(e) $x^3 + y^5$.

The singularities of curves listed in Exercise 7.2.5 are the so-called *simple (*or *Du Val) singularities.* The corresponding singularity in a double branched cover is usually called a *rational double point*, or a simple (canonical, Du Val, inessential) singularity. Later on it will be clear why are these singularities so simple (cf. Remark 7.2.13 and Theorem 7.4.4).

The following formula will be crucial in the subsequent text. Suppose that $\{E_i \mid i = 1, \ldots, n\}$ is the set of exceptional spheres in $\rho^{-1}(0)$, with squares $\{e_i \mid i = 1, \dots, n\}$ and multiplicities $\{m_i \mid i = 1, \dots, n\}$. The multiplicities of the components of $\widetilde{B}$ are 1. Let $|E_i \cap \widetilde{B}|$ denote the number of points where $\widetilde{B}$ intersects E_i.

Lemma 7.2.6. *For all i $(1 \le i \le n)$ we have*

$$m_i e_i + |E_i \cap \widetilde{B}| + \sum_{j \neq i,\, E_i \cap E_j \neq \emptyset} m_j = 0.$$

Proof. By Lemma 7.1.4 we have $[B] = [B'] = [\widetilde{B}] + \sum_j m_j [E_j] = 0$ in $H_2(D^4 \# m\overline{\mathbb{CP}^2}, \partial D^4; \mathbb{Z})$, and all curves (the exceptional spheres E_i and the proper transform $\widetilde{B}$) intersect transversally. Multiplying both sides by $[E_i]$, we get $[\widetilde{B}] \cdot [E_i] + \sum_j m_j [E_j] \cdot [E_i] = 0$; since by definition $[E_i] \cdot [E_i] = e_i$, $[E_j] \cdot [E_i]$ is 1 or 0 depending on whether E_j intersects E_i or not, and $[\widetilde{B}] \cdot [E_i] = |E_i \cap \widetilde{B}|$, we get $m_i e_i + |E_i \cap \widetilde{B}| + \sum_{j \neq i,\, E_i \cap E_j \neq \emptyset} m_j = 0$. □

To continue our discussion of resolving singularities of complex surfaces, consider the singular manifold $X = \{z^2 = f(x, y)\} \subset \mathbb{C}^3$. Suppose that the desingularization of the curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ has already been determined, so we have the decorated diagram describing $\rho^{-1}(0) \subset \mathbb{C}^2 \# n\overline{\mathbb{CP}^2}$. A recipe for determining the desingularization $\pi^{-1}(0) \subset X'$ is given in [**HKK**]. We will describe this algorithm and indicate why the process does what we expect. Before beginning the desingularization of X, first separate the curves with odd multiplicities in the diagram obtained by resolving the singularity of the curve $\{f(x, y) = 0\}$: If E_i and E_j in $\rho^{-1}(0)$ intersect each other and $m_i \equiv m_j \equiv 1 \pmod 2$, then blow up $E_i \cap E_j$. The new exceptional curve has even multiplicity (in fact, it is $m_i + m_j$) and $\widetilde{E}_i$ will be disjoint from $\widetilde{E}_j$, so after finitely many steps the new configuration will have the additional property that curves with odd multiplicities do not intersect each other. Obviously, Lemma 7.2.6 will hold for the configuration we have after performing the above additional blow-ups. (Recall that by our assumption the multiplicity of $\widetilde{B}$ is 1. In particular, in our final configuration any E_i with odd multiplicity m_i will be disjoint from $\widetilde{B}$.) The following two lemmas will be used in the algorithm for describing $\pi^{-1}(0) \subset X'$.

Lemma 7.2.7. *If the multiplicity m_i of E_i is odd, then e_i is even.*

Proof. By Lemma 7.2.6, the product $m_i e_i$ has the same parity as the sum $|E_i \cap \widetilde{B}| + \sum m_j$, where the summation runs over all $j \neq i$ with $E_i \cap E_j \neq \emptyset$. Since we separated odd multiplicities, all of these m_j's are even (in particular $E_i \cap \widetilde{B} = \emptyset$), hence the sum is even, as is $m_i e_i$. This, however, implies that e_i is even (since m_i is odd). □

Lemma 7.2.8. *If the multiplicity m_i of E_i is even, then the cardinality of the set $\mathcal{P}_i = \{P \in E_i \mid P \in E_i \cap E_j$ with m_j odd, or $P \in E_i \cap \widetilde{B}\}$ is even.*

Proof. Again, by applying the formula of Lemma 7.2.6 we see that the sum $|E_i \cap \widetilde{B}| + \sum m_j$ is even (since $m_i e_i$ is even by assumption), so the number of odd m_j's plus $|E_i \cap \widetilde{B}|$ is even. This proves the lemma. □

Now we are ready to describe the algorithm for determining the configuration of curves in $\pi^{-1}(0)$, where $\pi\colon X' \to X$ is the desingularization of the (singular) double branched cover X. Again, we use the local description of X and X' around the singular point P (which is assumed to be mapped to the origin). Every rational curve $E_i \subset \rho^{-1}(0)$ defines one or two curves (F_i or $\{F_i, F_i'\}$) in $\pi^{-1}(0)$ — according to the parity of the multiplicity m_i and the size of $\mathcal{P}_i$: If m_i is odd, then we take a rational curve F_i with self-intersection $\frac{1}{2}e_i$. If m_i is even, then we take a curve F_i which is the double branched cover of E_i branched along $\mathcal{P}_i$ (cf. Lemma 7.2.8). If $\mathcal{P}_i = \emptyset$, we take two copies of E_i for F_i and F_i' (the trivial double branched cover of E_i branched along $\mathcal{P}_i = \emptyset$). The self-intersection of F_i in the even m_i case is $2e_i$. (If $\mathcal{P}_i = \emptyset$, so we have two disjoint copies F_i and F_i', then each component has self-intersection e_i.) We have now described the curves (F_i or $\{F_i, F_i'\}$) in $\pi^{-1}(0)$ and also determined their self-intersections. We still have to describe how these curves intersect each other. If $E_i \cap E_j = \emptyset$, the corresponding F's will be disjoint as well. If E_i intersects E_j, and m_j is odd (hence m_i is even), then F_i has one component (since $\mathcal{P}_i \neq \emptyset$), and it will intersect F_j in a single point. If both E_i and E_j have even multiplicities, then the correponding F's will meet each other in two points. There are various ways for this to happen:

- if $\mathcal{P}_i \neq \emptyset$ and $\mathcal{P}_j = \emptyset$, then F_i will intersect F_j and F_j' each in one point;
- if $\mathcal{P}_i \neq \emptyset$ and $\mathcal{P}_j \neq \emptyset$, then F_i and F_j will intersect each other transversely in two points;
- finally, if $\mathcal{P}_i = \emptyset$ and $\mathcal{P}_j = \emptyset$, then F_i will intersect F_j and F_i' will intersect F_j' each in a point, otherwise these curves will be disjoint.

Hence we have determined the configuration of curves $\{F_i\}$ in $\pi^{-1}(0) \subset X'$ (where for some i we have two curves F_i and F_i'). Note, however, that F_i might be a curve different from $\mathbb{CP}^1$ — if m_i is even and $|\mathcal{P}_i| > 2$, then F_i will have genus at least 1. Note also that two curves may intersect each other transversely twice (cf. Exercise 7.2.12(e)). We will visualize the configuration of curves in $\pi^{-1}(0)$ using a convention similar to the one we used in the description of $\rho^{-1}(0)$. Every curve will be symbolized by an interval and these intervals will intersect each other according to the intersection pattern of the curves F_i. The intervals are decorated by two integers; the negative integer is the self-intersection of the curve F_i, while the positive one gives the genus of it. This latter number will not be written if $F_i \approx \mathbb{CP}^1$ (hence $g(F_i) = 0$). Note that we no longer need to keep track of multiplicities or the proper transform $\widetilde{B}$. We can also describe $\pi^{-1}(0)$ by the dual graph (cf. the text after Exercises 7.2.4); the graph will be a plumbing diagram decorated by the self-intersections and the genera of the curves F_i (as in Example 4.6.2). If the configuration of curves in $\pi^{-1}(0)$ contains a rational curve with self-intersection -1, then that component can be contracted to a point (cf. Proposition 2.2.10). Note that the contraction will change the intersection pattern (and the self-intersections) of the curves meeting that rational curve.

Warning 7.2.9. Although the above algorithm produces a configuration of curves in $\pi^{-1}(0)$ with the properties that every curve is smooth, there are only transverse intersections, and we have no triple intersections, these properties will be lost after contracting the rational -1-curves — when passing from the *canonical* desingularization to the *minimal* desingularization. Hence the minimal desingularization cannot be described as easily as the canonical one; because of the presence of possibly singular curves we might need a more detailed description of the diagram. Examples involving singular curves and nontransversal intersections in $\pi^{-1}(0)$ for the minimal resolutions are presented at the end of the section. (See Exercises 7.2.15.)

An easy argument shows that the above algorithm for constructing $\pi^{-1}(0)$ does exactly what we expect. We turn back to our global picture of the double branched cover $\phi\colon X \to Y$ with branch locus B. (As always, we assume that X is given by Construction 7.1.1 using Y and B as inputs.) If $P_1, \dots, P_k$ are the singular points of $B \subset Y$, then after repeated blow-ups we get a collection of curves $B' = \widetilde{B} \bigcup_{i=1}^{k} \cup_{j=1}^{n_i} E_{ij} \subset Y \# m\overline{\mathbb{CP}^2}$ — the total transform of B. (Recall that these curves are all smooth and have only transverse intersections, there are no triple intersections, and curves with odd multiplicities are disjoint.) A careful analysis of the algorithm shows that all we did was remove each E_{ij} with even multiplicity from the branch locus B', change all odd multiplicities to one, and then take the double branched cover (using Construction 7.1.1) along this new — and smooth — branch curve B''. Note that B'' is smooth because in B' components with odd multiplicities are disjoint. The homology class $[B]$ was divisible by 2, hence so is $[B']$. By the above description of B'' one can see that 2 divides $[B'] - [B'']$ in $H_2(Y \# m\overline{\mathbb{CP}^2}; \mathbb{Z})$, consequently $[B'']$ is also divisible by 2. It is now easy to see that the double branched cover X' of $Y \# m\overline{\mathbb{CP}^2}$ along B' is a complex surface satisfying Definition 7.2.1; consequently the algorithm described above gives a desingularization of the (singular) double branched cover X.

Remark 7.2.10. One can take the branched cover along B' as well, but then (as we already mentioned in Remark 7.2.2) the resulting surface $\widetilde{X}$ will not even be normal — it will have codimension-1 singularities corresponding to the curves in B' with multiplicities greater than 1. A general method of algebraic geometry (called normalization) produces a normal surface out of $\widetilde{X}$. Deleting curves with even multiplicity from B' (and reducing odd multiplicities to 1) does exactly what normalization would do with the (singular) branched cover $\widetilde{X}$ branched along B'. (This can be checked by applying Construction 7.1.1 to a local model $Y = \mathbb{C}^2$, $B = \{x^{2n} = 0\}$ or $Y = \mathbb{C}^2$, $B = \{x^{2n+1} = 0\}$. Details of the complete argument are left to the reader.)

In the following, we demonstrate this algorithm in practice. We do this by desingularizing $\{z^2 = x^2 + y^3\}$ and $\{z^2 = x^3 + y^6\}$ (since the first step of the desingularization process has already been performed on the curves corresponding to these double branched covers, cf. Examples 7.2.3).

Examples 7.2.11. (a) $\{z^2 = x^2 + y^3\}$

The starting configuration involves 4 curves, $\widetilde{B}$, $\widetilde{E}_1$, $\widetilde{E}_2$ and E_3, see Figure 7.1(c). It is clear that there are no curves with odd multiplicities intersecting each other, hence we do not need additional blow-ups. $\widetilde{E}_2$ gives the sphere F_2 with square -1, while E_3 gives the double branched cover F_3 branched along two points (which is also a sphere) with square -2. $\widetilde{E}_1$ has even multiplicity and does not intersect any curve with odd multiplicity (or $\widetilde{B}$), so $\mathcal{P}_1 = \emptyset$, hence it gives two rational curves, F_1 and F_1', each with square -3, and both intersect F_3 in a point; F_2 intersects F_3 in a point as well. Hence $\pi^{-1}(0)$ consists of four spheres plumbed according to the diagram given by Figure 7.3(a). Blowing down F_2 (which is a sphere of square -1) we are left with three spheres. One of them (the curve defined by F_3) has square -1, so we can contract it. The resulting configuration consists of two intersecting rational curves, each of square -2. This is the final picture of the minimal resolution we were looking for; for the dual graph see Figure 7.3(b). (Cf. also Exercise 7.2.12(a) with $k = 3$.)

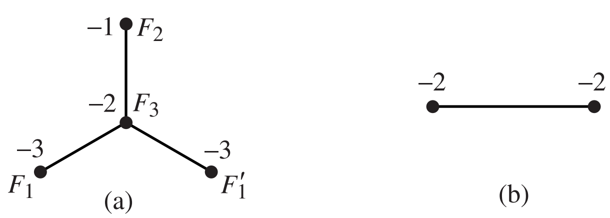


Figure 7.3. Canonical and minimal resolution of the cusp singularity.

(b) $\{z^2 = x^3 + y^6\}$

In this case we have two exceptional curves $\widetilde{E}_1$ and E_2 intersecting each other; moreover $\widetilde{B}$ intersects E_2 in three points (see Figure 7.2). Now F_2 is the branched cover of E_2 branched along 4 points — hence it is a torus; the self-intersection of F_2 equals -2. $\widetilde{E}_1$ gives the rational curve F_1 with square -1. After contracting F_1, we find that $\pi^{-1}(0)$ consists of a single torus with square -1. Note, however, that since this is not a rational curve, it cannot be contracted. Comparison with Exercises 7.2.5(c) and (e) shows that the above singularity is the "simplest" nonsimple one. We will make use of the resolution given here in Section 7.4.

Exercises 7.2.12. (a)* Use the solutions of Exercises 7.2.5(a) and (c) to resolve $z^2 = x^2 + y^k$ and $z^2 = x^3 + y^4$.

(b) Using the result of Exercise 7.2.5(e), resolve the singularity $z^2 = x^3 + y^5$.

(c)* More generally, resolve the singularity $z^2 = x^{2n-1} + y^{4n-3}$. (Note that Exercise 7.2.12(b) is the special case of this for which $n = 2$.)

(d) Resolve the singularity $z^2 = x^n + y^{2n}$. (*Hint*: Use the solution of Exercise 7.2.4(b).)

(e) Resolve the singularity $z^2 = (x^2+y^3)(x^2+y^2)$. (This singularity provides an example with curves intersecting transversally twice.)

Remark 7.2.13. There are two obvious methods for removing an isolated singularity $V = \{(x, y, z) \in \mathbb{C}^3 \mid g(x, y, z) = 0\}$: We can resolve it (using the generalization of the above algorithm) or we can *deform* it as $V_{def} = \{(x, y, z) \in \mathbb{C}^3 \mid g(x, y, z) = \varepsilon\}$ for some $\varepsilon \in \mathbb{C}$ (with $|\varepsilon| \ll 1$). The two ways usually give different smooth manifolds. It turns out that the resolution coincides with the deformation iff V can be defined by a polynomial of the form $g(x, y, z) = f(x, y) + z^2$, where f is one of the polynomials listed in Exercise 7.2.5 — so V has a *rational double point*. This can be seen from the fact that the intersection form of V_{def} is negative definite iff V is a rational double point [**Du**], while the resolution always gives a smooth manifold with negative definite intersection form [**La1**]. This rules out all other singularities; on the other hand, if V is a rational double point then the resolution and the deformation coincide [**HKK**]. (For the case of $z^2 = x^3 + y^5$, also see Exercises 7.2.12(b) and 8.3.4(c).) For more about rational double points see [**Du**].

Exercise 7.2.14. Prove that the minimal resolution V_{res} and the deformation V_{def} are diffeomorphic if $V = \{(x, y, z) \in \mathbb{C}^3 \mid z^2 = x^2 + y^k\}$. (*Hint*: Compare the solution of Exercise 7.2.12(a) with that of Exercise 6.3.11 for the Milnor fiber $M_c(2, 2, k)$. See also Proposition 7.3.13 for the case $k = 2$.)

An easy example of different V_{res} and V_{def} is provided by the singularity $z^2 = x^3 + y^6$. By Example 7.2.11(b) V_{res} is the disk bundle of Chern number -1 over the torus (hence $Q_{V_{res}}$ is odd), while the Milnor fiber $V_{def} = M(2, 3, 6)$ is an even (and simply connected) manifold.

An algorithm similar to the one described above works if one wants to desingularize triple branched covers. The crucial point is that by additional blow-ups we can always separate exceptional spheres in $\rho^{-1}(0)$ with multiplicity not divisible by three. (See [**HKK**], for example.) In general, however, the algorithm described for double covers will not work for larger d — for example, when $d = 5$ one can find cases where exceptional curves with multiplicities not divisible by 5 cannot be separated by additional blow-ups. Consequently, the general case needs more work (for example, the detailed analysis of the normalization procedure, cf. Remark 7.2.10), but since we will not use desingularizations of manifolds other than double branched covers, we will omit the discussion of the general case here. (For a detailed description see [**La1**] or [**Nm**].) We close this section by giving some examples of resolutions where the minimal resolution contains singular curves, triple and nontransversal intersections.

Exercises 7.2.15. Using the algorithm described above, resolve the following singularities:

(a)* $z^2 = x^3 + y^7$;

(b)* $z^2 = x^3 + xy^5$;

(c)* $z^2 = x^3 + y^8$.

7.3. Elliptic surfaces revisited

In the next two sections we will make use of the branched cover construction and the desingularization algorithm described in Sections 7.1 and 7.2. This section is devoted to showing that the different constructions of elliptic surfaces described in Sections 3.1 and 3.2 give diffeomorphic 4-manifolds, cf. Theorem 3.2.9. We will return to the discussion of elliptic surfaces again in Chapter 8. (For the topology of elliptic surfaces, see also [**FM1**], [**HKK**].) Recall that we defined

- $E(1)$ as the 9-fold blow-up of $\mathbb{CP}^2$,
- $E(n)$ as the n-fold fiber sum $\#_f nE(1)$,
- $X(n)$ as the resolution of the quotient $\Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1)$ (for σ_i as defined in Section 3.3), and
- $V(n) \subset \mathbb{CP}^1 \times \mathbb{CP}^2$ as the zero set of a bihomogeneous polynomial of bidegree $(n, 3)$.

We add one more construction to the list, and then begin to prove that the corresponding manifolds are diffeomorphic (Theorem 7.3.3). The complex curve $B_{2,n} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is defined as the union of 4 horizontal and $2n$ vertical spheres in the direct product $\mathbb{CP}^1 \times \mathbb{CP}^1$; more precisely, fix distinct points $p_1, \ldots, p_4, q_1, \ldots, q_{2n}$ in $\mathbb{CP}^1$ and consider the complex curve $B_{2,n} =$

$\cup_{i=1}^{4}(\mathbb{CP}^1 \times \{p_i\}) \bigcup \cup_{j=1}^{2n}(\{q_j\} \times \mathbb{CP}^1) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. Clearly $B_{2,n}$ is of class $(4, 2n)$ in $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z})$ (that is, $[B_{2,n}] = 4\alpha + 2n\beta$ with respect to the obvious generators $\alpha = [\mathbb{CP}^1 \times \{p\}]$ and $\beta = [\{q\} \times \mathbb{CP}^1]$), and $B_{2,n}$ has only normal crossings as singularities.

Definition 7.3.1. We define the complex surface $D'(n)$ as the desingularization of the double branched cover $D(2,n)$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$ along $B_{2,n}$.

Note that the (singular) complex surface $D(2,n)$ admits an elliptic fibration with $2n$ singular fibers (corresponding to the fiber components $\{q_j\} \times \mathbb{CP}^1$ of the branch curve $B_{2,n}$): Compose the branched covering map $D(2,n) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ with the projection $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ to the first factor. This implies that $D'(n)$ also admits an elliptic fibration.

Exercise 7.3.2. Determine the topology of a singular fiber of the elliptic fibration $D'(n) \to \mathbb{CP}^1$ defined above. (*Hint*: Compare with the proof of Proposition 7.3.7.)

The next few lemmas and propositions will give a proof of the following theorem:

Theorem 7.3.3. *For fixed $n \geq 1$, the smooth 4-manifolds $E(n), V(n), D'(n)$ and $X(n)$ are all diffeomorphic.*

Lemma 7.3.4. *The 4-manifold $X(n)$ is diffeomorphic to $D'(n)$.*

Proof. By taking the hyperelliptic actions σ_{n-1} and σ_1 on Σ_{n-1} and $\Sigma_1 = T^2$ respectively, we obtain an action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\Sigma_{n-1} \times \Sigma_1$. It is easy to see that the quotient of $\Sigma_{n-1} \times T^2$ by this action is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ (since $\Sigma_{n-1}/\sigma_{n-1} \approx \mathbb{CP}^1$). Hence $\Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1)$ inherits a $\mathbb{Z}_2$-action, and the quotient of $\Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1)$ by this further $\mathbb{Z}_2$-action is $\mathbb{CP}^1 \times \mathbb{CP}^1$. Consequently, $\Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is a double branched cover with branch curve equal to $B_{2,n}$ (for appropriate choices of $p_1, \ldots, p_4$ and $q_1, \ldots, q_{2n}$). This implies that $D(2,n) = \Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1)$. Since $X(n)$ is the desingularization of $\Sigma_{n-1} \times T^2/(\sigma_{n-1} \times \sigma_1)$ and $D'(n)$ is the desingularization of $D(2,n)$, the lemma follows. □

Remark 7.3.5. Following the same pattern, one can easily prove that the manifold $X(n,m)$ (cf. Section 3.2) can also be given as the resolution of a branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The corresponding branch curve in this case turns out to be $B_{n,m} = \cup_{i=1}^{2n}(\mathbb{CP}^1 \times \{p_i\}) \bigcup \cup_{j=1}^{2m}(\{q_j\} \times \mathbb{CP}^1)$ (where $p_1, \ldots, p_{2n}, q_1, \ldots, q_{2m}$ are distinct points of $\mathbb{CP}^1$). Recall that $X(n)$ is by definition the same as $X(n,2) \approx X(2,n)$, cf. Remark 3.2.6(b).

It is easy to prove that the fiber sum of the (singular) elliptic surfaces $D(2,n)$ and $D(2,m)$ (cf. Definition 7.3.1) is $D(2, n+m)$; take a disk D

in $\mathbb{CP}^1$ containing $q_1, \dots, q_{2n}$, with the remaining points $q_{2n+1}, \dots, q_{2n+2m}$ contained in the complementary disk D'. If $\pi\colon D(2, n+m) \to \mathbb{CP}^1$ is the map giving the elliptic fibration on $D(2, n+m)$ (cf. the text after Definition 7.3.1), then $\pi^{-1}(D)$ is simply $D(2,n) - \nu F$ and $\pi^{-1}(D') = D(2,m) - \nu F'$ (where F and F' are regular fibers of the corresponding fibrations). This description shows that $D(2,n)\#_f D(2,m) = D(2, n+m)$, and implies that the fiber sum of $D'(n)$ and $D'(m)$ is $D'(n+m)$. In particular, $D'(n)$ is the fiber sum of n copies of $D'(1)$.

Remark 7.3.6. The above observation can be extended to the manifolds $X(n,m)$ of Section 3.2; one only needs to replace the fiber sum operation by the generalized fiber sum given by Definition 7.1.11 (since the fibers of the fibrations on $X(n,m)$ are not necessarily tori). Consequently, the generalized fiber sum $X(n_1,m)\#_F X(n_2,m)$ is $X(n_1+n_2,m)$, and $X(n,m_1)\#_F X(n,m_2) \approx X(n, m_1+m_2)$. (In the first case we summed along a curve of genus $m-1$, while in the second case along a curve of genus $n-1$.) Note that $X(n)$ (as it is by definition the same as $X(n,2) \approx X(2,n)$) admits a fibration with complex curves of genus $n-1$. We also see that $X(2,n)$ is the generalized fiber sum of two copies of $X(1,n)$.

Proposition 7.3.7. *The manifold $D'(1)$ is diffeomorphic to the complex surface $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, and hence to $E(1)$.*

Proof. Note that the singular surface $D(2,1)$ (of Definition 7.3.1) admits a fibration over $\mathbb{CP}^1$ as

$$D(2,1) \to \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{CP}^1,$$

where $\mathrm{pr}_2\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ is the projection to the second factor. The generic fiber of this map $\phi\colon D(2,1) \to \mathbb{CP}^1$ is $\mathbb{CP}^1$ (since the generic fiber of pr_2 meets the branch locus $B_{2,1}$ in two points). There are 4 singular fibers originating from the fibers $\mathbb{CP}^1 \times \{p_i\}$ ($i = 1, \dots, 4$) of pr_2; each of these fibers contains two of the eight singular points of $D(2,1)$. When desingularizing $D(2,1)$, we blow up the eight singular points of $B_{2,1}$ and take the double branched cover along the smooth proper transform $\widetilde{B}_{2,1}$. (Compare this with the algorithm given in Section 7.2.) Hence it is easy to see that the desingularized manifold $D'(1)$ fibers over $\mathbb{CP}^1$ with generic fiber a rational curve ($\approx \mathbb{CP}^1$), and each of the four singular fibers consists of three spheres (by the desingularization algorithm). Two of these spheres are disjoint and have square -2, the third has square -1, and the latter intersects each of the first two curves transversally once (see Figure 7.4). Blowing down the four -1-curves of the singular fibers, we get a complex surface still admitting a fibration over $\mathbb{CP}^1$. Now the four singular fibers each consist of two transversally intersecting -1-spheres. Note that 4 of these curves intersect the inverse image of the subset $\{q_1\} \times \mathbb{CP}^1 \subset B_{2,1}$ of

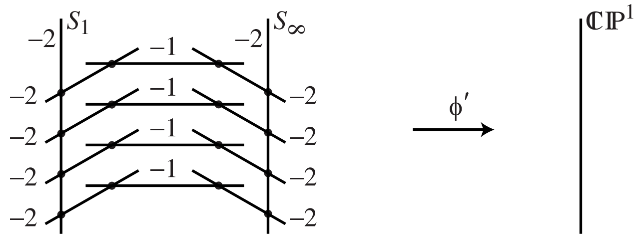


Figure 7.4. Singular fibers in the fibration $\phi' \colon D'(1) \to \mathbb{CP}^1$.

the branch locus (which will be denoted by S_1), and the remaining 4 intersect the inverse image of $\{q_2\} \times \mathbb{CP}^1 \subset B_{2,1}$ (denoted by S_∞). The square of $\{q_1\} \times \mathbb{CP}^1$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is 0; after the 4 blow-ups and the double branched cover, the self-intersection of the resulting curve S_1 (and similarly of S_∞) becomes -2. Now blowing down 3 of the -1-curves intersecting S_1 and one intersecting S_∞, we get a $\mathbb{CP}^1$-fibration with a section (originating from S_1) of square 1. (Note that S_∞ gives rise to another section, with square -1.) Consequently, blowing down $D'(1)$ eight times gives the Hirzebruch surface $\mathbb{F}_1$, implying that $D'(1) \approx \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \approx E(1)$. □

Exercises 7.3.8. (a)* Prove that a generic fiber of the elliptic fibration on $D'(1)$ corresponds to the 9-fold blow-up of a cubic curve in $\mathbb{CP}^2$, hence to a generic fiber of $E(1)$.

(b)* Prove that for $n \geq 1$ the 4-manifold $X(n,1)$ is diffeomorphic to the blown-up projective plane $\mathbb{CP}^2 \# (4n+1)\overline{\mathbb{CP}^2}$.

Now Proposition 7.3.7 and the solution of Exercise 7.3.8(a) prove that $E(n) \approx \#_f nE(1) \approx \#_f nD'(1) \approx D'(n)$. Consequently, in order to prove Theorem 7.3.3, we only need to show that $E(n) \approx V(n)$.

Remarks 7.3.9. (a) Note that although $E(1)$ and $E(1)_p$ are diffeomorphic for any p (cf. Theorem 8.3.11), the fiber sums $E(1)\#_f E(1) = E(2)$ and $E(1)_p \#_f E(1)_p = E(2)_{p,p}$ are obviously different for $p > 1$; for example, the fundamental groups are nonisomorphic. For this reason, showing that $D'(n) \approx E(n)$ requires the diffeomorphism between $D'(1)$ and $E(1)$ to map the generic fibers into each other (Exercise 7.3.8(a)). (The diffeomorphism $E(1) \approx E(1)_p$ does not map generic fiber to generic fiber.) In contrast, the fiber sum is independent of the choice of the gluing map on $\partial\nu F$ (Theorem 8.3.11).

(b) Note that by Remark 7.3.6, the manifold $X(n) = X(2,n)$ is the generalized fiber sum of two copies of $X(1,n)$. Since $X(n) \approx D'(n) \approx E(n)$, the solution of Exercise 7.3.8(b) implies that $E(n)$ is, in fact, the generalized fiber sum of two copies of $\mathbb{CP}^2 \# (4n+1)\overline{\mathbb{CP}^2}$ along a surface of genus

$n-1$. Note, in particular, that we get a decomposition of $E(1) \approx X(2,1)$ as $X(1,1)\#_f X(1,1) = (\mathbb{CP}^2\#5\overline{\mathbb{CP}^2})\#_f(\mathbb{CP}^2\#5\overline{\mathbb{CP}^2})$, where the fiber sum is taken along a sphere. This is, however, not very surprising: $E(1) \approx \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, as a blow-up of a Hirzebruch surface, admits a $\mathbb{CP}^1$-fibration with 8 singular fibers, and the above fiber sum decomposes this fibration (cf. also the proof of Proposition 7.3.7).

Lemma 7.3.10. *The elliptic surface $V(1) \subset \mathbb{CP}^1 \times \mathbb{CP}^2$ is diffeomorphic to $E(1)$. We can choose this diffeomorphism in such a way that fibers of the elliptic fibration $V(1) \to \mathbb{CP}^1$ map to fibers of $E(1) \to \mathbb{CP}^1$.*

Proof. Recall that $V(1)$ has been defined as the zero set of a generic bihomogeneous polynomial P of bidegree (1,3) in $\mathbb{CP}^1 \times \mathbb{CP}^2$. This implies that P has the form $P(x,y;z_0,z_1,z_2) = xp_0(z_0,z_1,z_2) + yp_1(z_0,z_1,z_2)$ for generic homogeneous cubic polynomials p_0 and p_1. Keeping this in mind, we can easily describe the projection of $V(1)$ to the second factor of the ambient space $\mathbb{CP}^1 \times \mathbb{CP}^2$: Fix a point $[z_0 : z_1 : z_2] \in \mathbb{CP}^2$; if $p_0(z_0,z_1,z_2) \neq 0$ or $p_1(z_0,z_1,z_2) \neq 0$, there is a unique solution of $xp_0(z_0,z_1,z_2) + yp_1(z_0,z_1,z_2) = 0$. If $p_0(z_0,z_1,z_2) = p_1(z_0,z_1,z_2) = 0$, all pairs $[x : y] \in \mathbb{CP}^1$ solve the above equation. Hence the projection $\mathrm{pr}_2\colon V(1) \to \mathbb{CP}^2$ is 1-1 except at the points of $\mathbb{CP}^2$ where $p_0 = p_1 = 0$; by blowing up these 9 points we get a diffeomorphism $\widetilde{\mathrm{pr}}_2\colon V(1) \to \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, and this proves the lemma (cf. also Remark 7.3.26). □

In order to prove that $E(n) \approx V(n)$ for $n > 1$ we need one more observation. The fiber sum of n copies of $E(1)$ can be described in the following way. Define the map $\varphi_n\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by the formula $z \mapsto z^n$, and take the pullback of the elliptic fibration $E(1) \to \mathbb{CP}^1$ via this map φ_n. Assume that over 0 and $\infty \in \mathbb{CP}^1$ the elliptic surface $E(1)$ has regular fibers, and denote the pullback by Y_{φ_n}.

Claim 7.3.11. *The manifold Y_{φ_n} defined above is the n-fold fiber sum of $E(1)$, hence it is $E(n)$.*

Proof. Assume that all the singular fibers of $\pi\colon E(1) \to \mathbb{CP}^1$ are mapped into a disk $D \subset \mathbb{CP}^1$ not containing 0 or ∞. Then $E(1)$ decomposes as $\pi^{-1}(D) \cup \pi^{-1}(\mathbb{CP}^1 - D)$. Note that by our assumption $\pi^{-1}(\mathbb{CP}^1 - D)$ is diffeomorphic to $D^2 \times T^2 \approx \nu F$ and $\pi^{-1}(D) \approx E(1) - \nu F$ (where F is the generic fiber of the elliptic fibration on $E(1)$). The inverse image $\varphi_n^{-1}(D) \subset \mathbb{CP}^1$ consists of n disjoint copies of the disk D, so the pullback of $\pi^{-1}(\mathbb{CP}^1 - D)$ via φ_n is diffeomorphic to $(\mathbb{CP}^1 - \{n \text{ disjoint copies of } D\}) \times T^2$. The pullback of $\pi^{-1}(D)$ via φ_n consists of n disjoint copies of itself, each diffeomorphic to $E(1) - \nu F$. By gluing the pieces together we complete the proof. Note that this claim, in fact, shows that $E(n)$ is the cyclic n-fold branched cover of $E(1)$

branched along $\pi^{-1}(\{0,\infty\})$. (The fundamental group of $E(1)-\pi^{-1}(\{0,\infty\})$ is $\mathbb{Z}$, hence the cyclic cover is determined by the branch locus and n.) □

Theorem 7.3.12. *The complex surface $V(n)$ is diffeomorphic to $E(n)$.*

Proof. Use the bihomogeneous polynomial $P_n = x^n p_0 + y^n p_1$ to define $V(n)$ in $\mathbb{CP}^1 \times \mathbb{CP}^2$. The result is the same as one gets by pulling back $V(1) \subset \mathbb{CP}^1 \times \mathbb{CP}^2$ via $\varphi_n\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ (defined above), that is, by taking the inverse image of $V(1)$ (given by $P = xp_0 + yp_1$) via $\tilde{\varphi}_n = \varphi_n \times \mathrm{id}_{\mathbb{CP}^2}\colon \mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^1 \times \mathbb{CP}^2$. As in the proof of Claim 7.3.11, this description shows that, in fact, $V(n)$ is the n-fold cyclic branched cover of $V(1)$ along two fibers. We conclude that $V(n)$ and $E(n)$ are diffeomorphic. □

The proof of Theorem 7.3.12 now completes the proof of Theorem 7.3.3. □

As a consequence of Theorem 7.3.3, we have found a description of the elliptic surface $E(n)$ as the resolution of a double branched cover of the Hirzebruch surface $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ (as $D'(n)$). Note that the branch locus $B_{2,n}$ is a curve with singular points; hence we have to desingularize the double branched cover. The singularities of $B_{2,n}$ are the "mildest possible", since these are all normal crossings modeled on $\{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$. Besides resolving, however, there is a different way to remove a singularity; we can smooth out the branch curve as we described in Section 2.1 and then take the branched cover along the new (and smooth) curve $\widetilde{B}_{2,n}$ — this alternative way corresponds to deformation as mentioned in Remark 7.2.13. Note that in this example we can do a complex deformation globally, by writing $B_{2,n}$ in the form $\{p_1(x)p_2(y) = 0\}$ and setting $\widetilde{B}_{2,n} = \{p_1(x)p_2(y) = \varepsilon\}$. Since the singularities of $B_{2,n}$ are transverse intersections of curves, deformation and resolution give diffeomorphic results (cf. Remark 7.2.13). Consequently, the resolutions of the singular double branched covers appearing in this section can be given as double branched covers along the smooth curves obtained by desingularizing in the manner discussed in Section 2.1. To make our presentation complete, we prove the statement of Remark 7.2.13 in the case we used in the above argument, namely when $f(x,y) = xy$ (which is the same as $g(u,v) = u^2 + v^2$); cf. also Exercise 7.2.14.

Proposition 7.3.13. *The desingularization of the double branched cover of the 4-ball branched along a normal crossing is diffeomorphic rel boundary to the double branched cover of the same 4-ball branched along the resolution of the normal crossing as discussed in Section 2.1.*

Proof. The latter branched cover is the disk bundle X over S^2 with Euler number -2 (Exercise 6.3.9(b)). The desingularization of the former is obtained by first blowing up to obtain a negative Hopf disk bundle, with branch locus becoming a pair of fibers, and then taking the branched cover to obtain the same disk bundle X as before (cf. Exercise 6.3.5(a)). In each case, either component of the branch locus in ∂D^4 lifts to a fiber of the circle bundle ∂X (cf. Exercise 6.3.9(a)); standard 3-manifold techniques now show that we can assume the diffeomorphism between the two 4-manifolds is rel ∂. (Actually, it is known that *any* orientation-preserving self-diffeomorphism of $\partial X = \mathbb{RP}^3$ is isotopic to the identity.) □

Consequently, we have found a presentation of the elliptic surface $E(n)$ as the double branched cover of $\mathbb{F}_0$ along a smooth complex curve $\widetilde{B}_{2,n}$ (which is a slight deformation of $B_{2,n}$). We have already seen (cf. Example 5.1.3(a)) that $\mathbb{F}_{2k}$ is diffeomorphic to $\mathbb{F}_0$; hence $E(n)$ can be thought of (up to diffeomorphism) as a double branched cover of $\mathbb{F}_{2k}$.

Remark 7.3.14. The same reasoning shows that the manifolds $X(n,m)$ are double branched covers of the Hirzebruch surface $\mathbb{F}_0$ along smooth complex curves. Furthermore, since $\mathbb{F}_0 \approx \mathbb{F}_{2k}$, we can think of $X(n,m)$ as the double branched cover of $\mathbb{F}_{2k}$. Since the diffeomorphism $\mathbb{F}_0 \approx \mathbb{F}_{2k}$ does not respect the complex structure, the branch locus of the double branched cover $X(n,m) \to \mathbb{F}_{2k}$ is not necessarily a complex curve — it is a smooth submanifold of $\mathbb{F}_{2k}$.

Exercise 7.3.15. Determine the homology class of the submanifold corresponding to $B_{2,n} \subset \mathbb{F}_0$ under the diffeomorphism $\mathbb{F}_{2n} \approx \mathbb{F}_0$ provided by Example 5.1.3(a), in terms of the fiber F_{2n} and the affine section S_{2n} of $\mathbb{F}_{2n}$. (*Answer*: $[F_0]$ maps to $[F_{2n}]$ and $[S_0]+n[F_0]$ maps to $[S_{2n}]$, so $[B_{2,n}]$ (which is equal to $4[S_0]+2n[F_0]$) maps to $4[S_{2n}]-2n[F_{2n}]$.)

One further construction of elliptic surfaces illuminates the topology of $E(n)$ and generalizes to produce interesting surfaces of general type. Recall that $\mathbb{F}_{2n}$ admits an infinity section S_∞ with square $-2n$; as we have seen, $[S_\infty] = [S_{2n}] - 2n[F_{2n}]$. Consequently $[B_{2,n}] = 3[S_{2n}] + [S_\infty]$; hence by taking three affine sections S, S', S'' and the infinity section S_∞ we get a complex curve C_n in $\mathbb{F}_{2n}$ homologous to $B_{2,n}$ (and to $\widetilde{B}_{2,n}$). Note that $\widetilde{B}_{2,n}$ is connected, while the curve $C_n = S \cup S' \cup S'' \cup S_\infty \subset \mathbb{F}_{2n}$ has two connected components (since any affine section of $\mathbb{F}_{2n}$ is disjoint from the infinity section S_∞). We can take the double branched cover along C_n; the desingularization of that branched cover will be denoted by $T(n)$. Note that S, S' and S'' can be chosen in such a way that C_n has only normal crossing singularities. Now using Proposition 7.3.13 we may suppose that the above branched cover $T(n) \to \mathbb{F}_{2n}$ is branched along a smooth curve — since C_n has only normal crossing singularities, a small perturbation turns it into a smooth curve $\widetilde{C}_n$ (with two connected components) but leaves the diffeomorphism type of the double branched cover unchanged. A priori it is not clear whether $T(n)$ is diffeomorphic to $E(n)$; the two complex surfaces have the same characteristic numbers, since $[\widetilde{C}_n] = [C_n] = [B_{2,n}] = [\widetilde{B}_{2,n}]$ (and the characteristic numbers of a double branched cover branched along a smooth curve depend only on the homology class of the branch locus, cf. Lemma 7.1.7), but they might be nondiffeomorphic. Observe that the above argument can be repeated for any curve $B_{m,(m-1)n} \subset \mathbb{F}_0$: The homology class $[B_{m,(m-1)n}] = 2m[S_0] + 2(m-1)n[F_0]$ maps to $2m[S_{2n}] - 2n[F_{2n}]$ in $\mathbb{F}_{2n}$, which is equal to $(2m-1)[S_{2n}]+[S_\infty]$. Consequently, a complex surface $U(m,n)$ can be obtained as the desingularization of the double branched cover of $\mathbb{F}_{2n}$ branched along $(2m-1)$ affine sections and the infinity section. By adapting the argument above it can be shown that $U(m,n)$ has the same characteristic numbers as $X(m,(m-1)n)$, and in many cases one can prove that $U(m,n)$ is homeomorphic to $X(m,(m-1)n)$. On the other hand, for example if $m = 3$ and $n = 1$, the corresponding 4-manifolds $X(3,2) \approx X(2,3) \approx E(3)$ and $U(3,1)$ are obviously nondiffeomorphic: As we have seen, $E(3)$ does not contain any smooth sphere with square -1; on the other hand, $U(3,1)$ (which is defined as the desingularization of a double branched cover of $\mathbb{F}_2$) has a -1-sphere originating from the infinity section of $\mathbb{F}_2$.

Exercises 7.3.16. **(a)*** Show that $X(n,m)$ is simply connected.

(b) Using Seiberg-Witten theory, show that $E(3)$ does not contain any smoothly embedded sphere with square -1. (*Hint*: Recall that $\mathcal{B}as_{E(3)} = \{\pm PD(f)\}$ with $f^2 = 0$; now apply the blow-up formula 2.4.9.)

(c) Show that $X(2,3)$ and $U(3,1)$ are homeomorphic 4-manifolds. (*Hint*: Using Remark 7.3.20(b) show that $U(3,1)$ is simply connected. Since the characteristic numbers coincide and the signature of $X(2,3)$ is equal to -24, both manifolds are nonspin, hence Theorem 1.2.27 completes the solution.)

(d) In general, prove that if n or m is odd, then $X(n,m)$ is nonspin. (*Hint*: Show that the two fibrations of $X(n,m)$ described earlier admit sections of square $-n$ and $-m$, respectively.) Prove that if n, m are both even, then $X(n,m)$ is spin. (*Hint*: Suppose that $n = 2k$, and apply induction on k; note that $k = 1$ produced spin elliptic surfaces.)

(e) Prove that $T(n) = U(2,n)$ admits an elliptic fibration and is the fiber sum of $T(n-1)$ and $T(1)$. (*Hint*: Show first that the composition of the double branched covering $T(n) \to \mathbb{F}_{2n}$ with the ruling $\mathbb{F}_{2n} \to \mathbb{CP}^1$ is an elliptic fibration on $T(n)$. Then use the fact that $\mathbb{F}_{2n}$ is the generalized fiber sum of $\mathbb{F}_{2n-2}$ and $\mathbb{F}_2$, and prove that one can choose $\widetilde{C}_{n-1} \subset \mathbb{F}_{2n-2}$ and $\widetilde{C}_1 \subset \mathbb{F}_2$ in such a way that these will be glued together to give $\widetilde{C}_n$ in $\mathbb{F}_{2n}$.)

(f) Using the same idea as above, show that $U(m,n)$ admits a fibration over $\mathbb{CP}^1$ with fibers of genus $m-1$. Decompose $U(m,n)$ as the generalized fiber sum of $U(m,n-1)$ and $U(m,1)$.

As we have now seen, double branched covers along homologous surfaces in a fixed smooth 4-manifold may result in homeomorphic but nondiffeomorphic 4-manifolds. As we will prove, however, the 4-manifolds $E(n)$ and $T(n) = U(2,n)$ are diffeomorphic, giving additional insight into the geometry of elliptic surfaces. Before proving that $E(n) \approx T(n)$, we will examine the 4-manifolds $T(n)$ more thoroughly. The infinity section $S_\infty \subset \widetilde{C}_n \subset \mathbb{F}_{2n}$ gives a section of square $-n$ of $T(n)$. Deleting a neighborhood of S_∞, we find that $T(n) - \nu(\text{section})$ is the double branched cover of the total space of the line bundle $L_{2n} \to \mathbb{CP}^1$ (with $c_1(L_{2n}) = 2n$) branched along the smoothing of the union of three sections. Suppose that F is a fiber of $L_{2n} \to \mathbb{CP}^1$ with the property that the fiber defined by F in $T(n)$ is a generic one (i.e., F intersects the branch locus $\widetilde{C}_n - S_\infty$ transversely in three points). Deleting a neighborhood of F and then taking the double branched cover, we see that $T(n) - \nu(\text{section} \cup \text{generic fiber})$ is the double branched cover of $L_{2n} - \nu F$, branched along (the smoothing of) the union of three sections. It is easy to see that $L_{2n} - \nu F \approx \mathbb{C}^2$, and a section can be described by the equation $y + z^{2n} = 0$. Hence the part of the smooth curve $\widetilde{C}_n$ in $L_{2n} - \nu F$ can be described by $y^3 + z^{6n} = \varepsilon$. As we will verify, the double branched cover of $\mathbb{C}^2$ along the curve $y^3 + z^{6n} = \varepsilon$ is the Milnor fiber $M(2,3,6n)$; consequently we see that $T(n)$ is the union of the compactified Milnor fiber $M_c(2,3,6n)$ and the neighborhood of a section and a regular fiber.

Exercise 7.3.17. * Draw a Kirby diagram for a neighborhood of the union of a section and a regular fiber.

Recall that the Milnor fiber $M(p,q,r)$ is by definition the 4-manifold $\{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = \varepsilon\}$. Since for y,z fixed, the equation $x^p = \varepsilon' = \varepsilon - y^q - z^r$ has p different solutions for $\varepsilon' \neq 0$, the projection $(x,y,z) \mapsto (y,z)$ presents $M(p,q,r)$ as the p-fold branched cover of $\mathbb{C}^2$ branched along the complex curve $\{(y,z) \in \mathbb{C}^2 \mid y^q + z^r = \varepsilon\}$. By projecting $M(p,q,r)$ to the line $\{x = 0,\ y = 0\}$ via the map $(x,y,z) \mapsto (0,0,z)$, we get a fibration $f_z \colon M(p,q,r) \to \mathbb{C}$. If $z^r \neq \varepsilon$, the fiber over z is the smooth complex curve $x^p + y^q = \varepsilon''$ $(= \varepsilon - z^r)$; if $z^r = \varepsilon$, we get singular fibers given by the equation $x^p + y^q = 0$. The $\mathbb{Z}_r$-action on $M(p,q,r)$ determined by $z \mapsto e^{\frac{2\pi i}{r}} z$ fixes the fiber F over $z = 0$ pointwise and cyclically permutes the singular fibers, showing that each singular fiber is attached to νF by the same gluing map (cf. the discussion of monodromies in Chapter 8). The Milnor fiber $M(p,q,r)$ admits a compactification $M_c(p,q,r) = M(p,q,r) \cap D^6$, where $D^6 \subset \mathbb{C}^3$ is a round ball (with sufficiently large radius). The presentation of $M(p,q,r)$ as a p-fold branched cover of $\mathbb{C}^2$ provides a p-fold branched covering map $M_c(p,q,r) \to D^4$, where $D^4 \subset \mathbb{C}^4$ is a ball of sufficiently large radius. Likewise, the above fibration of $M(p,q,r)$ obviously determines a fibration $M_c(p,q,r) \to D^2 \subset \mathbb{C}$ — choose $|\varepsilon| \ll 1$ so that all singular fibers of

$M(p,q,r) \to \mathbb{C}$ are in the unit disk $D^2 \subset \mathbb{C}$. Suppose that $L_1 = L \cap D^2 \subset D^2$ is a segment of the (real) line $L \subset \mathbb{C}$ separating D^2 as $D_1 \cup D_2$. Suppose, futhermore, that D_i contains r_i of the points solving $z^r = \varepsilon$. (Such a segment L_1 can easily be found for any triple (r, r_1, r_2) with $r_1 + r_2 = r$ and $r_i \geq 0$.) Since the inverse image of D_i via $f_z \colon M_c(p,q,r) \to D^2$ is diffeomorphic to $M_c(p,q,r_i)$, we deduce that for $r' \leq r$ the manifold $M_c(p,q,r')$ embeds in $M_c(p,q,r)$ (see also Exercises 6.3.11 and 6.3.13(b)). Note that the above reasoning proves, in particular, that $M_c(p,q,r)$ is the "boundary fiber sum" of $M_c(p,q,r_1)$ and $M_c(p,q,r_2)$ for $r = r_1 + r_2$.

Exercise 7.3.18. Making use of the fibrations given by the two other projections f_x and f_y, prove that if $p \leq p'$, $q \leq q'$ and $r \leq r'$, then $M_c(p,q,r)$ embeds in $M_c(p',q',r')$ (cf. Exercise 6.3.13(b)).

If $p = 2$ and $q = 3$ (the relevant case for examining $T(n)$), we have a fibration of $M(2,3,r)$ over $\mathbb{C}$ such that the generic fiber is a punctured torus and there are r punctured cusp fibers as singular fibers.

Corollary 7.3.19. *The 4-manifold $T(n)$ decomposes as the union of the nucleus $N(n)$ and $M_c(2,3,6n-1)$.*

Proof. Recall that the nucleus $N(n)$ has been defined as the tubular neighborhood of a section and a cusp fiber. Since $M_c(2,3,6n)$ can be decomposed as the boundary fiber sum of $M_c(2,3,6n-1)$ and $M_c(2,3,1)$, and since $M_c(2,3,1)$ is the neighborhood of a cusp fiber minus a section, the decomposition $T(n) = M_c(2,3,6n) \cup \nu(\text{section} \cup \text{generic fiber})$ implies the corollary. □

Remarks 7.3.20. (a) As Exercise 7.3.21(a) shows (cf. also Example 8.2.8), the nucleus $N(n)$ admits a handle decomposition with one 0-handle and two 2-handles. Hence Corollary 6.3.19 implies, in particular, that $T(n)$ admits a handle decomposition without 1- or 3-handles.

(b) Observe that the above argument decomposes the manifold $U(m,n)$ (defined as the double branched cover of $\mathbb{F}_{2n}$ branched along a certain disconnected branch curve) as the Milnor fiber $M_c(2, 2m-1, 2n(2m-1))$ union the tubular neighborhood of a section and a regular fiber. Moreover, in the same vein as Corollary 7.3.19 shows for $T(n)$, the 4-manifold $U(m,n)$ decomposes as the union of $M_c(2, 2m-1, 2n(2m-1)-1)$, a neighborhood of a section and $M_c(2, 2m-1, 1)$. The union of the latter two is frequently called a *generalized nucleus* and is denoted by $N(m,n)$ [**Fu3**]. (For $m = 2$ we get back the original description of the nucleus $N(n)$ in the elliptic surface $E(n)$.) Note that $M_c(2, 2m-1, 2n(2m-1)-1)$ is a simply connected spin 4-manifold with a homology sphere boundary.

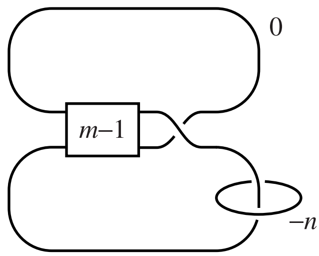


Figure 7.5. Kirby diagram for the generalized nucleus $N(m, n)$.

Exercises 7.3.21. (a)* Show that the generalized nucleus $N(m, n)$ is diffeomorphic to the manifold given by the Kirby diagram of Figure 7.5. Deduce from this picture that $N(m, n)$ is simply connected for all m, n, and it is spin iff n is even. (Note that since the Milnor fiber is simply connected and spin, the solution of this exercise implies that $U(m, n)$ is simply connected, and it is spin iff n is even.)

(b) Let $F = \{x^3 + y^2z = 0\} \subset \mathbb{CP}^2$ and $L = \{z = 0\} \subset \mathbb{CP}^2$. The complex curve F is a sphere with a cusp singularity at $[0 : 0 : 1]$ (cf. C_2 in Section 2.3), and $F \cap L = \{[0 : 1 : 0]\}$ with $F \cdot L = 3$. Blow up to resolve the singularity of $F \cup L$ at $[0 : 1 : 0]$, then continue blowing up $\widetilde{F}$ until $[\widetilde{F}]^2 = 0$. You should get the plumbing in Figure 7.6, where we have included the sphere $\widetilde{L}$. This example (of Friedman and Morgan) shows that $E(1) \approx N(1) \cup_\partial (-E_8)$-plumbing: Verify that the complement of a tubular neighborhood $\nu L \subset \mathbb{CP}^2$ is a 4-ball intersecting F in a cone on a right trefoil knot, so the union of $\widetilde{F}$ and the -1-sphere in Figure 7.6 is diffeomorphic to $N(1)$. Check that removing $N(1)$ leaves behind a $(-E_8)$-plumbing. (What happens to the ninth -2-sphere?) We have actually constructed an elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ by blowing up the pencil of cubic curves $t_0(x^3 + y^2z) + t_1z^3 = 0$. By perturbing the construction, we get a generic elliptic fibration on $E(1)$ as in Section 3.1. Thus, $E(1) - N(1)$ is a $(-E_8)$-plumbing. Since the latter is diffeomorphic to the Milnor fiber $M_c(2, 3, 5)$ (Exercise 8.3.4(c)) and every self-diffeomorphism of its boundary is isotopic to the identity (cf. Exercise 5.5.9(b)), we conclude that $E(1) \approx T(1)$ (cf. also the beginning of Section 8.3), and the diffeomorphism sends a generic fiber to a generic fiber, preserving its canonical normal framing. (We have also exhibited an $\widetilde{E}_8$*-fiber* of an elliptic fibration, given by the nine -2-framed spheres in Figure 7.6, and shown that $E(1)$ has an elliptic fibration with two singular fibers: a cusp and an $\widetilde{E}_8$-fiber.)

(c) Use the decomposition $E(1) \approx T(1) \approx N(1) \cup_\partial (-E_8$-plumbing) to verify by Kirby calculus that these manifolds are diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. (*Hint*: Exercise 5.5.9(b).) Compare the construction in (b) above with

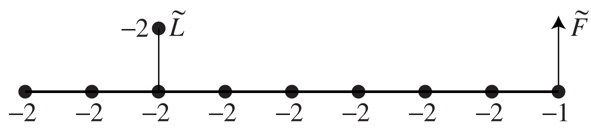


Figure 7.6. A decomposition of $E(1)$.

the solution of Exercise 5.5.9(b) using the (alternate) solution of Exercise 5.1.12(a), and convince yourself that the procedures correspond. (*Hint*: Blow down the plumbing in Figure 7.6, keeping track of a 0-framed meridian μ of the -1-framed circle. You should get S^3 with μ a 0-framed left trefoil knot. This verifies that we can add a 4-handle to the plumbing; the result is $E(1)$. Locate neighborhoods of $\widetilde{F}$ and $N(1)$ in the diagram. (Their 0-handle is the 4-handle of $E(1)$.) What are their complements? Compare each step of the blow-down with the corresponding diagram of the resolution.)

Now the solutions of Exercises 7.3.21(b) and 7.3.16(e) imply the following result.

Theorem 7.3.22. *The 4-manifold $T(n)$ constructed above is diffeomorphic to $E(n)$.* □

Thus we have given two different presentations of $E(n)$ as the double branched cover of $\mathbb{F}_{2n} \approx \mathbb{F}_0$, branched first along a connected, then along a disconnected branch locus. The decomposition of $T(n)$ as the union of the Milnor fiber $M_c(2,3,6n)$ and the tubular neighborhood of a section and a regular fiber now decomposes $E(n)$ as well, providing interesting corollaries concerning the elliptic surface $E(n)$.

Corollary 7.3.23. *The elliptic surface $E(n)$ admits an elliptic fibration with $6n$ singular fibers, each being a cusp fiber. Moreover, $E(n)$ can be decomposed as the union of the nucleus $N(2,n) = N(n)$ and $M_c(2,3,6n-1)$; and if $k \leq 6n$, the Milnor fiber $M_c(2,3,k)$ embeds in the elliptic surface $E(n)$. Furthermore, $E(n)$ admits a handle decomposition without 1- or 3-handles (cf. also the text preceding Exercise 8.3.1 and Corollary 8.3.17).*

Proof. All these statements obviously follow from Theorem 7.3.22 together with Corollary 7.3.19, Exercise 7.3.18 and Remark 7.3.20(a). □

We point out that Theorem 7.3.3 gives several different presentations of the $K3$-surface — as $E(2)$, $V(2)$, $X(2)$ and as $D'(2)$. There is one other standard way to define the $K3$-surface: as the double branched cover of $\mathbb{CP}^2$ along a smooth curve of degree six. Using Proposition 7.3.13, we now show that this last construction yields a manifold diffeomorphic to $E(2)$. Take smooth cubic curves $p_0, p_1 \subset \mathbb{CP}^2$ in general position. Since $[p_0] + [p_1] =$

$6h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is divisible by 2, we can take the double branched cover X of $\mathbb{CP}^2$ branched along $p_0 \cup p_1$ as given by Construction 7.1.1.

Lemma 7.3.24. *The desingularization of the singular surface X is diffeomorphic to the $K3$-surface $E(2)$.*

Proof. By the algorithm described in Section 7.2, we must blow up the 9 intersection points of the curves defined by p_0 and p_1, and then take the double branched cover along the proper transform. Consequently, the desingularization of X is simply the double branched cover of $E(1)$ along two fibers, which is (by Claim 7.3.11) diffeomorphic to $E(2)$. □

Now using Proposition 7.3.13 we conclude

Corollary 7.3.25. *The double cover of $\mathbb{CP}^2$ branched along a smooth sextic curve is diffeomorphic to $E(2)$, hence it is a $K3$-surface.*

Remark 7.3.26. If we define $V(2) \subset \mathbb{CP}^1 \times \mathbb{CP}^2$ as in Section 3.2 using the equation $x^2p_0(z_0,z_1,z_2) + y^2p_1(z_0,z_1,z_2) = 0$, then the projection to the second factor gives a map $\mathrm{pr}_2\colon V(2) \to \mathbb{CP}^2$, which is 2:1 away from $B = \{p_0 = 0\} \cup \{p_1 = 0\}$ and 1:1 in $B - \{p_0 = 0\} \cap \{p_1 = 0\}$; the inverse image of each point $P \in \{p_0 = 0\} \cap \{p_1 = 0\}$ is a sphere. A more detailed analysis shows that the above projection pr_2 is, in fact, the desingularization of the double branched cover of $\mathbb{CP}^2$ along B. (Since $V(2)$ is diffeomorphic to $E(2)$, the above observation yields another proof of Corollary 7.3.25.) The same argument shows that $V(n) = \{x^np_0(z_0,z_1,z_2) + y^np_1(z_0,z_1,z_2) = 0\}$ (which is diffeomorphic to $E(n)$) is the resolution of the n-fold branched cover of $\mathbb{CP}^2$ branched along the above B.

Exercises 7.3.27. (a) Using Proposition 7.3.13 prove that the complete intersection surface S_2 of Section 1.3 is diffeomorphic to $\mathbb{F}_2$. (*Hint*: Describe S_2 as the double branched cover of $\mathbb{CP}^2$ along a smooth quadric curve, then prove that $\mathbb{F}_2$ is the resolution of the double branched cover of $\mathbb{CP}^2$ branched along a pair of distinct lines. Make use of the branched cover $\mathbb{F}_2 \to \mathbb{F}_1$ branched along a pair of fibers; cf. also Exercise 6.3.16(b).)

(b)* More generally, prove that the double branched cover $W(d)$ of $\mathbb{CP}^2$ along a (smooth) curve of degree $2d$ can be decomposed as the generalized fiber sum of two copies of $\mathbb{CP}^2\#d^2\overline{\mathbb{CP}^2}$ along a complex curve of genus equal to $\frac{1}{2}(d-1)(d-2)$.

(c) Using the result of the above exercise, compute $c_1^2(W(d))$, $\chi_h(W(d))$ and show that $W(d)$ is simply connected. (*Answer*: $c_1^2(W(d)) = 2(d-3)^2$ and $\chi_h(W(d)) = \frac{1}{2}d(d-3)+2$; moreover, $c_2(W(d)) = 4d^2-6d+6$.) Determine the parity of $Q_{W(d)}$ using Remark 7.1.8(a).

For the double branched cover $X = X(n,m)$ of $\mathbb{F}_0$ branched along $\widetilde{B}_{n,m}$ we have $c_1^2(X) = 4(n-2)(m-2)$ and $\chi_h(X) = (n-1)(m-1)+1$. (This follows from the fact that $c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) = 2[\mathbb{CP}^1 \times \{\text{pt.}\}] + 2[\{\text{pt.}\} \times \mathbb{CP}^1]$; cf. Exercise 7.4.25.) From these formulae it is easy to see that the surfaces $X(3,n)$ satisfy $c_1^2(X(3,n)) = 4(n-2)$ and $\chi_h(X(3,n)) = 2n-1$, hence $c_1^2(X(3,n)) = 2\chi_h(X(3,n)) - 6$. This shows that the Noether inequality (Theorem 3.4.19) is, in fact, sharp. Surfaces satisfying $c_1^2 = 2\chi_h - 6$ are frequently called *Horikawa surfaces*. The examples $X(3,n)$ are simply connected and all admit genus 2 fibrations. In the following, these will also be denoted by $H(n)$. Recall that $X(3,n) = H(n)$ can be decomposed as the generalized fiber sum of three copies of $X(1,n) \approx \mathbb{CP}^2 \# (4n+1)\overline{\mathbb{CP}^2}$, cf. Exercise 7.3.8(b). By sewing the sections — the exceptional curves — of $X(1,n)$ together we can see that the surface $X(3,n)$ contains spheres of square -3; consequently its intersection form is odd, cf. also Exercise 7.3.16(d). (A -3-sphere in $H(n)$ can be seen directly from the desingularization algorithm if we consider $X(3,n)$ as the desingularization of the double branched cover of $\mathbb{F}_0$ along the complex curve $B_{3,n}$.) Above, we constructed a complex surface $U(3,n)$ (as a branched cover of $\mathbb{F}_{2n}$ along a disconnected curve) and observed that the characteristic numbers of $H(2n) = X(3,2n)$ and $U(3,n)$ are identical, hence the surface $H'(n) = U(3,n)$ is a Horikawa surface as well. By Remark 7.3.20(b) we see that $H'(n)$ is spin iff n is even. Since $U(m,n)$ and $X(n,m)$ are simply connected complex surfaces for any m and n (cf. Remark 7.3.20(b) and Exercise 7.3.16(a)), applying Theorem 1.2.27 and Exercise 7.3.21(a) shows

Corollary 7.3.28. *The Horikawa surfaces $H(2n)$ and $H'(n)$ are homeomorphic iff n is odd.* □

It was shown by Horikawa [**Hr1**] that the complex surfaces $H(2n)$ and $H'(n)$ are not deformation equivalent (cf. Theorem 7.4.20), which is obvious if n is even — in that case $H(2n)$ and $H'(n)$ are not even homotopy equivalent. On the other hand, it is still an open (and very interesting) question, whether for n odd (and $n > 1$) the homeomorphic, deformation inequivalent surfaces $H(2n)$ and $H'(n)$ are diffeomorphic or not, cf. Theorem 3.4.13 and Conjecture 3.4.21[1]. Note that since (for $n > 1$) the manifolds $H(2n)$ and $H'(n)$ are both minimal surfaces of general type, their Seiberg-Witten basic classes are equal to $\pm$ the first Chern class (with $SW(\pm c_1) = \pm 1$), hence we cannot distinguish the smooth structures of $H(2n)$ and $H'(n)$ by their Seiberg-Witten invariants. Using the complex surfaces $U(m,n)$ and $X(m,(m-1)n)$ described above for appropriate m and n, many similar examples can be found. The Horikawa surfaces $H(6) = X(3,6)$ and $H'(3) = U(3,3)$ have the smallest characteristic numbers in this family. (An easy computation shows that the simply connected surface $X(3,6)$ has Euler characteristic $\chi(X(3,6)) = 116$, signature $\sigma(X(3,6)) = -72$, hence

[1]See also footnote on Page 91

$b_2^+(X(3,6)) = 21$.) We will see Horikawa surfaces again in Sections 8.4 and 8.5. (For Kirby diagrams of $U(m,n)$ and $X(n,m)$, see Figures 8.31 through 8.34.)

7.4. Surfaces of general type

In this section a very fruitful method of constructing complex surfaces with various characteristic numbers will be described. The idea we outline here is due to Persson (see [**Pe**]); he used this construction to prove results concerning the geography problem for complex surfaces (cf. Section 3.4). Persson observed that by taking double branched covers of geometrically ruled surfaces along curves with various singularities, one can get (after resolving) minimal surfaces with almost all possible (χ_h, c_1^2)-invariants. After discussing this method in detail, we close the section by sketching a construction of complex surfaces with positive signature.

In order to have control of the change of topology of the double branched cover while resolving a singularity, we need a definition concerning singularities. Recall that if $B \subset Y$ is a smooth curve, then for a double branched cover X we have $c_1^2(X) = 2(c_1(Y) - \frac{1}{2}[B])^2 = 2(c_1^2(Y) - \chi(B)) - \frac{3}{2}[B]^2$ and $c_2(X) = 2c_2(Y) - \chi(B)$; hence an easy computation shows that $\chi_h(X) = \frac{c_1^2(X)+c_2(X)}{12} = 2\chi_h(Y) - \frac{\chi(B)}{4} - \frac{1}{8}[B]^2$.

Exercises 7.4.1. Compute $c_1^2(X)$ and $\chi_h(X)$ for the double branched cover X when

(a)* Y is the Hirzebruch surface $\mathbb{F}_n$ and B is a smooth, connected curve representing the homology class $2a[S_n] + 2b[F_n]$ — again, S_n denotes an affine section and F_n a fiber of $\mathbb{F}_n$. In particular, prove that if $Z(m)$ is the double branched cover of $\mathbb{F}_1$ along $6[S_1] + 2m[F_1]$, then $c_1^2(Z(m)) = 4m - 2$ and $\chi_h(Z(m)) = 2m + 2$; consequently $Z(m)$ is a Horikawa surface.

(b) Y is a geometrically ruled surface $\mathbb{G}_{n,g} \to \Sigma_g$ over the (real) 2-dimensional surface Σ_g of genus g, an affine section S_n of $\mathbb{G}_{n,g} \to \Sigma_g$ has square $[S_n]^2 = n$ and B is a smooth curve with $[B] = 2a[S_n] + 2b[F_n]$. (*Answer*: $c_1^2(X) = 2a(a-2)n + 4(a-2)(b+2g-2)$ and $\chi_h(X) = \frac{1}{2}a(a-1)n + (a-1)(b+g-1) + 1 - g$.) Recall that the complex surface $\mathbb{G}_{n,g}$ is constructed by projectivizing the bundle $L_n \oplus \underline{\mathbb{C}} \to \Sigma_g$, where $c_1(L_n)[\Sigma_g] = n$ (cf. Section 3.4).

Note that the expressions for $c_1^2(X)$ and $\chi_h(X)$ depend only on the homology class of B, so they make sense even when B is singular — of course, in that case X is singular as well, so $c_1^2(X)$ and $\chi_h(X)$ have no real meaning. We formally define $c_1^2(X)$ and $\chi_h(X)$ of the singular surface X by the above formulae (where we set $\chi(B) = c_1(Y)[B] - [B]^2$ as given by the adjunction formula). If B is singular, the formulae have to be modified to give c_1^2 and

χ_h of the (minimal) resolution X'; the correction terms depend only on the types of singularities B admits. Suppose that the unique singular point of B is given by the equation $\{f(x,y)=0\}$, $\phi\colon X \to Y$ is the double branched cover branched along B, and X' is the minimal resolution.

Definition 7.4.2. Suppose that $c_1^2(X') = 2(c_1(Y) - \frac{1}{2}[B])^2 - a = c_1^2(X) - a$ and $\chi_h(X') = 2\chi_h(Y) - \frac{\chi(B)}{4} - \frac{1}{8}[B]^2 - b = \chi_h(X) - b$. Then (a,b) is called the *specialization vector* of the singularity $\{f(x,y)=0\}$ of B.

(Note that X' denotes the *minimal* resolution of X, so in certain cases one has to blow down rational -1-curves in the canonical resolution provided by the algorithm given in Section 7.2.)

The vector (a,b) can easily be computed for a singularity $\{f(x,y)=0\}$. Here we give an outline of this computation; the specialization vectors of some interesting singularities will be computed later. Suppose that $B \subset Y$ has a singularity at $P \in B$ described by f. When we blow up Y at P, the total transform B' of B is equal to $\widetilde{B} + d_1E_1$, where (as always) $\widetilde{B}$ stands for the proper transform and E_1 for the exceptional curve. The branched cover along B' might not be smooth, and in fact it might be even more singular than X — if $d_1 > 1$, then it will not even be normal. To fix this (according to the algorithm described in Section 7.2) we change B' to $B_1 = \widetilde{B} + (d_1 - 2[\frac{d_1}{2}])E_1 = B' - 2n_1E_1$, where $n_1 = [\frac{d_1}{2}]$ is the greatest integer $\leq \frac{d_1}{2}$. The double branched cover along B_1 will be normal, and by repeating the above process we will construct the same canonical resolution as we found in Section 7.2. (Note that all we did above was to drop E_i from the branch locus if d_i was even and to reduce its multiplicity to 1 if d_i was odd. By Remark 7.2.10 this is the same as the algorithm described in Section 7.2.) The multiplicities d_i are obviously different from the m_i's we found in our algorithm. For m_i, all those previous multiplicities contribute for which the corresponding exceptional curve passes through the point blown up. With d_i, however, one only records the parities of the multiplicities of the previous exceptional curves; consequently d_i will be different from m_i. (Obviously $d_1 = m_1$, but d_i and m_i might be different for $i > 1$.) Since we are interested in the minimal resolution, we must also know the number of blow-downs required to reach the minimal resolution from the canonical one; in the following this number will be denoted by r. The specialization vector (a,b) can be computed from the above multiplicities n_i and r by the following formulae:

Lemma 7.4.3. ([**Ch1**]) *The specialization vector* (a,b) *of* $\{f(x,y)=0\}$ *is given by* $a = 2\sum(n_i-1)^2 - r$ *and* $b = \frac{1}{2}\sum n_i(n_i-1)$. □

The following theorem motivates the definition of simple (or inessential) singularities.

Theorem 7.4.4. ([**Pe**]) *The singularity* $\{f(x,y)=0\}$ *is simple iff the corresponding specialization vector is* $(0,0)$. □

(One direction of the above equivalence can be shown easily based on the solution of Exercises 7.2.5; the other direction needs more work.) Hence the introduction of simple singularities on B does not change the characteristic numbers of the desingularization of the double branched cover. Singularities other than simple ones are usually called *essential* singularities.

Example 7.4.5. The specialization vector of the *infinitely close triple point* $f(x,y) = x^3 + y^6$ can be computed in the following way (cf. Examples 7.2.3(b) and 7.2.11(b)): by our previous computation $d_1 = 3$, so $n_1 = [\frac{d_1}{2}] = 1$. Now B_1 can be descibed by the equation $t(s^3+t^3) = 0$ (as opposed to $B' = t^3(s^3+t^3)$) on U_1, so $d_2 = 4$ and $n_2 = 2$ (recall that $m_2 = 6$). After the second blow-up we got a smooth curve; the branched cover surface was, however, nonminimal: one rational -1-curve could have been blown down, so the specialization vector of the above singularity turns out to be $(1,1)$.

Exercise 7.4.6. Determine the specialization vector of an *infinitely close triple point of order* n defined by the equation $f(x,y) = x^3 + y^{3n+3}$. (For other examples of computations of specialization vectors see [**Ch1**].)

Suppose that for given n,k,a (with $0 \le k \le 2n + 2[\frac{2a}{3}]$) the curve $B = B_{n,k,a} \subset \mathbb{F}_n$ represents the homology class $6[S_n] + 2a[F_n]$ and B has exactly k infinitely close triple points and no other essential singularities. (Recall that S_n denotes an affine section and F_n a fiber of the Hirzebruch surface $\mathbb{F}_n \to \mathbb{CP}^1$. We will prove the existence of such B below.) The following result is straightforward from Exercise 7.4.1(a) and the specialization vector of an infinitely close triple point.

Lemma 7.4.7. *If* $X_{n,k,a}$ *is the minimal resolution of the double branched cover of* $\mathbb{F}_n$ *branched along* $B = B_{n,k,a}$, *then* $\chi_h(X_{n,k,a}) = 3n + 2a - 1 - k$ *and* $c_1^2(X_{n,k,a}) = 6n + 4a - 8 - k$. □

One can, in fact, prove that for $n \ge 2$ all these surfaces are minimal — for the proof of this statement see [**Pe**]. Now some elementary number theory shows that a certain region of the geography picture can be populated by minimal surfaces of general type, more precisely:

Theorem 7.4.8. *If* $(x,y) \in \mathbb{N} \times \mathbb{N}$ *satisfies* $0 \le 2x - 6 \le y \le 4x - 6$, *then there exists a triple* (n,k,a) *with* $a = 0, 1$ *or* 2 *such that* $\chi_h(X_{n,k,a}) = x$ *and* $c_1^2(X_{n,k,a}) = y$. *Moreover, for* $x \ge 7$ *the surface* $X_{n,k,a}$ *is minimal.*

Proof. We must solve the equations

$$x = 3n + 2a - 1 - k, \quad y = 6n + 4a - 8 - k$$

for every given pair (x, y) with $2x-6 \leq y \leq 4x-6$ in such a way that n, a, k are nonnegative integers subject to the constraint $k \leq 2n + 2[\frac{2a}{3}]$. One can easily see that $k = y - 2x + 6$ and $n = \frac{y-x+7-2a}{3}$ gives a solution; this latter expression gives an integer for the appropriate choice of a (solving the congruence $2a \equiv y - x + 1 \pmod 3$). An easy computation shows that the conditions $0 \leq 2x - 6 \leq y$ and $a \in \{0, 1, 2\}$ imply that $n \geq 2$ when $x \geq 7$; hence minimality follows from the remark after Lemma 7.4.7. □

Exercise 7.4.9. * Prove that the condition $y \leq 4x - 6$ ensures that k satisfies $k \leq 2n + 2[\frac{2a}{3}]$.

The validity of Theorem 7.4.8 rests on the fact that the curve $B_{n,k,a} \subset \mathbb{F}_n$ with the prescribed properties exists; this is, however, nontrivial. In proving the existence of $B_{n,k,a}$ we can make use of the double branched cover construction: Take the zero and infinity sections S_{2n}, S_∞ of $\mathbb{F}_{2n}$, let $b = [\frac{2a}{3}]$ and fix $2n+2b$ distinct points $q_1, \ldots, q_{2n+b} \in S_{2n}$, $q_{2n+b+1}, \ldots, q_{2n+2b} \in S_\infty$ (no two q_i on the same fiber). Fix k of the above $2n+2b$ points and call them $q'_1, \ldots, q'_k$. Take distinct smooth (connnected) complex curves C_1, C_2, C_3 in $\mathbb{F}_{2n}$ homologous to $[S_{2n}] + b[F_{2n}]$ such that their common intersections on $S_{2n} \cup S_\infty$ are $q'_1, \ldots, q'_k$ and all the other intersections are generic. (The existence of such curves is provided by [**Pe**], cf. also Exercise 7.4.10. Note that $[C_i] \cdot [C_j] = [C_i]^2 = 2n + 2b$.) Recall that there is a map $\pi\colon \mathbb{F}_n \to \mathbb{F}_{2n}$ which is the double branched cover branched along $S_{2n} \cup S_\infty$.

Exercise 7.4.10. Show that for $b = 0$ the curves C_1, C_2, C_3 with the above properties exist. (*Hint*: For $b = 0$ the curves are affine sections of the Hirzebruch surface $\mathbb{F}_{2n}$, hence each C_i can be given by a polynomial of degree $2n$ with k preassigned (common) zeros.) Note that for most cases needed in Theorem 7.4.8 the argument with $b = 0$ suffices ($a = 0, 1$ implies $b = 0$); we described the above construction for general b in order to assure simple connectivity of the resulting complex surfaces, cf. Remark 7.4.12.

Lemma 7.4.11. *The union of $\pi^{-1}(C_i)$ $(i = 1, 2, 3)$ with $(2a - 3b)$ generic fibers gives a curve $B_{n,k,a} \subset \mathbb{F}_n$ with $[B_{n,k,a}] = 6[S_n] + 2a[F_n]$; moreover $B_{n,k,a}$ has k infinitely close triple points and no other essential singularities.*

Proof. By Exercise 7.1.10(a) the homology class $[B_{n,k,a}] = [\bigcup_{i=1}^3 \pi^{-1}(C_i)] + (2a - 3b)[F_n] \in H_2(\mathbb{F}_n; \mathbb{Z})$ is equal to $6[S_n] + 2a[F_n]$. Hence the only thing remaining to be proved is that an ordinary triple point located on the branch curve (provided by $C_1 \cap C_2 \cap C_3$ on $S_{2n} \cup S_\infty$) becomes an infinitely close triple point in the double branched cover. This can be checked in a local model: In an appropriate chart $U(x, y)$ the branch curve is given by the equation $\{x = 0\}$ and the three lines meeting each other transversally can be chosen as $x - y$, $x - \zeta_1 y$ and $x - \zeta_2 y$ (where 1, ζ_1 and ζ_2 are three different third roots of unity). Now the double branched cover locally is described by $\{(x, y, z) \in \mathbb{C}^3 \mid z^2 = x\}$, and so the inverse images of the three lines become $z^2 - y$, $z^2 - \zeta_1 y$ and $z^2 - \zeta_2 y$, giving the infinitely close triple point $0 = (z^2 - y)(z^2 - \zeta_1 y)(z^2 - \zeta_2 y) = z^6 - y^3$. □

Remark 7.4.12. Each of the examples in Theorem 7.4.8 admits a fibration over $\mathbb{CP}^1$ with fibers of genus 2: The surface $X_{n,k,a}$ admits a map ϕ to $\mathbb{F}_n$, and the composition of ϕ with the ruling $\mathbb{F}_n \to \mathbb{CP}^1$ yields a (singular) fibration $X_{n,k,a} \to \mathbb{CP}^1$. The fiber originates from the fiber $F \subset \mathbb{F}_n$ (a sphere) intersecting the branch locus $B_{n,k,a}$ generically in six points, so the double branched cover of it is a curve of genus 2. The fibration will contain singular fibers — these are the fibers corresponding to fibers of $\mathbb{F}_n$ passing through singular points of $B_{n,k,a}$, tangent to or contained in $B_{n,k,a}$. It is easy to see that $c_1^2(X_{n,k,a+3}) = c_1^2(X_{n+2,k,a})$ and $\chi_h(X_{n,k,a+3}) = \chi_h(X_{n+2,k,a})$, hence in most cases we can arrange $a \geq 3$. This observation turns out to be important in computing the fundamental groups of the complex surfaces constructed.

The general construction of surfaces with characteristic numbers in the region $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid 2x - 6 \leq y \leq 8x\}$ follows a similar pattern; one just has to replace $\mathbb{F}_n$ with the geometrically ruled surface $\mathbb{G}_{n,g}$ (cf. Section 3.4). Here we restrict ourselves only to a short outline of the construction. Consider the complex surface $X^n_{g,c}$ which is the double branched cover of $\mathbb{G}_{n,g}$ along a smooth curve E_c with $[E_c] = 6[S_n] + 2c[F_n]$. Applying the solution of Exercise 7.4.1(b), one can easily compute the characteristic numbers of $X^n_{g,c}$:

Lemma 7.4.13. ([**Pe**]) *The characteristic numbers of $X^n_{g,c}$ are given by $c_1^2(X^n_{g,c}) = 8g - 8 + 2(2c + 3n)$ and $\chi_h(X^n_{g,c}) = g - 1 + 2c + 3n$. The subset of the region $R = \{(x, y) \mid 2x - 6 \leq y \leq 8x\}$ whose points have the form $x = \chi_h(X^n_{g,c})$, $y = c_1^2(X^n_{g,c})$ for some n, g, c forms a sublattice of R with coarea 6.* □

In order to fill up the gaps one has to introduce essential singularities (e.g., infinitely close triple points) on the curves E_c. Strategy similar to that of the proof of Lemma 7.4.11 will provide the required curve, so the following theorem can be proved.

Theorem 7.4.14. *If $(x, y) \in \mathbb{N} \times \mathbb{N}$ satisfies $2x - 6 \leq y \leq 8x$, then there is a minimal surface S of general type with $\chi_h(S) = x$ and $c_1^2(S) = y$.* □

(For details of the proof see [**Pe**]. Persson proved the above theorem for pairs satisfying $2x - 6 \leq y \leq 8x - 20$, then Xiao removed the constant -20 and proved Theorem 7.4.14 in the form it is stated above, see [**Xi**].) Note that the construction of S provides a fibration on it — in this case, however, the complex surface fibers over a complex curve Σ with possibly nonzero genus g.

The above theorem does not provide simply connected examples — although we are mainly interested in the simply connected case. As we have already mentioned, the fundamental group of a branched cover is sometimes hard to compute. The following lemma (essentially from [**Pe**]) gives a sufficient condition for a fibration to be simply connected. For the proof, see Exercise 8.1.10(b).

Lemma 7.4.15. *Let X be a closed 4-manifold with $\pi\colon X \to \mathbb{CP}^1$ holomorphic (or more generally, locally modeled at each point of X by a holomorphic map). If some fiber $\pi^{-1}(p)$ is simply connected and each fiber contains a point where $d\pi$ is surjective, then X is simply connected.* □

Exercise 7.4.16. * Using Lemma 7.4.15, show that the surfaces $X(n,m)$ are simply connected (cf. also Exercise 7.3.16(a)).

Applying this result to the geography of simply connected surfaces, Persson proved the following.

Theorem 7.4.17. *If $2x - 6 \leq y \leq 8(x - 5x^{2/3})$ and $y > 0$, then there exists a minimal, simply connected complex surface (of general type) such that $\chi_h(S) = x$ and $c_1^2(S) = y$.* □

These surfaces are rather large by the standards of topologists: It is easy to see that the inequalities imply $x \geq 294$, so $b_2^+(S) \geq 587$ and $b_2^-(S)$ is considerably larger (since the signature $\sigma(S)$ is negative). The geography of simply connected surfaces with small b_2 is still somewhat mysterious. Lemma 7.4.15 can be applied directly to the examples $X_{n,k,a}$ of Theorem 7.4.8 — by Remark 7.4.12 all these examples admit the required fibration over $\mathbb{CP}^1$, and if the branch curve contains at least one fiber of $\mathbb{F}_n$, then the fibration on $X_{n,k,a}$ will have a fiber satisfying the required hypotheses (cf. the solution of Exercise 7.4.16). It can be arranged that $B_{n,k,a}$ contains fiber components, for example, if a is not divisible by 3, or if we choose $a \geq 3$. On the other hand, Theorem 7.4.14 gives complex surfaces fibered over complex curves with nonzero genus, and these will never be simply connected. For more about the construction of simply connected surfaces, see [**Pe**].

One would also like to determine the parity of the intersection form of a double branched cover — recall that for a simply connected surface S the invariants $\chi_h(S)$, $c_1^2(S)$ and the parity of Q_S are enough information to determine the homeomorphism type of S. For a sufficient condition for a double cover S to be spin (i.e., Q_S even) see [**PPX**]. Recall that the invariants $(\chi_h(S), c_1^2(S))$ of a simply connected spin complex surface satisfy the relations $c_1^2(S) \equiv 0 \pmod 8$ and $\frac{c_1^2(S)}{8} \equiv \chi_h(S) \pmod 2$. (Cf. the text after Theorem 3.4.19.) Consequently, if $(x, y) \in \mathbb{N} \times \mathbb{N}$ does not satisfy the above constraints, the homeomorphism type of the corresponding simply connected complex surface is unique (and nonspin). For the geography of spin surfaces the following result has been obtained.

Theorem 7.4.18. ([**PPX**]) *Suppose that the pair* $(x, y) \in \mathbb{N} \times \mathbb{N}$ *satisfies* $y \equiv 0 \pmod 8$ *and* $\frac{y}{8} \equiv x \pmod 2$.

1. *If* $\frac{16}{5}(x-4) \leq y < 8x - 271x^{\frac{3}{4}}$, *then there exists a simply connected spin surface* S *with* $\chi_h(S) = x$ *and* $c_1^2(S) = y$.
2. *Suppose that* $3(x-5) \leq y < \frac{16}{5}(x-4)$ *and* $\frac{y}{8} + x \equiv 2 \pmod 4$. *Then there exists a simply connected spin surface* S *with* $\chi_h(S) = x$ *and* $c_1^2(S) = y$. *Hence in that region roughly half of the possible points correspond to simply connected spin surfaces (and nothing is said about the points satisfying* $\frac{y}{8} + x \equiv 0 \pmod 4$*)*.
3. *If* S *is a simply connected spin surface with* $2\chi_h(S) - 6 \leq c_1^2(S) < 3(\chi_h(S) - 5)$, *then either* $c_1^2(S) = 2(\chi_h(S) - 3) \equiv 8 \pmod{16}$ *or* $c_1^2(S) = \frac{8}{3}(\chi_h(S) - 4)$ *with* $\chi_h(S) \equiv 1 \pmod 3$. *The points allowed by these constraints can be realized by simply connected spin surfaces.* □

Remark 7.4.19. After seeing Theorem 7.4.14 (and Theorem 7.4.23), one has the impression that most points in the region defined by the Noether and the Bogomolov-Miyaoka-Yau inequalities correspond to complex surfaces (see Figure 7.7). As Theorem 7.4.18(3) shows, once one poses the same question for simply connected spin surfaces, there are "gaps" in the geography.

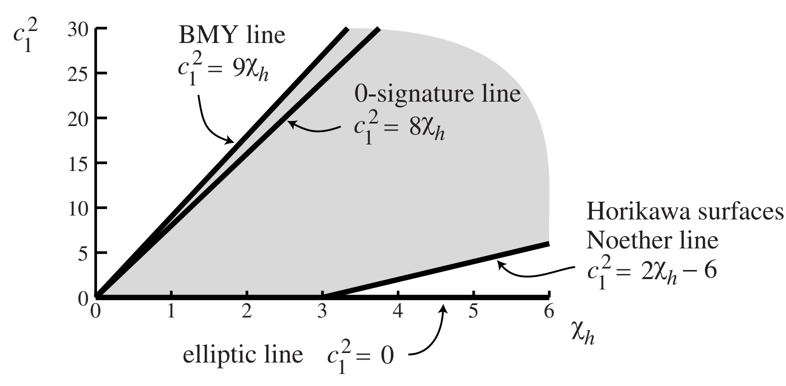


Figure 7.7. Geography of minimal complex surfaces of general type.

Recall that besides the geography, one would also like to get information about the *botany* of surfaces of general type: one would like to describe *all* minimal surfaces with a given pair of invariants (χ_h, c_1^2). Not much is known about the solution of this problem. By Theorem 3.4.17 the number of nondiffeomorphic (or even deformation inequivalent) minimal surfaces of general type within a given homeomorphism type is finite; due to a theorem of Salvetti [**Sv**], however, this number can be arbitrarily large (even in the simply connected case). Results of Horikawa describing all surfaces with $c_1^2 = 2\chi_h - 6 + k$ for $k = 0, 1, 2, 3$ give examples of the few botanical results [**Hr1**]. We only give one example of these results, when $k = 0$.

Theorem 7.4.20. (Horikawa, [**Hr1**]) *Suppose that* $y = 2x - 6 > 0$*. If* y *is not divisible by 8, then there is a unique deformation equivalence class of minimal complex surfaces of general type with the prescribed invariants* $(\chi_h, c_1^2) = (x, y)$*, given by the surface* $H(n) = X(3, n)$ *of Section 7.3 if* $y \equiv 0 \pmod 4$ *(with* $n = \frac{1}{4}y + 2$*) and* $Z(n)$ *of Exercise 7.4.1(a) for* $y \equiv 2 \pmod 4$ *(with* $n = \frac{1}{4}(y+2)$*). If* y *is divisible by 8, then there are exactly 2 deformation equivalence classes of surfaces representing the pair* (x, y)*; the surfaces* $H(2n)$ *and* $H'(n) = U(3, n)$ *(corresponding to* $n = \frac{1}{8}y + 1$*) discussed in Section 7.3. All these surfaces are simply connected and admit genus-2 fibrations.* □

Remark 7.4.21. Note that $W(4)$ and $W(5)$ (of Exercise 7.3.27) also satisfy $c_1^2 = 2\chi_h - 6$, hence by the above theorem $W(4)$ is deformation equivalent to $Z(1)$ of Exercise 7.4.1(a), and $W(5)$ (which is spin) is deformation equivalent to $H'(2)$.

We close this chapter by listing theorems and examples concerning surfaces with characteristic numbers $8\chi_h < c_1^2 \leq 9\chi_h$ — or equivalently $2c_2 < c_1^2 \leq 3c_2$. Note that since the signature $\sigma(S)$ of a complex surface S is given by the formula

$$\sigma(S) = \frac{1}{3}(c_1^2(S) - 2c_2(S)),$$

the surfaces in the "arctic region" $8\chi_h < c_1^2 \leq 9\chi_h$ are the ones with positive signature. We begin by listing a few theorems concerning the geography problem for surfaces with positive signature.

Theorem 7.4.22. (Chen, [**Ch2**]) *If* $8x \leq y \leq 9x - 347$*, then there is a minimal surface* S *of general type with* $x = \chi_h(S)$ *and* $y = c_1^2(S)$*.* □

Later in this section we will outline the strategy of the proof of Theorem 7.4.22. From the construction it will be obvious that the surfaces provided by Theorem 7.4.22 have large fundamental groups; most of these examples are out of reach for a topologist favoring, e.g., Kirby calculus. Putting Theorem 7.4.22 together with Theorem 7.4.14 we get the following:

Theorem 7.4.23. *If* $(x, y) \in \mathbb{N} \times \mathbb{N}$ *satisfies* $2x - 6 \leq y \leq 9x - 347$*, then there exists a minimal surface of general type corresponding to the given point* (x, y)*.* □

The upper borderline of the region $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 8x \leq y \leq 9x\}$ — where $y = 9x$ — is frequently called the *Bogomolov-Miyaoka-Yau (BMY) line.* Complex surfaces of general type satisfying $c_1^2(S) = 9\chi_h(S)$ are handled by the following theorem.

Theorem 7.4.24. ([**Hi1**], [**Y**], [**My**]) *If S is a complex surface of general type with $c_1^2(S) = 9\chi_h(S)$, then the universal cover of S is biholomorphic to the unit disk $D^4 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$, hence in particular $|\pi_1(S)| = \infty$. Conversely, if the universal cover of a compact complex surface is biholomorphic to the unit disk, then $c_1^2(S) = 9\chi_h(S)$.* □

Note that Theorem 7.4.22 does not guarantee the existence of *simply connected* surfaces in the arctic region. Constructions of Moishezon-Teicher [**MT**], Chen [**Ch1**] and Peters-Persson-Xiao [**PPX**] provide simply connected surfaces with positive signature; we will not discuss these constructions here. (For other results concerning the arctic region see [**BPV**].) In the following, we describe a few constructions for manifolds in the arctic region.

Exercise 7.4.25. * Consider the complex surface $X = C_1 \times C_2$ where the complex curves C_i have genus $g(C_i)$. Find a basis for $H_2(X; \mathbb{Z})$ and determine $c_1(X)$, $c_1^2(X)$, $c_2(X)$, $\chi_h(X)$ and the signature $\sigma(X)$.

Next we provide a surface on the Bogomolov-Miyaoka-Yau line $c_1^2 = 9\chi_h$. First we find a map $\varphi\colon G \to \mathbb{CP}^1$ from the (real 2-dimensional) surface G of genus 2 to $\mathbb{CP}^1$ with the property that φ is $5:1$ except over three points $Q_1, Q_2, Q_3 \in \mathbb{CP}^1$, each of which has only one inverse image. Take the (singular) curve $G_1 = \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0^5 - x_1^3 x_2(x_1 + x_2) = 0\}$ in $\mathbb{CP}^2$ and blow up $\mathbb{CP}^2$ at $[0 : 0 : 1]$ (the singular point of the curve G_1). The proper transform $\widetilde{G}_1$ still has one singular point, but the proper transform G of an additional blow-up will be smooth. Hence we have found a smooth curve G in $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$; restricting the blow-down map $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2} \to \mathbb{CP}^2$ to G and composing it with the projection $\mathbb{CP}^2 - [1 : 0 : 0] \to \{x_0 = 0\} \approx \mathbb{CP}^1$ (mapping $[x_0 : x_1 : x_2]$ to $[0 : x_1 : x_2]$), we get $\varphi\colon G \to \mathbb{CP}^1$ with the properties described above. Note that we have simply given an explicit description of a 5-fold cyclic branched cover $G \to \mathbb{CP}^1$ branched at the three points $Q_1, Q_2, Q_3 \in \mathbb{CP}^1$. The corresponding $\mathbb{Z}_5$-action on G is generated by $\gamma([x_0 : x_1 : x_2]) = [e^{2\pi i/5} x_0 : x_1 : x_2]$. The fixed points of γ are the inverse images of Q_k $(k = 1, 2, 3)$ in G (also denoted by $Q_k \in G$ $(k = 1, 2, 3)$). Obviously we have $G/\langle\gamma\rangle \approx \mathbb{CP}^1$.

Exercise 7.4.26. * Show that the complex curve G constructed above has genus 2.

Let Δ denote the diagonal of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Take the inverse image of Δ in $G \times G$ via $\varphi \times \varphi\colon G \times G \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and denote it by $F \subset G \times G$. Note that since $[\Delta] = [\mathbb{CP}^1 \times \{\text{pt.}\}] + [\{\text{pt.}\} \times \mathbb{CP}^1] \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z})$, we have that $[F] = 5([G \times \{\text{pt.}\}] + [\{\text{pt.}\} \times G]) \in H_2(G \times G; \mathbb{Z})$.

Lemma 7.4.27. *F consists of the union of 5 curves $F_1, \dots, F_5$, each isomorphic to G. Each F_i goes through the points (Q_1, Q_1), (Q_2, Q_2) and (Q_3, Q_3); F_i intersects F_j transversally in (Q_k, Q_k) $(k = 1, 2, 3)$, and otherwise these curves are disjoint. Moreover, $[F_i]^2 = -2$ for $1 \leq i \leq 5$.* □

Proof. The curve F_i can be given as the graph of the map $\gamma^i \colon G \to G$. (Recall that γ generates the $\mathbb{Z}_5$-action on G given by the cyclic branched cover $\varphi \colon G \to \mathbb{CP}^1$.) This implies that $[F_i]^2 = [F_j]^2$. Since F_5 is the diagonal of $G \times G$, we have $[F_5]^2 = \langle e(\nu F_5), [F_5] \rangle = \langle e(TF_5), [F_5] \rangle = -2$. Since $(\sum [F_i])^2 = (5([G \times \{\text{pt.}\}] + [\{\text{pt.}\} \times G]))^2 = 50$ and the F_i are distinct complex curves, the lemma follows. □

If we blow up $G \times G$ at the points (Q_k, Q_k) $(k = 1, 2, 3)$, the proper transform of F consists of 5 disjoint curves $\widetilde{F}_1, \dots, \widetilde{F}_5 \subset G \times G \# 3\overline{\mathbb{CP}^2}$. Since in the second homology group of $G \times G \# 3\overline{\mathbb{CP}^2}$ we have $[\widetilde{F}_1] + \ldots + [\widetilde{F}_5] = 5([G \times \{\text{pt.}\}] + [\{\text{pt.}\} \times G]) - 5e_1 - 5e_2 - 5e_3$ (where e_i is the exceptional sphere of the $i^{\underline{th}}$ blow-up), we can use Construction 7.1.1 with $d = 5$ to take the 5-fold cyclic branched cover of $G \times G \# 3\overline{\mathbb{CP}^2}$ along $\widetilde{F}_1 \cup \ldots \cup \widetilde{F}_5$. We denote the resulting smooth surface by H. By generalizing Lemma 7.1.7 to 5-fold covers (cf. Remark 7.1.8(b)) we can easily compute the characteristic numbers of H:

Lemma 7.4.28. *The Euler characteristic $\chi(H)$ of H is equal to 75 and $c_1^2(H) = 225$, hence $\chi_h(H) = \frac{1}{12}(c_2(H) + c_1^2(H)) = 25$. Consequently $c_1^2(H) = 9\chi_h(H)$, so H is on the Bogomolov-Miyaoka-Yau line.* □

From this basic example other arctic surfaces can be produced, leading to a proof of Theorem 7.4.22. First note that the composition of the maps $H \to G \times G \# 3\overline{\mathbb{CP}^2} \to G \times G \xrightarrow{\text{pr}} G$ gives a fibration of H over G, the regular fiber being a curve which is a 5-fold cover of G branched at 5 points (hence a curve of genus 16). For a given number n, take an n-fold (unbranched) cover $\tau \colon C' \to G$, and let X_n denote the pullback of $H \to G$ via τ. Next take a double branched cover $C \to C'$ branched at generic points $p_1, \dots, p_{2d} \in C'$. (Here generic means that the fiber of $H \to G$ over $\tau(p_i)$ is smooth.) Pulling the fibration $H \to G$ back via $C \to G$, we get a surface $X_{n,d} \to C$ with characteristic numbers

$$c_1^2(X_{n,d}) = 450n + 120d \quad \text{and} \quad \chi_h(X_{n,d}) = 50n + 15d,$$

obviously satisfying $8\chi_h(X_{n,d}) \leq c_1^2(X_{n,d}) \leq 9\chi_h(X_{n,d})$. If $d = 0$, i.e., we have an unbranched $2n$-fold cover $X_{n,0}$ of H, then $c_1^2(X_{n,0}) = 3c_2(X_{n,0}) = 6nc_2(H)$. (Note that for $d = 0$ the map $C \to G$ is a $2n$-fold cover, since the double cover $C \to C'$ is not branched in this case.) The pullback of

$X_n \to C'$ to the double branched cover $C \to C'$ is just a double branched cover $X_{n,d} \to X_n$ branched along the fibers over the points $p_1, \ldots, p_{2d}$; in particular, for $d = 1$ the manifold $X_{n,1}$ is just the generalized fiber sum $X_n \#_f X_n$.

Exercise 7.4.29. Verify the expressions for $c_1^2(X_{n,d})$ and $\chi_h(X_{n,d})$ given above.

Since H is a cyclic 5-fold branched cover, it admits a $\mathbb{Z}_5$-action mapping each fiber of $H \to G$ into itself. By construction, this $\mathbb{Z}_5$-action can be pulled back, defining a $\mathbb{Z}_5$-action α (with $\alpha^5 = \mathrm{id}_{X_{n,d}}$) on $X_{n,d}$. Note that this action has 5 fixed points in every fiber — the points corresponding to the intersections of the fiber with the inverse image of the branch locus of the branched cover $H \to G \times G\#3\overline{\mathbb{CP}^2}$. By pulling back $X_{n,d} \to C$ to a cyclic 5-fold branched cover $\varphi\colon \widetilde{C} \to C$, we obtain another $\mathbb{Z}_5$-action β (with $\beta^5 = \mathrm{id}_{X_{n,d}(\varphi)}$) on the resulting complex surface $X_{n,d}(\varphi)$:

$$\begin{array}{ccc} X_{n,d}(\varphi) & \longrightarrow & X_{n,d} \\ \downarrow & & \downarrow \\ \widetilde{C} & \xrightarrow{\varphi} & C \end{array}$$

(For this, regard $X_{n,d}(\varphi)$ as the pullback of $\widetilde{C} \to C$ by the map $X_{n,d} \to C$ and pull back the $\mathbb{Z}_5$-action provided by the 5-fold cyclic branched covering $\widetilde{C} \to C$.) We still have the fibration of $X_{n,d}(\varphi)$ with curves of genus 16 as fibers, and a $\mathbb{Z}_5$-action (still denoted by α) mapping each fiber into itself — this can be seen by regarding $X_{n,d}(\varphi)$ as the pullback of the fibration $X_{n,d} \to C$ by the map $\varphi\colon \widetilde{C} \to C$. Note that β has fixed points only over the branch points of $\varphi\colon \widetilde{C} \to C$. Since the two actions (α and β) commute, the product $\alpha\beta$ defines a $\mathbb{Z}_5$-action on $X_{n,d}(\varphi)$ with isolated fixed points (the intersections of the two fixed point sets). Resolving the singularities of the quotient of $X_{n,d}(\varphi)$ by the $\mathbb{Z}_5$-action generated by $\alpha\beta$ defines a complex surface $Z(n, d, \varphi)$. For appropriately chosen parameters n, d and φ, the computation of $\chi_h(Z(n, d, \varphi))$ and $c_1^2(Z(n, d, \varphi))$ proves Theorem 7.4.22. (For further details see [**Ch2**].)

Remark 7.4.30. Each singularity of the quotient $X_{n,d}(\varphi)/\mathbb{Z}_5$ is a cone over the quotient of S^3 by a $\mathbb{Z}_5$-action, which is a lens space $L(5, q)$. The value q depends on the type of the $\mathbb{Z}_5$-action at the point at hand; this is determined by the cyclic branched covering φ. Since $L(5, 2) \approx L(5, 3)$, we have 3 possibilities at each singular point.

Exercise 7.4.31. Using a continued fraction expansion of $-\frac{5}{q}$, find a smooth, negative definite 4-manifold with boundary $L(5, q)$ (cf. Exercise 5.3.9(b)). Using these plumbing manifolds, describe the topology of the resolutions in $Z(n, d, \varphi)$.

While these examples are generally not simply connected, Chen also described constructions of simply connected surfaces in the arctic region [**Ch1**]. The strategy he used is very similar to the one invented by Persson (and described earlier in this section) — Chen introduced various essential singularities on the branch curve and then resolved the corresponding branched covers; for details see [**Ch1**]. The comment made after Theorem 7.4.17 also applies to these surfaces: by topological standards these manifolds are enormously large (for example, $\chi_h > 10^8$).

Exercise 7.4.32. Show that the set $R = \{\frac{c_1^2(X_{n,d})}{\chi_h(X_{n,d})} \mid n \geq 1, \ d \geq 0\}$ is dense in the interval $[8, 9]$. (*Hint*: Recall that $c_1^2(X_{n,d}) = 450n + 120d$, $\chi_h(X_{n,d}) = 50n + 15d$ and show that for $n = 3(q - p)$, $d = 10p$ ($0 < p < q$ integers) the ratio $\frac{c_1^2(X_{n,d})}{\chi_h(X_{n,d})}$ is equal to $9 - \frac{p}{q}$. See also [**So**].)

Chapter 8

Elliptic and Lefschetz fibrations

In Chapters 3 and 7, we encountered elliptic surfaces in several different contexts. We saw that they were a sufficiently large class of complex surfaces to exhibit behavior such as infinitely many diffeomorphism types within a homeomorphism type. On the other hand, the fact that they admit elliptic fibrations gives us a way to understand their topology in detail. In this chapter, we will analyze the topological structure provided by an elliptic fibration. This will shed light on the classification of elliptic surfaces, and allow us to draw Kirby diagrams of these manifolds. Elliptic fibrations naturally generalize to structures called *Lefschetz fibrations* and *Lefschetz pencils*. As we will see, any projective surface admits a Lefschetz pencil. More generally (Theorem 10.2.28), any closed symplectic 4-manifold admits a Lefschetz pencil and conversely, so Lefschetz pencils topologically characterize an important geometric property, namely whether the manifold supports a symplectic structure. On the other hand, a sufficiently detailed understanding of a Lefschetz pencil on a 4-manifold allows one to draw a Kirby diagram of it. Our technique for drawing Kirby diagrams of Lefschetz and elliptic fibrations is generalized from [**HKK**]; for more on (holomorphic) Lefschetz pencils (in arbitrary dimensions) and their use in understanding the topology of projective varieties, see [**Lm**]. The theory of Lefschetz fibrations has been developing rapidly in recent months, perhaps in response to recent connections with symplectic geometry. We have attempted to include some of the most recent references, but the list will undoubtedly be outdated by the time this text reaches the bookshelves.

8.1. Lefschetz pencils and fibrations

We begin by constructing a fibration-like structure on an arbitrary (smooth) projective surface $S \subset \mathbb{CP}^N$; this will be our prototype of a Lefschetz pencil. Let $A \subset \mathbb{CP}^N$ be a generic linear subspace of (complex) codimension 2. Then A is transverse to S, so it intersects S in a finite number n of points. Since each intersection is positive, n equals $A \cdot S$ and is independent of A (and we have $[S] = n[\mathbb{CP}^2] \in H_4(\mathbb{CP}^N;\mathbb{Z}) \cong \mathbb{Z}$). Since A can easily be chosen to intersect S, we have $n > 0$. We call n the *degree* of S, A the *axis* of the pencil we will construct, and $B = A \cap S \neq \emptyset$ the *base locus* of the pencil. Now consider the set of all hyperplanes (complex codimension-1 linear subspaces) H_t of $\mathbb{CP}^N$ containing A. These are parametrized by $\mathbb{CP}^1$ — in fact, A is determined by a pair of homogeneous linear equations $p_0(z) = p_1(z) = 0$, and for $t = [t_0 : t_1] \in \mathbb{CP}^1$, we can take H_t to be the zero set of the linear polynomial $t_0p_0 + t_1p_1$. Setting $F_t = H_t \cap S$, we obtain a family of (possibly singular) complex curves in S parametrized by $\mathbb{CP}^1$. The union of these curves is S, and any two intersect precisely in B (since the analogous statements hold for the hyperplanes $H_t \subset \mathbb{CP}^N$). The family $\{F_t \mid t \in \mathbb{CP}^1\}$ is a Lefschetz pencil on S; see Figure 8.1. Note that it is not a singular fibration, since the projection to $\mathbb{CP}^1$ cannot be defined on B. However, the transversality of A implies that in a neighborhood of B each F_t is smoothly embedded, so we can blow up B to obtain a complex surface $S' \approx S \# n\overline{\mathbb{CP}^2}$ in which the proper transforms F'_t are disjointly embedded. The resulting holomorphic map $\pi\colon S' \to \mathbb{CP}^1$ defined by $\pi(F'_t) = \{t\}$ is our prototype for a Lefschetz fibration (Figure 8.1). Clearly, each of the n exceptional curves E_i is a section.

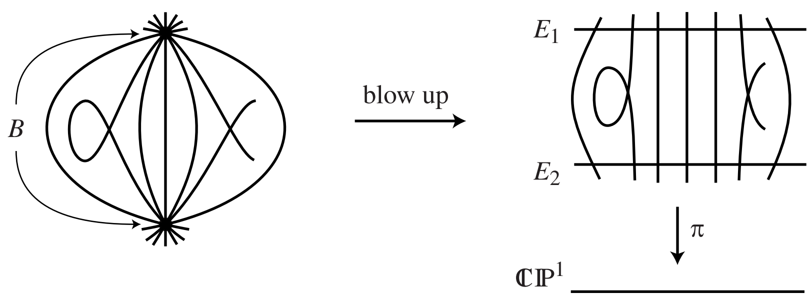


Figure 8.1. Lefschetz pencil and corresponding Lefschetz fibration.

Exercises 8.1.1. (a)* Apply this construction to the hypersurface S_d of degree d in $\mathbb{CP}^3$ to obtain a Lefschetz fibration $S_d \# d\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. What is the genus of a generic fiber? Identify the fibration when $d = 1, 3$. (*Hint*: Lemma 3.1.17.)

(b) Let $q_0, \dots, q_N$ denote all homogeneous monomials of degree d in the variables z_0, z_1, z_2. Show that the formula $q(z) = [q_0(z) : \dots : q_N(z)]$ gives a well-defined embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^N$. (This is called the *Veronese embedding*.) Show that preimages of hyperplanes in $\mathbb{CP}^N$ are precisely the same as (possibly singular) degree-d curves in $\mathbb{CP}^2$. Now show that the above construction of a Lefschetz pencil on $q(\mathbb{CP}^2)$ is equivalent to picking a generic pair of degree-d polynomials p_0, p_1 and considering the family of curves $F_t = \{z \in \mathbb{CP}^2 \mid t_0 p_0(z) + t_1 p_1(z) = 0\}$, $t = [t_0 : t_1] \in \mathbb{CP}^1$, as in Lemma 3.1.4. What is the degree of $q(\mathbb{CP}^2)$? Prove that the hypersurface S_d in $\mathbb{CP}^3$ (or more generally, any smooth hypersurface in $\mathbb{CP}^k$ ($k \geq 3$), as in Definition 1.3.7) is simply connected by constructing a suitable Veronese embedding $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^N$ and applying the Lefschetz Hyperplane Theorem 1.4.22.

As the last exercise shows, we have seen Lefschetz pencils and fibrations before — when we considered pencils of curves in $\mathbb{CP}^2$ in and preceding Lemma 3.1.4. In particular, these fibrations for $d = 1, 3$ are the usual fibration of the Hirzebruch surface $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and an elliptic fibration $E(1) \to \mathbb{CP}^1$, respectively.

We wish to understand the critical points of the Lefschetz fibration $\pi\colon S' \to \mathbb{CP}^1$ constructed above. For any nonconstant holomorphic map $f\colon S \to \Sigma$ from a compact complex surface to a connected complex curve, generic points of Σ will be regular values, by Sard's Theorem [**GP**]. Since the critical values form a projective subvariety of Σ with nonzero codimension, they must be a finite set. Away from this set, f will be a fiber bundle projection with connected base space, so all but finitely many fibers of f will be complex submanifolds of S with a fixed diffeomorphism type. The set of critical points of f in S, however, may have components with complex dimension 1 — that is, df may vanish everywhere along a curve C in a fiber of f. At any smooth point of C, there will be local coordinates in which f is given by $f(z_1, z_2) = z_1^m$ for some $m \geq 2$. (Consider a multiple fiber in an elliptic fibration or the singular fibers of $X(m, n)$ from Section 3.2.) Such a curve C cannot occur in the Lefschetz fibration π constructed above, however, since C would intersect some section E_i in a smooth point, and we would then have the contradiction that $E_i \cdot F \geq m \geq 2$ for a generic fiber F. (Note that any curve in a fiber $F_t \subset S$ must intersect B, since it has nonzero degree in $H_t \cong \mathbb{CP}^{N-1}$.) Thus, π has only finitely many critical points in S'. Since the axis A of the pencil was chosen generically, the critical points of π will be generic. Recall that a generic real-valued smooth function has Morse critical points that are quadratic in local coordinates. Similarly, a generic holomorphic isolated critical point has holomorphic local coordinates in which the function is given by $\pi(z_1, z_2) = z_1^2 + z_2^2$, or (after a linear change of coordinates) $\pi(z_1, z_2) = z_1 z_2$. (In contrast with the real-valued case, there is no index, since we can reverse signs by multiplying coordinates by i.)

Exercises 8.1.2. (a)* Define π near 0 in $\mathbb{C}^2$ by $\pi(z_1, z_2) = z_1^{m_1} + z_2^{m_2}$. Find a perturbation of π with only quadratic critical points. How many critical points does it have?

(b) Prove Proposition 3.1.5. (*Hint*: In the local coordinates given above, the zero locus of the map π at a generic critical point is the union of the lines $z_2 = \pm i z_1$. Use the obvious action of the connected group $GL(3, \mathbb{C})$ on $\mathbb{CP}^2$ to reduce to the case of a cubic curve in $\mathbb{CP}^2$ with a unique generic singularity at $[0 : 0 : 1]$, tangent to the lines $y = \pm x$ in the local coordinates $(x, y) \mapsto [x : y : 1]$. Show that the corresponding cubic equation is given by $zy^2 = zx^2 + p(x, y)$ for some homogeneous cubic polynomial p. Reduce to the case $p(x, y) = x^3$ as in Claim 1.3.11.)

Remarks 8.1.3. (a) We have already studied smooth multiple fibers in elliptic fibrations (Section 3.3). A similar phenomenon occurs in higher-genus singular fibrations, although the multiple fiber F_m will have a different genus from the generic fiber F (since $\chi(F) = m\chi(F_m)$). The key idea is that the 3-manifold $S^1 \times F_m$ can be realized as the associated S^1-bundle of a nontrivial $\mathbb{Z}_m$-bundle over F_m, and this exhibits a fibering of it by surfaces F m-fold covering F_m. There is no higher-genus analog of logarithmic transformations, however. In fact, it is impossible to change the diffeomorphism type of a 4-manifold by removing a closed, orientable genus-g surface with trivial normal bundle ($g \geq 2$) and gluing in $D^2 \times F$ by a diffeomorphism of the boundaries: The two surfaces would have the same genus (as determined by $b_1(S^1 \times F)$), and standard 3-manifold theory shows that any self-diffeomorphism of $S^1 \times F$ must preserve the framed circle $S^1 \times \{\text{pt.}\}$ up to isotopy and orientation reversal. (The fact that the circle is preserved up to *homotopy* and orientation reversal follows easily from the triviality of the center of $\pi_1(F)$.) Since gluing $D^2 \times F$ to a 4-manifold along its boundary is the same as attaching a 2-handle and 3- and 4-handles, the resulting diffeomorphism type is unique.

(b) Fibers of arbitrary holomorphic maps from surfaces to curves can be quite complicated. However, they have been classified up to fiber-preserving homeomorphisms of a regular neighborhood [**MtM**].

We are now ready to define Lefschetz pencils and fibrations on smooth 4-manifolds.

Definition 8.1.4. Let X be a compact, connected, oriented, smooth 4-manifold.

(a) For $\partial X = \emptyset$, a *Lefschetz pencil* on X is a nonempty finite subset B of X, called the *base locus*, together with a smooth map $\pi\colon X - B \to \mathbb{CP}^1$ such that each point $b \in B$ has an orientation-preserving coordinate chart in which π is given by projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$, and each critical point of π has an orientation-preserving chart on which $\pi(z_1, z_2) = z_1^2 + z_2^2$ relative to a suitable smooth chart on $\mathbb{CP}^1$. For $t \in \mathbb{CP}^1$, the *fiber* F_t is $\pi^{-1}(t) \cup B \subset X$.

(b) A *Lefschetz fibration* on X is a map $\pi\colon X \to \Sigma$, where Σ is a compact, connected, oriented surface and $\pi^{-1}(\partial\Sigma) = \partial X$, such that each critical point of π lies in int X and has a local coordinate chart as in (a).

We can assume without loss of generality that the given charts on Σ (or $\mathbb{CP}^1$) preserve orientation, since complex conjugation leaves π invariant and reverses orientation on $\mathbb{C}$ but not on $\mathbb{C}^2$. Clearly, each fiber F_t of a Lefschetz pencil or fibration is compact and canonically oriented, and in the former case F_t intersects some neighborhood of B in a smooth surface. Blowing up the base locus turns a Lefschetz pencil into a Lefschetz fibration over S^2, with each exceptional sphere a section. In either case, π has only finitely many critical points, and removing the corresponding singular fibers turns a Lefschetz fibration into a fiber bundle with a connected base space. Thus, all but finitely many fibers of a Lefschetz pencil or fibration are smooth, closed surfaces, all of which have the same diffeomorphism type. If a generic fiber is connected (as we will show is always true when $\Sigma = S^2$, Proposition 8.1.9), its genus will be called the *genus* of the Lefschetz pencil or fibration. By perturbing π if necessary, we can arrange it to be injective on its set of critical points, so that each singular fiber has a unique singularity. (Some authors include this in the definition.)

Examples 8.1.5. It is immediate that the pencil and singular fibration we constructed from an arbitrary projective surface S satisfy Definition 8.1.4. In particular, a generic pencil of degree-d curves on $\mathbb{CP}^2$ will be a Lefschetz pencil of genus $\frac{(d-1)(d-2)}{2}$, inducing a Lefschetz fibration of the same genus on $\mathbb{CP}^2 \# d^2 \overline{\mathbb{CP}^2}$. A generic elliptic fibration on $E(n)$ will have only fishtails as singular fibers (Proposition 3.1.5), so it will also be a Lefschetz fibration. In contrast, elliptic fibrations $E(n)_{p_1,\dots,p_k}$ with multiple fibers are not Lefschetz, and neither are the obvious fibrations on the manifolds $X(m,n)$ of Section 3.2 (since each singular fiber will have a component with multiplicity 2). Of course, these manifolds admit higher-genus Lefschetz pencils since they are projective. We can obtain an elliptic Lefschetz fibration over a surface Σ of any genus g by forming the fiber sum $E(n,g) = E(n) \#_f \Sigma \times T^2$.

In fact, the fiber sum operation generalizes immediately to Lefschetz fibrations of any fixed genus (cf. Definition 7.1.11), although we will see (Theorem 8.4.3) that a fiber sum of two holomorphic Lefschetz fibrations need not admit a complex structure.

Exercise 8.1.6. * Prove that a Lefschetz pencil or fibration on X determines an almost-complex structure, i.e., a complex vector bundle structure on TX. In particular, for a closed X with a Lefschetz pencil or fibration, $b_2^+(X) - b_1(X)$ must be odd (Theorem 1.4.13 and subsequent text). For more on these almost-complex structures, see the end of Section 8.4.

To understand the singular fibers, we analyze the critical points in more detail. Each critical point has a local coordinate chart in which $\pi(z_1, z_2) = z_1 z_2$. On this chart, the unique critical value is 0, and $\pi^{-1}(0) = \{(z_1, z_2) \mid z_1 = 0 \text{ or } z_2 = 0\}$ is a pair of intersecting planes (a *nodal singularity*). Thus, each singular fiber is a smoothly immersed surface, and each critical point corresponds to a positive transverse self-intersection. Nearby fibers $\pi^{-1}(t)$, $t \neq 0$, are nonsingular, and are obtained from $\pi^{-1}(0)$ by removing the intersection as in Section 2.1. That is, we perform surgery on a 0-sphere in the fiber (the pair of identified points) by removing the intersecting disks and replacing them with the annulus $z_1 z_2 = t$. Equivalently, each critical point corresponds to an embedded surgery circle called a *vanishing cycle* in a nearby regular fiber, and the singular fiber is obtained by collapsing the vanishing cycle to a point to create a transverse self-intersection. (Different critical points on the same singular fiber correspond to disjoint vanishing cycles.) If a vanishing cycle (or union of cycles with a given critical value) separates the generic fiber, the singular fiber will be the image of an immersion of a disconnected surface.

Since each fiber F_t of a Lefschetz fibration is a self-transverse immersed surface, its regular neighborhood νF_t is a plumbing, and all signs of intersection are positive. Thus, once we know the vanishing cycles that determine F_t, the diffeomorphism type of νF_t will be determined by the normal Euler numbers of the components of the surface whose image is F_t. To determine these Euler numbers, we first observe that away from the critical points of π, F_t has a canonical normal framing (obtained by pulling back a basis of $T_t\Sigma$). In the given coordinates around a critical point, this framing is determined on the surface $z_2 = 0$ by the vector field $(0, z_1^{-1})$ (as we see by differentiating π or considering a nearby regular fiber $\pi^{-1}(t)$). Since the vector field has a singularity of degree -1 at $z_1 = 0$, it follows that for a self-transverse immersion $i\colon F' \to X$ with image contained in a fiber and F' connected, the normal Euler number $e(\nu F')$ is minus the number of points of F' mapping into the critical set of π, and $[i(F')]^2$ is minus the number of critical points on $i(F')$ at which i is injective (cf. Exercise 6.1.1(a)). (Note that if we write the homology class $[F_t]$ of any fiber as the sum of such classes $[i(F')]$, we obtain $[F_t]^2 = 0$ as required.) In the most important case, a genus-g Lefschetz fibration with a single critical point on F_t, F_t will be either an immersed surface of genus $g-1$ and square 0 (a fishtail if $g=1$) or a transversely intersecting pair of embedded surfaces with square -1 and genera adding to g, depending on whether the vanishing cycle separates a generic fiber. In a Lefschetz pencil, the corresponding numbers $e(\nu F')$ and $[i(F')]^2$ for any F' as above are obtained from the above formulas by subtracting the number of points in $i(F') \cap B$.

For a simple example of a separating vanishing cycle, consider the effect of blowing up a regular point P in a Lefschetz fibration $\pi\colon X \to \Sigma$. Composing π with the blow-down map $X' \to X$, we obtain a map $\pi'\colon X' \to \Sigma$ which is a Lefschetz fibration. (See the first exercise below.) The fibers of π' are the same as those of π, except that the fiber of π' over $\pi(P)$ is the total transform of the corresponding fiber of π — in particular, it contains the exceptional sphere E_P. Conversely, if any Lefschetz fibration $\pi'\colon X' \to \Sigma$ has a fiber containing an exceptional sphere E, then we can blow down E to obtain a Lefschetz fibration $\pi\colon X \to \Sigma$. To see this, observe that by the previous paragraph, E contains a unique critical point of π' (since $[E]^2 = -1$). Using the given local coordinates at the critical point, it is routine to construct a fiber-preserving diffeomorphism from a neighborhood of E to a neighborhood of our previous model E_P; the map π is now easily constructed. We call a Lefschetz fibration *relatively minimal* if no fiber contains an exceptional sphere. (Some authors include this in the definition of Lefschetz fibrations.) Clearly, any Lefschetz fibration can be blown down to obtain a relatively minimal one. In a relatively minimal Lefschetz fibration, no vanishing cycle bounds a disk in a generic fiber F, so any vanishing cycle in F is nontrivial in $\pi_1(F)$. In particular, there can be no separating vanishing cycle if the genus is ≤ 1. We obtain:

Proposition 8.1.7. *A relatively minimal Lefschetz fibration of genus* 0 *is an S^2-bundle.* □

In the genus-1 case, after we perturb π to be injective on the critical points, all singular fibers must be fishtails. We will see (Theorem 8.3.12) that the only such fibrations without boundary are torus bundles and the generic elliptic fibrations $E(n,g)$ of Examples 8.1.5. The classification problem rapidly becomes more difficult as the genus of the fiber increases.

Exercises 8.1.8. (a) Consider the projection $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ by $\pi(z_1, z_2) = z_1$. Using local coordinates, blow up $0 \in \mathbb{C}^2$ and show that π pulls back to a map π' with a unique critical point, which satisfies the Lefschetz condition. Conclude that any blow-up (at regular points) of a Lefschetz fibration is Lefschetz.

(b)* Consider a generic pencil of quadric curves on $\mathbb{CP}^2$, Exercise 3.1.3. This is a Lefschetz pencil (Examples 8.1.5). Show that the corresponding Lefschetz fibration is obtained by blowing up either of the two S^2-bundles over S^2.

(c) Consider the elliptic Lefschetz fibration on $E(1)$ obtained by blowing up a generic pencil of cubic curves (Section 3.1). Show that this Lefschetz fibration is relatively minimal, although $E(1)$ is not a minimal complex surface. Compare with the degree-1 case, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \approx S^2 \tilde{\times} S^2$.

(d)* How many singular curves are there in a generic degree-d pencil on $\mathbb{CP}^2$? (*Hint*: Blow up to get a Lefschetz fibration, then compute the Euler characteristic.)

(e)* Let $\pi\colon X \to S^2$ be the Lefschetz fibration obtained as an n-fold fiber sum of copies of the Lefschetz fibration of degree-d curves on $\mathbb{CP}^2 \# d^2 \overline{\mathbb{CP}^2}$. Check that X is simply connected, and compute $\chi(X)$ and $\sigma(X)$. (Signature adds under fiber sum by Remark 9.1.7.) What is X for $d = 1, 2, 3$? (Compare with Exercises 7.3.27.)

Now we return to the issue of connectedness of the fiber of a Lefschetz pencil or fibration. We have the following analog of the long exact homotopy sequence of a (homotopy-theoretic) fibration.

Proposition 8.1.9. *For any fiber F of a Lefschetz fibration $\pi\colon X \to \Sigma$, the maps $F \hookrightarrow X \to \Sigma$ induce an exact sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0$.* □

(The first exercise below gives a proof.) It follows that if Σ is simply connected then each fiber of π is connected and carries $\pi_1(X)$. In particular, each fiber of a Lefschetz pencil is connected and carries $\pi_1(X)$, since we can blow up to obtain a Lefschetz fibration over S^2. (This generalizes the Lefschetz Hyperplane Theorem 1.4.22 from the projective case induced by a pencil of hyperplanes.) If a fiber of a Lefschetz fibration is connected, then π induces a surjection on π_1 and H_1. In this case, the condition $b_1(X) = 0$ implies that Σ is S^2 or D^2 (depending on whether $\partial X = \emptyset$). If a fiber is not connected, then $\pi_1(X)$ maps to a finite-index subgroup of $\pi_1(\Sigma)$. Passing to the corresponding finite cover $\widetilde{\Sigma}$ of Σ, we obtain a new Lefschetz fibration $\tilde{\pi}\colon X \to \widetilde{\Sigma}$ that is surjective in π_1, so $\tilde{\pi}$ has connected fibers. This construction allows us to restrict our attention to Lefschetz fibrations whose fibers are connected, without losing generality.

Exercises 8.1.10. (a)* Define the last map in the above sequence, and prove that the sequence is exact. (*Hint*: First show that a neighborhood of any critical value $t \in \Sigma$ admits a section intersecting any preassigned component of F_t. Now define the map by lifting a path of regular values and verify that it is well-defined.)

(b)* Let $\pi\colon X^4 \to \Sigma^2$ be a smooth map between closed, connected, oriented manifolds. Suppose that π has only finitely many critical values, and that the fiber over each critical value t is homeomorphic to a union of closed connected surfaces $\Sigma_1, \dots, \Sigma_n$ (depending on t), with finitely many finite subsets identified to points. Suppose also that away from finitely many points of X, π is locally modeled by $\pi(z, w) = z^m$ ($m \in \mathbb{N}$). The integer m is constant over each Σ_i, and is called the *multiplicity* m_i of Σ_i (cf. Section 7.1). Assume that each critical value t is the endpoint of some smoothly embedded arc $A \subset \Sigma$ for which $\pi^{-1}(A)$ is the mapping cylinder of a map $f\colon F_{t_0} \to F_t$ sending each component of the regular fiber F_{t_0} onto a connected component of F_t. Note that $f_*[F_{t_0}] = \sum_i m_i[\Sigma_i] \in H_2(F_t; \mathbb{Z})$. Define the *multiplicity* of F_t to be $\gcd(m_1, \dots, m_n)$, and define the multiplicity of each connected component of F_t similarly (letting i range over surfaces Σ_i in the given component). If each connected component of each singular fiber has multiplicity 1, prove that Proposition 8.1.9 still applies. Deduce Lemma 7.4.15 as a corollary.

Lefschetz fibrations can be described combinatorially by means of their *monodromy*. For a smooth fiber bundle $\pi\colon E \to B$ with fibers diffeomorphic to a manifold F, we let $\mathcal{M}(F)$ denote the set of isotopy classes of self-diffeomorphisms of F, and define the *monodromy representation* $\Psi\colon \pi_1(B) \to \mathcal{M}(F)$ of π relative to a fixed identification φ of F with the fiber over the base point of B: For each loop $\gamma\colon I \to B$ the bundle $\pi_\gamma\colon \gamma^*(E) \to I$ is canonically trivial, inducing a diffeomorphism $\pi_\gamma^{-1}(0) \to \pi_\gamma^{-1}(1)$ (up to isotopy). Using φ to identify $\pi_\gamma^{-1}(0)$ and $\pi_\gamma^{-1}(1)$ with F, we obtain the element $\Psi(\gamma) \in \mathcal{M}(F)$. It is routine to verify that Ψ is well-defined, and is an *anti*homomorphism relative to composition of functions on $\mathcal{M}(F)$ (since loops concatenate from left to right, whereas diffeomorphisms act on the left). We can make Ψ into a homomorphism by defining a new group structure on $\mathcal{M}(F)$, $\psi_1 * \psi_2 = \psi_2 \circ \psi_1$, so that diffeomorphisms act on the right. The new group structure is isomorphic to the old one via the inversion map $\psi \mapsto \psi^{-1}$, and it can be considered an attempt to rectify the centuries of confusion caused by the first mathematician to write "$f(x)$" instead of "$(x)f$". Changing the identification φ changes Ψ by an inner automorphism, i.e., conjugation by an element of $\mathcal{M}(F)$. For a relatively minimal, genus-g Lefschetz fibration $\pi\colon X \to \Sigma$, we define the monodromy to be that of $\pi|\Sigma^*$, where $\Sigma^* \subset \Sigma$ is the set of regular values. (The definition clearly extends to arbitrary singular fibrations.) We restrict to the subgroup $\mathcal{M}_g \subset \mathcal{M}(F)$ coming from orientation-preserving diffeomorphisms. This subgroup is called the *mapping class group* of the genus-g surface F. The group is finitely presented; for an explicit (complicated) presentation (with diffeomorphisms acting on the right) see [**Wj**]. For $\pi\colon X \to \Sigma$ as above with π injective on its critical points, the monodromy representation $\Psi\colon \pi_1(\Sigma^*) \to \mathcal{M}_g$ determines π up to isomorphism, except in the cases of sphere and torus bundles ($\Sigma^* = \Sigma$) over closed surfaces. (An isomorphism of Lefschetz fibrations $\pi\colon X \to \Sigma$ and $\pi'\colon X' \to \Sigma'$ is a pair of diffeomorphisms $X \to X'$ and $\Sigma \to \Sigma'$ commuting with π and π'.) In fact, two such Lefschetz fibrations are isomorphic if and only if their monodromy representations agree (up to inner automorphisms of $\mathcal{M}_g$ and isomorphisms $\pi_1(\Sigma^*) \to \pi_1(\Sigma'^*)$ induced by smooth maps $\Sigma \to \Sigma'$) [**Mt4**]. (The corresponding statement for arbitrary relatively minimal locally holomorphic fibrations follows from [**MtM**], provided that we exclude multiple fibers when $g = 1$.) As we will see in the next section, one can also characterize which monodromy representations can be realized by Lefschetz fibrations. Thus, the classification of Lefschetz fibrations reduces to a problem in finitely presented group theory.

8.2. The topology of Lefschetz fibrations

We have seen that Lefschetz critical points can be thought of as complex analogs of Morse critical points. We now carry the analogy further, using Lefschetz fibrations and pencils to obtain handle decompositions of 4-manifolds. For any Lefschetz fibration $\pi\colon X \to D^2$, the function $||\pi||^2\colon X \to [0,1]$ will be a Morse function away from 0, with the same critical points as π, allowing us to build X as a handlebody from a neighborhood of the fiber F_0. We will analyze the handle structure carefully, construct the corresponding Kirby diagrams, and finally consider more general base spaces and Lefschetz pencils. As we have seen, we lose no generality by assuming that each fiber of a Lefschetz fibration is connected and has at most one singularity.

First, we show that a Lefschetz critical point corresponds to a 2-handle, and determine its attaching map. Near the critical point, we can write $\pi(z_1, z_2) = z_1^2 + z_2^2$, so a regular fiber is given by $z_1^2 + z_2^2 = t$, and after multiplying π by a unit complex number we can assume $t > 0$. If we intersect the fiber with $\mathbb{R}^2 \subset \mathbb{C}^2$, we obtain the circle $x_1^2 + x_2^2 = t$ in $\mathbb{R}^2$ (where $z_j = x_j + iy_j$). This circle bounds a disk $D_t \subset \mathbb{R}^2$, which was called a *thimble* by Lefschetz. As t approaches 0, the thimble D_t shrinks to a point in $\mathbb{R}^2$. Thus, $\partial D_t = F_t \cap \mathbb{R}^2$ is the vanishing cycle of the critical point, and we explicitly see the singular fiber F_0 being created from F_t by the collapse of D_t. A regular neighborhood νF_0 of the singular fiber is obtained from the neighborhood νF_t by adding a regular neighborhood of D_t. This latter neighborhood is clearly a 2-handle h attached to νF_t. (In fact, a corresponding Morse function can be given locally by $f = -\mathrm{Re}\,\pi$, or $f(z) = y_1^2 + y_2^2 - x_1^2 - x_2^2$.) If $\partial\nu F_t$ is chosen to contain a fiber F_s, $0 < s < t$, then the core of h is D_s and the attaching circle is the vanishing cycle $\partial D_s \subset F_s$. We describe the framing of h by comparing it with the framing on $\partial D_s \subset \partial\nu F_t$

determined by F_s. At a point $(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta) \in \partial D_s \subset \mathbb{R}^2$, the vector $(-\sin\theta, \cos\theta)$ is tangent to ∂D_s. Since ∂D_s lies in F_s, which is holomorphic in the given local coordinates, the vector field $v(\theta) = (-i\sin\theta, i\cos\theta)$ on ∂D_s is also tangent to F_s. Since $v\colon \partial D_s \to i\mathbb{R}^2 - \{0\}$ has degree 1, it has one right twist relative to the normal framing of D in $\partial h \subset \mathbb{R}^2 \times i\mathbb{R}^2$. Equivalently, the product framing on h has one left twist relative to v. Now ∂h agrees with $\partial\nu F_t$ near ∂D_s, but the orientations require care. The above computation was performed on ∂h, with h oriented by $\mathbb{R}^2 \times i\mathbb{R}^2$. This is oriented oppositely from the usual orientation of $\mathbb{C}^2$ (since the standard bases of these vector spaces differ by a transposition). However, we must reverse orientation again to get from ∂h to $\partial\nu F$, so the above answer applies to $\partial\nu F$ with its usual orientation. That is, the 2-handle h is attached to $\partial D_s \subset \partial\nu F$ with framing -1 relative to the framing induced by the surface F_s containing ∂D_s. (Such orientation computations are notoriously tricky. A check of the sign is included in Exercise 8.2.2 below.)

Exercises 8.2.1. (a) Using the above description, draw $F_s \cap \partial h$ in $\partial h \approx S^3$, with orientation induced by the usual one on $h \subset X$. You should get Figure 2.1. Note that the boundary is a positive Hopf link, as required.

(b) For smooth $2n$-manifolds, a Lefschetz critical point is locally modeled by $\pi(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^2$. Prove that a singular fiber is locally (up to orientation) the cone on the unit tangent bundle of S^{n-1}, or equivalently, TS^{n-1} modulo the 0-section. (*Hint*: Write out the real and imaginary parts of π.) Verify that the critical point corresponds to an n-handle attached to $\partial\nu F_t$ along an $(n-1)$-sphere S in F_s whose normal bundle in F_s is isomorphic (up to orientation) to TS^{n-1}. Why is the normal bundle of S in $\partial\nu F_t$ trivial as required?

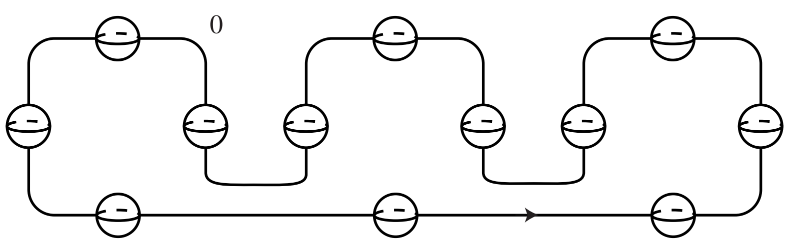


Figure 8.2. $D^2 \times F$, F a genus-3 surface.

To draw Kirby diagrams of Lefschetz fibrations, we begin with a neighborhood of a single fiber F, which we assume is connected. If $\pi\colon X \to D^2$ is a Lefschetz fibration without critical points, then it is the trivial bundle $X = D^2 \times F \to D^2$. It is convenient to draw X as in Figure 8.2 (the genus 3 case), which is obtained by connected summing the circles in copies

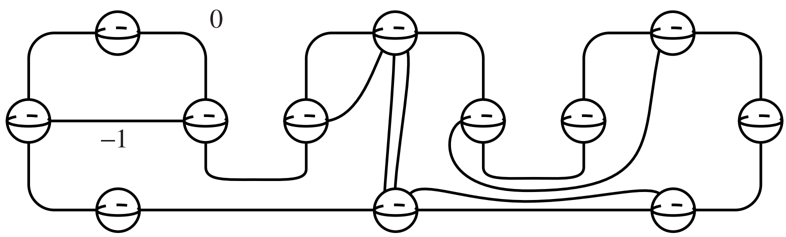


Figure 8.3. Two vanishing cycles in $D^2 \times F$.

of Figure 4.36. The advantage of this type of diagram is that it is easy to see the structure of ∂X as $S^1 \times F$: The obvious spanning disk of the closed curve drawn in ∂D^4 extends over the 1- and 2-handles to form $\{\text{pt.}\} \times F$, and the obvious fibration of the complement of the closed curve by spanning disks gives the entire S^1-family of copies of F (cf. Exercise 4.6.6(a)). The spanning disk inherits an orientation from $\mathbb{R}^2$ in the diagram, which orients the fibers F. The corresponding orientation of S^1 is then given by a right-handed meridian. To draw a neighborhood of a fiber with a unique critical point, it now suffices to locate the vanishing cycle in $\{\text{pt.}\} \times F$ in the previous diagram. This will be some circle drawn without overcrossings (but running over the 1-handles). Two such circles are shown in Figure 8.3. The required manifold is obtained by attaching a 2-handle to this vanishing cycle with framing -1 relative to the blackboard framing (the latter being induced by $\{\text{pt.}\} \times F$). Note that one framed attaching circle in Figure 8.3 cannot be assigned a framing coefficient unless we draw the relevant reference arcs for the 1-handles, cf. Section 5.4. In particular, the crossing of the reference arcs is crucial.

Exercise 8.2.2. Use Kirby moves to show that a neighborhood of a singular fiber with a unique critical point is a plumbing of the form described in Section 8.1. (*Hint*: Up to diffeomorphism of F, there is only one nonseparating curve, and a separating curve is determined by the genera of the surfaces it splits off.) Check that the sign of intersection is positive, which verifies the sign we computed for the framing of the 2-handle.

We can now determine the monodromy around a critical value. For any bundle with fiber F over an oriented circle, the monodromy $\pi_1(S^1) \to \mathcal{M}(F)$ (for a fixed identification of F with the fiber over $1 \in S^1$) is determined by a single diffeomorphism ψ representing the image of the canonical generator of $\pi_1(S^1)$ in $\mathcal{M}(F)$. The bundle is then canonically isomorphic to the fibration $I \times F/((1,x) \sim (0,\psi(x)) \to I/\partial I \approx S^1$. Given a Lefschetz fibration $\pi\colon X \to \Sigma$ and a disk $D \subset \Sigma$ (inheriting the orientation of Σ), we can consider the monodromy of the bundle $\pi|\partial D$ provided that the oriented circle ∂D avoids the critical values of π. If D contains no critical values then $\pi|D$

is trivial, as is the monodromy (ψ is isotopic to id_F). If D contains a unique critical value, however, the monodromy is nontrivial (provided $\pi|D$ is relatively minimal). To compute this monodromy, we return to our pictures of regular neighborhoods of a regular and singular fiber. The fibrations on the boundaries of these two neighborhoods correspond in the obvious way (as drawn) away from the vanishing cycle. (For example, the required fiber-preserving diffeomorphism can be constructed explicitly via a flow obtained by pulling back a suitable vector field on Σ.) There is essentially a unique way to extend the fibration across the surgery at the vanishing cycle, and it is not hard to verify that the resulting monodromy ψ is a right-handed Dehn twist on the vanishing cycle C. (See the following definition and exercise.)

Definition 8.2.3. A right-handed *Dehn twist* $\psi\colon F \to F$ on a circle C in an oriented surface F is a diffeomorphism obtained by cutting F along C, twisting 360° to the right and regluing (Figure 8.4, where F is oriented as the boundary of the solid cylinder). More formally, we identify $\nu C \subset F$ with $S^1 \times I$, set $\psi(\theta, t) = (\theta + 2\pi t, t)$ on νC and smoothly glue into $\mathrm{id}_{F-\nu C}$.

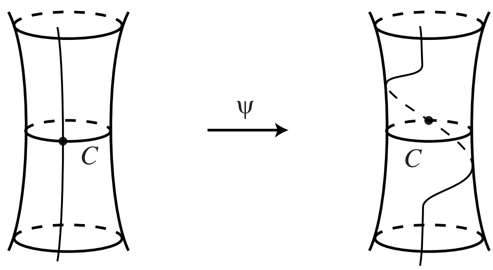


Figure 8.4. Right-handed Dehn twist.

Exercise 8.2.4. * Prove that Figure 8.5 represents the torus bundle over S^1 with monodromy ψ^n, where ψ is a right-handed Dehn twist. (*Hint*: The $-\frac{1}{n}$-surgery is equivalent to cutting $S^1 \times T^2$ along a fiber and regluing by ψ^n. Don't forget to check the orientation.) Now verify that the monodromy around any critical value of a Lefschetz fibration is a right-handed Dehn twist on the vanishing cycle (or cycles in the case of several critical points).

Remark 8.2.5. Note that the right-handed Dehn twist ψ along C is determined up to isotopy by C and is independent of the orientation on C. The effect of ψ on $H_1(F)$ is given by the *Picard-Lefschetz formula* $\psi_*(\alpha) = \alpha - (\alpha \cdot C)[C] = \alpha + (C \cdot \alpha)[C]$. The given sign may explain why some mathematicians refer to ψ as a "negative" Dehn twist. Since the authors prefer to consider right-handed Dehn twists to be "positive", we will avoid the issue by retaining the terminology "right-handed". The notation

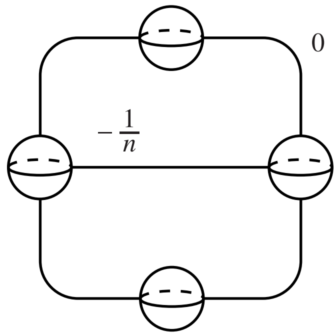


Figure 8.5. T^2-bundle over S^1.

$D(C)$ for the isotopy class of the Dehn twist ψ on C seems to be becoming standard.

We are now ready to analyze arbitrary Lefschetz fibrations over a disk. Given such a map $\pi\colon X \to D$ with n critical points lying on distinct fibers $F_i = \pi^{-1}(t_i)$, $i = 1, \dots, n$, choose a regular fiber $F_0 = \pi^{-1}(t_0)$ and let $A_1, \dots, A_n \subset D$ be embedded arcs, beginning at t_0 and otherwise disjoint, connecting t_0 to the respective critical values $t_1, \dots, t_n$. The arcs A_i are cyclically ordered by traveling counterclockwise around t_0; we choose the indexing to be compatible with this ordering. Each subset $\pi^{-1}(A_i) \subset X$ determines a map gluing νF_i to νF_0 along a pair of regular fibers, and the resulting union fills all of X except for a collar of ∂X. Thus, we can describe X as $D^2 \times F_0$ with n 2-handles $h_1, \dots, h_n$ attached. The attaching circles lie in consecutive fibers in $\partial D^2 \times F_0$, and h_i is attached with framing -1 to the vanishing cycle for F_i (where we have used $\pi^{-1}(A_i)$ to identify F_0 with the regular fiber carrying the vanishing cycle). The corresponding Kirby diagram is obtained from the picture of $D^2 \times F_0$ (cf. Figure 8.2) by adding n 2-handles with attaching circles in parallel levels (with index increasing toward the reader) and each framed attaching circle obtained from a vanishing cycle as in the case of a single critical point. (See Figure 8.6 for a genus-1 example with $n = 4$.) Working backwards, we see that any such diagram determines a Lefschetz fibration $\pi\colon X \to D^2$ and a collection of arcs $\{A_i\}$ as above. We can express this in terms of the monodromy representation $\Psi\colon \pi_1(D - \{t_1, \dots, t_n\}) \to \mathcal{M}_g$ for a fixed identification φ of F_0 with the standard genus-g surface F: We obtain an ordered basis for the domain of Ψ by connecting a counterclockwise circle around each t_i to the base point t_0 using A_i. Since a Dehn twist determines its generating circle up to isotopy (as can be seen via hyperbolic geometry, for example) the data $\pi\colon X \to D^2$ and $\{A_i\}$ (for fixed φ) are now equivalent to an ordered collection $(\psi_1, \dots, \psi_n)$ of right-handed Dehn twists of F (up to isotopy),

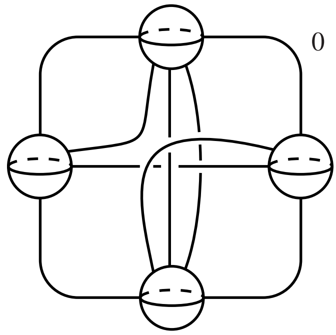


Figure 8.6. Genus-1 Lefschetz fibration over D^2.

where ψ_i is the monodromy around t_i. The monodromy around ∂D^2 is the product $\psi_1 * \cdots * \psi_n = \psi_n \circ \cdots \circ \psi_1$ (in the notation introduced at the end of the previous section). Note that a cyclic permutation of the indices $1, \ldots, n$ has the same effect on the monodromy around ∂D^2 as a certain change of φ, since it corresponds to changing how the circle ∂D is attached to the base point t_0.

Exercise 8.2.6. * Check the above formula for the monodromy around ∂D^2 by visualizing the bundle over ∂D^2 in, e.g., Figure 8.6. What does a cyclic permutation look like in such a diagram?

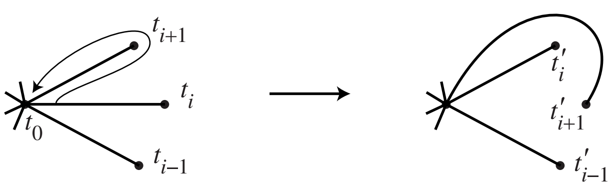


Figure 8.7. Elementary transformation.

A Lefschetz fibration $\pi \colon X \to D^2$ does not completely determine the ordered collection $(\psi_1, \ldots, \psi_n)$. Aside from cyclic permutations and being able to conjugate all elements ψ_i by a fixed (arbitrary) element of $\mathcal{M}_g$ (by changing φ), different choices of $\{A_i\}$ will give different monodromies. Given two choices of $\{A_i\}$, it is possible to get between them (up to changes of φ) by a sequence of moves as in Figure 8.7 and their inverses. (See Exercise 8.2.7(c).) These moves, which are called *elementary transformations*, can be thought of as the Lefschetz analog of handle slides in Morse theory. Each move interchanges the two corresponding vanishing cycles, and also acts on one of the two cycles by the monodromy of the other. Equivalently, the pair of Dehn twists (ψ_i, ψ_{i+1}) is replaced by $(\psi_{i+1}, \psi_{i+1}^{-1} * \psi_i * \psi_{i+1})$.

(Note that the product $\psi_1 * \cdots * \psi_n$ is preserved.) Thus, two relatively minimal Lefschetz fibrations over D^2 (with π injective on critical points) will be isomorphic if and only if it is possible to get between the corresponding ordered collections of monodromies by a sequence of elementary transformations (and their inverses), together with an inner automorphism of $\mathcal{M}_g$ (conjugating all ψ_i by a fixed element). In particular, we have reduced the classification of Lefschetz fibrations over D^2 to a group-theoretic problem; cf. the last paragraph of Section 8.1.

Exercises 8.2.7. **(a)*** Describe elementary transformations as handle slides in a Kirby diagram.

(b)* Show that any cyclic permutation of $(\psi_1, \dots, \psi_n)$ can be realized up to an inner automorphism of $\mathcal{M}_g$ by elementary transformations. If the monodromy around ∂D is trivial, show that cyclic permutations can be realized exactly (without an inner automorphism).

(c)* Let $\pi\colon X \to D$ be a Lefschetz fibration over a disk, and let $\{A_i\}, \{A_i'\}$ be two collections of arcs in D as above. Show that $\{A_i'\}$ can be transformed into $\{A_i\}$ by a sequence of elementary transformations. (*Hint*: Draw D so that the arcs A_i are radial. Now the arcs A_i' are complicated but have the same endpoints t_i as A_i (up to permutation). There is an isotopy of this picture that sends $\{A_i'\}$ to a family of radial arcs. Use the isotopy to guide your choice of moves.)

Example 8.2.8. The mapping class group $\mathcal{M}_1$ of a torus is given by $SL(2,\mathbb{Z})$ acting linearly on $\mathbb{R}^2/\mathbb{Z}^2$. (We take $SL(2,\mathbb{Z})$ to act as usual on the left on column vectors, although a case could be made for transposing everything to obtain a right action on row vectors so that $*$ corresponds to matrix multiplication.) A relatively minimal genus-1 Lefschetz fibration over D^2 with a unique critical point is a neighborhood of a fishtail fiber. This is shown in Figure 8.8, where we have drawn $D^2 \times T^2$ so that the vanishing cycle appears as simple as possible. The monodromy is given by $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in SL(2,\mathbb{Z})$ with respect to the obvious basis of $\mathbb{R}^2$. Rotating the figure 90° gives a different picture with monodromy given by $B = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$. A cusp fiber in an elliptic fibration has a unique singularity with $\pi(z_1, z_2) = z_1^2 + z_2^3$ locally. (See the curve C_2 of Section 2.3.) By Exercise 8.1.2, this splits into a pair of quadratic critical points, so a neighborhood of a cusp fiber is (after perturbing π) a Lefschetz fibration over D^2 with two fishtail fibers. Since the cusp fiber is simply connected, the vanishing cycles must form a basis of $\pi_1(T^2)$, so we obtain Figure 8.9. The monodromy around the cusp fiber is given by $C = A * B = BA = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Since the singular curve $z_1^2 + z_2^3 = 0$ is topologically a cone on a right-handed trefoil knot, a neighborhood of a cusp fiber must be obtained from D^4 by

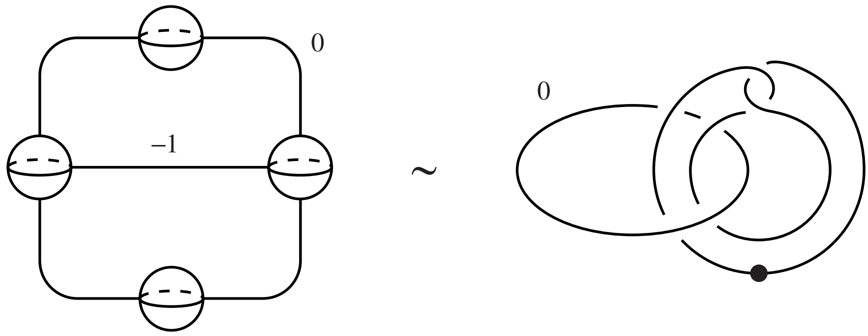


Figure 8.8. Fishtail fiber.

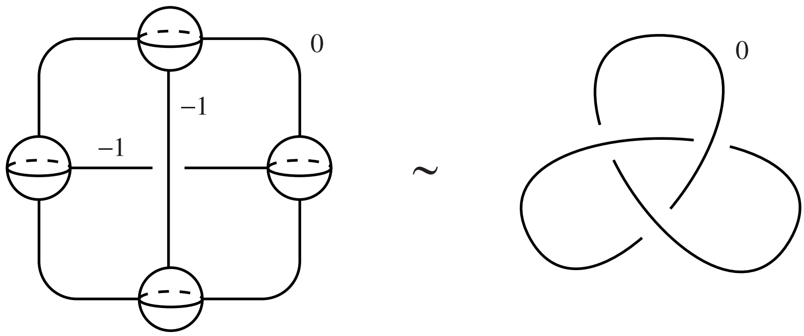


Figure 8.9. Cusp fiber.

adding a 2-handle to a right trefoil, as Figure 8.9 shows explicitly. Any singular fiber of a relatively minimal elliptic fibration can be split into a collection of fishtail and smooth multiple fibers [**Msh**]. (For higher-genus holomorphic maps, the corresponding "Morsification problem" of splitting complicated fibers into simpler ones is still poorly understood. See e.g. [**AA**], [**Hr2**], [**Rd**].) Note that since A and B generate $SL(2,\mathbb{Z})$, we can now draw a Kirby diagram for any T^2-bundle over S^1. For a different method, see the appendix of [**KMe2**].

Exercise 8.2.9. Check the diffeomorphisms of Figures 8.8 and 8.9 by Kirby calculus.

We now progress to Lefschetz fibrations over a sphere. Such a fibration canonically determines a Lefschetz fibration over a disk, by the removal of a neighborhood of a generic fiber. Conversely, a Lefschetz fibration $\pi\colon X \to D$ over a disk extends to one over a sphere if and only if the monodromy around ∂D is trivial. For genus $g \geq 2$, any extension over S^2 is unique, so the classification of such fibrations again reduces to group theory. (To prove uniqueness, note that as we did for disk bundles in Section 4.1, we can assume any other extension differs from the given one by an element of $\pi_1(\mathrm{Diff}(F))$, where $\mathrm{Diff}(F)$ denotes the group of self-diffeomorphisms of F. But each component of $\mathrm{Diff}(F)$ is contractible for $g \geq 2$ [**EE**].) A Kirby diagram of the resulting closed 4-manifold can in principle be drawn by first drawing X, then simplifying the resulting picture of ∂X so that it is clearly visible as $S^1 \times F$, and finally pulling back the framed circle $S^1 \times \{\mathrm{pt.}\}$ to the original diagram. (This framed circle is unique; cf. Remark 8.1.3(a). In practice, it may be more practical to construct the framed circle by following the product structure through the monodromy, as in Example 8.2.11 and Exercise 8.2.12 below.) The closed 4-manifold is then obtained from the diagram of X by adding a 2-handle, $2g$ 3-handles and a 4-handle. If $g = 1$, then the extension is no longer unique. The resulting Lefschetz fibrations over S^2 will be related to each other by multiplicity-1 logarithmic transformations on the last fiber. We will see in Lemma 8.3.6 that if X has a cusp fiber (or more precisely, a consecutive pair of orthogonal fishtails as in Figure 8.9) then these Lefschetz fibrations will still be isomorphic.

Exercise 8.2.10. * Express the condition that $\pi\colon X \to S^2$ splits as a fiber sum $X_1 \#_f X_2$ of nontrivial Lefschetz fibrations in terms of monodromy representations.

Example 8.2.11. The matrix C of Example 8.2.8 has order 6 in $SL(2,\mathbb{Z})$. (In fact, $C^3 = -I$.) Thus, the Lefschetz fibration over D^2 with $12n$ critical points ($n \geq 1$) and alternating monodromies given by $(A, B, \dots, A, B)$ (Figure 8.10) extends over S^2, and the resulting fibration $\pi_n\colon X_n \to S^2$ is unique up to isomorphism. Moishezon [**Msh**] showed that any ordered collection of conjugates of A in $SL(2,\mathbb{Z})$ whose product is the identity can be changed to $(A, B, \dots, A, B)$ by elementary transformations. Thus, any relatively minimal, genus-1 Lefschetz fibration over S^2 with at least one critical point is isomorphic to some π_n. In particular, a generic elliptic fibration $\pi\colon E(n) \to S^2$ is isomorphic to $\pi_n\colon X_n \to S^2$. That is, we have constructed the elliptic surface $E(n)$ with a generic elliptic fibration (or with $6n$ cusp fibers, cf. Corollary 7.3.23), without essential use of algebraic geometry.

To draw a Kirby diagram of $E(n)$, we merely need to locate the last framed attaching circle in Figure 8.10. An easy way to do this is to recall that $E(n)$ has a section with square $-n$. This section (after isotopy) must restrict to $D^2 \times \{0\} \subset D^2 \times T^2$, so in the initial diagram of $D^2 \times T^2$ it appears as a cocore of the 2-handle. Each vanishing cycle in T^2 can be isotoped into its standard position without crossing $0 \in T^2$. (Start with any ambient isotopy of T^2 moving the cycle as desired, then modify it by translations of $\mathbb{R}^2/\mathbb{Z}^2$ to keep 0 fixed.) Thus, the vanishing cycles in Figure 8.10 can be assumed not to tangle with the section, so the latter is obtained by attaching a 2-handle along a $-n$-framed meridian as in Figure 8.11 (and the

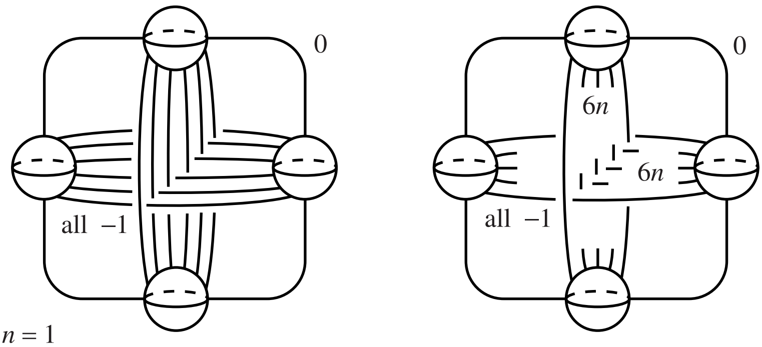


Figure 8.10. Elliptic surface $E(n)$ with a regular fiber deleted.

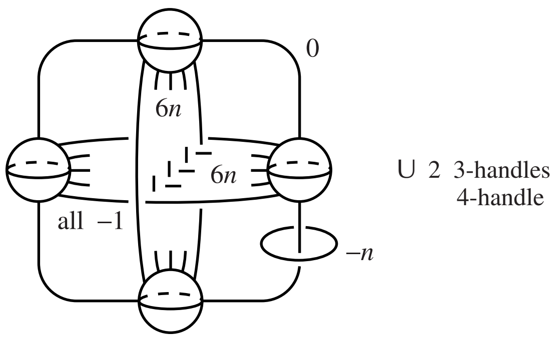


Figure 8.11. Elliptic surface $E(n)$.

obvious sphere is the section). Since the section intersects the last $D^2 \times T^2$ in $D^2 \times \{0\}$, it forms the core of the last 2-handle of $E(n)$, and we obtain the latter by attaching 3- and 4-handles as in Figure 8.11. (For simpler pictures of $E(n)$, see the next section.) For an alternate way to construct this picture of the Lefschetz fibration $X_n \approx E(n)$ that avoids our discussion of $E(n)$ (and shows directly that π_n has a section), we observe that the pictured meridian is clearly a section of the given T^2-bundle over S^1, and that any section can be written as $S^1 \times \{0\} \subset S^1 \times T^2$ for a suitable trivialization of the bundle. Thus, we can complete X_n by adding $D^2 \times T^2$ as a 2-handle, two 3-handles and a 4-handle, where the 2-handle is attached along this meridian. (For an example with a more complicated attaching circle, see Figure 8.33 and the text before Exercises 8.4.2.) It only remains to compute the framing, which corresponds to the product framing in $S^1 \times T^2$:

Exercise 8.2.12. Let A', B' be Dehn twists of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (supported near their generating circles) that fix a neighborhood of 0 and are isotopic to the linear transformations $A, B \in SL(2,\mathbb{Z})$ of Example 8.2.8. Let $C' = B' \circ A'$ (isotopic to C). Show that the isotopies of $A' \circ B' \circ A'$ and $B' \circ A' \circ B'$ to linear transformations each fix 0 but twist its tangent space 90° clockwise. Thus, $(C')^{6n}$ twists the tangent space through n full left turns when it isotopes to the identity. Conclude that the product framing of $S^1 \times \{0\} \subset S^1 \times T^2$ corresponds to the $-n$-framing on the given meridian. This also shows that if we consider diffeomorphisms rel boundary of a *punctured* torus, then $(C')^6$ is not isotopic rel boundary to the identity, but rather to a right-handed Dehn twist parallel to the boundary; cf. [**Wj**].

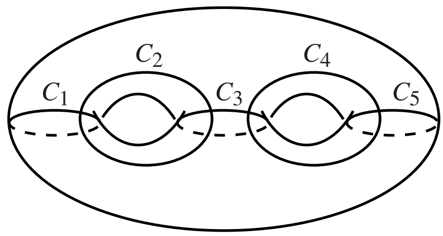


Figure 8.12. A family of vanishing cycles.

Since we now see that any relatively minimal, genus-1 Lefschetz fibration over S^2 with a critical point is a fiber sum of copies of $E(1)$ (via the identity map on a regular fiber), it is natural to ask the corresponding question about higher genus Lefschetz fibrations. In the genus-2 case, there are partial results. Chakiris [**Cha**] classifies relatively minimal genus-2 holomorphic Lefschetz fibrations for which at most $\frac{1}{19}$ of the vanishing cycles separate. In the case with no separating vanishing cycles, he shows that every such fibration is obtained as a fiber sum (via the identity on a fiber) of three basic building blocks. If $\psi_1, \ldots, \psi_5$ denote the Dehn twists on the curves $C_1, \ldots, C_5$, respectively, shown in Figure 8.12, then the monodromy representations of these building blocks are given by

$$\begin{aligned} \alpha &= (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_5, \psi_4, \psi_3, \psi_2, \psi_1)^2 \ , \\ \beta &= (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^6, \\ \gamma &= (\psi_1, \psi_2, \psi_3, \psi_4)^{10} \ , \end{aligned}$$

where the exponents denote concatenation of the n-tuples. In fact, Chakiris shows the above hypotheses guarantee that the monodromy can be reduced to $\alpha^m \beta^n$ or γ^n by elementary transformations (and change of the identification $\varphi\colon F \to F_{t_0}$ in the latter case). (See also [**Smi**] for a simpler proof.) All these manifolds lie on the Noether line $c_1^2 = 2\chi_h - 6$, as can be seen from the fact that each nonseparating vanishing cycle contributes $-\frac{3}{5}$ to the signature [**Mt4**] (whereas each separating vanishing cycle contributes $-\frac{1}{5}$; see also [**En**] for a higher-genus generalization). In particular, Theorem 7.4.20 lists the possible diffeomorphism types of these complex surfaces (except in a few nonminimal cases.) For example, the diffeomorphism types corresponding to β, α^n and γ^n, respectively, are $E(2)\#2\overline{\mathbb{CP}^2}$ [**Mt4**] and the Horikawa surfaces $H(n)$ and $H'(n)$ defined before Corollary 7.3.28 [**Fu2**]. (See also Exercise 8.4.2(e) and [**Smi**] for identifying these manifolds as double branched covers of rational surfaces.) In particular, Fuller observes that α^2 and γ represent homeomorphic but nondiffeomorphic manifolds (cf. the text before Exercises 7.3.16). In the genus-1 case, the number of critical points of a relatively minimal Lefschetz fibration over S^2 is divisible by 12. In the genus-2 case, the number is divisible by 10, provided that we count separating vanishing cycles with multiplicity 2. These statements follow from the fact that the abelianization of the mapping class group is $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$, respectively, in these cases, generated by a Dehn twist on a nonseparating curve (e.g., [**Wj**]). (Note that any two such Dehn twists are conjugate, and in the genus-2 case a separating Dehn twist is a product of 12 nonseparating ones by the previous exercise.) In the higher genus cases, the corresponding abelianizations are trivial, providing no corresponding information. Not much is known about Lefschetz fibrations of genus ≥ 3. See Section 8.4 and [**Smi**] for a few examples, and [**Oz**] for an algorithm for computing signatures.

To complete our discussion, we consider how to draw Lefschetz pencils as well as Lefschetz fibrations over arbitrary bases Σ. For the former, we simply observe that one can construct the corresponding Lefschetz fibration over S^2 and then blow down the resulting sections. In practice, it may be difficult to locate these sections, however — consider the 9 sections of $E(1)$ and Figure 8.11. For Lefschetz fibrations over general surfaces Σ, the extra complication arises from the 1-handles of Σ. We must determine the monodromy around these 1-handles; then we add a 1-handle, $2g$ 2-handles and a 3-handle for each (where g is the genus of the fiber); cf. Example 4.6.8. The classification of such Lefschetz fibrations can still be expressed in terms of the monodromy representation as in Section 8.1. In the genus-1 case, a complete list of relatively minimal Lefschetz fibrations over a closed surface Σ of genus g is given by $E(n,g) = E(n)\#_f\Sigma \times T^2$ $(n \in \mathbb{N})$ together with torus bundles over Σ [**Mt3**], cf. Theorem 8.3.12.

8.3. The topology of elliptic surfaces

In the last section, we saw that the manifolds $E(n)$ and $E(n) - \operatorname{int} \nu F$ (F a regular fiber) are given by Figures 8.11 and 8.10, respectively. In this section, we simplify the pictures and use them to study the topology of the elliptic surfaces $E(n)$. We also return to logarithmic transformations and the classification of elliptic surfaces. Much of the theory in this section is taken from [**HKK**], [**G9**] and other sources; some (notably Theorem 8.3.2) appears in this volume for the first time.

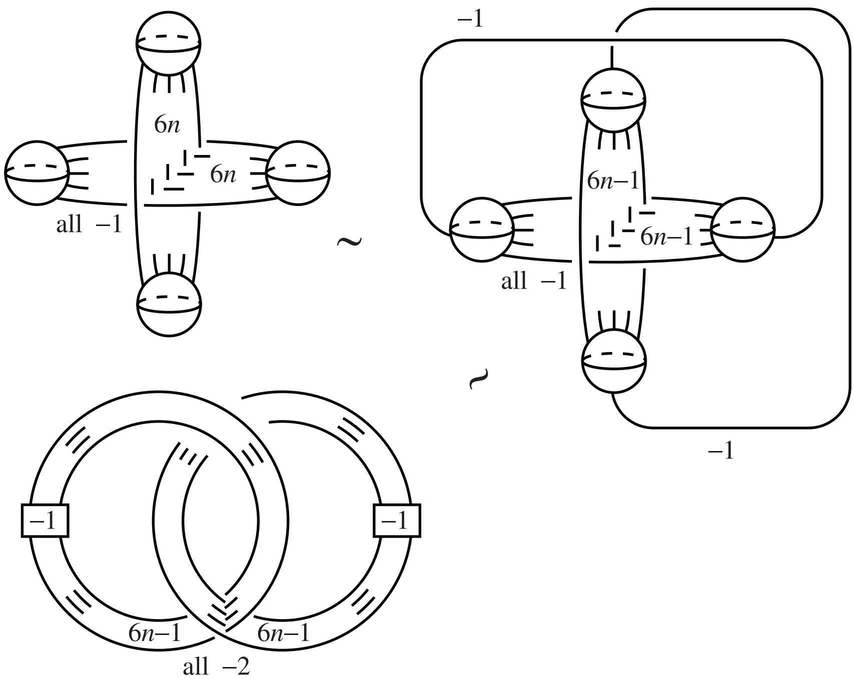


Figure 8.13. Milnor fiber $M_c(2,3,6n)$ inside $E(n)$.

We begin by examining the relationship between elliptic surfaces and Milnor fibers. We first encountered this relationship in Section 7.3, but now we gain additional insight by an independent approach using Kirby diagrams. Recall that the sphere S in Figure 8.11 given by the $-n$-framed meridian is a section of $E(n)$. Clearly, its intersection with $E(n) - \text{int } \nu F$ is a cocore of the 0-framed 2-handle in Figure 8.10. Thus, $E(n) - \text{int } (\nu F \cup \nu S)$ is obtained by removing the 0-framed 2-handle in Figure 8.10 to obtain Figure 8.13. Cancelling the 1-handles as indicated, we see that $E(n) - \text{int } (\nu F \cup \nu S)$ is diffeomorphic to the Milnor fiber $M_c(2,3,6n)$ (cf. Figure 6.45 and Corollary 7.3.23). Similar reasoning shows that for $r \in \mathbb{N}$, $M_c(2,3,r)$ is obtained by removing a section from the genus-1 Lefschetz fibration over D^2 with $2r$ singular fibers and monodromies $(A, B, \dots, A, B)$ as in Examples 8.2.8 and 8.2.11. (Recall that each pair (A, B) can be interpreted as a cusp fiber; also cf. the text following Exercise 7.3.17.) By suitably cutting the base S^2 into two disks, we can now decompose $E(n) - \text{int } \nu S$ as $M_c(2,3,r) \cup M_c(2,3,6n-r)$ for any $r \in \{1, 2, 3, \dots, 6n-1\}$. Similarly, $E(n) \approx M_c(2,3,6n-1) \cup_\partial N(n)$, where the nucleus $N(n)$ is νS union a regular neighborhood of a cusp fiber (Figure 8.14), so we have recovered Corollary 7.3.23 (at least, ignoring complex structures). In particular,

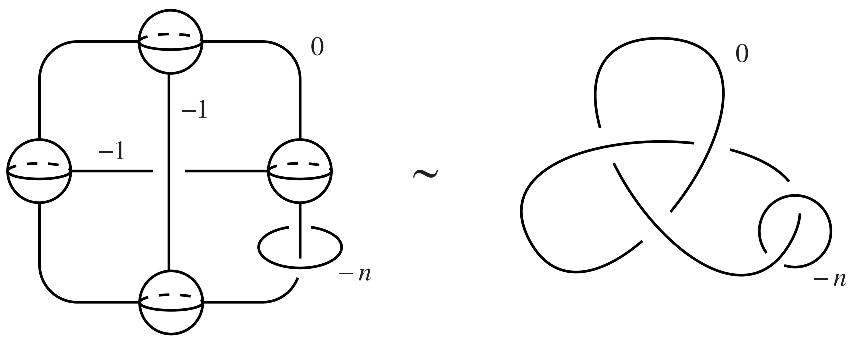


Figure 8.14. Nucleus $N(n)$.

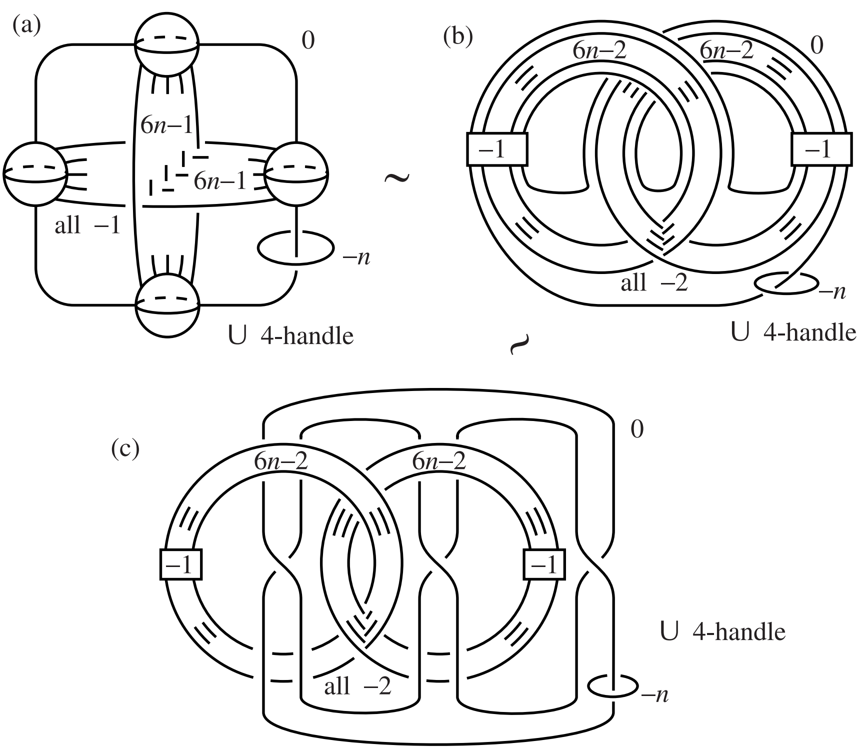


Figure 8.15. Elliptic surface $E(n)$.

$E(n)$ admits a handle decomposition without 1- or 3-handles (since both $M_c(2,3,6n-1)$ and $N(n)$ do). For an explicit such picture, remove a regular neighborhood of a cusp fiber from $E(n)$ to obtain a Lefschetz fibration over D^2 with monodromy $(A,B)^{6n-1}$ (Figure 8.10 with $6n$ replaced by $6n-1$). Filling the cusp fiber back in attaches a 2-handle and 4-handle, and the core of the 2-handle (a cocore of either 0-framed 2-handle in Figure 8.9) completes the section of $E(n)$, resulting in Figure 8.15 (where (c) is adapted from [**Fu3**]). In short, two -1-framed 2-handles cancel the 3-handles in Figure 8.11, and two others cancel the 1-handles.

Exercise 8.3.1. Prove directly that $E(1) \approx \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. (*Hint*: First blow down the -1-framed unknot in Figure 8.15(b), then slide the resulting $+1$-framed trefoil over one -2-framed circle so that it becomes a -1-framed unknot. Now blow this down to remove the two -1-twists.)

The following theorem gives a different Kirby diagram of $E(n)$ that will be useful in Section 9.3; its proof illustrates how induction can be applied in Kirby calculus computations for families of manifolds.

Theorem 8.3.2. *For any $r, n \geq 1$, the manifolds $M_c(2,3,r)$ and $E(n)$ are given by Figure 8.16.*

Note the embeddings $M_c(2,3,2m) \subset M_c(2,3,2m+1) \subset M_c(2,3,2m+2)$ of the Milnor fibers (the latter embedding visible after sliding a 2-handle over a -1-framed meridian to reduce a linking number from 3 to 2), and the embedding $M_c(2,3,6n-1) \subset E(n)$. There is also an obvious nucleus $N(n) \subset E(n)$, consisting of $D^4 \cup$ 1-handle together with the two leftmost 2-handles and a -1-framed handle cancelling the 1-handle.

Proof. The cases $r = 1, 2$ are easy (check!), so we assume $r \geq 3$. Figure 8.17(a) with $r = 6n-1$ and a 4-handle added represents $E(n)$. (Change the 1-handles in Figure 8.15(a) to dotted circle notation.) If we remove the pair of circles with framings 0 and $-n$, the same diagram represents $M_c(2,3,r)$ (cf. Figure 8.13). We will deal with both manifolds simultaneously by manipulating the diagram without sliding any handles over the pair with framings 0 and $-n$. We first set up for the induction by sliding handles as follows to obtain Figure 8.17(b). Note that the innermost circle in each ring in (a) is actually a meridian of a dotted circle; we move the meridian on the right out of the way. In the absence of the dotted circles, we could simplify further by blowing down. The dotted circles prevent this, but we can still perform the corresponding 2-handle slides, leaving the -1-framed unknots as meridians of the dotted circles. In this manner, we "blow down" the innermost remaining circle on the right. The outermost circle in the left ring now links the right dotted circle, and its framing has changed to 0. We slide the next outermost circle over this, obtaining a third -1-framed meridian to the right dotted circle. Again "blow down" the innermost circle on the right, obtaining (b) as required.

Now we come to the induction step. This is typically the hardest part of the argument to construct, but in our case a bit of experimenting yields Figure 8.18 and the lemma below. Let $Y(n,r,k)$, $n \geq 1$, $r \geq 3$, $k \geq 0$, denote the manifold shown in Figure 8.18, and let $Y_0(r,k)$ be the corresponding

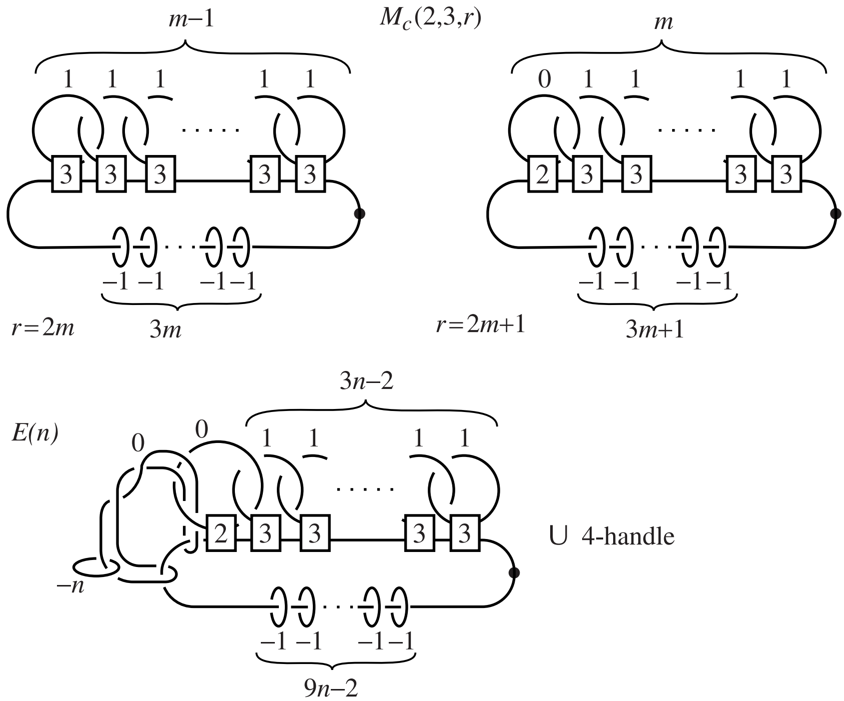


Figure 8.16. Milnor fiber $M_c(2,3,r)$ and elliptic surface $E(n)$.

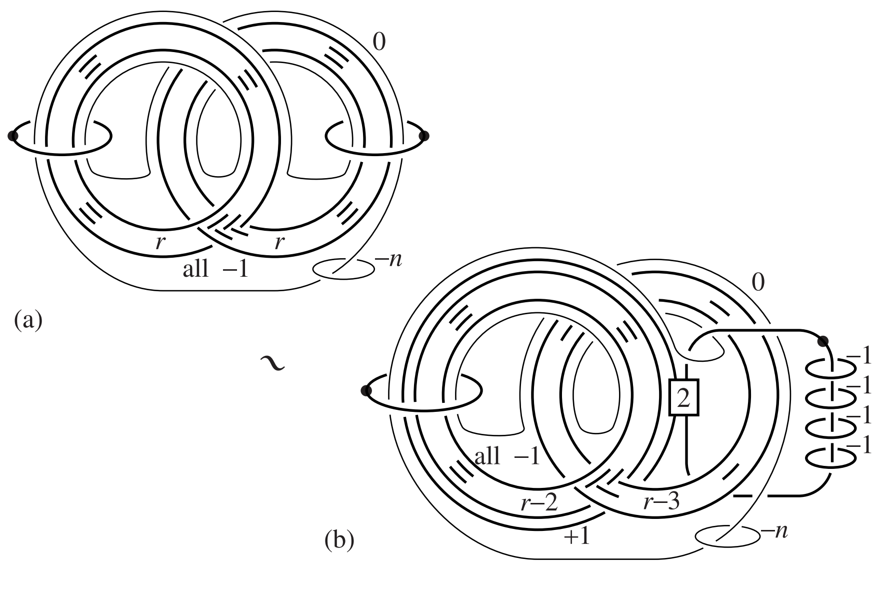


Figure 8.17. $E(n)$ and $M_c(2,3,r)$.

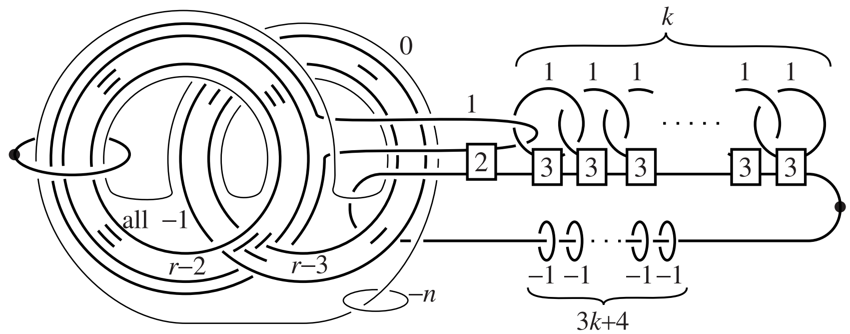


Figure 8.18. $Y(n,r,k)$ and $Y_0(r,k)$.

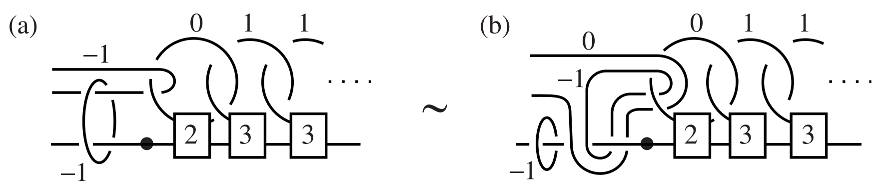


Figure 8.19

manifold without the pair of handles with framings 0 and $-n$. Note that by the above calculation, we have $E(n) \approx Y(n, 6n-1, 0) \cup$ 4-handle and $M_c(2,3,r) \approx Y_0(r,0)$.

Lemma 8.3.3. *For $n \geq 1$, $r \geq 5$ and $k \geq 0$ we have the diffeomorphisms $Y(n,r,k) \approx Y(n, r-2, k+1)$ and $Y_0(r,k) \approx Y_0(r-2, k+1)$.*

Proof. Draw Figure 8.18 on a blackboard. Slide the long 1-framed circle over the outermost -1-framed circle in the left ring. It becomes the 0-framed circle in (a) of Figure 8.19, where the -1-framed circles are the outermost circle of the left ring (of $r-2$ 2-handles) and the innermost circle of the right ring in the previous picture. "Blow down" the latter circle (the small -1 of Figure 8.19(a)) to change the remaining -1 to 0, then slide the outermost -1 in the left ring over this new 0-framed curve to obtain Figure 8.19(b). "Blow down" this -1-framed circle and the new innermost circle of the right ring; the result is $Y(n, r-2, k+1)$ exhibited as Figure 8.18. □

To complete the proof of Theorem 8.3.2, use the lemma to infer that

$$M_c(2,3,2m+1) \approx Y_0(2m+1,0) \approx Y_0(3,m-1) \text{ and}$$

$$E(n) \approx Y(n,3,3n-2) \cup \text{4-handle},$$

Figure 8.20 with $m = 3n-1$ in the latter case and two 2-handles deleted in the former. Cancel the left 1-handle with its -1-framed meridian, and

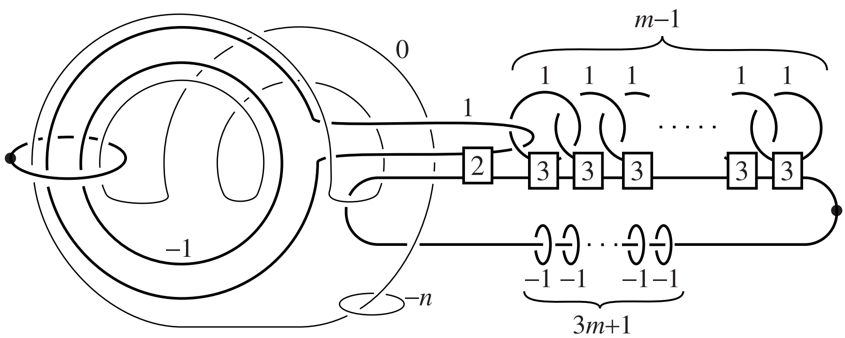


Figure 8.20. $Y(n, 3, m-1)$ and $Y_0(3, m-1)$.

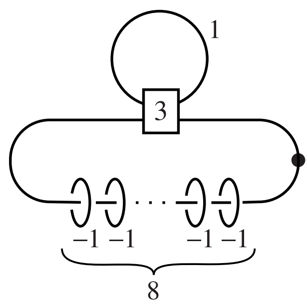


Figure 8.21. E_8-plumbing (negative).

isotope to obtain Figure 8.16 as required. The remaining case, $M_c(2, 3, 2m)$, is handled by the next exercise. □

Exercises 8.3.4. (a)* Prove that $M_c(2, 3, 2m)$, $m \geq 2$, is given by Figure 8.16.

(b)* Prove that the negative E_8-plumbing is given by Figure 8.21. (*Hint*: Exercise 5.1.12(a).)

(c)* Prove that $M_c(2, 3, 5)$ is diffeomorphic to the (negative) E_8-plumbing. (*Hint*: Figures 8.16 ($r = 5$) and 8.21 differ by a single handle slide.) A different Kirby calculus proof will appear in [**KMe3**].

(d) Prove that $E(n)\#\mathbb{CP}^2$ is diffeomorphic to $2n\mathbb{CP}^2\#(10n-1)\overline{\mathbb{CP}^2}$. (*Hint*: Starting with Figure 8.16, blow up a +1-framed meridian to one of the −1-framed meridians, then cancel the resulting 0 against the 1-handle; the net result is that the dotted circle becomes a +1-framed 2-handle and one −1-framed meridian is gone. Blow down the remaining −1's and eliminate the $-n$-framed curve and the 0 it links (Proposition 5.1.4), obtaining Figure 8.22. Use the 0-framed circle to eliminate the adjacent 1, then blow down the next 1 and obtain Figure 8.22 again with n reduced by 1. Now

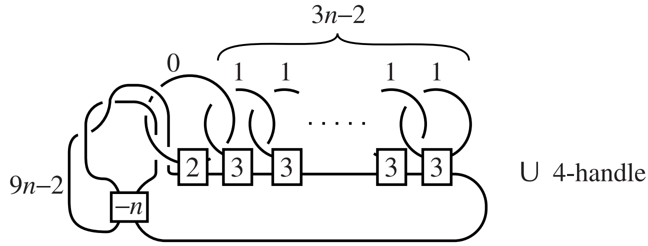


Figure 8.22. Blowing down $E(n)\#\mathbb{CP}^2$.

by induction assume $n = 1$ and finish the computation. One can also start from Figure 8.15(b) by blowing up to reverse the crossing of the bands of the punctured torus Seifert surface and inductively blowing down the resulting -1's — cf. the proof that $E(n)_0 \approx (2n-1)\mathbb{CP}^2\#(10n-1)\overline{\mathbb{CP}^2}$ in [**G9**].)

Next we examine logarithmic transformations in more detail (cf. also Sections 3.3 and 8.5). In the fullest generality, a logarithmic transformation consists of locating a torus T embedded in a 4-manifold X with trivial normal bundle $\nu T \approx T \times D^2$, removing int νT from X and gluing in $T^2 \times D^2$ by some diffeomorphism $\varphi\colon T^2 \times S^1 \to \partial\nu T$. (Compare with Dehn surgeries on 3-manifolds, Section 5.3.) Since the last gluing is the same as attaching a 2-handle, two 3-handles and a 4-handle, the resulting diffeomorphism type is determined by the framed attaching circle $\varphi(\{\text{pt.}\} \times S^1)$ of the 2-handle. The self-diffeomorphisms of $T^2 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$ are given up to isotopy by $GL(3;\mathbb{Z})$. Since any element of $GL(3,\mathbb{Z})$ that preserves $\{0\} \times S^1$ also preserves its normal framing as determined by the product structure, it follows that the circle $\varphi(\{\text{pt.}\} \times S^1)$ is canonically framed, and so this circle determines the diffeomorphism type resulting from the logarithmic transformation. The circle $\varphi(\{\text{pt.}\} \times S^1)$ is in turn determined by its homology class $\alpha \in H_1(\partial\nu T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, which can be any primitive element. To specify α, we first fix a projection $\pi\colon \nu T \to D^2$ onto D^2 with its standard orientation. (If T is a generic fiber in an elliptic fibration, we take π to be the given projection.) We now have a fixed diffeomorphism $\nu T \approx T \times D^2$, and hence, an identification $H_1(\partial\nu T;\mathbb{Z}) \cong H_1(T;\mathbb{Z}) \oplus \mathbb{Z}$, with the last summand generated by a (positively oriented) meridian of T. Now α is specified by an element α' of $H_1(T;\mathbb{Z})$ and an integer p (depending on π in general) called the *multiplicity* of the logarithmic transformation. We arrange $p \geq 0$ by reversing the sign of α if necessary. The element α' can be written as $q\alpha_0$ with α_0 primitive in $H_1(T;\mathbb{Z})$ and $q \geq 0$ relatively prime to p; α_0 and q are called the *direction* and *auxiliary multiplicity* of the logarithmic transformation. To summarize, for fixed T and π, a logarithmic transformation is specified by a direction $\alpha_0 \in H_1(T;\mathbb{Z})$ and a pair of relatively prime nonnegative integers p, q. In general, all of these data affect the resulting diffeomorphism type, but we will now show that for an elliptic fibration with a cusp fiber, only the multiplicity p is significant (proving Theorem 3.3.3). More generally, if X is any 4-manifold and $T \subset X$ is a torus lying in a cusp neighborhood, then the manifold $X_{p_1,\dots,p_k}$ obtained from X by logarithmic transformations on parallel copies of T (determined by π) only depends on (X, T), π and the given multiplicities $p_1, \dots, p_k$. (We say that T *lies in a cusp neighborhood* if there is a cusp neighborhood N (Figure 8.9) embedded in X such that T is a regular fiber of N and $\pi\colon \nu T \to D^2$ is determined by the elliptic fibration on N. We use similar terminology for fishtail neighborhoods (Figure 8.8) and nuclei.) Recall that by Example 8.2.8, a cusp fiber can be perturbed into a pair of fishtail fibers with monodromies (A, B) in a suitable basis, so it suffices to prove the following:

Theorem 8.3.5. ([**G9**], cf. also [**Msh**]) *Let $\pi\colon X \to D^2$ be a genus-1 Lefschetz fibration with two singular fibers, and monodromies (A, B) as in Example 8.2.8. Let X_1 and X_2 each be obtained from X by logarithmic transformations on k regular fibers, with the same k multiplicities $p_1, \dots, p_k$ in each case. Then X_1 and X_2 are diffeomorphic* rel ∂.

The proof depends on the following lemma of Moishezon [**Msh**], who attributes the idea of the proof below to D. Mumford. (An explicit formula for the required diffeomorphism appears in [**HKK**], proof of Theorem 1.27.)

Lemma 8.3.6. ([**Msh**]) *Let T be a regular fiber of the above map $\pi\colon X \to D^2$. Then any orientation- and fiber-preserving self-diffeomorphism φ of $\partial\nu T$ extends to a fiber-preserving self-diffeomorphism of $X - \operatorname{int} \nu T$* rel ∂X.

Proof. The manifold $X' = (X - \operatorname{int} \nu T) \cup_\varphi \nu T$ is obtained from X by a multiplicity-1 logarithmic transformation on T, and the Lefschetz fibration on X extends over X' with no new singular fibers. Clearly, it suffices to find a fiber-preserving diffeomorphism $X \to X'$ rel ∂ sending νT to itself by the identity map. First note that there is a fiber-preserving self-diffeomorphism of X rel ∂ that realizes any preassigned orientation-preserving self-diffeomorphism ψ on T: Since the monodromies A, B generate $SL(2,\mathbb{Z})$, we can write ψ as a word in $A^{\pm1}$, $B^{\pm1}$, then lift an isotopy of D^2 obtained by moving the point $\pi(T)$ around the corresponding loop in $\operatorname{int} D^2 - \{\text{critical values of } \pi\}$. (The monodromy representation of X is unchanged since the monodromy around T is trivial.) Now let $\alpha, \beta \in H_1(T;\mathbb{Z})$ be a basis such that $\alpha + \beta$ is the direction of the given logarithmic transformation, and let $q \geq 0$ be the auxiliary multiplicity. We can think of X' as being constructed in two steps: First, we construct a manifold X'' by a multiplicity-1 logarithmic transformation on $T \subset X$ with direction α and auxiliary multiplicity q, then we perform a second logarithmic transformation on the same fiber T in X'' with the same multiplicities 1 and

q, but direction β. Since the first of these transformations has multiplicity 1, X'' inherits a Lefschetz fibration with T a regular fiber. Thus, by isotoping T in X'' before the second transformation, we can modify T by a self-diffeomorphism, changing the direction of the second transformation from β to $-\alpha$. Now the second transformation is the inverse of the first, so we have exhibited the required diffeomorphism between X and X'. □

Proof of Theorem 8.3.5. By lifting suitable diffeomorphisms of D^2, we can arrange the logarithmic transformations producing X_1 and X_2 from X to be on the same k fibers $T_1, \dots, T_k$, and for these to lie in a collar $C \approx [0,1) \times \partial X$ of ∂X. We can assume that for both X_1 and X_2, the multiplicity of the transformation on each T_i is p_i. By the lemma, any self-diffeomorphism of $\bigcup \partial\nu T_i$ preserving components, orientations and fibers extends to a fiber-preserving self-diffeomorphism of $X - \bigcup \operatorname{int} \nu T_i \operatorname{rel} \partial X$ — simply push one T_i at a time into $X - C$ and apply the lemma there $\operatorname{rel} \partial(X - C)$. Now for each T_i, let μ_i be the meridian and let $\alpha_i, \beta_i \in H_1(T_i; \mathbb{Z})$ be a basis with α_i the direction of the logarithmic transformation used to produce X_1. Thus, the attaching circle of the new 2-handle is $q_i\alpha_i + p_i\mu_i \in H_1(\partial\nu T_i; \mathbb{Z})$. Since $\gcd(p_i, q_i) = 1$, there are integers ℓ_i and m_i with $\ell_i p_i + m_i q_i = 1$. Now apply any fiber-preserving map sending $\alpha_i \mapsto \alpha_i + m_i\beta_i$, $\mu_i \mapsto \mu_i + \ell_i\beta_i$. This sends the attaching circle to $q_i\alpha_i + \beta_i + p_i\mu_i$. The corresponding projection $q_i\alpha_i + \beta_i$ in $H_1(T_i; \mathbb{Z})$ is primitive, so we have changed the auxiliary multiplicities of the logarithmic transformations producing X_1 to 1. After applying the same argument to X_2, we can use the lemma again to make the directions determining X_1 and X_2 correspond, and the proof is complete. □

Corollary 8.3.7. *The elliptic surface $E(n)_{p_1,\dots,p_k}$ is determined up to diffeomorphism by n and $p_1, \dots, p_k$. The same holds for the nucleus $N(n)_{p_1,\dots,p_k}$ and for the elliptic surface $E(n,g)_{p_1,\dots,p_k}$ (with base a surface of fixed genus g) obtained by logarithmic transformations on the genus-1 Lefschetz fibration $E(n,g)$ of Examples 8.1.5.* □

Corollary 8.3.8. *For a regular fiber F of $E(n)_{p_1,\dots,p_k}$, any orientation- and fiber-preserving self-diffeomorphism of $\partial\nu F$ extends over the complement $E(n)_{p_1,\dots,p_k} - \operatorname{int} \nu F$ (and similarly for $N(n)_{p_1,\dots,p_k}$ and $E(n,g)_{p_1,\dots,p_k}$).* □

Exercise 8.3.9. For $i = 1, 2$, let $\pi_i \colon E(n_i) \to S^2$ be an elliptic fibration with a regular fiber F_i with meridian μ_i and tubular neighborhood $\nu F_i = \pi_i^{-1}(D_i)$. Let $\varphi \colon \partial\nu F_2 \to \partial\nu F_1$ be an orientation-reversing diffeomorphism, and define $p, q \geq 0$ by the isomorphisms $H_1(\partial D_1; \mathbb{Z})/\pi_*\varphi_* H_1(F_2; \mathbb{Z}) \cong \mathbb{Z}_p$ and $H_1(\partial D_1; \mathbb{Z})/\pi_*\varphi_*\langle\mu_2\rangle \cong \mathbb{Z}_q$. Let $Z(n_1, n_2; p, q)$ denote the simply connected 4-manifold $(E(n_1) - \operatorname{int} \nu F_1) \cup_\varphi (E(n_2) - \operatorname{int} \nu F_2)$. Prove that such a manifold is uniquely determined up to diffeomorphism by $n_1, n_2 \in \mathbb{N}$ and relatively prime integers $p, q \geq 0$, that $Z(n_1, n_2; p, q) \approx Z(n_1, n_2; p, q')$ provided that $q' \equiv \pm q \bmod p$ (so we can assume $0 \leq q \leq \frac{1}{2}p$ or $p = 0$, $q = 1$), and that $Z(n_2, n_1; p, q') \approx Z(n_1, n_2; p, q)$ when $qq' \equiv \pm 1 \bmod p$. (*Hint*: Choose bases for $H_1(\partial \nu F_i; \mathbb{Z})$ so that φ is conveniently given by a matrix in $SL(3; \mathbb{Z})$. Simplify using Lemma 8.3.6 and an involution of $E(n_2)$ (Figure 8.15) reversing the orientation of F_2.) What is $Z(n_1, n_2; 0, 1)$? By Theorem 8.3.11 below, $Z(n_1, 1; p, q) \approx E(n_1 + 1)$.

Lemma 8.3.10. (cf. [**G9**]) *Any self-diffeomorphism of* $\partial M_c(2, 3, 6n - 1)$ *extends over* $M_c(2, 3, 6n - 1)$.

Proof (sketch). The manifolds $\Sigma(p, q, r) = \partial M_c(p, q, r)$ have been extensively studied. It is known that the group of self-diffeomorphisms of $\Sigma(2, 3, 6n - 1)$ up to isotopy is trivial for $n = 1$ and $\mathbb{Z}_2$ for $n > 1$ [**BO**]. The nontrivial self-diffeomorphism is the obvious involution of Figure 8.14 (essentially a 180° rotation) that preserves the 0- and $-n$-framed circles but reverses their orientations. This involution clearly extends to $E(n)$ (Figure 8.15), hence, to $M_c(2, 3, 6n - 1)$. (In fact, the involution can be identified with complex conjugation if we use real equations to define $E(n)$ or $M_c(2, 3, 6n - 1)$.) See [**G9**] for further details. □

Theorem 8.3.11. (cf. [**Mat**], [**G9**], [**GuM**] page 46) *Let* $F \subset N(n) \subset E(n)$ *be a regular fiber in a nucleus. For* $n = 1$*, any orientation-preserving self-diffeomorphism of* $\partial \nu F$ *extends over* $N(n) - \operatorname{int} \nu F$ *and* $E(n) - \operatorname{int} \nu F$*. Hence, there are diffeomorphisms* $N(1)_p \approx N(1)$ *and* $E(1)_p \approx E(1)$ *for any* $p \geq 0$*. For* $n \geq 2$*, a self-diffeomorphism of* $\partial \nu F$ *extends over* $N(n) - \operatorname{int} \nu F$ *or* $E(n) - \operatorname{int} \nu F$ *if and only if it preserves fibers and orientation.*

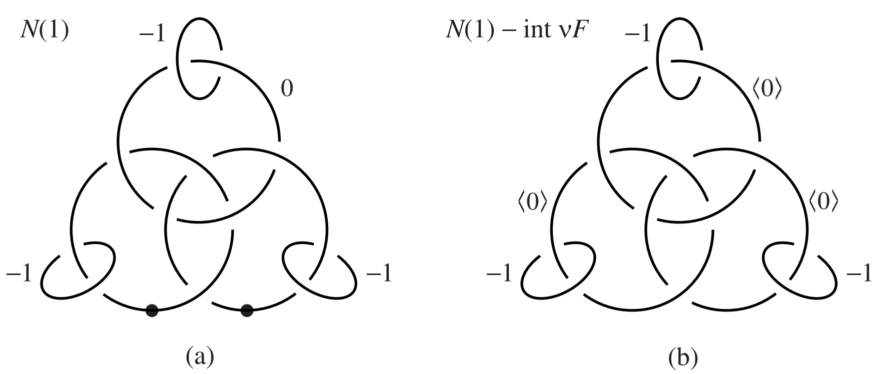


Figure 8.23

Proof. Changing Figure 8.14 of $N(1)$ to dotted circle notation, we obtain (a) of Figure 8.23. Removing $\operatorname{int} \nu F$ produces (b) (in the notation of Section 5.5). In that figure, there is an obvious $\mathbb{Z}_3$-action of $N(1) - \operatorname{int} \nu F$

(120° rotation) that cyclically permutes the factors of $\partial\nu F \approx F \times S^1 \approx S^1 \times S^1 \times S^1$. Now recall that the orientation-preserving diffeomorphisms of $\partial\nu F$ up to isotopy are given by $SL(3;\mathbb{Z})$. It is easy to verify that this is generated by the fiber-preserving diffeomorphisms of Lemma 8.3.6 and their conjugates under the above cyclic permutation; these generators extend over $N(1) - \operatorname{int}\nu F$ as required and fix $\partial N(1)$, so they extend over $E(1) - \operatorname{int}\nu F$. The diffeomorphisms $N(1)_p \approx N(1)$ and $E(1)_p \approx E(1)$ are immediate. If any self-diffeomorphism of $N(n) - \operatorname{int}\nu F$ or $E(n) - \operatorname{int}\nu F$ ($n \geq 2$) failed to preserve fibers on $\partial\nu F$ (up to isotopy), then we could reduce to the latter case by Lemma 8.3.10, so some circle in a fiber of $E(n) - \operatorname{int}\nu F$ would be sent to a circle with nontrivial projection to ∂D^2. We would then have $E(n)_p \approx E(n)_0 \approx \#(2n-1)\mathbb{CP}^2\#(10n-1)\overline{\mathbb{CP}^2}$ for some $p \neq 0$ (cf. Exercise 8.3.15), violating (1) of Theorems 2.4.6 and 2.4.7. No orientation-reversing diffeomorphism extends, since $\sigma(E(n)) \neq 0$ and $\partial N_n = \overline{\Sigma}(2,3,6n-1)$ admits no orientation-reversing diffeomorphism. □

We are now ready for the classification of elliptic surfaces. We reduce to the case of *relatively minimal* elliptic surfaces by blowing down any sphere of square -1 contained in a fiber, just as we did for Lefschetz fibrations. There are now two cases: If every fiber of an elliptic surface X is a torus, then $\chi(X) = 0$ and X is obtained from a torus bundle over an orientable surface by logarithmic transformations on fibers. If X has other singular fibers then $\chi(X) > 0$; this is the case we consider in detail.

Theorem 8.3.12. (Classification of minimal elliptic surfaces with $\chi \neq 0$). *Let X be a relatively minimal elliptic surface with $\chi(X) \neq 0$. Then X is diffeomorphic to $E(n,g)_{p_1,\dots,p_k}$ for exactly one choice of $n, g, k, p_1, \dots, p_k$ with $n \geq 1$, $g, k \geq 0$, $2 \leq p_1 \leq \cdots \leq p_k$ and $k \neq 1$ if $(n,g) = (1,0)$. No two of these manifolds $E(n,g)_{p_1,\dots,p_k}$ become diffeomorphic after blow-ups.*

Proof (sketch). Recall that $E(n,g)_{p_1,\dots,p_k}$ is well-defined by Corollary 8.3.7. To prove that any relatively minimal elliptic surface with $\chi \neq 0$ is diffeomorphic to some $E(n,g)_{p_1,\dots,p_k}$, perturb the projection so that all singular fibers are fishtails and smooth multiple fibers ([**Msh**] Theorems 8, 8a), then eliminate the latter by inverse logarithmic transformations. Matsumoto [**Mt3**] proved that the resulting elliptic surface is diffeomorphic to some $E(n,g)$ by manipulating the monodromies as in Section 8.2. Clearly, we can write the multiplicities in increasing order and cancel any that equal 1, and replace any $E(1,0)_p$ by $E(1,0) = E(1)$ (Theorem 8.3.11). To distinguish the remaining manifolds, note that the relations $\chi(E(n,g)_{p_1,\dots,p_n}) = 12n$ and $\sigma(E(n,g)_{p_1,\dots,p_k}) = -8n$ determine n (even after blow-ups) and $b_1(E(n,g)_{p_1,\dots,p_k}) = 2g$ determines g. According to Ue [**U**], the manifolds $E(n,g)_{p_1,\dots,p_k}$ with n fixed (n, g, k, p_i as above) are distinguished by their fundamental groups, except for those with cyclic π_1. The latter case consists of the elliptic surfaces $E(n,0)_{p,q}$ $(1 \le p \le q)$, for which $\pi_1 \cong \mathbb{Z}_{\gcd(p,q)}$, and these are distinguished by gauge theory, which is not significantly affected by blowing up. (See Corollary 3.3.7, Theorem 3.3.8 and Remark 3.3.9(a).) □

Remark 8.3.13. Suppose we consider all relatively minimal elliptic surfaces except for those with cyclic fundamental groups and those with $\chi = 0$ for which Σ is spherical when considered as an orbifold (i.e., $\Sigma = S^2$, $k \le 3$ and (when $k = 3$) $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1$). Ue shows the remaining manifolds are distinguished by χ and π_1. In particular, this set of elliptic surfaces does not realize any exotic smooth structures, in contrast to the case $\chi > 0$, π_1 cyclic.

Corollary 8.3.14. *No two of the nuclei $N(n)_{p,q}$, $n \ge 1$, $0 \le p \le q$, are diffeomorphic (or become diffeomorphic after blowing up), except for the cases $N(1)_{1,q} \approx N(1)_{1,1} \approx N(1)_{0,1}$ for all q.*

Proof. The boundaries $\partial N(n)_{p,q} = \overline{\Sigma}(2,3,6n-1)$ are distinct for different values of n, so we may assume n is fixed. If $N(n)_{p,q} \approx N(n)_{p',q'}$, then by Lemma 8.3.10 there is a diffeomorphism $E(n)_{p,q} \approx E(n)_{p',q'}$, so $p = p'$, $q = q'$ unless $n = p = p' = 1$ or $pp' = 0$. The nuclei $N(n)_{0,q}$ are distinguished from each other by their fundamental groups $\mathbb{Z}_q$, and from the other nuclei by the fact that each $N(n)_{0,q}$ has a $\overline{\mathbb{CP}^2}$ connected summand (Exercise 8.3.16(d) below), whereas for $p \ge 1$ (or $p \ge 2$ if $n = 1$) $N(n)_{p,q}$ lies in $E(n)_{p,q}$ and the latter is irreducible. A similar argument applies after blowing up, since $E(n)_{p,q} \# k\overline{\mathbb{CP}^2}$ does not split off $\#(k+1)\overline{\mathbb{CP}^2}$. □

Finally, we discuss the effect of logarithmic transformations on Kirby diagrams. We give two approaches [**HKK**] to drawing a logarithmic transformation of X on an embedded torus T. First, we can draw X as $X - \operatorname{int} \nu T$ with a 2-handle, two 3-handles and a 4-handle attached. If we can visualize the T^3-structure of $\partial \nu T$ in the picture, then we can apply the gluing diffeomorphism φ of the logarithmic transformation to the framed attaching circle of the 2-handle to obtain the required manifold. For example, for F a regular fiber of $E(n)$, $E(n) - \operatorname{int} \nu F$ is given by Figure 8.10, and Figure 8.11 shows how to add the extra handles to obtain $E(n)$ (Example 8.2.11). The T^3-structure of the 3-manifold $\partial \nu F$ shown in Figure 8.10 is clearly visible: the obvious torus is a fiber, and the $-n$-framed circle in Figure 8.11 is a section with the product framing (Exercise 8.2.12). To construct $E(n)_p$, we reglue νF so that the attaching circle of the 2-handle runs once along a fiber and p times along the section, using the product framing of νF. The result is Figure 8.24(a), where the spiral has p strands. To compute the framing on the last 2-handle, first draw the picture in $\partial(T^2 \times D^2)$, Figure 8.24(b), to eliminate the effect of the $-n$-twist. The $p = 0$ case has framing 0 (the product framing in $\partial \nu F$). The diffeomorphism wrapping the circle p times

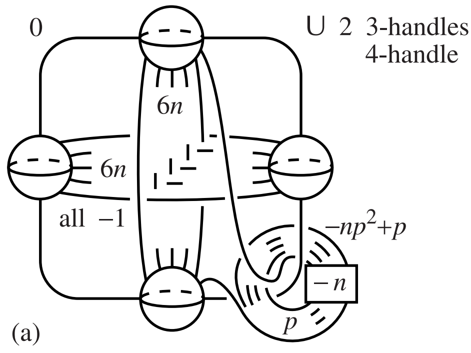


Figure 8.24. (a) Elliptic surface $E(n)_p$ and (b) framing computation.

around F adds a twist to the framing for each turn around F. (Visualize this.) When we transport the curve back to (a), the additional $-n$-twist lowers the framing by np^2. Note that the 1-handles of $E(n)_p$ can be cancelled. It is less obvious that the 3-handles can be cancelled — we return to this question below.

Exercise 8.3.15. Prove that $E(n)_0$ has the form $X\#6n\overline{\mathbb{CP}^2}$ for some manifold X. Thus, $E(n)_0$ is not diffeomorphic to the elliptic surface $E(n)_{p,q}$ for $p, q \geq 1$ unless p or q equals $n = 1$. For a harder exercise, prove that $E(n)_0 \approx (2n-1)\mathbb{CP}^2\#(10n-1)\overline{\mathbb{CP}^2}$. (See [**G9**] for an answer.)

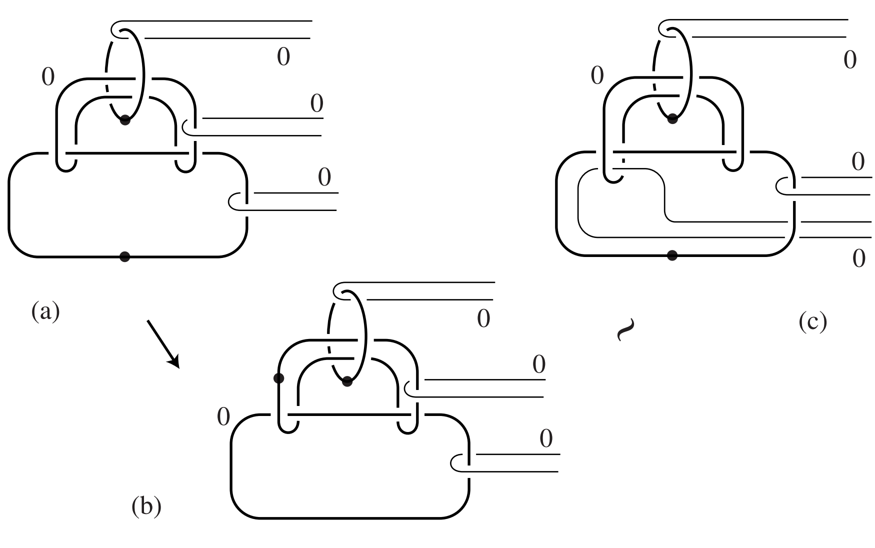


Figure 8.25. A multiplicity-0 logarithmic transformation of $T^2 \times D^2$.

The second approach to drawing logarithmic transformations is to draw X as $\nu T \cup$ handles and apply the gluing diffeomorphism to the handles. We can construct a general solution for this problem as in Remark 5.5.10(a). First we draw $\nu T = T^2 \times D^2$ as in Figure 8.25(a), with a 1-handlebody H of the obvious genus-3 Heegaard splitting of $\partial\nu T$ represented by three framed arcs attached to a sphere at infinity that we hold fixed during the computation. (Verify that removing the arcs and ball at ∞ leaves behind $\natural 3 S^1 \times D^2$, by expanding H to include the solid tori glued in during the surgery (cf. slam-dunks) and unknotting the resulting handlebody in S^3.) We perform a multiplicity-0 logarithmic transformation on νT, fixing the diagram of $\partial\nu T$, by interchanging a dot and 0-framing as in (b). An easy isotopy of the diagram returns us to the original picture of νT (c), showing how the gluing diffeomorphism moves H in $\partial\nu T$. Next, we exhibit a multiplicity-p logarithmic transformation of νT, which completes the solution provided that T lies in a cusp neighborhood (so that the resulting diffeomorphism type does not depend on the auxiliary multiplicity or direction we have chosen). First, we visualize a torus fiber of $\partial\nu T$ in Figure 8.25(a) as the obvious spanning disk of the 0-framed circle, surgered to avoid two punctures by the lower dotted circle (cf. Exercise 5.3.3(d)). Clearly, the intersection number of any circle with this fiber is given by its linking number with the 0-framed circle in (a), or equivalently, with the lower dotted circle in (c). Thus, we change the multiplicity from 0 to p by applying any diffeomorphism to $T^2 \times D^2$ that wraps the meridian of the 0-framed circle in (c) p times around the lower dotted circle. To visualize such a diffeomorphism, locate a torus $T' = T^2 \times \{\text{pt.}\} \subset T^2 \times D^2$ in (c) (spanning the 0-framed circle). We cut $\partial\nu T$ along T' and reglue so that any arc passing through T' is wrapped p times around the lower dotted circle. (Compare with a Dehn twist, Definition 8.2.3, in the 2-dimensional case.) Applying such a diffeomorphism (or more precisely, its inverse) to the boundary curves in Figure 8.25(c), we obtain Figure 8.26.

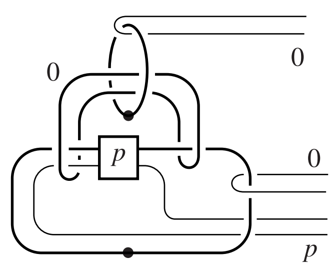


Figure 8.26. A multiplicity-p logarithmic transformation of $T^2 \times D^2$.

Figure 8.26 allows us to draw a logarithmic transform X_p of any handlebody X built on νT. For example, suppose Q is a fishtail neighborhood of T with vanishing cycle a -1-framed meridian of the lower dotted circle in Figure 8.25(a). For convenience, we slide the lower 1-handle of H over this meridian as in Figure 8.27(a); now (b) is Q_p, obtained by substituting (a) into Figure 8.26. (Note that H still represents a Heegaard splitting in ∂Q_p.) For a cusp neighborhood N, we apply the above construction to both 1-handles, obtaining Figure 8.28 after cancelling a handle pair. For a nucleus $N(n)$ we also add a $-n$-framed meridian along the remaining (middle) 1-handle of H, obtaining Figure 8.29 (where we have erased H since any self-diffeomorphism of $\partial N(n)_p$ extends over $N(n)_p$). The same method can be applied to Figure 8.24(a) to construct a diagram of $E(n)_{p,q}$. See [**HKK**] for a detailed discussion of $E(1)_{2,3}$, and [**G9**] for drawing logarithmic transformations on more than two parallel tori (or more than one torus and $\partial X \neq \emptyset$, e.g., $N(n)_{p,q}$).

Exercises 8.3.16. (a)* Show that the logarithmic transformation in Figure 8.27 has auxiliary multiplicity 1 and direction given by the vanishing cycle of Q (oriented as in the figure when $p \geq 0$ and the meridian of T is oriented counterclockwise as shown).

(b)* Verify that Q has a fiber-preserving self-diffeomorphism (preserving fiber orientations) that reverses the orientation of the vanishing cycle. Since reversing the direction of a logarithmic transformation is equivalent to reversing the sign of its multiplicity, it follows that Q_{-p} and Q_p are diffeomorphic. Check this directly by Kirby calculus. (Note that the diffeomorphism does not fix ∂Q_p; it is not clear whether this direction reversal can change the diffeomorphism type of a logarithmic transform of a manifold containing Q but no cusp neighborhood. However, Remark 8.5.10(b) shows that $Q_{-2} \approx Q_2 \operatorname{rel} \partial$, eliminating the ambiguity when $p = 2$.)

(c) Draw $E(n)_{p,q}$. (The picture depends on your choice of directions for the logarithmic transformations, although the diffeomorphism type does not.)

(d)* Prove that the manifolds $E(n)_{0,p_1,\dots,p_k}$ and $N(n)_{0,p_1,\dots,p_k}$ can be written in the forms $X \# 6n\overline{\mathbb{CP}^2}$ and $Y \# \overline{\mathbb{CP}^2}$, respectively, for suitable manifolds X, Y.

Finally, we consider Figure 8.29(b) of $N(n)_p$ in more detail. Since we have now shown that $E(n)_p = N(n)_p \cup_\partial M_c(2,3,6n-1)$ is a union of two 2-handlebodies (Corollary 6.3.19), we have:

Corollary 8.3.17. *The elliptic surface $E(n)_p$ has a handle decomposition without 1- or 3-handles.* □

An earlier (different) proof is sketched in [**Ma2**]. For an explicit picture of $E(n)_p$ as 2-handlebody ∪ 4-handle, see [**G9**]. For any fixed $n \geq 1$ and relatively prime $p, q \geq 2$, it is not known whether $E(n)_{p,q}$ admits a handle decomposition without 1-handles (cf. [**HKK**]).

Theorem 8.3.18. ([**G9**]). *Fix $n \geq 2$ and let p vary over the nonnegative integers; for n even fix the* mod 2 *residue of p. Then the 2-handlebodies*

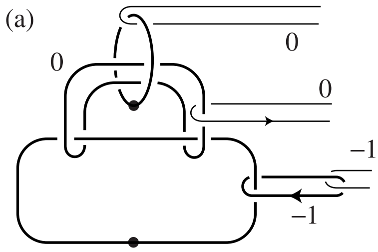


Figure 8.27. Logarithmic transformation from a fishtail neighborhood Q to Q_p.

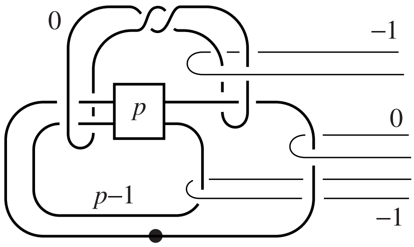


Figure 8.28. Logarithmic transform N_p of a cusp neighborhood.

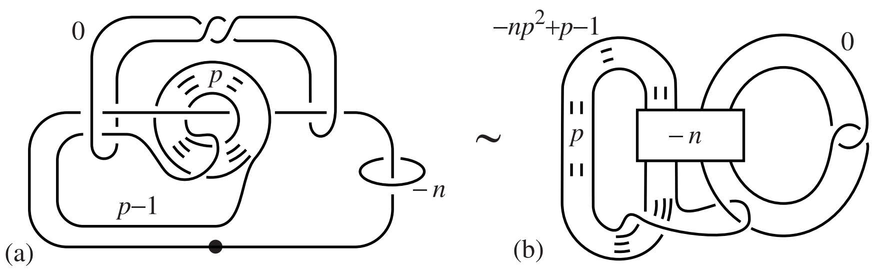


Figure 8.29. Nucleus $N(n)_p$.

$N(n)_p$ shown in Figure 8.29(b) are all homeomorphic to each other, but no two are diffeomorphic. In fact, no two become diffeomorphic under connected sum with copies of $\overline{\mathbb{CP}^2}$.

Proof. The manifolds in question all have the same intersection form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -np^2+p-1 \end{smallmatrix}\right]$ (check the parity) and the same homology sphere boundary $\partial N(n) = \overline{\Sigma}(2,3,6n-1)$. Freedman's Classification Theorem 1.2.27 extends without change to compact manifolds with boundary a fixed oriented homology sphere [**FQ**], so the manifolds are all homeomorphic. However, no two are diffeomorphic (or become so after blowing up) by Corollary 8.3.14. □

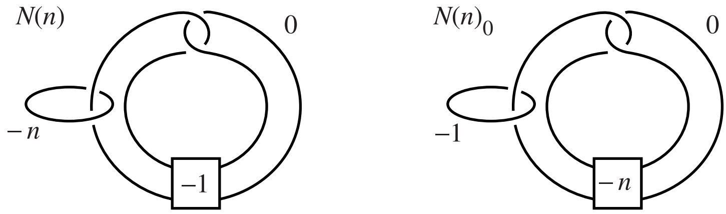


Figure 8.30. Homeomorphic but nondiffeomorphic manifolds $N(n)$, $N(n)_0$ (fixed odd $n \geq 3$).

A particularly simple example is the pair $N(n)$, $N(n)_0$ shown in Figure 8.30 ($n \geq 3$ odd). It is easy to see directly that their boundaries are diffeomorphic by slam-dunking the meridians and Rolfsen twisting to remove the given twists, obtaining a Whitehead link with framings $\frac{1}{n}$ and 1 in each case. For an even simpler pair of homeomorphic but nondiffeomorphic 2-handlebodies see Theorem 11.4.8.

Exercise 8.3.19. Prove directly that the manifolds $N(n)_p$ as given by Theorem 8.3.18 all have diffeomorphic boundaries, and that the manifolds in the corresponding family $N(n)_p \# S^2 \times S^2$ are all diffeomorphic. Conclude (by Lemma 8.3.10) that $E(n)_p \# S^2 \times S^2 \approx E(n)_q \# S^2 \times S^2$ provided that $p \equiv q \pmod 2$ if n is even, and so (by Exercise 8.3.15) $E(n)_p \# S^2 \times S^2 \approx \# 2n\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2}$ if n is odd or p is even, cf. Theorem 9.1.15 and the subsequent text. (*Hint*: Back up to Figure 8.29(a), surger the dotted circle into a 0-framed 2-handle, then surger the other 0-framed 2-handle to a dotted circle and cancel it (cf. Figure 12.65). The boundary will be $\frac{1}{n}$-surgery on the right trefoil.)

8.4. Higher genus and generalized fibrations

We briefly discuss some examples of simply connected Lefschetz fibrations of higher genus. Then we consider the effect of allowing the coordinate charts at singularities of a Lefschetz pencil (Definition 8.1.4) to reverse orientation.

First, we generalize elliptic surfaces $E(n)$ to obtain higher genus Lefschetz fibrations on surfaces of general type, following Fuller [**Fu2**], [**Fu3**]. Recall that (preceding Exercises 7.3.16) we defined $U(m, n)$ to be the (desingularized) double branched cover of the Hirzebruch surface $\mathbb{F}_{2n}$, branched along the union of $2m-1$ (affine) sections of self-intersection $2n$ and one (the infinity section) of self-intersection $-2n$. The genus-0 bundle structure on $\mathbb{F}_{2n}$ lifts to a genus-$(m-1)$ singular fibration that becomes Lefschetz after smooth perturbation. Thus $U(2, n) \approx E(n)$ is elliptic. For $m \geq 3$, $U(m, n)$ has general type, with $U(3, n) = H'(n)$ a Horikawa surface (cf. Corollary 7.3.28 and preceding). Fuller [**Fu3**] used the branched-cover description of $U(m, n)$ (in the manner of Examples 6.2.7 and 6.3.10) and Kirby

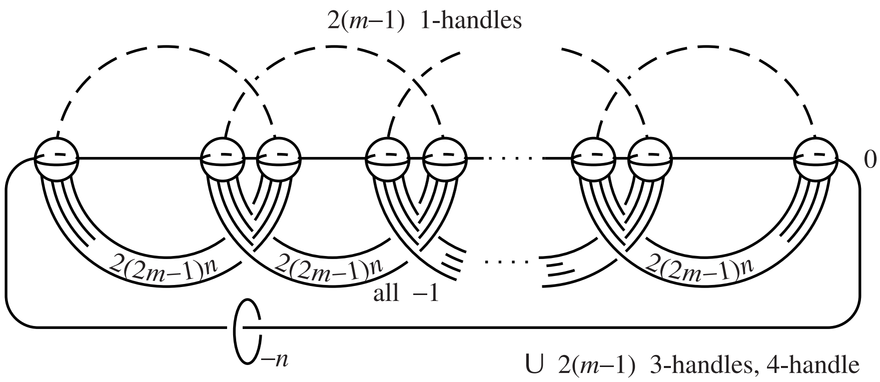


Figure 8.31. Lefschetz fibration $U(m,n) \to S^2$.

moves to obtain the diagram of $U(m,n)$ given in Figure 8.31. Note that we recover Figure 8.11 when $m = 2$.

Exercises 8.4.1. (a) Verify that the 1-handles and 0-framed 2-handle in Figure 8.31 represent $F \times D^2$, for F a surface of genus $m-1$ (cf. Example 4.6.5). Check that removing the $-n$-framed 2-handle and the 3- and 4-handles from the diagram yields a genus-$(m-1)$ Lefschetz fibration over D^2 with monodromy representation $(\psi_1, \dots, \psi_{2(m-1)})^{2(2m-1)n}$, where the Dehn twists ψ_i on F are generated by circles C_i generalizing Figure 8.12 in the obvious way; cf. γ^n at the end of Section 8.2 when $m = 3$. Fuller shows that these removed handles form a copy of $F \times D^2$, so the above monodromy around ∂D^2 gives a trivial word in $\mathcal{M}_{m-1}$ and the entire diagram represents a Lefschetz fibration $U(m,n) \to S^2$.

(b) Verify that the Lefschetz fibration over D^2 given by $(\psi_1, \dots, \psi_{2(m-1)})$ is a generalized cusp neighborhood, obtained by adding a 2-handle to D^4 along the 0-framed torus knot $T_{2,2m-1}$ (but beware that Lemma 8.3.6 does not generalize). Thus, $U(m,n)$ admits a singular fibration whose singular fibers are $2(2m-1)n$ generalized cusps. Verify by Kirby calculus that $U(m,n) \approx M_c(2, 2m-1, 2(2m-1)n-1) \cup_\partial N(m,n)$, where $N(m,n)$ is a generalized nucleus (Remark 7.3.20(b)), and that $U(m,n)$ is given by Figure 8.32. (*Hint*: Imitate the beginning of Section 8.3.)

A similar approach [**Fu2**] (see also [**Fu1**]) yields a diagram of $X(m,n)$, the double cover of $S^2 \times S^2$ branched along a smoothing of the curve $B_{m,n} = (S^2 \times \{2m \text{ points}\}) \cup (\{2n \text{ points}\} \times S^2)$, cf. Remark 7.3.5. Recall that $X(m,n) \approx X(n,m)$ has a genus-$(m-1)$ singular fibration over S^2, $X(2,n) \approx E(n)$, and $X(3,n)$ is the Horikawa surface $H(n)$ on the Noether line (cf. Corollary 7.3.28 and preceding). For $m,n \geq 3$, $X(m,n)$ is a complex surface of general type. Recall (Corollary 7.3.28 and subsequent text)

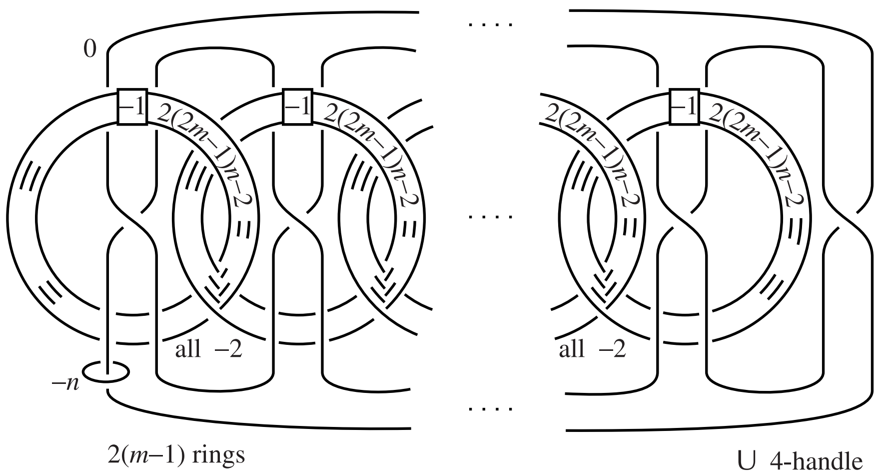


Figure 8.32. Complex surface $U(m,n)$ — general type for $m \geq 3$.

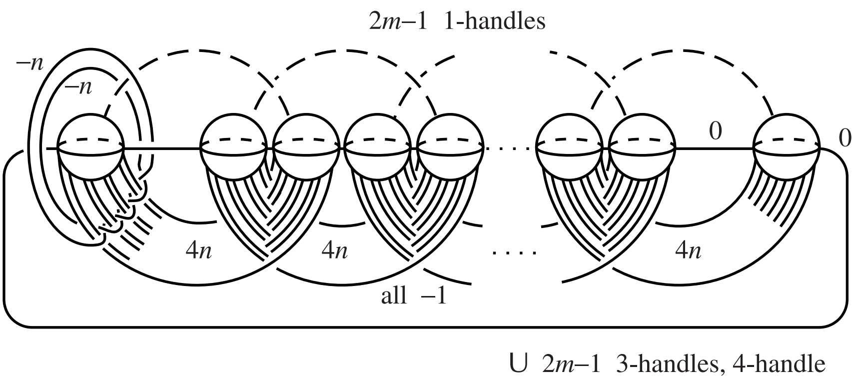


Figure 8.33. Lefschetz fibration $X(m,n) \to S^2$.

that it is a major unsolved problem to determine whether (for example) $H(2n) = X(3,2n)$ and $H'(n) = U(3,n)$ are diffeomorphic for odd $n > 1$. As before, the branched-cover description of $X(m,n)$ leads to a Kirby diagram, although the different pattern of intersections in the branch locus leads to a different configuration of 1-handles in the smoothed branch set (cf. Figures 6.32 and 6.33), and hence a different configuration of 2-handles in $X(m,n)$, Figure 8.33. As before, the $-n$-framed handles (with linking number n), together with the 3- and 4-handles, form a copy of $F \times D^2$ for F a surface of genus $m-1$, with cores of the 2-handles representing $\{\text{pt.}\} \times D^2$. Thus, we can cancel (either) one of the $-n$-framed 2-handles against a 3-handle, and the other will give a section of the Lefschetz fibration.

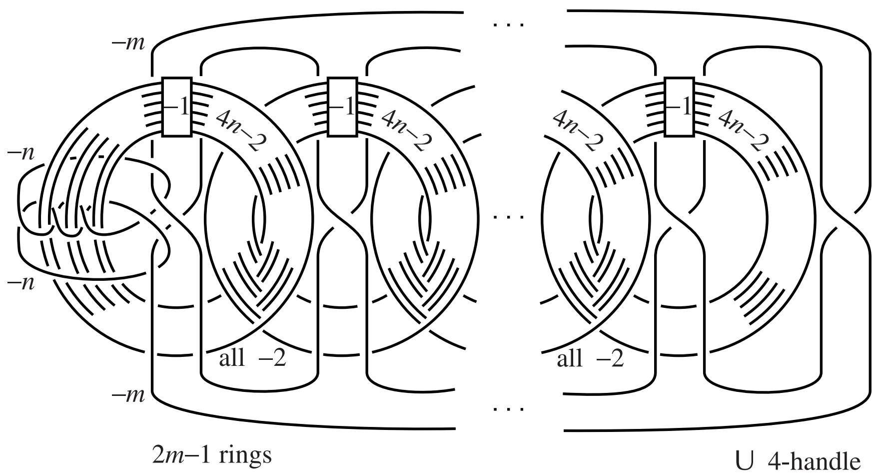


Figure 8.34. Complex surface $X(m,n)$ — general type for $m, n \geq 3$.

Exercises 8.4.2. (a) Verify that the 1-handles and two 0-framed 2-handles give a copy of $F \times D^2$ (and correspond to a handle decomposition of F with two 2-handles). Note that each 0-framed circle goes homologically zero times over $m-1$ of the 1-handles, so the 0-framing is the blackboard framing. Check that removing the two $-n$-framed handles and 3- and 4-handles yields a Lefschetz fibration over D^2 with monodromy $(\psi_1, \dots, \psi_{2m-1}, \psi_{2m-1}, \dots, \psi_1)^{2n}$ with ψ_i as before, so the whole diagram represents a Lefschetz fibration $X(m,n) \to S^2$ with the same monodromy; cf. α^n at the end of Section 8.2 when $m = 3$.

(b) Verify that the monodromy $(\psi_1, \dots, \psi_{2m-1})$ gives a Lefschetz fibration $X \to D^2$ with X a 2-handlebody on the torus link $T_{2,2m}$ with framings $-m$. Thus, $X(m,n)$ has a singular fibration with $4n$ singular fibers whose neighborhoods are diffeomorphic to X (and glued nontrivially along the fibers). Conclude that $X(m,n)$ is given by Figure 8.34. (*Hint*: The cocores of the 2-handles of X come from $\{\text{pt.}\} \times D^2 \subset F \times D^2$, so attaching the last copy of X is the same as attaching the $-n$-framed 2-handles with linking number n and the 4-handle in Figure 8.34.)

(c)* Show that the monodromy $(\psi_1, \dots, \psi_{2m-1}, \psi_{2m-1}, \dots, \psi_1)$ gives the plumbing P of spheres shown in Figure 8.35, with the homology class of F given by twice the sphere S of square $-m$ plus the sum of the other spheres (oriented so that all intersection numbers are ≥ 0). Thus, $X(m,n)$ has a singular fibration with $2n$ singular fibers given by P (with multiplicity two on S), and these are glued by the identity on the fiber over the base point. Now deduce this directly from the branched cover description of $X(m,n)$.

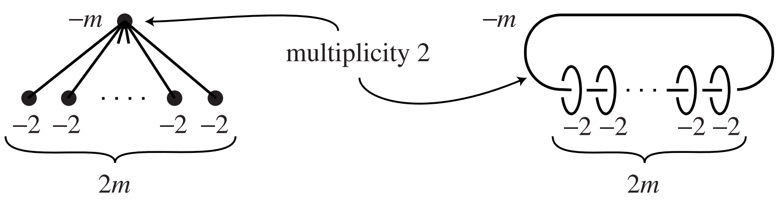


Figure 8.35. Singular fiber of $X(m,n) \to \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$.

(d) By breaking each -1-twist of Figure 8.34 into two $-\frac{1}{2}$-twists (cf. the solution of Exercise 6.3.13(a)), exhibit an involution of $X(m,n)$. (See Remark 6.3.15.) Show that the quotient is $S^2 \times S^2$ and draw the branch locus $B \subset S^2 \times S^2$ (cf. Exercise 6.3.9(a)). With more work, one can show that B is isotopic to the smoothing of the complex curve $B_{m,n}$, so we have exhibited the defining branched covering $X(m,n) \to S^2 \times S^2$.

(e) Show that the complex surface $Z(n)$ of Exercise 7.4.1(a) and Theorem 7.4.20 admits a genus-2 Lefschetz fibration with monodromy $\alpha^n\beta$ (in the notation given at the end of Section 8.2). (*Hint*: Show that when $n = 0$ you get a genus-2 Lefschetz fibration on $K3\#2\overline{\mathbb{CP}^2}$, where $K3$ arises as the double cover of $\mathbb{CP}^2$ branched along a sextic curve, cf. Corollary 7.3.25. By comparing Euler characteristics, verify that the monodromy must be β. For $n > 0$, compare with (c) above.)

Since Lefschetz fibrations $\pi\colon X \to \Sigma$ originated in algebraic geometry, it is natural to ask whether they are always projective (for $\partial\Sigma = \emptyset$), or at least whether there is always a complex (or Kähler) structure on X making them holomorphic. We have seen that genus-0 Lefschetz fibrations are all blow-ups of ruled surfaces (Proposition 8.1.7 and Section 3.4), so they are Kähler with holomorphic projection maps. Genus-1 Lefschetz fibrations have also been classified (Theorem 8.3.12). For $\Sigma = S^2$ these are all holomorphic, and the additional hypothesis of at least one singular fiber (X relatively minimal) guarantees a Kähler structure. For higher genus, the situation is different.

Theorem 8.4.3. *For any $g \geq 2$, there is a genus-g Lefschetz fibration $X_g \to S^2$ for which X_g admits no complex structure, although the fibration is a fiber sum of two Lefschetz fibrations, each obtained by deforming a holomorphic map. For $g \geq 3$, we can assume X_g is simply connected.* □

The genus-2 case of this theorem is a recent observation of Ozbagci and the second author [**OzS**] and (independently) I. Smith [**Smi**]. One begins with a Lefschetz fibration $S^2 \times T^2\#4\overline{\mathbb{CP}^2} \to S^2$ of Matsumoto [**Mt4**], which has monodromy $(\zeta_1, \dots, \zeta_4)^2$ for Dehn twists ζ_i given by the (symmetrical) circles C_i in Figure 8.36. By fiber summing two copies of this using a

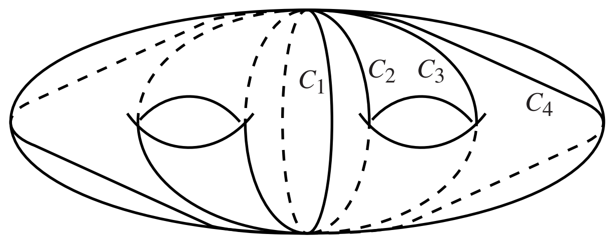


Figure 8.36. Vanishing cycles of Matsumoto's fibration.

suitable identification of the fibers, one can arrange the resulting manifold X_2 to have (for example) $\pi_1(X_2) \cong \mathbb{Z}$. Since $b_1(X_2)$ is odd, X_2 cannot be Kähler. Computing the characteristic numbers $\chi(X_2) = 12$, $b_+(X_2) = 2$ and $c_1^2(X_2) = 0$, we find that X_2 is not even homotopy equivalent to a complex surface, by the Enriques-Kodaira classification (cf. Theorem 3.4.32). Note that the fibration on X_2 has separating vanishing cycles — it is currently an open question whether a genus-2 fibration without separating vanishing cycles must be holomorphic.

The Matsumoto fibration $(\zeta_1, \dots, \zeta_4)^2$ can be generalized to a genus-g fibration with monodromy $(\zeta_1, \dots, \zeta_{g+2})^2$ for any even g, by generalizing Figure 8.36 in the obvious way. Note that the fibration has only $2g+4$ critical points. For odd g, a similar construction gives a fibration with $2g+10$ critical points. See Cadavid [**Ca**] for details of these constructions. It can be shown [**S7**] that the total space of these fibrations is diffeomorphic to $S^2 \times \Sigma_h \# 4\overline{\mathbb{CP}^2}$ ($g = 2h$) when g is even and to $S^2 \times \Sigma_h \# 8\overline{\mathbb{CP}^2}$ ($g = 2h + 1$) when g is odd. Fiber summing two copies of these genus-g Lefschetz fibrations using a gluing map similar to the one used in the genus-2 case above, we find a genus-g Lefschetz fibration $X_g \to S^2$ with $b_1(X_g) = 1$. Generalizing the above argument shows that X_g is a noncomplex genus-g Lefschetz fibration. For simply connected genus-g Lefschetz fibrations not admitting complex structures, see Remark 8.5.7.

Now we generalize Lefschetz pencils to allow singularities with the opposite orientation.

Definition 8.4.4. Let X be a compact, connected, oriented, smooth 4-manifold. An *achiral Lefschetz pencil* or *fibration* on X is defined as in Definition 8.1.4, except that the given coordinate charts around critical points of π and points of the base locus are allowed to reverse orientation. (We still require the surface Σ to be oriented.)

Clearly, every Lefschetz pencil (fibration) is automatically an achiral Lefschetz pencil (fibration). In fact, the set of 4-manifolds admitting achiral Lefschetz pencils is much larger than the set admitting ordinary (chiral)

Lefschetz pencils. We will see (Corollary 10.2.23) that a Lefschetz pencil on X (or fibration if $\partial X = \emptyset$ and $b_1(X) = 0$; Remark 10.2.22(a)) determines a symplectic structure. Thus, $b_2^+(X) > 0$, X admits an almost-complex structure so $b_2^+(X) - b_1(X)$ is odd (Exercise 8.1.6), and X has nontrivial Seiberg-Witten invariants (Theorem 2.4.7) so it cannot split as $X_1 \# X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$ (Theorem 2.4.6). For achiral Lefschetz pencils and fibrations, we will provide counterexamples to all of the corresponding statements. We will see that if X admits an achiral Lefschetz fibration (or pencil) for which π has at least one critical point, then so does $X \# S^2 \times S^2$. Furthermore, S^4 admits both a genus-1 achiral Lefschetz fibration and a genus-0 achiral Lefschetz pencil (without critical points but with nonempty base locus). These examples raise the question of whether there are *any* obstructions to the existence of such structures. We will see that while such an X need not be almost-complex, $X \# q\mathbb{CP}^2$ does admit an almost-complex structure, where q is the number of base locus and critical points whose charts reverse orientation. A consequence is that $\# mS^3 \times S^1$ admits no achiral Lefschetz pencil or fibration if $m \geq 2$ (Corollary 8.4.14). There are no known obstructions for simply-connected 4-manifolds.

Remark 8.4.5. Achiral Lefschetz fibrations were first studied by Harer [**H2**], who also allowed fibers to have boundary. He showed that such a singular fibration $X \to D^2$ existed if and only if X had a handle decomposition without handles of index ≥ 3, and produced a "cobordism" classification for $\partial X = \emptyset$.

Example 8.4.6. – Connected sums. Recall that if X admits a Lefschetz fibration, then so does $X \# \overline{\mathbb{CP}^2}$ — simply blow up a point and compose the blow-down map with the fibration. In the achiral case, the same argument in an orientation-reversing chart shows that $X \# \mathbb{CP}^2$ admits an achiral Lefschetz fibration when X does. This already provides counterexamples to achiral generalizations of most of the above statements about Lefschetz fibrations. To deal with $X \# S^2 \times S^2$, we observe that achiral Lefschetz fibrations have monodromy representations just as ordinary Lefschetz fibrations do, the only difference being that critical points with the new orientation (negative self-intersections of fibers) will correspond to left-handed Dehn twists (so the corresponding framings in Kirby diagrams are $+1$ relative to the fibers). If two critical points with the same vanishing cycle $C \subset F$ are adjacent and have opposite orientations, then their monodromies will cancel. Thus, we can insert such a pair of critical points anywhere in an achiral Lefschetz fibration on a closed X to obtain a new closed manifold X' with an achiral Lefschetz fibration. The new manifold X' is obtained from X by surgery on $C \subset X$ with framing opposite the one induced by $F \subset X$ and $\pi\colon X \to \Sigma$ (Figure 8.37). Thus, for C nullhomotopic in X, the manifold X' will be diffeomorphic to either $X \# S^2 \times S^2$ or $X \# \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. If an achiral

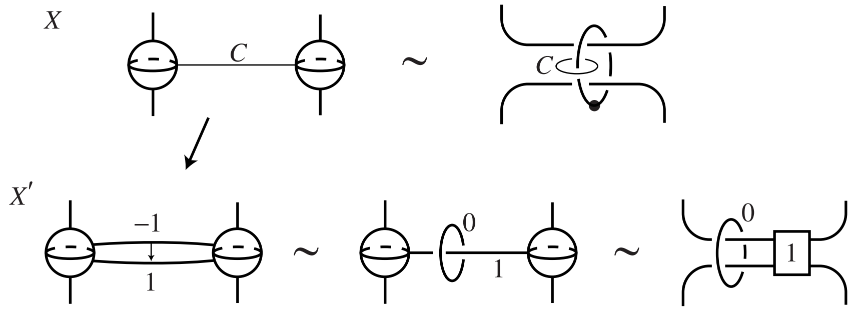


Figure 8.37

Lefschetz fibration $\pi\colon X \to \Sigma$ has at least one critical point, then we can choose C to be this vanishing cycle and conclude that $X' = X\#S^2 \times S^2$ has an achiral Lefschetz fibration. Similar reasoning shows that $X\#S^2\times S^2$ admits an achiral Lefschetz *pencil* with a critical point whenever X does. Since any simply connected, closed X satisfies $X\#kS^2 \times S^2 \approx S\#\ell S^2 \times S^2$ (up to orientation) for some $k, \ell \in \mathbb{N}$ and projective surface S (e.g., by letting S range over $S^2\times S^2$, $E(2n)$ and blow-ups of $\mathbb{CP}^2$; see Theorem 9.1.12), we conclude that any simply-connected, closed 4-manifold admits an achiral Lefschetz pencil after summing with enough copies of $S^2 \times S^2$.

Example 8.4.7. Matsumoto [**Mt1**] constructed a genus-1 achiral Lefschetz fibration on S^4 with two critical points of opposite orientation — that is, two fishtail fibers with one oriented incorrectly (a "fishhead" fiber?). We can easily construct such a fibration using the previous example. We begin with the projection $\pi\colon S^2 \times T^2 \to S^2$. A multiplicity-1 logarithmic transformation with auxiliary multiplicity 1 yields $\pi\colon S^3 \times S^1 \to S^2$, projection to S^3 followed by the Hopf fibration. We add a pair of oppositely oriented critical points as above, with vanishing cycle $C = \{\text{pt.}\} \times S^1 \subset S^3 \times S^1$. The resulting surgery produces S^4 with the required achiral fibration.

Exercise 8.4.8. Verify that this construction is depicted by Figure 8.38, and show by Kirby calculus that the pictured manifold is S^4. Compare with Exercise 6.2.6(b).

Example 8.4.9. To construct a genus-0 achiral Lefschetz pencil without critical points on S^4, one-point compactify $\mathbb{C}^2$ with its pencil of complex lines through 0. Equivalently, restrict that pencil in $\mathbb{C}^2$ to D^4 and then double. The base locus is $B = \{0, \infty\}$, and the fibration is oppositely oriented at the two points. If we blow up to eliminate the base locus, we obtain $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with its usual fibration by spheres.

For one more example to illustrate the effect of allowing achirality in Lefschetz fibrations, we state a theorem of Matsumoto [**Mt2**] (sharpened

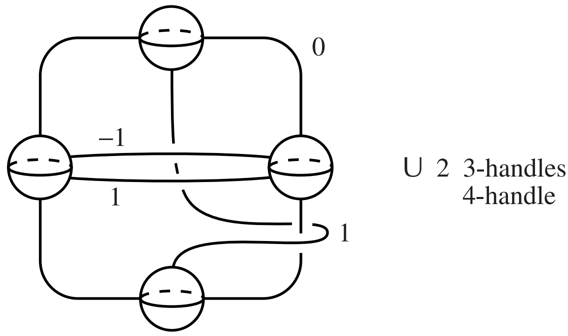


Figure 8.38. Achiral Lefschetz fibration on S^4.

by [**Iw**], [**G10**]) which was proved by analyzing monodromy as in Example 8.2.11.

Theorem 8.4.10. *Let $\pi\colon X \to S^2$ be a genus-1 achiral Lefschetz fibration. Suppose that X is simply connected and π has critical points of both orientations. Then X is diffeomorphic (possibly reversing orientation) to a connected sum of copies of either $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ or $E(2)$ and $S^2 \times S^2$. The same holds if we also allow smooth multiple fibers, provided that $\sigma(X) \neq 0$.* □

Remark 8.4.11. It seems likely that the signature restriction is unnecessary. For work in this direction, see [**Mt2**] (Theorem 3.7, $\nu = 2\ell$), [**Iw**], [**G10**] (Theorem 16, $n = k$).

To find an obstruction to the existence of achiral Lefschetz pencils and fibrations, we consider characteristic classes. For X closed, let $\pi\colon X - B \to \Sigma$ be an achiral Lefschetz pencil or fibration (so either $\Sigma = S^2$ or $B = \emptyset$). Let $Q \subset X$ be the set of points in B and the critical set of π for which the coordinate charts specified by Definition 8.4.4 reverse orientation, and let q be the number of points in Q. As in Exercise 8.1.6, $X - Q$ inherits an almost-complex structure J from π. Although J does not extend over X, its Chern class $c_1(X, J) \in H^2(X - Q; \mathbb{Z}) \cong H^2(X; \mathbb{Z})$ does extend uniquely. (Beware that the square $c_1^2[X, J]$ of this class need not equal $c_1^2[X]$, which we defined to be $3\sigma(X) + 2\chi(X)$ in Section 1.4.) Let $X' = X \# q\mathbb{CP}^2$ be the result of reversed orientation blow-ups of the points of Q; let $e_i \in H^2(X'; \mathbb{Z})$, $i = 1, \dots, q$, be the corresponding exceptional classes $[\mathbb{CP}^1] \in \mathbb{CP}^2$, so $e_i^2 = 1$. (Note that while X' is essentially independent of the choices of orientation-reversing charts, the identification of a summand with $\mathbb{CP}^2$ can be conjugated by changing the chart by conjugation. We fix the choice by requiring proper transforms of fibers of π to intersect each e_i positively.)

Lemma 8.4.12. *The almost-complex structure J on $X - Q$ extends (after a homotopy) to a structure J' on X' with $c_1(X', J') = c_1(X, J) + 3\sum e_i$. Thus, $c_1^2[X, J] = 3\sigma(X) + 2\chi(X) - 4q = c_1^2[X] - 4q$.*

Proof. Near any point in $Q \cap B$, the pencil agrees with that of Example 8.4.9 near $\infty \in S^4$, and J agrees with the standard structure on $\mathbb{C}^2 = S^4 - \{\infty\}$. If we blow up at $\infty \in S^4$ to obtain $\mathbb{CP}^2$, J extends to the standard complex structure J'' on $\mathbb{CP}^2$, and $c_1(\mathbb{CP}^2, J'') = 3[\mathbb{CP}^1]$. This local model shows us how to extend J when we blow up points in $Q \cap B$. For any other point in Q, a neighborhood can be identified with that of the self-intersection p of the fishhead fiber of Example 8.4.7. Now J agrees with the almost-complex structure on $S^4 - \{p\}$ constructed from that achiral Lefschetz fibration. Since $S^4 - \{p\}$ is contractible, it admits only one almost-complex structure up to homotopy, so when we blow up p to obtain $\mathbb{CP}^2$, J agrees up to homotopy with the standard structure on $\mathbb{CP}^2$. Thus, J on $X - Q$ extends to X' after a homotopy near the critical points of π in Q, and $c_1(X', J') = c_1(X, J) + 3\sum e_i$ (since J' restricts to J on $X - Q$ and the standard structure on each $\mathbb{CP}^2 - \{\text{pt.}\}$). Now $c_1^2[X', J'] = c_1^2[X, J] + 9q$, but by Theorem 1.4.15 $c_1^2[X', J'] = 3\sigma(X') + 2\chi(X') = 3\sigma(X) + 2\chi(X) + 5q$, so the last formula of the lemma follows. □

The above lemma provides an obstruction to the existence of achiral Lefschetz pencils. Consider a closed, oriented 4-manifold X with Q_X positive definite. Recall that by Donaldson's Theorem 1.2.30 (which is true regardless of $\pi_1(X)$, cf. Remark 2.4.30), we must have $Q_X \cong b_2(X)\langle 1 \rangle$.

Theorem 8.4.13. *Suppose that X is a closed 4-manifold with an achiral Lefschetz pencil or fibration, and that $Q_X \cong b_2(X)\langle 1 \rangle$. Then we must have $1 - b_1(X) + b_2(X) \geq q \geq 0$ (for q as defined above).*

Proof. Since $c_1(X, J)|_2 = w_2(X)$, $c_1(X, J)$ projects (into $H^2(X; \mathbb{Z})/\text{torsion}$) to a characteristic element of the pairing Q_X (Proposition 1.4.18 or the text preceding Corollary 5.7.6). In particular, its components in the above splitting of Q_X are all odd. Thus, $c_1^2[X, J] \geq \sigma(X)$. The lemma now implies that $\sigma(X) + \chi(X) \geq 2q$, or $1 - b_1(X) + b_2(X) \geq q$. □

Corollary 8.4.14. *For $m \geq 1$, the manifold $\#mS^3 \times S^1$ admits no achiral Lefschetz pencil or fibration, except for the standard torus bundle when $m = 1$ (that is, projection to S^3 followed by the Hopf fibration).*

Proof. The case $m > 1$ follows immediately from the above theorem. For $m = 1$ we conclude that any such structure satisfies $q = 0$, and after reversing orientation the same argument shows that $B = \emptyset$ and π has no critical points. Since $\chi(S^3 \times S^1) = 0$ and $b_1(S^3 \times S^1) < 2$, we must have a torus bundle over S^2. Since the bundle is trivial over each hemisphere, it is obtained from $S^2 \times T^2$ by a multiplicity-1 logarithmic transformation, and the auxiliary multiplicity is 1 (since $\pi_1(S^3 \times S^1) \cong \mathbb{Z}$). It is now easy to see that the bundle is the standard one. □

Exercises 8.4.15. (a)* Classify achiral Lefschetz pencils and fibrations on S^4. What can you say about such structures on a homology 4-sphere X?

(b)* For $\pi\colon X - B \to \Sigma$ an achiral Lefschetz pencil or fibration as above, let $P = B \cup \{\text{critical points of } \pi\}$. There is an induced complex line bundle $L = \ker d\pi$ on $X - P$ (tangent to the fibers), and this extends uniquely over X (since $H^2(X - P; \mathbb{Z}) \cong H^2(X; \mathbb{Z})$ classifies line bundles). Prove that $c_1(X, J) = c_1(L) + \chi(\Sigma)\pi^*[\Sigma]$ in $H^2(X; \mathbb{Z})$. For an achiral Lefschetz fibration with connected, rationally nullhomologous fibers, prove that the genus is 1.

8.5. Rationally blowing down

We now discuss the rational blow-down process discovered by Fintushel and Stern [**FS2**]. Among other applications, this describes the logarithmic transformations studied in Sections 3.3 and 8.3 from a different point of view. At the same time, the gauge theoretic invariants (cf. Section 2.4) of manifolds constructed by the rational blow-down process are easy to determine from the knowledge of those invariants of the original manifold (cf. Theorem 8.5.12). In conclusion, for example, the Seiberg-Witten invariants of the elliptic surfaces $E(n)_{p,q}$ $(n \geq 2)$ can be determined; this latter result can be applied to prove Theorem 3.3.7 and more generally Theorem 8.3.12.

We begin our discussion by defining the 4-manifold C_p $(p \geq 2)$ as the plumbing manifold according to the tree given by Figure 8.39.

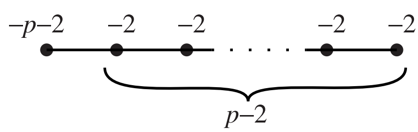


Figure 8.39. Plumbing manifold C_p $(p \geq 2)$.

Exercises 8.5.1. (a)* Draw a Kirby diagram for C_p. Using Kirby calculus, prove that C_p is given by Figure 8.40. (*Hint*: Use the -1-framed meridians in the latter figure to eliminate $p-1$ twists.)

(b)* Show that the 4-manifold C_p embeds in $\#(p-1)\overline{\mathbb{CP}^2}$. (Try this both with and without Kirby calculus.)

Lemma 8.5.2. *The boundary ∂C_p is the lens space $L(p^2, p-1)$; hence $\pi_1(\partial C_p) \cong \mathbb{Z}_{p^2}$.*

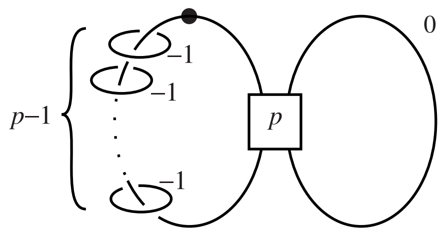


Figure 8.40. Kirby diagram for C_p.

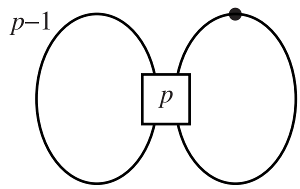


Figure 8.41. Rational 4-ball B_p.

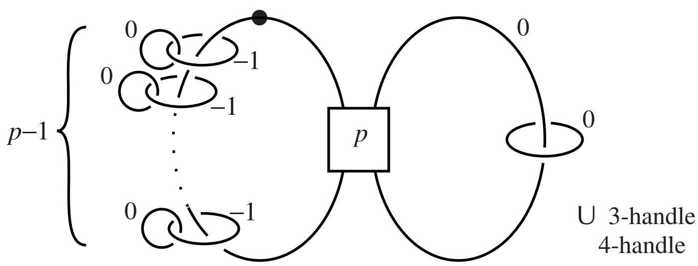


Figure 8.42. Double DC_p.

Proof. This follows from Exercise 5.3.9(b), the plumbing diagram of Figure 8.39 and the fact that the continued fraction expansion of $\frac{p^2}{p-1}$ is given by $[p+2, 2, \ldots, 2]$ with $p-2$ many 2's. □

Let B_p be the 4-manifold given by Figure 8.41. We will show that $B_p \cup_\partial \overline{C}_p = \#(p-1)\mathbb{CP}^2$; so, in particular, ∂C_p is diffeomorphic to ∂B_p. As we saw in Exercise 8.5.1(a), C_p is given by the Kirby diagram drawn in Figure 8.40; hence the double $DC_p = C_p \cup_\partial \overline{C}_p$ is given by Figure 8.42. Surger inside $C_p \subset DC_p$ twice (interchanging the dot and the 0-framing of the two circles linking each other p times) and blow down the -1-framed circles (still in C_p) to obtain Figure 8.43(a) — a Kirby diagram of $B_p \cup_\partial \overline{C}_p$. ($B_p$ is given by the two circles linking each other p times, while we did not do anything in the upper part giving $\overline{C}_p$ as a relative handlebody on ∂C_p.) Sliding the $(p-1)$-framed circle over its meridians, we end up with $p-1$ unknots with framing 1 (see Figure 8.43(b)), and using the 0-framed

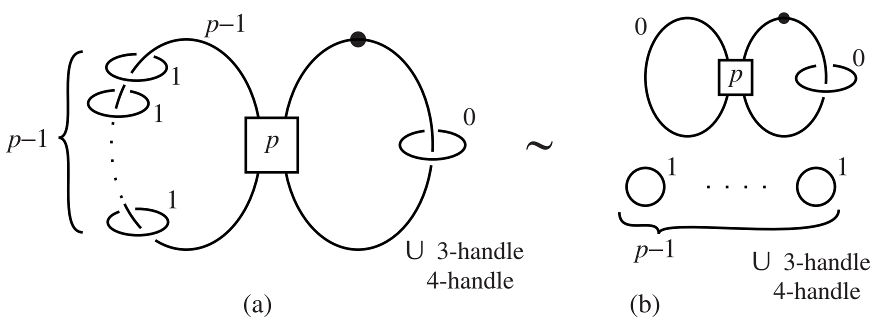


Figure 8.43. $B_p \cup_\partial \overline{C}_p = \#(p-1)\mathbb{CP}^2$.

meridian of the dotted circle to unlink the two circles defining B_p, we get cancelling 1-handle/2-handle and 2-handle/3-handle pairs. Consequently, Figure 8.43(b) is $\#(p-1)\mathbb{CP}^2$, which proves that the manifold B_p given by Figure 8.41 can be given as the complement of $\overline{C}_p$ in $\#(p-1)\mathbb{CP}^2$. It is not very hard to see that the above embedding of $\overline{C}_p$ in $\#(p-1)\mathbb{CP}^2$ — after reversing orientation — coincides up to diffeomorphism with the embeddings provided by the solutions of Exercise 8.5.1(b) (cf. Theorem 8.5.3 and Exercise 8.5.8(a) below). From the link description it is obvious that B_p has trivial rational homology and $\pi_1(B_p) \cong \mathbb{Z}_p$. The symmetry of the diagram in Figure 8.41 (180° rotation around the y-axis) shows that ∂B_p admits a self-diffeomorphism not homotopic to the identity. (In fact, the diffeomorphism inverts a meridian to the dotted circle, which corresponds to a meridian of the sphere of square $-p-2$ in C_p and hence generates $H_1(\partial B_p) \cong \mathbb{Z}_{p^2}$.) Moreover, the above description of this self-diffeomorphism also shows that it extends to B_p. Combining this observation with a theorem of Bonahon [**Bon**] (stating that $\pi_0(\mathrm{Diff}(L(p^2, p-1))) \cong \mathbb{Z}_2$) we get

Theorem 8.5.3. *Any self-diffeomorphism of ∂B_p extends to B_p.* □

Now we are ready to define the rational blow-down of a 4-manifold X.

Definition 8.5.4. Assume that the plumbing C_p embeds in X, that is, $X = C_p \cup_{L(p^2,p-1)} X^o$. The 4-manifold $X_{(p)} = B_p \cup_{L(p^2,p-1)} X^o$ is by definition the *rational blow-down* of X along the given copy of C_p.

By Theorem 8.5.3 the 4-manifold $X_{(p)}$ is well-defined up to diffeomorphism for a fixed pair (X, C_p). Thus, in a Kirby diagram of X built on C_p as in Figure 8.40, we can perform the rational blow-down by our previous procedure of interchanging the dot and 0-framing and blowing down the -1-framed meridians. Note that the net effect is to remove a negative definite submanifold from X and replace it by a rational ball, so the operation preserves b_2^+ and b_1 while decreasing b_2^-, just as with ordinary blowing down. (The operation may create torsion in H_1, however.)

The power of the rational blow-down process rests on two facts. On one hand, interesting constructions — like logarithmic transformation and blow-down of a -4-sphere (cf. Theorem 10.2.14) — can be formulated in terms of rational blow-down. On the other hand, in many cases the Seiberg-Witten function $SW_{X_{(p)}}$ can easily be computed in terms of SW_X; hence a detailed analysis of the smooth topology of the resulting 4-manifold $X_{(p)}$ is possible. Next we describe some examples of manifolds containing copies of C_p for some p.

Examples 8.5.5. **(a)** Suppose that the homology class $f \in H_2(X;\mathbb{Z})$ with $f^2 = 0$ can be represented by an immersed sphere Σ with a unique, positive double point. (Thus, Σ determines a fishtail neighborhood, Figure 8.8.) Then $X\#(p-1)\overline{\mathbb{CP}^2}$ contains a copy of C_p in the following way: Blow up the double point of Σ to obtain a sphere $\widetilde{\Sigma}$ of square -4 in $X\#\overline{\mathbb{CP}^2}$ (with $[\widetilde{\Sigma}] = f - 2e_1$); a tubular neighborhood of $\widetilde{\Sigma}$ is a copy of C_2 in $X\#\overline{\mathbb{CP}^2}$. Blowing up one of the two intersection points of $\widetilde{\Sigma}$ with the exceptional sphere, we get $C_3 \subset X\#2\overline{\mathbb{CP}^2}$. If we keep blowing up the intersection point of the proper transform of Σ with the last exceptional sphere (as in the proof of Exercise 8.5.1(b)), we get C_p in $X\#(p-1)\overline{\mathbb{CP}^2}$. (The homology classes of the spheres in C_p are $f-2e_1-e_2-\ldots-e_{p-1}, e_1-e_2, e_2-e_3, \ldots, e_{p-2}-e_{p-1}$.) This construction can be applied, e.g., when Σ is a fishtail fiber in an elliptic fibration $E(n)$.

(b) The nine sections of $E(4)$ give nine disjoint copies of C_2 in $E(4)$, hence we can blow down these C_2-configurations one by one. More generally, for $n \geq 4$ the decomposition of $E(n)$ as the generalized fiber sum of two copies of $X(1,n) \approx \mathbb{CP}^2\#(4n+1)\overline{\mathbb{CP}^2}$ (cf. Remark 7.3.9(b)) shows that $E(n)$ contains 2 disjoint copies of C_{n-2}, each $-n$-sphere being a section of $E(n)$ (cf. Exercise 3.1.12(a)), while the -2-spheres in C_{n-2} are disjoint from a generic fiber [**FS2**]. These copies of C_{n-2} can be seen by observing that the resolution of the singularity $z^2 = x^{2n-1} + y^{4n-3}$ contains C_{n-2} (cf. Exercise 7.2.12(c)), and it can be shown that this resolution is contained in $\mathbb{CP}^2\#(4n+1)\overline{\mathbb{CP}^2}$ disjoint from the fiber along which the generalized fiber sum $X(1,n)\#_f X(1,n) = E(n)$ is taken. Let $F(n)$ and $G(n)$ denote the results of rationally blowing down one or both of these configurations, respectively.

Exercise 8.5.6. Prove that $F(n)$ and $G(n)$ are simply connected 4-manifolds with $\chi_h(F(n)) = \chi_h(G(n)) = \chi_h(E(n)) = n$, $c_1^2(F(n)) = n-3$ and $c_1^2(G(n)) = 2n-6$.

We have already seen the 4-manifold $G(n)$, since by a theorem of Fintushel and Stern [**FS2**], it is diffeomorphic to the corresponding Horikawa surface defined as the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a connected curve. (In particular, in the notation of Corollary 7.3.28 and Exercise 7.4.1(a), $G(n)$ is either $H(\frac{n+1}{2})$ or $Z(\frac{n-2}{2})$, depending on the parity of n.) By determining the basic classes of $F(n)$, one can show that it is an irreducible 4-manifold. (For the definition of irreducibility see Definition 10.1.17.) Since $0 < c_1^2(F(n)) < 2\chi_h(F(n)) - 6$, the manifold $F(n)$ violates the Noether inequality, hence it is a (symplectic) 4-manifold not homotopy equivalent to a minimal complex surface.

Remark 8.5.7. Making use of the double branched cover construction of Horikawa surfaces, Fintushel and Stern showed that $G(n)$ admits a genus-$(n-1)$ Lefschetz fibration, and this fibration decomposes as a fiber sum $Q(n)\#_f Q(n)$ with $Q(n)$ a rational surface [**FS4**]. Recall that $E(n)$ admits a genus-$(n-1)$ Lefschetz fibration which decomposes as $X(1,n)\#_f X(1,n)$, moreover $X(1,n) \approx \mathbb{CP}^2\#(4n+1)\overline{\mathbb{CP}^2}$ (Exercise 7.3.8(b)). Applying the rational blow-down process, Fintushel and Stern showed that for $n \geq 4$ the manifold $F(n)$ is diffeomorphic to the fiber sum $X(1,n)\#_f Q(n)$ [**FS4**]. Since $F(n)$ is an irreducible, simply connected (symplectic) 4-manifold not admitting a complex structure, the above description provides examples of simply connected, noncomplex Lefschetz fibrations (with genus ≥ 3), and fiber sums of holomorphic Lefschetz fibrations which are noncomplex. This example completes the proof of Theorem 8.4.3.

The next theorem shows that, under suitable assumptions, logarithmic transformation can be performed by doing ordinary blow-ups followed by a rational blow-down. As we will see, this description enables us to determine the Seiberg-Witten invariants of simply connected elliptic surfaces. For the proof of Theorem 8.5.9 we need a little preparation.

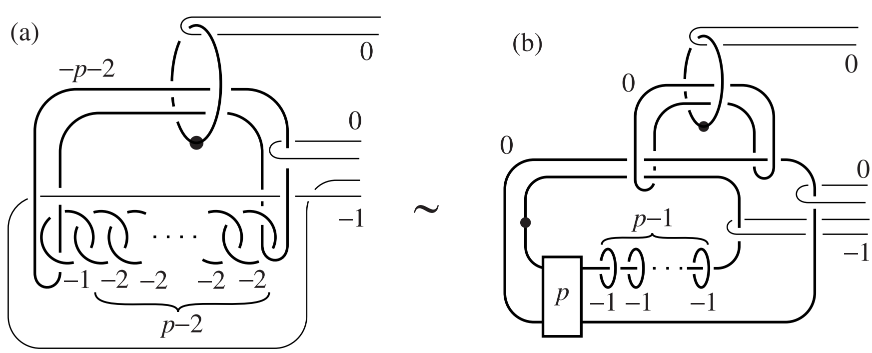


Figure 8.44. Blown-up fishtail neighborhood $Q\#(p-1)\overline{\mathbb{CP}^2}$.

Exercises 8.5.8. (a)* Prove that blowing up a fishtail neighborhood as in Example 8.5.5(a) results in the manifold given by Figure 8.44(a), where the fine curves are carried along from Figure 8.27(a). Locate the resulting copy of C_p explicitly.

(b)* Show that there is a diffeomorphism between (a) and (b) of Figure 8.44, sending the given C_p to the subhandlebody of (b) equivalent to Figure 8.40. (*Hint*: Imitate the solution of Exercise 8.5.1(a), but slide over one -1-framed meridian a second time to eliminate the $p^{\underline{th}}$ twist.)

Assume now that the 4-manifold X contains a fishtail neighborhood Q, so that Example 8.5.5(a) provides a copy of C_p embedded in $X\#(p-1)\overline{\mathbb{CP}^2}$ for all $p \geq 2$.

Theorem 8.5.9. ([**FS2**]) *The 4-manifold obtained by rationally blowing down the above $C_p \subset X\#(p-1)\overline{\mathbb{CP}^2}$ is diffeomorphic to the result of a logarithmic transformation of multiplicity p (and auxiliary data described by Exercise 8.3.16(a) and Remark 8.5.10(b)) along the torus lying in the fishtail neighborhood Q.*

Proof. By Exercises 8.5.8, the given embedding $C_p \subset Q\#(p-1)\overline{\mathbb{CP}^2}$ is exhibited by Figure 8.44(b) and its subhandlebody that is obviously equivalent to Figure 8.40. To rationally blow down, we replace C_p by B_p as before, interchanging the 0-framed and dotted circles of C_p and blowing down the -1-framed meridians. The result is Figure 8.27(b), the multiplicity-p logarithmic transform Q_p of Q (with auxiliary multiplicity 1 and direction given by the vanishing cycle, oriented as in Figure 8.27(a), cf. Exercise 8.3.16(a)). Now recall that the fine curves represent a handlebody $H \subset \partial Q$ inducing a Heegaard splitting. Thus, any remaining 1- and 2-handles of $X\#(p-1)\overline{\mathbb{CP}^2}$ can be assumed to lie in H. Since the rational blow-down and logarithmic transformation have the same effect on H, we obtain a diffeomorphism between the resulting 4-manifolds (cf. Remark 5.5.10). □

Remarks 8.5.10. (a) Recall that when Q lies in a cusp neighborhood, the auxiliary data do not affect the resulting diffeomorphism type (Theorem 8.3.5), so the above construction yields the unique multiplicity-p logarithmic transform of X along the given torus. For simple examples, consider the cusp N itself and the nucleus $N(n)$; performing the above construction on these yields the manifolds N_p and $N(n)_p$, respectively (Figures 8.28 and 8.29). Since any regular fiber in an elliptic surface lies in a cusp neighborhood (up to diffeomorphism), the theorem presents $E(n)_p$ (and more generally $E(n)_{p_1,\dots,p_k}$) as the (appropriate) rational blow-down of the blown-up elliptic surface $E(n)\#(p-1)\overline{\mathbb{CP}^2}$.

(b) Recall from Exercise 8.3.16(b) that for $Q \subset X$ not lying in a cusp neighborhood there is a possible $\mathbb{Z}_2$-ambiguity in the diffeomorphism type of X_p requiring a choice of orientation of both the meridian and vanishing cycle of Q (Figure 8.27(a)). In the above rational blow-down construction, the corresponding choice is that of which sheet of Σ at the double point should be repeatedly blown up. (Note that in Figure 8.44(a), orienting μ determines an orientation of the sphere of square $-p-2$, and with the orientations of Figure 8.27(a) we have blown up repeatedly at the sheet of Σ linking the vanishing cycle negatively.) Since no such choice is necessary when $p=2$, it follows that $Q_{-2} \approx Q_2$ rel ∂.

Next we will state some theorems for computing the Seiberg-Witten invariants of $X_{(p)}$ in terms of SW_X — for the proofs see [**FS2**]. The first proposition gives a correspondence between characteristic cohomology elements of X and $X_{(p)}$.

Proposition 8.5.11. *Assume that $C_p \subset X$, and $X_{(p)}$ is the rational blow-down of X along C_p. Then for every characteristic element $\overline{K} \in \mathcal{C}_{X_{(p)}}$ there is an element $K \in \mathcal{C}_X$ such that $\overline{K}|X^o = K|X^o$ and $K^2 - \overline{K}^2 = -(p-1)$. The class K is called a lift of $\overline{K}$.* □

By computing the dimensions of the corresponding Seiberg-Witten moduli spaces $\mathcal{M}_{\overline{K}}(X_{(p)})$ and $\mathcal{M}_K(X)$, it becomes obvious that the above relation $K^2 - \overline{K}^2 = -(p-1)$ simply means that these dimensions are equal. Assume that X and $X_{(p)}$ are both simply connected. The next theorem relates Seiberg-Witten invariants of X and $X_{(p)}$.

Theorem 8.5.12. ([**FS2**]) *Suppose that X and $X_{(p)}$ are simply connected 4-manifolds as above. Choose $\overline{K} \in \mathcal{C}_{X_{(p)}}$, and fix a lift $K \in \mathcal{C}_X$ for it. If $K^2 \geq 3\sigma(X) + 2\chi(X)$ (so $\dim \mathcal{M}_K \geq 0$), then $SW_{X_{(p)}}(\overline{K}) = SW_X(K)$. Consequently, SW_X determines the Seiberg-Witten invariants of $X_{(p)}$. In particular, if X has Seiberg-Witten simple type, then so does $X_{(p)}$.* □

Remark 8.5.13. Recall that when X and $X_{(p)}$ are simply connected, spin^c structures are determined by their determinant line bundles — a correspondence used in the proof of Theorem 8.5.12. In fact, the same argument provides an identical result if $H_1(X;\mathbb{Z})$ and $H_1(X_{(p)};\mathbb{Z})$ have no 2-torsion. In general, however, we only get the corresponding equality for the sum of Seiberg-Witten values on spin^c structures with a fixed determinant line bundle on X^o.

Since logarithmic transformation in a fishtail neighborhood can be formulated in terms of rationally blowing down, the basic classes of the manifolds $E(n)_p$ (and more generally $E(n)_{p,q}$ with $\gcd(p,q)=1$) can be computed using Theorem 8.5.12. In the following, we will carry out this computation in the more general setting described in Example 8.5.5(a) — with the assumptions that X is simply connected and has Seiberg-Witten simple type, and that all basic classes of X evaluate trivially on the homology class $[\Sigma]$. (These assumptions are obviously satisfied for the elliptic surface $E(n)$ with $[\Sigma] = f$ a fishtail fiber.) By the blow-up formula (Theorem 2.4.9) we know the basic classes of $X\#(p-1)\overline{\mathbb{CP}^2}$: these are of the form $L \pm e_1 \pm \ldots \pm e_{p-1}$ ($L \in \mathcal{B}as_X$). For $J \in \{\pm 1\}^{p-1}$ we denote the corresponding characteristic element $L + \sum J(i)e_i$ by L_J; the sum $\sum J(i)$ will be denoted by $|J|$. An easy argument shows the following.

Lemma 8.5.14. *If X^o denotes the complement of* int C_p *in $X\#(p-1)\overline{\mathbb{CP}^2}$, then the unique extension of $L_J|X^o$ to $X_{(p)}$ equals $L + |J|\frac{[\Sigma]}{p} \in H^2(X_{(p)};\mathbb{Z})$.* □

(We have identified the homology element $\frac{[\Sigma]}{p}$ with its Poincaré dual. Note that although $[\Sigma]$ might be a primitive element in $H_2(X;\mathbb{Z})$, it will be divisible by p in $X_{(p)}$, since in algebraic terms, performing the rational blow-down mods out the relations $e_i - e_{i+1} \equiv 0$ ($i = 1, \ldots, p-2$) and $[\Sigma] - 2e_1 - e_2 - \ldots e_{p-1} \equiv [\Sigma] - pe_1 \equiv 0$.) Now applying Theorem 8.5.12 yields the following.

Theorem 8.5.15. *If the simply connected 4-manifolds X and $X_{(p)}$ are as above, and $SW_{X_{(p)}}(\overline{K}) \neq 0$ (i.e. $\overline{K} \in \mathcal{B}as_{X_{(p)}}$), then $\overline{K}$ is the extension of $L_J|X^o$ for some $J \in \{\pm 1\}^{p-1}$. Consequently, the basic classes of $X_{(p)}$ are of the form $L + |J|\frac{[\Sigma]}{p}$ (where L is a basic class of X).* □

Since we know the basic classes of $E(n)$ (cf. Section 3.1), this theorem allows us to determine the basic classes of $E(n)_{p,q}$ ($n \geq 2$ and $\gcd(p,q) = 1$). For the description of basic classes of $E(n)_{p,q}$ see Theorem 3.3.6.

There is another special case of the rational blow-down process when the properties of the embedding of C_p make the computation of the basic classes of $X_{(p)}$ particularly simple.

Definition 8.5.16. Denote the spheres in C_p by s_i ($i = 1, \ldots, p-1$), with $[s_1]^2 = -p-2$. We call a configuration $C_p \subset X$ *tautly embedded* if for each basic class K we have $K(s_i) = 0$ ($i = 2, \ldots, p-1$) and $|K(s_1)| \leq p$.

Exercises 8.5.17. (a) Check that a section of $E(4)$ gives a tautly embedded copy of $C_2 \subset E(4)$. More generally, show that the two copies of $C_{n-2} \subset E(n)$ found above are tautly embedded. (*Hint*: Recall the computation of basic classes of $E(n)$ (cf. Section 3.1) and apply the adjunction equality of Theorem 1.4.17.)

(b) Show the embedding $C_p \subset X\#(p-1)\overline{\mathbb{CP}^2}$ that we used in Theorem 8.5.9 is not a taut embedding. (*Hint*: Using the blow-up formula, find basic classes of $X\#(p-1)\overline{\mathbb{CP}^2}$ which evaluate nontrivially on s_i for $i \geq 2$.)

Theorem 8.5.18. ([**FS2**]) *Suppose that X is a simply connected 4-manifold. If $C_p \subset X$ is tautly embedded, and $\overline{K}$ is a Seiberg-Witten basic class of $X_{(p)}$, then the corresponding lift $K \in \mathcal{C}_X$ satisfies $|K(s_1)| = p$.* □

The above theorem enables us to determine the basic classes of many 4-manifolds given as rational blow-downs of other (well-known) manifolds.

Exercises 8.5.19. (a) Determine the basic classes of the 4-manifold obtained by rationally blowing down one section of $E(4)$. Conclude that the resulting manifold is irreducible and not homotopy equivalent to a minimal complex surface, therefore not diffeomorphic to *any* complex surface.

(b)* More generally, determine the basic classes of the 4-manifold P_j obtained by blowing down j sections of $E(4)$ $(1 \leq j \leq 9)$. Show that P_j is irreducible.

(c) Determine the basic classes of the manifold $F(n)$. (Recall that $F(n)$ was defined as the rational blow-down of the elliptic surface $E(n)$ along one copy of C_{n-2}.)

(d)* Draw a Kirby diagram for the 4-manifold $P_1 = F(4)$ described in (b). This 4-manifold is simply connected, irreducible, symplectic, and does not admit a complex structure; cf. Section 10.2.

Chapter 9

Cobordisms, h-cobordisms and exotic $\mathbb{R}^4$'s

This chapter begins with the study of *cobordisms* of manifolds. As we will see, we have a complete classification of 3- and 4-dimensional (oriented, closed, smooth) manifolds — up to cobordism. One consequence is that homeomorphic 4-manifolds become diffeomorphic when summed with enough $S^2 \times S^2$'s, which leads us to consider nonuniqueness of connected sums in general. We will also describe some relevant results for higher dimensional manifolds and study embeddings in Euclidean spaces. In Section 9.2 we will focus on *h-cobordisms*, with special attention to the 4-dimensional case — in which dimension the celebrated h-Cobordism Theorem fails to hold for smooth manifolds. We will study this failure in detail and examine some partial results. For example, simply connected h-cobordant 4-manifolds are homeomorphic, and while they need not be diffeomorphic, they become so when a certain compact contractible submanifold is removed from each. A similar construction yields exotic $\mathbb{R}^4$'s — manifolds homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to it. We will study these constructions in Section 9.3 and prove that the manifold of Figure 6.16 is an exotic $\mathbb{R}^4$. In the final section, we give a more general survey of exotic $\mathbb{R}^4$'s, including various constructions of uncountably many smooth structures on $\mathbb{R}^4$ and other manifolds. For an additional reference on much of this material see [**K2**].

9.1. Cobordism groups

Definition 9.1.1. Two n-dimensional, closed, oriented manifolds X_-, X_+ are *(oriented) cobordant* (in notation $X_- \sim_c X_+$) if there exists an $(n+1)$-dimensional compact, oriented manifold W with boundary such that $\partial W = \overline{X}_- \amalg X_+$. The manifold W is called a *cobordism* between X_- and X_+.

It is easy to see that $\sim_c$ is an equivalence relation. The equivalence classes of n-dimensional manifolds form an abelian group Ω_n with disjoint union $\amalg$ as addition. The equivalence class of the n-dimensional empty manifold $\emptyset$ plays the role of 0, and $\overline{X}$ (the manifold X with the opposite orientation) is an inverse for X. The class represented by X will be denoted by $[X] \in \Omega_n$. The manifolds cobordant to $\emptyset$ are called *nullcobordant*. By the above description, the manifold X^n is nullcobordant iff there exists W^{n+1} such that $\partial W = X$.

Examples 9.1.2. **(a)** The product $I \times X$ gives a (trivial) cobordism of X to itself.

(b) A less trivial example can be given by surgery on a manifold X^n. Attach a $(k+1)$-handle to $I \times X^n$ along $\hat{\varphi}\colon S^k \times D^{n-k} \to \{1\} \times X \subset \partial(I \times X)$ (cf. Definition 5.2.1). As a result we get a cobordism between $X = \{0\} \times X$ and the surgered manifold $X_{\hat{\varphi}}$. (See also the text after Definition 5.2.1.)

(c) For X, Y disjoint manifolds, start with $I \times (X \amalg Y)$ and then construct $I \times X \cup_\varphi I \times Y$ as in Definition 1.3.2 with $U \subset X$, $V \subset Y$ and $\varphi\colon U \to V$ (thinking of $X \amalg Y$ as $\{1\} \times (X \amalg Y) \subset I \times (X \amalg Y)$). As one can see, $X \amalg Y$ and $(X - \operatorname{int} U) \cup_{\varphi|\partial U} (Y - \operatorname{int} V)$ are cobordant. As special cases, this observation implies that the connected sum $X \# Y$ and the generalized fiber sum $X \#_F Y$ of the 4-manifolds X and Y along the surface F (as described in Definition 7.1.11) are cobordant to $X \amalg Y$.

(d) By the fact that $S^1 = \partial D^2$ is the unique connected, closed 1-manifold (up to diffeomorphism), we have that $\Omega_1 = 0$.

Exercises 9.1.3. **(a)** Prove that $\Omega_0 \cong \mathbb{Z}$, and define the isomorphism.

(b) Prove that $\Omega_2 = 0$. (*Hint*: Use the classification of 2-dimensional surfaces described in Section 1.1; cf. also Exercise 4.2.14.)

It is somewhat harder to compute Ω_3 and Ω_4.

Theorem 9.1.4. *The cobordism group Ω_3 is trivial, while $\Omega_4 \cong \mathbb{Z}$. The class of $\mathbb{CP}^2$ generates Ω_4.* □

The statement about Ω_3 is essentially Theorem 5.3.4. For a proof that $\Omega_4 \cong \mathbb{Z}$ see, e.g., [**K2**]; in the following, we merely outline a proof. The most common way to exhibit the isomorphism $\Omega_4 \cong \mathbb{Z}$ is to associate the signature $\sigma(X^4)$ to the 4-manifold X.

Lemma 9.1.5. *If X is the boundary of a 5-dimensional compact, oriented manifold W, then $\sigma(X) = 0$.* □

Note that Lemma 9.1.5 implies that the signature of a 4-manifold X depends only on the cobordism class of X. For the disjoint union $X_- \amalg X_+$ we have $\sigma(X_- \amalg X_+) = \sigma(X_-) + \sigma(X_+)$, so the map $\sigma\colon \Omega_4 \to \mathbb{Z}$ is a group-homomorphism. It is clearly surjective: observe that $\sigma(\#n\mathbb{CP}^2) = n$ and $\sigma(\#m\overline{\mathbb{CP}^2}) = -m$ $(n, m \geq 0)$. Hence, to prove that Ω_4 is in fact isomorphic to $\mathbb{Z}$, one only needs the following converse of Lemma 9.1.5. We will not present a proof for this here, see [**K2**].

Theorem 9.1.6. *If $\sigma(X^4) = 0$, then there exists a compact, oriented 5-manifold W such that $\partial W = X$.* □

Remark 9.1.7. Note that Lemma 9.1.5 implies, in particular, that for a connected sum we have $\sigma(X\#Y) = \sigma(X) + \sigma(Y)$ (which also follows from Exercise 1.3.5(a)); moreover it implies that the signature $\sigma(X\#_F Y)$ of the generalized fiber sum of X and Y along the surface F is equal to $\sigma(X)+\sigma(Y)$ (cf. Example 9.1.2(c)). It is true in general that if we decompose a closed, oriented 4-manifold X along a 3-manifold N as $X = X_1 \cup_N X_2$, then we have $\sigma(X) = \sigma(X_1) + \sigma(X_2)$. This latter equation is usually referred to as *Novikov additivity*. (For the proof see [**K2**].)

The next proposition can easily be proved using the material discussed in Part 2 of this volume.

Proposition 9.1.8. *A closed, oriented 4-manifold X is cobordant to a handlebody $Y =$ 0-handle$\cup$ 2-handles$\cup$ 4-handle. In particular, there is a simply connected 4-manifold cobordant to X.*

Proof. Take a handle decomposition of X. Surgery on the 1-handles (changing the dotted circles in the corresponding Kirby diagram to 0-framed unknots) gives a manifold Y_1, which is (by Example 9.1.2(b)) cobordant to X. The manifold Y_1 has no 1-handles, hence it is simply connected. Turning the handle decomposition upside down and repeating the argument concludes the proof. □

The following result (due to Thom) gives information about the higher dimensional cobordism groups. Take the infinite direct sum $\Omega_* = \bigoplus_n \Omega_n$. A ring structure on Ω_* is defined in the following way: For X_i representing $[X_i] \in \Omega_{n_i}$ $(i = 1, 2)$ take the Cartesian product $X_1 \times X_2$. The element represented by this $(n_1 + n_2)$-manifold in $\Omega_{n_1+n_2}$ will not depend on our choice of the representatives of the classes $[X_i]$, hence this operation gives a multiplication $\Omega_{n_1} \times \Omega_{n_2} \to \Omega_{n_1+n_2}$. The group Ω_* equipped with this multiplication is easily seen to be a graded ring. The following theorem describes Ω_* up to torsion (see, e.g., [**MS**]).

Theorem 9.1.9. *The graded ring* $\Omega_*\otimes\mathbb{Q}$ *is isomorphic to the polynomial algebra over* $\mathbb{Q}$ *generated by independent generators* $x_4, x_8, \dots, x_{4n}, \dots$, *where* $x_{4n} = [\mathbb{CP}^{2n}] \in \Omega_{4n}$. *Consequently,* $\Omega_n \otimes \mathbb{Q}$ *is 0 if* n *is not divisible by 4 and* $\Omega_{4n} \otimes \mathbb{Q} \cong \mathbb{Q}^{p(n)}$, *where* $p(n)$ *is the number of partitions of* n. □

Next we list some generalizations of the definition of the cobordism groups Ω_n. Since a spin structure on X induces a spin structure on its boundary ∂X (cf. Section 5.6), the *spin cobordism group* Ω_n^{spin} can be defined in the obvious way: the closed spin manifolds $(X_-, s_-), (X_+, s_+)$ (with spin structures s_- and s_+) are *spin cobordant* if there is a compact spin manifold (W^{n+1}, s) such that its boundary ∂W with its induced spin structure $s|\partial W$ is $(\overline{X}_-, s_-) \amalg (X_+, s_+)$. We have $\Omega_0^{spin} \cong \mathbb{Z}$ and $\Omega_1^{spin} \cong \Omega_2^{spin} \cong \mathbb{Z}_2$ [**K2**]. By Theorem 5.7.14, a spin 3-manifold is the spin boundary of a spin 4-manifold, hence $\Omega_3^{spin} = 0$. In the 4-dimensional case, however, we have already seen spin 4-manifolds representing nonzero classes of Ω_4 — for example, the elliptic surfaces $E(2n)$. (These are not nullcobordant, since $\sigma(E(2n)) = -16n$.) This observation already shows that $\Omega_4^{spin} \neq 0$; for the proof of the following theorem, see [**K2**].

Theorem 9.1.10. *The fourth spin cobordism group* Ω_4^{spin} *is isomorphic to* $\mathbb{Z}$. *The map* $[X] \mapsto \frac{\sigma(X)}{16}$ *gives the isomorphism* $\Omega_4^{spin} \cong \mathbb{Z}$, *so the class of the* $K3$*-surface generates* Ω_4^{spin}. □

Remark 9.1.11. For comparison, we just mention two related results: The cobordism group of *topological* 4-manifolds, Ω_4^{top}, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. The isomorphism can be given by associating the pair $(\sigma(X), ks(X))$ to X, where the latter invariant $ks(X)$ is called the *Kirby-Siebenmann invariant* of X, and it is a characteristic class that vanishes exactly when the "topological tangent bundle" of X reduces to a vector bundle — in particular, it vanishes if X admits a smooth structure. (We have seen the Kirby-Siebenmann invariant before as the invariant distinguishing the two closed, simply connected, topological 4-manifolds with a given odd intersection form (Theorem 1.2.27), and as $\frac{\sigma}{8}$ modulo 2 in the even case.) The analogue of the spin cobordism group Ω_4^{spin} can also be defined for topological manifolds, and it turns out that $\Omega_4^{topspin}$ is isomorphic to $\mathbb{Z}$. The isomorphism can be given by $X \mapsto \frac{\sigma(X)}{8}$. For details and definitions of the above, see [**FQ**], [**KS**].

Before proceeding further, let us study cobordisms between 4-manifolds more thoroughly.

Theorem 9.1.12. ([**W2**]) *If two closed, smooth, simply connected 4-manifolds* X *and* Y *are homeomorphic, then* $X\#kS^2 \times S^2$ *is diffeomorphic to* $Y\#kS^2 \times S^2$ *for some* $k \geq 0$.

Proof. Homeomorphic 4-manifolds have isomorphic intersection forms, hence $\sigma(X) = \sigma(Y)$; consequently we have a cobordism W between X and Y. By surgering the 1- and 4-handles we may assume that W is built on $I \times X$ by adding 2- and 3-handles only. We first attach the 2-handles to X; as usual, the resulting relative handlebody will be denoted by W_2. Since X is simply connected (and 4-dimensional), Proposition 5.2.3 implies that $\partial_+ W_2$ is diffeomorphic to $X \# kS^2 \times S^2$. (If X (and hence Y) is spin, W can be chosen to be spin (cf. Theorem 9.1.10), hence W_2 and its other boundary $\partial_+ W_2$ are spin as well; if X is not spin, then we use the fact that $X \# S^2 \tilde{\times} S^2 \approx X \# S^2 \times S^2$ (Proposition 5.2.4).) Now turn the handlebody upside down and repeat the same argument with Y. The resulting handlebody W_2^* will have $\partial_- W_2^* \approx Y$ and $\partial_+ W_2^* = Y \# lS^2 \times S^2$. Note that the 2-handles of the handlebody W_2^* are the 3-handles of W, hence $W = W_2 \cup \overline{W}_2^*$, giving that $X \# kS^2 \times S^2 \approx \partial_+ W_2 = \partial_- \overline{W}_2^* = \partial_+ W_2^* \approx Y \# lS^2 \times S^2$. Since X and Y are homeomorphic (in particular $\operatorname{rk}(H_2(X;\mathbb{Z})) = \operatorname{rk}(H_2(Y;\mathbb{Z}))$), it follows that $k = l$, and this concludes the proof. □

Remark 9.1.13. Theorem 9.1.12 holds for arbitrary compact orientable 4-manifolds X and Y with possibly nonempty boundary. It can be extended to nonorientable manifolds by allowing connected sums with $S^2 \tilde{\times} S^2$ as well. (See [**G3**].) Note, however, that summands of the form $S^2 \tilde{\times} S^2$ are needed in the nonorientable case; for example, there is an exotic smooth structure on $\mathbb{RP}^4$ that never becomes standard under sum with $S^2 \times S^2$ [**CS**].

Theorem 9.1.12 can be applied to show how nonunique the decomposition of a 4-manifold can be.

Corollary 9.1.14. *For any two simply connected, smooth, closed 4-manifolds X and Y there are integers k_1, l_1, k_2, l_2 such that $X \# k_1 \mathbb{CP}^2 \# l_1 \overline{\mathbb{CP}^2}$ is diffeomorphic to $Y \# k_2 \mathbb{CP}^2 \# l_2 \overline{\mathbb{CP}^2}$.*

Proof. Choose $n_1, m_1, n_2, m_2 \geq 1$ such that $X \# n_1 \mathbb{CP}^2 \# m_1 \overline{\mathbb{CP}^2}$ is homeomorphic to $Y \# n_2 \mathbb{CP}^2 \# m_2 \overline{\mathbb{CP}^2}$. (Recall the classification of odd indefinite forms and apply Theorem 1.2.27.) Now apply Theorem 9.1.12 and the fact that $S^2 \times S^2 \# \overline{\mathbb{CP}^2} \approx \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$. □

There are no general theorems concerning the minimum value of k for which a given pair X, Y satisfies Theorem 9.1.12. There are many examples for which $k = 1$ suffices, and it is possible that $k = 1$ is always sufficient. Pairs requiring $k > 1$ would be difficult to detect, due to the lack of suitable invariants. We proceed to list some families for which $k = 1$ suffices, as well as related results on nonuniqueness of connected sum decompositions. Following [**G10**], we say that a simply connected 4-manifold X *dissolves* if it is diffeomorphic either to the connected sum $\# n\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ or to $\pm(\# nK3 \# mS^2 \times S^2)$ for some $n, m \geq 0$.

Theorem 9.1.15. ([**Ma1**], [**Msh**]) *If S is a complex surface which is either a complete intersection (e.g., S_d of Section 1.3) or a simply connected elliptic surface, then $S\#\mathbb{CP}^2$ dissolves.* □

Note that Exercise 8.3.4(d) implies $E(n)\#\mathbb{CP}^2$ dissolves. Recall that if X has odd intersection form, then the connected sum $X\#\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is diffeomorphic to $X\#S^2 \times S^2$ (cf. Proposition 5.2.4); hence, the above theorem gives examples of homeomorphic, nondiffeomorphic pairs $(X, \#n\mathbb{CP}^2\#m\overline{\mathbb{CP}^2})$ for which the required k of Theorem 9.1.12 is equal to 1.

Proposition 9.1.16. *Assume that Y admits a handle decomposition with no 1- or 3-handles. Then the connected sum $Y\#\overline{Y}$ dissolves.*

Proof. Take the union Y_2 of the 0- and 2-handles of Y in the handle decomposition with no 1- or 3-handles. The double of Y_2 is obviously $Y\#\overline{Y}$ (cf. Example 4.6.3), hence Corollary 5.1.6 implies the proposition. □

Since the elliptic surfaces $E(n)_p$ and the complete intersections S_d (cf. Section 1.3) admit handle decompositions without 1- or 3-handles (cf. Corollaries 6.3.19 and 8.3.17), the above proposition implies that the connected sums $E(n)_p\#\overline{E(n)_p}$ and $S_d\#\overline{S_d}$ dissolve. If a simply connected Y has a handle decomposition with no 3-handles and l 1-handles, then $Y\#\overline{Y}\#lS^2 \times S^2$ dissolves by Exercise 5.1.10(b).

Remark 9.1.17. One might hope that any closed, simply connected 4-manifold has a handle decomposition with no 1- or 3-handles, or at least without 1-handles. This problem is still open (for $E(n)_{p,q}$, $p, q \geq 2$, for example); for manifolds with boundary there are compact, contractible counterexamples (cf. the problem list [**K4**], Problem 4.18).

Additional results about dissolving connected sums can be found in [**Ma1**], [**G5**], [**G10**] (cf. also Theorem 8.4.10). For example, if S and S' are simply connected elliptic surfaces, then the manifolds $S\#S^2 \times S^2$ and $S\#\overline{S'}$ dissolve (cf. Exercise 8.3.19). The same holds if S, S' or both are complete intersections, provided that Q_S is odd. Nothing is known about dissolving manifolds of the form $S\#S'$ when the summands are compatibly oriented complex surfaces different from blow-ups of $S^2 \times S^2$ or $\mathbb{CP}^2$.

Since cobordism theory can be applied to the study of low codimensional embeddings, we close this section by stating some classical facts about embeddings of manifolds in Euclidean spaces. First of all, by a theorem of Whitney an n-dimensional manifold (as defined in Definition 1.1.1) can always be assumed to be a submanifold of the Euclidean vector space $\mathbb{R}^{2n}$. For infinitely many values of n, the exponent $2n$ cannot be improved. (For n-manifolds which cannot be embedded in $\mathbb{R}^{2n-1}$ see [**MS**].) Here we state the relevant results for manifolds of dimension ≤ 4. First, it can be easily seen that if $\dim X = n \leq 2$, then an orientable X embeds in $\mathbb{R}^{n+1}$.

Theorem 9.1.18. *Any closed, orientable 3-manifold M admits an embedding in $\mathbb{R}^5$.*

Proof. As we have seen (Remark 1.4.27(b) and Theorem 5.7.14), an orientable 3-manifold M bounds a spin 4-manifold X, which can be chosen to be a 2-handlebody (i.e., $D^4 \cup$ 2-handles). The double DX (in which M embeds) is diffeomorphic to $\# nS^2 \times S^2$ (cf. Corollary 5.1.6). Since $\# nS^2 \times S^2 = \partial(\natural nS^2 \times D^3)$ embeds in $\mathbb{R}^5$, the theorem follows. (Cf. also Exercise 5.7.15(b).) ☐

Remark 9.1.19. We have already seen that not every oriented 3-manifold embeds in $\mathbb{R}^4$. For example, the Poincaré homology sphere (or any homology sphere with nontrivial Rohlin invariant) does not embed smoothly in $\mathbb{R}^4$, although for any homology sphere M, $I \times M$ embeds topologically in $\mathbb{R}^4$ (Exercise 5.7.17(b)). For 3-manifolds with torsion in $H_1(M;\mathbb{Z})$, the linking form provides an obstruction (Exercises 4.5.12(c,d), 5.3.3(c) and 5.3.13(g)); the same reasoning applies in the topological category to obstruct homeomorphic embeddings $I \times M \hookrightarrow \mathbb{R}^4$.

Exercise 9.1.20. * Let X be any compact 4-manifold with Q_X unimodular, negative definite and not isomorphic to $n\langle -1\rangle$ (e.g., any such form is realized by a 2-handlebody). Prove that ∂X does not embed smoothly in S^4. (For related examples, see [**CG**].)

Note that the questions of embedding X^n in $\mathbb{R}^k$ or in the sphere $S^k = \mathbb{R}^k \cup \{\infty\}$ are equivalent. By generalizing the method of finding a Seifert surface for a knot $S^1 \hookrightarrow S^3$ (cf. Section 4.5) to an embedding $X^n \hookrightarrow S^{n+2}$, and observing that a codimension-1 oriented submanifold of a spin manifold inherits a spin structure, one can prove the following.

Theorem 9.1.21. *If a smooth, closed, oriented manifold X^n can be embedded in $\mathbb{R}^{n+2}$, then X is nullcobordant and admits a spin structure that is trivial in Ω_n^{spin}. Consequently, if such a 4-manifold X embeds in $\mathbb{R}^6$, then X is spin and $\sigma(X) = 0$.* ☐

Remark 9.1.22. In fact, the converse of the above theorem also holds for 4-manifolds: X^4 embeds in $\mathbb{R}^6$ iff X^4 is spin and $\sigma(X^4) = 0$. (See, for example, [**Rb1**].)

Theorem 9.1.23. *Any orientable, smooth, closed 4-manifold X admits an embedding in $\mathbb{R}^7$.* ☐

Remark 9.1.24. By a theorem of Boéchat and Haefliger, X^4 embeds in $\mathbb{R}^7$ iff there exists a characteristic element $\alpha \in H_2(X;\mathbb{Z})$ such that $Q_X(\alpha,\alpha) =$

$\sigma(X)$ ([**BH**]). Such an α can be found if Q_X is indefinite (cf. Exercise 1.2.23(a)), and by Donaldson's Theorem 1.2.30 (and Remark 2.4.30) it is easy to find for definite intersection forms of smooth 4-manifolds. In fact, the property that the shortest characteristic vector has length rk (which is equal to σ for a positive definite intersection form) characterizes $n\langle 1\rangle$ among positive definite intersection forms. (See Theorem 2.4.28.) Note that the exponent 7 in Theorem 9.1.23 cannot be improved in general, since there are 4-manifolds which are not nullcobordant (cf. Theorem 9.1.21). Theorem 9.1.23 was the last case in proving that any orientable n-dimensional manifold ($n \geq 2$) embeds in $\mathbb{R}^{2n-1}$. For nonorientable 4-manifolds, one can show that $\mathbb{RP}^4$ does not embed in $\mathbb{R}^7$ [**MS**].

9.2. h-cobordisms

One of the most important ingredients of the smooth classification program for manifolds with dimension $n \geq 5$ is the h-Cobordism Theorem of Smale. In the following we will give an outline of the proof of this theorem, and then list some relevant results in dimension 4.

Definition 9.2.1. Two simply connected, closed, oriented n-dimensional manifolds X_-, X_+ are *h-cobordant* if there is a cobordism W between them such that the inclusions $i_\pm \colon X_\pm \hookrightarrow W$ are homotopy equivalences between $X_\pm$ and W.

For the proof of the following theorem see, e.g., [**M4**], [**RS**] or [**Sm**]; here we restrict ourselves to outlining the main ideas involved in the proof.

Theorem 9.2.2. (The h-Cobordism Theorem) *If W is an h-cobordism between the simply connected n-dimensional manifolds X_- and X_+, and $n \geq 5$, then W is diffeomorphic to the product $I \times X_-$. In particular, X_- is diffeomorphic to X_+.* □

Fix the convention that $\partial_- W = X_-$ and $\partial_+ W = X_+$. Starting with a relative handle decomposition of (W, X_-), one wants to prove that all handles of this decomposition can be cancelled, so W is in fact diffeomorphic to $I \times X_-$. This program involves two major steps. As the first step, the following proposition can be proved.

Proposition 9.2.3. *Consider a relative $(n+1)$-dimensional handlebody (W^{n+1}, X^n_-) with $n \geq 4$ and $X^n_\pm$ nonempty and connected; assume furthermore that $\pi_1(W) = 1$. Then we can modify the handle decomposition of W so that it involves no 0-,1-,n- or (n+1)-handles.*

Proof. By the trick of turning the handlebody upside down, it is clear that we have to prove the statement only for the 0- and 1-handles. By Proposition

4.2.13, all 0-handles can be cancelled (against some 1-handles). For every remaining 1-handle, we will introduce a cancelling 2-3 handle pair such that the 2-handle cancels with the 1-handle. (Thus we "trade" a 1-handle for a 3-handle. Note that since $n \geq 4$, this procedure introduces no handles of index $\geq n$.) Take an arc α in $\partial_+ W_2$ parallel to the core of the 1-handle h. (As usual, W_2 is the handlebody obtained from $I \times X_-$ by adding the 1- and 2-handles of W.) By attaching an arc to α in $\{1\} \times X_-$, we get a circle K in $\partial_+ W_2$. If a 2-handle is attached to W_2 along K (with arbitrary framing), it will cancel h. On the other hand, for the unknot K_0 lying in X_-, a 2-handle attached to $I \times X_-$ along K_0 with the trivial framing can be cancelled by a 3-handle. (For cancelling pairs, see the explanation after Exercise 4.2.8.) Hence we only have to prove that K can be isotoped in $\partial_+ W_2$ to K_0. (Note that to cancel h the framing of K can be arbitrary, so we can choose it to correspond to the trivial framing on K_0 under the isotopy.) Since $\dim \partial_+ W_2 \geq 4$, homotopy implies isotopy (cf. Example 4.1.3), hence it is enough to find a homotopy in $\partial_+ W_2$ between K and K_0. The assumption that $\pi_1(W) = 1$ implies that W_2 is simply connected (since the 2-handles must kill the fundamental group generated by X_- and the 1-handles), hence there is a homotopy between K and K_0 in W_2. This homotopy can be easily pushed into the boundary $\partial_+ W_2$: In fact, for dimensional reasons, we may assume that the homotopy in W_2 (which is a map of an annulus) is disjoint from the cores of the 1- and 2-handles (extended down to $\partial_- W$), so we can push off of $[0, 1) \times X_-$ and then radially away from the cores into $\partial_+ W_2$. This concludes the proof of the proposition. □

Remark 9.2.4. Note that the proposition holds for $n = 4$ as well. (In fact, we will use Proposition 9.2.3 for analyzing h-cobordisms between simply connected 4-manifolds.) A modification of the proposition works in the case $n + 1 = 4$ as well, giving a (weaker) result about the possible handle cancellations in a 4-dimensional handlebody (cf. also Remark 9.1.17). This version will be given later; see Lemma 9.2.17.

Definition 9.2.5. Assume that Y_1^n, Y_2^m are transversally intersecting, oriented, smooth submanifolds of complementary dimensions in the oriented manifold X^{n+m}. The *geometric* intersection number of Y_1 and Y_2 is simply the cardinality of the set $Y_1 \cap Y_2$. The *algebraic* intersection number of Y_1 and Y_2 is by definition the sum of the signs of the intersection points (cf. the text before Proposition 1.2.5). We will denote the algebraic intersection number by $Y_1 \cdot Y_2$. The number $Y_1 \cdot Y_2$ is not necessarily equal to $Y_2 \cdot Y_1$ — the absolute values of the two, however, will be the same. Assume that h_1 and h_2 are two handles in a handle decomposition and index(h_1) =index$(h_2) - 1$. If the attaching sphere A of h_2 intersects the belt sphere B of h_1 transversally in a single point, then the pair h_1, h_2 can be cancelled (cf. Proposition 4.2.9).

If we have only $A \cdot B = \pm 1$, we say that h_2 *algebraically cancels* h_1. More generally, we define the algebraic intersection $h_1 \cdot h_2$ of the two handles as $B \cdot A$.

We now turn back to the h-cobordism $(W, X_\pm)$, which (by Proposition 9.2.3) has no 0-, 1-, n- or $(n+1)$-handles. Since $H_*(W, X_-; \mathbb{Z}) = 0$, the chain groups $C_i(W, X_-)$ admit bases $\{k_1^i, \dots k_{n_i}^i\}$ with the following property: The elements of the set $\{k_j^i \mid 2 \leq i \leq n-1,\ 1 \leq j \leq n_i\}$ can be paired up as $\{(k_{l_1}^i, k_{l_2}^{i-1})\}$ such that $\partial_* k_{l_1}^i = k_{l_2}^{i-1}$. According to the description at the end of Section 4.2, a handle decomposition always defines bases of the chain groups $C_i(W, X_-)$; moreover, any basis change can be achieved by appropriate handle slides. Consequently, we may assume that the handle decomposition defines bases of the groups $C_i(W, X_-)$ with the property described above. More precisely:

Proposition 9.2.6. *The relative handlebody (W, X_-) admits a handle decomposition (with no 0-, 1-, n- or (n+1)-handles) such that for each handle h_{i_1} there is a unique handle h_{i_2} (with $|\text{index}(h_{i_1}) - \text{index}(h_{i_2})| = 1$) such that $h_{i_1} \cdot h_{i_2} = 1$ and the algebraic intersection of h_{i_1} and h_{i_2} with each further handle is 0.* □

Now we would like to cancel these pairs (h_{i_1}, h_{i_2}), which would finish the proof of Theorem 9.2.2. Note, however, that to cancel a pair we need the intersection number of the corresponding attaching and belt sphere to be *geometrically* 1. In showing that A and B can be isotoped to achieve that their algebraic intersection number equals their geometric intersection number, one uses the *Whitney trick*.

Theorem 9.2.7. (Whitney trick) *Let Y_1^n, Y_2^m be transversally intersecting, connected, smooth submanifolds of complementary dimensions in the simply connected $(n+m)$-manifold X^{n+m}. Assume furthermore that $m \geq 3$ and $n \geq 2$ (and when $n = 2$, $\pi_1(X - Y_2) = 1$). If $p, q \in Y_1 \cap Y_2$ are intersection points with opposite signs, then there exists an isotopy φ_t ($t \in [0,1]$) of id_X such that $\varphi_1(Y_1) \cap Y_2 = Y_1 \cap Y_2 - \{p, q\}$.*

Proof (sketch). The proof of Theorem 9.2.7 goes roughly in the following way. Connecting p and q by arcs in Y_1 and Y_2 respectively, one defines a circle C in X. Since X is simply connected, there is a map $f\colon D^2 \to X$ with $f(\partial D^2) = C$. By the dimension assumptions, f can be assumed to be an embedding (since $\dim X \geq 5$) with $f(\text{int } D^2)$ disjoint from Y_1 and Y_2. (The circle C is usually referred to as the *Whitney circle* corresponding to p and q, while $f(D)$ is their *Whitney disk.*) To define the isotopy φ_t, one merely needs to identify a neighborhood of the Whitney disk D with the local model suggested by Figure 9.1. This is easy, provided that the normal bundles of

C in Y_1 and Y_2 match up with the corresponding bundles in the model. Essentially, one needs to identify D as the core of a 2-handle attached to a neighborhood of $Y_1 \cup Y_2$ with a suitable framing. Such an identification can be constructed using the hypotheses on dimension and the assumption that the intersection points p and q have opposite signs. □

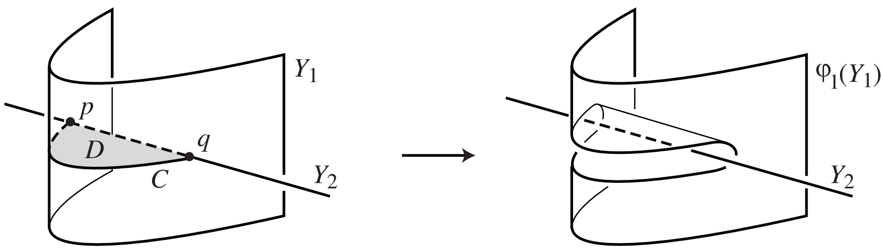


Figure 9.1. Whitney trick.

Exercise 9.2.8. Show that Theorem 9.2.7 can be applied when $X = \partial_+ W_k$, Y_1 is the belt sphere and Y_2 is the attaching sphere of a k- and $(k+1)$-handle, respectively; i.e., check that $\pi_1(X) = 1$, $\pi_1(X - Y_2) = 1$ if $k = n - 2$ and $\pi_1(X - Y_1) = 1$ for $k = 2$. (*Hint*: Show that for $k = 2$, the complement of the belt spheres in X is diffeomorphic to X_- minus the attaching circles of the 2-handles, using the fact that the decomposition of W does not contain 1-handles. Since $\pi_1(X_-) = 1$, this completes the $k = 2$ case; the $k = n - 2$ case proceeds similarly.)

Proof of Theorem 9.2.2. Now the proof of the h-Cobordism Theorem can be finished without any further difficulty. Note first that since W has no 0-, 1-, n or $(n + 1)$-handles, the paired attaching and belt spheres have dimensions ≥ 2. Isotope the spheres of the algebraically cancelling pairs of Proposition 9.2.6 using the Whitney trick and cancel each pair. At the end of this process we are left with a decomposition of W with no handles, proving that W is, in fact, diffeomorphic to $I \times X_-$. □

The h-Cobordism Theorem has resulted in many important theorems in manifold topology, among which we mention only one.

Theorem 9.2.9. (Smale, [**Sm**]) *If X^n is a closed, simply connected, smooth manifold of dimension $n \geq 5$ and $H_*(X^n; \mathbb{Z}) \cong H_*(S^n; \mathbb{Z})$, then X^n is homeomorphic to S^n.* □

Remark 9.2.10. The above result is often called the *Generalized (Topological) Poincaré Conjecture*. Note that for $n = 4$ the theorem of Freedman (Theorem 1.2.27) gives the same result; the $n = 3$ case is still open, cf. Conjecture 1.1.7. One can ask the corresponding question for diffeomorphism as well: Is a smooth manifold X^n that is homeomorphic to S^n actually diffeomorphic to it? The answer to this question is yes in a few dimensions, e.g. 3,5,6, but no in general. The question is still open in dimension 4. In high dimensions, exotic spheres (manifolds homeomorphic but not diffeomorphic to S^n) have been extensively studied (cf. [**KeM1**]). For $n \geq 6$, Theorem 9.2.9 can be proved by removing the interiors of two disks from X^n to obtain an h-cobordism between two copies of S^{n-1}. It follows immediately that X can be decomposed as a union of a 0-handle and an n-handle, cf. Example 4.1.4(c). For $n = 5$, the same decomposition occurs since X^5 is h-cobordant (hence diffeomorphic) to S^5. For $n \leq 4$, however, such a simple handlebody is necessarily diffeomorphic to S^n, so an exotic 4-sphere would be qualitatively different: a 4-handle attached to an exotic 4-ball (which would be a smooth 4-manifold homeomorphic but not diffeomorphic to D^4).

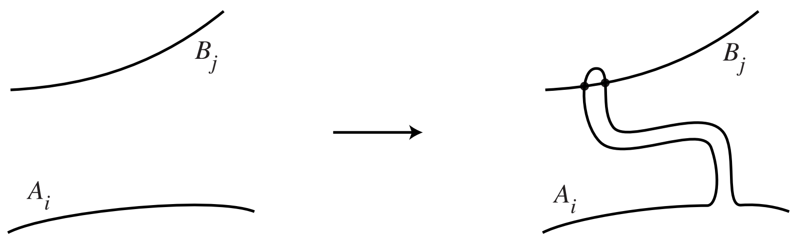


Figure 9.2. Finger move.

The proof of the Whitney trick cannot be carried out for 4-dimensional manifolds, so we cannot simplify the corresponding h-cobordisms in this way. As we will see, not only does the proof of Theorem 9.2.7 fail, but the statement itself is false in dimension 4. In the following, we will outline what one can do in the 4-dimensional case; cf. also [**K2**]. Recall that if $(W, X_\pm)$ is a given h-cobordism between the 4-manifolds $X_\pm$, then (by Proposition 9.2.3) we can assume that the handle decomposition of W has only 2- and 3-handles; moreover (by Proposition 9.2.6) the attaching 2-spheres of the 3-handles (denoted by A_i) and the belt 2-spheres of the 2-handles (denoted by B_j) intersect algebraically according to the Kronecker symbol δ_{ij}. The manifolds (X, Y_1, Y_2) for which we would like to apply the Whitney trick are $\partial_+ W_2$ (the 4-dimensional boundary of the 5-dimensional relative 2-handlebody in W) and the spheres A_i and B_j described above. The main problem arises from the fact that the Whitney disk D cannot be embedded, only immersed in dimension 4. In fact, it requires some additional work just to prove that int D can be chosen to be disjoint from the 2-spheres A_i and B_j — it involves applying *finger moves* to the spheres (see [**C**] and Figure 9.2) to arrange for the complement of the union of all attaching and belt spheres to be simply connected. By applying finger moves (which can be regarded as "inverse Whitney tricks") we increase the geometric intersection of the spheres A_i and B_j while leaving $A_i \cdot B_j$ unchanged. This seems to run contrary to our original goal (of making algebraic and geometric intersections equal), but in this way we can at least achieve that $\partial_+ W_2 - (\cup A_i \cup B_j)$ is simply connected, hence the above program for proving the h-Cobordism Theorem can be begun. That is, the usual Whitney circles and (immersed) Whitney disks can be specified for the pairs of intersection points p, q in $A_i \cap B_j$ with opposite signs. If the Whitney disk D appearing in the proof of Theorem 9.2.7 can be embedded (with suitable normal framing), we may think of it as the core of a (suitably framed) 2-handle, and then by the method of proof of Theorem 9.2.7 the points p and q can be removed from $A_i \cap B_j$. In dimension four, however, D is only immersed, hence it only defines a *kinky* handle (cf. Example 6.1.3).

Given an h-cobordism between simply connected 4-manifolds, we can do several things. First, we can attempt to turn the Whitney disks into suitably embedded disks. By a delicate argument [**C**], one can at least arrange for them to be disjointly immersed, such that the corresponding kinky handles are attached to a neighborhood of the spheres with the correct framings. One can now attempt to transform the kinky handles into 2-handles by ambiently attaching 2-handles to the 0-framed meridians of their dotted circles as in Example 6.1.3. Again this fails, but a similar delicate argument allows one to attach disjoint kinky handles to the dotted circles to obtain 2-stage *Casson towers*. Continuing in this manner, one can construct n-stage Casson towers for each n, and taking the union (after removing part of the boundary) will give a *Casson handle* attached to each Whitney circle C (cf. also Example 6.1.3). Applying Freedman's fundamental result that a Casson handle is homeomorphic to $D^2 \times \text{int}\, D^2$, we obtain homeomorphically embedded Whitney disks and conclude:

Theorem 9.2.11. *An h-cobordism between simply connected 4-manifolds is* topologically *trivial.* □

As we will see, this implies that smooth 4-manifolds with isomorphic intersection forms are homeomorphic (cf. Theorem 9.2.13), which is a major step in Freedman's Classification Theorem 1.2.27. On the other hand, we know that many h-cobordisms are smoothly nontrivial. We will exploit this fact and a closer analysis of the above argument to prove the existence of exotic $\mathbb{R}^4$'s, and exhibit an explicit family of nontrivial h-cobordisms which show that the manifold R (given in Figure 6.16) is an exotic $\mathbb{R}^4$ (see Section 9.3). Proceeding in a different direction, if we stop after specifying the first (immersed) Whitney disks for the Whitney circles (hence we have only 1-stage Casson towers), then an appropriate version of Proposition 9.2.3 will show that any h-cobordism is trivial away from a compact, contractible submanifold called *Akbulut's cork* (Theorem 9.2.18). Our discussion of these topics, commencing with Lemma 9.2.17, will begin with the proof of this latter statement, and we will return to the existence of exotic $\mathbb{R}^4$'s in the next section.

First we would like to find conditions guaranteeing that smooth 4-manifolds are h-cobordant. An h-cobordism W between the simply connected 4-manifolds X_- and X_+ naturally induces an isomorphism φ_W between the intersection forms $(H_2(X_+;\mathbb{Z}), Q_{X_+})$ and $(H_2(X_-;\mathbb{Z}), Q_{X_-})$, by the formula $\varphi_W = (i_-)_*^{-1} \circ (i_+)_*\colon H_2(X_+;\mathbb{Z}) \to H_2(X_-;\mathbb{Z})$ (where $i_\pm\colon X_\pm \hookrightarrow W$). In fact, the converse of this statement also holds:

Theorem 9.2.12. *If φ is an isomorphism between the intersection forms Q_{X_-} and Q_{X_+}, then there is an h-cobordism W between X_- and X_+ such that $\varphi_W = \varphi$.* □

We will indicate the proof of a slightly weaker result, namely

Theorem 9.2.13. (Wall, [**W3**]) *If two simply connected, smooth 4-manifolds have isomorphic intersection forms, then these manifolds are h-cobordant.*

Coupling this with our previous statement that such an h-cobordism is *homeomorphic* to the trivial h-cobordism, we recover Freedman's result that smooth, closed, simply connected 4-manifolds with isomorphic intersection forms are homeomorphic. Inverting this reasoning for our present purposes, we obtain that simply connected, smooth 4-manifolds are h-cobordant iff they are homeomorphic. In particular, the existence of homeomorphic but nondiffeomorphic 4-manifolds proves the failure of the smooth h-Cobordism Theorem in dimension 4.

Proof of Theorem 9.2.13 (sketch). Consider the relative handlebodies W_2 and W_2^* found in the proof of Theorem 9.1.12. We only need to find a diffeomorphism f to glue $\partial_+ W_2$ to $\partial_+ W_2^*$ in such a way that the resulting handlebody $W' = W_2 \cup_f \overline{W}_2^*$ is homotopy equivalent to X_- (or X_+), and so gives the desired h-cobordism. To achieve this, however, we only have to match up the 2- and 3-handles to cancel each other *algebraically*. Thus, it suffices to find a diffeomorphism $f\colon \partial_+ W_2 \to \partial_+ W_2^*$ such that (after we identify $\partial_+ W_2$ and $\partial_+ W_2^*$ with $X_- \# k S^2 \times S^2$) the induced map f_* is a suitably prescribed automorphism of $(H_2(X_- \# k S^2 \times S^2;\mathbb{Z}), Q_{X_- \# k S^2 \times S^2})$. The existence of this diffeomorphism is guaranteed by the following theorem.

Theorem 9.2.14. (Wall [**W2**]; see also [**K2**]) *Assume that the smooth, simply connected 4-manifold Y has an indefinite intersection form. Then any automorphism of $(H_2(Y \# S^2 \times S^2), Q_{Y \# S^2 \times S^2})$ is induced by a diffeomorphism of $Y \# S^2 \times S^2$.* □

Adding a further cancelling 2-3 handle pair to W if necessary (to ensure that $X_-\#(k-1)S^2 \times S^2$ is indefinite), we can apply Theorem 9.2.14 in our case, and this concludes the proof of Theorem 9.2.13. □

Example 9.2.15. Using Theorem 9.2.12, a nontrivial h-cobordism W can be constructed as follows. Take the $K3$-surface $E(2)$ and define $\varphi\colon H_2(E(2);\mathbb{Z}) \to H_2(E(2);\mathbb{Z})$ as multiplication by -1. This is obviously an automorphism of $Q_{E(2)}$ — hence it gives rise to an h-cobordism W of $E(2)$ with itself. The triviality of W would imply that $\varphi = \varphi_W$ is induced by a self-diffeomorphism $f\colon E(2) \to E(2)$. We use Theorem 2.4.3 and Remark 2.4.4(c) to show that such an f cannot exist. Since f^* is the map -1 on $H^2(E(2);\mathbb{Z})$, it reverses the orientation of the rank-3 subspace $H^+(E(2);\mathbb{Z})$. Hence (by Theorem 2.4.3 and Remark 2.4.4(c)), $SW_{E(2)}(K) = -SW_{E(2)}(f^*K)$ for a basic class K. The facts that $f^*(0) = 0$ and $SW_{E(2)}(0) \neq 0$ contradict the existence of f. (Note that this argument can be extended without change to the 4-manifolds $E(2n)_{p,q}$ with p, q odd.)

Remark 9.2.16. In fact, there is only one nonproduct h-cobordism (up to diffeomorphism) from $E(2)$ to itself (see [**Lw1**]). More generally, if X is a closed, smooth, simply connected 4-manifold and Q_X is not definite, then there is a unique nonproduct h-cobordism W_{np} from X to itself, and W_{np} can be built on $I \times X$ by attaching a single 2- and 3-handle pair. Lawson [**Lw1**] also found similar results for Dolgachev surfaces $E(1)_{p,q}$ (with $p, q \geq 2$, $\gcd(p,q) = 1$): There is a unique h-cobordism $W_{p,q}$ between $E(1)$ and $E(1)_{p,q}$ (which is trivial iff $p = 1$ or $q = 1$). If it is nontrivial, $W_{p,q}$ can be built with one 2- and one 3-handle on $I \times E(1)$. On the other hand, there are infinitely many nondiffeomorphic h-cobordisms between any fixed pair of nondiffeomorphic Dolgachev surfaces. Using similar techniques, one can prove the "converse" of Theorem 9.1.12: If $X_-\#kS^2 \times S^2$ is diffeomorphic to $X_+\#kS^2 \times S^2$, then there exists an h-cobordism W between X_- and X_+ which is built on $I \times X_-$ with k 2-handles and k 3-handles.

Next we will give the counterpart of Proposition 9.2.3 in the case $n = 3$. (Throughout the following arguments the distinction between algebraic and geometric intersections will be crucial. For a better understanding of the difference, the link shown in Figure 4.29 might be helpful; it shows two unknots linked algebraically once but geometrically three times.)

Lemma 9.2.17. *Suppose that $(Z^4, \partial_- Z^4)$ is a 4-dimensional relative handlebody with $\partial_- Z^4$ connected and $\pi_1(Z) = 1$. For every 1-handle h of this decomposition, one can introduce a cancelling 2-3 handle pair such that the 2-handle cancels h algebraically (but not necessarily geometrically). In fact, one can arrange the attaching circles of the new 2-handles to represent the* *canonical basis for the free factor of $\pi_1(Z_1)$ determined by the 1-handles (suitably attached to the base point).*

Proof. We adopt the notation of Proposition 9.2.3 with the change that the cobordism is now denoted by Z. Fix an unknot $K_0 \subset \partial_+ Z_2$ (so that K_0 bounds an embedded disk D^2 in the 3-manifold $\partial_+ Z_2$); since Z^4 is simply connected, we can find a homotopy F between K and K_0 in Z (as in the proof of Proposition 9.2.3, cf. also the solution of Exercise 5.1.10(b)). In the 4-dimensional case we consider now, however, the homotopy F cannot be pushed into the boundary (as we did in the higher dimensional case). The annulus F can be assumed to miss the cores of the 1-handles, but it might intersect cores of 2-handles. For such an intersection point P, form the band-sum of the boundary circle of a small disk around P in F with the circle K using a band contained in F; the resulting circle will be denoted by K', cf. Figure 9.3.

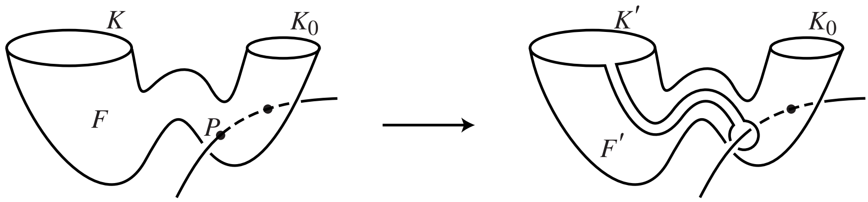


Figure 9.3. Removing the intersection of an annulus and the core of a 2-handle.

By deleting the band and the normal disk of P from F we get a homotopy F' between K' and K_0 which now intersects the cores of the 2-handles in one less point than F did. Repeating this process, we end up with a circle K_1' in Z_2 which is homotopic to K_0 by a homotopy missing the cores of the 1- and 2-handles. Pushing this homotopy into $\partial_+ Z_2$, we get a circle K_1 (the image of K_1') together with a homotopy between K_1 and K_0 in $\partial_+ Z_2$. It is clear that K_1 goes over the 1-handle h algebraically once: If D denotes the cocore of the 1-handle h, then obviously $K_1 \cdot D = K \cdot D = \pm 1$ (so a 2-handle attached along K_1 cancels h algebraically); the geometric intersection of K_1 and D, however, might differ from 1 (since the bands might run over 1-handles). The dimension of $\partial_+ Z_2$ is 3, so homotopy does not imply isotopy. In fact, the homotopy fails to be an isotopy at finitely many times when the knot crosses through itself (cf. the solution of Exercise 5.1.7(b)). Let K_ε be the knot obtained from K_0 by the first such crossing change. Clearly, we can recover K_0 from K_ε by band-summing the latter with a meridian of K_ε along a suitable band. Now continue the homotopy of K_ε, dragging along the band and meridian (isotoping them so that they never intersect the knot). Continuing this procedure for the other crossing changes, we obtain the knot K_1 connected to meridians of itself by a finite collection of bands (possibly running over 1-handles) in $\partial_+ Z_1 \cap \partial_+ Z_2$. Let $\widetilde{K}$ be the result of performing all of the given band-sums on K_1. Then $\widetilde{K}$ is homotopic to K_1 in $\partial_+ Z_1$, so it goes over h algebraically once, and in fact represents the basis element of $\pi_1(Z_1)$ given by h. In particular, a 2-handle h_2 attached to Z_2 along $\widetilde{K}$ will algebraically cancel h. However, $\widetilde{K}$ is also isotopic to K_0 in $\partial_+ Z_2$, so h_2 can be cancelled by a 3-handle. The same argument produces a cancelling 2-3 handle pair for any circle representing an element of the fundamental group $\pi_1(Z_1)$ given by the core of a 1-handle; that element of the fundamental group will evidently be killed by the 2-handle. □

We close this section by discussing one further result about h-cobordisms of 4-manifolds. We have seen that if W^5 is an h-cobordism between the simply connected 4-manifolds X_- and X_+, then W^5 is not necessarily the trivial cobordism $I \times X_-$. It turns out, however, that W is "not far" from being trivial. The main statement of the following theorem is due to Curtis and Hsiang; variations and addenda of it were proved by Freedman and Stong, Kirby and Bižaca, and by Matveyev. (See also [**K3**].) We extend the notion of cobordism in the obvious way to manifolds with boundary.

Theorem 9.2.18. ([**CFHS**]) *Suppose that W^5 is an h-cobordism between X_- and X_+. Then there is a subcobordism $V \subset W$ between the compact 4-manifolds $Y_\pm \subset X_\pm$ such that $W - \operatorname{int} V$ is the product cobordism (i.e. $(W - \operatorname{int} V, X_+ - \operatorname{int} Y_+, X_- - \operatorname{int} Y_-)$ is diffeomorphic as a triple to the product $I \times (X_- - \operatorname{int} Y_-)$) and V, $Y_\pm$ are contractible.*

Proof. According to Proposition 9.2.3, the h-cobordism W has a handle decomposition with only 2- and 3-handles. As usual, the union of $I \times X_-$ with the 2-handles is denoted by W_2, the belt spheres of the 2-handles are denoted by $B_j \subset \partial_+ W_2$, and $A_i \subset \partial_+ W_2$ are the attaching spheres of the 3-handles. We can assume that the algebraic intersection number of A_i with B_j is δ_{ij} by Proposition 9.2.6. Since this intersection number is not necessarily the geometric intersection of A_i and B_j, there are extra pairs of intersections $\{p_1, q_1, \dots, p_k, q_k\} \subset A_i \cap B_j$, with a corresponding (immersed) Whitney disk D_l for each pair (p_l, q_l). The interiors of these disks can be assumed to be disjoint from the spheres A_i, B_j (cf. the text before Theorem 9.2.11), but are generally only disjointly immersed and not embedded. Note that surgery of $\partial_+ W_2$ on the spheres A_i turns it into X_+ while surgery on the spheres B_j turns $\partial_+ W_2$ into X_-. Take a (closed) regular neighborhood $C \subset \partial_+ W_2$ of the union of the spheres A_i, B_j (for all i, j) and the Whitney disks. The neighborhood of the union of the spheres A_i, B_j is obviously a plumbing manifold, hence we get C by adding kinky handles to a plumbing manifold. Using this observation we can easily find a Kirby diagram representing C. This description shows, in particular, that $\pi_1(C)$ is a free group, whose generators correspond to circles given by arcs connecting $p_n, q_m \in A_i \cap B_j$ ($n \neq m$, hence without a Whitney disk corresponding to the pair of intersections) and to the self-intersections of the Whitney disks. Take a relative (4-dimensional) handle decomposition of $(\partial_+ W_2 - \text{int } C, \partial C)$, and consider $C_1 = C \cup$ 1-handles of that relative handlebody. By adding 1-handles to C we simply add more generators and no relators to the fundamental group $\pi_1(C)$. Consequently, $\pi_1(C_1)$ is a free group of some rank n; note that $H_2(C_1; \mathbb{Z}) \cong H_2(C; \mathbb{Z})$ is generated by the attaching and belt spheres $\{A_i\}$ and $\{B_j\}$. Let $\ell_1, \ldots, \ell_n$ be circles in ∂C_1 representing a basis of $\pi_1(C_1)$. By adding all the remaining 2-handles of $\partial_+ W_2$ to C_1, we get a simply connected 4-manifold C'. Now by applying the method of the proof of Lemma 9.2.17 to each ℓ_i, we can introduce cancelling 2-3 handle pairs (h_i, h_i') ($i = 1, \ldots, n$) in C' in such a way that the 2-handles h_i kill the circles ℓ_i in $\pi_1(C')$. Define $Y \subset \partial_+ W_2$ as the union of C_1 with the n 2-handles h_i found above ($i = 1, \ldots, n$). By construction Y is simply connected, and the second homology of Y is generated by the spheres A_i and B_j. Define $Y_+ \subset X_+$ as the surgery of Y on the attaching spheres A_i; similarly we get $Y_- \subset X_-$ by performing surgery on the belt spheres B_j. The subcobordism V from Y_- to Y_+ is given by reversing the surgery on the belt spheres B_j and then performing the surgery on A_i — recall that surgery corresponds to cobordism (Example 9.1.2(b)). Since it contains no handles at all, the cobordism $W - \text{int } V$ is trivial.

The only thing remaining to prove is that Y_- and Y_+ are contractible. (This implies that V is contractible: The condition $\pi_1(V) = 1$ follows from the fact that we get V by adding 2- and 3-handles to Y_-, and since these handles cancel each other algebraically, we also obtain $H_*(V; \mathbb{Z}) = 0$.) Since the second homology of Y is generated by the spheres A_i and B_j, doing surgery on one set of these spheres (which turns Y into Y_- or Y_+) gives an acyclic 4-manifold. Hence we only need to prove that Y_- and Y_+ are simply connected. For proving that $\pi_1(Y_+) = 1$, it is enough to show that for each A_i there is an immersed sphere S_i in Y which intersects A_i geometrically once and is disjoint from all other spheres A_j — the corresponding result for B_j will show that $\pi_1(Y_-) = 1$. (This immersed sphere will show that the meridian of A_i can be contracted after the surgery is performed.) The immersed sphere S_i will be constructed using B_i as follows. Consider a pair $p_l, q_l \in B_i \cap A_j$ of extra intersection points with a fixed immersed Whitney disk in Y. Dragging B_i along this disk, we end up with an immersed sphere not intersecting A_j in the chosen points p_l, q_l. Repeating this process for all pairs of extra intersections, we end up with the desired immersed sphere S_i, which proves that $\pi_1(Y_\pm) = 1$, concluding the proof of the theorem. □

Remark 9.2.19. By a slight modification of the above arguments one can achieve that V and $I \times Y_\pm$ are in fact diffeomorphic to D^5, and that Y_- is diffeomorphic to Y_+ by a diffeomorphism inducing an involution on the homology sphere $\partial Y_- = \partial Y_+$. One can also arrange the choice of $Y_\pm$ in $X_\pm$ in such a way that the complements $X_\pm - \operatorname{int} Y_\pm$ are simply connected. (For these addenda see [**CFHS**], [**K3**] and [**Mtv**].) Consequently, if X_- is homeomorphic to X_+ (and $X_\pm$ are smooth, closed and simply connected), then by cutting a contractible piece Y_- out of X_- and regluing it by an involution of its boundary, we get a smooth 4-manifold diffeomorphic to X_+. This contractible piece $Y_- \subset X_-$ also depends on the manifold X_+; the compact submanifold Y_- is called an *Akbulut cork* corresponding to the pair $X_\pm$. Explicit examples of this construction will be given in the next section.

9.3. Akbulut corks and exotic $\mathbb{R}^4$'s

At the end of the previous section, we saw that given a homeomorphic pair $X_\pm$ of smooth, closed, simply connected 4-manifolds, we could transform X_- to X_+ by removing an Akbulut cork, a certain contractible, smooth, compact 4-manifold $Y_- \subset X_-$, and regluing it by an involution of ∂Y_-. We will now see that the argument can be modified to yield a contractible *open* subset $R_- \subset X_-$ with similar properties, such that R_- will be an exotic $\mathbb{R}^4$ if X_+ is not diffeomorphic to X_-. This latter argument predates the previous one — in fact, much of it is due to Casson [**C**] in the 1970's, with the rest filled in by Freedman in the 1980's. We begin by sketching the construction in general (cf. also [**K2**]), and then restrict to an explicit family of manifold pairs where the Akbulut cork is visible by Kirby calculus. The explicit construction, which is a simplification of [**BG**], will also show that the manifold R drawn in Figure 6.16 is an exotic $\mathbb{R}^4$.

Theorem 9.3.1. *Let W be a smooth h-cobordism between closed, simply connected 4-manifolds X_- and X_+. Then there is an open subset $U \subset W$ homeomorphic to $I \times \mathbb{R}^4$ with a compact subset $K \subset U$ such that the pair $(W - K, U - K)$ is diffeomorphic to a product $I \times (X_- - K, U \cap X_- - K)$. The subsets $R_\pm = U \cap X_\pm$ (homeomorphic to $\mathbb{R}^4$) are diffeomorphic to open subsets of $\mathbb{R}^4$. If X_- and X_+ are not diffeomorphic, then there is no smooth 4-ball in $R_\pm$ containing the compact set $K \cap R_\pm$, so both $R_\pm$ are exotic $\mathbb{R}^4$'s.*

Remark 9.3.2. Actually, the last statement holds whenever the h-cobordism W is nontrivial. We can also arrange for the manifolds $R_\pm$ to be diffeomorphic to each other in such a way that the product structure on $U - K$ determines an involution of R_- minus a compact subset (cf. Remark 9.2.19). We can easily change K to make it homeomorphic to $I \times D^4$ (as a triple) with $X_\pm - K$ homeomorphic to $X_\pm - \{\text{pt.}\}$. (Take a sufficiently large topological S^3 in R_- and extend it to $I \times S^3 \subset W$ using the product structure outside of the original K, then fill in to get the new K.) For more on these and other exotic 4-manifolds, see the next section (and also [**K2**]).

Proof of Theorem 9.3.1. As before, we find a handle decomposition of (W, X_-) with only 2- and 3-handles, whose belt and attaching spheres intersect as $A_i \cdot B_j = \delta_{ij}$ in $\partial_+ W_2$ (Proposition 9.2.6). Recall (preceding Theorem 9.2.11) that after modifying the spheres B_j if necessary, we can find Casson handles in $\partial_+ W_2$ determining topological Whitney disks, then perform the Whitney trick to trivialize W up to homeomorphism. Let $N \subset \partial_+ W_2$ denote the union of these Casson handles with an open regular neighborhood (which we assume is connected) of the attaching and belt spheres. Now $\pi_1(N)$ is free, with generators given by the extra pairs of intersections of attaching and belt spheres. Using the Whitney trick, one can easily verify that N is homeomorphic to $\#kS^2 \times S^2 - \{\text{pt.}\}$ with an additional 1-handle attached for each generator of π_1 (and with boundary removed). To eliminate these 1-handles, one returns to the construction of the Casson handles and verifies that the algorithm also allows us to simultaneously construct additional Casson handles determining *accessory disks*. These are characterized by the condition that the new Casson handles should topologically cancel the 1-handles. Thus, if N' denotes the union of N with the new Casson handles, then N' is homeomorphic to $\#kS^2 \times S^2 - \{\text{pt.}\}$. We see immediately that surgery on either set of spheres $\{A_i\}$ or $\{B_j\}$ turns N' into a manifold R_+ or R_- homeomorphic to $\mathbb{R}^4$, and that the corresponding open subset $U \subset W$ is a topologically trivial h-cobordism from R_- to R_+ (cf. Theorem 9.2.18). If we take $K \subset U$ to be the union of all cores and cocores of handles of W, together with points of W below each attaching sphere and above each belt sphere, then $W - K$ is essentially the result of removing all handles and their attaching regions (and points below) from W, so it is a trivial (noncompact) cobordism.

Exercise 9.3.3. Describe $K \cap X_\pm$. This will be a 2-complex in $X_\pm$ determined by the attaching spheres of the (5-dimensional) handles of W, so it can be thought of as a relative Kirby diagram of $(W, X_\pm)$ one dimension higher than usual.

To understand $R_\pm$ as smooth manifolds, first observe that U is diffeomorphic to an open subset of $I \times S^4$, such that the given smooth product structure on $U - K$ extends over $I \times S^4 - K$. To construct such a structure, simply mimic the construction of N' in W by adding cancelling 2-handle/3-handle pairs to $I \times S^4$ and making finger moves in the resulting belt spheres to match the original geometric intersection pattern of the spheres A_i and B_j in $\partial_+ W_2$. By trivially adding double points to the resulting embedded Whitney and accessory disks in $I \times S^4$, one begins to build Casson

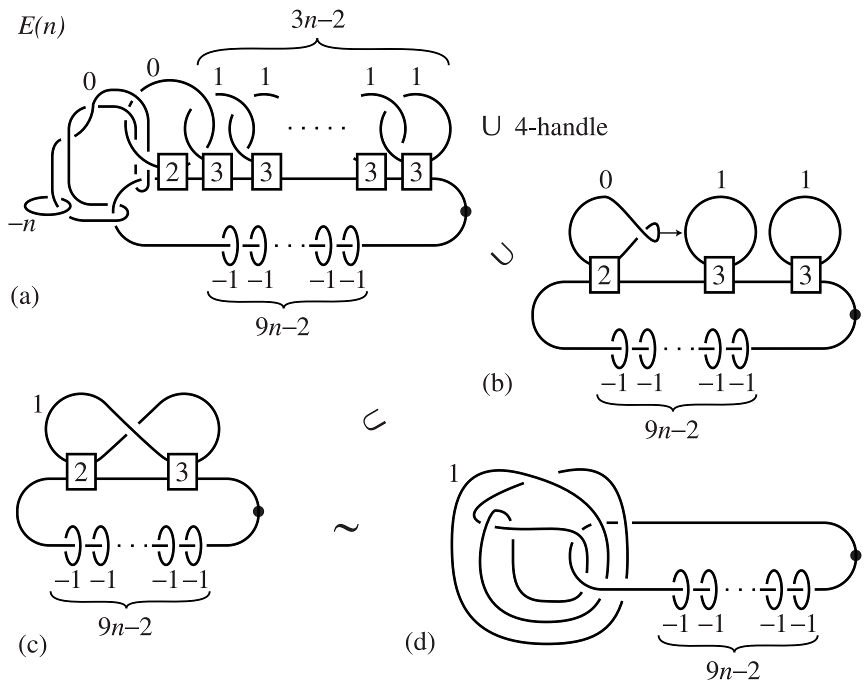


Figure 9.4. A submanifold of $E(n)$.

handles in the middle level to match those of N'. Continuing in this manner produces the required open subset of $I \times S^4$. We immediately obtain $R_\pm$ exhibited as open subsets of $\mathbb{R}^4$. Now suppose that there is a smooth, closed 4-ball $D_- \subset R_-$ with $K \cap R_- \subset \text{int } D_-$. (Such a ball obviously exists if R_- is diffeomorphic to $\mathbb{R}^4$.) The product structure on $U - K$ sends ∂D_- to a smooth 3-sphere in R_+ bounding a compact manifold D_+ with $K \cap R_+ \subset D_+ \subset R_+$, with $X_- - \text{int } D_-$ diffeomorphic to $X_+ - \text{int } D_+$ and similarly $S^4 - \text{int } D_- \approx S^4 - \text{int } D_+$. Since any two smooth, orientation-preserving embeddings $D^4 \to X^4$ are isotopic (as one can show by approximating them by their derivatives at 0), the closure of the complement of any ball in S^4 is diffeomorphic to D^4. Thus, $D^4 \approx S^4 - \text{int } D_- \approx S^4 - \text{int } D_+$ and $D_+ \approx D^4$. Hence, X_+ and X_- are both obtained by adding 4-handles $D_\pm$ to the manifold $X_- - \text{int } D_- \approx X_+ - \text{int } D_+$, so X_+ and X_- are diffeomorphic. □

To construct explicit examples of Akbulut corks and exotic $\mathbb{R}^4$'s determined by h-cobordisms, begin with an elliptic surface $E(n)$, $n \geq 2$. In Theorem 8.3.2 we constructed the Kirby diagram of $E(n)$ shown in (a) of Figure 9.4 (which is the same as Figure 8.16). Now we cut down to a subhandlebody (b), make the indicated handle slide and remove two more handles

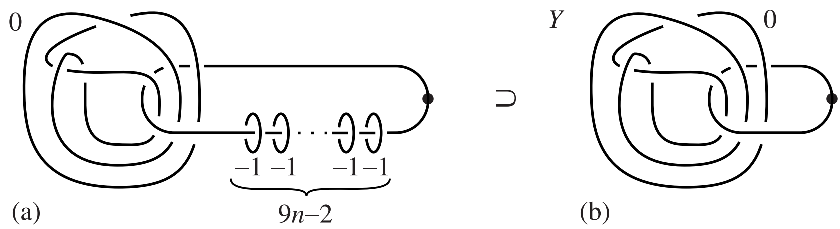


Figure 9.5. Akbulut cork Y in $E(n)\#\overline{\mathbb{CP}^2}$.

to obtain (c), which is isotopic to (d). We reduce the framing 1 to 0 by blowing up a -1-framed meridian and then removing the new 2-handle to get (a) of Figure 9.5. The subhandlebody Y shown in (b) is our example of an Akbulut cork in $E(n)\#\overline{\mathbb{CP}^2}$. To see that Y has the required properties, first note that Y is contractible. (For example, up to homotopy we may undo the clasp and then cancel the handles. Such a contractible manifold made from an algebraically cancelling 1-2 pair is often called a *Mazur manifold.*) Next, observe that ∂Y is given by 0-surgery on the given link, and this link is *symmetric* — i.e., there is an isotopy interchanging the two link components. Let $\varphi\colon \partial Y \to \partial Y$ denote the resulting diffeomorphism (which we will see can be taken to be an involution). We wish to show that the manifold X_n obtained from $E(n)\#\overline{\mathbb{CP}^2}$ by cutting out Y and regluing it using φ is not diffeomorphic to the complex surface $E(n)\#\overline{\mathbb{CP}^2}$ (although the two are necessarily homeomorphic by Freedman's Theorem 1.2.27). Our construction of Y essentially exhibited $E(n)\#\overline{\mathbb{CP}^2}$ as $Y \cup$ handles, and regluing by φ can be realized by interchanging the roles of the two circles in Figure 9.5(b) (cf. Example 5.5.8). Thus, the dotted circle becomes a 0-framed 2-handle. Tracing back to Figure 9.4(b), we see that the rightmost $+1$-framed unknot is no longer linked with the dotted circle, so we can blow it down. Thus, X_n splits off a $\mathbb{CP}^2$-summand, so by Theorems 2.4.6(1) and 2.4.7(1) it cannot be a complex surface. We conclude that the diffeomorphism type of $E(n)\#\overline{\mathbb{CP}^2}$ is changed by cutting out the contractible submanifold Y and regluing it by the involution φ on ∂Y, so Y is indeed an Akbulut cork. (Furthermore, $E(n)\#\overline{\mathbb{CP}^2} - \operatorname{int} Y$ is simply connected, since we could turn its handle decomposition upside down to obtain a 2-handlebody.) Note that while the proof of Theorem 9.2.18 allows no control of the complexity of the resulting Akbulut corks, we have obtained an infinite family of pairs $E(n)\#\overline{\mathbb{CP}^2}, X_n$ ($n \geq 2$) for which the Akbulut cork is a fixed Mazur manifold, and is the simplest nontrivial contractible manifold. (This construction is a simplification of [**BG**], and Akbulut's original example [**A3**] is essentially the special case $n = 2$.)

Exercise 9.3.4. Prove that $X_n \approx \#(2n-1)\mathbb{CP}^2\#10n\overline{\mathbb{CP}^2}$. (*Hint*: Construct a Kirby diagram for X_n. Its dotted circle has a -1-framed meridian coming from the blow-up of $E(n)$. Simplify the diagram as in Figure 8.22. This liberates a $\mathbb{CP}^2$-summand, so the dotted circle can be eliminated by blowing up its -1-framed meridian as in Exercise 8.3.4(d). Now a handle slide yields Figure 8.22.)

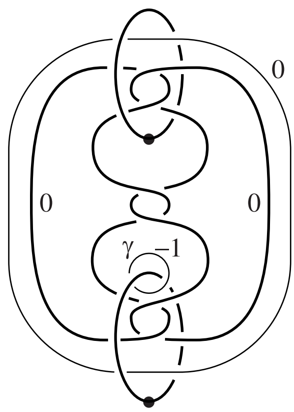


Figure 9.6. h-cobordism from Y to itself.

To relate our cork Y to the h-cobordism description of Theorem 9.2.18, consider Figure 9.6. Ignoring the two fine curves, we see a plumbing of two spheres A, B with trivial normal bundles, $A \cdot B = 1$ and one extra pair of intersections (cf. Section 6.1). We can take this extra pair to correspond to the middle and bottom clasp. (Note that these clasps have opposite sign as required.) Now it is not hard to see that the meridian γ is a Whitney circle (i.e., it is obtained from a circle in $A \cup B$ consisting of two arcs connecting the given pair of intersections, by pushing into the boundary of the plumbing). If we attach a 2-handle to γ with framing 0 instead of -1, then we obtain a correctly framed Whitney disk, as we see by cancelling the 1-handle and obtaining $(S^2 \times S^2 - \text{int } D^4) \cup 1$-handle. The same cancellation shows that the remaining fine framed circle gives an accessory disk (since it cancels the remaining 1-handle).

Exercise 9.3.5. * Show that if we surger out either A or B in Figure 9.6 (including fine curves), the resulting 4-manifold will be given by Figure 9.7, where δ denotes the image of a 0-framed meridian parallel to γ. Verify that this is diffeomorphic to Y and show that the above diffeomorphism $\varphi\colon \partial Y \to \partial Y$ is the involution obtained by $180°$ rotation about a vertical axis.

We immediately see that Figure 9.6 can be interpreted as the middle level of an h-cobordism V from Y to itself, relative to a product structure $I \times \partial Y$

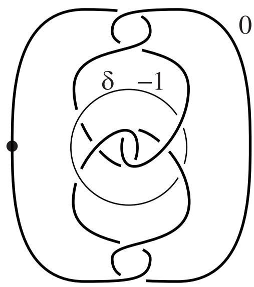


Figure 9.7. The Akbulut cork Y.

on the lateral boundary of V that induces the map $\varphi\colon \partial Y \to \partial Y$ between the boundaries of the two copies of Y. The induced handle structure on V has a unique (5-dimensional) 2-handle/3-handle pair, and this algebraically cancels but has a single extra pair of intersections between the attaching sphere A and belt sphere B, and there is an embedded accessory disk. If we glue together V and $I \times (E(n)\#\overline{\mathbb{CP}^2} - \mathrm{int}\, Y)$ along their lateral boundaries $I \times \partial Y$, we obtain a nontrivial h-cobordism W_n between the closed manifolds $E(n)\#\overline{\mathbb{CP}^2}$ and $X_n \approx \#(2n-1)\mathbb{CP}^2\#10n\overline{\mathbb{CP}^2}$, with a handle decomposition again satisfying the above description. The pair (W_n, V) satisfies the conclusion of Theorem 9.2.18 with $Y_\pm = Y$.

Exercise 9.3.6. Verify that this example satisfies Remark 9.2.19. (Use the fact that a compact contractible 5-manifold is diffeomorphic to D^5 if its boundary is diffeomorphic to S^4.)

We now wish to exhibit an exotic $\mathbb{R}^4$ by applying the method of Theorem 9.3.1 to each of our examples W_n. Since there is a unique extra pair of intersections in each middle level, and the pair has a smoothly embedded accessory disk, it suffices to find a Casson handle attached to the correctly framed Whitney circle. That is, it suffices to find a Casson handle in $E(n)\#\overline{\mathbb{CP}^2} - \mathrm{int}\, Y$ whose framed attaching circle in ∂Y is given by δ in Figure 9.7. Although such a Casson handle can be found directly by Casson's original work, we wish our Casson handles to be as simple as possible. Thus, we invoke the following lemma of Bižaca [**BG**], which we prove below.

Lemma 9.3.7. *The manifold $E(n)\#\overline{\mathbb{CP}^2} - \mathrm{int}\, Y$ contains a $(9n-3)$-stage Casson tower T_{9n-3} attached to the framed circle δ in ∂Y, such that each stage of T_{9n-3} is a kinky handle with a single self-plumbing, whose sign is positive.*

Work of Freedman [**F**] (or Casson if we suitably deal with π_1) now implies that after we modify the top two stages of T_{9n-3} we can extend to an entire Casson handle CH in $E(n)\#\overline{\mathbb{CP}^2} - \mathrm{int}\, Y$ whose first $9n-5$ stages T_{9n-5} agree

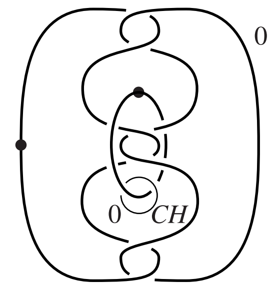


Figure 9.8. Exotic $\mathbb{R}^4$.

with those of T_{9n-3}. Thus, we obtain an exotic $\mathbb{R}^4$ as in Theorem 9.3.1. This is obtained from a neighborhood of $A \cup B \cup$ (accessory disk) by adding CH and surgering out B, so it is given by Figure 9.6 with the -1-framed 2-handle deleted, CH attached to γ with framing 0, and a dot added to one circle. Simplifying as in Exercise 9.3.5, we obtain Figure 9.8, where the fine circle denotes the Casson handle CH, and this Casson handle has a single self-intersection at each of its first $9n-5$ stages. Unfortunately, this picture of an exotic $\mathbb{R}^4$ is still not completely explicit, since the number of intersections at each higher stage of CH will increase superexponentially. Fortunately, however, we can now easily conclude that Figure 9.8 is an exotic $\mathbb{R}^4$ when CH is the Casson handle CH_+ with a single (positive) self-intersection at each stage: For each n, the compact subset $K \subset W_n$ defined in the proof of Theorem 9.3.1 intersects $\partial_- W_n$ only inside the compact submanifold K_0 of Figure 9.8 obtained by removing CH and a collar of the boundary (since K descends from $A \cup B$ in Figure 9.6). If Figure 9.8 were diffeomorphic to $\mathbb{R}^4$ for $CH = CH_+$, then in that case we could find a smooth ball in the figure containing K_0. By compactness, the ball would still lie in the figure if we replaced CH by the subtower T_{9n-5} for n sufficiently large. We would then obtain the contradiction $E(n)\#\overline{\mathbb{CP}^2} \approx X_n$ for this n as in Theorem 9.3.1. We conclude:

Theorem 9.3.8. ([**BG**]) *The interior R of the manifold shown in Figure 6.16, which is the same as Figure 9.8 with $CH = CH_+$, is an exotic $\mathbb{R}^4$.* □

Remarks 9.3.9. (a) To construct an h-cobordism U from R to itself that is topologically trivial but smoothly nontrivial relative to the induced product structure near infinity (cf. Theorem 9.3.1), add a (5-dimensional) 2-handle to $I \times R$ through the big 1-handle of Figure 9.8, turning the dotted circle into a 0-framed 2-handle, then add a (5-dimensional) 3-handle to the opposite sphere A, turning the rightmost 2-handle into a dotted circle. The resulting manifold is diffeomorphic to R by 180° rotation about a vertical axis. The same rotation gives an involution of $R - \operatorname{int} K_0$ (explicitly seen by putting

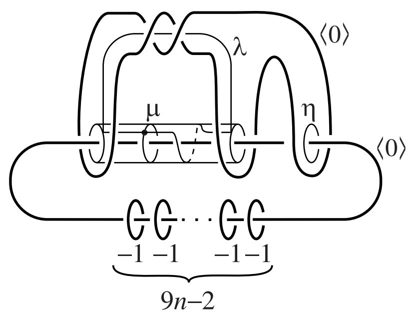


Figure 9.9. Casson tower factory.

the coefficient $\langle 0 \rangle$ on the three heavy circles as in Section 5.5); the involution cannot be smoothly extended over R. Compare with Remark 9.3.2 and the involution $\varphi\colon \partial Y \to \partial Y$ of the Akbulut cork, Exercise 9.3.5.

(b) For an explicit identification of R with an open subset of $\mathbb{R}^4$, see Exercise 6.2.5(a) and use the fact that any Casson handle arises as an open subset of a 2-handle. To be explicit, define the *Whitehead continuum* in $\partial D^2 \times D^2$ to be the intersection of the infinite nested family of solid tori obtained by iterated positive (untwisted) Whitehead doubling of the circle $\partial D^2 \times \{0\}$ (Remark 6.1.2). If $C \subset D^2 \times D^2$ is the cone on this Whitehead continuum, and we embed $(D^2 \times D^2, \partial D^2 \times D^2) \hookrightarrow (D^4, S^3)$ as a tubular neighborhood of the obvious ribbon disk of the $(-3,-3,3)$ pretzel knot (Figure 6.24), then int D^4 minus the image of C will be diffeomorphic to R (cf. [**K2**], the Appendix to Lecture 2 in [**C**], and the introduction of [**BG**]).

(c) We will see (Theorem 11.2.7) that the manifold R admits a *Stein structure*, that is, it is diffeomorphic to a closed, holomorphic submanifold of $\mathbb{C}^N$ for some N.

(d) Note that while we have proved that R is exotic, we did not actually construct it inside an h-cobordism. The exotic $\mathbb{R}^4$'s we actually constructed in $E(n)\#\overline{\mathbb{CP}^2}$ had more complicated Casson handles. (We will see in the next section that uncountably many diffeomorphism types of exotic $\mathbb{R}^4$'s can be obtained from Figure 9.8 by choosing CH to be sufficiently complicated.) In practice, Casson handles constructed in 4-manifolds where smooth 2-handles cannot be found tend to have many self-intersections in each kinky handle (which makes Lemma 9.3.7 seem more remarkable). It is an open question whether the Casson handle CH_+ can ever occur in such a situation.

Proof of Lemma 9.3.7. Recall that Figure 9.7 is isotopic to Figure 9.5(b), which lies in the submanifold Z of $E(n)\#\overline{\mathbb{CP}^2}$ pictured in Figure 9.5(a). We isotope the 0-framed 2-handle and remove int Y as in Section 5.5 to obtain Figure 9.9 of $Z - \text{int}\, Y$, which Bižaca refers to as a "tower factory." We will build T_{9n-3} in $Z - \text{int}\, Y$, attached to a 0-framed meridian η in ∂Y. Since δ bounds a disk in Figure 9.7 that is punctured twice by the dotted circle (whose meridian is η), we can then change our embedding of T_{9n-3} into the desired one by boundary-summing its first-stage core disk with the core of one of the $9n-2$ 2-handles attached to -1-framed meridians. This leaves $9n-3$ available -1-framed 2-handles for the tower construction.

To construct T_{9n-3}, first consider the punctured torus visible in Figure 9.9, whose boundary is η. A basis of its homology is given by the circles λ and μ. Note that both of these are isotopic to the meridian η in ∂Y (check this for λ!) and the normal framing of the torus induces the 0-framings on λ, μ and η. Let $\tau_1, \ldots, \tau_{9n-3}$ be disjoint parallel copies of this punctured torus, so the circles $\partial\tau_i$ are (unlinked) meridians parallel to η. We can assume that the corresponding parallel copies of μ are the attaching circles of the $9n-3$ available -1-framed 2-handles. Use each 2-handle to ambiently surger the corresponding punctured torus τ_i. That is, replace an annulus of τ_i with a pair of oppositely oriented core disks of the 2-handle. Since the framing is -1 rather than 0, the two disks will be forced to intersect once, and since they are oppositely oriented, the intersection number will be $+1$. Thus, each τ_i is transformed to an immersed disk D_i with a single (positive) self-intersection. Figure 9.10 shows explicitly how a neighborhood of τ_i transforms into a kinky handle when the -1-framed 2-handle is attached. Since λ becomes a 0-framed meridian of the dotted circle of the kinky handle, we will obtain a $(9n-3)$-stage tower if we can glue each 0-framed ∂D_i onto λ on D_{i-1}. But since λ and η are isotopic framed circles in ∂Y, it is routine to fit the disks together as required; see the following exercise. □

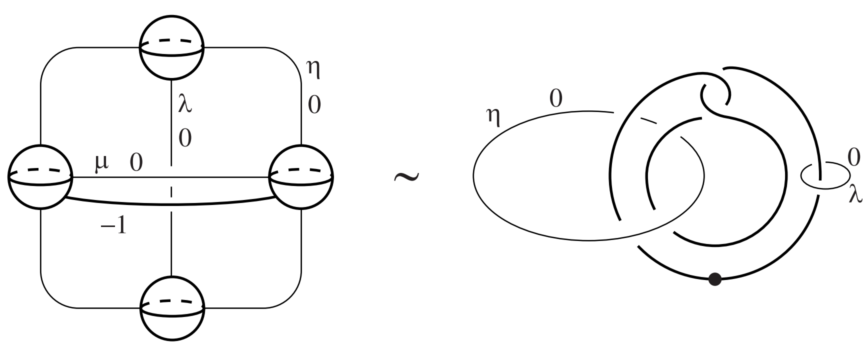


Figure 9.10. Transforming a punctured torus into a kinky handle.

Exercise 9.3.10. A common source of error in proofs involving embeddings of 2-complexes is that hidden intersections are frequently overlooked. To be sure that this proof is correct, visualize T_{9n-3} in a level picture of $Z - \text{int}\, Y$ as in Section 6.2, and verify that it can be constructed without additional intersections. Why does the 2-handle core used at the end of the first paragraph of the proof connect to ∂Y without intersecting T_{9n-3}? (What behavior must the disks in T_{9n-3} avoid for this to work?) Convince yourself that the kinky handles in T_{9n-3} are really attached with the correct framings. (Note that if we were allowed to attach at a given stage with framing -1, then no self-intersections would be required.)

9.4. More exotica

We conclude this chapter by surveying the current state of the theory of exotic smooth structures on $\mathbb{R}^4$ and other noncompact 4-manifolds. For more details of some proofs and a more careful discussion of the early history of the subject, see [**K2**].

The known smooth structures on $\mathbb{R}^4$ fall into two types, with rather different properties. Both arise through the interplay between Freedman's theory of topological 4-manifolds and gauge-theoretic results on smooth 4-manifolds, and the genesis of both types can be traced back further, to Casson's work [**C**] in the early 1970's. In the previous section, we encountered one type of exotic $\mathbb{R}^4$, which arises from the topological success and smooth failure of the h-Cobordism Theorem in dimension 4. These exotic $\mathbb{R}^4$'s are diffeomorphic to open subsets of $\mathbb{R}^4$, and they have (infinite) handle decompositions without handles of index ≥ 3. As we have seen, it is possible to draw these handle structures, although there are complications regarding how many self-intersections etc. are required. This type of exotic $\mathbb{R}^4$ was predated by the other type, arising from the topological success and smooth failure of high-dimensional surgery theory applied to dimension 4. Such exotic $\mathbb{R}^4$'s have 4-dimensional compact submanifolds that cannot be smoothly embedded in $\mathbb{R}^4$. We will call an exotic $\mathbb{R}^4$ *large* if it contains such a submanifold and *small* otherwise. (Clearly, the exotic $\mathbb{R}^4$'s in the previous section were small.) The known large exotic $\mathbb{R}^4$'s require infinitely many 3-handles in any handle decomposition, and there is presently no clue as to how one might draw explicit handle diagrams of them (even after removing their 3-handles). Certainly, their construction seems much too complicated for a direct approach.

Exercises 9.4.1. (a)* Suppose there were an exotic $\mathbb{R}^4$ arising as the interior of a finite handlebody. Conclude that the smooth Poincaré Conjecture would fail in either dimension 3 or 4 — i.e., there would be a simply connected homology 3- or 4-sphere not diffeomorphic to S^3 or S^4. (*Hint*: See Exercise 6.1.4(c).)

(b)* Suppose that $X = \bigcup_{i=1}^{\infty} D_i$, where each D_i is diffeomorphic to D^4 and $D_i \subset \text{int } D_{i+1}$. Prove that X is diffeomorphic to $\mathbb{R}^4$. Thus, every exotic $\mathbb{R}^4$ contains a compact subset not lying in a smoothly embedded 4-ball.

(c) Prove that the interior of any Casson handle is diffeomorphic to $\mathbb{R}^4$. (*Hint*: Apply (b) and the fact that any n-stage tower is diffeomorphic to $\# mS^1 \times D^3$ for some m.)

We next sketch the construction of some large exotic $\mathbb{R}^4$'s. (See also [**K2**] or [**G1**], [**G4**].) The following key lemma is proved below.

Lemma 9.4.2. *There exist pairs* (X, Y) *and* (L, K) *of smooth, oriented 4-manifolds with* Y, K *compact and* X, L *open (i.e., noncompact and boundaryless),* L *homeomorphic to* $\mathbb{R}^4$ *and* Q_X *a negative definite form not isomorphic to* $n\langle -1\rangle$*, such that* $X - \operatorname{int} Y$ *and* $L - \operatorname{int} K$ *are orientation-preserving diffeomorphic.*

Theorem 9.4.3. *Any* L *as above is a large exotic* $\mathbb{R}^4$*. In fact, there is a compact 4-manifold* $K' \subset L$ *that cannot be smoothly embedded in any closed, negative definite 4-manifold.*

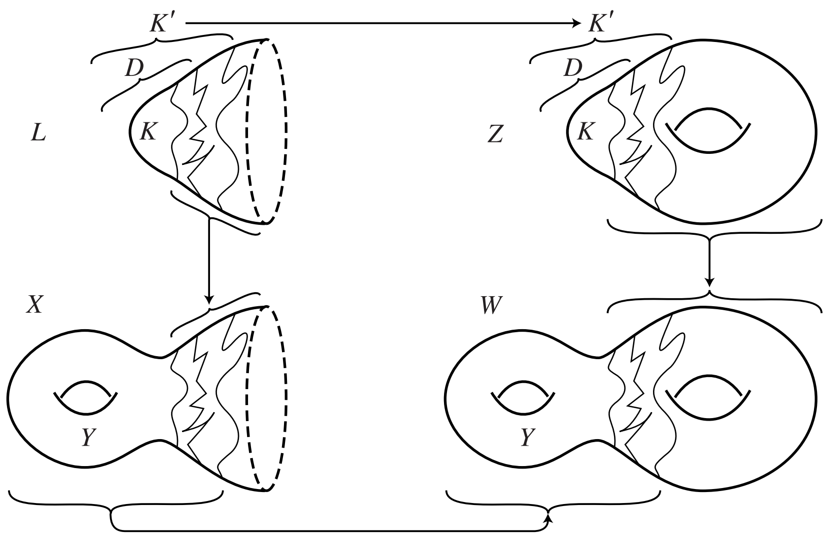


Figure 9.11. Proving L is a large exotic $\mathbb{R}^4$.

Proof. Fix a homeomorphism of L with $\mathbb{R}^4$, and let $D \subset L$ correspond to a round ball in $\mathbb{R}^4$ of sufficiently large radius that the given K lies in int D (top left of Figure 9.11). Then $L - D$ is homeomorphic to $S^3 \times \mathbb{R}$. The projection to $\mathbb{R}$ can be perturbed to a smooth, proper map f, and we can set $K' = D \cup f^{-1}(-\infty, a]$ for any regular value a. If K' embeds in a closed, negative definite Z, then we can remove the image of $L - K'$ from X and replace it by $Z - K$ to obtain a closed, smooth 4-manifold W (Figure 9.11). Topologically, we are just forming a connected sum along

∂D, so $Q_W = Q_X \oplus Q_Z \neq n\langle -1\rangle$ (as we can see by counting elements with square -1), contradicting Donaldson's Theorem (Remark 2.4.30). $\square$

Addendum 9.4.4. *For X, L as Lemma 9.4.2, we can assume that X is simply connected and either*

(a) $L \subset \mathbb{CP}^2$, *or*

(b) $L \subset \#3S^2 \times S^2$ *and* $Q_X \cong -2E_8$.

Proof of Lemma 9.4.2 and Addendum 9.4.4. For (a), note that the element $x = 3h - \sum_{i=1}^{8} e_i \in \langle 1\rangle \oplus 8\langle -1\rangle$ is a characteristic element with square 1, so $\langle x\rangle^{\perp} \cong -E_8$ (cf. Proof of Proposition 2.1.4 in Section 2.2), and in $\langle 1\rangle \oplus 9\langle -1\rangle$ we have $\langle x\rangle^{\perp} \cong -E_8 \oplus \langle -1\rangle \not\cong 9\langle -1\rangle$. Thus $x \in H_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2};\mathbb{Z})$ cannot be represented by a smoothly embedded sphere (since blowing down would contradict Donaldson's Theorem, cf. Exercises 2.3.6(c,d)). However Casson's work [**C**], [**K2**] allows us to represent x by a Casson handle attached to D^4 along a 1-framed unknot. By Freedman, the interior U of this subset is homeomorphic to $\mathbb{CP}^2 - \{\text{pt.}\}$, and the corresponding topological sphere $S \subset U$ represents x. Since any Casson handle can be realized as a subset of a 2-handle, we also have $S \subset U \subset \mathbb{CP}^2$. Now set $X = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2} - S$ (so $Q_X \cong -E_8 \oplus \langle -1\rangle$), $Y = X - \text{int } C$ for C any smooth, compact submanifold of U containing S in its interior, $L = \mathbb{CP}^2 - S$ and $K = \mathbb{CP}^2 - \text{int } C$. Clearly, L is contractible and simply connected at infinity (cf. Exercise 6.1.4(c)), so Freedman's work implies that L is homeomorphic to $\mathbb{R}^4$. The proof of (b) is similar, using the $K3$-surface in place of $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, six Casson handles, and $\#3S^2 \times S^2$ in place of $\mathbb{CP}^2$. $\square$

Remark 9.4.5. With just a bit more care in (b), one can obtain a large exotic $\mathbb{R}^4$ in $S^2 \times S^2$ and a smoothing of the thrice-punctured $-2E_8$-manifold such that near each of the three punctures the smooth structure agrees with that on $L - \text{int } K$. (That is, $X - \text{int } Y \approx \coprod 3(L - \text{int } K)$.) Theorem 9.4.3 applies as before.

Although it is not clear whether the manifolds L given by (a) and (b) of the addendum can be diffeomorphic to each other, we can easily show that there are many large exotic $\mathbb{R}^4$'s. We use the following operation [**G1**].

Definition 9.4.6. Let X_1, X_2 be noncompact, oriented 4-manifolds that are simply connected at infinity (e.g., any exotic $\mathbb{R}^4$'s). Choose proper embeddings $\gamma_i\colon [0,\infty) \to X_i$, remove a tubular neighborhood of $\gamma_i(0,\infty)$ from each X_i and glue the resulting $\mathbb{R}^3$ boundaries together (respecting the orientations on X_i) to obtain an oriented manifold that we will call the *end sum* $X_1 \natural X_2$.

The hypothesis that X_i be simply connected at infinity guarantees that the ray γ_i is unique up to ambient isotopy (Example 4.1.3 in the setting of proper maps), so $X_1 \natural X_2$ is well-defined. End summing is the noncompact analog of boundary summing. An equivalent way to think of end summing is to attach X_1 to X_2 with a piece of tape: Glue $I \times \mathbb{R}^3$ to X_1 and X_2 by identifying $[0, \frac{1}{2}) \times \mathbb{R}^3$ with a tubular neighborhood of γ_1 and $(\frac{1}{2}, 1] \times \mathbb{R}^3$ with a neighborhood of γ_2. It is not hard to generalize to countable end sums $\natural_{i=1}^{\infty} X_i$ by summing with $\mathbb{R}^4$ along a countable collection of rays [**G4**], and to verify that these are independent of the order of the summands. (The end sum of $k \in \{0, 1, 2, \ldots, \infty\}$ copies of X will be denoted $\natural kX$, with $\natural 0X = \mathbb{R}^4$.) Since $X \natural \mathbb{R}^4 \approx X$ in both the smooth and topological settings, the set $\mathcal{R}$ of orientation-preserving diffeomorphism types of smooth structures on $\mathbb{R}^4$ inherits a commutative monoid structure (with identity $\mathbb{R}^4$), and this acts on the set of orientation-preserving diffeomorphism types of smoothings of any fixed X as above. The monoid $(\mathcal{R}, \natural)$ is far from being a group, however, since infinite sums are defined: Any homomorphism $h\colon \mathcal{R} \to G$ into a group is trivial, since for any $R \in \mathcal{R}$, $h(\natural\infty R) = h(R \natural \infty R) = h(R)h(\natural\infty R)$, implying $h(R)$ is the identity. Similarly, no exotic $\mathbb{R}^4$ has an inverse under $\natural$, since R invertible implies $R = R\natural(R^{-1}\natural R)\natural(R^{-1}\natural R)\natural \cdots = (R\natural R^{-1})\natural(R\natural R^{-1})\natural \cdots = \mathbb{R}^4$.

Corollary 9.4.7. *For L as in (a) of Addendum 9.4.4, no two of the manifolds $\mathbb{R}^4$, L, $\overline{L}$ and $L\natural\overline{L}$ are orientation-preserving diffeomorphic. For L as in (b) and $k, \ell \in \{0, 1, 2, \ldots, \infty\}$, $\natural kL$ and $\natural \ell L$ are not diffeomorphic unless $k = \ell$; furthermore, $\natural kL$ is an open subset of $\#3kS^2 \times S^2$ but has a compact submanifold that cannot embed in $\#2kS^2 \times S^2$ ($k \neq 0, \infty$). (Thus $\natural\infty L$ does not embed in any $\#nS^2 \times S^2$, n finite.)*

Proof. In case (a), L embeds in $\mathbb{CP}^2$ but not $\overline{\mathbb{CP}^2}$ (since the latter is negative definite); $\overline{L}$ is the opposite. Thus $L\natural\overline{L}$ embeds in neither $\mathbb{CP}^2$ nor $\overline{\mathbb{CP}^2}$. In (b), the manifolds $\natural kL$ and $\natural kX$ ($k \in \mathbb{N}$) become diffeomorphic when we remove suitable compact subsets. If every compact submanifold of $\natural kL$ embedded in $\#2kS^2 \times S^2$, then the method of proof of Theorem 9.4.3 would produce a closed, smooth, simply connected 4-manifold with intersection form $kQ_X \oplus 2kQ_{S^2\times S^2} = -2kE_8 \oplus 2kH$, contradicting Furuta's Theorem 1.2.31. However, $\natural kL$ can be embedded in $\#3kS^2 \times S^2$ by Exercise 9.4.8(a) below. If $\natural kL$ and $\natural \ell L$ were orientation-preserving diffeomorphic for some $k < \ell < \infty$ then $\{\natural nL \mid n = 0, 1, 2, \ldots\}$ would contain at most ℓ diffeomorphism types (because of the monoid structure), all of which would embed in $\#3kS^2 \times S^2$, contradicting our previous assertion for sufficiently large n. A similar argument rules out orientation-reversing diffeomorphisms for $k \neq l$ (although L can be constructed with an orientation-reversing self-diffeomorphism so that $\natural kL \approx \natural k\overline{L}$). The infinite end sum $\natural\infty L$ is different from the rest, since it embeds in no $\#kS^2 \times S^2$. □

Exercises 9.4.8. (a)* For $i = 1, 2$, let R_i be an exotic $\mathbb{R}^4$ contained in a connected 4-manifold X_i. Prove that $R_1 \natural R_2$ embeds in $X_1 \# X_2$. (*Hint*: You may need to modify the inclusions $R_i \hookrightarrow X_i$.) Now suppose that each X_i is as in Definition 9.4.6 and $R_i = \operatorname{int} D_i$, where D_i is a flat topologically embedded 4-ball in X_i with ∂D_i smooth near some point $p_i \in \partial D_i$. (Flat means that there is a topological embedding $[-1, 1] \times S^3 \hookrightarrow X_i$ with ∂D_i the image of $\{0\} \times S^3$.) Prove that $R_1 \natural R_2$ embeds in $X_1 \natural X_2$.

(b)* For $R \subset \mathbb{R}^4$ as in Theorem 9.3.8, prove that no two of $\mathbb{R}^4$, R, $\overline{R}$ and $R\natural\overline{R}$ are orientation-preserving diffeomorphic. (*Hint*: What happens if you form the connected sum with infinitely many copies of $\mathbb{CP}^2$ along a properly embedded infinite disjoint union of balls? See Exercise 9.4.1(b) and Theorem 2.4.9.)

(c)* Prove that for L as in (b) of Addendum 9.4.4, the submanifold K' constructed in the proof of Theorem 9.4.3 admits no orientation-preserving embedding in $L - K'$. Conclude (after L. Taylor) that L cannot be a nontrivial covering space. (Note that it suffices to show L cannot be an infinite cover, since a contractible manifold cannot cover anything with torsion in π_1: Any $\mathbb{Z}_p$-subgroup of covering transformations would have $K(\mathbb{Z}_p, 1)$ as quotient, contradicting the fact that the latter has homology in infinitely many dimensions.)

One of the surprising peculiarities of 4-manifolds is that, unlike in other dimensions, open 4-manifolds with finitely generated (or trivial) homology can admit uncountably many smooth structures. To distinguish uncountably many diffeomorphism types of exotic $\mathbb{R}^4$'s, we begin with the following construction. For any smooth manifold R homeomorphic to $\mathbb{R}^4$, fix a homeomorphism $h\colon \mathbb{R}^4 \to R$. Let $R_t \subset R$ be the image of the open ball of radius t centered at 0 in $\mathbb{R}^4$, and let $R_\infty = R$. Each R_t inherits a smooth structure as an open subset of R. By work of Freedman and Quinn (e.g., [**FQ**]) any homeomorphism between smooth 4-manifolds is isotopic to one which is a local diffeomorphism near a preassigned 1-complex; it is convenient to assume h has been smoothed in this way near the nonnegative x_1-axis (cf. Exercise 9.4.8(a)).

Definition 9.4.9. A family $\{R_t \mid 0 < t \leq \infty\}$ as above will be called a *radial family* of $\mathbb{R}^4$'s in R.

Theorem 9.4.10. *Let (X, Y) and (L, K) be as in Lemma 9.4.2 with X simply connected. Let $L_t \subset L$ be a corresponding radial family, with $K \subset L_r$. Then $\{L_t \mid r \leq t \leq \infty\}$ is an uncountable family of nondiffeomorphic large*

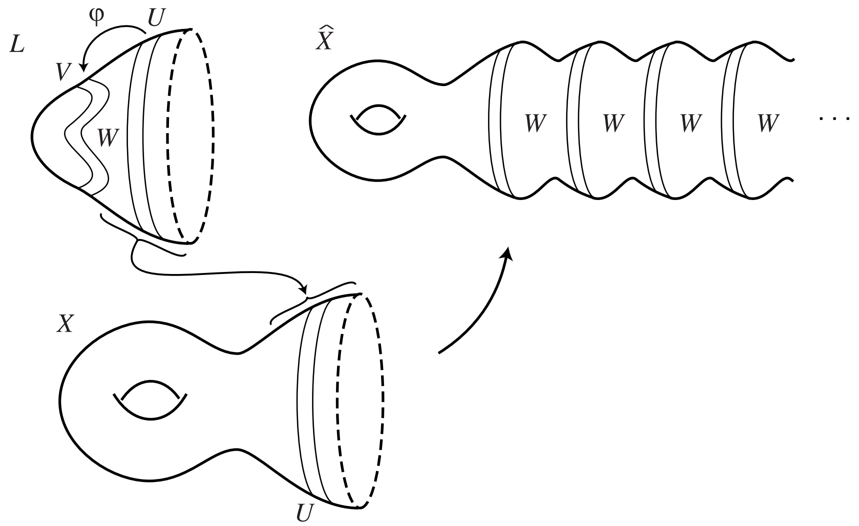


Figure 9.12. Distinguishing uncountably many exotic $\mathbb{R}^4$'s.

exotic $\mathbb{R}^4$'s. In fact, for $r \le s < t \le \infty$, there is a compact 4-manifold $K' \subset L_t$ that cannot be embedded in L_s.

Proof. Fix $t' \in (s, t)$ and let $U = L_{t'} - c\ell(L_s)$. Suppose that there is a diffeomorphism $\varphi\colon U \to V \subset L_s$, with the outward ends of U and V corresponding. Let $W \subset L$ be the ring consisting of U, V and the region in between (Figure 9.12). Trim the end of X by removing the image of $L - L_{t'}$ from X to expose the copy of $U \subset X$, then add an infinite stack of copies of W to the end, using φ to glue each V to the previous U. The resulting smooth manifold $\widehat{X}$ is homeomorphic to X but has a *periodic end*, that is, $\widehat{X}$ minus a compact set is diffeomorphic to half of the universal cover of some smoothing of $S^1 \times S^3$ (namely W with its ends glued together by φ). In [**T1**], Taubes showed that much of gauge theory can be extended from closed 4-manifolds to open 4-manifolds with periodic ends. (The basic idea is that by requiring all functions to suitably converge to periodic functions near infinity, one gains sufficient control there to adapt analytical techniques from a compact setting, much as boundary conditions make Laplace's equation tractable on compact bounded domains.) In particular, Taubes extended Donaldson's Theorem 1.2.30 to the case of end-periodic 4-manifolds, showing that the above manifold $\widehat{X}$ cannot exist. We conclude that the diffeomorphism φ does not exist. As in the proof of Theorem 9.4.3, we construct a compact K' with $U \subset L_{t'} \subset K' \subset L_t$, and the proof is complete. □

Exercises 9.4.11. (a)* Show that for $r \le s < t \le \infty$, L_s and L_t cannot have diffeomorphic ends, i.e., it is impossible to find compact 4-manifolds $K_s \subset L_s$ and $K_t \subset L_t$ with $L_s - \operatorname{int} K_s$ diffeomorphic to $L_t - \operatorname{int} K_t$. Prove that there are uncountably many diffeomorphism types of manifolds homeomorphic to $\mathbb{CP}^2 - \{\text{pt.}\}$. (*Hint*: Consider the manifold U in the proof of Addendum 9.4.4(a).)

(b)* Prove that if L is as in Addendum 9.4.4(b), then for any orientation-preserving diffeomorphism $\psi\colon U \to V' \subset L$ (U as above) we must have $U \cap V' \neq \emptyset$. In particular, for any $q > r$ there is no way to smoothly isotope the topological sphere ∂L_q off of itself.

Theorem 9.4.12. *For R as in Theorem 9.3.8, a radial family $R_t \subset R$ contains uncountably many diffeomorphism types of small exotic $\mathbb{R}^4$'s. These can be chosen to have Kirby diagrams given by Figure 9.8 for suitably chosen Casson handles CH.*

Proof (sketch). The basic idea is the same as before; see [**DF**] for the full proof. Fix r with $K_0 \subset R_r$, for $K_0 \subset R$ as given before Theorem 9.3.8. An argument based on Freedman theory shows that we can assume that for a Cantor set $C \subset [r, \infty]$, all R_t with $t \in C$ can be assumed to have the required form (differing from R only in the complexity of the Casson handle). Now suppose that for some s, t with $r \le s < t < \infty$ we can find a diffeomorphism $\varphi\colon R_t \to R_s$ fixing K_0. Since R_t has compact closure in R, it is associated to our h-cobordism W_n from $E(n)\#\overline{\mathbb{CP}^2}$ to X_n for some fixed n. Then iterates φ^i of φ produce a periodic end structure on the open manifold $E_\infty = E(n)\#\overline{\mathbb{CP}^2} - \bigcap_{i=1}^\infty \varphi^i(R_t)$, and the product structure of W_n over $E(n)\#\overline{\mathbb{CP}^2} - K_0$ sends this to an open submanifold $X_\infty \subset X_n$ diffeomorphic to E_∞. End-periodic gauge theory produces invariants for E_∞ and X_∞ that must agree since $E_\infty \approx X_\infty$, but disagree since they are inherited from $E(n)\#\overline{\mathbb{CP}^2}$ and X_n, respectively. This contradiction implies that no two of the pairs (R_t, K_0) are diffeomorphic $\operatorname{rel} \operatorname{id}_{K_0}$. But there are only countably many isotopy classes of embeddings of K_0 in R_t (as can be seen, for example, by passing to the PL-category and counting subcomplexes), so each R_t is diffeomorphic to only countably many others. Since the Cantor set C is uncountable, we are done. □

Exercises 9.4.13. (a) Prove that the above family represents uncountably many diffeomorphism types of ends. (*Hint*: Show that only countably many exotic $\mathbb{R}^4$'s can realize a given end.)

(b)* Prove that there are uncountably many diffeomorphism types of Casson handles.

To find yet larger families of exotic $\mathbb{R}^4$'s, we introduce more structure on the set $\mathcal{R}$ of smoothings of $\mathbb{R}^4$ (up to orientation-preserving diffeomorphism).

Definition 9.4.14. ([**G4**]) For $R_1, R_2 \in \mathcal{R}$ we write $R_1 \leq R_2$ if every compact 4-manifold in R_1 smoothly embeds (preserving orientation) in R_2. If $R_1 \leq R_2 \leq R_1$ we say R_1 and R_2 are *compactly equivalent*. The set of compact equivalence classes in $\mathcal{R}$ is denoted $\mathcal{R}_\sim$.

Thus, R_1 and R_2 are compactly equivalent iff they share all compact submanifolds. $R_1 \not\approx \mathbb{R}^4$ is compactly equivalent to $\mathbb{R}^4$ iff it is a small exotic $\mathbb{R}^4$, so the family $\{R_t\}$ of Theorem 9.4.12 maps to a point in $\mathcal{R}_\sim$, namely the equivalence class of $\mathbb{R}^4$. It is easy to see that the relation $\leq$ descends to a partial ordering on $\mathcal{R}_\sim$, with a unique minimal element (the equivalence class consisting of $\mathbb{R}^4$ and all small exotic $\mathbb{R}^4$'s); there is also a unique maximal element which we discuss below. Clearly, the family $\{L_t \mid r \leq t \leq \infty\}$ of Theorem 9.4.10 has the order type of an interval ($L_s \leq L_t$ iff $s \leq t$), so $\mathcal{R}_\sim$ is uncountable. We can now construct several subsets of $\mathcal{R}$ that are naturally indexed by $I \times I$. Presumably, one should be able to construct families of distinct diffeomorphism types with higher-dimensional index sets, but this seems to exceed our current capacity for distinguishing exotic $\mathbb{R}^4$'s. (It is also possible to give $\mathcal{R}_\sim$ a metrizable topology with countable basis [**G7**], although the usefulness of this topology is not presently clear. The topology has the peculiar property that every increasing sequence converges — for the limit, see the first exercise below.)

Exercises 9.4.15. (a) Let $R_1 \leq R_2 \leq R_3 \leq \cdots$ be an increasing sequence of exotic $\mathbb{R}^4$'s. Construct R_∞, an exotic $\mathbb{R}^4$ such that each $R_n \leq R_\infty$ but any compact submanifold of R_∞ embeds in some R_n. This property characterizes R_∞ up to compact equivalence. (Note that for $R_1 \subset R_2 \subset \cdots$ we can set $R_\infty = \bigcup_{n=1}^{\infty} R_n$. What do we get for sequences in the set $\{L_t\}$ of Theorem 9.4.10?) See [**G7**] for a solution.

(b)* If $R_1 \leq R_2$ and $R_3 \leq R_4$, prove that $R_1 \natural R_3 \leq R_2 \natural R_4$. Thus, the monoid structure on $\mathcal{R}$ descends to $\mathcal{R}_\sim$.

Theorem 9.4.16. *Let (L, K) be as in Addendum 9.4.4(a) with a radial family $L_t \subset L$, $K \subset L_r$. Let $R_t \subset R \subset \mathbb{R}^4$, $K_0 \subset R_r$, be as in Theorem 9.4.12.*

(a) ([**G4**]) *The family of exotic $\mathbb{R}^4$'s $L_{s,t} = L_s \natural \overline{L}_t$, $r \leq s, t \leq \infty$, satisfies $L_{s,t} \subset L_{s',t'}$ iff $L_{s,t} \leq L_{s',t'}$ iff $s \leq s'$ and $t \leq t'$. Thus, the map $(s,t) \mapsto L_{s,t}$ defines an order-preserving injection $[r,\infty] \times [r,\infty] \to \mathcal{R}_\sim$.*

(b) ([**G11**]) *For fixed $t \in [r,\infty]$, the large exotic $\mathbb{R}^4$'s $R_s \natural \overline{L}_t$ represent uncountably many diffeomorphism types. Thus, the family $\{L_t \mid r \leq t \leq \infty\}$ determines an uncountable family of compact equivalence classes (with the order type of $[r,\infty]$ in $\mathcal{R}_\sim$), each of which contains uncountably many diffeomorphism types.*

Proof. We combine our radial family arguments with our previous use of orientations. (The required inclusions $L_{s,t} \subset L_{s',t'}$ follow by Exercise 9.4.8(a).) Suppose that $s > s' \geq r$ and any compact submanifold of $L_{s,t}$ embeds (preserving orientation) in $L_{s',t'}$. Then after shrinking s slightly, we may assume that L_s embeds in $L_{s',t'}$. Thus, we obtain embeddings $L_s \hookrightarrow L_{s',t'} \hookrightarrow L_s \# \overline{\mathbb{CP}^2}$ (since $L_{s'} \subset L_s$ and $\overline{L}_{t'} \subset \overline{L} \subset \overline{\mathbb{CP}^2}$, see Exercise 9.4.8(a)), and the composite embedding i has image with compact closure. Now we proceed as in the proof of Theorem 9.4.10, setting $W = L_s \# \overline{\mathbb{CP}^2} - i(K')$ for a suitably large compact subset $K' \subset L_s$, and gluing copies of W to create a manifold $\widehat{X}$ with periodic end modeled on half the universal cover of some smoothing of $S^1 \times S^3 \# \overline{\mathbb{CP}^2}$. Since the work of Taubes is unaffected by negative definite homology in the end, and the intersection form $Q_{\widehat{X}} = Q_X \oplus \infty\langle -1 \rangle$ is not diagonalizable (because there is no basis of elements with square -1), we have the required contradiction. To prove (b), first suppose that $s > s' \geq r$ and that there is a diffeomorphism $R_s \natural \overline{L}_t \to R_{s'} \natural \overline{L}_t$ restricting to the identity on K_0 in the first summand. As before, we obtain an embedding $R_s \hookrightarrow R_s \# \overline{\mathbb{CP}^2}$ restricting to id_{K_0} and having image with compact closure. Applying the method of proof of Theorem 9.4.12, blowing up whenever necessary, we obtain a contradiction via end-periodic manifolds E_∞ and X_∞ homeomorphic to $(E(n) - \{\text{pt.}\}) \# \infty \overline{\mathbb{CP}^2}$, and again conclude uncountability. See [**G11**] for further discussion. □

Exercises 9.4.17. **(a)** Prove that no two of the above manifolds $L_{s,t}$ can have diffeomorphic ends, cf. Exercise 9.4.11(a). Prove that two of the above manifolds $R_s \natural \overline{L}_t$ cannot have diffeomorphic ends unless they have the same value of t, and for each t there are uncountably many ends, cf. Exercise 9.4.13(a).

(b) Prove the following theorem, cf. Exercise 9.4.8(b). (See [**BG**] Proposition 5.6 for a solution.)

Theorem 9.4.18. ([**BG**]) *For R_t as above with $K_0 \subset R_r$, let $R_{s,t} = R_s \natural \overline{R}_t$. Then any given diffeomorphism type is realized by at most countably many pairs (s, t), $r \leq s, t \leq \infty$. In fact, there are at most countably many such pairs for which $R_{s,t}$ is both $\mathbb{CP}^2$- and $\overline{\mathbb{CP}^2}$-stably diffeomorphic to a given R_{s_0,t_0}.* □

Next, we define a simple invariant on $\mathcal{R}_\sim$.

Definition 9.4.19. ([**Ta**]) For $R \in \mathcal{R}$, define $\gamma(R) \in \{0, 1, 2, \ldots, \infty\}$ to be $\sup_K\{\min_X\{\frac{1}{2} b_2(X)\}\}$, where K ranges over compact 4-manifolds embedding in R and X ranges over closed, spin 4-manifolds with signature 0 in which K smoothly embeds.

Note that any $K \subset R$ embeds in the spin manifold $X = DK$. Clearly, for $R_1 \leq R_2$ we must have $\gamma(R_1) \leq \gamma(R_2)$, so γ is well-defined on compact equivalence classes and gives an order-preserving function $\gamma\colon \mathcal{R}_\sim \to \{0, 1, 2, \dots, \infty\}$. Taylor [**Ta**] actually defines γ for all smooth 4-manifolds, but the definition and properties are more complicated in general. By Theorem 9.4.3, any L as in Lemma 9.4.2 must have $\gamma(L) > 0$. It is unknown whether any large exotic $\mathbb{R}^4$ can have $\gamma = 0$ (although every small $\mathbb{R}^4$ obviously does). A striking application by Taylor (see the first exercise below) is that for L any exotic $\mathbb{R}^4$ with $\gamma(L) > 0$, any handle decomposition of L must have infinitely many 3-handles, in contrast with our examples of small exotic $\mathbb{R}^4$'s, which were built without 3-handles. In particular, consider *Stein surfaces*, which are properly embedded complex surfaces in $\mathbb{C}^N$. (We will discuss these in more detail in Chapter 11.) Since any Stein surface has a handle decomposition without 3-handles, it follows that no known large exotic $\mathbb{R}^4$ admits a Stein structure. In contrast, we will show (Theorem 11.2.7) that uncountably many small exotic $\mathbb{R}^4$'s do admit Stein structures, namely R from Theorem 9.3.8 and the elements of a (suitably constructed) radial family $R_t \subset R$ indexed by a Cantor set (as in the proof of Theorem 9.4.12).

Exercises 9.4.20. **(a)*** If $R \in \mathcal{R}$ has a handle decomposition with only finitely many 3-handles, show that $\gamma(R) = 0$. (*Hint*: Exhibit R as a nested union of compact submanifolds K_n with $b_2(K_n) = 0$.)

(b)* Show that if R_1 and R_2 have diffeomorphic ends then $\gamma(R_1) = \gamma(R_2)$.

(c)* Show that any end sum satisfies $\sup_n\{\gamma(R_n)\} \leq \gamma(\natural_n R_n) \leq \sum_n \gamma(R_n)$.

(d)* Show that $\gamma\colon \mathcal{R}_\sim \to \{0, 1, 2, \dots, \infty\}$ is onto. (*Hint*: Use Remark 9.4.5 and the fact that Furuta's Theorem 1.2.31 holds for spin manifolds with arbitrary π_1.)

(e) Show that R_∞ as in Exercise 9.4.15(a) satisfies $\gamma(R_\infty) = \sup_n\{\gamma(R_n)\}$.

(f)* Prove that for any $n \in \mathbb{N}$ the preimage $\gamma^{-1}(n) \subset \mathcal{R}_\sim$ is uncountable. Taylor also shows that for infinitely many n including $n = \infty$ there are exotic $\mathbb{R}^4$'s $L_n \subset \mathbb{CP}^2$ with $\gamma(L_n) = n$. Conclude from this that $\gamma^{-1}(\infty)$ is uncountable.

(g)* Suppose $\gamma(R) = \infty$. Show that R does not embed in any compact spin 4-manifold (possibly with boundary). (Compare with (f) above.) Prove that there is no embedding of R in *any* 4-manifold such that $c\ell(R)$ is a flat topological 4-ball D as defined in Exercise 9.4.8(a).

Now we return to the maximal element of $\mathcal{R}_\sim$. In [**FT2**], Freedman and Taylor proved:

Theorem 9.4.21. *There is a unique diffeomorphism type U of exotic $\mathbb{R}^4$ such that for any $R \in \mathcal{R}$, the end sum $U \natural R$ is diffeomorphic to U.* ◻

Uniqueness is easy, since for any other U' satisfying this property we would have $U' \approx U \natural U' \approx U$. U is called the *universal* $\mathbb{R}^4$, since it contains all other exotic $\mathbb{R}^4$'s. Clearly U is large — in fact, its compact equivalence class is the unique maximal element of $\mathcal{R}_\sim$ under $\leq$. (Nothing is known about the cardinality of this equivalence class.) In particular, $\gamma(U) = \infty$, so U cannot embed in any closed spin 4-manifold (or definite manifold by Theorem 9.4.3), or with $c\ell(U)$ a flat topological ball in *any* 4-manifold, and any handle decomposition of U requires infinitely many 3-handles. Clearly, U admits many discrete group actions. For example, if X is any noncompact, smooth 4-manifold covered by $\mathbb{R}^4$ (e.g., $\mathbb{R} \times M^3$ for M^3 flat or hyperbolic), then $X \natural U$ is covered by $\mathbb{R}^4 \natural \infty U \approx U$, so $\pi_1(X)$ acts on U by covering translations. (Compare with Exercise 9.4.8(c). In fact, translating compact subsets shows that if $R \in \mathcal{R}$ is a nontrivial cover then $\natural \infty R$ is compactly equivalent to R, so $\gamma(\natural \infty R) = \gamma(R)$, cf. [**Ta**].) The basic idea for constructing U is to build it with as much complexity as possible, using Exercise 9.4.23 below. Specifically, if (W^5, X^4_+, X^4_-) is a noncompact (proper) h-cobordism that is both simply connected and simply connected at infinity, then we can find (infinitely many) spheres and Casson handles in $\partial_+ W_2$ as in the proof of Theorem 9.2.11; we require U to contain enough complexity so that all Casson handles can be replaced by 2-handles in $\partial_+ W_2 \natural U$. Thus, summing W with $I \times U$ along I produces a trivial h-cobordism that shows that $X_+ \natural U$ and $X_- \natural U$ are diffeomorphic. Since any $R \in \mathcal{R}$ is properly h-cobordant to $\mathbb{R}^4$ (by shaving all but a single coordinate chart off of one boundary component of $I \times R$), we conclude that $R \natural U \approx \mathbb{R}^4 \natural U \approx U$.

One more amusing construction involves 2-fold branched covers. Begin with the simplest example, $\mathbb{R}^4 = \mathbb{C}^2$ branched-covering itself by $f(z,w) = (z^2, w)$, with branch locus $\{0\} \times \mathbb{C}$. What examples can we obtain by changing the smooth structures on the domain and range but requiring the given $f \colon R_1 \to R_2$ to be smooth? As in the previous paragraph, we can end sum with any $R \in \mathcal{R}$ to obtain a branched covering $f \colon R \natural R \to R$ (e.g., $f \colon U \to U$) that is topologically standard. This example might lead us to conjecture that $R_1 \geq R_2$ in general. One might also expect that requiring R_2 to be standard would force R_1 to be standard. Surprisingly, both of these guesses are false. First, one can construct examples with R_1 exotic and $R_2 \approx \mathbb{R}^4$, and R_1 can be chosen to be either large or small. That is, there are smooth, proper embeddings $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ that are topologically ambiently isotopic to the standard embedding, but for which the corresponding double branched covers are exotic (and there are uncountably many such diffeomorphism types of large or small branched covers). Second, it is possible to have R_1 a small exotic $\mathbb{R}^4$ but R_2 large. (This R_1 seems to be a good candidate for a small exotic $\mathbb{R}^4$ requiring infinitely many 3-handles.) Again, there are uncountably many examples. It follows that R_1 and R_2 can be independently chosen to be large or small. It is an interesting open question whether we can have R_1 standard and R_2 exotic (large or small). That is, does $\mathbb{R}^4$ admit a smooth, topologically standard involution whose quotient is exotic? One might also ask about p-fold branched coverings; nothing nontrivial is known for $p > 2$. The proofs of the above assertions appear in [**G11**]. The basic idea is to embed the manifold X of Figure 9.6, with the fine circles removed and a Casson handle attached to a 0-framed meridian of each dotted circle, into $S^2 \times S^2$ so that it is equivariant under the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action of Exercise 6.3.18, and take suitable quotients. We have $X = S^2 \times S^2 \# R_-$ for some R_- as in Theorem 9.3.1, and a branched covering $R_- \to \mathbb{R}^4$ can be located. The remaining branched coverings $R^* \to L \to \mathbb{R}^4$ come from a neighborhood of $S^2 \times S^2 - X$: R^* is a small exotic $\mathbb{R}^4$ "complementary" to R_- in the sense that $R^* \cup R_- = S^4$ and $R^* \cap R_-$ is homeomorphic to $\mathbb{R} \times S^3$ but contains no smoothly embedded 3-sphere generating its homology. R^* inherits a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action with quotient $\mathbb{R}^4$, and by Kirby calculus one can identify one $\mathbb{Z}_2$-quotient L with a large $\mathbb{R}^4$ as in Addendum 9.4.4(a), so we have the required maps.

Finally, we consider smooth structures on other noncompact, connected topological 4-manifolds. It seems likely that all such 4-manifolds admit uncountably many smooth structures. Some caution is required, however — it still seems plausible that there could be an open 4-manifold with connected end, whose topology is so complicated that any smoothing is forced to be universal, $X \approx X \natural U$. Such a manifold might have only one smooth structure. We now sketch the broad (but still incomplete) results showing that large families of noncompact manifolds admit many smooth structures. Recall that the boundary of a 4-manifold X (being a 3-manifold) is uniquely smoothable up to isotopy. It is useful to consider *stable isotopy classes* of smoothings on X (or more precisely, on the pair $(X, \partial X)$), that is, isotopy classes of smoothings on $\mathbb{R} \times X$ that are standard on $\mathbb{R} \times \partial X$. By high-dimensional smoothing theory [**KS**], these are classified by $H^3(X, \partial X; \mathbb{Z}_2)$ (except for compact X with $ks(X) \neq 0 \in H^4(X, \partial X; \mathbb{Z}_2)$, for which no such stable smoothings exist). The first theorem comes from Freedman theory, with the last major piece filled in by Quinn.

Theorem 9.4.22. (e.g., [**FQ**]). *Any connected, noncompact 4-manifold X admits a smooth structure. In fact, each stable isotopy class of smoothings is realized by a smoothing of X.* □

Exercise 9.4.23. * Let $K \subset S^3$ be a topologically slice knot (Definition 6.2.3) that is not smoothly slice. (For example, the positive Whitehead double of the right trefoil (Figure 6.13) works, cf. Exercise 11.4.11(e).) Let X_K be D^4 with a 2-handle attached along K with framing 0. Prove that

X_K embeds smoothly in some $R \in \mathcal{R}$, and this latter is necessarily a large exotic $\mathbb{R}^4$.

In [**G11**] it was proved that for any connected topological 4-manifold X (not necessarily compact or orientable), $X - \{\text{pt.}\}$ admits uncountably many diffeomorphism types of smooth structures. This is equivalent to saying that if Y is a 4-manifold with one boundary component M homeomorphic to S^3, then $Y - M$ has uncountably many smooth structures. The proof involved a radial family of smoothings defined using a topological collar $V \approx \mathbb{R} \times M$ of M in $Y - M$. (That is, $c\ell(V) \subset Y$ was homeomorphic to $[-\infty, \infty] \times M$ with $\{\infty\} \times M$ corresponding to $M \subset Y$.) Various results have since been proved for more general M [**Di**], [**Fa**]. The following theorem, which we prove below, unifies and extends what is known about 4-manifolds with uncountably many smoothings. Recall that a map is *proper* if the preimage of any compact set is compact. Given proper maps $f, g \colon X \to Y$ agreeing on $Z \subset X$ outside a compact subset of X, we will call f, g *homotopic at infinity rel* Z if there is a compact $K \subset X$ such that $f, g \colon X - \text{int } K \to Y$ are properly homotopic rel $Z - \text{int } K$ (i.e., they are related by a homotopy $I \times (X - \text{int } K) \to Y$ that is a proper map and fixes $Z - \text{int } K$).

Theorem 9.4.24. *Let X be a noncompact, connected topological 4-manifold, and let $V \subset X$ be an open subset with $c\ell(V)$ noncompact but the point-set boundary $Bd(V) \subset X$ compact. If $X - V$ is compact, assume V is orientable. Suppose that V admits a smooth structure such that some finite cover $\widetilde{V}$ of V embeds smoothly in $\#n\mathbb{CP}^2$ for some $n \in \mathbb{N}$. Then there is an uncountable family of nonisotopic smoothings Σ_t on X which agree on $X - V$ and ∂X. No two of these are related by a diffeomorphism (or even a proper diffeomorphic embedding $(c\ell(V - K), \Sigma_t) \hookrightarrow (X, \Sigma_s)$, $K \subset X$ compact) fixing $\partial V = \partial X \cap V$ outside a compact subset of X, such that the restriction to $c\ell(V)$ is homotopic at infinity rel ∂V to the inclusion $c\ell(V) \hookrightarrow X$. All stable isotopy classes on X extending a particular one on V (induced by the given smoothing on V unless $ks(X, V) \neq 0$) are realized by such families, and for V orientable with $H^3(V, \partial V; \mathbb{Z}_2)$ finite, any stable isotopy class of X is so realized.*

Corollary 9.4.25. *Let Y be a connected topological 4-manifold with $M \subset \partial Y$ a (nonempty) compact 3-manifold whose boundary is flat in ∂Y (and possibly empty). If M has a flat topological embedding in $\#n\mathbb{CP}^2$ that is smooth near ∂M, then $Y - M$ has uncountably many diffeomorphism types of smooth structures in each stable isotopy class. For example, this holds for any orientable M with $b_1(M) = 0$ and linking form avoiding a certain small class of nondiagonalizable forms. The same holds if M has an orientable finite cover $\widetilde{M}$ that smoothly embeds in $\#n\mathbb{CP}^2$, e.g., if M is a Seifert fibered* *space, provided that for nonorientable M we assume Y is noncompact and range over stable isotopy classes of $Y - M$ induced by Y.*

Proof. We may assume M is connected, by removing all but one component of M from Y. Let $V \approx \mathbb{R} \times M \subset X = Y - M$ be a topological collar of M. The flat embedding $M \hookrightarrow \# n\mathbb{CP}^2$ extends to a topological embedding of $V \approx \mathbb{R} \times M$ (which is therefore orientable). Set $\widetilde{V} = V$ with the smooth structure inherited from $\# n\mathbb{CP}^2$. The theorem provides an uncountable family of smooth structures Σ_t on X in each stable isotopy class. Since there are only countably many homotopy classes of maps $(M, \partial M) \to (M, \partial M)$, only countably many of these smoothings Σ_t can be related to any given one by a proper embedding $c\ell(V - K) \to c\ell(V)$. But the end of X has at most countably many components collared by $\mathbb{R} \times M$. (Take the endpoint compactification of X [**Fre**], adding a compact, totally disconnected set E to X to obtain a compact metric space in which X is dense, such that any connected open subset of $X \cup E$ has connected intersection with X. Each $\mathbb{R} \times M$ collar as above determines an isolated point in E (with neighborhood $(-\infty, \infty] \times M/\{\infty\} \times M$ in $X \cup E$), and there can only be countably many such points.) The smoothings Σ_t agree on $X - V$, resulting in at most countably many diffeomorphism types of $\mathbb{R} \times M$ collars in $X - V$, so after throwing away countably many smoothings Σ_t, we can assume that no $(c\ell(V - K), \Sigma_t)$ embeds properly in $X - V$. Clearly, the remaining smoothings represent uncountably many diffeomorphism types, as required. Edmonds [**Ed**] proved that rational homology spheres admit such flat embeddings in $\# n\mathbb{CP}^2$, except in the cases of a few linking forms. For the case with cover $\widetilde{M}$ as above, put the product smooth structure on $V \approx \mathbb{R} \times M$ and set $\widetilde{V} = \mathbb{R} \times \widetilde{M}$. It is not hard to show that every Seifert fibered space is finitely covered by an orientable S^1-bundle over an orientable surface, so the following exercise completes the proof. □

Exercise 9.4.26. Show that every orientable D^2-bundle over a compact orientable surface embeds (possibly reversing orientation) in $\# n\mathbb{CP}^2$ for sufficiently large n.

Remark 9.4.27. Fang [**Fa**] (who introduced finite covers and nonorientable ends into the theory) also found a large class of oriented 3-manifolds (with $b_1(M) \geq 30$) that admit no flat embedding in any definite manifold. The obstruction involves the cup product $H^1 \otimes H^1 \otimes H^1 \to \mathbb{Z}$. As of this writing, it is unknown whether all rational homology spheres embed flatly in $\# n\mathbb{CP}^2$, although for certain linking forms such an embedding could not have simply connected complements [**Ed**].

Proof of Theorem 9.4.24. The first part of the proof follows [**G11**]; we refer there for additional details. For consistency of exposition, we reverse

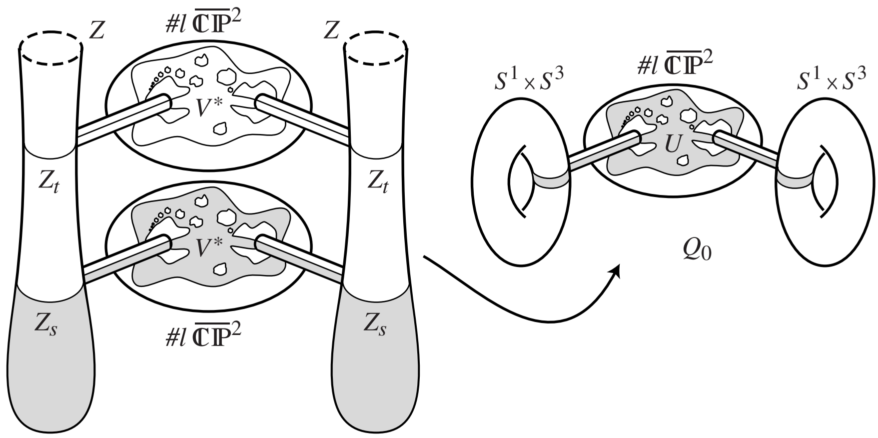


Figure 9.13. A periodic end construction for distinguishing smooth structures on a large class of 4-manifolds.

orientation so that $\widetilde{V} \subset \#n\overline{\mathbb{CP}^2}$. The first step is (after shrinking V) to extend the given smoothing of V over all of $X - \{p\}$ so that near p the smoothing agrees with the end of some smooth, negative definite manifold Z with Q_Z nondiagonalizable (cf. Lemma 9.4.2). To do this, we shrink V by deleting a compact subset of $c\ell(V)$ and restricting to a single component of the result, so that $Bd(V)$ becomes a smooth, connected submanifold (with boundary in ∂X), and let $Y = X - V$ (so $Bd(V) \subset \partial Y$). If the topological manifold Y is noncompact, we smooth it by Theorem 9.4.22 to obtain a smoothing on X extending the given one on V. Then we take the connected sum with some L as in Addendum 9.4.4(a) to get the required smoothing of $X - \{p\}$. If Y is compact and the Kirby-Siebenmann obstruction $ks(Y)$ vanishes, then for some m we can smooth $Y\#mS^2 \times S^2$ ([**FQ**] §8.6). Using the fact that any two Casson handles have a common refinement contained in both of them, we can find a smoothing U of $\#mS^2 \times S^2 - \{\text{pt.}\}$ that embeds smoothly in both $Y\#mS^2 \times S^2$ and $K3\#(m-3)S^2 \times S^2$. (We used a similar construction with $\#mS^2 \times S^2$ and $K3\#(m-3)S^2 \times S^2$ both replaced by $\mathbb{CP}^2$ in the proof of Addendum 9.4.4.) Removing a suitable compact subset from both copies of U, we obtain smoothings of $Y - \{p\}$ (hence $X - \{p\}$) and Z with the same end, where Z is simply connected with $Q_Z \cong -2E_8$. Finally, if $ks(Y) \neq 0$, a trick involving the Poincaré homology sphere Σ produces a similar smoothing with $Q_Z \cong -3E_8$: The contractible topological manifold Δ with $\partial\Delta = \overline{\Sigma}$ embeds in $\mathbb{R}^4 \subset Y$ (Exercise 5.7.17(b)). Then $ks(Y - \text{int}\ \Delta) = 0$, so the previous argument applies to it. Smooth $\Delta - \{\text{pt.}\}$, obtaining the same end as a smoothing of the ($-E_8$-manifold)$-\{\text{pt.}\}$ (= ($-E_8$-plumbing)$\cup_\Sigma \Delta - \{\text{pt.}\}$) and fuse together the two components of the end of $Y - \{2 \text{ points}\}$ by deleting a line connecting them.

Now that we have the smoothing of $X - \{p\}$ agreeing with $c\ell(V)$ and Z at their ends, we can define the required smoothings Σ_t on X. Observe that a neighborhood of p in X is homeomorphic to D^4 (with p mapping to 0), and we can assume the homeomorphism is smooth on an open radial arc (cf. the text preceding Definition 9.4.9). By identifying $X - \{p\}$ with $X - D$ for all sufficiently small round balls D about p, we obtain a radial family of smooth structures on $X - \{p\}$. Extending the smooth radial arc to a properly embedded line $\ell \subset X - \{p\}$ whose other end lies in V, and identifying X with $X - (\ell \cup \{p\})$, we get a family of smooth structures Σ_t on X with the end of each Σ_t diffeomorphic at V to the end of $c\ell(V)\natural Z_t$ for a radial family $Z_t \subset Z$. (The end sum may depend on the choice of defining ray in V, but we can assume the orientations match as given.) Note that if $V \approx \mathbb{R} \times S^3$, the proof of Theorem 9.4.10 shows that no two of these smooth structures are isotopic. Our main task is now to suitably generalize this argument.

Suppose that $\varphi\colon (c\ell(V), \Sigma_t) \hookrightarrow (X, \Sigma_s)$ is a proper diffeomorphic embedding as specified in the theorem (with V shrunk to replace $V - K$). Inverting φ outside of a compact subset if necessary, we can assume $t > s$. We wish to construct an end-periodic manifold contradicting Taubes' Theorem. The end of $(c\ell(V), \Sigma_t)$, or equivalently of $c\ell(V)\natural Z_t$, is finitely covered by the corresponding end of $V^*\natural mZ_t$, where $V^* = \widetilde{V}$ or $\widetilde{V}\natural m\overline{Z}_t$ minus a compact set, the latter case only occurring when V is nonorientable (so Y is noncompact by hypothesis and $\mathrm{end}(Z_t) \approx \mathrm{end}(L_t)$ with $L \subset \mathbb{CP}^2$). Thus V^* smoothly embeds in $\#\ell\,\overline{\mathbb{CP}^2}$, where $\ell = n$ or $n+m$. Form the connected sum of $\#l\overline{\mathbb{CP}^2}$ with m copies of Z, choosing the balls in $Z_t - c\ell(Z_s)$ and $\#l\overline{\mathbb{CP}^2}$ so as to obtain an embedding $\natural mZ_s\natural V^* \hookrightarrow \#mZ\#l\overline{\mathbb{CP}^2}$ as in Exercise 9.4.8(a) (for our given choice of the end sum rays). To modify the latter negative definite manifold to make its end periodic, we first perform m surgeries on 0-spheres connecting it to a second copy of $\#l\overline{\mathbb{CP}^2}$, using balls in each $Z - c\ell(Z_t)$ to create an embedding $\natural mZ_t\natural V^*\#l\overline{\mathbb{CP}^2}\#(m-1)S^1 \times S^3 \hookrightarrow \#mZ\#2l\overline{\mathbb{CP}^2}\#(m-1)S^1 \times S^3$ (Figure 9.13). Using the topological $\mathbb{R} \times S^3$ structure of the end of Z, we can roll up the region between the ends of each Z_s and Z_t, and identify the two copies of $\#l\overline{\mathbb{CP}^2}$ with each other, to obtain a topological manifold Q_0 homeomorphic to $\#l\overline{\mathbb{CP}^2}\#mS^1 \times S^3$. This inherits a smooth structure everywhere except near the end of each Z_s, where the gluing map fails to be smooth. We can assume this nonsmooth region lies in the image U of $\natural mZ_s\natural V^*$ minus a compact set, where the two disagreeing smooth structures are the lifts of Σ_s and Σ_t from $c\ell(V)$. To transform Q_0 into a smooth manifold Q_1, we use the cover $\tilde{\varphi}$ of φ to change the gluing on U. Choose a proper continuous map $f\colon U \to \mathbb{R}$ so that f approaches ∞ at the lift of the end of $c\ell(V)$, and $-\infty$ at boundary points of U corresponding to $Bd(V)$. Fix $q, r \in \mathbb{R}$ so that $\tilde{\varphi}(f^{-1}[q,\infty)) \subset U$ and $q < r$, remove the sets $f^{-1}[r, \infty)$ and $U - \tilde{\varphi}(f^{-1}(q, \infty))$ from their respective copies of $\# l\overline{\mathbb{CP}^2}$ (upper and lower in Figure 9.13) and let Q_1 be the smooth manifold obtained from Q_0 by gluing by $\tilde{\varphi}$ instead of id on $f^{-1}(q, r)$. (For $\partial V \neq \emptyset$ this is well-defined since $\tilde{\varphi} = \mathrm{id}$ on ∂U.) For r sufficiently large, $\tilde{\varphi}$ is homotopic to the identity in $\tilde{\varphi}(q, \infty)$ on a neighborhood of $f^{-1}(r)$ by hypothesis, allowing us to construct a continuous degree-1 map $\psi\colon Q_0 \to Q_1$. Similarly, moving farther toward ∞ we can construct maps $\chi\colon Q_1 \to Q_0$ and $\psi' \simeq \psi\colon Q_0 \to Q_1$ that are homotopy inverses, so Q_1 is homotopy equivalent to Q_0. (In fact, by surgering out m topological 3-spheres parallel to $Bd(Z_s)$, invoking Freedman's Theorem 1.2.27 and using the fact that a topological 4-ball has a unique orientation-preserving flat topological embedding in any connected 4-manifold [**FQ**], we can now easily verify that Q_1 is homeomorphic to $\# l\overline{\mathbb{CP}^2} \# mS^1 \times S^3$.) Now surger out $m - 1$ circles (representing differences of generators of the obvious $S^1 \times S^3$-summands) from Q_1 to obtain a smooth manifold Q_2 homeomorphic to $\# l\overline{\mathbb{CP}^2} \# S^1 \times S^3$. Attaching half of the universal cover of Q_2 to $\# mZ \# l\overline{\mathbb{CP}^2}$ by the above gluing and surgery procedure, we obtain the required simply connected end-periodic manifold and contradiction.

Note that the construction allows us to create families as above whose stable isotopy classes differ from a given one by any element of $H^3(Y, \partial Y; \mathbb{Z}_2)$ (since Theorem 9.4.22 and the smoothing of $Y \# mS^2 \times S^2$ allow such freedom). Thus, we can realize any stable isotopy class on X extending the one we used on V, which is inherited from the original smoothing of V except in the case $ks(Y) \neq 0$ (when the trick with the Poincaré homology sphere changed the stable isotopy class on V). Now if $H^3(V, \partial V; \mathbb{Z}_2)$ is finite, we can shrink V so that the end of $c\ell(V)$ becomes connected and the image of $H^3(X, \partial X; \mathbb{Z}_2) \to H^3(V, \partial V; \mathbb{Z}_2)$ is 0 for Y compact and $\mathbb{Z}_2$ otherwise. (When the end of $c\ell(V)$ is connected, the finite group $H_3(V, \partial V; \mathbb{Z}_2)$ will be carried by a compact $K \subset c\ell(V)$ containing $Bd(V)$, with $Bd(K)$ in $c\ell(V)$ a connected 3-manifold. Any class $\alpha \in H^3(X, \partial X; \mathbb{Z}_2)$ that vanishes on $Bd(V)$ must also vanish on $Bd(K)$, so after we add a suitable class in $H^3(K, \partial K; \mathbb{Z}_2)$, it will vanish on $H_3(V, \partial V; \mathbb{Z}_2)$. Thus any such α restricts to $0 \in H^3(V - K, \partial(V - K); \mathbb{Z}_2)$.) Then for Y compact, $H^3(Y, \partial Y; \mathbb{Z}_2) \cong H^3(X, V \cup \partial X; \mathbb{Z}_2)$ maps onto $H^3(X, \partial X; \mathbb{Z}_2)$, so all stable isotopy classes on X are realized in this manner. If Y is noncompact, the image is an index-2 subgroup of $H^3(X, \partial X; \mathbb{Z}_2)$, but when V is orientable we can reach the remaining stable isotopy classes using the trick with the Poincaré homology sphere. □

Remark 9.4.28. If an open 4-manifold X embeds topologically in $\#n\mathbb{CP}^2$, then one can find uncountably many diffeomorphism types of smooth structures on X that all embed in $\#n\mathbb{CP}^2$. This follows as in Fang [**Fa**] by end summing $\overline{X}$ with R_t and proceeding as in Theorem 9.4.16(b).

Some open 4-manifolds not covered by Theorem 9.4.24 are still known to admit (at least countably) infinite families of distinct smooth structures:

Theorem 9.4.29. *Let X be an open, connected, orientable 4-manifold; X is necessarily homeomorphic to the interior of a (possibly infinite) handlebody.*

(a) *If there is such a handlebody with only finitely many 3-handles (or more generally, if X has a smooth structure exhausted by smooth, compact 4-manifolds W_i with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$) and if $H_2(X)$ has finite dimension with both $\mathbb{Z}_2$- and $\mathbb{Q}$-coefficients, then each stable isotopy class on X contains infinitely many diffeomorphism types of smooth structures.*

(b) *If there is such a handlebody without 3-handles (hence, without 4-handles) and $H_2(X; \mathbb{Z}) \neq 0$, then X admits infinitely many isotopy classes of smooth structures (realizing the unique stable isotopy class).* □

For example, (b) applies to $\mathbb{R}^4 \# \infty S^2 \times S^2$, whereas Theorem 9.4.24 does not. For an example about which nothing appears to be known, consider an infinite end sum of $\mathbb{R}^2$-bundles over $\mathbb{RP}^2$ (e.g. with orientable total spaces and odd Euler numbers). Part (a) is a special case of a theorem of Taylor [**Ta**] extending work of Bižaca and Etnyre [**BE**]. Part (b) is proved in Exercise 11.4.11(d). (In this case $H^3(X; \mathbb{Z}_2) = 0$, so there is only one stable isotopy class of smoothings.)

Exercise 9.4.30. Let X be a noncompact, connected topological 4-manifold with an embedding into $\#nS^2 \times S^2$ (n finite) whose restriction to ∂X is smooth. Prove that X admits infinitely many diffeomorphism types of smooth structures. (*Hint*: The embedding provides one smooth structure. Now sum with copies of L as in Addendum 9.4.4(b); cf. Corollary 9.4.7.)

Chapter 10

Symplectic 4-manifolds

In studying the differential topology of 4-manifolds, it turns out to be very useful to assume the existence of some extra structure on the manifolds under examination — cf. results about complex surfaces in Section 3.4. This chapter is devoted to the discussion of *symplectic manifolds*.

First we will describe the connection between almost-complex and symplectic geometry. We will also quote the most important results concerning Seiberg-Witten invariants of symplectic 4-manifolds (cf. Section 2.4 as well as Section 10.4); for the proofs of these statements see [**T4**] or [**KKM**]. In Section 10.2 several constructions of symplectic manifolds will be presented; Section 10.3 is devoted to constructing certain manifolds not admitting symplectic structures. We will also review the geography problem for symplectic and for irreducible 4-manifolds. Finally, in Section 10.4 the Seiberg-Witten equations will be briefly studied on symplectic 4-manifolds.

10.1. Symplectic and almost-complex manifolds

A 2-form η on $\mathbb{R}^n$ is *nondegenerate* if for every nonzero vector $v \in \mathbb{R}^n$ there exists $w \in \mathbb{R}^n$ such that $\eta(v, w) \neq 0$. The existence of a nondegenerate 2-form implies that n is even. A 2-form $\eta \in \Omega^2(X)$ on the smooth manifold X is nondegenerate if it is nondegenerate on each tangent space T_pX $(p \in X)$. It is easy to show that on the $2n$-manifold X the 2-form η is nondegenerate iff the $n^{\underline{th}}$ wedge power $\eta^n = \eta \wedge \ldots \wedge \eta \in \Omega^{2n}(X)$ is nowhere zero.

Definition 10.1.1. A 2-form ω on a smooth $2n$-dimensional manifold X is called a *symplectic structure* if it is nondegenerate and closed; the pair (X, ω) is called a *symplectic manifold*. A diffeomorphism $f\colon X_1 \to X_2$

between symplectic manifolds (X_1, ω_1) and (X_2, ω_2) is a *symplectomorphism* if $\omega_1 = f^*(\omega_2)$.

The nondegeneracy assumption assures that ω gives an isomorphism between TX and T^*X — exactly like a Riemannian metric does. A symplectic structure ω induces an orientation on X by the condition $\omega^n > 0$. The $2n$-dimensional Euclidean vector space $\mathbb{R}^{2n}$ admits a canonical symplectic structure: If $(x_1, y_1, \ldots, x_n, y_n)$ gives the standard coordinates on $\mathbb{R}^{2n}$ then $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is a symplectic form. This example, in fact, can be regarded as the "typical symplectic manifold" locally, since — by the Darboux Theorem — every symplectic $2n$-manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. We will omit the discussion of this type of result here (see, e.g., [**McS1**]), since our primary interest is the effect of the existence of a symplectic structure on the topology and differential topology of a 4-manifold X. We will mainly investigate the existence question for symplectic structures on 4-manifolds. By a theorem of Gromov, every open almost-complex $2n$-manifold admits a symplectic structure [**McS1**]; consequently we will concentrate on closed manifolds. Note that if $n = 1$ (so we are working with a 2-dimensional manifold), then a volume form of a closed surface Σ is a symplectic form; consequently an oriented, closed 2-manifold Σ always admits a symplectic structure. Moreover, if (X_i, ω_i) $(i = 1, 2)$ are symplectic manifolds, then the sum $\pi_1^*\omega_1 + \pi_2^*\omega_2$ gives a symplectic structure on $X_1 \times X_2$. In this way we get our first examples of symplectic 4-manifolds, namely the products $\Sigma_n \times \Sigma_m$ of 2-dimensional manifolds Σ_n and Σ_m. Generalizations of this observation will be given in Section 10.2; we will see that most Lefschetz fibrations also admit symplectic structures.

Before turning to the discussion of the topology of symplectic 4-manifolds, we describe the relation between symplectic and almost-complex structures on a 4-manifold X. The structure group of the tangent bundle $TX \to X$ of an oriented, smooth 4-manifold X is isomorphic to $GL^+(4; \mathbb{R}) = \{A \mid A \text{ is a } 4 \times 4 \text{ real matrix with } \det(A) > 0\}$. The introduction of an almost-complex structure J (see Definition 1.4.14) reduces the structure group $GL^+(4; \mathbb{R})$ to $GL(2; \mathbb{C})$. A Riemannian metric g reduces $GL^+(4; \mathbb{R})$ to $SO(4)$; finally, the introduction of a nondegenerate 2-form ω (representing the orientation) reduces the structure group $GL^+(4; \mathbb{R})$ to the Lie group $Sp(4) = \{A \in GL(4; \mathbb{R}) \mid \omega_0(x, y) = \omega_0(Ax, Ay) \text{ for all } x, y \in \mathbb{R}^4\}$. (Recall that $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ is the standard symplectic form on $\mathbb{R}^4$.) It turns out that any two of the above three structures J, g and ω determine the third if the given two are compatible.

Definition 10.1.2. An almost-complex structure J is *compatible* with the Riemannian metric g if J is an orthogonal map fiberwise. A 2-form ω is *compatible* with g if ω is a self-dual 2-form with respect to g (as defined in

Section 2.4) and $||\omega|| = \sqrt{2}$. A 2-form ω and an almost-complex structure J are *compatible* if $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ and $\omega(v, Jv) > 0$ for all nonzero v in TX. (In this case we also say that J *calibrates* ω. If we only assume the latter condition (that $\omega(v, Jv) > 0$ for all $v \neq 0$), we say that ω *tames* J.)

A 2-form ω that is compatible with some g or J is automatically nondegenerate and respects the given orientation. It is not hard to see that in $GL(4;\mathbb{R})$ we have

$$SO(4) \cap Sp(4) = SO(4) \cap GL(2;\mathbb{C}) = Sp(4) \cap GL(2;\mathbb{C}) = U(2),$$

and this implies that once we fix two compatible structures from among g, J and ω, the third one is automatically defined. For a given J and g, the corresponding 2-form ω is given by the formula $\omega(x, y) = g(Jx, y)$. If compatible structures ω and J are fixed, then g is determined by the formula $g(x, y) = \omega(x, Jy)$. Finally, for fixed (compatible) g and ω we get J by the definition $J\tilde{x} = -*(\omega \wedge \tilde{x})$. (The symbol $*$ denotes the Hodge star-operator induced by g (cf. Section 2.4) and $\tilde{x}$ denotes the cotangent vector corresponding to $x \in TX$ via the identification $TX \cong T^*X$ given by g.) All the above statements are pointwise, so can be checked in $\mathbb{R}^4$; we did not assume anything about the global behavior of J, g or ω on X. In particular, the 2-form ω determined by the compatible structures g and J is nondegenerate, but not necessarily closed. Recall from Section 1.4 that the existence of an almost-complex structure on X depends only on homological properties of X. Here is another way to see it: Fix a metric g; by the above considerations, g-compatible structures J and ω determine each other. Hence J exists iff the bundle $\Lambda^+ \to X$ of self-dual 2-forms has a section with constant length $\sqrt{2}$. Obviously such a section exists iff a trivial real line bundle $\underline{\mathbb{R}}$ can be split off of Λ^+.

Exercises 10.1.3. **(a)** The (real) 3-dimensional bundle Λ^+ splits as the sum $\Lambda^+ = \underline{\mathbb{R}} \oplus K$ for some complex line bundle K iff there exists h in $H^2(X;\mathbb{Z})$ such that $h \equiv w_2(X) \pmod 2$ and $h^2 = 3\sigma(X) + 2\chi(X)$. (*Hint*: Prove that $w_2(\Lambda^+) = w_2(X)$ and $\langle p_1(\Lambda^+), [X]\rangle = 3\sigma(X) + 2\chi(X)$. Taking $c_1(K) = h$ and applying Theorem 1.4.20(c) yields the solution. Compare with Theorem 1.4.15 and Exercise 1.4.21(c).)

(b) Prove that for any finitely presented group G there exists an almost-complex closed 4-manifold (X, J) such that $\pi_1(X) \cong G$. (*Hint*: Take a smooth (closed) 4-manifold Y with $\pi_1(Y) \cong G$ (cf. Exercises 4.6.4(b) and 5.2.2(c)). Taking the connected sum of Y with $\mathbb{CP}^2$ if necessary, we can assume that $\chi(Y) + \sigma(Y) \equiv 0 \pmod 4$. Now using (a) show that $Y \# 4\mathbb{CP}^2$ or $Y \# 4\overline{\mathbb{CP}^2}$ admits an almost-complex structure — use the fact that a positive integer congruent to 4 mod 8 can be written as the sum of four odd squares. For the complete argument see [**Ko1**].)

A complex structure on X^4 naturally induces an almost-complex structure $J\colon TX \to TX$ by the formula $Jz = iz$. An almost-complex structure is called *integrable* if it arises in this manner from a complex structure. If ω, J and g are compatible structures on X, J is integrable and $d\omega = 0$, then (X, ω, J, g) is a *Kähler surface*. If one only assumes the latter condition $d\omega = 0$, then (X, ω, J, g) is called *almost-Kähler*. A complex surface S (with the induced almost-complex structure J) is called *Kähler* if it admits a metric g such that (S, ω, J, g) is a Kähler surface. Since a complex submanifold of a Kähler manifold is Kähler, and the projective space $\mathbb{CP}^n$ admits a Kähler metric (e.g., the Fubini-Study metric [**GH**]), all projective manifolds are Kähler. Consequently, our examples S_d in Section 1.3 (or more generally, the complete intersection surfaces defined in the same section) are all Kähler. We have already mentioned that a complex surface with even first Betti number b_1 is deformation equivalent to a projective surface (Theorem 3.4.32); hence if $b_1(S)$ is even, then S is at least diffeomorphic to a Kähler surface. More is true, however:

Theorem 10.1.4. ([**BPV**]) *A (closed) complex surface S is Kähler iff the first Betti number $b_1(S)$ is even.* □

Remark 10.1.5. It is a standard fact that the odd-degree Betti numbers (so in particular b_1) of a Kähler manifold are even. The surprising fact in Theorem 10.1.4 is that for surfaces the converse also holds. For more about the topology of Kähler manifolds see [**ABCKT**].

By Theorem 10.1.4, a simply connected complex surface is Kähler, hence can be equipped with a symplectic structure (provided by the complex structure and the Kähler metric). We will see later that the converse of this statement does not hold; examples of symplectic non-Kähler manifolds will be given in Section 10.2. On the other hand, every symplectic manifold admits almost-Kähler structures:

Proposition 10.1.6. *Any symplectic manifold (X, ω) admits a compatible almost-complex structure J and (hence) a Riemannian metric g such that (X, ω, J, g) is almost-Kähler. The space of compatible almost-complex structures is contractible.*

Proof. The proof rests on the fact that the space of almost-complex structures on $\mathbb{R}^4$ compatible with the fixed form ω_0 is nonempty and contractible (see e.g. [**McS1**]). It follows that the compatible almost-complex structures on (X, ω) are sections of a bundle with contractible fibers, so they form a nonempty, contractible space. □

Remark 10.1.7. The existence of an almost-Kähler structure on a symplectic manifold (X, ω) also follows from the fact that the obvious embedding

$U(n) \subset Sp(2n)$ is a homotopy equivalence, hence the structure group of TX (which is $Sp(2n)$ for a symplectic manifold) can be reduced to $U(n)$. Note that the space of *all* almost-complex structures on X is not connected — for example, varying $c_1(X, J) \in H^2(X; \mathbb{Z})$ results in almost-complex structures realizing different components. On the other hand, the space of ω-compatible almost-complex structures (where ω is a fixed symplectic form on X) is contractible.

Since there is a compatible almost-complex structure J on (X, ω), we may define the Chern classes $c_i(X, \omega, J) \in H^{2i}(X; \mathbb{Z})$. Because the space of compatible J's is connected (in fact contractible), these cohomology classes do not depend on J, hence $c_i(X, \omega) \in H^{2i}(X; \mathbb{Z})$ is well-defined. In the same way, one can define the canonical bundle K of the symplectic manifold (X, ω) — choose a compatible almost-complex structure J and take $K = \Lambda^2_J T^*X$; obviously $c_1(K) = -c_1(X, \omega)$.

Let J be a fixed almost-complex structure on the 4-manifold X. A (real 2-dimensional) submanifold $\Sigma \subset X$ is a *pseudo-holomorphic submanifold* (or *pseudo-holomorphic curve*) if J maps $T\Sigma \subset TX$ into itself. Since the adjunction formula (Theorem 1.4.17) holds for almost-complex manifolds as well, the Euler characteristic of a (closed) pseudo-holomorphic submanifold Σ is determined by its homology class: $-\chi(\Sigma) = [\Sigma]^2 - c_1(X, \omega)[\Sigma]$. This formula determines the genus of Σ provided that Σ is connected. A 2-dimensional submanifold $\Sigma \subset X$ is a *symplectic submanifold* if $(\Sigma, \omega|\Sigma)$ is a symplectic manifold. Note that if Σ is a (closed) symplectic submanifold, then (since $\omega|\Sigma$ is a volume form, so $\langle [\omega], [\Sigma] \rangle > 0$) the homology class $[\Sigma]$ is not zero in $H_2(X; \mathbb{Z})$, or even in $H_2(X; \mathbb{R})$. If (X, ω, J, g) is an almost-Kähler manifold, then a pseudo-holomorphic submanifold Σ is always a symplectic submanifold; moreover an embedded surface $\Sigma \subset X$ is a symplectic submanifold iff there is a compatible almost-complex structure J such that $\Sigma \subset (X, J)$ is pseudo-holomorphic. A 2-dimensional submanifold $\Sigma \subset X$ is *Lagrangian* if $\omega|\Sigma = 0$. The normal bundle of a Lagrangian submanifold is always isomorphic to its cotangent bundle, so if Σ is closed and oriented it satisfies $-\chi(\Sigma) = [\Sigma]^2$.

Exercise 10.1.8. Prove that a transverse intersection of pseudo-holomorphic curves in (X, J) is always positive (cf. Section 1.2). Find a pair of symplectic 2-planes in $(\mathbb{R}^4, \omega_0)$ that intersect negatively. (Note, however, that if a transverse intersection of two symplectic submanifolds is *orthogonal* (with respect to ω), then it is positive.)

Remark 10.1.9. Applying more delicate arguments, one can show that (as in the complex case) *any* intersection of pseudo-holomorphic curves is positive, i.e., the above transversality hypothesis is unnecessary [**Mc2**]. As the second statement of Exercise 10.1.8 asserts, symplectic submanifolds

$\Sigma_1, \Sigma_2 \subset X$ might intersect negatively. Hence — although there exists a compatible almost-complex structure J_i for which $\Sigma_i \subset X$ is a pseudo-holomorphic curve ($i = 1, 2$) — there might be no almost-complex structure on X for which *both* Σ_1 and Σ_2 are pseudo-holomorphic submanifolds.

Now we turn to studying the topology of a closed symplectic 4-manifold. As a consequence of Proposition 10.1.6, we find our first restriction (cf. Theorem 1.4.13):

Corollary 10.1.10. *Suppose that (X, ω) is a (closed) symplectic 4-manifold. Then $1 - b_1(X) + b_2^+(X)$ is even. In particular, a simply connected symplectic 4-manifold has odd $b_2^+(X)$.* □

Note that if (X, ω) is symplectic, then ω is a nondegenerate 2-form, so $[\omega] \in H^2(X; \mathbb{R})$ is nonzero ($[\omega] \wedge [\omega] = [\omega \wedge \omega] \neq 0$), and in particular, $b_2(X) \neq 0$. Moreover, since $\omega \wedge \omega$ is a volume form (so $\langle \omega \wedge \omega, [X] \rangle > 0$), we see that $b_2^+(X) > 0$ without assuming $\pi_1(X) = 1$. (In the above argument the condition $d\omega = 0$ is used in the implicit statement that $[\omega] \in H^2(X; \mathbb{R})$ is defined.) The existence of a nondegenerate 2-form by itself does not imply $b_2(X) > 0$; for example $S^1 \times S^3$ can be equipped with a complex structure (cf. Section 3.4), but the choice of a compatible Riemannian metric on $S^1 \times S^3$ defines only a nondegenerate 2-form ω which cannot satisfy the condition $d\omega = 0$.

The main breakthrough in the study of the topology of symplectic 4-manifolds came with the theorems of Taubes regarding Seiberg-Witten invariants of symplectic 4-manifolds.

Theorem 10.1.11. (Taubes, [**T2**]) *If (X, ω, J, g) is an almost-Kähler manifold with $b_2^+(X) > 1$, then the classes $\pm c_1(X, \omega)$ are Seiberg-Witten basic classes; moreover, $SW_X(\pm c_1(X, \omega)) = \pm 1$. In addition, for every Seiberg-Witten basic class K we have $|K \cdot [\omega]| \leq |c_1(X, \omega) \cdot [\omega]|$, and equality holds iff $K = \pm c_1(X, \omega)$.* □

Remark 10.1.12. Theorem 10.1.11 is proved by taking an appropriate perturbation of the Seiberg-Witten equations (see Sections 2.4 and 10.4). Taubes proved that the moduli space $\mathcal{M}^{\mu}_{c_1(X)}(g)$ consists of one (smooth) point if the perturbation form μ is equal to $r\omega$ for $r \in \mathbb{R}$ large enough; this implies that $SW_X(c_1(X, \omega)) = \pm 1$. The proof of the second part of Theorem 10.1.11 is based on a similar perturbation argument. Analogous statements hold in the $b_2^+(X) = 1$ case as well [**T2**].

Theorem 10.1.11 is frequently used to prove that certain manifolds do *not* admit symplectic structures.

Proposition 10.1.13. *The 4-manifold $\#3\mathbb{CP}^2$ admits no symplectic structure.*

Proof. By the vanishing theorem (Theorem 2.4.6) the 4-manifold $\#3\mathbb{CP}^2$ has vanishing Seiberg-Witten invariants, so there is no class $K \in H^2(\#3\mathbb{CP}^2;\mathbb{Z})$ which could play the role of $c_1(X,\omega)$ for a symplectic structure ω. □

Note that on the other hand, $\#3\mathbb{CP}^2$ does admit almost-complex structures (cf. Exercise 2.4.12); by Proposition 10.1.13 the corresponding forms ω fail to be closed. Of course, the argument in the proof of Proposition 10.1.13 can be adapted to prove the following more general statement:

Theorem 10.1.14. *If the symplectic 4-manifold (X,ω) decomposes as a connected sum $X = X_1\# X_2$ with $b_2^+(X_1) > 0$, then X_2 has a negative definite intersection form.* □

Our aim is to describe the topology of symplectic 4-manifolds; hence we would like to understand the above connected sum decompositions with $b_2^+(X_2) = 0$ as well. It turns out that symplectic manifolds can always be symplectically blown up: If (X,ω) is a symplectic manifold, then $X' = X\#\overline{\mathbb{CP}^2}$ carries a symplectic structure ω', so the blow-up process can be generalized to the symplectic setting [**Mc1**]. Following the algebraic geometric analogy, one can define *minimal* symplectic manifolds: We will say that a symplectic 4-manifold (X,ω) is *minimal* if there is no embedded symplectic sphere in X with square -1. As we will see in the next section (Theorem 10.2.3), a symplectic -1-sphere can be symplectically blown down, so any symplectic 4-manifold has a (possibly nonunique) minimal model — just as in the complex geometric case. To describe the possible connected sum decompositions of minimal symplectic manifolds, we will use the remarkable results of Taubes [**T3**], [**T4**] relating Seiberg-Witten invariants and pseudo-holomorphic submanifolds of an almost-Kähler 4-manifold (X,ω,J,g).

Theorem 10.1.15. (Taubes, [**T3**], [**T4**]; see also [**Ko2**]) *Suppose that (X,ω) is a symplectic manifold with $b_2^+(X) > 1$ and $SW_X(K) \neq 0$ for a given $K \in \mathcal{C}_X$. Assume furthermore that the class $c = \frac{1}{2}(K - c_1(X,\omega))$ is nonzero in $H^2(X;\mathbb{Z})$. Then for a generic compatible almost-complex structure J on X, the class $PD(c) \in H_2(X;\mathbb{Z})$ can be represented by a pseudo-holomorphic submanifold.* □

Remarks 10.1.16. (a) In fact, Taubes proved much more. By defining a rather delicate way of counting pseudo-holomorphic submanifolds representing a fixed homology class $PD(c) \in H_2(X;\mathbb{Z})$, he proved that this number and $SW_X(c_1(X,\omega)+2c)$ are equal. In our applications, however, we will only use the direction that a nonvanishing Seiberg-Witten invariant implies the existence of pseudo-holomorphic curves. Note that the curve Σ representing $PD(c)$ is not given to be connected. This observation becomes important if one wants to apply the adjunction formula to compute the genus of Σ.

(b) Since Theorem 10.1.11 shows that $-c_1(X, \omega) \in \mathcal{B}as_X$, Theorem 10.1.15 implies, in particular, that the Poincaré dual of $-c_1(X, \omega)$ can be represented by a pseudo-holomorphic submanifold (assuming it is nonzero). Since a pseudo-holomorphic submanifold is always symplectic, the above reasoning shows that $-c_1(X, \omega) \cdot [\omega] > 0$ for manifolds with $b_2^+(X) > 1$ and $c_1(X, \omega)$ nonzero. Furthermore, it can be shown that if $b_2^+(X) > 1$, then a class e with $e^2 = -1$, $c_1(X, \omega) \cdot PD(e) = 1$ and $SW_X(c_1(X, \omega) + 2PD(e)) \neq 0$ can be represented by a symplectic sphere; consequently X is nonminimal. (The fact that $c_1(X, \omega) + 2PD(e) \in \mathcal{B}as_X$ guarantees the existence of a pseudo-holomorphic representative for e. The two other assumptions — together with the adjunction formula — ensure that this representative is a sphere.) As a further application of Theorem 10.1.15, one can show that a symplectic 4-manifold with $b_2^+ > 1$ has Seiberg-Witten simple type, cf. [**Ko2**] and Exercise 10.1.20(d).

(c) Theorem 10.1.15 also proves the inequality in Theorem 10.1.11: If K is a basic class, then $c = \frac{1}{2}(K - c_1(K, \omega))$ can be represented by a pseudo-holomorphic (in particular symplectic) submanifold (unless $c = 0$), hence $c \cdot [\omega] \geq 0$. Reversing the sign of K if necessary, we can assume $K \cdot [\omega] \leq 0$, so that $c_1(X, \omega) \cdot [\omega] \leq K \cdot [\omega] \leq 0$, which proves the inequality. Note that equality implies $c \cdot [\omega] = 0$, hence $c = 0$, and consequently, $K = c_1(X, \omega)$ (or $K = -c_1(X, \omega)$).

(d) Above we only dealt with the case of $b_2^+(X) > 1$; recall that for manifolds with $b_2^+(X) = 1$ the Seiberg-Witten invariants depend on the chosen metric and perturbation. After the appropriate modifications, the theorems and properties discussed above extend to the case of $b_2^+(X) = 1$. For the sake of brevity, however, we will omit the discussion of these extensions and advise the reader to turn to [**Sa**], [**McS2**] or [**Liu**].

Before listing further consequences of Theorem 10.1.15, we give the definition of *irreducibility*; this notion will be examined more thoroughly in Section 10.3.

Definition 10.1.17. A smooth 4-manifold X is *irreducible* if for every smooth connected sum decomposition $X = X_1 \# X_2$, either X_1 or X_2 is homeomorphic to S^4.

Combining Theorem 10.1.15 with Theorems 1.2.30 and 1.2.27, one can deduce the following theorem (see also [**Ko2**], [**Ko3**] and Exercises 10.1.20(d) and (e)):

Theorem 10.1.18. *Suppose that (X, ω) is a minimal symplectic 4-manifold with $b_2^+(X) > 1$. If X decomposes as $X = X_1 \# N$ with $b_2^+(N) = 0$, then N is an integral homology 4-sphere. In particular, if X is simply connected, then so is N; hence it is homeomorphic to S^4. Consequently, a simply connected,* *minimal, symplectic 4-manifold with* $b_2^+ > 1$ *is irreducible. Furthermore, a minimal symplectic 4-manifold* X *with* $b_2^+(X) > 1$ *satisfies* $c_1^2(X, \omega) \geq 0$. □

Moreover, by combining Taubes' result 10.1.15 and a theorem of McDuff [**Mc3**] (asserting that a minimal symplectic 4-manifold containing a symplectic sphere of nonnegative square is either $\mathbb{CP}^2$ or a $\mathbb{CP}^1$-bundle over a Riemann surface), Liu and Li proved the following:

Theorem 10.1.19. ([**Liu**], [**LiL**]) *If* X *is a minimal symplectic 4-manifold with* $c_1^2(X) < 0$*, then* X *is a* $\mathbb{CP}^1$*-bundle over a Riemann surface (of genus* > 1*), i.e.,* X *is diffeomorphic to an irrational ruled surface. In particular, if the minimal symplectic 4-manifold* (X, ω) *has* $c_1^2(X, \omega) < 0$*, then* $b_2^+(X) = 1$*. Moreover, the symplectic structure on such a ruled surface is unique up to diffeomorphism and symplectic deformation.* □

Exercises 10.1.20. (a)* Show that if Σ is an embedded sphere with $[\Sigma]^2 = -1$ in the symplectic 4-manifold (X, ω) (with $b_2^+(X) > 1$), then for $e = [\Sigma]$ or $-[\Sigma]$ we have $c_1(X, \omega) \cdot PD(e) = 1$ and $SW_X(c_1(X, \omega) + 2PD(e)) \neq 0$. Note that these facts (together with the argument outlined in Remark 10.1.16(b) above) show that the homology class of a smooth -1-sphere in a 4-manifold X with $b_2^+(X) > 1$ can always be represented by a pseudo-holomorphic sphere.

(b)* Show that a symplectic 4-manifold (X, ω) with $b_2^+(X) > 1$ is minimal iff there is no basic class $K \in \mathcal{B}as_X$ with $(c_1(X, \omega) - K)^2 = -4$.

(c)* Prove that if $b_2^+(X) > 1$ and (X, ω) is a minimal symplectic manifold, then $c_1^2(X, \omega) \geq 0$.

(d) Show that a symplectic 4-manifold (X, ω) with $b_2^+(X) > 1$ has Seiberg-Witten simple type.

(e)* Suppose that a simply connected, minimal symplectic 4-manifold (X, ω) with $b_2^+(X) > 1$ decomposes as $X = X_1 \# N$. Show that if $b_2^+(X_1) > 0$, then N is homeomorphic to S^4.

10.2. Constructions of symplectic manifolds

The previous section presented obstructions to putting a symplectic structure on a 4-manifold (cf. Corollary 10.1.10 and Theorems 10.1.11, 10.1.14). In the following, we would like to describe ways to construct examples of symplectic 4-manifolds. We will also address the geography question for symplectic manifolds (cf. the same question for complex surfaces in Sections 3.4 and 7.4). We will start by describing the normal connected sum operation — a generalization of the fiber sum operation given in Section 7.1. As we will see (Theorem 10.2.1), the normal connected sum operation is well suited for symplectic manifolds. After discussing the geography question, we will turn to examining the relation between symplectic structures and Lefschetz fibrations.

Suppose that X_1, X_2 are symplectic 4-manifolds and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected symplectic submanifolds in them. Suppose furthermore that $[F_1]^2 + [F_2]^2 = 0$ and that the genera of F_1 and F_2 are equal. Take an orientation-preserving diffeomorphism $\psi\colon F_1 \to F_2$ and lift it to an orientation-reversing diffeomorphism $\Psi\colon \partial\nu F_1 \to \partial\nu F_2$ between the boundaries of the tubular neighborhoods νF_i. Using Ψ we can apply Definition 1.3.2 and construct $X_\Psi = (X_1 - \nu F_1) \cup_\Psi (X_2 - \nu F_2)$. The manifold X_Ψ is called the *(symplectic) normal connected sum* of X_1 and X_2 along F_1 and F_2, and is also denoted by $X_1 \#_\Psi X_2$ or $X_1 \#_F X_2$.

Theorem 10.2.1. ([**G12**]) *Under the above circumstances, the 4-manifold $X_\Psi = X_1 \#_\Psi X_2$ admits a symplectic structure. Moreover, this structure can be chosen in such a way that the symplectic (Lagrangian) submanifolds of $X_i - \nu F_i$ are symplectic (Lagrangian) in X_Ψ.* □

Note that the diffeomorphism type of X_Ψ might depend on the particular choice of the diffeomorphism Ψ — in fact, there are examples of such dependence. If $F \subset X$ is a Lagrangian submanifold in a symplectic manifold (X, ω) and $[F] \neq 0 \in H_2(X; \mathbb{R})$, then F can be made into a symplectic submanifold by a small global perturbation of ω, see [**G12**]. Hence, the above theorem can be extended to the case when the submanifolds $F_i \subset X_i$ are Lagrangian (with the caveat that Lagrangian submanifolds of $X_i - \nu F_i$ may become symplectic in X_Ψ). Note that the same operation can be performed in the smooth category without assuming that the manifolds F_i or even X_i are symplectic. Needless to say, in this latter case the normal connected sum X_Ψ is not necessarily symplectic.

Example 10.2.2. The fiber sum operation (introduced in Chapter 3) is a special case of the normal connected sum — the case when $g(F_i) = 1$ and $[F_i]^2 = 0$. More generally, the generalized fiber sum described in Definition 7.1.11 is also a special case of the normal connected sum operation — when we only assume that $[F_i]^2 = 0$ $(i = 1, 2)$ and have no restriction on the genus $g(F_i)$ of F_i.

Next we will give applications of the above construction. First we show that a symplectic -1-sphere can always be symplectically blown down.

Theorem 10.2.3. *If (X, ω) contains a symplectic sphere Σ with square -1, then X is symplectomorphic to the blow-up of a symplectic manifold Y.*

Proof. Recall that as a smooth 4-manifold, X decomposes as $Y \# \overline{\mathbb{CP}^2}$ and Σ becomes the exceptional sphere $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ (see Proposition 2.2.11). By assumption Σ is symplectic; hence, if we take the symplectic normal connected sum of (X, Σ) with $(\mathbb{CP}^2, H)$ (where $H = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_2 = 0\}$), i.e., replace $\nu\Sigma$ with $\mathbb{CP}^2 - \operatorname{int} \nu H \approx D^4$, Theorem 10.2.1 provides the desired symplectic structure on Y. □

Remarks 10.2.4. (a) Assuming $b_2^+(X) > 1$, Remark 10.1.16(a) and the solution of Exercise 10.1.20(a) show that the homology class of a smoothly embedded sphere $\Sigma \subset X$ with $[\Sigma]^2 = -1$ can be represented by a pseudo-holomorphic, hence symplectic, -1-sphere. Consequently, a symplectic 4-manifold X (with $b_2^+(X) > 1$) is minimal iff it does not contain any *smoothly* embedded sphere with self-intersection -1. Recall that every symplectic 4-manifold admits a (possibly nonunique) minimal model. As a further consequence of Taubes' Theorem 10.1.15, Li [**Li1**] showed that the minimal model is unique unless X is the blow-up of $\mathbb{CP}^2$ or a ruled surface — in analogy with complex geometry.

(b) Following the analogous notion in complex geometry, the *Kodaira dimension* $\kappa(X)$ of a minimal symplectic 4-manifold X can be defined in the following way [**McS2**]. We say that $\kappa(X) = -\infty$ if $c_1(X,\omega)[\omega] > 0$, $\kappa(X) = 0$ if $c_1(X,\omega)[\omega] = 0$, $\kappa(X) = 1$ if $c_1(X,\omega)[\omega] < 0$ and $c_1^2(X,\omega) = 0$, and finally $\kappa(X) = 2$ if $c_1(X,\omega)[\omega] < 0$ and $c_1^2(X,\omega) > 0$. (Note that according to Theorem 10.1.19 [**Liu**] the condition $c_1(X,\omega)[\omega] < 0$ implies $c_1^2(X,\omega) \geq 0$ for a minimal symplectic 4-manifold X.) For a general (nonminimal) symplectic 4-manifold (X,ω) we define $\kappa(X)$ to be the Kodaira dimension of its minimal model. It can be shown that a minimal symplectic 4-manifold X with $\kappa(X) = -\infty$ is diffeomorphic to a Kähler surface [**Liu**]. For a discussion about symplectic 4-manifolds with $\kappa = 0$ see [**McS2**]. Note that the manifolds $K_{p,q}$ of Corollary 10.2.8 and the examples in Theorem 10.2.14 are symplectic manifolds with Kodaira dimensions 1 and 2 admitting no complex structure.

Exercises 10.2.5. (a)* Show that if X is a minimal symplectic 4-manifold with $\kappa(X) = 0$ and $b_2^+(X) > 1$, then $\mathcal{B}as_X = \{0\}$. Conclude that if X is minimal with $\kappa(X) = 0$ and $b_2^+(X) > 1$, then X is spin. (Note that if we drop the assumption on b_2^+, the above statements become false, as the Enriques surface $E(1)_{2,2}$ shows.)

(b) Show that if X is a minimal symplectic 4-manifold with $\kappa(X) = 0$ and $b_2^+(X) = 1$, then $c_1(X,\omega)$ is a torsion class. (*Hint*: Note that $\kappa(X) = 0$ implies $c_1^2(X,\omega) \geq 0$ (cf. Theorem 10.1.19); now represent $c_1(X,\omega)$ and ω by harmonic 2-forms, and from the conditions $b_2^+(X) = 1$ and $c_1(X,\omega)[\omega] = 0$ deduce that $c_1(X,\omega)$ is 0 in $H^2(X;\mathbb{R})$.) In fact, it can be shown [**McS2**] that $2c_1(X,\omega) = 0$. (Recall that the Enriques surface has Kodaira dimension 0 and nonzero first Chern class.)

(c) Find a way to symplectically "blow down" a symplectic -4-sphere. (*Hint*: Consider a quadric curve in $\mathbb{CP}^2$.) There is no holomorphic analog of this construction. Recall that a neighborhood of a -4-sphere is a copy of C_2 of Section 8.5.

As we showed in Theorem 8.5.9, a logarithmic transformation of multiplicity 2 (and appropriate gluing data, cf. Exercise 8.3.16(a)) near a fishtail fiber Σ can be reformulated as an ordinary blow-up and the rational blow-down of the resulting -4-sphere $\widetilde{\Sigma} = \Sigma - 2e$. The solution of Exercise 10.2.5(c) shows that such a logarithmic transformation of multiplicity 2 is, in fact, a symplectic operation when the -4-sphere is a symplectic submanifold. This observation can be generalized to arbitrary $p \geq 2$, using the following:

Theorem 10.2.6. ([**Sy**]) *Assume that (X, ω) is a symplectic 4-manifold. Suppose that for $i = 1, \dots, p-1$ there are embedded spheres Σ_i in X intersecting each other according to the plumbing diagram given by Figure 8.39. If these spheres are symplectic and intersect positively, then the rational blow-down $X_{(p)}$ of X along the spheres Σ_i admits a symplectic structure.* □

Combining this with Theorem 8.5.9, we obtain:

Corollary 10.2.7. *Consider a multiplicity p (≥ 2) logarithmic transformation performed along a torus lying in a symplectic fishtail neighborhood (with the auxiliary gluing data described in Exercise 8.3.16(a)). Then the resulting 4-manifold admits a symplectic structure.* □

Examples of closed symplectic manifolds originally emerged from complex geometry; as we saw, Kähler surfaces are all symplectic 4-manifolds. The first examples of simply connected, symplectic but non-Kähler (hence noncomplex) 4-manifolds were constructed in [**G12**], by using the symplectic normal connected sum operation of Theorem 10.2.1. One of these families, which we now describe, can be regarded as a reformulation of the original construction of irreducible, noncomplex manifolds [**GM**]. Take the simply connected elliptic surfaces $E(1)_p$ and $E(1)_q$ with $p, q > 1$ (cf. Section 3.3). Specify generic fibers $F_1 \subset E(1)_p$ and $F_2 \subset E(1)_q$; taking the fiber sum along these tori gives the elliptic surface $E(2)_{p,q}$. On the other hand, if we choose the gluing map $\Psi \colon \partial(E(1)_p - \nu F_1) \to \partial(E(1)_q - \nu F_2)$ carefully, we get a symplectic non-Kähler manifold: Start with the identifications of the boundaries $\partial(E(1)_p - \nu F_1)$ and $\partial(E(1)_q - \nu F_2)$ with $S^1 \times S^1 \times \pm\partial D^2$ provided by the elliptic fibrations. The fiber sum operation corresponds to an identification $\Phi \colon \partial(E(1)_p - \nu F_1) \to \partial(E(1)_q - \nu F_2)$ generating the identity matrix on $H_1(S^1 \times S^1 \times S^1; \mathbb{Z})$. If $\Psi \colon \partial(E(1)_p - \nu F_1) \to \partial(E(1)_q - \nu F_2)$ generates $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ on $H_1(T^3; \mathbb{Z})$ (corresponding to a different lift of the given diffeomorphism $F_1 \to F_2$ to the normal circle bundles), then the resulting 4-

manifold $K_{p,q}$ is still symplectic (by Theorem 10.2.1), and by the Seifert-Van Kampen theorem it is simply connected. The Seiberg-Witten basic classes of $K_{p,q}$ are computed in [**FS2**], and we have that $\mathcal{B}as_{K_{p,q}}$ is equal to the set

$$\left\{ k_1 \frac{[F_1]}{p} + k_2 \frac{[F_2]}{q} \right\},$$

where $k_1 \equiv p-1 \pmod 2$, $k_2 \equiv q-1 \pmod 2$, $|k_1| \leq p-1$ and $|k_2| \leq q-1$. (Note that $[F_1] \neq [F_2]$ in $H_2(K_{p,q};\mathbb{Z})$.) It can be proved that $K_{p,q}$ is homeomorphic to $E(2)$ if pq is odd and to $\#3\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$ if pq is even. Comparing the set $\mathcal{B}as_{K_{p,q}}$ to the set of basic classes of complex surfaces (described in Section 3.3), one can easily deduce:

Corollary 10.2.8. *For $p, q > 1$ the manifold $K_{p,q}$ is a minimal symplectic 4-manifold which is not diffeomorphic to a complex surface.* □

Exercise 10.2.9. Prove that $K_{p,q}$ is minimal. (*Hint*: Since $b_2^+(K_{p,q}) = 3$, Exercise 10.1.20(b) applies. Since $[F_1]^2 = [F_2]^2 = [F_1] \cdot [F_2] = 0$, the above description of $\mathcal{B}as_{K_{p,q}}$ yields the solution.)

The above construction can be generalized to other elliptic surfaces and more logarithmic transformations. Many similar examples have been found for the same phenomenon (see [**FS2**], [**Sz1**], [**S2**]); in most of these cases the proof (that the manifold is noncomplex) involves the computation of gauge-theoretic invariants. (To avoid gauge theory, one can construct spin examples with $c_1^2 \neq 0$ violating the Noether inequality [**G12**]; these will not even be homotopy equivalent to complex surfaces.) One has to be careful in saying that a 4-manifold does not carry a complex structure. By reversing orientations it is easy to find 4-manifolds which are not orientation-preserving diffeomorphic to any complex surface (with its induced orientation as a complex manifold). For example, take a simply connected 4-manifold X with even b_2^-; then $\overline{X}$ (the same smooth 4-manifold with the opposite orientation) will have even b_2^+, so it will not admit any (almost) complex structure. The above examples $K_{p,q}$ have the property that they are not complex with *either* orientation. In the following, when we say "X does not admit a complex (or symplectic) structure", we will mean that the statement holds for both orientations of X. Most of the examples discussed in this section have the property that $SW_{\overline{X}} \equiv 0$ (mainly because X contains a sphere with square -2, hence $\overline{X}$ has a sphere of square 2, cf. Theorem 2.4.6). For these examples, X with the opposite orientation clearly has no complex (or symplectic) structure, cf. Theorem 2.4.7.

Next we address the question of determining the homeomorphism types of symplectic (minimal) 4-manifolds. First we show that the symplectic structure imposes no restriction on the fundamental group. There are various properties that the fundamental group of a Kähler surface must satisfy (see, e.g., [**ABCKT**]); for example, the rank of the abelianization must be even (cf. Remark 10.1.5). Thus, Theorem 10.2.10 illustrates the difference between Kähler and almost-Kähler manifolds. (Recall that every finitely presented group can be thought of as the fundamental group of an almost-complex 4-manifold, cf. Exercise 10.1.3(b).)

Theorem 10.2.10. ([**G12**]) *For any finitely presented group G there is a closed symplectic 4-manifold (X, ω) with $\pi_1(X) \cong G$.*

Proof. Take a finite presentation $G = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ with generators g_i and relators r_j. Let F be a surface with genus k and fix a collection of circles $\alpha_i, \beta_i \subset F$ representing a basis of $H_1(F;\mathbb{Z})$ (with $\alpha_i \cdot \beta_j = \delta_{ij}$, $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$). Take immersed curves γ_j ($j = 1, \dots, l$) in F representing the relators r_j in the free group $\pi_1(F)/\langle \beta_1, \dots, \beta_k \rangle$, with g_i replaced by α_i. The choices $\gamma_{l+i} = \beta_i$ ($i = 1, \dots, k$) complete the set $\{\gamma_i \mid i = 1, \dots, k+l\}$ of curves on F. Take a torus T^2 with generating circles α, β, and consider the collection of tori $T_i = \gamma_i \times \alpha \subset F \times T^2$ ($i = 1, \dots, k+l$) and $T_0 = \{\text{pt.}\} \times T^2$ in $F \times T^2$. It can be shown [**G12**] that (if the presentation for G was chosen carefully) these tori and the product symplectic form ω on $F \times T^2$ can be perturbed in such a way that the resulting collection $\{T_i'\}_0^{k+l}$ of tori are disjoint symplectic submanifolds of $(F \times T^2, \omega')$. Now take the symplectic normal connected sum of $F \times T^2$ and $(k+l+1)$ copies of the rational elliptic surface $E(1)$ along the tori $T_i' \subset F \times T^2$ and a generic fiber $F \subset E(1)$. Since $E(1) - \nu F$ is simply connected, the Seifert-Van Kampen Theorem shows that the resulting symplectic 4-manifold X has $\pi_1(X) = \langle \alpha_1, \beta_1, \dots ., \alpha_k, \beta_k \rangle / \langle \gamma_1, \dots, \gamma_{k+l} \rangle \cong G$. □

Remark 10.2.11. Using simple geometric ideas, one can prove that every finitely presented group can be given as the fundamental group of a Lefschetz fibration [**ABKP**]. In light of Theorem 10.2.18, this result provides a different proof for Theorem 10.2.10. Note that by coupling Theorems 10.2.10 and 10.2.25, it is straightforward to find a Lefschetz fibration $X \to \mathbb{CP}^1$ for each finitely presented group G, with $\pi_1(X) \cong G$. The proof of Theorem 10.2.25, however, is highly nonconstructive — in contrast to the proof given in [**ABKP**].

Restricting ourselves to the simply connected case, we would now like to determine the intersection forms corresponding to minimal symplectic 4-manifolds. Recall that the intersection form of a simply connected, smooth, closed 4-manifold is determined by its parity, rank and signature. Associate $c_1^2(X) = 3\sigma(X) + 2\chi(X)$ and $\chi_h(X) = \frac{1}{12}(c_1^2[X] + c_2[X]) = \frac{1}{4}(\sigma(X) + \chi(X))$ to the simply connected 4-manifold X as in Section 3.4. (Recall that $\chi(X)$ stands for the Euler characteristic of X, while χ_h generalizes the holomorphic Euler characteristic of a complex surface.) As we have already seen, if X is a smooth (simply connected) 4-manifold, then the intersection form Q_X is determined up to parity by $\chi_h(X)$ and $c_1^2(X)$.

Definition 10.2.12. By the *geography* of simply connected, minimal, closed symplectic 4-manifolds we mean determining the set of pairs $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ such that there is a minimal, simply connected, symplectic 4-manifold X with $a = \chi_h(X)$ and $b = c_1^2(X)$.

Remark 10.2.13. Compare the question of geography of symplectic manifolds to that of geography of simply connected complex surfaces (described in Sections 3.4 and 7.4). The latter geography is included in the former, since Kähler implies symplectic. Note that $\chi_h(X)$ is not necessarily an integer for a 4-manifold X. If X admits a symplectic (in particular almost-complex) structure, then Theorem 1.4.13 implies that $\chi_h(X) \in \mathbb{Z}$.

The invariants χ_h and c_1^2 do not determine the homeomorphism type of a simply connected 4-manifold completely (cf. the text after Theorem 3.4.19). In the case that c_1^2 is divisible by 8 and $\frac{c_1^2}{8} \equiv \chi_h \pmod 2$ (and only in this case), there is an even intersection form with the prescribed invariants in addition to the odd one. Hence, to fully answer the geography question for such pairs (χ_h, c_1^2), one should give two examples, a spin and a nonspin one. In the following, however, we will omit this subtlety. In the simply connected case (on which we are focusing now) $\chi_h(X) \geq 1$ and (as a consequence of Taubes' work quoted in Theorem 10.1.18) $c_1^2(X) \geq 0$ for a simply connected minimal symplectic manifold. As a consequence of Theorem 10.1.19 we have that if X has $\kappa(X) = -\infty$ then X is diffeomorphic to $\mathbb{CP}^2$ or $\mathbb{F}_n$. Minimal simply connected symplectic 4-manifolds with $\kappa(X) = 0$ are known to be homeomorphic to the $K3$-surface [**MSz**]. An easy argument shows that (simply connected) symplectic 4-manifolds with $\kappa = 1$ are homeomorphic to elliptic surfaces. As the examples of Corollary 10.2.8 (and the knot construction given in Section 10.3, cf. Remark 10.3.5) indicate, there are noncomplex manifolds with $\kappa(X) = 1$. Recall that a simply connected, minimal complex surface with $\kappa(X) = 2$ has invariants satisfying $c_1^2(X) > 0$ and $2\chi_h(X) - 6 \leq c_1^2(X) \leq 9\chi_h(X)$ (cf. Theorem 3.4.19). The situation is different for symplectic manifolds:

Theorem 10.2.14. *If $0 < b < 2a - 6$, then there is a minimal, simply connected, symplectic 4-manifold (with $\kappa(X) = 2$) with $\chi_h(X) = a$ and $c_1^2(X) = b$.*

Proof. There are many ways to use the normal summing operation to fill this region with simply connected symplectic manifolds [**G12**], [**S3**] — in fact, in the same vein one can fill the larger region $0 \leq b \leq 8a - 11$. The hard part of the proof of Theorem 10.2.14 is to prove irreducibility for the examples found (since one must compute gauge-theoretic invariants). The following family was shown to be irreducible by Fintushel and Stern. (See also Section 8.5.) Recall that the elliptic surface $E(4)$ contains 9 disjoint sections, which are spheres of square -4; moreover, these spheres are symplectic submanifolds. The complex surface $E(4)$ also contains two Lagrangian tori T_+ and T_- of square 0 disjoint from the 9 sections (see Exercise 3.1.9). Now write a in the form $a = 4k + l$ $(0 \leq l \leq 3)$ and take k copies of $E(4)$ and l copies of $E(1)$. Apply the normal connected sum construction for $(E(4)_i, T_+)$ and $(E(4)_{i+1}, T_-)$ $(i = 1, \dots, k-1)$ and finish the construction by taking the normal connected sum of the l copies of $(E(1), F)$ with the $k^{\underline{th}}$ $E(4)$ along l parallel copies of T_+. (Here $F \subset E(1)$ stands for the generic fiber of $E(1)$.) The resulting simply connected symplectic manifold Y has $\chi_h(Y) = a$ and $c_1^2(Y) = 0$. Moreover, Y has $9k$ embedded symplectic spheres with square -4. Since $b < 2a - 6 = 2(4k + l) - 6 = 8k + 2l - 6 \leq 8k$, we can construct X by summing Y along b symplectic -4-spheres with b copies of $(\mathbb{CP}^2, 2H)$, where $2H = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid \sum z_i^2 = 0\}$. Since $b < 8k$, we can assume that we sum along at most 8 spheres in each $E(4)$. It is easy to prove that X is simply connected (see below) and has the desired characteristic numbers. Adapting, e.g., the method developed in [**S2**] and using Theorem 8.5.12, one can determine the Seiberg-Witten basic classes of X. The proof of the minimality of X is now easy arithmetic based on the knowledge of $\mathcal{B}as_X$ and Exercise 10.1.20(b). □

Exercise 10.2.15. Prove that $\pi_1(X) = 1$. (*Hint*: What is the complement of 8 sections in $E(1) - \nu F$?)

Remarks 10.2.16. (a) This construction always produces manifolds with odd intersection forms (since $\mathbb{RP}^2 \subset \mathbb{CP}^2$ has nonzero mod 2 square and is disjoint from $2H$). By similar methods, it is also possible to realize all allowable even intersection forms in the above region (or in the larger region $0 \leq b \leq 8a - 32$) by simply connected, closed, spin, symplectic 4-manifolds [**G12**]. In the spin case minimality is obvious without computing the gauge-theoretic invariants.

(b) To complete the geography picture for simply connected symplectic manifolds, one also has to find a symplectic analogue of the Bogomolov-Miyaoka-Yau inequality $c_1^2(X) \leq 9\chi_h(X)$ (known for complex surfaces of general type). Note that by definition $c_1^2(X) + c_2(X) = 12\chi_h(X)$, hence if $c_2(X) \geq 0$ then $c_1^2(X) \leq 12\chi_h(X)$. In fact, by assuming simple connectivity we can prove that $c_1^2(X) < 5c_2(X)$, or equivalently $c_1^2(X) < 10\chi(X)$: We have $c_1^2(X) - 2c_2(X) = 3\sigma(X) = 3(c_2(X) + 2b_1(X) - 2b_2^-(X) - 2) < 3c_2(X)$ (since $b_1(X) = 0$). These arguments only use the assumption that $c_2(X)$ is nonnegative (or that $b_1(X) = 0$), and make no use of the presence of a symplectic structure on X. For this reason, one might expect sharper bounds for $c_1^2(X)$ in terms of $\chi_h(X)$ or $c_2(X)$ for a simply connected symplectic 4-

manifold (X, ω), but nothing further is currently known. For examples of simply connected symplectic 4-manifolds with $c_1^2 > (9-\varepsilon)\chi_h$ (for arbitrary $\varepsilon > 0$) see [**S4**].

(c) Based on the analogy with complex geometry, it seems plausible to expect that symplectic 4-manifolds with Kodaira dimension $\kappa(X) \geq 0$ have nonnegative Euler characteristic. (Note that this would imply $\chi_h(X) \geq 0$ as well.) It is expected that the invariants of symplectic 4-manifolds with $\kappa(X) = 2$ satisfy an inequality analogous to the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 9\chi_h$.

The first example of a symplectic manifold admitting no Kähler structure was given by Thurston [**Th1**] and was also known to Kodaira — this example has nontrivial fundamental group. The construction is the following: Take $\mathbb{R}^4$ with its standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and take the quotient of it by the group G generated by the unit translations parallel to the first three coordinate axes, together with the map $(x_1, y_1, x_2, y_2) \mapsto (x_1 + y_1, y_1, x_2, y_2 + 1)$. The resulting quotient Z inherits the symplectic structure of $\mathbb{R}^4$ but has $b_1(Z) = 3$; hence, it is not diffeomorphic to a Kähler surface (cf. Remark 10.1.5). Note that Z is a T^2-bundle over T^2. In fact, in most cases a surface bundle over a surface can be equipped with a symplectic structure.

Theorem 10.2.17. ([**Th1**], see also [**McS1**]) *Assume that Σ_g, Σ_h are closed, oriented, 2-dimensional surfaces. If $X \to \Sigma_h$ is a bundle with fiber Σ_g and the homology class of the fiber is nonzero in $H_2(X;\mathbb{R})$, then X admits a symplectic structure.* □

This theorem can be generalized as follows to manifolds admitting Lefschetz fibrations. (For more about Lefschetz fibrations see Chapter 8.)

Theorem 10.2.18. ([**G16**]) *Assume that the closed 4-manifold X admits a Lefschetz fibration $\pi\colon X \to \Sigma$, and let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers iff $[F] \neq 0$ in $H_2(X;\mathbb{R})$. If $e_1, \dots, e_n$ is a finite set of sections of the Lefschetz fibration, the symplectic form ω can be chosen in such a way that all these sections are symplectic.*

Proof. One direction of the above equivalence is easy to prove. If X carries a symplectic structure ω such that a fiber F is symplectic, then $\langle[\omega], [F]\rangle > 0$, hence $[F] \neq 0$ in $H_2(X;\mathbb{R})$. The other direction of the theorem is more complicated. To construct a symplectic structure on X, we can assume without loss of generality that the fibers are connected (cf. Proposition 8.1.9 and the subsequent paragraph), and perturb π so that it becomes injective on the set C of critical points. We now show how to construct a symplectic form on X that is symplectic on the resulting fibers. (More care is required to guarantee that ω is symplectic on the fibers when $\pi|C$ is not injective; see [**G16**] for details.)

Exercise 10.2.19. * For X as above, show that there exists a closed 2-form ζ on X such that if F_0 is a closed surface contained in a fiber (with the induced orientation) then $\int_{F_0} \zeta > 0$.

Let F_y denote the fiber $\pi^{-1}(y)$ ($y \in \Sigma$); we will next construct a 2-form ω_y on F_y. Choose disjoint open balls U_j in X around the elements of C, such that on each U_j the projection π is given as $\pi(z_1, z_2) = z_1^2 + z_2^2$. (Such neighborhoods exist by the definition of a Lefschetz fibration.) Now take $\omega_{U_j} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on U_j. Since $F_y \cap U_j$ is a holomorphic curve for $y \in \pi(U_j)$, the form $\omega_{U_j}|U_j \cap F_y$ is a symplectic form. Extend it to the fiber F_y as a symplectic form; in this way we get symplectic forms for all $y \in \pi(\bigcup_j U_j)$. Define ω_y for the remaining points $y \in \Sigma$ in such a way that ω_y is symplectic along F_y. By rescaling away from $\bigcup U_j$, we can assume that the condition $[\omega_y] = [\zeta|F_y]$ holds in $H^2_{dR}(F_y)$ for each y.

In the next step we will extend the above forms ω_y ($y \in \Sigma$) to closed 2-forms η_y defined on some neighborhood of each F_y, then glue the 2-forms η_y together to get a closed 2-form η on X. Take a neighborhood $W_y \subset \Sigma$ of y in Σ containing no critical values except (possibly) y. Take $\widetilde{W}_y = \pi^{-1}(W_y)$, and fix a retraction r_y of $\widetilde{W}_y$ to F_y or $F_y \cup c\ell(U_j)$, depending on whether $F_y \cap C = \emptyset$ or it consists of one point. (We assume that in the case $F_y \cap C \neq \emptyset$, $\widetilde{W}_y$ contains $c\ell(U_j)$; this can be achieved by shrinking each U_j.) Now define η_y as the pull-back $r^*(\omega_y)$ if $F_y \cap C = \emptyset$, and $r^*(\omega_y$ or $\omega_{U_j})$ otherwise. To glue the forms η_y together, choose a finite subcover of the cover $\{W_y\}$ of Σ. Rather than indexing this subcover by the corresponding points of Σ, we will index it by $i = 1, \ldots, n$ for some n; we reindex the corresponding F_y, ω_y and η_y by i accordingly. Fix a partition of unity $\{\rho_i\}$ subordinate to the cover $\{W_i\}$. Since the forms η_i and $\zeta|\widetilde{W}_i$ represent the same cohomology class on F_i, and $H^2_{dR}(\widetilde{W}_i) \cong H^2_{dR}(F_i)$, we have that $[\eta_i - \zeta|\widetilde{W}_i] = 0 \in H^2_{dR}(\widetilde{W}_i)$. Consequently $\eta_i - \zeta|\widetilde{W}_i$ is exact, i.e., there exists a 1-form $\theta_i \in \Omega^1(\widetilde{W}_i)$ such that $d\theta_i = \eta_i - \zeta|\widetilde{W}_i$. Now take $\eta = \zeta + d(\sum_i (\rho_i \circ \pi)\theta_i)$ on X. (Recall that $\{\rho_i\}$ is a partition of unity on Σ, hence we need π to pull it back to X for summing the forms $\theta_i \in \Omega^1(\widetilde{W}_i)$.) Since $d\eta = d\zeta = 0$, the 2-form $\eta \in \Omega^2(X)$ is closed. Moreover, by its construction, we have $\eta|F_y = \zeta|F_y + \sum_i \rho_i(y) d\theta_i|F_y = \zeta|F_y + \sum_i \rho_i(y)(\eta_i|F_y - \zeta|F_y) = \sum_i \rho_i(y)\eta_i|F_y$ for every $y \in \Sigma$. The last sum is a convex combination of compatibly oriented area forms on F_y, hence it is symplectic. Consequently η is a closed 2-form on X which is symplectic along the fibers.

Let ω_Σ be a symplectic form on the 2-dimensional surface Σ, and for $t > 0$ define $\omega_t = t\eta + \pi^*\omega_\Sigma$.

Proposition 10.2.20. *For sufficiently small t, the 2-form ω_t is a symplectic form on X.*

Proof. The form ω_t is obviously closed for any t, since $d\eta = 0$ and $d\omega_\Sigma = 0$ (implying $d\pi^*\omega_\Sigma = 0$). Hence we only need to show that ω_t is nondegenerate. In the fiber direction, nondegeneracy holds for every t, since $\omega_t|F_y = t\eta|F_y$ is symplectic on F_y for all $y \in \Sigma$. Away from C, nondegeneracy can be proved in the following way: Fix $y \in \Sigma$ and $x \in F_y$ such that x is not in C. The orthogonal complement of the tangent plane T_xF_y in T_xX is the same with respect to ω_t as with respect to η; it is also complementary to T_xF_y since $\eta|T_xF_y$ is nondegenerate. Thus, $\pi^*\omega_\Sigma$ is nondegenerate on $(T_xF_y)^{\perp_\eta}$, hence for t small enough the form ω_t will also be nondegenerate at x. (Recall that nondegeneracy is an open property, and for small t the term $\pi^*\omega_\Sigma$ dominates in ω_t on $(T_xF_y)^{\perp_\eta}$.) Now by the compactness of $X - \bigcup_j U_j$, we can choose an overall t_0 such that if $0 < t < t_0$, the form ω_t is nondegenerate on $X - \bigcup_j U_j$. In a neighborhood U_j, however, ω_t has the standard form $\omega_t|U_j = t\omega_{U_j} + \pi^*\omega_\Sigma$. (Recall that on the charts U_j the projection π has a standard form.) For a nonzero tangent vector $v \in TU_j$ we have $\omega_t(v, iv) > 0$ by the following exercise, proving that ω_t is nondegenerate everywhere. □

Exercise 10.2.21. Prove that under the above circumstances $\omega_t(v, iv) = t||v||^2 + \omega_\Sigma(\pi_*v, i\pi_*v) > 0$. (*Hint*: Use the definition of ω_{U_i} and the fact that $\pi_{U_i}\colon U_i \to \Sigma$ is given explicitly as $\pi(z_1, z_2) = z_1^2 + z_2^2$, hence it is holomorphic.)

Taking t even smaller, we can assume that a finite collection of sections will be symplectic, since the pull-back $\pi^*\omega_\Sigma$ is symplectic along sections, and for t small enough it dominates in ω_t. This observation completes the proof of Theorem 10.2.18. □

Remarks 10.2.22. (a) Note that the assumption on the homology class of the fiber ($[F] \neq 0 \in H_2(X;\mathbb{R})$) cannot be eliminated. For example, $S^1 \times S^3$ is obviously a T^2-bundle over the sphere S^2 (by taking the Hopf fibration of S^3 with circle fibers, and then multiplying it by S^1), but since $H_2(S^1 \times S^3;\mathbb{R}) = 0$, the 4-manifold $S^1 \times S^3$ admits no symplectic structure (cf. the text after Corollary 10.1.10). However, this situation only occurs for genus-1 Lefschetz fibrations, by Exercise 8.4.15(b). (Recall that Lefschetz fibrations are a special case of achiral Lefschetz fibrations, Definition 8.4.4.) The classification of elliptic surfaces without multiple fibers now shows that the hypothesis $[F] \neq 0$ on a Lefschetz fibration is automatic unless X is a torus bundle (so $\chi(X) = 0$ and $b_1(X) \neq 0$) or a blow-up of such.

(b) Suppose that Σ is a Riemann surface. A map $f\colon X \to \Sigma$ is called *locally holomorphic* if for each $x \in X$ there exist a neighborhood U_x and an orientation-preserving diffeomorphism $\varphi_x\colon U_x \to V_x \subset \mathbb{C}^2$ such that $f \circ \varphi_x^{-1}$ is a holomorphic map. Theorem 10.2.18 now can be extended in the following way: If the (closed) 4-manifold X admits a locally holomorphic map onto a Riemann surface Σ and for generic $p \in \Sigma$ the homology class $[f^{-1}(p)] \in H_2(X;\mathbb{R})$ is nonzero, then X can be equipped with a symplectic form (with the fibers of f symplectic wherever they are smooth) [**G16**]. The latter homological assumption can be replaced by the equivalent assumption that $f^*([\Sigma]) \in H^2(X;\mathbb{R})$ is nonzero.

(c) The geography question can be raised for (relatively minimal) Lefschetz fibrations as well. Suppose that $f\colon X \to \Sigma$ is a relatively minimal genus-g Lefschetz fibration with $g \geq 2$ and $g(\Sigma) = h > 0$. Equipping X with a symplectic structure provided by Theorem 10.2.18 and applying Taubes' work [**T4**], one can show that relative minimality implies minimality once $g(\Sigma) > 0$ [**S5**]. This implies, in particular, that $2(g-1)(h-1) \leq c_1^2(X) \leq 5c_2(X)$ for those Lefschetz fibrations [**Li2**], [**S5**]. (Note that since $g(\Sigma) \geq 1$, we have $b_1(X) \geq 2$, hence the argument given in Remark 10.2.16(b) does not apply.) As for genus-g Lefschetz fibrations over S^2, we know that $4 - 4g \leq c_1^2(X) \leq 5c_2(X) + 12(g-1)$ unless $X \to S^2$ is the trivial fibration $S^2 \times \Sigma_g \to S^2$ (in which case $c_1^2(X) = c_1^2(S^2 \times \Sigma_g) = 8 - 8g$) [**S6**]. All known examples of Lefschetz fibrations satisfy $c_1^2(X) \leq 3c_2(X)$.

By the addendum of Theorem 10.2.18 about sections $e_1, \ldots, e_n$, the statement of Theorem 10.2.18 can be extended to Lefschetz pencils as well:

Corollary 10.2.23. *If a 4-manifold X admits a Lefschetz pencil, then it has a symplectic structure.*

Proof. By blowing up X in the n points of the base locus B we get a Lefschetz fibration $X\#n\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. Since $B \neq \emptyset$ (cf. Definition 8.1.4(a)), there are sections (the exceptional spheres) intersecting each fiber nontrivially. Hence the homology class of the fiber is nontrivial, so by Theorem 10.2.18 the blown-up manifold $X\#n\overline{\mathbb{CP}^2}$ admits a symplectic structure for which the exceptional spheres (a finite set of sections) are symplectic. Now symplectically blowing down the exceptional spheres results in a symplectic structure on the manifold X. □

Coupling Corollary 10.2.23 with a recent result of Donaldson, we obtain a topological description of symplectic 4-manifolds. Before stating Donaldson's Theorem 10.2.25, we first quote the theorem on which its proof rests.

Theorem 10.2.24. ([**D2**]) *Suppose that $[\omega] \in H^2(X;\mathbb{R})$ lifts to an integral cohomology class $h \in H^2(X;\mathbb{Z})$. Then for $k \in \mathbb{N}$ sufficiently large, the*

Poincaré dual of $kh \in H^2(X;\mathbb{Z})$ can be represented by a symplectic surface in X. □

Theorem 10.2.25. (Donaldson, [**D3**]) *Any symplectic 4-manifold X admits a Lefschetz pencil.* □

Exercise 10.2.26. * Prove that every symplectic manifold (X, ω) admits a symplectic form ω' such that $[\omega'] \in H^2(X;\mathbb{R})$ lifts to an integral cohomology class, i.e., it is in the image of the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Remark 10.2.27. Let $L \to X$ be the complex line bundle with $c_1(L) = h \in H^2(X;\mathbb{Z})$. To prove Theorem 10.2.24, Donaldson showed that if k is large enough, then $L^{\otimes k} \to X$ admits a section s such that $s^{-1}(0) \subset X$ is a symplectic submanifold. Using the same basic idea, he also showed that for k large enough there are linearly independent sections $s_0, s_1 \in \Gamma(L^{\otimes k})$ such that the submanifolds $\{(t_0 s_0 + t_1 s_1)^{-1}(0) \subset X \mid [t_0 : t_1] \in \mathbb{CP}^1\}$ are symplectic and form a Lefschetz pencil on X. The proof is based on a technique of Kodaira for embedding Kähler manifolds in $\mathbb{CP}^N$ (cf. Remark 3.4.3(a)), although the analytical details are much more subtle in the symplectic case. Specifically, it was proved that the map $x \mapsto [s_0(x) : s_1(x)] \in \mathbb{CP}^1$ (defined on $X - \{s_0^{-1}(0) \cap s_1^{-1}(0)\}$) provides a Lefschetz fibration on some blow-up of X. Developing these ideas further, Auroux [**Au**] showed that for sufficiently large k the symplectic submanifolds $s_k^{-1}(0) \subset X$ (with $s_k \in \Gamma(L^{\otimes k})$) are unique up to isotopy. The corresponding uniqueness of Lefschetz fibrations gives an interesting, yet unexplored invariant of symplectic 4-manifolds.

Combining Corollary 10.2.23 and Theorem 10.2.25 gives the following topological characterization of symplectic 4-manifolds:

Theorem 10.2.28. *A 4-manifold X admits a symplectic structure iff it admits a Lefschetz pencil.* □

Remark 10.2.29. Recall that a Lefschetz fibration $\pi\colon X \to \mathbb{CP}^1$ can be described by the genus of the generic fiber and the monodromies of the singular fibers. Since Theorem 10.2.28 implies that for every symplectic 4-manifold X, the blow-up $X\#n\overline{\mathbb{CP}^2}$ admits a Lefschetz fibration for some $n \in \mathbb{N}$, in principle $X\#n\overline{\mathbb{CP}^2}$ can be described by the combinatorial data consisting of the genus of the fibration $X\#n\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ and the monodromies of the singular fibers — a word in the mapping class group of the generic fiber. In practice, however, these data are very hard to determine for a given X. Moreover, there is no effective way to determine when two sets of data describe diffeomorphic 4-manifolds. Furthermore, to recover X we must face the difficult task of locating the exceptional spheres of the n blow-ups. Both Corollary 10.2.23 and Theorem 10.2.25 admit generalizations to

$2n$-dimensional manifolds with arbitrary n; this leads to a topological characterization of symplectic $2n$-manifolds similar to the description given in Theorem 10.2.28. The details are much more complicated, however.

10.3. 4-manifolds with no symplectic structure

As we saw in Theorem 10.1.18, a simply connected, minimal symplectic 4-manifold is irreducible. (For irreducibility see Definition 10.1.17.) It is easy to see that every closed 4-manifold X admits a decomposition $X = X_1\#\dots\#X_n$ into irreducible summands. (Recall that by Grushko's Theorem, a group generated by m elements cannot split as a free product of more than m factors.) Note, however, that (as we saw in Section 9.2) this decomposition is not necessarily unique, for example $X = \mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ has another decomposition as $(\mathbb{CP}^1\times\mathbb{CP}^1)\#\overline{\mathbb{CP}^2}$ — there are many other examples for this nonuniqueness (cf. Corollary 9.1.14). Since simply connected, minimal symplectic manifolds are irreducible, it seemed possible until recently that every irreducible, simply connected 4-manifold ($\neq S^4$) admitted a symplectic structure (for at least one choice of orientation). At the same time, the remarkable results discussed in the previous section gave hope for describing minimal symplectic 4-manifolds from the differential topological point of view. 4-manifolds, however, turned out to be more complicated than the above scheme would suggest; this section is devoted to the description of certain irreducible manifolds not admitting symplectic structures.

Exercise 10.3.1. Prove that $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^1\times\mathbb{CP}^1$ are irreducible. (Note that according to Theorem 10.1.18 a simply connected symplectic 4-manifold with $b_2(X) > 1$ (in particular, a simply connected complex surface S with $b_2^+(S) > 1$) is irreducible.)

Our first example of an irreducible 4-manifold with no symplectic structure is constructed in the following way: Consider the hypersurface $S_{2d} = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid \sum z_i^{2d} = 0\}$ in $\mathbb{CP}^3$. The group $\mathbb{Z}_2$ acts freely on S_{2d} by complex conjugation; consider the quotient manifold $X = S_{2d}/\mathbb{Z}_2$.

Theorem 10.3.2. ([**Wa**]) *If $d > 2$, then X is an irreducible 4-manifold with no symplectic structure.*

Proof. In [**Wa**] a simple computation for the Seiberg-Witten function of X is given. It is proved that $SW_X \equiv 0$, implying that X admits no symplectic structure. Irreducibility can be proved using the following *covering trick*: If X decomposes as $X = X_1\#X_2$, then (since $\pi_1(X)\cong\mathbb{Z}_2$) we can assume $\pi_1(X_1) = 1$ and $\pi_1(X_2) = \mathbb{Z}_2$. Taking the double cover, we get $S_{2d} = X_1\#\widetilde{X}_2\#X_1$, which contradicts the fact that S_{2d} is a simply connected minimal complex surface (of general type), hence irreducible. □

Note that the above examples of Wang are not simply connected. The first examples of simply connected irreducible 4-manifolds with no symplectic structure were found by Szabó [**Sz2**], [**Sz3**]; here we give a more general construction — called the *knot construction* — due to Fintushel and Stern [**FS3**]. For any knot $K \subset S^3$, 0-surgery on K produces a 3-manifold N_K; consider the 4-manifold $M_K = S^1 \times N_K$. A meridian μ of K gives a circle in N_K, hence a torus $T = S^1 \times \mu$ in M_K with $[T]^2 = 0$ (and complement $\approx S^1 \times (S^3 - \nu(K))$). Assume that the simply connected 4-manifold X contains a torus F with $[F]^2 = 0$, and that the complement of F is simply connected. Now, using a suitable identification Ψ of the boundaries $\partial(X - \nu F)$ and $\partial(M_K - \nu T)$, apply the normal connected sum operation to (X, F) and (M_K, T); the resulting manifold $X \#_F M_K$ is denoted by X_K. (The diffeomorphism type of X_K might depend on the chosen identification Ψ. The statements about the homeomorphism type or Seiberg-Witten basic classes of X_K hence should be understood to apply for any choice of identification lifting a diffeomorphism $T \approx F$.) Note that in the above construction we did not assume that any of the manifolds (or submanifolds) are symplectic.

Exercise 10.3.3. Prove that X_K is homeomorphic to X. (*Hint*: The Seifert-Van Kampen Theorem shows that $\pi(X_K) = 1$. Compute $\sigma(X_K)$, $\chi(X_K)$ and the parity of X_K; Freedman's Theorem 1.2.27 concludes the solution.)

Theorem 10.3.4. ([**FS3**]) *Assume that F lies in a cusp neighborhood in X, and that the leading coefficient of the Alexander polynomial $\Delta_K(t)$ of K is not ± 1. Then X_K admits no symplectic structure.* □

Fintushel and Stern proved the above theorem by computing the Seiberg-Witten invariants of X_K in terms of SW_X and $\Delta_K(t)$. They proved that (under the circumstances described in Theorem 10.3.4) there is no basic class of X_K which has the properties required by Theorem 10.1.11 for $c_1(X, \omega)$ of a symplectic structure ω. We also want to prove that $\overline{X}_K$ is not symplectic. This can be achieved by assuming that X contains a sphere of negative square disjoint from the torus F, implying that $\overline{X}_K$ has vanishing Seiberg-Witten invariants. The knowledge of SW_{X_K} allows one to prove that X_K is irreducible in many cases, for example, if X is a minimal complex surface with $b_2^+(X) \geq 3$ or if X is symplectic and spin. As an example of a manifold X_K for which Theorem 10.3.4 applies, we can choose X to be the $K3$-surface $E(2)$ with a regular fiber as F, and the twist knot K given by Figure 10.1; if $|n| \geq 2$, the corresponding 4-manifold X_K will admit no symplectic structure by Theorem 10.3.4. A Kirby diagram of this manifold is given in Figure 10.2. (The dotted ribbon knot represents $I \times (S^3 - \nu K)$ as in Exercise 6.2.4(b), and the additional 1-handle, 0-framed 2-handles and 3-handle extend this to $S^1 \times (S^3 - \nu K)$. $X - \nu F$ has then been added by the method of Example 5.5.8, as can be seen by removing int $S^1 \times (S^3 - \nu K)$, performing a $-n$-fold Rolfsen twist in the resulting 3-manifold to remove the n twists, sliding the resulting $-\frac{1}{n}$-framed unknot over the 0-framed 2-handles to the bottom of the picture, and using it to cancel the $-n$-twist.) The above

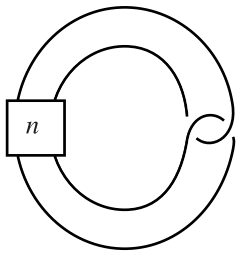


Figure 10.1. Twist knot.

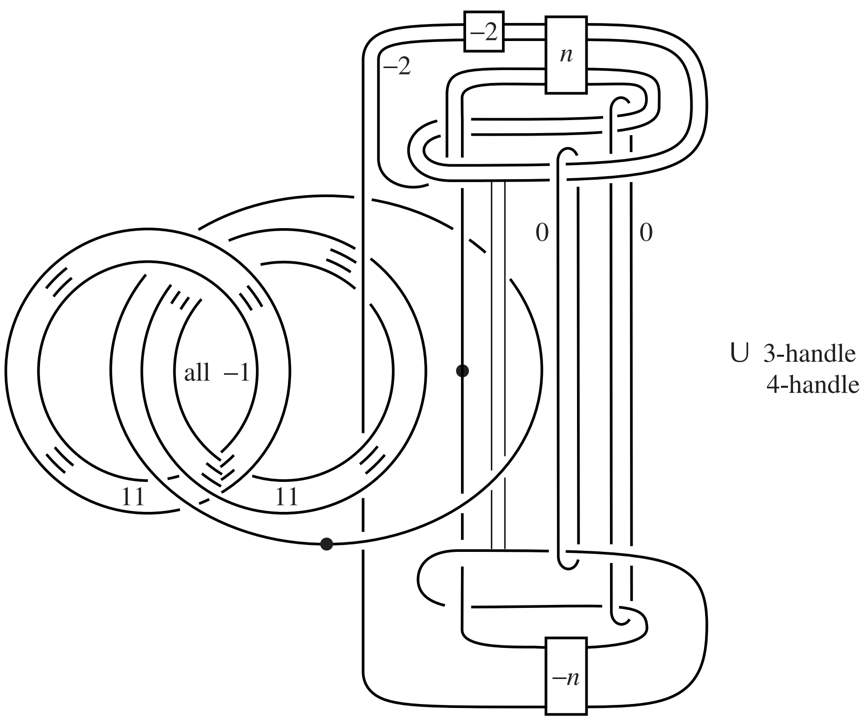


Figure 10.2. Irreducible, nonsymplectic manifold X_K ($|n| \geq 2$).

construction can be generalized from knots K to links with m components; in this case one should choose 4-manifolds $X_1, \ldots, X_m$ and mimic the same construction. For more details see [**FS3**]. In constructing the first simply connected nonsymplectic 4-manifolds, Szabó performed a logarithmic transformation along a symplectic torus lying in a fishtail neighborhood. The auxiliary gluing data of this logarithmic transformation, however, were not the same as the data described in Exercise 8.3.16(a) (cf. Corollary 10.2.7).

Remark 10.3.5. If $K \subset S^3$ is a *fibered knot* (that is, the complement $S^3 - K$ admits a fibration $S^3 - K \to S^1$ with a punctured genus-g surface as a fiber), then N_K fibers over S^1 by construction, hence $M_K = S^1 \times N_K$ admits a genus-g fibration over T^2. Now Theorem 10.2.17 provides a symplectic structure on M_K, which can be chosen in such a way that the torus $T \subset M_K$ (described in the text following Theorem 10.3.2) is a symplectic submanifold. Assuming that X is symplectic and $F \subset X$ is a symplectic submanifold of it, Theorem 10.2.1 provides a symplectic structure on X_K. (It is known that the Alexander polynomial of a fibered knot is monic, that is, its leading coefficient is ± 1; consequently Theorem 10.3.4 does not apply to fibered knots.) On the other hand, the computation of the Seiberg-Witten invariants given by Fintushel and Stern [**FS3**] is valid for any knot, hence the knot construction performed with fibered knots along a fiber of the $K3$-surface, for example, results in infinitely many nondiffeomorphic symplectic 4-manifolds, each homeomorphic to the $K3$-surface.

A different type of example — found by Morgan, Kotschick and Taubes [**MKT**] — shows that even the existence of a class K with $SW_X(K) = \pm 1$ does not imply the existence of a symplectic structure on X. Assume that Y is a simply connected symplectic 4-manifold and M^3 is a 3-dimensional rational homology sphere (for example, the Poincaré homology sphere $\Sigma(2,3,5)$). A surgery on $S^1 \times M$ killing $\pi_1(S^1)$ produces a rational homology 4-sphere N with $\pi_1(N) \cong \pi_1(M)$; let us consider $X = Y \# N$. Assume that the 4-manifold X admits an n-fold connected cover $\pi \colon \widetilde{X} \to X$ ($n > 1$). (For example, if M is the Poincaré homology sphere, then the universal cover will do.) It is clear that if ω is a symplectic form on X, then the pull-back $\pi^*\omega$ will be a symplectic form on $\widetilde{X}$. But since Y is simply connected and $b_2^+(Y) > 0$, $\widetilde{X} = \# nY \# \widetilde{N}$ has vanishing Seiberg-Witten invariants, so it cannot have a symplectic structure. Consequently X cannot admit a symplectic structure either. On the other hand, standard pull-apart arguments show that $SW_X(K) = SW_Y(K)$ for all $K \in H^2(Y;\mathbb{Z}) \subset H^2(X;\mathbb{Z})$; in particular, $SW_X(c_1(Y,\omega)) = \pm 1$ (where we regard $c_1(Y,\omega)$ as an element of $H^2(X;\mathbb{Z})$). If, in addition, N is an integral homology 4-sphere (as when M is the Poincaré homology sphere), then $SW_X(K) = SW_Y(K)$ for all $K \in H^2(Y;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$. Hence, in this case X has the same Seiberg-Witten invariants as a symplectic 4-manifold (namely Y), but X itself admits no symplectic structure.

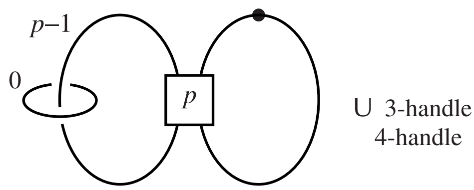


Figure 10.3. Kirby diagram for H_p.

Remark 10.3.6. Applying the rational blow-down process to $Y\#(p-1)\overline{\mathbb{CP}^2}$ (with the copy of $C_p \subset \#(p-1)\overline{\mathbb{CP}^2}$ provided by the solution of Exercise 8.5.1(b)), we get the 4-manifold $Y\#H_p$, where H_p is just the double of B_p. (Figure 10.3 gives a Kirby picture for H_p.) Note that $\pi_1(H_p) \cong \mathbb{Z}_p$, H_p is a rational homology 4-sphere and (by the argument described above) $Y\#H_p$ does not admit any symplectic structure — although $SW_{Y\#H_p} = SW_Y$ on $H^2(Y;\mathbb{Z}) \subset H^2(Y\#H_p;\mathbb{Z})$. If Y is symplectic, then so is $Y\#(p-1)\overline{\mathbb{CP}^2}$, but we conclude that the embedding of C_p cannot be symplectic as in Theorem 10.2.6. Note that in these last examples the nontriviality of the fundamental group plays a crucial role.

Having asked the geography question for complex, then for symplectic manifolds, we can ask the same question for irreducible 4-manifolds: Which pairs $(a, b) \in \mathbb{Z}\times\mathbb{Z}$ correspond to simply connected, irreducible 4-manifolds? Again, X corresponds to (a, b) if $a = \chi_h(X) = \frac{1}{4}(\sigma(X) + \chi(X))$ and $b = c_1^2[X] = 3\sigma(X)+2\chi(X)$. Assume for a moment that $\sigma(X)+\chi(X)$ is divisible by 4. (Observe that the sum $\sigma(X)+\chi(X)$ is divisible by 4 iff $b_1(X)-b_2^+(X)$ is odd.) Very little is known about the answer to this question, beyond the examples provided by complex geometry and the symplectic constructions described earlier. The answers to the following questions would be very interesting:

q1: Does the inequality $c_1^2(X) \geq 0$ hold for every irreducible (simply connected) 4-manifold?

q2: Is there a bound on $c_1^2(X)$ in terms of $\chi_h(X)$ for an irreducible 4-manifold X, similar to the Bogomolov-Miyaoka-Yau inequality for complex surfaces? (Cf. also Remark 10.2.16(b).) Note that on reversing the orientation, the inequality $c_1^2(X) \geq 0$ becomes $c_1^2(\overline{X}) \leq 4c_2(\overline{X})$, i.e., $c_1^2(\overline{X}) \leq (9\frac{3}{5})\chi_h(\overline{X})$.

Exercises 10.3.7. (a)* Show that if $c_1^2(X) \geq 0$ for every irreducible (simply connected) 4-manifold, then the $\frac{11}{8}$-Conjecture is true. (The conjecture that $c_1^2(X) \geq 0$ for every irreducible 4-manifold X is frequently called the $\frac{3}{2}$-*Conjecture*.)

(b)* Show that $c_1^2(\overline{X}) = 4c_2(X) - c_1^2(X)$. Note that for the elliptic surfaces $E(n)_{p,q}$ we have $c_1^2(E(n)_{p,q}) = 0$; hence by reversing the orientation we find irreducible 4-manifolds $\overline{E(n)_{p,q}}$ satisfying $c_1^2(\overline{E(n)_{p,q}}) = 4c_2(\overline{E(n)_{p,q}})$. All of the known examples of simply connected, irreducible 4-manifolds satisfy $0 \le c_1^2(X) \le 4c_2(X)$, or equivalently $0 \le c_1^2(X) \le (9\frac{3}{5})\chi_h(X)$. (A simply connected 4-manifold satisfies $c_1^2(X) \le 10\chi_h$; cf. Remark 10.2.16(b).)

The geography of irreducible manifolds is completely uncharted when $b_1 - b_2^+$ is even. All of our above results concerned only the case when $b_1 - b_2^+$ is odd, due to the lack of a sufficiently sensitive diffeomorphism invariant when $b_1 - b_2^+$ is even. Of course, since irreducibility does not depend on the orientation, one can easily construct an irreducible 4-manifold with even b_2^+ (and $b_1 = 0$): Take a simply connected, irreducible manifold X with even b_2^- and reverse the orientation. When we extend the geography question to manifolds with even $b_1 - b_2^+$, we are looking for less trivial examples — for example, manifolds with even $b_1 - b_2^+$ *and* $b_1 - b_2^-$. A possible candidate for a simply connected, irreducible 4-manifold with even b_2^+ and b_2^- can be constructed in the following way. Take two copies of the $K3$-surface $E(2)$, delete a neighborhood of a sphere of square -2 from each and glue the resulting manifolds together via the (unique) orientation-reversing diffeomorphism of the boundaries. We will denote the resulting manifold by $K3\#_2K3$. A Kirby diagram of this manifold is given by Figure 10.4.

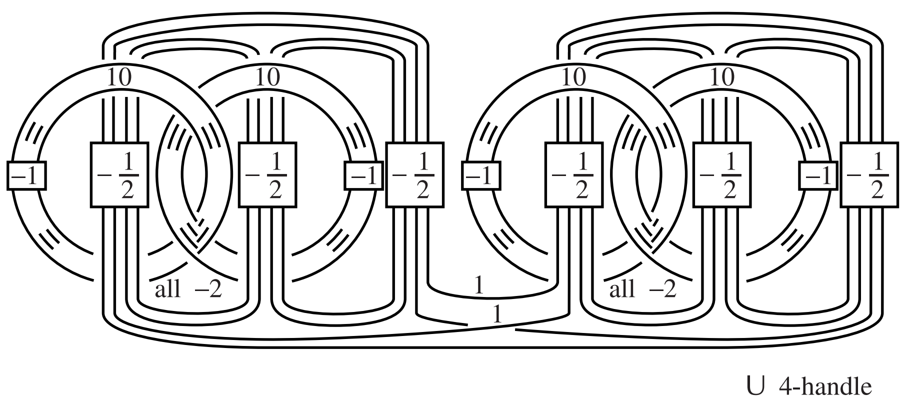


Figure 10.4. $K3\#_2K3$ — Is this irreducible?

Exercises 10.3.8. (a)* Prove that the boundary 3-manifold of a tubular neighborhood of a sphere with square -2 admits an orientation-reversing diffeomorphism. What is this 3-manifold? Prove also that $K3\#_2K3$ is simply connected, has even b_2^+ and b_2^-, and $c_1^2(K3\#_2K3) < 0$.

(b)* Let X and Y be handlebodies whose diagrams each contain a -2-framed unknot. Describe a diagram for the corresponding manifold $X\#_2Y$. Derive Figure 10.4 as a special case.

Even the full answer for the geography question would not describe irreducible 4-manifolds completely. One also must determine the different manifolds corresponding to a given pair (χ_h, c_1^2); research in this direction is called *botany*. Many results are known about the number of different irreducible manifolds corresponding to a fixed pair $(\chi_h, c_1^2) \in \mathbb{N} \times \mathbb{N}$ (e.g., Theorem 10.3.9); the description of *all* manifolds with preassigned characteristic numbers is, however, still a mystery. It can be proved that a compact, topological manifold can carry only countably many different smooth structures. (In dimensions ≤ 6, for example, the question is equivalent to counting PL-structures, but there are only countably many finite simplicial complexes. Note the contrast with $\mathbb{R}^4$ having continuously many smooth structures.) The following result indicates that most (simply connected, closed) smoothable topological manifolds carry infinitely many smooth structures.

Theorem 10.3.9. ([**Pa**]) *All but finitely many points $(\chi_h, c_1^2) \in \mathbb{N} \times \mathbb{N}$ with $0 \leq c_1^2 \leq 8\chi_h$ can be realized as characteristic numbers of infinitely many distinct, smooth, irreducible, simply connected 4-manifolds.* □

10.4. Gauge theory on symplectic 4-manifolds

The strong relation between almost-complex and spin^c structures on a 4-manifold simplifies the Seiberg-Witten equations in the presence of an almost-complex structure J. If, in addition, the manifold is almost-Kähler (that is, $d\omega = 0$), the solutions of the Seiberg-Witten equations can be compared to almost-complex geometric objects in the manifold, leading us to the celebrated results of Taubes (Theorems 10.1.11 and 10.1.15). In this appendix we outline the simplifications in the gauge theory allowed by the presence of an almost-complex structure, and briefly indicate the effect of the condition $d\omega = 0$ on the Seiberg-Witten equations. We then outline the proofs of some of the key theorems on the Seiberg-Witten invariants of Kähler and almost-Kähler manifolds. The interested reader is advised to turn to [**A5**], [**Ko2**], [**KKM**], [**Mr1**], [**Sa**] or [**T2**] for more details.

We begin by discussing the relation between almost-complex and spin^c structures. Recall that a spin^c structure is (by Definition 2.4.15) a lift of the cocycle structure of the tangent bundle TX into $Spin^c(4)$. If (X, J) is an almost-complex manifold, then the structure group of TX reduces to $U(2) \subset SO(4)$, and since there is a canonical lift $U(2) \to Spin^c(4)$ given by

$$A \mapsto \left(\left[\begin{smallmatrix} \det A & 0 \\ 0 & 1 \end{smallmatrix}\right], A\right) \in Spin^c(4) \subset U(2) \times U(2)$$

for an element $A \in U(2)$, an almost-complex structure canonically defines a spin^c structure on X. Using the presentation of $Spin^c(4)$ given by Exercise 2.4.14, we can describe the above map as follows. If $[e^{i\theta}, q]$ is an element of $S^1 \times SU(2)/\mathbb{Z}_2 = U(2)$, then take

$$[e^{i\theta}, q] \mapsto [e^{i\theta}, \left[\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right], q] \in S^1 \times SU(2) \times SU(2)/\mathbb{Z}_2.$$

From the first description it is clear that for the $U(2)$-bundles $W^\pm$ (associated to the principal $Spin^c(4)$-bundle $P_{Spin^c(4)}$ via $\mu^\pm$, cf. Section 2.4) we have complex bundle isomorphisms $W^+ \cong \det TX \oplus \underline{\mathbb{C}} = K_{X,J}^{-1} \oplus \underline{\mathbb{C}}$ and $W^- \cong TX$. The complex line bundle $K_{X,J}$ above is the *canonical line bundle* of the almost-complex manifold (X, J); recall that $c_1(K_{X,J}) = -c_1(X, J)$. Note that the determinant line bundle $\det W^+$ of the spin^c structure induced by the almost-complex structure J is isomorphic to the inverse of the canonical line bundle of J.

Claim 10.4.1. *A given spin^c structure is induced by an almost-complex structure iff $c_2(W^+) = 0$ for the correponding positive spinor bundle.*

Proof. Since $c_2(K_{X,J}^{-1} \oplus \underline{\mathbb{C}}) = 0$, one direction is obvious. Now if $c_2(W^+) = 0$ for a spin^c structure, then (by the classification of $U(2)$-bundles over 4-manifolds, Theorem 1.4.20(a)) $W^+ = L \oplus \underline{\mathbb{C}}$ for some complex line bundle L. Applying $\sigma\colon \Gamma(W^+) \to \Gamma(\Lambda^+)$ (defined in the text preceding Remark 2.4.22) to a constant section of $\underline{\mathbb{C}}$ we get $\omega \in \Gamma(\Lambda^+)$ with constant length; this ω induces the desired almost-complex structure J. □

Remarks 10.4.2. (a) For X closed, it is not hard to see that $c_2(W^+) = \frac{1}{4}(c_1^2(W^+) - 3\sigma(X) - 2\chi(X))$, hence (by Theorem 2.4.24) $c_2(W^+)$ is equal to the dimension of the moduli space given by the Seiberg-Witten equations. Using this observation and Claim 10.4.1, we can reformulate the simple type condition in the following way: The simply connected 4-manifold X is of simple type if every basic class of X is the first Chern class of some almost-complex structure.

(b) The above correspondence shows that an almost-complex structure determines a spin^c structure together with a section $\psi \in W^+$ of unit length (trivializing the $\underline{\mathbb{C}}$-factor); conversely, such a pair (W^+, ψ) determines an almost-complex structure. For the complete discussion of this correspondence (and its 3-dimensional analog) see [**KM2**].

There is an alternative way to define $W^\pm$ in terms of J; this other definition turns out to be more suitable for our present purposes. Recall [**GH**] that J splits $\Lambda^1_{\mathbb{C}} = T^*X \otimes \mathbb{C}$ into the sum $\Lambda^{0,1} \oplus \Lambda^{1,0}$, where $\Lambda^{1,0} = \{v \in T^*X \otimes \mathbb{C} \mid Jv = iv\}$ and $\Lambda^{0,1} = \{v \in T^*X \otimes \mathbb{C} \mid Jv = -iv\}$; the bundle $\Lambda^{p,q}$ is defined as $\bigwedge^p \Lambda^{1,0} \bigwedge^q \Lambda^{0,1}$. The space of sections of $\Lambda^{p,q}$ is frequently denoted by $\Omega^{p,q}$; note that $\Gamma(\Lambda^i X\otimes\mathbb{C}) = \Omega^i_{\mathbb{C}} = \sum_{p+q=i}\Omega^{p,q}$. Two operators are naturally associated to the splitting of $\Omega^1_{\mathbb{C}}$ as $\Gamma(T^*X\otimes\mathbb{C}) = \Omega^{1,0}\oplus\Omega^{0,1}$: Define $\overline{\partial}\colon \Omega^{0,0}\to\Omega^{0,1}$ as the composition of $d\colon \Omega^0_{\mathbb{C}}\to\Omega^1_{\mathbb{C}}$ with the projection $\Omega^1_{\mathbb{C}}\to\Omega^{0,1}$; similarly, $\partial\colon \Omega^{0,0}\to\Omega^{1,0}$ is given as the composition of d with the projection $\Omega^1_{\mathbb{C}}\to\Omega^{1,0}$ to the first factor. In the same fashion the operators $\overline{\partial}\colon \Omega^{p,q}\to\Omega^{p,q+1}$ and $\partial\colon\Omega^{p,q}\to\Omega^{p+1,q}$ can be defined. Once p or q is not zero, however, the equation $d=\partial+\overline{\partial}$ fails to hold for a generic almost-complex structure J: we have $d=\partial+\overline{\partial}+N+\overline{N}$, where the map $N\colon\Omega^{p,q}\to\Omega^{p+2,q-1}$ is a tensor — called the *Nijenhuis tensor* of the almost-complex structure J. (In fact, the integrability of J is equivalent to the condition that $N=0$, i.e., $d=\partial+\overline{\partial}$. This last identity is equivalent to either of the conditions $\partial^2=0$ or $\overline{\partial}^2=0$.)

Take the $U(2)$-bundles $W^+\cong\Lambda^{0,0}\oplus\Lambda^{0,2}$ and $W^-\cong\Lambda^{0,1}$ with the map $\rho\colon TX\otimes\mathbb{C}\to \mathrm{Hom}_{\mathbb{C}}(W^+,W^-)$ defined in the following way: if $\alpha\in\Omega^{0,0}$ and $\beta\in\Omega^{0,2}$, and x is a tangent vector, then

$$\rho(x)(\alpha,\beta)=\sqrt{2}((\tilde{x}+iJ\tilde{x})\alpha - *((\tilde{x}+iJ\tilde{x})\wedge *\beta))\in\Omega^{0,1}.$$

(As before, $\tilde{x}$ stands for the cotangent vector corresponding to x via the metric g and $*$ denotes the Hodge star operator.) In this way we get a triple $(W^\pm,\rho)$, which turns out to be a spin^c structure.

Exercise 10.4.3. Prove that the above two definitions of spin^c structures induced by the almost-complex structure J give isomorphic structures.

If $(W^\pm,\rho)$ is the spin^c structure induced by the almost-complex structure J, and $L_a\to X$ is the line bundle with $c_1(L_a)=a$, then $(W^\pm\otimes L_a,\rho\otimes\mathrm{id}_{L_a})$ gives a spin^c structure with $c_1(W^+\otimes L_a)=c_1(W^+)+2a\in H^2(X;\mathbb{Z})$. It can be shown that the map $a\mapsto(W^\pm\otimes L_a,\rho\otimes\mathrm{id}_{L_a})$ gives an isomorphism between the elements of $H^2(X;\mathbb{Z})$ and $\mathcal{S}^c(X)$ — even in the presence of 2-torsion in $H^2(X;\mathbb{Z})$. (Recall that in Section 2.4 we saw that the map $\mathcal{S}^c(X)\to\mathcal{C}_X\subset H^2(X;\mathbb{Z})$ associating the first Chern class of the determinant bundle to $(W^\pm,\rho)$ was not a monomorphism unless $H^2(X;\mathbb{Z})$ had no 2-torsion. Note that associating the first Chern class to a spin^c structure does not require additional choices, while the identification of $\mathcal{S}^c(X)$ with $H^2(X;\mathbb{Z})$ described above needs the choice of a "base spin^c structure". This identification is very similar to the correspondence between $\mathcal{S}(X)$ — the set of spin structures on X — and $H^1(X;\mathbb{Z}_2)$.)

Let $\omega\in\Omega^+$ denote the nondegenerate 2-form induced by J. The presence of a compatible pair consisting of an almost-complex structure J and a metric g defines two decompositions of $\Lambda^2_{\mathbb{C}}=\Lambda^2T^*X\otimes\mathbb{C}$: J decomposes it as $\Lambda^{2,0}\oplus\Lambda^{1,1}\oplus\Lambda^{0,2}$, while g (via the Hodge star-operator $*_g$) provides the decomposition $\Lambda^+_{\mathbb{C}}\oplus\Lambda^-_{\mathbb{C}}$. A local computation gives the following relation between the two splittings of $\Lambda^2_{\mathbb{C}}=\Lambda^2\otimes\mathbb{C}$.

Lemma 10.4.4. *The 2-form ω induced by J and g is in $\Omega^{1,1}$. Moreover, $\Lambda^+_{\mathbb{C}} = \Lambda^+ \otimes \mathbb{C} = \Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \mathbb{C} \cdot \omega$ and $\Lambda^-_{\mathbb{C}} = \langle \omega \rangle^{\perp} \subset \Lambda^{1,1}$.* □

Now we turn to the discussion of the Seiberg-Witten equations on closed, almost-complex manifolds. Recall that these equations read as follows. If $(W^{\pm}, \rho)$ is a given spin^c structure, $L = \det W^+$ and $(A, \psi) \in \mathcal{A}_L \times \Gamma(W^+)$, then we have

$$\partial\!\!\!/_A \psi = 0 \quad \text{and} \quad F_A^+ = i\sigma(\psi).$$

Suppose that (X, J) is an almost-complex 4-manifold with a compatible metric g. (Recall that J and g determine a nondegenerate 2-form ω.) First we restrict ourselves to the spin^c structure induced by J. Using the decomposition of W^+ as $\Lambda^{0,0} \oplus \Lambda^{0,2}$, we decompose the spinor $\psi \in \Gamma(W^+)$ as $\psi = (\alpha, \beta)$, where $\alpha \in \Omega^{0,0}$ and $\beta \in \Omega^{0,2}$. The components of $\sigma(\psi) = \sigma(\alpha, \beta) \in \Omega^2_{\mathbb{C}}$ in $\Omega^{0,2}$, $\Omega^{2,0}$ and $\Omega^{1,1}$ can be determined explicitly, cf. [**Mr1**].

Lemma 10.4.5. *The component of $\sigma(\psi)$ in the ω-direction is equal to $\frac{1}{4}(|\alpha|^2 - |\beta|^2)\omega$. The $(0,2)$ part of $\sigma(\psi)$ is $\frac{1}{2}\overline{\alpha}\beta$, while the $(2,0)$ part is equal to $\frac{1}{2}\alpha\overline{\beta}$.* □

If we decompose F_A^+ according to Lemma 10.4.4, the second Seiberg-Witten equation becomes

$$(F_A^+)^{\omega} = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega \;\text{ and }\; (F_A^+)^{0,2} = \frac{1}{2}\overline{\alpha}\beta \;\; (\text{and } (F_A^+)^{2,0} = \frac{1}{2}\alpha\overline{\beta} \text{ resp.}),$$

where $(F_A^+)^{\omega}$ and $(F_A^+)^{0,2}$ ($(F_A^+)^{2,0}$ resp.) are the components of F_A^+ in the ω-direction and in $\Lambda^{0,2}$ ($\Lambda^{2,0}$ resp.).

The determinant line bundle $L = \det W^+$ — which in this case is isomorphic to the inverse of the canonical line bundle $K_{X,J} = \Lambda^2_{\mathbb{C}} T^*X$ — admits a preferred connection A_0 (the one induced on $K^{-1}_{X,J}$ by the Levi-Civita connection of (X, g)). Coupling a connection on the trivial bundle $\underline{\mathbb{C}}$ with A_0 gives a connection on L. In this way we get an identification of the spaces $\mathcal{A}_L$ and $\mathcal{A}_{\underline{\mathbb{C}}}$; the connection in $\mathcal{A}_{\underline{\mathbb{C}}}$ corresponding to $A \in \mathcal{A}_L$ will be denoted by $\hat{A} \in \underline{\mathbb{C}}$. Recall that the almost-complex structure J induces the operators ∂ and $\overline{\partial}$; coupling these with the connection $\hat{A}$, the operator $\overline{\partial}_{\hat{A}}$ and its adjoint $\overline{\partial}^*_{\hat{A}}$ can be defined. Note that until now we have not used any special properties of the almost-complex structure J or the associated 2-form ω. The following lemma demonstrates the advantage of having an almost-Kähler (that is, symplectic) structure on X as opposed to having only an almost-complex structure. For the proof of Lemma 10.4.6 see [**Sa**] or [**Mr1**].

Lemma 10.4.6. *If $d\omega = 0$ (i.e., we are working with an almost-Kähler manifold), then the Dirac operator $\partial\!\!\!/_{\hat{A}}$ associated to the connection $\hat{A}$ satisfies the equation $\partial\!\!\!/_{\hat{A}} = \sqrt{2}(\overline{\partial}_{\hat{A}} + \overline{\partial}^*_{\hat{A}})\colon \Omega^{0,0} \oplus \Omega^{0,2} \to \Omega^{0,1}$.* □

Suppose now that (X, ω, J, g) is Kähler, that is, in addition to the condition $d\omega = 0$, J is integrable. This assumption implies that $\overline{\partial}_{\hat{A}}^2 = (F_{\hat{A}}^+)^{0,2}$; moreover it can be shown that $F_{\hat{A}}^{0,2} = \frac{1}{2}F_A^{0,2}$ (cf. [**Mr1**]), hence $\overline{\partial}_{\hat{A}}^2 = (F_{\hat{A}}^+)^{0,2} = \frac{1}{4}\overline{\alpha}\beta$. Applying the $\overline{\partial}_{\hat{A}}$-operator to the first Seiberg-Witten equation and using this latter identity, we get

$$\overline{\partial}_{\hat{A}}(\overline{\partial}_{\hat{A}}\alpha) + \overline{\partial}_{\hat{A}}(\overline{\partial}_{\hat{A}}^*\beta) = \frac{1}{4}\overline{\alpha}\beta\alpha + \overline{\partial}_{\hat{A}}(\overline{\partial}_{\hat{A}}^*\beta) = 0.$$

Pairing this equation with $\overline{\beta}$, we find that $\frac{1}{4}|\alpha|^2|\beta|^2 + |\overline{\partial}_{\hat{A}}^*\beta|^2 = 0$, implying that either $\alpha = 0$ or $\beta = 0$; moreover $\overline{\partial}_{\hat{A}}^*\beta = 0$. This observation takes us into the holomorphic category, since the equation $(F_{\hat{A}}^+)^{0,2} = \frac{1}{4}\overline{\alpha}\beta = 0$ means that $\hat{A}$ defines a holomorphic structure on $\underline{\mathbb{C}}$. Note that above we only dealt with the spin^c structure induced by J. In fact, these arguments extend to arbitrary spin^c structures, and show that for a solution $(\hat{A}, (\alpha, \beta))$ in the product $\mathcal{A}_{L_a} \times \Omega^{0,0}(L_a) \times \Omega^{0,2}(L_a)$ (where the spin^c structure $(W^\pm, \rho)$ under examination satisfies $\det W^+ = K_{X,J}^{-1} \otimes L_a^{\otimes 2}$) we have $\overline{\partial}_{\hat{A}}^*\beta = 0$ and $\overline{\alpha}\beta = 0$, hence $\hat{A}$ equips L_a with a holomorphic structure.

Based on the first equation, $\overline{\partial}_{\hat{A}}^*\beta = 0$ implies that $\overline{\partial}_{\hat{A}}\alpha = 0$, hence α is a holomorphic section of L_a. Because ω is a $(1,1)$-form, we have

$$\langle c_1(\det W^+) \cup [\omega], [X]\rangle = \frac{i}{2\pi}\int_X F_A \wedge \omega = \frac{i}{2\pi}\int_X (F_A^+)^{1,1} \wedge \omega.$$

By the Seiberg-Witten equations, however, this quantity is equal to the integral $-\frac{1}{8\pi}\int_X(|\alpha|^2 - |\beta|^2)\mathrm{vol}(M)$; hence the sign of $\langle c_1(\det W^+) \cup [\omega], [X]\rangle$ determines whether $\alpha = 0$ or $\beta = 0$. The quantity $\langle c_1(\det W^+) \cup [\omega], [X]\rangle$ is frequently called the *degree* $\deg(\det W^+)$ of the line bundle $\det W^+$.

Assume now that for the given spin^c structure $(W^\pm, \gamma)$ we have $\deg(\det W^+) < 0$. Applying results of Kazdan and Warner [**KW**], it can be shown that if α is a holomorphic section of the line bundle L_a satisfying $\det W^+ = K_{X,J}^{-1} \otimes L_a^{\otimes 2}$, then the connection $\hat{A}$ solving the Seiberg-Witten equations with α is unique (up to gauge equivalence). Associating $\alpha^{-1}(0) \subset X$ to the solution $(\hat{A}, (\alpha, 0))$, we reach the following conclusion.

Theorem 10.4.7. *Under the above circumstances, the moduli space of solutions of the Seiberg-Witten equations corresponding to the spin^c structure $(W^\pm, \gamma)$ can be identified with the moduli space of holomorphic divisors corresponding to the line bundle L_a (with $\det W^+ = K_{X,J}^{-1} \otimes L_a^{\otimes 2}$).*

A similar argument shows that if $\deg(\det W^+) > 0$ (i.e., for a solution we have $\alpha = 0$), then the moduli space can be identified with the moduli space of holomorphic divisors corresponding to the line bundle $K_{X,J} \otimes L_a^{-1}$.

Now fairly standard arguments of algebraic geometry compute the Seiberg-Witten invariants of Kähler surfaces with $b_2^+ > 1$ [**FM2**]. These computations show that the basic classes of minimal elliptic surfaces are (well-described) multiples of the Poincaré dual of the fiber, while a minimal surface of general type admits only $\pm c_1$ as basic classes. The blow-up formula completes the description of basic classes of Kähler surfaces with $b_2^+ > 1$.

In the almost-Kähler case (i.e., when J is not integrable, consequently $\overline{\partial}^2_{\hat{A}} = (F^+_{\hat{A}})^{0,2}$ fails to hold), Taubes introduced a suitable perturbation to study the Seiberg-Witten equations. For the perturbation $\delta = F^+_{A_0} - \frac{ir}{4}\omega$, the perturbed equations

$$\partial\!\!\!/_A \psi = 0 \quad \text{and} \quad F^+_A = F^+_{A_0} - \frac{ir}{4}\omega + i\sigma(\psi)$$

for the canonical spin^c structure admit a unique solution once r is large enough. Since the signed number of solutions is independent of the perturbation (at least for manifolds with $b_2^+ > 1$), we conclude Theorem 2.4.7. Using similar perturbations, Taubes also studied the equations for other spin^c structures. Considering the zero set of a suitable component of the spinor of a solution (exactly as in the Kähler case), he showed that these submanifolds "converge" to a pseudo-holomorphic curve when $r \to \infty$. This argument led him to the proof of Theorem 10.1.15. A more refined analysis — together with a construction of solutions to the Seiberg-Witten equations starting from a pseudo-holomorphic curve — allowed Taubes to construct a correspondence between Seiberg-Witten solutions associated to a particular spin^c structure and pseudo-holomorphic curves representing a related homology class, cf. Remark 10.1.16(a). (For more details, see [**T5**] and [**Sa**].) Based on these theorems, remarkable results in the topology of symplectic 4-manifolds have emerged in the past few years.

Chapter 11

Stein surfaces

A *Stein manifold* is a complex manifold that admits a (proper) biholomorphic embedding in some $\mathbb{C}^N$. Since a holomorphic function never maximizes its norm on an open set, it is easy to see that Stein manifolds can never be compact. However, the Stein condition constrains the behavior of the manifold near infinity, so that Stein manifolds share many properties with closed Kähler manifolds such as the projective surfaces we have discussed in previous chapters. For example, we will see that in complex dimension 2, Seiberg-Witten theory can be applied, yielding strong constraints on the genus function of a Stein surface. A much more surprising development, however, is that the question of which smooth manifolds admit Stein structures can be completely reduced to a problem in handlebody theory. This is remarkable, since a corresponding reduction for closed Kähler manifolds seems unlikely to exist. For 4-manifolds, one can express the Stein condition in terms of Kirby diagrams. As a direct consequence, one obtains genus bounds and diffeomorphism invariants for Kirby diagrams. Since Stein structures are intimately related to geometric structures called *contact structures*, we also obtain a powerful tool for constructing contact 3-manifolds. Much of this chapter is based on [**G13**]; a more expository version of that paper, along with a list of open problems, appears as [**G14**].

11.1. Contact structures

We begin by considering smooth, oriented 2-plane fields ξ on an orientable 3-manifold M. It is easy to verify that such a plane field can be described as the kernel of a nowhere-zero 1-form α on M, and that ξ determines α uniquely up to multiplication by nowhere-zero functions $M \to \mathbb{R}$. The condition that $\alpha \wedge d\alpha$ be identically zero is equivalent to specifying that ξ be *integrable*, or locally equivalent to the horizontal plane field $\ker dz$ in $\mathbb{R}^3$. (It follows that ξ determines a *foliation*, or decomposition of M as a union of disjoint surfaces that can be described in local coordinates as the planes $z =$ constant in $\mathbb{R}^3$.) Contact structures are specified by the opposite condition:

Definition 11.1.1. A *contact structure* on M^3 is a 2-plane field $\xi = \ker \alpha$ for which $\alpha \wedge d\alpha$ is nowhere zero. The pair (M, ξ) is a *contact manifold*. A *contactomorphism* between contact manifolds (M_i, ξ_i) is a diffeomorphism $f : M_1 \to M_2$ such that $f_* \xi_1 = \xi_2$. Two contact structures on M are *isotopic* if there is a contactomorphism between them that is smoothly isotopic to the identity on M.

Remark 11.1.2. Foliations have long been an important tool in 3-manifold topology. Contact structures have related, but less well explored, connections to the topology of the ambient 3-manifolds. A recent generalization [**ET**] includes both these notions: A plane field ξ is called a *confoliation* if $\alpha \wedge d\alpha \geq 0$ (relative to a fixed orientation of M). This theory is still virtually uncharted territory.

A contact structure ξ determines an orientation of M via the volume form $\alpha \wedge d\alpha$. It is easy to see that this orientation is independent of the choice of α, and of the orientation of ξ. Any contact structure is locally contactomorphic to the standard structure given by $\alpha = dz + x\,dy$ on $\mathbb{R}^3$, and two contact structures ξ_0 and ξ_1 on a closed manifold M are isotopic if they are connected by a path ξ_t $(0 \leq t \leq 1)$ of contact structures.

Remark 11.1.3. The definition of contact structures can be extended to arbitrary odd dimensions by the condition that $\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ never vanish. In dimensions congruent to $3 \bmod 4$, we can also allow ξ to be nonorientable by only requiring α to exist locally. In this case, ξ still defines an orientation on M (since the volume form is independent of the sign of α). Since we will not need these generalizations, we refer the reader to [**ABKLR**] for further details. The numerous relations between contact and symplectic structures, including the above local triviality and formal similarity of the contact condition with the symplectic condition $\omega \wedge \cdots \wedge \omega \neq 0$, suggest that contact structures should be thought of as the odd-dimensional analogs of symplectic structures.

For any $m \geq 0$, the manifold $\#mS^1 \times S^2$ admits a canonical contact structure ξ_c. The precise sense in which ξ_c is canonical will be discussed at the end of this section. For $m = 0$, we construct ξ_c by identifying S^3 as the boundary of the unit disk D^4 in $\mathbb{C}^2$. Each tangent space to S^3 contains a unique complex line of $\mathbb{C}^2$, namely $T_pS^3 \cap iT_pS^3$. Thus, we obtain a field of oriented real 2-planes on S^3; this is the required contact structure ξ_c. (Note that ξ_c is also the normal 2-plane field to the Hopf fibration obtained by intersecting the complex lines through 0 in $\mathbb{C}^2$ with S^3.) It can be shown that $(S^3 - \infty, \xi_c)$ is contactomorphic to $\mathbb{R}^3$ with the standard structure $\ker(dz + x\,dy)$. For $m \geq 1$, it is possible to construct $(\#mS^1 \times S^2, \xi_c)$ in a similar manner, by putting a suitable complex (in fact, Stein) structure on $D^4 \cup m$ 1-handles and taking the induced 2-plane field on the boundary [**E2**]. Note that any 3-manifold M embedded in a complex surface inherits an oriented 2-plane field $TM \cap iTM$. (In fact, an almost-complex structure is sufficient here.) We will return to the question of when this 2-plane field is a contact structure in the next section.

Exercise 11.1.4. Write down a 1-form α on $\mathbb{C}^2 = \mathbb{R}^4$ whose restriction to S^3 generates ξ_c. Show directly that ξ_c is a contact structure determining the boundary orientation on $S^3 = \partial D^4$. (*Hint*: Obtain α by composing $d(r^2)$ with multiplication by i on $T_p\mathbb{C}^2$. Then show that $d(r^2) \wedge \alpha \wedge d\alpha$ is a positive volume form on $\mathbb{C}^2 - \{0\}$.)

Next, we consider links that are suitably compatible with a given contact structure.

Definition 11.1.5. Let (M, ξ) be a contact 3-manifold. A *Legendrian link* L in (M, ξ) is a link in M whose tangent vectors all lie in ξ. A *Legendrian isotopy* of L is an isotopy through Legendrian links. The *canonical framing* on L is the framing (up to isotopy) induced by any vector field on L transverse to ξ.

It can be shown that any Legendrian isotopy extends to an ambient isotopy through contactomorphisms of (M, ξ), so for practical purposes, Legendrian isotopic links are essentially the same. In particular, Legendrian isotopies obviously preserve the canonical framing. (Recall that strictly speaking, the framing of the normal bundle determined by a vector field is a basis for each normal plane, so it also depends on a choice of orientation of L — we continue to ignore this ambiguity since it has no effect on framing coefficients or attached handles.)

It is not hard to draw Legendrian links in (S^3, ξ_c). First, we remove a point to obtain $(\mathbb{R}^3, \ker(dz + x\,dy))$. Then, we orthogonally project into the y-z plane. A Legendrian curve $\gamma(t) = (x(t), y(t), z(t))$ is characterized by the equation $x = -\frac{dz}{dy}$, that is, its x-coordinate is determined by the slope of its projection, so that at each crossing the curve of greater slope passes behind the other. The projection can never have vertical tangencies, since $x = -\frac{dz}{dy}$ is always finite. Instead, a generic Legendrian link projection has cusp singularities (isotopic to $z^2 = y^3$ or $z^2 = -y^3$) as in Figure 11.1(a), where the tangent line to L is parallel to the x-axis. As for ordinary links, we can assume that the only self-crossings of a Legendrian link projection are double points, and these must be transverse (since the coordinates $x =$

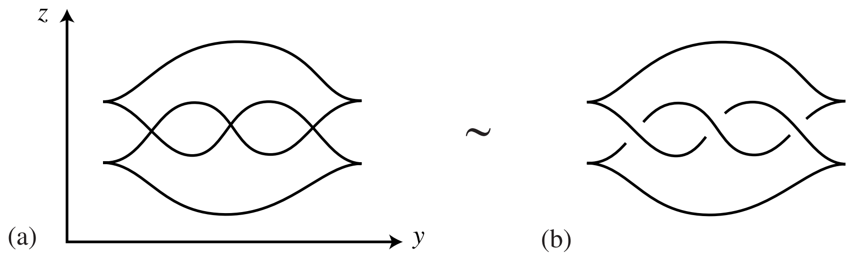


Figure 11.1. Legendrian knot.

$-\frac{dz}{dy}$ must be different). Thus, Figure 11.1 shows a typical Legendrian knot projection. Since the curve of more positive slope always crosses behind at a double point, the projection completely determines the Legendrian knot (as in (b) of the figure), and we continue to draw the undercrossings only for the sake of clarity. It is easy to see that any immersed collection of circles in $\mathbb{R}^2$ with only transverse double point singularities generates a Legendrian link — we simply replace its vertical tangencies by cusps and reconstruct the x-coordinate by the formula $x = -\frac{dz}{dy}$.

Exercise 11.1.6. * Prove that any link in (S^3, ξ_c) is isotopic to a Legendrian link. (This is actually true in any contact 3-manifold.)

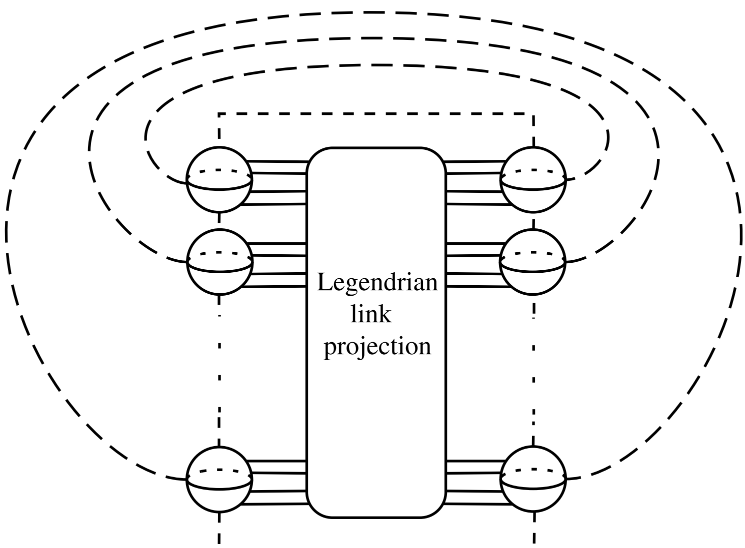


Figure 11.2. Legendrian link diagram in standard form.

As the next theorem shows, the above description of Legendrian links in (S^3, ξ_c) generalizes to $(\#mS^1 \times S^2, \xi_c)$ for any m. We begin by representing

$\#mS^1 \times S^2$ as $\partial(D^4 \cup m$ 1-handles), where the attaching balls of each 1-handle are aligned horizontally as in Figure 11.2. We think of the contact manifold $(\#mS^1 \times S^2, \xi_c)$ as being obtained from (S^3, ξ_c) by removing the interiors of the attaching balls and gluing along the resulting boundaries by a contactomorphism. (Beware that while such a contactomorphism exists, it is somewhat complicated — the balls cannot quite be round, and the gluing map will have a twist in it. One of the main implications of the following theorem is that these complications can be ignored.) Now suppose we draw a link projection as before, with cusps instead of vertical tangencies, and only transverse double point singularities. We allow the curves to run over the 1-handles (using the usual identification by reflection) but require the projection to lie in the region between the attaching balls (as indicated by the box in Figure 11.2).

Definition 11.1.7. A diagram as described above (Figure 11.2) is called a *Legendrian link diagram in standard form.*

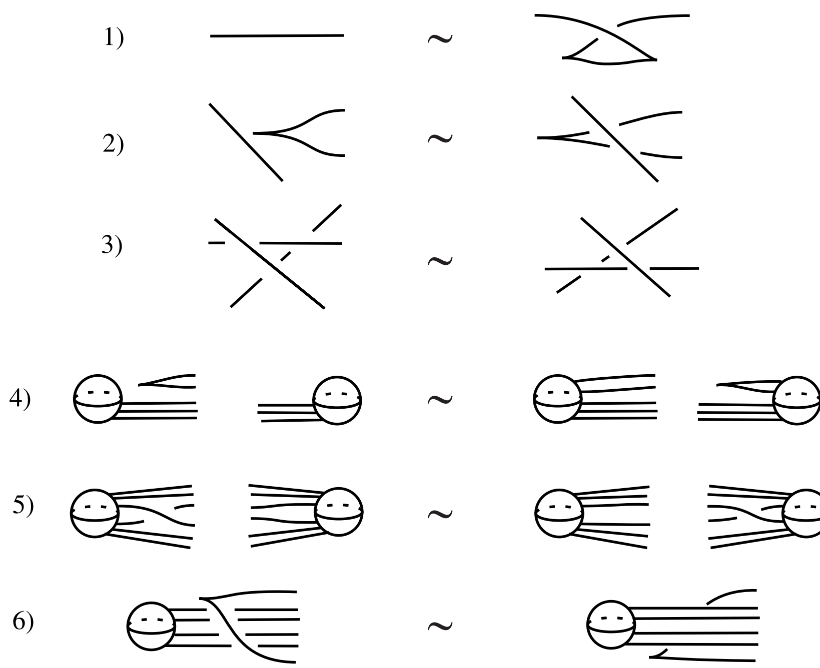


Figure 11.3. Legendrian Reidemeister moves in $\# mS^1 \times S^2$.

Theorem 11.1.8. ([**G13**]) *A Legendrian link diagram determines a Legendrian link in* $(\#mS^1 \times S^2, \xi_c)$ *(via the above identification of* $(\#mS^1 \times S^2, \xi_c)$ *with* (S^3, ξ_c) *surgered on the balls). Any Legendrian link in* $(\#mS^1 \times S^2, \xi_c)$ *is Legendrian isotopic to one given by such a diagram. Two Legendrian links* *in standard form are Legendrian isotopic in* $\#mS^1 \times S^2$ *if and only if they are related by a sequence of the 6 moves shown in Figure 11.3 (and their images under* $180°$ *rotation about each coordinate axis), together with isotopies within the box of Figure 11.2 that introduce no vertical tangencies.* □

Informally, the theorem says that the complications of the gluing maps on the 2-spheres can be ignored, and that (after a bit of trickery) strands of a link that wrap around attaching balls can be pulled into the box. The theorem also gives a complete reduction of Legendrian link theory in $(\#mS^1 \times S^2, \xi_c)$ to the theory of diagrams in standard form, by supplying a complete set of moves of the diagrams corresponding to Legendrian isotopies. The first 3 of these moves are precise analogs of the Reidemeister moves of ordinary link diagrams (Figure 4.26), and are well-known to be a complete set of moves in (S^3, ξ_c). The remaining moves are required for sliding cusps and crossings over 1-handles, and for swinging a strand of the link around an attaching ball.

We also wish to understand the canonical framing of a Legendrian knot K in standard form. It suffices to compute its framing coefficient, which is called the *Thurston-Bennequin invariant* $tb(K) \in \mathbb{Z}$. Recall from Section 5.4 (Figure 5.37) that framing coefficients are well-defined in the presence of 1-handles, provided that we connect each pair of balls by an arc, and recompute the framing coefficient of any knot that isotopes through such an arc. We choose the obvious family of dashed arcs avoiding the box in Figure 11.2; without ambiguity we can suppress these from the notation. The framing coefficient $tb(K)$ will then be invariant under the first five moves in Figure 11.3, but under Move 6 it will change by twice the algebraic number of times K crosses the relevant 1-handle. To compute $tb(K)$ explicitly for a Legendrian knot K in $\mathbb{R}^3$, first observe that the vector field $\frac{\partial}{\partial z}$ is transverse to the contact structure $\ker(dz + x\,dy)$ everywhere, so $tb(K) = \ell k(K, K')$, where K' is obtained from K by a small vertical displacement. The proof of Theorem 11.1.8 shows that for Legendrian knots in standard form in $(\#mS^1 \times S^2, \xi_c)$, the canonical framing is still determined by a vertical displacement of K. It is now easy to see that the canonical framing differs from the blackboard framing by a left half-twist for each cusp (Figure 11.4). If $\lambda(K)$ and $\rho(K)$ denote the numbers of left and right cusps of K, and $w(K)$ is its writhe (Proposition 4.5.8), we obtain

$$tb(K) = w(K) - \frac{1}{2}(\lambda(K) + \rho(K)) = w(K) - \lambda(K)$$

for any Legendrian knot K in standard form. (The last equality follows because $\lambda(K) = \rho(K)$, since left and right cusps alternate as we travel around K.) For example, the right trefoil knot in Figure 11.1 has $tb(K) = 1$.

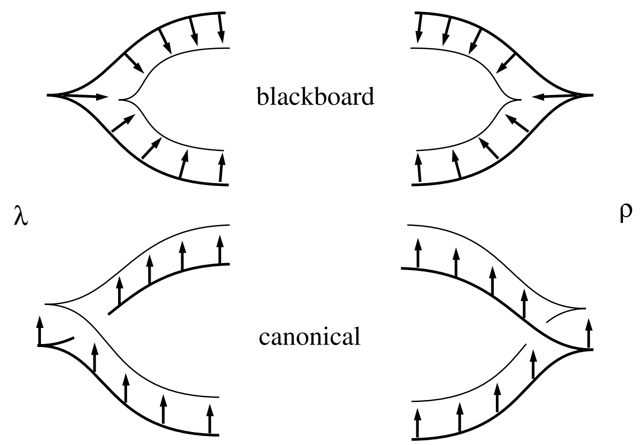


Figure 11.4. Computing the coefficient $tb(K)$ of the canonical framing.

There is one more basic invariant for an oriented Legendrian knot K in standard form. Note that the vector field $\frac{\partial}{\partial x}$ on $\mathbb{R}^3$ lies in $\ker(dz + x\,dy)$ everywhere. The proof of Theorem 11.1.8 shows that if we restrict $\frac{\partial}{\partial x}$ to the box in Figure 11.2, it fits together in the obvious way on the 1-handles. (In fact, it uniquely determines a nowhere-zero vector field in ξ_c over all of $\#mS^1 \times S^2$, since nowhere-zero vector fields in trivial complex line bundles are classified by $H^1(\,\cdot\,,\mathbb{Z})$, cf. Exercise 5.6.4(d).) Since the tangent vector field τ to K lies in ξ_c, it has a well-defined winding number with respect to $\frac{\partial}{\partial x}$ as we travel once around K in the direction specified by its orientation. (Positive winding corresponds to right-handed twisting about the z-axis.) This winding number is called the *rotation number* $r(K) \in \mathbb{Z}$. It is related to tb by the formula that $tb(K)+r(K)+1$ is congruent mod 2 to the number of times K goes over 1-handles (Exercise 11.3.11). To compute $r(K)$, we count (with sign) how many times τ passes $\frac{\partial}{\partial x}$ as we traverse K. Note that such passing only occurs at cusps. Let $\lambda_+(K)$ (resp. $\lambda_-(K)$) be the number of left cusps at which K is oriented upward (downward), and define $\rho_\pm$ similarly (Figure 11.5). Let $t_\pm = \lambda_\pm + \rho_\pm$ be the total number of upward (downward) cusps. Since each downward left cusp represents a positive crossing of τ past $\frac{\partial}{\partial x}$ and each upward right cusp gives a negative crossing, we have

$$r(K) = \lambda_- - \rho_+ = \rho_- - \lambda_+ = \frac{1}{2}(t_- - t_+),$$

where the second equality is obtained by replacing $\frac{\partial}{\partial x}$ by $-\frac{\partial}{\partial x}$, and the third is obtained by averaging the first two.

Exercises 11.1.9. (a) Use Figure 11.3 and the formula for $r(K)$ to prove directly that the latter is invariant under Legendrian isotopy. Similarly,

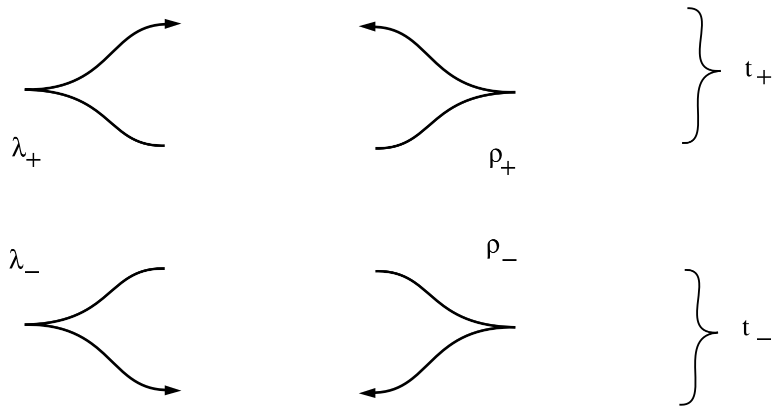


Figure 11.5. Convention for computing $r(K)$.

prove that $tb(K)$ is invariant under the first five moves and changes as required under Move 6. Show that reversing the orientation of K reverses the sign of $r(K)$. Explain this in terms of the definition of $r(K)$. Show that for any integer $n \leq tb(K)$ there is a Legendrian knot K' smoothly isotopic to K with $tb(K') = n$. Show that K is smoothly isotopic to a Legendrian K' realizing any preassigned value of $r(K')$.

(b) Suppose that K is nullhomologous in $\#mS^1 \times S^2$ and let F be any Seifert surface, oriented so that $\partial F = K$. Prove that $r(K)$ is the Chern number $\langle c_1(\xi_c, \tau), F\rangle$ of ξ_c relative to the trivialization over K induced by τ. (Note that $c_1(\xi_c, \tau)$ lies in $H^2(\#mS^1 \times S^2, K; \mathbb{Z})$, cf. Exercise 5.6.2(c).) This characterization of $r(K)$ generalizes to a definition of $r(K, F)$ for all nullhomologous K in arbitrary contact 3-manifolds. If $c_1(\xi) \neq 0$, then $r(K, F)$ depends on the choice of $[F] \in H_2(M, K; \mathbb{Z})$.

We now return to the discussion of arbitrary contact 3-manifolds (M, ξ). Since framing coefficients can be defined for any nullhomologous knot in an oriented 3-manifold (cf. Section 4.5), the Thurston-Bennequin invariant $tb(K) \in \mathbb{Z}$ is well-defined for any nullhomologous Legendrian knot K. As we saw for (S^3, ξ_c), any knot $K \subset M$ is isotopic to a Legendrian knot K' (in fact, by a C^0-small isotopy). In the nullhomologous case, we can arrange for $tb(K')$ to be any sufficiently small integer. It is less clear, however, when we can increase $tb(K')$. Any (M, ξ) obviously contains a Legendrian unknot with $tb(K) = -1$. (Draw this in $\mathbb{R}^3$ and use the fact that ξ is locally $\ker(dz + x\,dy)$.) If it contains a Legendrian unknot with $tb(K) = 0$, then ξ is called *overtwisted*; otherwise it is called *tight*. In an overtwisted contact manifold, any nullhomologous knot K can be made Legendrian with $tb(K)$ realizing any preassigned integer, whereas in a tight contact manifold, the Legendrian knots smoothly isotopic to a fixed nullhomologous knot K realize only those integers less than or equal to some maximal value $TB(K) \in \mathbb{Z}$. The overtwisted contact structures on a fixed closed, oriented 3-manifold M are sufficiently "flexible" that they can be completely classified: There is exactly one (up to isotopy) for each homotopy class of oriented plane fields on M [**E1**]. (These latter homotopy classes have been classified; see Section 11.3. For any closed M there are infinitely many such classes.) Tight contact structures, however, are much more subtle, and are the subject of much current research. It is not known if every closed, oriented 3-manifold admits a tight contact structure, although at present this seems unlikely. If we impose the stronger condition of "holomorphic fillability" (see the text preceding Proposition 11.2.8), the corresponding statement is false: A recent result of Lisca [**Ls2**] shows that the Poincaré homology sphere Σ with reversed orientation has no holomorphically fillable contact structure (see the text following Exercises 11.2.11). This same manifold may admit no tight structure, in which case $\Sigma \# \overline{\Sigma}$ would admit no tight structure with either orientation. It is known that a fixed closed 3-manifold admits at most finitely many homotopy classes of plane fields containing holomorphically fillable contact structures (Theorem 11.4.3, [**KM2**]), but the 3-torus admits infinitely many isotopy classes of such contact structures within a single homotopy class of plane fields [**EP**]. Each of the manifolds $\# m S^1 \times S^2$, $m \geq 0$, admits a unique tight contact structure respecting the given orientation [**Be**], [**E4**]; this is the (holomorphically fillable) canonical structure ξ_c discussed above. See, e.g., the problem list of [**G14**] for further discussion.

11.2. Kirby diagrams of Stein surfaces

Recall that a Stein manifold is a complex manifold with a proper biholomorphic embedding in $\mathbb{C}^N$ — that is, a smooth, affine analytic variety. There is a corresponding class of compact, complex manifolds with boundary, called *Stein domains*, which will be characterized by the theorem below. In complex dimension 2, we will abuse terminology slightly by referring to both classes of objects as *Stein surfaces* (open and compact, respectively).

Theorem 11.2.1. ([**Gr**], see also [**E6**].) *A complex surface S (compact with boundary or open) is a Stein surface if and only if it admits a proper Morse function $f\colon S \to [0,\infty)$ (with $\partial S = f^{-1}(1)$ and $f(S) \subset [0,1]$ in the compact case) such that away from the critical points each subset $f^{-1}(t)$, with the plane field induced by the complex structure, is a contact 3-manifold whose contact orientation agrees with its orientation as $\partial f^{-1}[0,t]$.* □

The corresponding theorem holds in higher dimensions if we replace the contact condition by the stronger condition "strictly pseudoconvex"; these conditions are equivalent in the dimension of interest. For a Stein manifold properly embedded in $\mathbb{C}^N$, the distance to a generic point of $\mathbb{C}^N$ will provide the required Morse function. The theorem shows that for any Stein manifold S and f as above with regular value t, $f^{-1}[0,t]$ will be a Stein domain, and that the interior of any Stein domain will be a Stein manifold. Thus, we can think of Stein domains as compact analogs of Stein manifolds. Clearly, the boundary of any Stein domain has an inherited contact structure. In complex dimension 2, this structure is tight [**E3**], providing an important source of tight contact manifolds. (For example, consider the unit ball in $\mathbb{C}^2$ with Morse function $\|z\|^2$; this is a Stein domain with boundary (S^3, ξ_c).) Conversely, any compact complex surface with nonempty (correctly oriented) contact boundary can be deformed into a blow-up of a Stein surface [**Bog**]. (Note that Stein surfaces are always minimal, since their interiors embed holomorphically in $\mathbb{C}^N$ and thus contain no closed complex curves.)

A theorem of Eliashberg [**E2**] characterizes those smooth manifolds that admit Stein structures. It has long been known that the Morse functions given in the previous theorem have no critical points of index larger than the complex dimension of S (cf. Chapter 7 of [**M2**]). Eliashberg proved the converse: For $n > 2$, a smooth, almost-complex manifold X of real dimension $2n$ (compact with boundary or open) admits a Stein structure if it has a proper Morse function $f\colon X \to [0,\infty)$ (with $\partial X = f^{-1}(1)$ and $f(X) \subset [0,1]$ in the compact case) without critical points of index $> n$. The same technique applies in the case $n = 2$ (cf. [**E5**]), but a delicate condition arises on the framings of the 2-handles. (The corresponding condition in higher dimensions is satisfied automatically.) In the compact case, the condition can be stated easily using Theorem 11.1.8:

Theorem 11.2.2. ([**G13**]) *A smooth, compact, connected, oriented 4-manifold X admits a Stein structure (inducing the given orientation) if and only if it can be presented as a handlebody by attaching 2-handles to a framed link in $\partial(D^4 \cup$ 1-handles$) = \#mS^1 \times S^2$, where the link is drawn in standard form (Definition 11.1.7) and the framing coefficient on each link component K is given by $tb(K) - 1$.* □

Similarly, an open Stein surface is characterized as being the interior of a (possibly infinite) handlebody without 3- or 4-handles, where the 2-handles are attached to Legendrian knots with framings twisted -1 relative to the canonical framings. These are more awkward to draw in general, but any finite subhandlebody can be described as in the theorem.

Exercise 11.2.3. (a) Prove that any even-dimensional oriented handlebody without handles of index > 2 admits an almost-complex structure. (*Hint*: The set of complex vector space structures on the tangent space T_pX^{2n} is given by $SO(2n)/U(n)$, which is simply connected. Use obstruction theory (Section 5.6) to find a section of the corresponding bundle.)

(b)* Using the above discussion of Stein manifolds, prove the Lefschetz Hyperplane Theorem 1.4.22, assuming H is transverse to X.

Eliashberg proves his theorem by using the given Morse function to build X as a handlebody, guided by the given almost-complex structure (which automatically exists when $n = 2$ by Exercise 11.2.3(a)) to make the gluing maps holomorphic, and trimming back the boundary after each handle is attached to recover the contact (or pseudoconvexity) condition. This explains the framing coefficient $tb(K) - 1$: When $n = 2$, each 2-handle is defined to be a neighborhood of $D^2 \times 0 \subset i\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C}^2$. (Note that the equality reverses the natural orientations.) The attaching circle $S^1 \times 0$ is glued to the given Legendrian knot K in $\#mS^1 \times S^2$, so its tangent vector field maps into ξ_c. Since ξ_c is a complex line field, it follows that $i\tau$ also maps into ξ_c, and it gives the canonical framing on K. But $i\tau$ differs from the product framing on $S^1 \times 0 \subset S^1 \times \mathbb{R}^2$ by one twist, since $\tau\colon S^1 \to S^1$ has degree 1. Thus, the framing coefficient on K, corresponding to the product framing on $S^1 \times 0$, is given by $tb(K) - 1$. (The sign is related to the orientation-reversal mentioned above.) Note that the same -1-twist appeared when we analyzed 2-handles corresponding to critical points of Lefschetz fibrations; see Section 8.2.

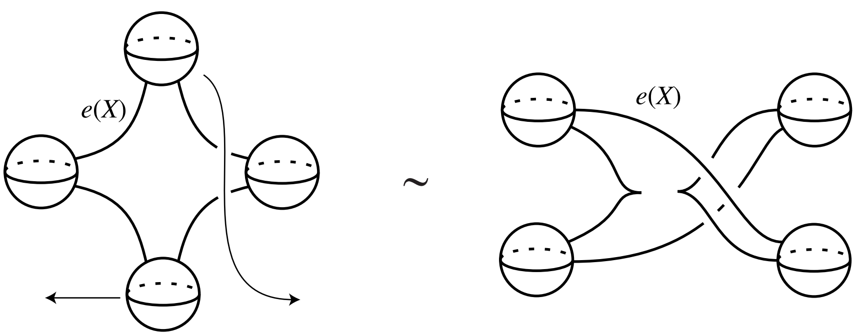


Figure 11.6. Putting a diagram into standard form.

Example 11.2.4. We analyze which disk bundles X over the torus T^2 admit Stein structures. First, we move the usual picture of X into standard form by lowering the uppermost attaching ball as in Figure 11.6, and replacing vertical tangencies by cusps. Note that the crossings in the resulting diagram have the correct form for a Legendrian projection; otherwise we would have to modify the picture (as in Exercise 11.1.6, for example). Our Legendrian knot has $tb = 0$, and this can be reduced by any integer (by adding zig-zags to lower λ, cf. Exercise 11.1.9(a)), so Theorem 11.2.2 realizes X as a Stein surface whenever $e(X) < 0$. As is often the case, we can improve on our initial answer slightly by more cleverly constructing our

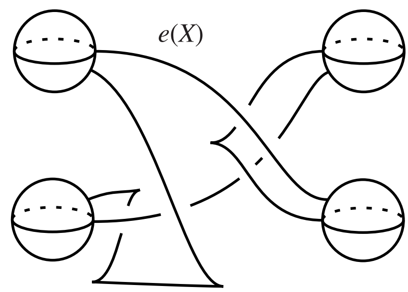


Figure 11.7. Stein structure on D^2-bundle X over T^2 with $e(X) \leq 0$.

diagram. By wrapping one strand around the lower left attaching ball, we obtain a smoothly isotopic picture of X, Figure 11.7. Now we have $tb = 1$, but the framing coefficient is still $e(X)$, so we conclude that X admits a Stein structure whenever $e(X) \leq 0$. We will see in Section 11.4 that this result is optimal; when $e(X) > 0$, X admits no Stein structure.

Exercises 11.2.5. **(a)*** Prove that a disk bundle $X^4 \to F$ (with X oriented and F a closed, connected surface) admits a Stein structure provided that $e(X)+\chi(F) \leq 0$ (cf. the text preceding Exercises 4.6.7 for F nonorientable). We will see in Exercise 11.4.11(c) that this is optimal.

(b)* Figure 11.6 shows a Stein structure on the disk bundle over T^2 with $e(X) = -1$. Show that the same Stein surface (up to "Stein homotopy" [**E6**]) can be obtained by adding a zig-zag to Figure 11.7. (*Hint*: It suffices to show [**E6**] that the attaching circles are Legendrian isotopic. Use Figure 11.3 — specifically, arrange to make Move 6 at the lower left attaching ball.)

(c)* Let h be a 2-handle of a Stein surface X presented in standard form. For both values of the sign $\pm$, let $X_\pm$ denote the manifold obtained from X by putting a $(\pm)$-self-plumbing in h. (This adds ± 2 to its framing coefficient if we use the convention of Example 6.1.3). Prove that $X_\pm$ admits a Stein structure. (*Hint*: Figures 6.7 and 6.10.)

This last exercise can be used to show that the high-dimensional version of Eliashberg's characterization of Stein manifolds works up to *homeomorphism* in dimension 4.

Theorem 11.2.6. ([**G13**]) *An open, oriented topological 4-manifold X is (orientation-preserving) homeomorphic to a Stein surface if and only if it is homeomorphic to the interior of a (possibly infinite) handlebody H without 3- or 4-handles.*

Proof. By subtracting sufficiently large even numbers from the framings of the 2-handles of H, we obtain a handlebody H' whose interior is Stein. By Exercise 11.2.5(c), we can return to the original framings by putting positive self-plumbings in the 2-handles of H'. The resulting manifold H'' has Stein interior, and it is obtained from the gluing data for H by attaching kinky handles instead of 2-handles. It is now easy to extend the kinky handles of H'' to Casson handles (cf. Example 6.1.3), preserving the Stein structure: At each stage, we can attach kinky handles to the required framed circles by repeating the previous argument. (In fact, a close look at the required framed circles in Figure 12.78 shows that they can be drawn with $tb = 0$, so after the first stage we can use any kinky handles with more positive than negative self-plumbings.) We now have a Stein surface S obtained from H by replacing all 2-handles by Casson handles and removing any remaining boundary. Since Casson handles are homeomorphic to open 2-handles, S is homeomorphic to X. □

Note that even when H is a finite handlebody, the corresponding Stein surface S will usually have infinitely many handles. We should expect a typical S not to be diffeomorphic to the interior of any compact manifold, and that a typical X generates uncountably many diffeomorphism types of such Stein surfaces (since we can use arbitrarily complicated Casson handles in the construction; cf. Exercise 9.4.13(b)). For example, consider $\mathbb{R}^2$-bundles $X \to S^2$, corresponding to 2-handlebodies H on the unknot. By the theorem, these are all orientation-preserving homeomorphic to Stein surfaces, but we will see (Theorem 11.4.7) that such an X (with its usual smoothing) admits a Stein structure if and only if $e(X) \leq -2$. For example, if $e(X) = k = 0$ (resp. -1), the open manifold shown in Figure 6.14 admits a Stein structure, so it is homeomorphic but not diffeomorphic to $S^2 \times \mathbb{R}^2$ (resp. $\overline{\mathbb{CP}^2} - \{\text{pt.}\}$), and it contains no sphere generating its homology. For any fixed $e(X)$, X will generate infinitely many diffeomorphism types of such Stein surfaces S, and we can arrange for the minimal genus of a generator of $H_2(S) \cong \mathbb{Z}$ to be arbitrarily large (Exercise 11.4.11(d)). For $e(X) = \pm 1$ (and one would expect in general) there are uncountably many diffeomorphism types of such S which are not interiors of closed manifolds (cf. the solution of Exercise 9.4.13(b) and the method of proof of Theorem 11.2.7). A related example of "exotic" Stein surfaces is the following:

Theorem 11.2.7. ([**G13**]) *There is a Stein structure on R, the exotic $\mathbb{R}^4$ of Figure 6.16 and Theorem 9.3.8. There are uncountably many diffeomorphism types of Stein surfaces homeomorphic to $\mathbb{R}^4$.*

Proof. It is routine to transform Figure 6.16 of R into Figure 11.8, where the fine curve is the attaching circle of the Casson handle CH. (To see this easily, change Figure 11.8 to dotted circle notation in the obvious way, then isotope to get Figure 6.16.) The fine curve has $tb = 0$, so we can attach the required 0-framed Casson handle (with one positive self-plumbing at each stage), obtaining a Stein surface. However, the other knot has $tb = -2$, so

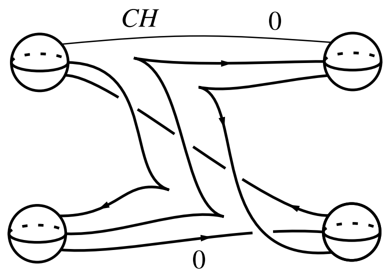


Figure 11.8

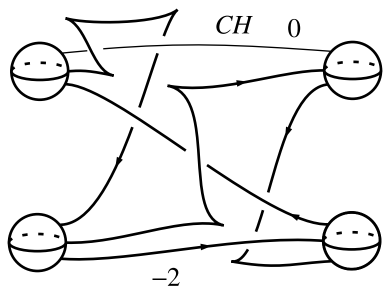


Figure 11.9. Stein exotic $\mathbb{R}^4$.

we must increase its canonical framing by 3 units by a smooth isotopy before we can attach the required 2-handle. This is easily accomplished by passing 2 strands around 1-handles to obtain Figure 11.9. The new Legendrian knot has $tb = -1$ and the required framing coefficient is -2, so we can attach the required 2-handle and obtain R as a Stein surface. By Theorem 9.4.12 we can obtain uncountably many diffeomorphism types of exotic $\mathbb{R}^4$'s by varying the choice of Casson handle in Figure 11.9. Freedman's argument for constructing the required nested family of Casson handles allows us the freedom to add extra positive self-plumbings whenever necessary, so we can arrange for the resulting exotic $\mathbb{R}^4$'s to be Stein. □

We next address the question of which oriented 3-manifolds admit tight contact structures. Recall that the boundary of a compact Stein surface inherits a tight contact structure — such contact structures are called *holomorphically fillable*. (There exist tight structures on T^3 that are not holomorphically fillable; i.e., they are not given by Stein surfaces [**E7**].) We sharpen the above question by asking which 3-manifolds bound Stein surfaces. Since it is often convenient to describe 3-manifolds by rational, rather than integral, surgery, the following proposition is useful. (We use $-\infty$ in place of the surgery coefficient ∞, for compatibility with the ordering of $\mathbb{Q}$.)

Proposition 11.2.8. *Suppose that an oriented 3-manifold M is given by surgery on a Legendrian link L in standard form in $\#mS^1 \times S^2$, with surgery coefficients in $\{-\infty\} \cup \mathbb{Q}$. Suppose that the coefficient of each component K of L is less than $tb(K)$. Then M is the oriented boundary of a Stein surface.* □

Exercise 11.2.9. * Prove the above proposition. (*Hint*: Exercise 5.3.9(b).)

Now let L be an n-component link in S^3. (The case of an arbitrary M^3 is similar.) The 3-manifolds obtained by rational surgery on L are indexed by the corresponding ordered n-tuples of surgery coefficients in $(\{-\infty\} \cup \mathbb{Q})^n$. It is natural to ask which of these oriented 3-manifolds bound Stein surfaces (allowing for the fact that such Stein surfaces may have no obvious relation to the link L). A variant of the proof of the previous proposition shows:

Proposition 11.2.10. *For any link $L \subset S^3$, the subset of $(\{-\infty\} \cup \mathbb{Q})^n$ corresponding to oriented 3-manifolds bounding Stein surfaces is open, provided that we use the lower limit topology on each $\{-\infty\} \cup \mathbb{Q}$ factor. That is, whenever rational surgery on L bounds any Stein surface, we can increase the surgery coefficients by any sufficiently small nonnegative rational numbers (and change any $-\infty$ to any sufficiently negative rational number) and still have a Stein boundary.*

Proof (sketch). The basic idea is that a small rational increase of one surgery coefficient corresponds to surgery on a suitable circle with a very negative coefficient. (In the integral case the circle is a meridian and we slam-dunk to see the correspondence; the general case follows by expanding as in Exercise 5.3.9(b) and surgering a meridian of the last unknot.) If the original manifold bounds a Stein surface S, the new manifold is now obtained by rational surgeries in ∂S with suitably negative coefficients. By the method of proof of the previous proposition, we can replace this surgery by attaching handles to S preserving the Stein condition. (See [**G13**] for details.) □

Exercises 11.2.11. (a)* Prove that all lens spaces bound Stein surfaces (with the boundary orientation).

(b)* Prove the same for all rational surgeries on the left trefoil knot. (*Hint*: Start with the case of -1-surgery, the Poincaré homology sphere, Exercise 5.1.12(a).)

As the exercises show for the unknot and left trefoil, it is sometimes possible to realize all rational surgeries on a given knot as Stein boundaries, although for large values of the coefficient the resulting diagrams may bear no resemblance to the original knot. The examples in the above exercises are special cases of rational surgeries on the Borromean rings, which were analyzed in [**G13**]. It was shown that except for a "small" region in $\mathbb{Q}^3$ these could be realized as Stein boundaries. (Of course, the meaning of "small" in a countably infinite set is rather vague.) There is an infinite family of knots and links obtained from the Borromean rings by putting a coefficient with integral reciprocal on up to 2 components and then removing these by Rolfsen twists — e.g., the left trefoil is obtained by setting 2 coefficients equal to -1. All of these knots and links have the property that if a finite set of coefficients is excluded for each component, all remaining integral surgeries will be Stein boundaries. (It is an open question how common this property is among arbitrary links.) An infinite family of these knots shares with the left trefoil the property that all rational surgeries are Stein boundaries. A similar analysis of oriented Seifert fibered spaces (such as circle bundles over surfaces and the homology spheres $\Sigma(p,q,r)$) showed that all such 3-manifolds are Stein boundaries after possibly reversing orientation, and "most" bound with both orientations. These examples suggest that there may be a sense in which "most" closed, oriented 3-manifolds are Stein boundaries. (Perhaps "most" surgeries on "most" links, for example?) Not all oriented 3-manifolds are Stein boundaries, however — Lisca [**Ls2**] has recently shown that the Poincaré homology sphere Σ with its orientation reversed, i.e., $+1$ surgery on the right trefoil, is not a Stein boundary. Since $M_1 \# M_2$ is a Stein boundary iff M_1 and M_2 are [**E3**], it follows that $\Sigma \# \overline{\Sigma}$ does not bound a Stein surface with either orientation. It is still unknown whether these manifolds admit any tight contact structures.

While holomorphically fillable contact structures are always tight, their finite covers may be overtwisted, as the next proposition shows. We can use tightness of finite covers to distinguish fillable contact structures, even if they are homotopic as 2-plane fields, as the subsequent example demonstrates. (We will discuss other ways of distinguishing contact structures in the rest of this chapter.)

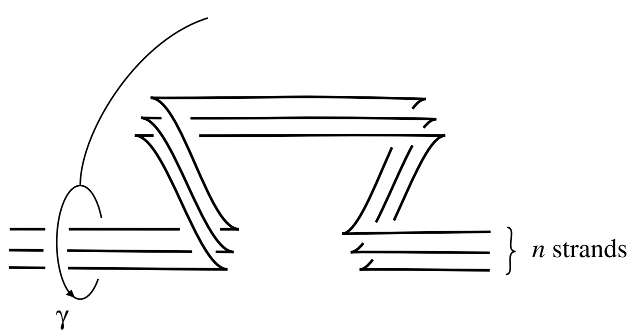


Figure 11.10. Stein boundary with overtwisted cover.

Proposition 11.2.12. *Let (M, ξ) be a contact 3-manifold exhibited as the boundary of a Stein surface in standard form. Suppose that the diagram intersects some disk in $\mathbb{R}^2$ in a collection of $n \geq 1$ parallel strands in the configuration shown in Figure 11.10, and let γ be a loop surrounding the strands as shown. Let $(\widetilde{M}, \tilde{\xi}) \to (M, \xi)$ be any locally contactomorphic covering map such that some conjugate of γ in $\pi_1(M)$ is not in the image of $\pi_1(\widetilde{M})$. Then $\tilde{\xi}$ is an overtwisted contact structure.*

For example, if L is any Legendrian link in $\#m\, S^1 \times S^2$, and we modify L by adding zig-zags to some component K to decrease $tb(K)$ by 2 without changing $r(K)$, then the proposition applies to the manifold (M, ξ) obtained by contact surgery on the modified link, with γ a meridian of K. Applying this to the unknot in S^3, we see that any lens space of the form $L(p, 1)$, $p \geq 4$, admits fillable structures all of whose covers are overtwisted. These same lens spaces bound holomorphic disk bundles over S^2 with negative Euler numbers, and all covers of the resulting contact structures are tight (since they are fillable by negative holomorphic disk bundles by branched covering).

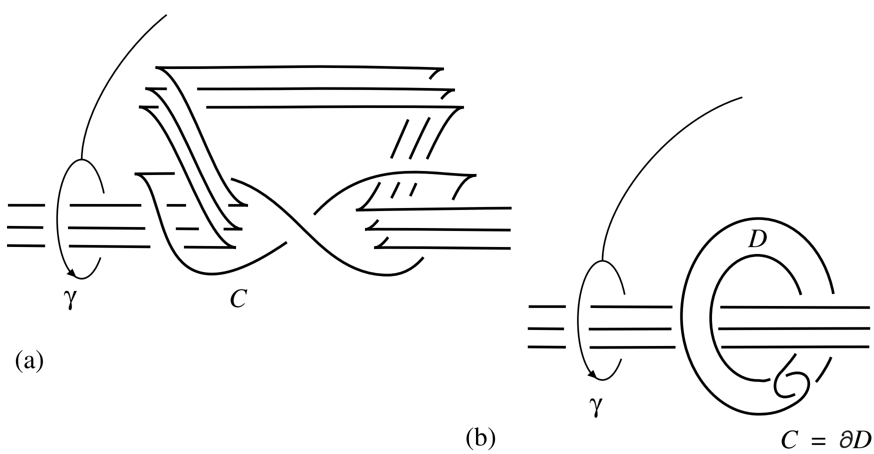


Figure 11.11. Immersed overtwisted disk D.

Proof of Proposition 11.2.12. Let C be the Legendrian curve shown in Figure 11.11(a), with $tb(C) = -2$. As a smooth knot in the complement of the given link, C is isotopic to the negative Whitehead double of γ, so it bounds an immersed disk D disjoint from the link, as is clearly visible in Figure 11.11(b). The framing induced by D on $C = \partial D$ is the blackboard framing in Figure 11.11(b), corresponding to the writhe $w(C) = -2 = tb(C)$. Thus, D induces the canonical framing on C. The π_1-condition on γ guarantees that some lift $\widetilde{D}$ of D is an embedded disk in $\widetilde{M}$. Since $\widetilde{D}$ still

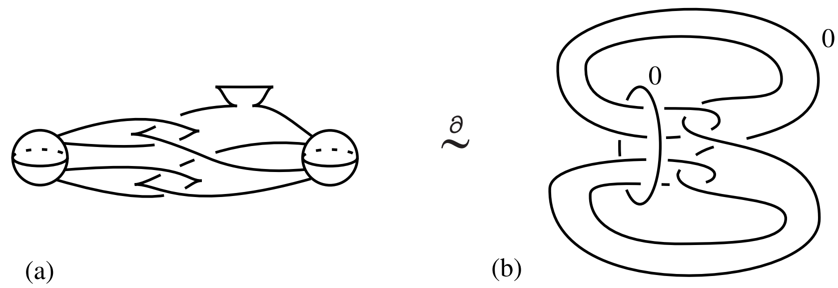


Figure 11.12. Construction of contactomorphic but nonisotopic contact structures.

induces the canonical framing on its boundary, it is an overtwisted disk in $\widetilde{M}$, i.e., its boundary is an unknot with $tb(\partial\widetilde{D}) = 0$. $\square$

Example 11.2.13. Let (M, ξ) be the boundary of the Stein surface S shown in Figure 11.12(a). The 2-handle is attached along a knot K with $tb(K) = 1$, so its framing coefficient is 0. By changing to dotted circle notation and surgering out the 1-handle, we realize M as 0-surgery on the 2-component link of Figure 11.12(b). It is routine to check that this link is symmetric, i.e., there is an isotopy interchanging its 2 components. This gives an orientation-preserving self-diffeomorphism $\varphi\colon M \to M$ interchanging the 2 meridians. We obtain a contact structure $\varphi_*\xi$ on M that is obviously contactomorphic to ξ (via φ). We will see later (Exercise 11.3.13) that ξ and $\varphi_*\xi$ are also homotopic as 2-plane fields, but we will now show that they are not isotopic contact structures. (That is, ξ and $\varphi_*\xi$ are connected by a continuous family ξ_t of 2-plane fields, but we cannot arrange for all ξ_t to be contact structures. The first such example of a pair of tight contact structures is due to Giroux [**Gi**].) Since $\pi_1(S) \cong \mathbb{Z}$, S has a unique double cover $\widetilde{S}$. The corresponding double cover $(\widetilde{M}, \tilde{\xi})$ of (M, ξ) bounds $\widetilde{S}$, so it is tight. However, the lift of $\varphi_*\xi$ to $\widetilde{M}$ is contactomorphic to a double cover of (M, ξ) with index 2 along K, so it is overtwisted by Proposition 11.2.12.

Exercise 11.2.14. Prove that a finite cover of a Stein surface is Stein. Draw an explicit diagram for the above Stein surface $\widetilde{S}$.

11.3. Invariants of Stein and contact structures

Stein and contact structures are special cases of almost-complex structures and plane fields, respectively. These latter structures can be classified up to homotopy using obstruction theory. In this section, we will discuss invariants arising from the homotopy classification of such structures.

We begin with Stein surfaces. For an almost-complex 4-manifold (X, J) with $H^3(X) = H^4(X) = 0$, the one obvious invariant is its Chern class

$c_1(X, J) \in H^2(X; \mathbb{Z})$. This completely determines the homotopy type of the almost-complex structure, provided that there is no 2-torsion in $H^2(X; \mathbb{Z})$, e.g. for X simply connected. (For arbitrary $H^2(X; \mathbb{Z})$ and $H^3 = H^4 = 0$, almost-complex structures on X correspond bijectively to spin^c structures, and the bijection preserves c_1. Thus, *twice* the difference obstruction between two almost-complex structures is the difference between their Chern classes, and c_1 classifies such structures in the absence of 2-torsion. See the beginning of Section 10.4.) Now suppose that S is a Stein surface exhibited as a handlebody on a Legendrian link L in $\#mS^1 \times S^2$ in standard form. To specify $c_1(S)$, we compute cohomology using the handles of S as bases for the chain groups as at the end of Section 4.2. To fix the signs, we orient L and use the induced orientation (as preceding Proposition 4.5.11) on each 2-handle h attached along a component K of L. (Thus, $\partial_* h = -[K] \in C_1(S)$, since for nullhomologous K, the orientation of the homology class of h should come from a Seifert surface of K.)

Theorem 11.3.1. *For S and L as above, the class $c_1(S) \in H^2(S; \mathbb{Z})$ is represented by a cocycle whose value on each oriented 2-handle h attached along a component K of L is given by $r(K)$.* □

For a proof, see Proposition 2.3 of [**G13**]. The basic idea is to define a complex trivialization of the tangent bundle of $S_1 = D^4 \cup$1-handles starting with $\frac{\partial}{\partial x}$ and a normal vector field to the 3-manifold in the box of Figure 11.2, then compute that the obstruction to extending over each h is the corresponding $r(K)$. Recall that we already observed a connection between rotation numbers and Chern classes in Exercise 11.1.9(b).

Exercises 11.3.2. (a)* Let X be an oriented disk bundle over a closed, orientable surface F. Prove that X can be realized as a Stein surface with any Chern class satisfying the conditions $|\langle c_1(X), F\rangle| \leq -\chi(F) - e(X)$ and $\langle c_1(X), F\rangle \equiv e(X) \pmod 2$. (See Exercise 11.2.5(a).) By Exercise 11.4.11(c), this result is optimal for $F \neq S^2$. Why must the second condition be satisfied by all Stein structures on X? What happens when F is nonorientable?

(b) For $X = \operatorname{int} H$ as in Theorem 11.2.6, prove that any integral lift of $w_2(X)$ can be realized as $\varphi^* c_1(S)$ for some (orientation-preserving) homeomorphism $\varphi\colon X \to S$ onto a Stein surface. (See [**G13**] for the answer.)

Now we consider oriented 2-plane fields ξ on closed, oriented 3-manifolds M, up to homotopy (i.e., we allow ξ to vary in continuous families ξ_t of 2-plane fields). The most obvious invariant is again the Chern or Euler class $c_1(\xi) = e(\xi) \in H^2(M; \mathbb{Z})$, where we consider ξ to be an abstract complex line bundle or oriented real 2-plane bundle over M. If ξ is the induced contact structure on the boundary M of a Stein surface S, then $c_1(\xi)$ is easy to compute: Since ξ is a complex line bundle in $TS|M$, the latter splits as $\xi \oplus \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is the trivial summand spanned by the outward normal to S in $TS|M$. Thus, $c_1(\xi) = c_1(\xi \oplus \underline{\mathbb{C}}) = c_1(TS|M) = c_1(S)|M$, and we can compute the latter using the previous theorem. The Chern class $c_1(\xi)$ determines ξ as an abstract complex line bundle. In fact, it determines ξ up to isomorphisms of TM, since the latter is isomorphic to $\xi \oplus \underline{\mathbb{R}}$. However, if we consider ξ up to homotopy inside TM, the classification problem becomes more subtle; the ensuing discussion constitutes the remainder of this section.

The simplest approach to classifying oriented 2-plane fields ξ up to homotopy on a closed, oriented 3-manifold M is as follows. Recall that the tangent bundle TM is trivial (Remark 1.4.27(b)), and fix a trivialization τ. Since an oriented plane in $\mathbb{R}^3$ has a unique positive unit normal vector, plane fields ξ now correspond to maps $M \to S^2$, and the desired classification reduces to understanding the set $[M, S^2]$ of homotopy classes of maps into S^2. This set was first computed by Pontrjagin around 1940 [**Po**], using obstruction theory (cf. Section 5.6). (For a simpler approach using the Thom-Pontrjagin construction, see [**G13**].) Since S^2 is simply connected with $\pi_2(S^2) \cong \mathbb{Z}$, the obstruction Γ to uniqueness over the 2-skeleton $M_2 = M - \operatorname{int} D^3$ lies in the group $H^2(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$. While it is tempting to try to identify Γ with the Chern class $c_1(\xi)$, the latter does not determine the former when $H_1(M;\mathbb{Z})$ has 2-torsion, e.g. for lens spaces $L(2p, q)$ — in fact, $c_1(\xi) = 2\Gamma$. (One way to explain this is to observe that ξ determines a complex structure on $TM \oplus \underline{\mathbb{R}}$, and hence, a spin^c structure on M. In fact, 2-plane fields over M_2 correspond bijectively to spin^c structures on M, so as in the previous situation, *twice* the difference class of two plane fields is the difference of their Chern classes.) For a fixed $\Gamma \in H^2(M;\mathbb{Z})$, the corresponding map $M_2 \to S^2$ always extends over M, but since $\pi_3(S^2) \cong \mathbb{Z}$, there is a secondary uniqueness obstruction. Difference classes for these extensions lie in $H^3(M;\mathbb{Z}) \cong \mathbb{Z}$, making the set of all extensions a $\mathbb{Z}$-space isomorphic to $\mathbb{Z}$, but different extensions may actually represent the same plane field on M, related by a nontrivial self-equivalence over M_2. In fact, the actual obstruction group is the cyclic group $\mathbb{Z}_{2d(\Gamma)}$ whose order $2d(\Gamma)$ is twice the divisibility of Γ in $H^2(M;\mathbb{Z})/\text{torsion}$ (with $d(\Gamma) = 0$ if Γ is a torsion class). Hence, we have a surjection $[M, S^2] \to H^2(M;\mathbb{Z})$ such that each $\Gamma \in H^2(M;\mathbb{Z})$ has preimage isomorphic to $\mathbb{Z}_{2d(\Gamma)}$ as a $\mathbb{Z}$-space, and $[M, S^2]$ has been computed.

For our purposes, however, this approach has a serious drawback. To reduce the classification of 2-plane fields to understanding $[M, S^2]$, we had to choose a trivialization of TM. The resulting invariants depend in a crucial way on this trivialization. (If we allow the trivialization to vary, then we are only classifying plane fields up to isomorphisms of TM, and we lose everything but the Chern class.) Since it is hard to keep track of a trivialization (particularly over the last 3-handle) when we do Kirby calculus, we devote the remainder of the section to a discussion of the invariants of [**G13**], which provide the same information without dependence on a trivialization.

The simplest of the invariants of [**G13**] determines the 3-dimensional uniqueness obstruction when $c_1(\xi)$ (hence Γ) is a torsion class. The definition is analogous to that of the Rohlin invariant (Definition 5.7.16), using the fact that for any closed, almost-complex 4-manifold (X, J) the quantity $c_1^2[X, J] - 2\chi(X) - 3\sigma(X)$ vanishes (Theorem 1.4.15). It is not hard to show (using obstruction theory) that any (M, ξ) as above can be realized as the boundary of a compact, almost-complex 4-manifold (X, J). We would like to define the above quantity for this latter manifold and show that it is an invariant of (M, ξ). The difficulty arises in defining the first term, since $H^2(X; \mathbb{Z}) \cong H_2(X, M; \mathbb{Z})$ does not have a well-defined intersection pairing on it in general. When $c_1(\xi)$ is a torsion class, we can solve the problem by using rational coefficients, for then $c_1(\xi)$ vanishes. Since the image of $H_2(M; \mathbb{Q}) \to H_2(X; \mathbb{Q})$ is annihilated by the intersection pairing, the long exact sequence of (X, M) shows that $\ker(\partial_* \colon H_2(X, M; \mathbb{Q}) \to H_1(M; \mathbb{Q}))$ inherits a pairing. Since $\partial_* PDc_1(X, J) = PDc_1(\xi)$ vanishes over $\mathbb{Q}$, the above expression is well-defined for (X, J).

Definition 11.3.3. For (M, ξ) and (X, J) as above with $c_1(\xi)$ a torsion class, define $\theta(\xi)$ to be $(PDc_1(X, J))^2 - 2\chi(X) - 3\sigma(X) \in \mathbb{Q}$.

Theorem 11.3.4. *For (M, ξ) as above with $c_1(\xi)$ a torsion class, $\theta(\xi) \in \mathbb{Q}$ depends only on (M, ξ), and it reverses sign if the orientation of M is reversed.*

Proof. Let (M, ξ) and $(\overline{M}, \xi)$ bound almost-complex manifolds (X_0, J_0) and (X_1, J_1), respectively, and let θ_0 and θ_1 denote the resulting values of the above invariant. Clearly, the manifold $Y = X_0 \cup_M X_1$ inherits an almost-complex structure J, and $c_1^2[Y, J] - 2\chi(Y) - 3\sigma(Y) = 0$. It is easy to verify that all 3 terms add under this gluing (cf. Remark 9.1.7), so we obtain $\theta_0 + \theta_1 = 0$. Since X_0 and X_1 were defined independently, it follows that $\theta(\xi) = \theta_0$ is independent of the choice of (X_0, J_0), and $\theta_1 = -\theta_0$ is the invariant for $(\overline{M}, \xi)$. □

Example 11.3.5. Let S be a Stein surface obtained by adding a 2-handle to D^4 along a Legendrian knot K with $tb(K) = 0$, and let (M, ξ) be the resulting holomorphically fillable contact manifold. Since the framing coefficient is $tb(K) - 1 = -1$, M is a homology sphere, so $c_1(\xi) = 0$ and $\theta(\xi)$ is defined. Orienting K determines a canonical generator $\alpha \in H_2(S; \mathbb{Z})$, and by Theorem 11.3.1 we have $\langle c_1(S), \alpha \rangle = r(K)$, so $PDc_1(S) = -r(K)\alpha$. Since $\alpha^2 = -1$, Definition 11.3.3 gives $\theta(\xi) = -(r(K))^2 - 1$. Now consider a Legendrian knot K' with $tb(K') \geq 2$. By adding zig-zags to K', we can create Legendrian knots K with $tb(K) = 0$, and by using the freedom to zig-zag upward or downward, we can obtain distinct values of $|r(K)|$. Thus, we obtain noncontactomorphic holomorphically fillable contact structures on the same smooth 3-manifold M. Since M is a homology sphere, however, the 2-dimensional invariant must vanish, so these structures will be homotopic over M_2 but not over M. We can construct 3-manifolds admitting arbitrarily large finite families of such structures, by starting with knots K' with $tb(K')$ arbitrarily large, for example torus knots $T_{p,q}$ with $p \gg q > 1$ (Example 6.2.7) or connected sums of right trefoils.

Exercises 11.3.6. **(a)** Show by example that the intersection pairing need not be well-defined on $H_2(X, \partial X; \mathbb{Z})$.

(b)* Compute $\theta(\xi)$ (when defined) for any (M, ξ) bounding a Stein surface made with a single 2-handle attached to D^4 along a Legendrian knot K. Note that $\theta(\xi)$ need not be an integer if $tb(K) \neq 0, 2$.

(c)* For $i = 0, 1$, let (M_i, ξ_i) be obtained from K_i as in (b). Suppose there is a contactomorphism $\varphi\colon M_0 \to M_1$ (or more generally, an orientation-preserving diffeomorphism with $\varphi_*\xi_0$ homotopic to ξ_1). Prove that $tb(K_0) = tb(K_1)$ and $|r(K_0)| = |r(K_1)|$. In particular, the framing coefficients $n_i = tb(K_i) - 1$ have the same sign (cf. Exercise 5.3.7(a)). (*Hint*: If μ_i is the meridian of K_i, then $\varphi_*[\mu_0] = k[\mu_1] \in H_1(M_1; \mathbb{Z})$ for some integer k. What do the linking forms of M_0 and M_1 say about k (Exercise 5.3.13(g))? For the endgame, consider mod 3 reductions.)

Next we examine the obstruction Γ to uniqueness of 2-plane fields over the 2-skeleton M_2 of M. This depends on a choice of trivialization of TM, but only through its restriction to M_2. That is, Γ is determined by a spin structure $s \in \mathcal{S}(M)$ (cf. Sections 5.6, 5.7), so we denote it by $\Gamma(\xi, s)$. (This invariant can be interpreted in terms of spin^c structures: Note that both ξ and s determine spin^c structures on M, the former as above and the latter through the inclusion $Spin(3) \hookrightarrow Spin^c(3)$. It can be shown [**G13**] that $\Gamma(\xi, s) \in H^2(M; \mathbb{Z})$ is the difference class of these spin^c structures.) We now give a different definition of $\Gamma(\xi, s)$ which will be useful in our discussion of the general 3-dimensional invariant $\widetilde{\Theta}$ (when we allow $c_1(\xi)$ to have infinite order). The subsequent proposition asserts that $\Gamma(\xi, s)$ is well-defined and depends in a simple way on s. This is followed by an explicit formula for $\Gamma(\xi, s)$ when (M, ξ) is a Stein boundary.

Definition 11.3.7. Let ξ be an oriented plane field on a closed, oriented 3-manifold M. Let v be a vector field in ξ whose zero locus is an oriented link γ in M with multiplicity 2 (so $2[\gamma] = PDc_1(\xi)$). Then v in $\xi \subset T(M - \gamma)$ determines a trivialization of $T(M - \gamma)$, and this extends uniquely to a spin structure s on M because v vanishes with even multiplicity on γ. Define $\Gamma(\xi, s)$ to be $[\gamma] \in H_1(M;\mathbb{Z})$.

Proposition 11.3.8. *For (M, ξ) as above, the previous definition determines a map $\Gamma(\xi, \cdot)\colon \mathcal{S}(M) \to H_1(M;\mathbb{Z})$ that depends only on (M, ξ), and is $H^1(M;\mathbb{Z}_2)$-equivariant. Its image is $\{x \in H_1(M;\mathbb{Z}) \mid 2x = PDc_1(\xi)\}$. For fixed s, reversing the orientation of either ξ or M reverses the sign of $\Gamma(\xi, s)$.* □

For a proof (by obstruction theory) see [**G13**]. To understand the equivariance, recall that $H^1(M;\mathbb{Z}_2) \cong H_2(M;\mathbb{Z}_2)$ acts on $\mathcal{S}(M)$ through difference classes (Proposition 5.6.3), and on $H_1(M;\mathbb{Z})$ via the Bockstein homomorphism $\beta\colon H_2(M;\mathbb{Z}_2) \to H_1(M;\mathbb{Z})$ of the long exact sequence induced by the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. This equivariance shows that $\Gamma(\xi, \cdot\,)$ is determined by its value on any one spin structure, hence, by any vector field as in Definition 11.3.7.

To compute $\Gamma(\xi, s)$ for a Stein boundary $(M, \xi) = \partial S$, it is convenient to express spin structures as characteristic sublinks via Proposition 5.7.11. Thus, given S in standard form, we switch to dotted circle notation using the dashed arcs in Figure 11.2, then surger the 1-handles to 0-framed 2-handles, obtaining a 2-handlebody X on a framed link $L = K_1 \cup \cdots \cup K_n$. Let $L_0 \subset L$ denote the unlink coming from the 1-handles. Orient L to obtain a canonical basis $\{\alpha_1, \dots, \alpha_n\}$ for $H_2(X;\mathbb{Z})$ corresponding to $\{K_1, \dots, K_n\}$.

Theorem 11.3.9. ([**G13**]) *For $(M, \xi) = \partial S$ and X as above, let $s \in \mathcal{S}(M)$ be a spin structure with corresponding characteristic sublink $L' \subset L$. Then $PD(\Gamma(\xi, s))$ is the restriction to M of the class $\rho \in H^2(X;\mathbb{Z})$ whose value on each α_i is the integer*

$$\langle \rho, \alpha_i \rangle = \frac{1}{2}(r(K_i) + \ell k(K_i, L_0 + L')),$$

where $r(K_i)$ equals 0 for K_i in L_0, and otherwise equals the rotation number of the oriented Legendrian knot corresponding to K_i. □

Remark 11.3.10. We could replace $L_0 + L'$ in the above expression by any fixed formal linear combination of components of L with the same mod 2 reduction as $L_0 + L'$, since the resulting change in ρ would have trivial restriction to M. A similar argument shows that $2\rho|M = c_1(\xi)$, as required.

Exercise 11.3.11. Let $K \subset \# m S^1 \times S^2$ be a Legendrian knot in standard form. Prove that $tb(K) + r(K) + 1$ is congruent mod 2 to the number of times K runs over 1-handles. (*Hint*: The formula in Theorem 11.3.9 gives an integer.)

Example 11.3.12. Consider the Stein surface S_p ($p \geq 1$) shown in Figure 11.13(a). This is obtained by adding a 2-handle to $S^1 \times D^3$ along a Legendrian knot K that runs $2p$ times over the 1-handle. Thus, $w(K) = 2p - 1$.

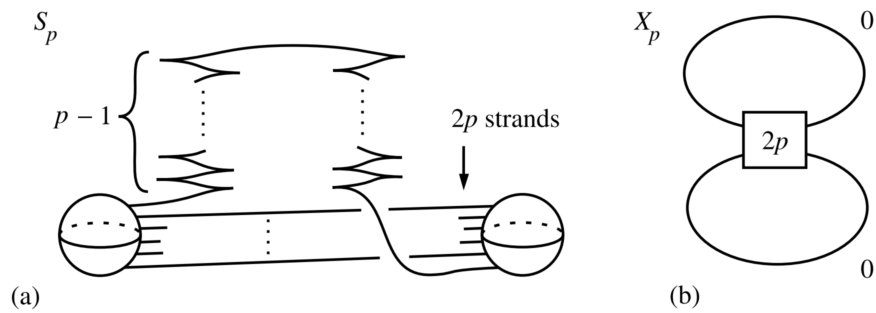


Figure 11.13. Nonisotopic contact structures distinguished by Γ.

There are $2p-2$ left cusps, half oriented upward and the other half downward, so $\lambda_+ = \lambda_- = \rho_+ = \rho_- = p-1$. Thus, the framing of the 2-handle is $tb(K)-1=0$, and $r(K)=0$. The corresponding 2-handlebody X_p of Figure 11.13(b) (obtained by surgery as above) admits a unique spin structure, whose restriction s to $M_p = \partial X_p = \partial S_p$ is given by the empty characteristic sublink. Using Theorem 11.3.9, it is easy to calculate that $\Gamma(\xi, s) = p\mu$, where μ is the meridian of K in $H_1(M_p;\mathbb{Z}) \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_{2p}$ (which is generated by the two meridians in ∂X_p). Now observe that X_p admits an involution φ that interchanges the two 2-handles but preserves the orientation and s. Since $\varphi_*\Gamma(\xi, s) \neq \Gamma(\xi, s)$, we conclude that the holomorphically fillable contact structures ξ and $\varphi_*\xi$ on M_p are not homotopic as 2-plane fields. In particular, they are not isotopic (although they are contactomorphic via φ). However, they are not distinguished by the other homotopy invariants, since $c_1(\xi) = c_1(\varphi_*\xi) = 0$ and $\theta(\xi) = \theta(\varphi_*\xi) = -2$. Thus, this example and Example 11.3.5 show that Γ and θ are independent invariants, even for holomorphically fillable contact structures with $c_1 = 0$.

Exercise 11.3.13. Prove that the contact structures ξ and $\varphi_*\xi$ defined in Example 11.2.13 are homotopic as 2-plane fields. (It suffices to show that their values of Γ and θ agree.)

Finally, we consider the 3-dimensional obstruction in full generality. Returning to our previous strategy for defining θ, we wish to interpret the term $c_1^2[X,J]$ for an almost-complex manifold (X,J) whose boundary (M,ξ) has Chern class $c_1(\xi)$ with possibly infinite order. Let z be a relative integral cycle in (X,M) with ∂z carried by a link L in M, and let f be a framing on L. We can define the self-intersection number of z relative to f by adding 2-handles to X along (L,f), extending z in the obvious way to a cycle $\hat{z}$ in the resulting 4-manifold, and setting $Q_f(z) = \hat{z}^2 \in \mathbb{Z}$. This self-intersection number depends on the choices of z and f, although its residue in $\mathbb{Z}_{2d([\partial z])}$ (where $d([\partial z])$ is the divisibility of $[\partial z] \in H_1(M;\mathbb{Z})/\text{torsion}$) depends only on $[z] \in H_2(X, M; \mathbb{Z})$ and f. To make sense of this last statement, we interpret f as a framing on the homology class $[\partial z] = \partial_*[z] \in H_1(M; \mathbb{Z})$, which is well-defined up to adding any multiple of $2d([\partial z])$ twists. (See the following exercises.)

Exercises 11.3.14. (a)* Prove that for fixed L, $Q_f[z]$ is defined up to (and only up to) multiples of $2d([\partial z])$. (*Hint*: What happens if you replace $\hat{z}$ by $\hat{z} + z'$ with z' a cycle in M?)

(b) Define the notion of a framing modulo $2d(x)$ on a homology class x in $H_1(M; \mathbb{Z})$. (*Hint*: If γ_0 and γ_1 are nonempty oriented links representing x, then there is a connected, oriented surface $F \subset I \times M$ with $\partial F = 1 \times \gamma_1 - 0 \times \gamma_0$. Use the normal bundle of F to transport a framing from γ_0 to γ_1. For the $2d(x)$ ambiguity, see (a).) How does a framing on a representative of x change under ribbon moves (cf. Section 6.2)? How is a framing affected by a relative maximum or minimum of F?

(c) If $x = \sum k_i[K_i] \in H_1(M; \mathbb{Z})$ is given by an integral linear combination of components of an oriented link L, then a framing f on L determines one on x — simply push off $|k_i|$ framed parallel copies of each K_i using the framing f. Show that this correspondence preserves $Q_f(z)$, and that adding a twist to f along K_i adds k_i^2 twists to the induced framing on x (increasing $Q_f(z)$ by k_i^2).

(d) Prove that $Q_f[z] \in \mathbb{Z}_{2d([\partial z])}$ depends only on $[z] \in H_2(X, M; \mathbb{Z})$ and the framing f on $[\partial z]$. (*Hint*: By (c), it suffices to consider relative cycles z with ∂z a link (with all coefficients $= 1$). Now add a collar $I \times M$ to X along ∂X and use (b) to reduce to (a).)

Definition 11.3.15. Let M, ξ, v, γ and s be as in Definition 11.3.7 with $(M, \xi) = \partial(X, J)$ and a framing f specified on γ. Then v and the outward normal of X define a complex trivialization of $TX|(M - \gamma)$. Let z be a relative 2-cycle in (X, M) that is Poincaré dual to the relative Chern class $c_1(X, v) \in H^2(X, M - \gamma; \mathbb{Z})$ (so that $\partial z = 2\gamma$). Define $\widetilde{\Theta}(\xi, s, f)$ to be $Q_f(z) - 2\chi(X) - 3\sigma(X) \in \mathbb{Z}_{4d(\xi)}$ (where $d(\xi) = d(c_1(\xi))$), and let $\Theta_f(\xi)$ be the corresponding residue in $\mathbb{Z}_{2d(\xi)}$.

Note that since $[\partial z] = PDc_1(\xi)$, the first term of $\Theta_f(\xi)$ depends (mod $2d(\xi)$) only on $[z] = PDc_1(X) \in H_2(X, M; \mathbb{Z})$ and f (as a framing on $PDc_1(\xi)$), so $\Theta_f(\xi)$ is independent of s and its definition extends to all framings on $PDc_1(\xi)$. However, for $d(\xi) \neq 0$ we must lift to $\widetilde{\Theta}$ to obtain a complete invariant for 2-plane fields, and $\widetilde{\Theta}$ does depend on the choice of s. The spin structure also affects $\Theta_f(\xi)$ indirectly through f: If we add a twist to f on $[\gamma] = \Gamma(\xi, s)$ then the corresponding framing on $PDc_1(\xi) = 2\Gamma(\xi, s)$ changes by 4 twists (Exercise 11.3.14(c)). Now $\Theta_f(\xi)$ varies over all elements of $\mathbb{Z}_{2d(\xi)}$ as we vary f on $PDc_1(\xi)$, but a choice of s determines $\Gamma(\xi, s)$, and hence a preferred mod 4 coset of induced framings on $PDc_1(\xi)$, resulting in a preferred mod 4 residue of $\Theta_f(\xi)$.

Theorem 11.3.16. ([**G13**]) *Let M, ξ, s and f on $\Gamma(\xi, s)$ be as in Definition 11.3.15. Then $\widetilde{\Theta}(\xi, s, f) \in \mathbb{Z}_{4d(\xi)}$ depends only on M, s, f and the homotopy class of ξ. The invariant $\widetilde{\Theta}$ varies with s and f by the formula $\widetilde{\Theta}(\xi, s_1, f_1) = \widetilde{\Theta}(\xi, s_0, f_0) + Q_{f_0,f_1}(z_M)$, where z_M is any relative integral 2-cycle in $(I \times M, \{0,1\} \times M)$ such that ∂z_M has the form $2(1 \times \gamma_1 - 0 \times \gamma_0)$ with $[\gamma_i] = \Gamma(\xi, s_i)$ and for which the* mod 2 *reduction $z_M|_2$ is Poincaré dual to the difference class $\Delta(s_0, s_1)$. For fixed s and f, $\widetilde{\Theta}(\xi, s, f)$ is independent of the orientation of ξ and reverses sign if the orientation of M is reversed. If ξ_0 and ξ_1 are oriented 2-plane fields on a connected M, then they are homotopic if and only if for some (hence, any) choice of s and f, $\Gamma(\xi_0, s) = \Gamma(\xi_1, s)$ and $\widetilde{\Theta}(\xi_0, s, f) = \widetilde{\Theta}(\xi_1, s, f)$. If $c_1(\xi_0)$ is a torsion class, then the same is true with $\theta(\xi_i)$ or $\Theta_f(\xi_i)$ in place of each $\widetilde{\Theta}(\xi_i, s, f)$.* □

The proof is a more sophisticated version of that of Theorem 11.3.4. See [**G13**] for details and for additional properties of the invariants and corollaries. Note that the theorem asserts that Γ and $\widetilde{\Theta}$ (or Γ and θ if $b_1(M) = 0$) are a complete set of invariants for homotopy classes of oriented plane fields on M. Thus, they contain all of the information given by the obstructions described earlier, allowing us to avoid keeping track of a trivialization of TM. Recall that the previous viewpoint gave the set of plane fields extending a fixed ξ over M_2 as a $\mathbb{Z}$-space isomorphic to $\mathbb{Z}_{d(\xi)}$. From our new viewpoint, the generator of the $\mathbb{Z}$-action subtracts 4 from $\widetilde{\Theta}$, so the required $\mathbb{Z}$-space is a mod 4 coset of $\mathbb{Z}_{4d(\xi)}$. This shows that for $d(\xi) \neq 0$, $\Theta_f \in \mathbb{Z}_{2d(\xi)}$ classifies plane fields only up to a $2:1$ ambiguity.

We now give recipes from [**G13**] for computing the above quantities Θ_f, $\widetilde{\Theta}$ and $Q_{f_0,f_1}(z_M)$ for the boundary of a Stein surface S in standard form. If D_i denotes the cocore of the 2-handle attached to K_i (with $\mu_i = \partial D_i$ a right-handed meridian of K_i) then by Theorem 11.3.1 $PDc_1(S)$ is represented by $\sum r(K_i)D_i$. Thus, $\Theta_f(\xi)$ is easily computed using the observation that $Q_f(PDc_1(S)) = 0 \in \mathbb{Z}_{2d(\xi)}$ when f is the 0-framing on $\bigcup \mu_i$. Given two Stein surfaces with a diffeomorphism preserving $c_1(\xi)$ between their boundaries, we can now compare the corresponding values of Θ by comparing the two framings on the class $c_1(\xi)$ as in Exercise 11.3.14(b). (If the diffeomorphism does not preserve $c_1(\xi)$, then we cannot compare the invariants Θ since we cannot compare framings, but then the plane fields are distinguished by $c_1(\xi)$.) A recipe for $\widetilde{\Theta}$ can be obtained by the method of Theorem 11.3.9: Pass to the 2-handlebody X as in that theorem, and let c be any integral 2-chain in M such that $\partial c = 2\gamma - \sum r(K_i)\mu_i$ for some oriented link γ. Let $s \in \mathcal{S}(M)$ be the spin structure whose characteristic sublink is obtained from the sublink representing $[c] \in H_2(X, D^4; \mathbb{Z}_2)$ by adding the sublink L_0 coming from the 1-handles of S. If we set $z = \sum r(K_i)D_i + c$, then $\widetilde{\Theta}(\xi, s, f) = Q_f(z) - 2\chi(X) - 3\sigma(X) \in \mathbb{Z}_{4d(\xi)}$ for any framing f on $[\gamma] = \Gamma(\xi, s)$, where the first term is computed as usual by adding 2-handles to X (or S) along (γ, f). Given a diffeomorphism preserving Γ between two Stein boundaries, we would like to compare the corresponding invariants $\widetilde{\Theta}$ for a fixed s and f. Given the above recipe for $\widetilde{\Theta}$, it suffices to compute the change $Q_{f_0,f_1}(z_M)$ induced by a change of s and f (since we already know how to pull a given s and f through a diffeomorphism realized by Kirby moves). Fortunately, this is straightforward. We choose z_M representing $PD(\Delta(s_0, s_1))$ (which is given by the difference of the corresponding characteristic sublinks, cf. the proof of Proposition 5.7.11), with ∂z_M as required and γ_0 connected. By adding the same number of twists to f_0 and f_1, we can arrange f_0 to be the 0-framing on $\gamma_0 \subset \partial X$ (leaving $Q_{f_0,f_1}(z_M)$ unchanged). Then we eliminate γ_0 from ∂z_M by adding 2 copies of a surface in D^4 with boundary γ_0 (in lieu of a 0-framed 2-handle attached along $0 \times \gamma_0$ in $0 \times M$), and compute $\hat{z}_M$ by adding handles along (γ_1, f_1) as before.

We close the section with some relations between the invariants we have defined for plane fields. For $x \in H_1(M; \mathbb{Z})$ a torsion class with framing f, let $q_f(x) \in \mathbb{Q}$ denote the rational framing coefficient defined in Exercise 4.5.12(c). Its mod 1 residue $q(x) \in \mathbb{Q}/\mathbb{Z}$ is the square of x under the linking form of M (same exercise). When $\theta(\xi)$ is defined (i.e., $c_1(\xi)$ is a torsion class), then $d(\xi) = 0$, so the invariants $\widetilde{\Theta}(\xi, s, f)$ and $\Theta_f(\xi)$ are integers (and equal when the second framing is induced by the first). By [**G13**] we then have

$$\Theta_f(\xi) = \theta(\xi) + q_f(PDc_1(\xi)).$$

In particular, the mod 1 residue of $\theta(\xi)$ is $-q(PDc_1(\xi)) \in \mathbb{Q}/\mathbb{Z}$. Bilinearity of the linking form now implies that $\theta(\xi)$ is an integer divided by the (finite) order of $c_1(\xi)$. We can also compute the mod 4 residues of the invariants θ and Θ, the latter depending on a choice of spin structure (through the framing f) as described previously. We have mod 4 congruences (cf. the last paragraph of [**G14**])

$$\widetilde{\Theta}(\xi, s, f) \equiv \Theta_f(\xi) \equiv 2(b_0(M) + b_1(M)) - \mu(M, s)$$

and

$$\theta(\xi) \equiv 2(b_0(M) + b_1(M)) - \mu(M, s) - 4q(\Gamma(\xi, s)),$$

where $\Theta_f(\xi)$ is given for a framing f induced by s as above, we assume $d(\xi) = 0$ in the second formula, and μ denotes the Rohlin invariant (Definition 5.7.16). Since $\theta(\xi)$ does not depend on s, the right-hand side of the second equation must also be independent of s mod 4. These mod 4 congruences follow from the fact [**G13**] that (M, ξ, s) bounds an almost-complex spin manifold (X, J, s_X) with X a union of 0- and 2-handles. Since $c_1(X, v)|_2 = w_2(X, s) = 0$, we can apply Definition 11.3.15 with $z = 2z_0$ for a suitable relative integral cycle z_0, obtaining $\widetilde{\Theta}(\xi, s, f) = 4Q_f(z_0) - 2\chi(X) - 3\sigma(X) \equiv -2(\chi(X) + \sigma(X)) - \sigma(X) \equiv 2(b_0(X) + n(X)) - \mu(M, s) \pmod 4$, where $n(X)$ is the nullity of Q_X and equals $b_1(M)$ by Corollary 5.3.12.

11.4. Stein surfaces and gauge theory

We end this chapter by outlining the relation of Seiberg-Witten theory to manifolds having nonempty boundary; in particular, we obtain information about the topology of Stein surfaces. (Recall that in Section 2.4 we defined SW_X only for closed 4-manifolds satisfying some additional constraints.) Various approaches to this problem have been worked out. In one (developed by Kronheimer and Mrowka [**KM2**]) a contact structure ξ on ∂X is fixed and a diffeomorphism invariant is defined, namely the Seiberg-Witten invariant $SW_{X,\xi}$ of the pair (X, ξ) mapping from the set $\mathcal{S}^c_{X,\xi}$ to $\mathbb{Z}$. (Here $\mathcal{S}^c_{X,\xi}$ denotes the set of spin^c structures on X inducing the spin^c structure on ∂X provided by ξ. Note that a contact structure reduces the structure group of TM to $SO(2)$, hence — as in the case of an almost-complex structure on a 4-manifold — specifies a spin^c structure on a 3-manifold [**KM2**].) Surprisingly enough, this approach not only gives information about the smooth topology of X, but also provides contact geometric results about the 3-manifold $M = \partial X$. This theory is easy to handle when X admits a symplectic, or in particular, a Stein structure. (In the following we will restrict ourselves to listing the relevant theorems about $SW_{X,\xi}$ and will discuss its definition only very briefly, cf. Remark 11.4.4.) Another approach (developed by Lisca and Matić) provides an embedding of a Stein surface in a closed complex surface of general type, and uses earlier gauge theoretic results about such complex surfaces. We will return to this approach in Theorem 11.4.5.

Let ξ be a fixed contact structure on the 3-manifold $M = \partial X$. A symplectic structure ω on the compact manifold $(X, \partial X)$ is *compatible* with the contact structure ξ on $M = \partial X$ if ω is positive on the oriented 2-plane field ξ. Suppose that γ_0 is a spin^c structure on (X, ξ) generated by a compatible symplectic structure ω. The following theorem generalizes Theorem 2.4.7(2) of Taubes and Theorem 3.4.22 about minimal surfaces of general type to the case $\partial X \neq \emptyset$.

Theorem 11.4.1. *Fix a contact structure ξ on $M = \partial X$. If ω is a compatible symplectic structure on $(X, \partial X)$, then $SW_{X,\xi}(\gamma_0) = \pm 1$. If, in addition, ω is an exact form, then γ_0 is the only spin^c structure on X (inducing ξ on ∂X) for which $SW_{X,\xi}$ is nonzero.* □

Examples of symplectic manifolds with exact sympletic forms are given by Stein surfaces: If S is Stein and f is the proper Morse function provided by Theorem 11.2.1, then $\omega = i\partial\overline{\partial} f$ gives an exact Kähler form on S. One corollary of Theorem 11.4.1 is the following:

Theorem 11.4.2. ([**LM1**], [**KM2**]) *Let J_i $(i = 1, 2)$ be two Stein structures on a 4-manifold X with induced spin^c structures γ_i and induced contact structures ξ_i on ∂X. If ξ_1 is isotopic to ξ_2 (preserving the given orientations on the 2-plane fields), then γ_1 is isomorphic to γ_2. This isomorphism implies that the almost-complex structures J_1 and J_2 are homotopic (cf. the beginning of Section 11.3); in particular, $c_1(X, J_1) = c_1(X, J_2)$.* □

Coupling Theorem 11.4.1 with gauge theoretic arguments (cf. [**Frø**] and [**MMR**]), one can derive the following contact geometric information about the boundary ∂X.

Theorem 11.4.3. *For any 3-manifold M there are only finitely many homotopy classes of 2-plane fields which correspond to holomorphically fillable contact structures.* □

Remark 11.4.4. Recall from Section 10.4 that an almost-complex structure J determines a spinor $\psi_0 \in \Gamma(W^+)$, where W^+ corresponds to the canonical spin^c structure provided by J. It can be shown that there is a unique connection A_0 with $\partial\!\!\!/_{A_0}\psi_0 = 0$ [**KM2**]. Now considering solutions (A, ψ) of the Seiberg-Witten equations on (X, ξ) such that $A - A_0$ and $\psi - \psi_0$ are in L^2, a moduli space and corresponding invariant can be defined. Generalizing Taubes' arguments to this setting, Kronheimer and Mrowka found a proof of Theorem 11.4.1 for these invariants.

Theorem 11.4.2 was obtained earlier by Lisca and Matić; in the following we will use their approach — see also [**LM1**].

Theorem 11.4.5. ([**LM1**]) *Suppose that S is a Stein surface and $f\colon S \to \mathbb{R}$ is the proper Morse function provided by Theorem 11.2.1. Then for any regular value $r \in \mathbb{R}$ there is a Kähler embedding of the submanifold $S_r = \{x \in S \mid f(x) < r\} \subset S$ into a minimal surface T of general type with $b_2^+(T) > 1$.* □

The fact that a Stein surface S_r embeds in a projective surface was known for some time; the above theorem asserts that the target space T can be chosen to be a minimal surface of general type and the embedding is symplectic.

Exercise 11.4.6. Deduce the conclusion $b_2^+(T) > 1$ from the rest of the theorem. (*Hint*: By adding 2-handles to S (keeping the Stein condition) we can embed S in a Stein surface S' with $b_2^+(S') > 1$. Now embedding S'_r in some T, we obviously have $b_2^+(T) > 1$.)

Recall that if T is a minimal surface of general type (with $b_2^+(T) > 1$) then $SW_T(K) = 0$ unless $K = \pm c_1(T)$, and $SW_T(\pm c_1(T)) = \pm 1$. Applying the embedding provided by Theorem 11.4.5, we can prove the following version of the adjunction inequality.

Theorem 11.4.7. ([**LM1**]) *If S is a Stein surface and $\Sigma \subset S$ is a connected, smooth, orientable surface with $g(\Sigma) > 0$, then*

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |c_1(S)([\Sigma])|.$$

Moreover, if $\Sigma \subset S$ is an embedded sphere and $[\Sigma] \neq 0$, then $[\Sigma]^2 \leq -2$.

Proof. Since Σ is compact, it lies in some S_r. Consider the given holomorphic embedding $\varphi\colon S_r \to T$ of S_r into the minimal surface of general type with $b_2^+(T) > 1$. First we show that if $[\Sigma] \neq 0$ in S, the same can be assumed in T: Since S_r has no 3-handles, $H_2(S_r;\mathbb{Z})$ has no torsion, so $[\Sigma]$ pairs nontrivially with some element of $H^2(S_r;\mathbb{Z}) \cong H_2(S_r, \partial S_r;\mathbb{Z})$. By adding a 2-handle to S_r (keeping the Stein condition), we can assume that Σ pairs nontrivially with a closed surface in $S_r \cup$2-handle, hence in the minimal surface of general type corresponding to this new Stein surface (still called T). Since $c_1(T)|\varphi(S_r) = c_1(S_r)$, the generalized adjunction formula 2.4.8 implies that $2g(\Sigma) - 2 = 2g(\varphi(\Sigma)) - 2 \geq [\varphi(\Sigma)]^2 + |c_1(T)([\varphi(\Sigma)])| = [\Sigma]^2 + |c_1(S_r)([\Sigma])|$, proving the first statement of the theorem. (Note that since T is a Kähler surface, it is of simple type.) Now Theorem 2.4.6 (together with Theorem 2.4.7) shows that a sphere cannot have nonnegative self-intersection. Finally, a sphere $\Sigma \subset S$ with $[\Sigma]^2 = -1$ would give rise to a sphere of square -1 in T. A minimal surface of general type, however, does not contain any smoothly embedded -1-sphere. (T has basic classes $\pm c_1(T)$ with $c_1^2(T) > 0$, so the blow-up formula shows that T cannot be decomposed as $T' \# \overline{\mathbb{CP}^2}$.) □

Using the above observation, we obtain a particularly simple example of two nondiffeomorphic smooth structures on a compact manifold [**AM1**], each obtained by attaching a single 2-handle to D^4. (The example is originally due to Akbulut [**A4**], by different methods.)

Theorem 11.4.8. *The compact 4-manifolds S and X given as the two leftmost diagrams of Figure 11.14 are homeomorphic but not diffeomorphic. (Even their interiors cannot be diffeomorphic.)*

Proof. Figure 11.14 shows that ∂S and ∂X are diffeomorphic homology spheres. (Blow up S, interchange link components by an isotopy, blow up again and surger a 0-framed unknot to a dotted circle, then isotope and cancel a handle pair.) Freedman's Theorem 1.2.27 generalizes without change to compact 4-manifolds with oriented boundary a fixed homology sphere [**FQ**]. Since S and X are smooth and simply connected with the same intersection form $\langle -1 \rangle$, it follows that they are homeomorphic. But S is exhibited as a Stein surface, so $H_2(S;\mathbb{Z})$ cannot be generated by an embedded sphere. On the other hand, X is obtained from D^4 by attaching a 2-handle to a slice knot, as the ribbon move in Figure 11.14 shows, and the slice disk union the core of the 2-handle is a sphere generating $H_2(X;\mathbb{Z})$. □

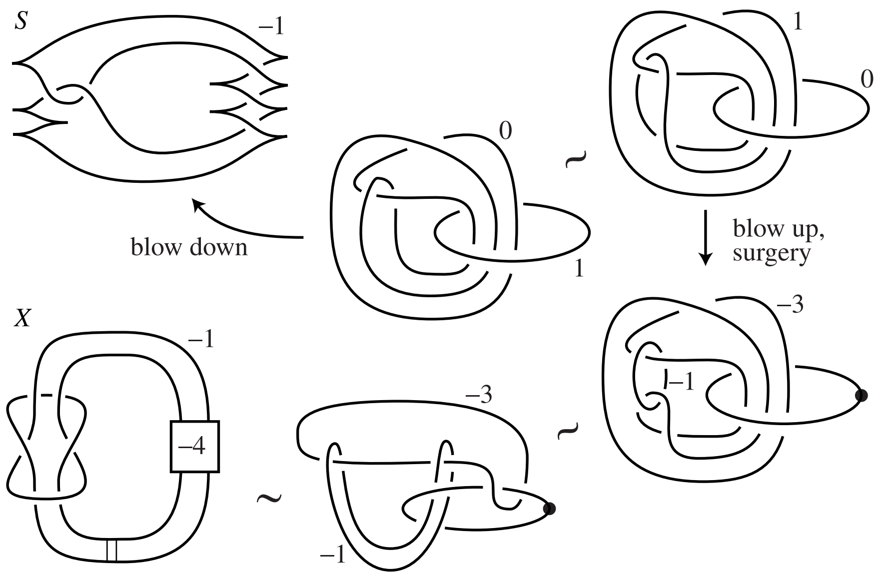


Figure 11.14. Homeomorphic but nondiffeomorphic 4-manifolds S, X.

The adjunction inequality given above has an important application to knot theory. Two classical invariants of a knot $K \subset S^3$ are the *genus* $g(K)$ and the *slice genus* $g_s(K)$, which are, respectively, the minimal genus of a Seifert surface for K in S^3 and the minimal genus of an embedded orientable surface $F \subset D^4$ with $\partial F = K$. Clearly, $g(K) \geq g_s(K)$, and upper bounds for both can be obtained by explicitly constructing surfaces. For example, $g_s(K) = 0$ iff K is (smoothly) slice. Lower bounds are much more difficult to obtain, particularly for g_s. Even determining which knots have $g_s(K) > 0$ (or equivalently, are not slice) is still a subject of current research. The following corollary allows us to compute a lower bound on $g_s(K)$ simply by drawing K as a Legendrian link. (The original proof of the last sentence of the corollary is due to Rudolph [**Ru2**] via Theorem 2.4.6(1).)

Corollary 11.4.9. *Suppose that $K \subset S^3$ is a Legendrian knot and $F \subset D^4$ is a smooth surface with $\partial F = K$. Then $2g(F) - 1 \geq tb(K) + |r(K)|$; thus $g(K) \geq g_s(K) \geq \frac{1}{2}(tb(K) + 1 + |r(K)|)$. In particular, any knot $K \subset S^3$ satisfies $g_s(K) \geq \frac{1}{2}(TB(K) + 1)$, where $TB(K)$ is the maximal Thurston-Bennequin invariant of (Legendrian representatives of) the knot K.*

Proof. Summing K with a Legendrian trefoil (Figures 11.1 and 12.76) increases $tb(K)$ by 2 and $g(F)$ by 1, so we may assume $g(F) > 0$. Attach a 2-handle along K with framing $tb(K) - 1$ to get the Stein surface S_K; let Σ denote the surface obtained by sewing F together with the core of the

2-handle. Then by Theorem 11.4.7 we have $2g(\Sigma)-2 \geq [\Sigma]^2+|c_1(S_K)([\Sigma])|$. Since $tb(K) = [\Sigma]^2 + 1$ and $r(K) = c_1(S_K)([\Sigma])$, the corollary follows. □

Remark 11.4.10. Using essentially the same idea, one can generalize the above corollary [**AM2**]: Assume that F is a 2-dimensional submanifold of a Stein surface S in standard form such that $K = \partial F \subset \partial S$ is a Legendrian knot in the contact 3-manifold ∂S. If the framing of K induced by F has coefficient n and the rotation number with respect to F is $r(K, F)$, then $tb(K) - n + |r(K, F)| \leq -\chi(F)$.

Exercises 11.4.11. (a) Determine the possible Chern classes of Stein structures on the nucleus $N(n)$. (*Hint*: Repeat the argument given before Corollary 3.1.15 and conclude that $c_1 = PD(kf)$ for some $k \equiv n \pmod 2$, $|k| \leq n-2$; recall that f stands for the homology class of the fiber in $N(n)$.)

(b)* Realize $PD((2k-n)f) \in H^2(N(n);\mathbb{Z})$ for $1 \leq k \leq n-1$ as first Chern class of a Stein structure on $N(n)$.

(c)* Prove that if the disk bundle $X^4 \to F$ (as described in Exercise 11.2.5(a)) admits a Stein structure then $e(X) + \chi(F) \leq 0$. For F oriented, show that when X admits a Stein structure, the corresponding first Chern class satisfies $|\langle c_1(X), F\rangle| \leq \max(-\chi(F), 0) - e(X)$ and $\langle c_1(X), F\rangle \equiv e(X) \pmod 2$. Note that this proves optimality for $F \neq S^2$ in Exercise 11.3.2(a).

(d)* Prove that any oriented $\mathbb{R}^2$-bundle over an orientable surface is homeomorphic to Stein surfaces S for which the minimum genus of a smooth surface generating $H_2(S;\mathbb{Z})$ is arbitrarily large. (*Hint*: Exercise 11.3.2(b).) Now let X be the interior of an oriented handlebody without 3- or 4-handles, and suppose $H_2(X;\mathbb{Z}) \neq 0$. Prove that X admits infinitely many distinct smooth structures (up to isotopy), each of which is Stein. (This proves Theorem 9.4.29(b).)

(e) Let K be a Legendrian knot in S^3 with $tb(K) \geq 0$. Prove that the positive, untwisted Whitehead double DK of K (Remark 6.1.2) can be drawn as a Legendrian knot with $tb(DK) = 1$. Conclude that no iterated double $D^nK = D(D^{n-1}K)$ is smoothly slice. (This was originally proved for knots such as the right trefoil by Rudolph [**Ru1**] by a different method.) Recall (Definition 6.2.3 and the subsequent text) that the double of any knot is topologically slice.

(f) Prove that Theorem 11.4.7 and Corollary 11.4.9 remain true if the relevant surfaces are allowed to be immersed but with only negative double points. Conclude that a generically immersed disk in D^4 with Legendrian boundary K must have at least $\frac{1}{2}(tb(K)+1+|r(K)|)$ positive double points. (*Hint*: Blowing up negative double points of a surface preserves its homology class, by Proposition 2.3.5. When $g(\Sigma) = 0$, use the blow-up formula

Theorem 2.4.9 (which applies even though T may not be simply connected, cf. Remark 2.4.11) and the fact that $c_1^2(T) > 0$ for T a minimal surface of general type. Positive double points can be eliminated by adding genus, cf. the beginning of Section 2.1.)

We close this section by describing a result illustrating that Seiberg-Witten theory on a Stein surface S provides information about the contact geometry of ∂S; Corollary 11.4.13 can be regarded an application of Theorem 11.4.2.

Theorem 11.4.12. ([**LM1**], see also Example 11.3.5) *The 3-manifold $\partial N(n)$ admits at least $[\frac{n}{2}]$ noncontactomorphic contact structures.*

Proof. The Stein structures of $N(n)$ constructed in Exercise 11.4.11(b) have different Chern classes. Hence Theorem 11.4.2 implies that the corresponding $n-1$ contact structures on the boundary $\partial N(n)$ are nonisotopic (preserving the given orientations on the 2-plane fields). The unique nontrivial self-diffeomorphism of $\partial N(n)$ (cf. the proof of Lemma 8.3.10) is given by 180° rotation about the z-axis in Figure 12.81(a), so it identifies these contact structures in pairs with opposite $c_1(N(n), J)$. Any additional contactomorphism (even reversing orientation on the 2-plane fields) would provide a forbidden isotopy. □

Since any Stein structure J on the nucleus $N(n)$ has $c_1^2[N(n), J] = 0$ (Exercise 11.4.11(a)), any contact structure ξ on the homology sphere $\partial N(n)$ induced by a Stein structure J on $N(n)$ has $\theta(\xi) = -6$ (and $\Gamma(\xi, s) = 0$). Consequently all contact structures provided by Theorem 11.4.12 above are homotopic.

Corollary 11.4.13. ([**LM1**], cf. Example 11.3.5) *For any $n \in \mathbb{N}$ there exists a homology 3-sphere M which admits at least n noncontactomorphic, homotopic, holomorphically fillable contact structures.* □

Part 4

Appendices

Chapter 12

Solutions

12.1. Solutions of some exercises in Part 1

Solution of Exercise 1.2.10: Fix a basis $\{\alpha_1, \ldots, \alpha_n\}$ for $H_2(X;\mathbb{Z})/\operatorname{Tor}$; we denote the dual basis in $H^2(X;\mathbb{Z})/\operatorname{Tor} \cong (H_2(X;\mathbb{Z})/\operatorname{Tor})^*$ by $\{b_1, \ldots, b_n\}$ (i.e., $b_i(\alpha_j) = \delta_{ij}$). Taking the Poincaré dual $\{\beta_1, \ldots, \beta_n\}$ of the above dual basis, we find that $Q_X(\alpha_i, \beta_j) = \delta_{ij}$. Consequently the matrix representing Q_X in the basis $\{\alpha_1, \ldots, \alpha_n\}$ equals the matrix corresponding to the change of basis $\{\alpha_1, \ldots, \alpha_n\} \mapsto \{\beta_1, \ldots, \beta_n\}$. This latter matrix is invertible over $\mathbb{Z}$, proving the unimodularity of Q_X. □

Solution of Exercise 1.2.17(a): Take the basis $\{x, y - \frac{1}{2}Q(y,y)x\}$ for $\operatorname{span}(x,y)$; the matrix of Q in this basis is exactly H. □

Solution of Exercise 1.2.17(b): Suppose that $Q(y,y) = 2k+1$. In the basis $\{kx - y, (k+1)x - y\}$ the bilinear form Q is represented by the matrix $\langle 1\rangle \oplus \langle -1\rangle$. □

Solution of Exercise 1.2.17(c): If $\{x,y,z\}$ is a basis in which Q is represented by $H \oplus \langle -1\rangle$, then take $\{x-z, y-z, x+y-z\}$; in this new basis, Q is represented by $2\langle -1\rangle \oplus \langle 1\rangle$. □

Solution of Exercise 1.3.1(a): Observe that $\mathbb{CP}^n$ can be given as the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1} - \{0\}$ by the action of the circle S^1. Now the formula $\pi_1(\mathbb{CP}^n) = 1$ follows from the homotopy exact sequence of this S^1-fibration; this description also shows the compactness of $\mathbb{CP}^n$. □

Solution of Exercise 1.3.1(e): Take $H' = \{[x : y : z] \in \mathbb{CP}^2 \mid y = 0\}$. Clearly $H \cap H' = \{[0 : 0 : 1]\}$, and it is a transverse intersection, consequently $Q_{\mathbb{CP}^2}([H], [H']) = \pm 1$. In fact, this number is $+1$, since it is a transverse intersection of two complex submanifolds. Hence $[H]$ cannot be a multiple of any other class, so it generates $H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$. Since $H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$, and two generic lines intersect each other in a point with positive sign, we have that $Q_{\mathbb{CP}^2} = \langle 1 \rangle$. (Recall that the complex structure gives a canonical orientation for $\mathbb{CP}^2$ in which transverse complex submanifolds intersect positively.) □

Solution of Exercise 1.3.5(a): Applying the Mayer-Vietoris sequence for the triple $(X_1 \# X_2, X_1 - \text{int } D_1, X_2 - \text{int } D_2)$, we see that $H_2(X_1 \# X_2; \mathbb{Z}) \cong H_2(X_1; \mathbb{Z}) \oplus H_2(X_2; \mathbb{Z})$. Since any class in the $i^{\underline{th}}$ summand is represented by a closed, oriented surface in $X_i - \text{int } D_i$, it is easy to see that the pairings add as required. □

Solution of Exercise 1.3.5(b): The same argument as in Exercise 1.3.5(a) works here, since if $H_1(N; \mathbb{Z}) = H_2(N; \mathbb{Z}) = 0$, then the Mayer-Vietoris sequence gives $H_2(X; \mathbb{Z}) \cong H_2(X_1; \mathbb{Z}) \oplus H_2(X_2; \mathbb{Z})$. The rest of the argument applies without change. □

Solution of Exercise 1.3.12(a): Take $Q \cong 2kE_8 \oplus lH$. By assumption $l \geq |3k|$, hence Q splits as $(2kE_8 \oplus |3k|H) \oplus (l - |3k|)H$. If $k \leq 0$, then take $X = \#|k|S_4 \#(l - |3k|)S^2 \times S^2$; if $k > 0$, then take $X = \#k\overline{S}_4 \#(l - 3k)S^2 \times S^2$. (Recall that S_4 is the $K3$-surface and $\overline{S}_4$ denotes the same 4-manifold with the opposite orientation.) Now Exercise 1.3.5(a) proves that $Q_X \cong Q$. □

Solution of Exercise 1.3.12(b): Since $b_2^+(X) = 0$, Theorem 1.2.30 implies that $Q_X \cong n\langle -1 \rangle$. Using the fact that Q_X is even, we conclude that $n = 0$, hence Theorem 1.2.27 completes the proof. □

Solution of Exercise 1.4.11(b): Let Σ' be a section of the normal bundle $\nu\Sigma$ intersecting the 0-section transversely. The Tubular Neighborhood Theorem identifies $\nu\Sigma$ with a neighborhood of Σ in X, so by the definition of $e(\nu\Sigma)$ we have $e(\nu\Sigma)[\Sigma] = Q_X([\Sigma], [\Sigma']) = [\Sigma]^2$. □

Solution of Exercise 1.4.16(b): If X is a simply connected, smooth, closed 4-manifold, then (by Theorems 1.2.21 and 1.2.30) Q_X is isomorphic either to $n\langle 1 \rangle \oplus m\langle -1 \rangle$ or to $2kE_8 \oplus lH$ $(m, n, l \geq 0,\ k \in \mathbb{Z})$. By the Noether formula (Theorem 1.4.13) we know that if X is almost-complex, then $b_2^+(X)$ (which is equal to n and l respectively) is odd. For the converse, assume first that the intersection form Q_X is odd and n is equal to $2p + 1$. Choose the basis $\{\alpha_1, \ldots, \alpha_{2p+1}, \beta_1, \ldots, \beta_m\} \subset H^2(X; \mathbb{Z})$ giving Q_X as $n\langle 1 \rangle \oplus m\langle -1 \rangle$.

The cohomology element $h = 3\alpha_1 + \sum_{i=1}^{p}(3\alpha_{2i} + \alpha_{2i+1}) + \sum_{j=1}^{m}\beta_j$ satisfies $h^2 = 9 + 9p + p - m = 10p + 9 - m = 5n - m + 4 = 3\sigma(X) + 2\chi(X)$ and it is congruent to $w_2(X) \pmod 2$; consequently (by Theorem 1.4.15) it corresponds to an almost-complex structure. If $Q_X = 2kE_8 \oplus lH$ and l is odd, then $c_1^2(X) = 3\sigma(X) + 2\chi(X) = 48k + 32|k| + 4l + 4$ is divisible by 8. If $\{a, b\}$ is the basis of the first hyperbolic pair H in Q_X and $c_1^2(X) = 8q$, then for $h = 2qa + 2b$ we have $h^2 = 3\sigma(X) + 2\chi(X)$ and $h \equiv 0 = w_2(X) \pmod 2$, i.e., h corresponds to an almost-complex structure on X (again through Theorem 1.4.15). □

Solution of Exercise 1.4.21(c): Suppose that $E \to X$ is a $U(2)$-bundle with $c_1(E) = h$ and $c_2[E] = \chi(X)$. Note that $e[E] = c_2[E] = \chi(X) = e[TX]$ and $p_1[E] = c_1^2[E] - 2c_2[E] = h^2 - 2\chi(X) = 3\sigma(X) = p_1[TX]$; moreover $w_2(E) \equiv c_1(E) = h \equiv w_2(TX)$. Consequently the bundles $TX \to X$ and $E \to X$ (regarded as an $SO(4)$-bundle) are isomorphic by Theorem 1.4.20(b). Since E is a $\mathbb{C}^2$-bundle, this isomorphism yields an almost-complex structure J on TX with $c_1(TX, J) = c_1(E) = h$. □

Solution of Exercise 1.4.26(a): It is known that the nontrivial double covers of X are in 1-1 correspondence with subgroups of $\pi_1(X)$ of index 2. Note that such a subgroup G is always normal, consequently it is determined by a (nontrivial) homomorphism $\varphi_G\colon \pi_1(X) \to \mathbb{Z}_2$. Conversely, each nontrivial homomorphism $\varphi\colon \pi_1(X) \to \mathbb{Z}_2$ determines a subgroup of index 2 (by taking $G = \ker\ \varphi$), hence a nontrivial double cover of X. The map $\varphi\colon \pi_1(X) \to \mathbb{Z}_2$ is obviously determined by the map $\overline{\varphi}\colon H_1(X;\mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)] \to \mathbb{Z}_2$ defined on the first homology. Note, however, that nontrivial maps $\overline{\varphi}\colon H_1(X;\mathbb{Z}) \to \mathbb{Z}_2$ are in 1-1 correspondence with nonzero elements of $H^1(X;\mathbb{Z}_2)$. Associating the trivial double cover $(X \amalg X \to X)$ to $0 \in H^1(X;\mathbb{Z}_2)$ completes the required correspondence. □

Solution of Exercise 2.2.4: The sections $y = 1$ and $x = \bar{v}$ over the two respective hemispheres agree at the equator (unit circle); splice these together with a partition of unity. □

Solution of Exercise 2.2.12(b): If d is even, then Q_{S_d} is even (see Section 1.3), hence there is no homology class of square -1 in $H_2(S_d;\mathbb{Z})$; in particular there is no rational -1-curve in S_d. □

Solution of Exercise 2.3.6(c): Repeat the proof of Proposition 2.1.4 to obtain a manifold Y' with intersection form $(-E_8)\oplus\langle -1\rangle$. Applying Exercise 1.2.24 and Donaldson's Theorem 1.2.30 completes the solution. □

Solution of Exercise 2.3.6(d): Blow up the double points of the immersion f. If there is no positive double point, then the proper transform will represent the class $3h \in H_2(\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}; \mathbb{Z})$ by an embedded sphere. Extending the solution of Exercise 2.3.6(c) to n blow-ups now gives the required contradiction. $\square$

Solution of Exercise 2.4.1: Since $K \in \mathcal{C}_X$, we have $K^2 \equiv \sigma(X) \pmod 8$. Hence $\dim \mathcal{M}_K^\delta(g) = \frac{K^2-\sigma(X)}{4} - \frac{\sigma(X)+\chi(X)}{2} \equiv 1 - b_1(X) + b_2^+(X) \pmod 2$; now the solution easily follows. $\square$

Solution of Exercise 2.4.12(b): Represent a basis $\{\alpha_1, \dots, \alpha_n\}$ by surfaces $\Sigma_i \subset X$. Then the adjunction formula provides bounds for $K(\alpha_i)$ in terms of $g(\Sigma_i)$ for a basic class $K \in \mathcal{C}_X$. If X is not of simple type, however, we need $\alpha_i^2 \geq 0$ — this can be achieved by an appropriate basis change. $\square$

Solution of Exercise 2.4.13: Based on our computations in Section 1.3, it is easy to see that $Q_{S_4 \# \overline{\mathbb{CP}^2}} = 2(-E_8) \oplus 3H \oplus \langle -1 \rangle \cong 3\langle 1 \rangle \oplus 20\langle -1 \rangle$. Since $Q_{3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}} = 3\langle 1 \rangle \oplus 20\langle -1 \rangle$ and both 4-manifolds are smooth and simply connected, Freedman's Theorem 1.2.27 implies that X and Y are homeomorphic. If we decompose Y as $\mathbb{CP}^2 \# (\# 2\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2})$, Theorem 2.4.6(1) shows that $SW_Y \equiv 0$. Since X is the blow-up of the simply connected complex surface S_4, Theorem 2.4.7(1) shows that $SW_X(\pm c_1(X)) \neq 0$. (In fact, in this case $SW_X(K) = 0$ unless $K = \pm c_1(X)$; moreover, $SW_X(\pm c_1(X)) = \pm 1$.) Now Theorem 2.4.3 shows that X and Y are nondiffeomorphic. $\square$

Solution of Exercise 2.4.14: For $(A, B) \in Spin^c(4) \subset U(2) \times U(2)$, choose λ such that $\lambda^2 = \det(A)$ $(= \det(B))$. Then $(\lambda, A \cdot \left[\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right], B \cdot \left[\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right])$ is a well-defined element of $S^1 \times SU(2) \times SU(2)/\{\pm(1, I, I)\}$ (independent of the choice of λ), and the map $(A, B) \mapsto (\lambda, A \cdot \left[\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right], B \cdot \left[\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right])$ gives the desired isomorphism. $\square$

Solution of Exercise 2.4.17: The double cover $G \to SO(2) \times SO(4)$ extends over $SO(6)$ iff $\pi_1(G) \xrightarrow{\varphi_*} \pi_1(SO(2) \times SO(4)) \xrightarrow{\vartheta_*} \pi_1(SO(6)) \cong \mathbb{Z}_2$ is exact. (This is because the above composition should be the same as $\pi_1(G) \to \pi_1(Spin(6)) \to \pi_1(SO(6))$, which is the 0 homomorphism; the index of $\mathrm{Im}(\varphi_*)$ in $\pi_1(SO(2) \times SO(4))$ is 2 because it corresponds to a double cover.) Now if $\pi_1(SO(2)) \cong \mathbb{Z} = \langle x \rangle$, $\pi_1(SO(4)) \cong \mathbb{Z}_2 = \langle a \rangle$ and $\pi_1(SO(6)) \cong \mathbb{Z}_2 = \langle b \rangle$, then $\vartheta_*(lx, ka) = (l + k)b$, and this proves the assertion. $\square$

Solution of Exercise 2.4.21: Recall that by Hodge theory, $H^2(X;\mathbb{R})$ can be identified with the space of g-harmonic 2-forms $\mathcal{H}^2(X)$ [**We**]. It is easy to see that a self-dual (or ASD) closed 2-form is harmonic (recall that $d^* = -*d*$), hence $\mathcal{H}^2(X)$ can be decomposed as the sum $\mathcal{H}^+(X)\oplus\mathcal{H}^-(X)$, where $\mathcal{H}^\pm(X)$ is the space of self-dual (resp. ASD) 2-forms. Since for $\omega,\eta\in\mathcal{H}^2$ we have $Q_X([\omega],[\eta]) = \int_X \omega\wedge\eta$, and the norm $||\omega||^2 = \int_X \omega\wedge *\omega$ is positive definite, it is easy to see that $Q_X|\mathcal{H}^+(X)$ is positive definite and $Q_X|\mathcal{H}^-(X)$ is negative definite. This observation implies $\dim\mathcal{H}^+(X) = b_2^+(X)$ (and similarly $\dim\mathcal{H}^-(X) = b_2^-(X)$). □

Solution of Exercise 2.4.26(b): If (A,ψ) is a solution, then (since $\partial\!\!\!/_A\psi = 0$ and $F_A\psi = F_A^+\psi = \frac{1}{2}|\psi|^2\psi$)

$$0 = \partial\!\!\!/_A(\partial\!\!\!/_A\psi) = \nabla_A^*\nabla_A\psi + \frac{1}{4}s\psi + \frac{1}{4}|\psi|^2\psi.$$

Taking the pointwise inner product with ψ, we get

$$0 = \langle\nabla_A^*\nabla_A\psi,\psi\rangle + \frac{1}{4}s|\psi|^2 + \frac{1}{4}|\psi|^4.$$

Let $x_0\in X$ be the global maximum for $|\psi|$, so $\langle\nabla_A^*\nabla_A\psi(x_0),\psi(x_0)\rangle\geq 0$, hence $|\psi(x_0)|^4 \leq -s(x_0)|\psi(x_0)|^2$. Now either $\psi(x_0) = 0$ (in which case $\psi\equiv 0$) or $|\psi(x)|^2\leq|\psi(x_0)|^2\leq -s(x_0)\leq s_{(X,g)}$, concluding the solution. □

Solution of Exercise 3.1.2: The trivial S^2-bundle over S^2 has total space $S^2\times S^2$, which is a spin manifold ($Q_{S^2\times S^2} = H$). On the other hand, $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ has odd intersection form; hence the solution is complete. In fact, $S^2\times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ are the only two S^2-bundles over S^2; for more about Hirzebruch surfaces see Section 3.4. □

Solution of Exercise 3.1.12(a): The adjunction formula gives $c_1(E(n))([C]) = 2-n\neq 0$, and since $c_1(E(n)) = PD((2-n)f)$, it implies $[f]\cdot[C] = 1$. Since each fiber is a complex curve, and complex curves intersect each other only in points of positive sign, C intersects every fiber in a unique point. Consequently C is a section of the fibration. Note that the square of any rational curve in $E(n)$ is equal to $(2-n)k-2$ for some nonnegative integer k. In particular, a rational curve in the $K3$-surface has square -2. Note that in the above argument it is essential that C is a complex submanifold of $E(n)$; there are smoothly embedded spheres of square -4 in $E(4)$, for example, which are disjoint from the generic fiber. (Tube two disjoint spheres of square -2 in the complement of the nucleus $N(4)\subset E(4)$ together.) □

Solution of Exercise 3.1.12(b): Fix a cusp fiber F and a section S in $E(n)$. We can find a sphere S' of square -2 in $E(n)$ disjoint from F and S. (Consider, for example, $F,S\subset N(n)$ and $S'\subset\Phi(n)$.) By tubing S and S'

together we get a sphere Σ with $[\Sigma]^2 = -n-2$. Now the tubular neighborhood of $F \cup \Sigma$ provides a copy of $N(n+2)$ in $E(n)$. When we blow up S, the proper transform $\widetilde{S} \subset E(n)\#\overline{\mathbb{CP}^2}$ together with F gives rise to $N(n+1) \subset E(n)\#\overline{\mathbb{CP}^2}$. □

Solution of Exercise 3.1.16(a): Suppose that $2n-1$ disjoint copies of $N(2)$ are embedded in $E(n)$. By the adjunction formula we know that a basic class evaluates trivially on the homology classes represented by each $N(2) \subset E(n)$. On the other hand, since $b_2^+(E(n)) = 2n-1$, the complement of the $2n-1$ copies of $N(2)$ has a negative definite intersection form; hence the condition $K^2 = 0$ (which is satisfied by all basic classes of $E(n)$) implies $K = 0$. This argument shows that each basic class is equal to 0, which is a contradiction for $n > 2$. Since $b_2^+(E(2)) = 3$ and $b_2^+(N(2)) = 1$ (and $\partial N(2)$ is a homology sphere), $E(2)$ does not contain 4 disjoint copies of $N(2)$. □

Solution of Exercise 3.1.16(b): By Corollary 3.1.15 we have that $\mathcal{B}as_{E(3)} = \{\pm PD(f)\}$. If $E(3)$ contains an embedded -1-sphere, i.e., it decomposes as $X\#\overline{\mathbb{CP}^2}$, then — by the blow-up formula 2.4.9 — we get that $\{\pm PD(f)\} = \{K \pm E \mid K \in \mathcal{B}as_X\}$. This implies $K = 0$, hence $PD(f) = \pm E$, leading to a contradiction (since $0 = PD(f)^2 \neq E^2 = -1$). □

Solution of Exercise 3.2.10(b): Assume that $V(n)$ is a complete intersection $S(d_1, \ldots, d_{n-2})$. Since $V(n) \approx E(n)$ and $c_1^2(E(n)) = 0$, we get that $\sum_{i=1}^{n-2} d_i = n+1$ (cf. Exercise 1.3.13(b)). Since each $d_i \geq 2$, the inequality $2(n-2) \leq \sum d_i = n+1$ follows, hence $n \leq 5$. Now the equation $\sum_{i=1}^{n-2} d_i = n+1$ reads as $d_1 = 4$, $d_1+d_2 = 5$ and $d_1+d_2+d_3 = 6$ (for the cases $n = 3, 4, 5$ resp.), resulting in complex surfaces with signature -16 (cf. Exercise 1.3.13(c)). This leads to a contradiction, since $\sigma(V(n)) = -8n \neq -16$ unless $n = 2$. In fact, the complex surfaces $S(4)$, $S(2,3)$ and $S(2,2,2)$ (the only complete intersection surfaces with $c_1^2 = 0$) are all $K3$-surfaces. □

12.2. Solutions of some exercises in Part 2

Solution of Exercise 4.3.1(a): See Figure 12.1. The attaching disks of the 1-handles are identified by horizontal and vertical reflections in the obvious way, except for the pair at 0 and ∞, whose boundaries are identified by a dilation. The disk at ∞ is not drawn. □

Solution of Exercise 4.3.1(b): To identify the manifold, cancel a handle pair to obtain $\mathbb{RP}^3\#S^1 \times S^2$. □

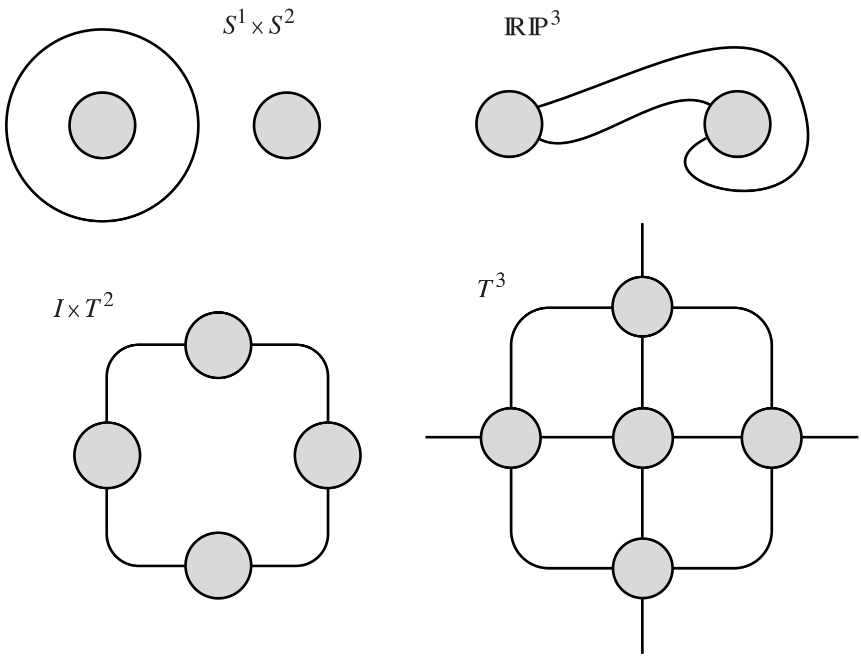


Figure 12.1. Heegaard diagrams.

Solution of Exercise 4.3.1(c): $\pi_1(L(5,q)) \cong \mathbb{Z}_5$; $H_k(L(5,q)) \cong \mathbb{Z}_5$ $(k = 1)$, $\mathbb{Z}$ $(k = 0, 3)$, 0 otherwise; $H^k(L(5,q)) \cong H_{3-k}(L(5,q))$ (Poincaré duality). The diagram generalizes in the obvious way to give manifolds (lens spaces, cf. Section 5.3) with $\pi_1(X) \cong \mathbb{Z}_n$. □

Solution of Exercise 4.4.4: The isotopy is given in Figure 12.2. For a more abstract description, think of the manifold as being the trivial S^2-bundle over S^1, with K the section given by the north pole of each fiber. Let φ be the self-diffeomorphism that rotates the fiber over each $\theta \in S^1$ by 2θ. Then φ fixes K but puts two twists in any given framing on it. But φ can be interpreted as twice the generator of $\pi_1(O(3))$. Since (by Remark 4.1.5) $\pi_1(O(3)) \cong \mathbb{Z}_2$, φ is (fiber-preserving) isotopic to the identity, and this isotopy removes the twists from the framing on K. As for Philippine dancing, stand holding a glass of wine in the palm of one hand, and without changing your grip or moving your feet, rotate the glass through 720° about a fixed axis without spilling the wine. □

Solution of Exercise 4.5.9: The coefficient in Figure 4.19 is 6. The 0-framing is obtained by reversing the handedness of the 3 twists. By redistributing these as a left half-twist at each over- or undercrossing, one can visualize the 0-framing as the outward normal to the punctured-torus Seifert surface

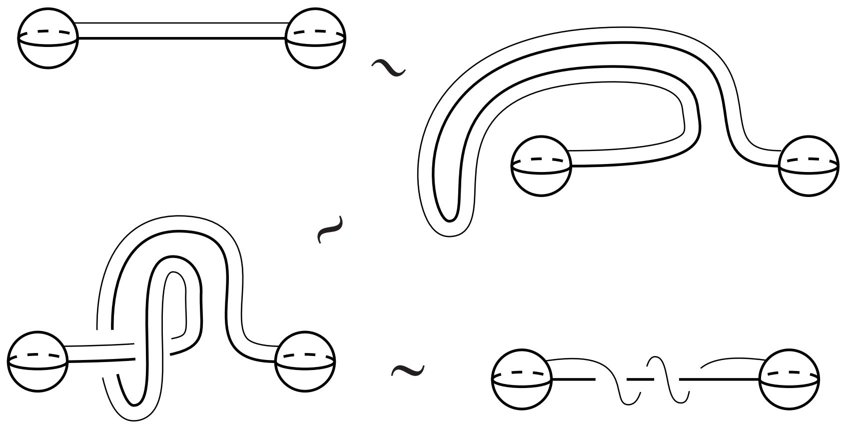


Figure 12.2

consisting of the central triangle and the region containing ∞. The Möbius band induces the 6-framing. □

Solution of Exercise 4.5.12(a): Replace each crossing by a pair of embedded arcs in $\mathbb{R}^2$ as in Figure 12.3(a). Note that orientations are preserved. We obtain an oriented collection of circles embedded in $\mathbb{R}^2$. Each circle is the oriented boundary of an embedded (suitably oriented) disk in $\mathbb{R}^2$. Imagine these disks disjointly stacked in $\mathbb{R}^3$ like pancakes, with the smallest disks on top. Now recover the original oriented link by attaching half-twisted bands to the disks (Figure 12.3(b)), and verify that the resulting surface is oriented. (Note that there are four possibilities for where the disks may lie in the figure; one of these is immediately ruled out by checking orientations. Why don't additional disks cut through the band?) □

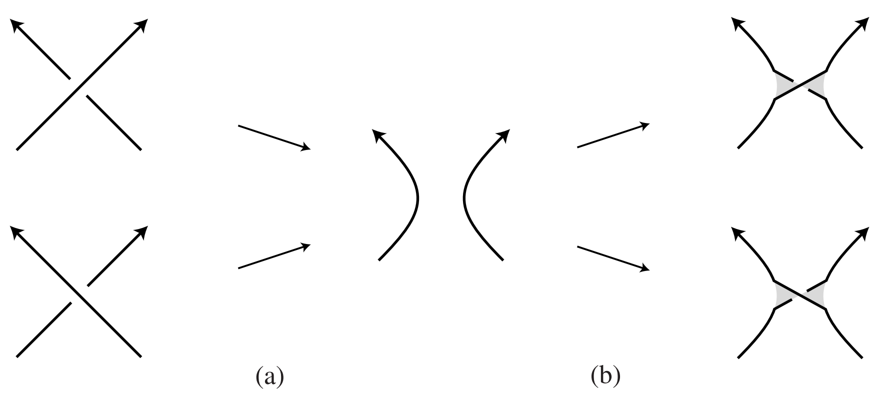


Figure 12.3

Solution of Exercise 4.5.12(b): First, consider $H_2(X;\mathbb{Z})$. Since every homology class is compactly supported, it suffices to work in a compact submanifold of X in the noncompact case. Represent z by a disjoint collection of suitably oriented core disks $D^2 \times \{\text{pt.}\}$ of 2-handles. The condition $\partial_* z = 0$ means that the resulting oriented link in ∂X_1 is homologically trivial, i.e., the strands running over each 1-handle can be grouped in pairs with opposite orientation (Figure 12.4). If we form an oriented surface F by attaching bands to the core disks as in the figure, the resulting boundary can clearly be slid off each 1-handle into ∂D^4. If F' denotes a Seifert surface for the resulting oriented link in ∂D^4, then $F \cup \overline{F'}$ is the required surface. In the case of $H_2(X;\mathbb{Z}_2)$, the same argument works with unoriented links. (Now the condition $\partial_* z = 0$ implies that the number of strands in the figure is even, but we may need to band together strands with parallel orientations and get a nonorientable F.) □

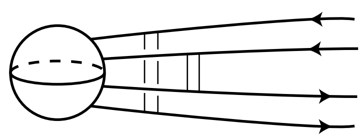


Figure 12.4

Solution of Exercise 4.5.12(c): By the long exact homology sequence, we see that $\ker(i_* : H_1(M - K_1;\mathbb{Q}) \to H_1(M;\mathbb{Q}))$ is generated by the meridian μ of K_1. The meridian is nontrivial in $H_1(M - K_1;\mathbb{Q})$, since a nullhomology of μ could be combined with the disk D in M bounded by μ to obtain a class $\alpha \in H_2(M;\mathbb{Q})$ satisfying $K_1 \cdot \alpha = 1$, contradicting the hypothesis that $[K_1] = 0 \in H_1(M;\mathbb{Q})$. Since $[K_2] \in \ker i_*$, it is a rational multiple $r\mu$ of μ; set $\ell k_{\mathbb{Q}}(K_1, K_2) = r$. Alternatively, let c be a rational 2-chain with $\partial c = K_1$. (For example, if $q[K_1] = 0 \in H_1(M;\mathbb{Z})$, there is a surface $F \subset M - \operatorname{int} \nu K_1$ with $\partial F = q\lambda + p\mu$ for some λ on $\partial \nu K_1$; set $c = \frac{1}{q}F$.) Define $\ell k_{\mathbb{Q}}(K_1, K_2) = c \cdot K_2 = \frac{1}{q}F \cdot K_2$. This is well-defined, since for any other chain c' with $\partial c' = K_1$ we have $(c - c') \cdot K_2 = 0$ because the first factor is a cycle and the second vanishes in $H_1(M;\mathbb{Q})$. Since the latter definition only depends on K_2 through its class in $H_1(M - \operatorname{int} \nu K_1;\mathbb{Q})$, and since it gives $\ell k_{\mathbb{Q}}(K_1, \mu) = 1$, it agrees with the former definition. For a third definition, find rational chains c_i in $I \times M$ with $\partial c_i = \{1\} \times K_i$. We can assume that these are rational multiples of transverse surfaces, and set $\ell k_{\mathbb{Q}}(K_1, K_2) = c_1 \cdot c_2$. This is well-defined (even though there is no well-defined intersection form on $H_2(X^4, \partial X^4)$ in general) since each ∂c_i is fixed: For a different choice c_1', the cycle $c_1 - c_1'$ can be made disjoint from c_2 by pushing the former down to $\{0\} \times M$. (The pairing on $H_2(X) \times H_2(X, \partial X)$

is well-defined.) This last definition is equivalent to the previous one by the method of Proposition 4.5.11, and it shows that $\ell k_{\mathbb{Q}}$ is symmetric. We can extend $\ell k_{\mathbb{Q}}$ to formal rational linear combinations of disjoint knots in the obvious way. A framing on a knot K determines a parallel push-off K'; for $[K] = 0 \in H_2(M;\mathbb{Q})$ define the framing coefficient to be $\ell k_{\mathbb{Q}}(K, K') \in \mathbb{Q}$. The coefficients of framings on K comprise a coset of $\mathbb{Z}$ in $\mathbb{Q}$. Now for torsion classes $\gamma_1, \gamma_2 \in H_1(M;\mathbb{Z})$, define $\ell k_{\mathbb{Q}/\mathbb{Z}}(\gamma_1, \gamma_2)$ to be the mod 1 reduction of $\ell k_{\mathbb{Q}}(K_1, K_2)$ for disjoint oriented knots K_i with $[K_i] = \gamma_i$. If we choose a different representative K_2' of γ_2 then there is an integral chain c in M with $\partial c = K_2 - K_2'$, and $\ell k_{\mathbb{Q}}(K_1, K_2)$ changes by the integer $K_1 \cdot c$ (by the first definition), so $\ell k_{\mathbb{Q}/\mathbb{Z}}$ is a well-defined, symmetric bilinear form. Since the framing coefficients defined above reduce mod 1 to $\ell k_{\mathbb{Q}/\mathbb{Z}}([K],[K])$, bilinearity implies that they lie in $\frac{1}{q}\mathbb{Z}$, where q is the order of $[K] \in H_1(M;\mathbb{Z})$.

To prove that $\ell k_{\mathbb{Q}/\mathbb{Z}}$ is nonsingular when M is closed, let K_1 represent a torsion class with order $q > 1$ in $H_1(M;\mathbb{Z})$ and consider the exact sequence (with integer coefficients)

$$H^1(M) \xrightarrow{j^*} H^1(M - K_1) \xrightarrow{\delta^*} H^2(M, M - K_1) \to H^2(M).$$

Now $H^2(M, M-K_1) \cong \mathbb{Z}$, with a generator α mapping to $PD[K_1] \in H^2(M)$. By exactness, $H^1(M - K_1) = \operatorname{Im} j^* \oplus \langle\beta\rangle$ for some class β with $\delta^*\beta = q\alpha$. Since $H_1(M - K_1)/\text{torsion}$ is the dual space of $H^1(M - K_1)$ there is an oriented knot $K_2 \subset M - K_1$ with $\langle\beta, K_2\rangle = 1$ and $\operatorname{Im} j^*$ vanishing on K_2. The latter condition implies $[K_2] \in H_1(M)$ is a torsion class, so $[K_2] = r\mu \in H_1(M - K_1;\mathbb{Q})$ for $\mu = [\partial D]$ a meridian of K_1, and $r = \ell k_{\mathbb{Q}}(K_1, K_2)$. But $1 = \langle\beta, r\mu\rangle = r\langle\delta^*\beta, D\rangle = rq\langle\alpha, D\rangle = rq$, so $r = \frac{1}{q} \notin \mathbb{Z}$, and $\ell k_{\mathbb{Q}/\mathbb{Z}}([K_1],[K_2]) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. □

Solution of Exercise 4.5.12(d): Let X_1, X_2 be the closures of the components of $S^4 - M^3$. The Mayer-Vietoris isomorphism $H_1(M;\mathbb{Z}) \to H_1(X_1;\mathbb{Z}) \oplus H_1(X_2;\mathbb{Z})$ splits the torsion of $H_1(M;\mathbb{Z})$ as $G_1 \oplus G_2$, where G_i injects into $H_1(X_i;\mathbb{Z})$ and vanishes in the other X_j. Given $\gamma_1, \gamma_2 \in G_1$, we represent γ_i by a knot K_i in M. Then K_i bounds a rational chain c_i in $I \times M \subset X_1$ and an oriented surface F_i in X_2. Let z_i be the rational cycle $c_i - F_i$ in S^4. Since Q_{S^4} is trivial, we have $0 = z_1 \cdot z_2 = c_1 \cdot c_2 + F_1 \cdot F_2$. Since $F_1 \cdot F_2 \in \mathbb{Z}$, so is $c_1 \cdot c_2$, i.e., $\ell k_{\mathbb{Q}/\mathbb{Z}}(\gamma_1, \gamma_2) = 0$. A direct calculation shows that the nonzero element of $H_1(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}_2$ has self-linking $\frac{1}{2}$ $(= \frac{1}{2}\mathbb{RP}^2 \cdot \mathbb{RP}^1)$, so the required splitting does not exist for $\mathbb{RP}^3$. □

Solution of Exercise 4.6.1 Cutting S^1 and S^3 into pairs of disks shows that $S^3 \times S^1$ has a handle decomposition with four handles, whose indices are 0,1,3 and 4. For $S^2 \tilde{\times} S^2$, see Example 4.6.3. □

Solution of Exercise 4.6.4(a): $D(X \natural Y) \approx DX \# DY$, so $D(\natural m S^k \times D^{n-k}) \approx \# m S^k \times S^{n-k}$. □

Solution of Exercise 4.6.4(b): Given a finite presentation of G, it is easy to construct an oriented handlebody X (of any fixed dimension $n \geq 4$) with a 1-handle for each generator and a 2-handle for each relator, so that $\pi_1(X) \cong G$. (What fails when $n = 3$?) Now DX is the required closed manifold, since it is obtained from X by adding handles of index $\geq n-2$. For $n \geq 5$, these handles never affect the fundamental group, and when $n = 4$, the 2-handles are attached along meridians, which are trivial in $\pi_1(X)$. To draw a Kirby diagram of such a 4-manifold, start with a 1-handle for each generator, for each relator draw any framed knot realizing the given word in $\pi_1(X_1)$, then add a 0-framed meridian to each knot, a 3-handle for each generator and a 4-handle. □

Solution of Exercise 4.6.6(a): We begin with the boundary, $T^2 \times S^1$. A single torus $T^2 \times \{\text{pt.}\} \subset T^2 \times S^1$ is given by a spanning disk of the circle in Figure 4.36, together with a pair of 1-handles and a 2-handle lying in the boundaries of the corresponding 4-dimensional handles. Next visualize an unknot complement fibered by disks. This picture describes the S^1-family of 0-handles of tori in Figure 4.36. These 0-handles intersect each sphere in Figure 4.36 (bounding an attaching ball of a 1-handle) in an S^1-family of arcs, which give the attaching regions of an S^1-family of 1-handles. We obtain an S^1-family of punctured tori, whose boundaries fiber the boundary of the attaching region of the 2-handle. The remainder of the 3-manifold is a solid torus $D^2 \times S^1$ in the boundary of the 2-handle, and the disks $D^2 \times \{\text{pt.}\}$ are the 2-handles of the S^1-family of tori. We can see all of $T^2 \times D^2$ by the same technique, first visualizing the 0-handle as $D^2 \times D^2$, where the disks fibering the interior intersect the boundary of the 0-handle in a D^2-family of circles forming a tubular neighborhood of the circle in Figure 4.36(a). The 4-dimensional 1-handles comprise two D^2-families of 2-dimensional 1-handles whose attaching regions are D^2-families of arcs filling the attaching balls, and the 4-dimensional 2-handle is a D^2-family of 2-dimensional 2-handles whose attaching regions fiber the attaching region $S^1 \times D^2$ of the 4-dimensional 2-handle. □

Solution of Exercise 4.6.6(b): A D^2-bundle $X \to F$ with F a genus-3 surface and a plumbing of two D^2-bundles $X_1, X_2 \to T^2$ are given in Figures 12.5 and 12.6, respectively; see Section 6.1 for further discussion. Since S^2-bundles over surfaces are doubles of D^2-bundles, we draw them by adding a 0-framed meridian to the attaching circle in a picture of the appropriate D^2-bundle (along with 3- and 4-handles). □

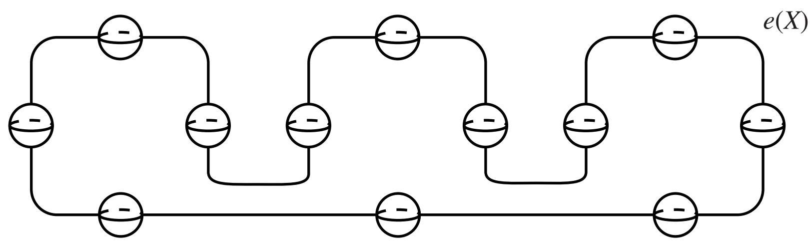


Figure 12.5. D^2-bundle X over a genus-3 surface.

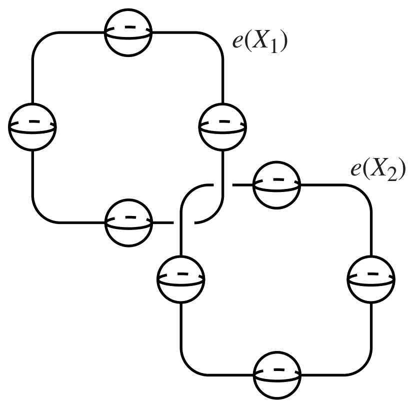


Figure 12.6. Plumbing of two D^2-bundles X_1, X_2 over T^2.

Solution of Exercise 4.6.9(d): $\mathbb{RP}^3 - \operatorname{int} D^3$ is a D^1-bundle over $\mathbb{RP}^2$, so its product with I is a D^2-bundle over $\mathbb{RP}^2$. The Euler number is 0 since the 0-section can be pushed off itself in the I-direction. □

Solution of Exercise 5.1.2(b): Figure 12.7 shows part of a Kirby diagram of a handle slide followed by the reverse slide. The resulting diagram is clearly isotopic to the original one. If we orient K_1 and K_2 so that the first slide is a handle addition, the resulting framing will be $n_1 + n_2 + 2\ell k(K_1, K_2)$ and the new linking number will be $\ell k(K_1, K_2) + n_2$. Plugging these values into the handle subtraction formula, we recover n_1 as the final coefficient of K_1, as required. □

Solution of Exercise 5.1.2(c): The discussion of handle slides generalizes to the setting of many strands isotoping across parallel disks $D^2 \times \{p_\ell\} \subset \partial(Y \cup h_2)$. We band-sum each strand with a parallel copy of K_2 determined by the framing. (These parallel copies lie disjointly on the boundary of a tubular neighborhood of K_2.) For each K_i that moves during the isotopy, the new framing coefficient will be given by $(\alpha_i + k_i\alpha_2)^2 = n_i + k_i^2 n_2 + 2k_i \cdot \ell k(K_i, K_2)$, where k_i is the signed number of strands of K_i being slid.

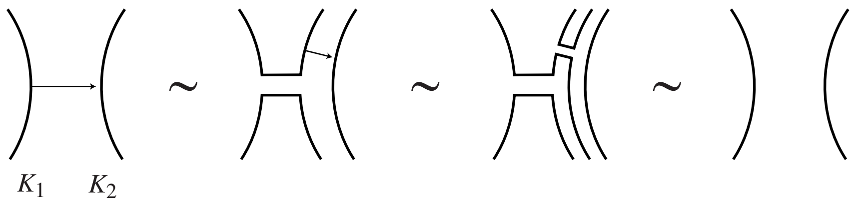


Figure 12.7. Reversing a handle slide.

(The linking matrix transforms as in Example 5.1.3(a), but with k times the corresponding row and column being added.) Note that the framing does not change if $k_i = 0$. □

Solution of Exercise 5.1.7(b): A generic homotopy of a knot in a 3-manifold is an isotopy except at finitely many times when the knot crosses through itself, so the method of Proposition 5.1.4 still applies when K_1 is nullhomotopic in ∂X_1. For a counterexample with K_1 nullhomologous, consider the double $D(T^2 \times D^2)$ (cf. Figure 4.36, $n = 0$). □

Solution of Exercise 5.1.10(a): The slide in Figure 12.8 followed by a 2-3 cancellation exhibits Figure 5.16 as $\overline{\mathbb{CP}^2}$. □

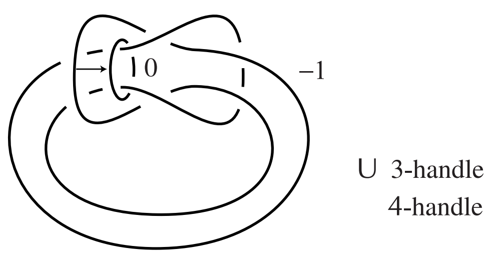


Figure 12.8

Solution of Exercise 5.1.10(b): For each 1-handle h_i of X, $i = 1, \ldots, \ell$, there is a knot K_i in ∂X intersecting its belt sphere once (transversely), so that a 2-handle attached along K_i with any framing would cancel h_i. Since X is simply connected, there is a homotopy $F_i \colon I \times S^1 \to X$ from K_i to a small unknotted circle K_0 in ∂X. By general position, F_i can be assumed to miss the wedge of circles in X determined by the cores of the 0- and 1-handles of X, so after a homotopy rel $0, 1$, we can assume F_i maps $I \times S^1$ into $\partial X_1 \cup$ 2-handles. Similarly, $F_i(I \times S^1)$ can be assumed to intersect the cores of the 2-handles in finitely many points, so we can assume that $F_i(I \times S^1)$ lies in ∂X except for finitely many disks which are cocore disks of 2-handles. Thus we can assume F_i appears in a Kirby diagram of X as a sequence of isotopies of K_i in ∂X (including slides over 2-handles), together with moves where K_i crosses through itself (cf. solution of Exercise 5.1.7(b)) and through attaching circles of 2-handles (the latter moves corresponding to cocore disks contained in $F_i(I \times S^1)$). Now add 0-framed meridians and 3- and 4-handles to the diagram to obtain DX, and add ℓ 0-framed Hopf links to realize $DX \# \ell S^2 \times S^2$. For each i, let K_i' denote one component of the $i^{\underline{th}}$ Hopf link. Then each F_i determines a sequence of Kirby moves sending the corresponding K_i' to K_i, since we can realize crossings of K_i through itself and other attaching circles as in Figure 5.11, using the 0-framed meridian to K_i'. Now we can cancel each 1-handle, leaving behind a 0-framed unknot (the meridian to K_i). Each of the m 2-handles of X still has a 0-framed meridian, and with the 1-handles eliminated, these split off as $S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$-summands (Proposition 5.1.4). We are left with a 0-framed unlink of ℓ components; this cancels the 3-handles. (For a different perspective on this construction, see Exercise 5.2.2(c).) □

Solution of Exercise 5.1.10(c): Given a finite group presentation with ℓ generators and m relators, Exercise 4.6.4(b) realizes the group as $\pi_1(DX)$ for a suitable handlebody X with ℓ 1-handles and m 2-handles. By Exercise 5.1.10(b) (above), $\pi_1(DX)$ is trivial if and only if $DX \# \ell S^2 \times S^2$ is diffeomorphic to $\# m S^2 \times S^2$ or $\# m\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$. Thus, the hypothesized algorithm for 4-manifolds would give an algorithm for determining the triviality of a group given by a finite presentation. □

Solution of Exercise 5.1.10(d): Let X be a handlebody realizing the presentation P, and suppose there is a sequence of Andrews-Curtis moves changing P to the empty presentation. Since $D(D^4) = S^4$, it suffices to realize these AC-moves by handle moves of DX, preserving its structure as the double of a handlebody realizing the corresponding presentation. The first few moves leave the handlebody unchanged — inversion and permutation correspond to changing the orientation and indexing of handles, and conjugating a relator corresponds to making a different choice of arc connecting an attaching circle to the base point. Multiplying a generator (relator) by another corresponds to a 1- (2-) handle slide in X. (In DX, the sliding 2-handle has a 0-framed meridian that allows us to unlink it from the meridian of the other 2-handle.) Adding a generator/relator pair corresponds to introducing a cancelling 1-handle/2-handle pair in X (and a corresponding 2-3 pair in DX). It remains to consider the effect of deleting a generator from P, together with a relator equal to it. Since the relator equals the generator, the corresponding attaching circle is homotopic in ∂X_1 to a curve cancelling the 1-handle. The 0-framed meridian allows us to follow the homotopy by

Kirby moves, so that we can cancel the 1-2 pair as required. The 0-framed meridian can then be separated from the other curves (since they have their own 0-framed meridians), and it cancels a 3-handle, restoring the handlebody to its form DX. Note that this last Andrews-Curtis move may change X — in fact, ∂X need not be simply connected in general. □

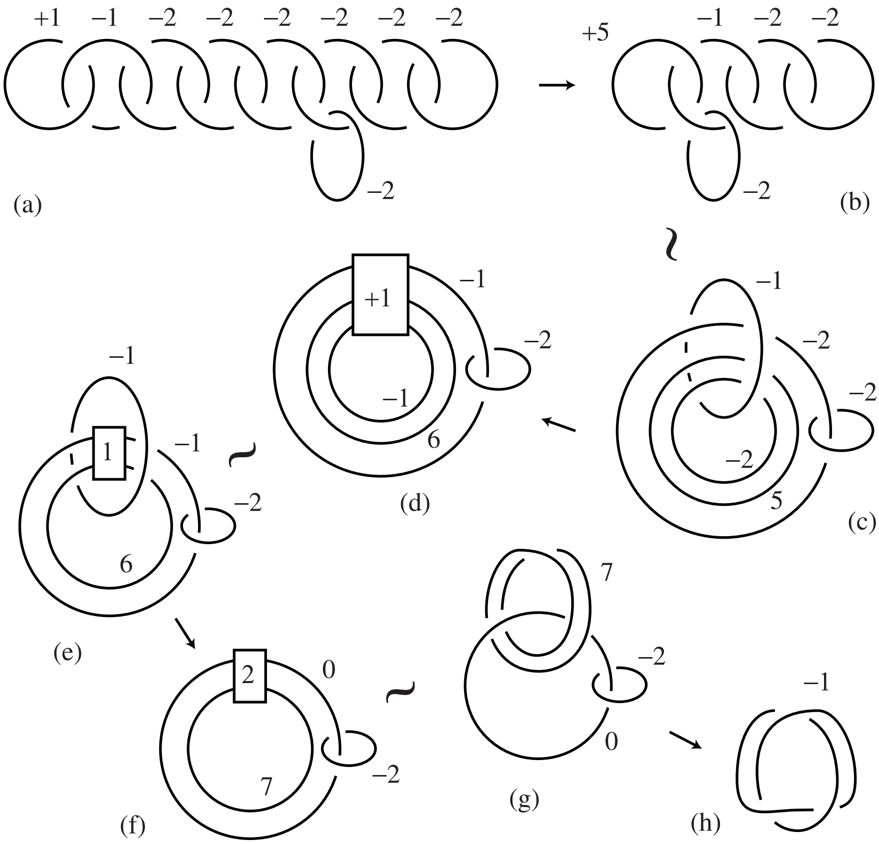


Figure 12.9. Blowing down $(-E_8\text{-plumbing}) \# \mathbb{CP}^2$.

Solution of Exercise 5.1.12(a): Figure 12.9 gives one possible answer. After blowing up $\mathbb{CP}^2$ (a), blow down 4 $\overline{\mathbb{CP}^2}$'s (b), (c), and then another (d). Redraw the twist so that one -1 does not run through it (e) and then blow down the -1 (f). Redraw the twist (g). The -2-framed curve has a 0-framed meridian; to split off the $S^2 \times S^2$ summand we first unlink the 7-framed circle from this meridian by sliding both strands over the -2-framed curve, obtaining a left trefoil (h). (For an alternate solution, separate out the other -1 in (d) and blow it down instead.) The same computation works when the long arm of the plumbing has $n + 4$ components with framing -2

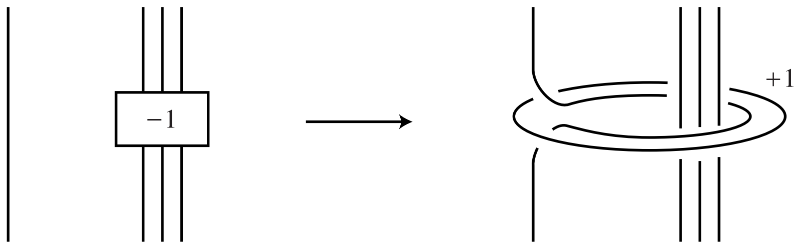


Figure 12.10

$(n \geq -4)$; we obtain an $(n-1)$-framed left trefoil and $n+7$ summands $\overline{\mathbb{CP}^2}$. □

Solution of Exercise 5.1.12(b): For the special case, blow down the 1-framed unknot in Figure 5.23, then blow it back up around all strands (Figure 12.10); this is equivalent to the handle slide. Note that the framing on the sliding curve gains a twist as required. Alternatively, the framing coefficient increases by $(k+1)^2 - k^2 = 2k+1$ as required, where k is the linking number of the two relevant handles in Figure 5.23 (oriented to correspond to handle addition). Now to slide over an arbitrary knot K, first unknot K by changing crossings as in Figure 5.19, then change its framing to 1 by blowing up meridians. Now perform the slide using the special case, then blow down the circles we just blew up. It is routine to verify that the result is equivalent to the required handle slide over K. □

Solution of Exercise 5.2.2(b): For surjectivity, note that any loop in M can be smoothed and then homotoped off of N by transversality. To understand $\ker i_*$, let γ be a loop in $M-N$ that is nullhomotopic in M. We may assume that γ and the nullhomotopy are smooth. By transversality, we may assume the nullhomotopy $F\colon I \times S^1 \to M$ has image in $M-N$ if $k \leq n-3$, proving i_* is injective. For $k = n-2$, we may assume that $\operatorname{Im} F$ intersects N in finitely many points, near which F appears as an embedding of normal disks to N. By precomposing F with a suitable homotopy of $\mathrm{id}_{1\times S^1}$ (as indicated by the downward arrows in Figure 12.11), we homotope γ to a product of meridians. □

Solution of Exercise 5.2.2(c): Given a presentation P of G with ℓ generators and m relators, perform ℓ surgeries on 0-spheres in S^n to obtain $\#\ell S^1 \times S^{n-1}$, whose fundamental group is free on a canonical set of ℓ generators (determined by the surgery). Now perform m surgeries on circles, one corresponding to each relator of P. By the previous exercise, removing the circles does not change π_1. When we glue in the m simply connected

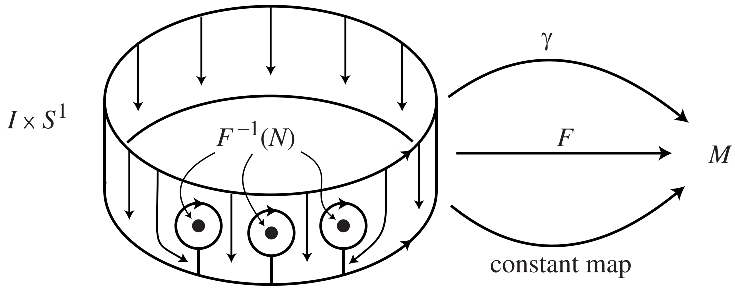


Figure 12.11

manifolds $D^2 \times S^{n-2}$, the resulting manifold M has $\pi_1(M) \cong G$ as required. Now let $X = D^n \cup$ 1-handles $\cup$ 2-handles be given by P, so that DX is the manifold with $\pi_1(DX) \cong G$ constructed in Exercise 4.6.4(b). Then $I \times X = D^{n+1} \cup$ 1-handles $\cup$ 2-handles is the corresponding handlebody with one higher dimension. By the definition of surgery, we have $M = \partial(I \times X) = DX$, so our new construction is equivalent to the old one. Our previous solutions to Exercises 5.1.10(b)–(d) become simpler (and generalize to higher dimensions) when we think of DX as the boundary of the handlebody $I \times X$. For (b), note that summing DX with $S^2 \times S^2$ is the same as adding a trivial 2-handle to $I \times X$, which immediately cancels a 1-handle (cf. Proposition 5.2.3). After we cancel the 1-handles, the remaining 2-handlebody immediately splits as required. The nontrivial moves in Exercise 5.1.10(d) correspond to sliding and cancelling handles of $I \times X$, and the cancellation is easy because the attaching circles cannot be knotted or linked in high dimensions. We conclude that under these hypotheses $I \times X^n \approx D^{n+1}$. □

Solution of Exercise 5.2.6(b): We prove (i) ⇒ (ii) ⇒ (iii) ⇒ (i). Let $\widetilde{M}$ denote the universal cover of M. Then the universal covers of the manifolds in (i) are obtained from $\widetilde{M}$ by summing with a (possibly infinite) collection of copies of $S^2 \times S^{n-2}$ and $S^2 \tilde{\times} S^{n-2}$, respectively. Condition (i) implies that these covers are diffeomorphic. Since $S^2 \times S^{n-2}$ is spin and $S^2 \tilde{\times} S^{n-2}$ is not, it follows that $\widetilde{M}$ cannot be spin, condition (ii). To see that (ii) implies (iii), assume that $\widetilde{M}$ is nonspin. Since $\widetilde{M}$ is simply connected, the proof of Proposition 5.2.4 shows that $w_2(\widetilde{M})$ is nonzero on some immersed 2-sphere in $\widetilde{M}$, or equivalently, this sphere has a twisted normal bundle (with odd Euler number if $n = 4$). Since the normal bundle is preserved when we push down into M, we conclude (iii) as required. The proof of Proposition 5.2.4 shows that (iii) implies (i). □

Solution of Exercise 5.2.7(b): Surger out S, then slide an arc of the resulting circle over a sphere with odd normal Euler number as in the proof of Proposition 5.2.4. □

Solution of Exercise 5.3.3(c): It is easy to compute that $H_1(L(p,q);\mathbb{Z}) \cong \mathbb{Z}_p$, generated by a meridian μ. To compute the linking form, note that the class $\alpha = p\mu - q\lambda$ is homologous to $p\mu$ in the unknot complement and bounds a disk in the complementary solid torus in $L(p,q)$. These two integer 2-chains fit together to give a chain c in $L(p,q)$ with $\partial(\frac{1}{p}c) = \mu$ over $\mathbb{Q}$. By counting intersections, we see that $\frac{1}{p}c \cdot \mu = \frac{q}{p}$ (cf. Exercise 5.3.13(g)). Thus, any class $n\mu \in H_1(L(p,q);\mathbb{Z})$ has square $\frac{n^2 q}{p} \in \mathbb{Q}/\mathbb{Z}$ under the linking form. If this square vanishes, then n^2 is divisible by p. In particular, n is a multiple of any preassigned prime factor p_0 of p. Since multiples of $p_0\mu$ do not generate $H_1(L(p,q);\mathbb{Z})$, there can be no splitting as required by Exercise 4.5.12(d) if $L(p,q)$ embeds in S^4. □

Solution of Exercise 5.3.3(d): The obvious sphere formed by one surgery is punctured twice during the remaining surgeries, but the punctures can be joined to form a torus, Figure 12.12(a). To see the S^1-family of tori, first imagine the S^1-family of spheres in 0-surgery on the unknot. (Push the disk out from the page, through ∞ and around the back to its original position.) Now consider the two additional surgeries and the tube. As the tube begins to thicken, we can think of it as a tube (b) around the other component, on the opposite side of the disk. (This reverses the roles of the meridian and longitude of the tube.) We push this tube across the surgery (c). The disk passes through ∞ (d), and then we again think of the tube as surrounding the opposite link component (e) as we did in passing from (a) to (b). As the disk approaches its original position, we push the tube across the other surgery to return to (a).

The 3-fold symmetry of the Borromean rings (120° rotation) corresponds to cyclic permutation of the coordinates of $T^3 = S^1 \times S^1 \times S^1$. Thus, the additional tori asked for in the exercise are the images of Figure 12.12(a) under the symmetry. Visualize the resulting intersections yourself and verify that they are as required. □

Solution of Exercise 5.3.7(a): For $k = 0$, the two required knots can be obtained by blowing down one or the other of the unknots in Figure 5.26. (Of course, ± 1-surgery on the unknot also works.) For general k, replace Figure 5.26 by any link of two unknots with linking number k. □

Solution of Exercise 5.3.7(b): First, we eliminate the 1- and 3-handles of X by surgery as in the proof of Corollary 5.3.5. By Propositions 5.2.3 and 5.2.4,

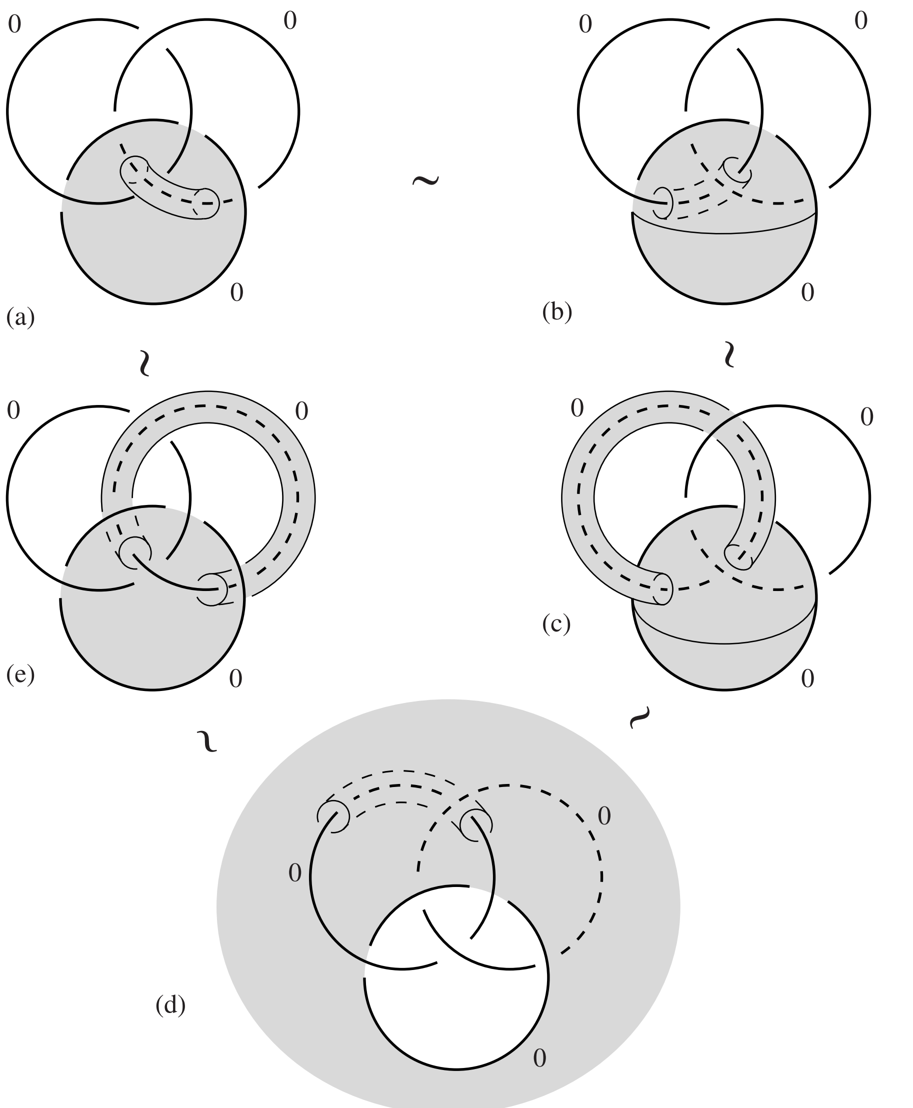


Figure 12.12. Fibering the 3-torus by 2-tori.

this changes X by connected summing with copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ (after we sum X with $\mathbb{CP}^2$ if necessary to make the intersection form odd). Now we have a 4-manifold obtained by adding a 4-handle to a 2-handlebody Y, so Y is given by a framed link L in S^3 and $\partial Y \approx S^3$. By Theorem 5.3.6, we can transform L into the empty link by blowing up and down. (Alternatively, we can surger just the 1-handles, apply the theorem to $\partial Y_2 \approx \# k S^1 \times S^2$, then cancel the 3-handles with the resulting 0-framed unlink.) $\square$

Solution of Exercise 5.3.8(a): To blow down a ± 1-framed unknot, perform a Rolfsen twist on it with $n = \mp 1$, so that its coefficient becomes ∞ and we can erase it. Blowing up is the reverse procedure. Figure 5.29 gives special cases of this. □

Solution of Exercise 5.3.8(b): Perform a $-n$-fold Rolfsen twist on the unknot in Figure 5.24 to obtain $L(p, q + np)$. □

Solution of Exercise 5.3.9(a): Slam-dunking a 0-framed meridian of an integer-framed knot changes the knot's surgery coefficient to ∞ so that we may erase it. Applying this procedure to the diagram of DX, we see that the required 3-manifold is diffeomorphic to the boundary of X_1. For the given Hopf link, if we slam-dunk the n-framed component, we obtain $-\frac{1}{n}$-surgery on the unknot, which can be erased after an n-fold Rolfsen twist. □

Solution of Exercise 5.3.9(b): Slam-dunking from left to right gives an unknot with coefficient

$$r = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{a_{n-2} \cdots - \frac{1}{a_1}}};$$

slam-dunking from the right reverses the order of the subscripts. Any rational number r has such continued fraction expansions. These can be constructed by repeatedly rounding (up or down) to an integer and taking the negative reciprocal of the remainder. The procedure must terminate (when the remainder is 0) since the numerator and denominator of the fraction decrease in absolute value at each step. It follows that any lens space can be expressed as the boundary of a plumbing on a linear graph. More generally, we can replace any rational surgery by integral surgeries as in Figure 12.13. □

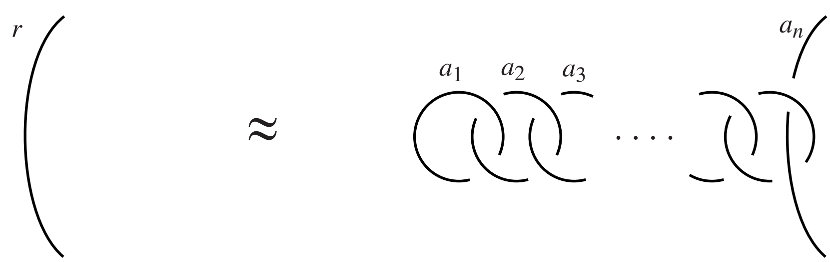


Figure 12.13. Changing rational surgery to integral surgery.

Solution of Exercise 5.3.9(c): In Figure 5.32, a $(1 - n)$-fold Rolfsen twist about K_1 changes the coefficient on K_2 from n to 1. After we blow down

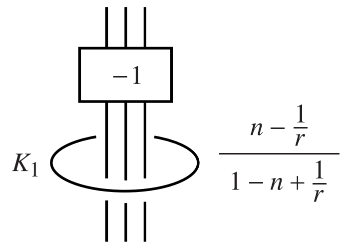


Figure 12.14

as in Exercise 5.3.8(a), K_1 encircles the twist as in Figure 12.14, and a +1-Rolfsen twist on it gives the same result as a slam-dunk in the original picture. Check that the coefficient on K_1 changes from r to $n-\frac{1}{r}$ as required. For the general case, we unknot K_2 by blowing up as in Figure 5.19, apply the previous construction and blow the extra unknots back down to recover K_2. □

Solution of Exercise 5.3.13(e): Let A be a matrix representing Q. It is easy to construct a framed link with linking matrix A; the corresponding 4-manifold X has intersection form Q (Proposition 4.5.11). By Corollary 5.3.12, ∂X is a homology sphere, so Freedman's result gives a contractible topological manifold Δ with $\partial\Delta = \overline{\partial X}$, and $X \cup_{\partial X} \Delta$ is the required closed 4-manifold. If ∂X bounds a *smooth*, contractible manifold, the same argument realizes Q by a smooth, closed, simply connected 4-manifold. Thus, we can guarantee that ∂X bounds no smooth, contractible manifold by choosing Q to be unrealizable by a smooth, closed 4-manifold (Theorems 1.2.29-1.2.31). The simplest example is the Poincaré homology sphere, Exercise 5.1.12(a). □

Solution of Exercise 5.3.13(f): $H_2(X,M;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$ has the same torsion as $H_1(X;\mathbb{Z})$ (by the Universal Coefficient Theorem); this vanishes by hypothesis. Similarly, $H_2(X;\mathbb{Z}) \cong H^2(X,M;\mathbb{Z})$ has the same (vanishing) torsion as $H_1(X,M;\mathbb{Z})$. Now the long exact homology sequence of (X,M) gives

$$H_2(X;\mathbb{Z}) \xrightarrow{\varphi} H_2(X,M;\mathbb{Z}) \xrightarrow{\partial_*} H_1(M;\mathbb{Z}) \longrightarrow 0,$$

where the first two terms are torsion-free, so φ presents $H_1(M,\mathbb{Z})$. As in the solution of Exercise 1.2.10, let $\{\alpha_1,\dots,\alpha_m\}$ be any basis for $H_2(X;\mathbb{Z})$ and let $\{\beta_1,\dots,\beta_m\}$ in $H_2(X,M;\mathbb{Z})$ be the Poincaré dual of the corresponding dual basis, so $\alpha_i\cdot\beta_j = \delta_{ij}$. Let $A=[a_{ij}]$ be the presentation matrix given by φ in these bases, so $\varphi(\alpha_i) = \sum a_{ij}\beta_j$. Then $\alpha_i\cdot\alpha_k = \varphi(\alpha_i)\cdot\alpha_k = \sum a_{ij}\beta_j\cdot\alpha_k = a_{ik}$, so A is also the intersection matrix of X with respect to $\{\alpha_1,\dots,\alpha_m\}$. When X is a 2-handlebody and $\{\alpha_i\}$ is the obvious basis, then $\{\beta_i\}$ is represented by the cocores of the 2-handles, so the corresponding generators of $H_1(M;\mathbb{Z})$ are the meridians $\mu_i = \partial_*\beta_i$. The relators $\varphi(\alpha_i)$ will correspond precisely to those we constructed when proving Proposition 5.3.11. (See this geometrically by isotoping the canonical surface representing α_i into M except on some copies of cocore disks.) Thus, we recover our previous proof in the case of integral surgery. □

Solution of Exercise 5.3.13(g): For the given (X, M) with $H_1(X;\mathbb{Z}) = 0$, use the notation of the previous solution and let $\{F_1, \ldots, F_m\}$ be disjoint oriented surfaces (with boundary) with F_i representing $\beta_i \in H_2(X, M;\mathbb{Z})$. (Disjointness is easily obtained by pushing intersections into $\partial X = M$.) Let F_i' be parallel to F_i. Now the rational matrix $B = [b_{ij}]$ given by $b_{ij} = \ell k_{\mathbb{Q}}(\partial F_i, \partial F_j')$ reduces mod 1 to a matrix for the linking form of M with respect to the spanning set $\{\partial_*\beta_1, \ldots, \partial_*\beta_m\}$ of $H_1(M;\mathbb{Z})$. By the exact homology sequence of (X, M), each basis element $\alpha_i \in H_2(X;\mathbb{Z})$ is represented by a cycle of the form $\sum a_{ij}F_j + c_i$, with a_{ij} as in the previous solution and c_i an integral chain in M with $\partial c_i = -\sum a_{ij}\partial F_j$. Now $\delta_{ik} = \alpha_i \cdot \beta_k = c_i \cdot F_k'$ (since $F_j \cap F_k' = \emptyset$), and the latter equals $c_i \cdot \partial F_k'$ in M. Thus $\delta_{ik} = \ell k_{\mathbb{Q}}(\partial c_i, \partial F_k') = -\sum a_{ij}\ell k_{\mathbb{Q}}(\partial F_j, \partial F_k') = -\sum a_{ij}b_{jk}$, and so $B = -A^{-1}$ as required. When X is a 2-handlebody, we can take the surfaces F_i, F_i' to be core disks, so the generating circles ∂F_i are meridians μ_i, and the pushoffs $\mu_i' = \partial F_i'$ are given by the 0-framing on μ_i. The chains c_i correspond to the relators of Proposition 5.3.11. (See the previous solution.) In this form, the proof generalizes to Dehn surgeries: The chain c_i is obtained by puncturing a Seifert surface (with oriented boundary K_i) and gluing it to $\frac{1}{q_i}D$ in M, where D is the disk $\{\text{pt.}\} \times D^2 \subset S^1 \times D^2$ of the solid torus attached to $\partial\nu K_i$. Thus, $c_i \cdot \mu_k' = \delta_{ik}$ by construction, and $\partial c_i = -\frac{p_i}{q_i}\mu_i - \sum_{j\neq i}\ell k(K_i, K_j)\mu_j = -\sum a_{ij}\mu_j$. (Check the signs!) The previous final computation now applies without further change. □

Solution of Exercise 5.4.1: In the first two pictures, we have an S^1-family of disks fibering the unknot complement. (The one through ∞ appears as a plane minus a disk.) These disks connect with the S^1-family of disks fibering the solid torus attached during the surgery, to form an S^1-family of spheres. In the remaining picture, the complement of the balls is $I \times S^2$, fibered by an I-family of spheres including the two boundary spheres and one through ∞ (appearing as a plane separating the balls). We get the S^1-family of spheres by identifying the boundary components of $I \times S^2$. The families of spheres in the three pictures all look the same outside of a ball containing the circle or attaching balls. To understand the correspondence inside the ball, see the next solution. □

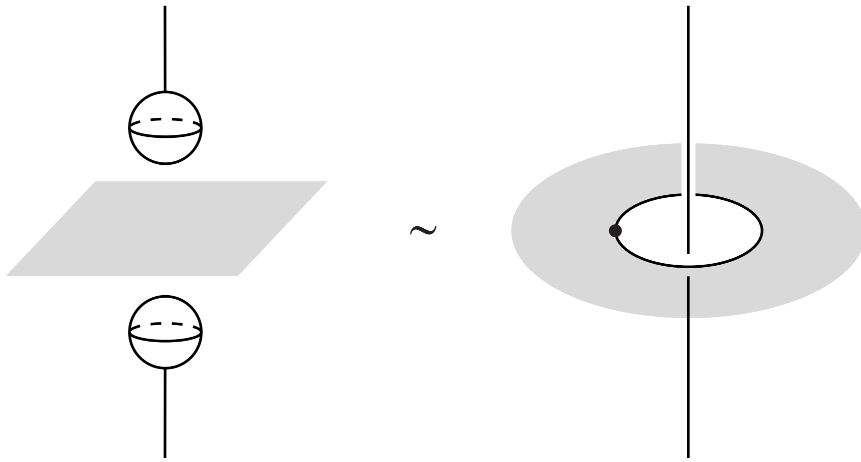


Figure 12.15

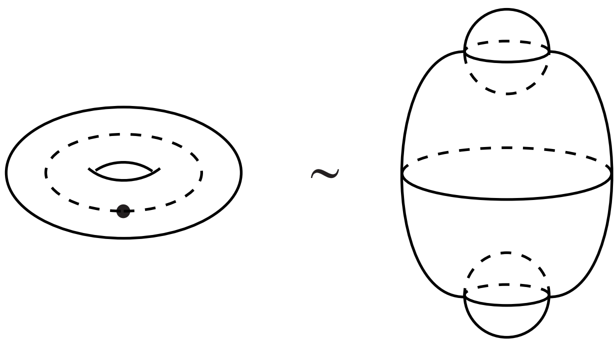


Figure 12.16. Heegaard torus in $S^1 \times S^2$.

Solution of Exercise 5.4.2(a): The curve and surface transform as in Figure 12.15. This figure should clarify the correspondence between families of spheres in the previous exercise. The spheres visible in ball notation appear as disks in dotted circle notation; their remaining hemispheres have disappeared under the 1-handle into the surgery solid torus. □

Solution of Exercise 5.4.2(b): See Figure 12.16. The Heegaard splitting in the left picture is the same as the decomposition into solid tori induced by the 0-surgery, or equivalently, by doubling the unknot complement. In the right picture, the outer region becomes a solid torus when we glue the two spheres together, and this is identified in the obvious way with the unknot complement. The inner region also becomes a solid torus, and this disappears into D^4 in the other picture, becoming the complementary solid torus produced by the surgery. □

Solution of Exercise 5.4.3(b): If we draw the slide as in Figure 5.36, then it preserves the blackboard framing. The corresponding framing coefficient is given by $w(K)$, which changes by $2\ell k(K, K_0)$ as required, since each crossing in the figure changes sign and the signed number of such crossings is $\ell k(K, K_0)$. The coefficient of any framing must change by the same number as that of the blackboard framing. □

Solution of Exercise 5.4.3(d): See Figure 12.17. The Gluck twist is performed by first surgering S, then changing the framing on the resulting framed circle and surgering it. The first step is realized by changing the 0-framed unknot to a dotted circle. The framing change corresponds to adding a ± 1-twist (as if we had blown down a ∓ 1-framed unknot parallel to the dotted circle). The final surgery changes the dotted circle back to a 0-framed unknot. Note that we can actually use any odd number of twists here, since we can change the diagram by any even number of twists by sliding all strands over the 0-framed unknot (cf. Example 5.1.3 and Exercise 4.4.4). We can also divide the twists as in Figure 5.42. □

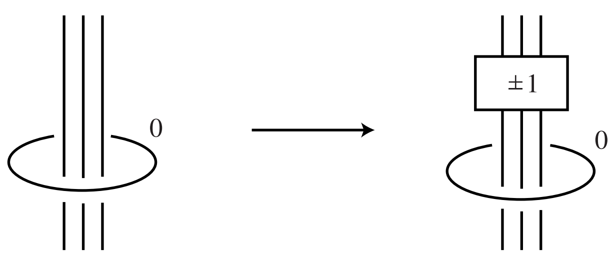


Figure 12.17. Gluck twist.

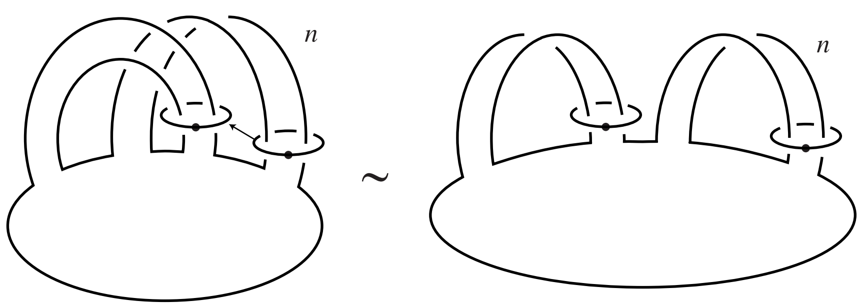


Figure 12.18

Solution of Exercise 5.4.4: For the bundle with Euler number e over F, $H_1(M) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_e$ for $F = \#gT^2$, and for $F = \#g\mathbb{RP}^2$ $(g > 0)$ we have $H_1(M) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (e even) or $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_4$ (e odd). □

Solution of Exercise 5.4.5(a): The manifold is S^4. (Cancel the 1-handle with the 2-framed circle by first sliding the other pair of strands off the 1-handle. The remaining 0-framed unknot cancels the 3-handle.) □

Solution of Exercise 5.4.5(b): One solution is given by Figure 12.18. The slide of dotted circles corresponds to the 1-handle slide in the 2-dimensional case, Figure 5.1. □

Solution of Exercise 5.5.2: See Figure 12.19, and recall that there is a unique way to glue in D^4 along an S^3 boundary component. □

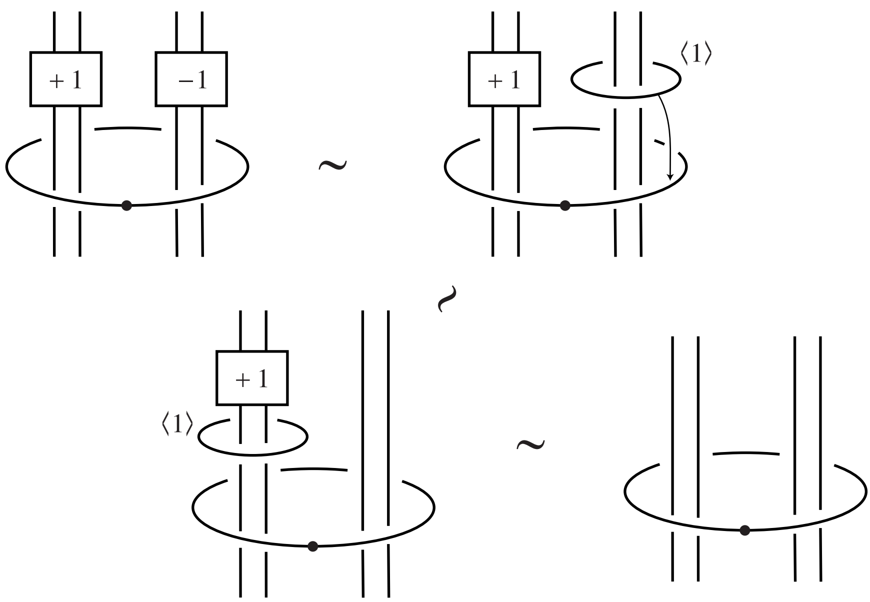


Figure 12.19

Solution of Exercise 5.5.6: Figure 12.20 shows $(X, \overline{\partial_+ X})$. □

Solution of Exercise 5.5.7(a): See Figure 12.21. For $\mathbb{RP}^2$, there are two solutions, differing by a reflection, corresponding to the cases $e(X) = \pm 2$. The case $e(X) = -2$ is drawn. Note that the complement of the $e(X) = \pm 2$ bundle is the $e(X) = \mp 2$ bundle; cf. Exercise 6.2.4(c). (These are the only possible Euler numbers of embeddings $\mathbb{RP}^2 \hookrightarrow S^4$.) □

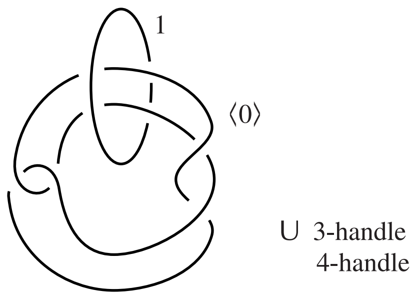


Figure 12.20

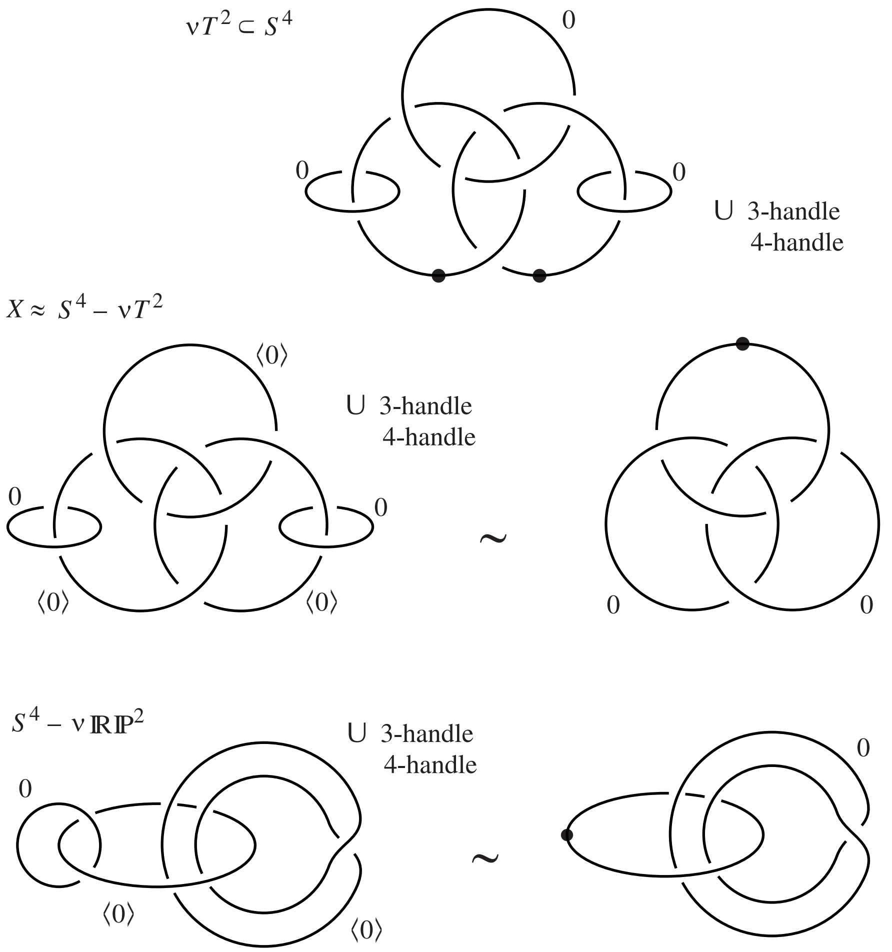


Figure 12.21

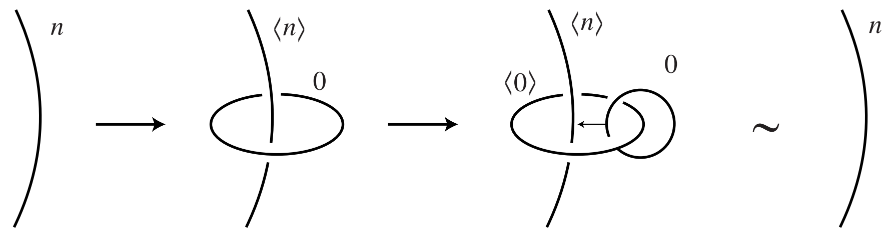


Figure 12.22

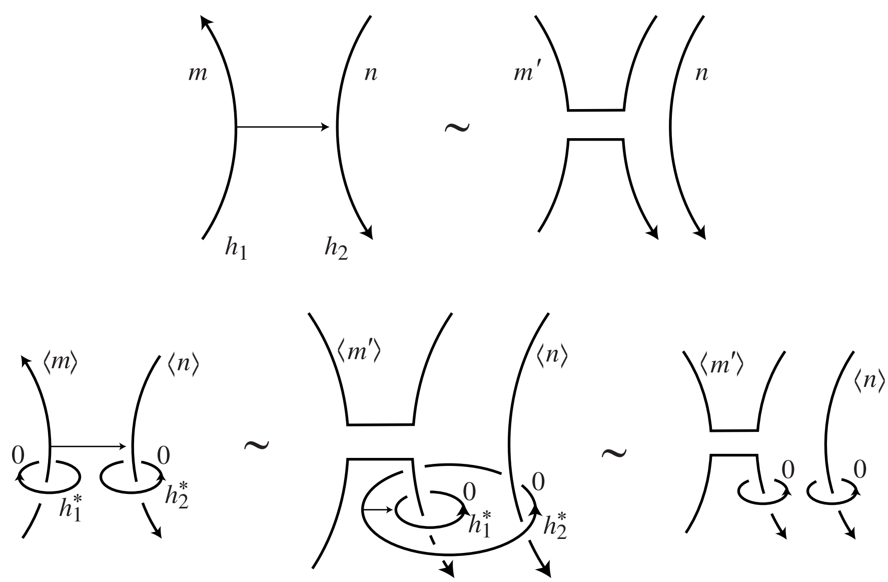


Figure 12.23. Dual handle slide.

Solution of Exercise 5.5.7(b): The main point is that a pair of consecutive applications of the dualization algorithm transforms each 2-handle as in Figure 12.22. After a handle slide and a slam-dunk, we recover the original picture. □

Solution of Exercise 5.5.7(c): The dual slide restores the dual 2-handles to their original form as 0-framed meridians. Figure 12.23 shows that h_2^* slides over h_1^* and that addition corresponds to subtraction. One can also see this algebraically. If (for example) X is closed and without 1- or 3-handles, then $\{h_i\}$ and $\{h_i^*\}$ determine bases for $H_2(X;\mathbb{Z})$ that are dual with respect to the intersection form, $h_i \cdot h_j^* = \delta_{ij}$. A simple computation now shows that changing basis by adding h_2 to h_1 corresponds to subtracting h_1^* from h_2^*. (The choice of signs of the meridians in the figure does not matter, as long as we are consistent. Here we have fixed the signs in $\partial_+ X$ before dualizing, recalling that the orientation on an attaching circle is determined by a Seifert surface, or *minus* the core disk.) □

Solution of Exercise 5.5.9(a): The manifolds are $S^1 \times S^3$, $S^1 \times S^3 \# S^2 \times S^2$ and $S^2 \times S^2$, respectively; see Figure 12.24. □

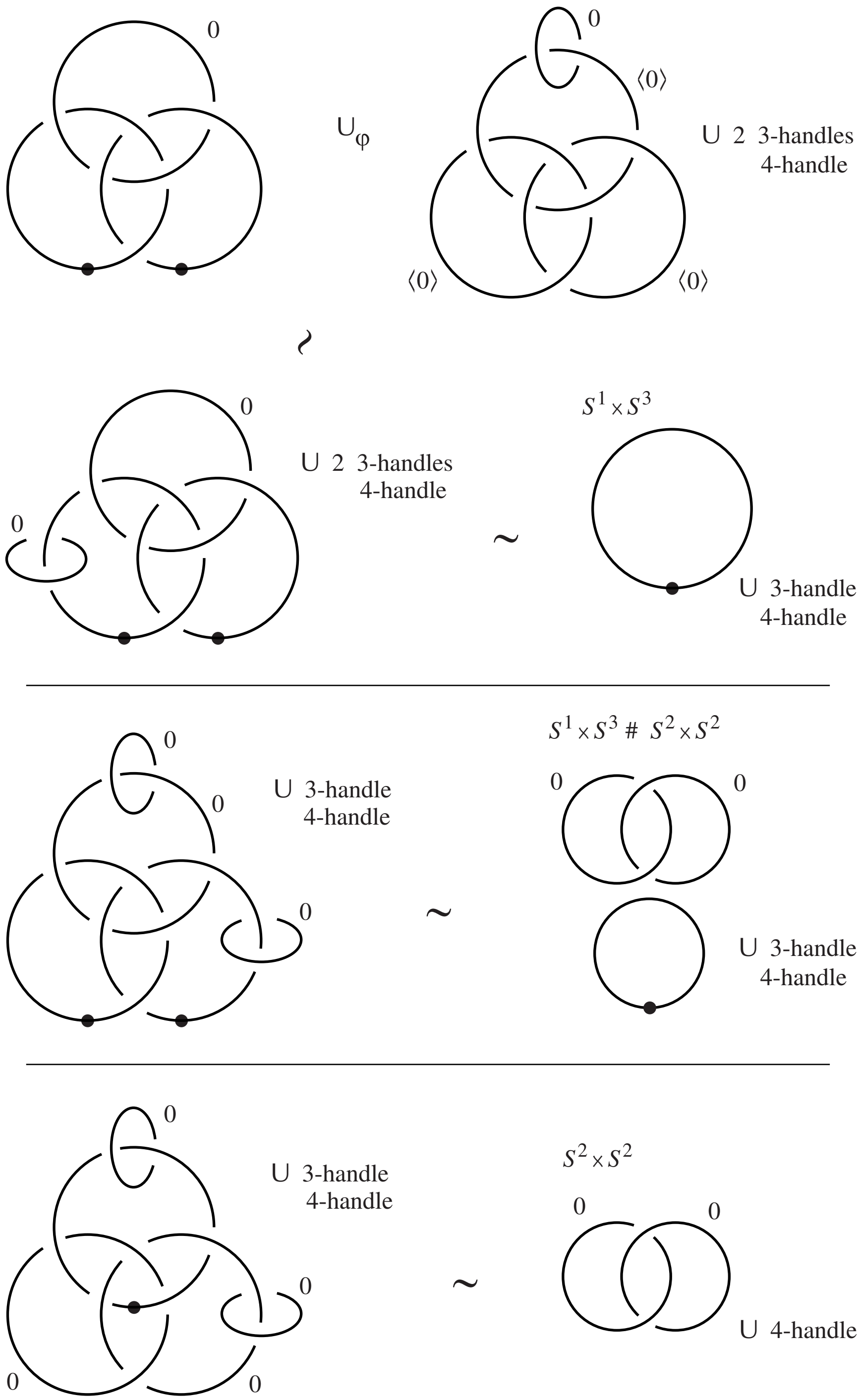
Solution of Exercise 5.5.9(b): Reversing the computation in Figure 12.9, we see that a 0-framed meridian of the knot in (h) pulls back to a 0-framed meridian of the 7-framed curve in (g), the 5-framed curve (b) and the +1-framed curve (a). Blowing down the +1, we recover the E_8-plumbing with an extra -1-framed meridian, Figure 12.25, which realizes $P \cup_\partial \overline{Q}$. Blowing down reduces this to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. □

Solution of Exercise 5.6.2(a): Repeat the previous construction using trivializations τ of $E|X_1$ whose restrictions to Y_1 determine the given spin structure s on $E|Y$. Then each cochain $c(\tau)$ vanishes on Y, so it can be interpreted as a relative cochain in $C^2(X, Y; \mathbb{Z}_2)$. Given two such trivializations τ and τ', we can assume they are equal over $X_0 \cup Y$, and obtain a difference cochain $d(\tau, \tau') \in C^1(X, Y; \mathbb{Z}_2)$. As before, we obtain a well-defined class $w_2(E, s) = [c(\tau)] \in H^2(X, Y; \mathbb{Z}_2)$. □

Solution of Exercise 5.6.4(a): $H^1(X, Y; \mathbb{Z}_2)$ acts freely and transitively on the set of spin structures on E extending s on $E|Y$, provided that the set is nonempty ($w_2(E, s) = 0$). This follows as in the absolute case $Y = \emptyset$, using the relative difference cocycles $d(\tau, \tau') \in C^1(X, Y; \mathbb{Z}_2)$ defined in the previous solution. These are defined up to arbitrary relative coboundaries, since τ and τ' determine the same spin structure on $E|Y$. □

Solution of Exercise 5.6.4(b): Choose a handle decomposition of $I \times X$ determined by some decomposition of X and the decomposition of I as two 0-handles $\cup$ 1-handle. Let τ, τ' be trivializations of $E|X_1$ determining s, s', respectively. Assume τ and τ' are equal over X_0. Then there is an induced trivialization τ^* of $E^*|(I \times X)_1$ that is constant on each 1-handle coming from a 0-handle of X and agrees with τ and τ' on $\{0\} \times X$ and $\{1\} \times X$, respectively. The cocycle $c(\tau^*) \in C^2(I \times X, \partial I \times X; \mathbb{Z}_2)$ represents $w_2(E^*, s^*)$, and it is nonzero precisely on those 2-handles that come from 1-handles of X on which τ and τ' disagree, i.e., on which $d(\tau, \tau') \neq 0$. That is, $w_2(E^*, s^*) = [c(\tau^*)] = [I \times d(\tau, \tau')] = [I] \times \Delta(s, s')$. Intuitively, s and s' differ by a twist along the Poincaré dual of $\Delta(s, s')$, and $w_2(E^*, s^*)$ measures that same twist. □

Solution of Exercise 5.6.4(d): $H^1(X; \mathbb{Z})$ acts freely and transitively on the set $\mathcal{C}(E)$ of complex trivializations of $E|X_2$, provided that it is nonempty


0
∪φ
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∪ 2 3-handles
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$S^1 \times S^3$
∪ 3-handle
4-handle
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$S^1 \times S^3 \# S^2 \times S^2$
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$S^2 \times S^2$
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Figure 12.24

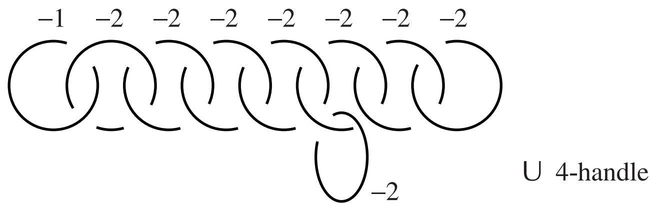

Figure 12.25

($c_1(E) = 0$). Such a trivialization clearly determines a spin structure, and the group actions on $\mathcal{C}(E)$ and $\mathcal{S}(E)$ correspond under the coefficient homomorphism $H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z}_2)$ (since $\pi_1(U(m)) \cong \mathbb{Z}$ maps onto $\pi_1(O(2m)) \cong \mathbb{Z}_2$ ($m \geq 2$) under the map induced by the inclusion $U(m) \to O(2m)$). Equivalently, choosing a base point $\tau \in \mathcal{C}(E)$ identifies it with $H^1(X;\mathbb{Z})$, so that the map $\mathcal{C}(E) \to \mathcal{S}(E)$ corresponds to the coefficient homomorphism (where we identify $\mathcal{S}(E)$ with $H^1(X;\mathbb{Z}_2)$ using the spin structure induced by τ). Analogous statements hold in the relative case, using $H^1(X,Y;\mathbb{Z})$ and $H^1(X,Y;\mathbb{Z}_2)$. □

Solution of Exercise 5.6.8(a): Map $S^1 \times D^n$ to itself by $(\theta, x) \mapsto (\theta, r_\theta(x))$, where r_θ is rotation through the angle θ in the first two coordinates of D^n. It is easy to see that this diffeomorphism is trivial in homology (both absolute and rel ∂), but it cannot be isotopic to the identity since it interchanges the two spin structures of $S^1 \times D^n$. □

Solution of Exercise 5.6.8(b): The manifold $S^2 \times S^n$ admits a spin structure since each factor does. (For example, $S^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ has a trivial tangent bundle.) However, the twisted bundle over S^2 contains the twisted disk bundle $S^2 \tilde{\times} D^n$ made from the gluing map of the previous solution. By definition, $w_2(S^2 \tilde{\times} D^n) \neq 0$, so $S^2 \tilde{\times} D^n$ and $S^2 \tilde{\times} S^n$ do not admit spin structures (and $w_2(S^2 \tilde{\times} S^n)$ has nonzero value on sections $S^2 \times \{p\}$, p a fixed point of r_θ). □

Solution of Exercise 5.7.3: Represent x by a closed surface with tubular neighborhood $N \subset X$. Then $\langle w_2(X), x\rangle$ and x^2 equal the corresponding quantities in N. Describe N by a handlebody as in Example 4.6.5. Add 2-handles to cancel the 1-handles; the above quantities are still preserved. We are left with $D^4 \cup$ 2-handle, and Corollary 5.7.2 shows that both quantities equal the framing coefficient mod 2. □

Solution of Exercise 5.7.7(a): $H_2(X;\mathbb{Z}) \cong \mathbb{Z}_2$, so Q_X is trivial. (For example, $H_1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and in the notation of the proof of Proposition 5.7.4 we have $T_2 \cong T^3 \cong T_1 \cong \mathbb{Z}_2$. However, the equations $\chi(X) = 2$ and $b_1 = b_3 = 0$

imply $b_2 = 0$.) The nonzero element $\alpha \in H_2(X;\mathbb{Z})$ is represented by the 0-framed 2-handle (which generates $\ker \partial_*$), which shows directly that $\alpha^2 = 0$. Over $\mathbb{Z}_2$, the boundary operators vanish and the intersection form is given by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & n \end{smallmatrix}\right]$. By Corollary 5.7.2 or the Wu formula, $w_2(X)$ vanishes on α and has value $n \pmod 2$ on the $\mathbb{Z}_2$-homology class β of the n-framed 2-handle. Thus, for $n = 0$ we have $w_2(X) = 0$, and for $n = 1$, $w_2(X)$ is Poincaré dual to α (reduced mod 2). In the latter case, the obvious sphere representing α is the required closed surface. (Also note that this is the only nontrivial $\mathbb{Z}_2$-class with an integer lift — in particular, β is not represented by an orientable surface.) Although w_2 lifts to the class $\alpha \in H_2(X;\mathbb{Z})$ that is not divisible by 2, this does not contradict the fact that Q_X is even, since α is a torsion class and represents $0 \in H_2(X;\mathbb{Z})/T_2$. The attaching sphere of the 3-handle is obtained from the disk spanning the dotted circle by repairing the two punctures using core disks of the 0-framed 2-handle as in Figure 12.26. This shows explicitly that $2\alpha = 0$. □

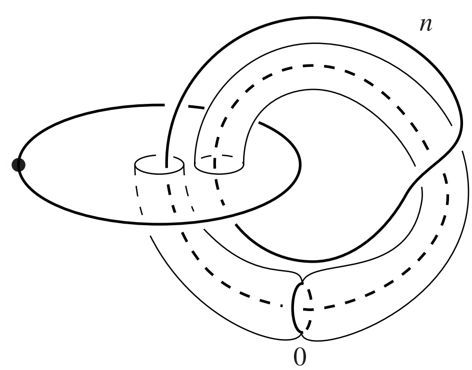

Figure 12.26. 3-handle attaching sphere in an S^2-bundle over $\mathbb{RP}^2$.

Solution of Exercise 5.7.7(b): Blowing up a point on S (with either orientation) preserves $\sigma(X) - [S]^2$, and the Wu formula (or Corollary 5.7.2) shows that S remains dual to $w_2(X)$. Thus, we can assume $[S]^2 = 1$. Blowing down S gives a manifold Y with signature $\sigma(Y) = \sigma(X) - 1 = \sigma(X) - [S]^2$. Since S is dual to $w_2(X)$, its complement admits a spin structure, and so must Y. Thus, Rohlin's Theorem implies that $\sigma(Y)$ is divisible by 16. □

Solution of Exercise 5.7.9: By Exercise 5.7.7(a), only one of the two S^2-bundles X over $\mathbb{RP}^2$ admits spin structures, namely Figure 5.46 with $n = 0$. Since $H^1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$, there are two spin structures. The cocycle c is identically 0, and both cochains Δ in $C^1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ satisfy $\delta\Delta = 0$. Thus, one spin structure is given by Figure 5.46 with the canonical framing on X_1, and the other is obtained by twisting the 1-handle. If we twist as in

Figure 5.42 with a +1-twist on one strand and a −1-twist on the other, the diagram will be unchanged, showing that there is a self-diffeomorphism of X interchanging the spin structures. (Note that this diffeomorphism acts trivially on homology.) Alternatively, we can exhibit the twist as in Figure 4.38 (which becomes Figure 6.2 in dotted circle notation, displaying the two spin structures via the canonical framing on X_1) and realize the diffeomorphism between these by sliding the 2-handle under the 1-handle as in Section 4.6. □

Solution of Exercise 5.7.12(a): The previous proof for 2-handlebodies still shows that $\mathcal{S}(\partial X)$ maps surjectively to the preimage of $w_2(X)$ in $H_2(X;\mathbb{Z}_2)$ for arbitrary compact, oriented X, but the map need not be injective in general. In fact, s and s' have the same image in $H_2(X;\mathbb{Z}_2)$ if and only if the dual of $\Delta(s,s')$ lies in $\ker(i_*\colon H_2(\partial X;\mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2))$. For disk bundles over $\mathbb{RP}^2$, $H_2(\partial X;\mathbb{Z}_2)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $e(X)$ even and $\mathbb{Z}_2$ for $e(X)$ odd. In either case, $\ker i_* \cong \mathbb{Z}_2$, so the map $s \mapsto w_2(X,s)$ is $2:1$. The nonzero element of $\ker i_*$ is represented by a Klein bottle K in ∂X (Figure 12.27), and $w_2(X,s) = w_2(X,s')$ if and only if s and s' agree on each circle intersecting K an even number of times, or equivalently, if they agree up to a twist on the 1-handle. For example, for $e(X)$ even, X admits two spin structures, whose restrictions to ∂X are distinct but both have $w_2(X,s) = 0$. If we surger out the 1-handle to obtain a 0-framed 2-handle, the above condition guaranteeing $w_2(X,s) = w_2(X,s')$ is that s and s' should have characteristic sublinks that differ only at the new 2-handle. □

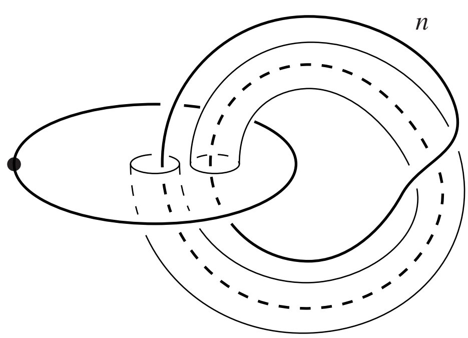

Figure 12.27

Solution of Exercise 5.7.12(b): There are two spin structures, with characteristic sublinks $\{K_1, K_2, K_3\}$ and $\{K_1, K_4\}$, respectively. (Note, for example, that linking with K_1 shows that K_2 and K_3 are either both in or both not

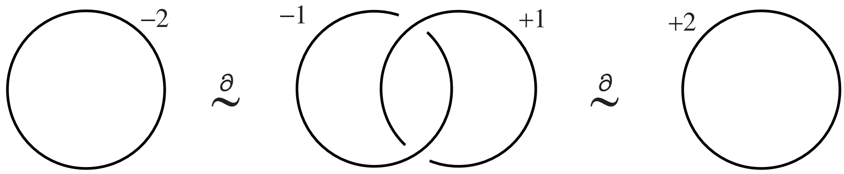

Figure 12.28. $\mathbb{RP}^3 \approx \overline{\mathbb{RP}^3}$.

in the characteristic sublink.) The spin structures extend over 2-handles attached to odd-framed meridians of K_1, K_2, K_3 and of K_1, K_4, respectively, and to even-framed meridians of the remaining components. □

Solution of Exercise 5.7.13: To get the diffeomorphism, blow up and down as in Figure 12.28. The empty characteristic sublink on one side corresponds to the nonempty one on the other — that is, the spin structures extending over the two disk bundles do not correspond. Alternatively, a meridian on one side goes to a meridian on the other, but its framing changes by a twist. □

Solution of Exercise 5.7.15(a): Represent $L(p,1)$ as $-p$-surgery on an unknot K. For p even, this already realizes one spin structure. For any p, it remains to deal with the spin structure for which K is characteristic. Following Kaplan's algorithm, we blow up $p-1$ meridians to raise the framing on K to -1, then blow down K. If we choose each blow-up to unlink the previous meridian from K, the final result is a plumbing on a linear graph of $p-1$ vertices with all framings 2 (cf. the solution of Exercise 5.1.12(a)).

Zero-surgery on the trefoil is given to us as a spin boundary. To realize the other spin structure, recall that Exercise 5.1.12(a) realized 0-surgery on the left trefoil as the boundary of the plumbing (sometimes called E_9 or $\widetilde{E}_8$) obtained from the (negative) E_8-plumbing by adding a -2 vertex to the long arm of the graph. We can see this by Kaplan's algorithm as in Figure 12.29. First, unknot the trefoil by blowing up (a–c). One blow-up as in Figure 5.21 unlinks the characteristic sublink (d–e). Blowing up meridians as before and blowing down the characteristic sublink (whose two components have framings 2 and 8 in (e)), we obtain diagram (f). The indicated slide exhibits this 4-manifold as the required plumbing (g). □

Solution of Exercise 5.7.15(b): Double the manifold X of Theorem 5.7.14 and apply Corollary 5.1.6. □

Solution of Exercise 5.7.17(a): Exercise 5.7.15(a) shows that for $p > 0$, $\mu(L(p,1), s) = p-1$ when s is given by the nonempty characteristic sublink

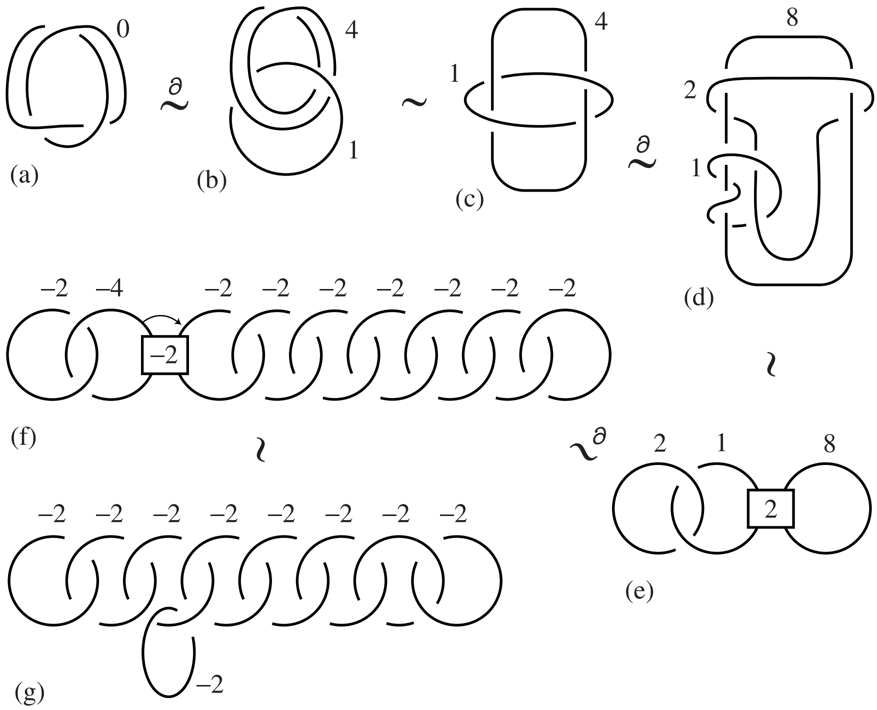

Figure 12.29

of $-p$-surgery on the unknot. (Check that the given intersection matrix is positive definite by diagonalizing over $\mathbb{Q}$.) For the other spin structure when p is even, $\mu(L(p,1),s) = -1$. For negative p reverse the signs (to get $1 - |p|$ and 1), and for $L(0,1) = S^1 \times S^2$ note that both spin structures spin bound $S^1 \times D^3$, so $\mu = 0$. If an orientation-preserving diffeomorphism of $L(p,1)$ interchanges the two spin structures (p even), then their Rohlin invariants must be equal, so $p \equiv 0 \pmod{16}$. For $|p| > 2$, the lens space $L(p,1)$ has no orientation-reversing self-diffeomorphisms. We can prove this when $|p| \not\equiv 1, 2, 9 \pmod{16}$ by observing that such a diffeomorphism would send a spin structure s to one with opposite Rohlin invariant, implying $|p| - 1 \equiv 1 - |p|$ or $1 \pmod{16}$. For $|p| \equiv 2 \pmod{16}$, we conclude that any such diffeomorphism must interchange the two spin structures. When $p = 2$, such a diffeomorphism is given by Exercise 5.7.13. □

Solution of Exercise 5.7.17(b): Since Σ bounds the E_8-plumbing, we have $\mu(\Sigma) = 8$. If Σ bounded an acyclic manifold X, then X would be spin ($H^2(X;\mathbb{Z}_2) = 0$) and we would have the contradiction $\mu(\Sigma) = 0$. If Σ embedded in $\mathbb{R}^4$, then the closure of one component of its complement would be a compact, acyclic 4-manifold bounded by Σ (Mayer-Vietoris). If Δ denotes

Freedman's contractible topological 4-manifold with $\partial\Delta = \Sigma$, then $\Delta \cup_\Sigma \overline{\Delta}$ is a closed, simply connected topological 4-manifold with the homology of S^4, so Freedman's Classification Theorem 1.2.27 implies that it is homeomorphic to S^4. Removing a point gives $\mathbb{R}^4$ with a topologically embedded copy of $I \times \Sigma$. □

Solution of Exercise 5.7.17(c): Represent T^3 as 0-surgery on the Borromean rings. Then every sublink is characteristic, and we obtain the 8 spin structures on T^3. The 7 spin structures corresponding to proper sublinks have $\mu = 0$. To see this, identify the diagram with $T^2 \times D^2$ so that the given sublink consists of dotted circles, then pass to the empty sublink by twisting the 1-handles as in Figure 5.42. (This actually shows that the diffeomorphisms of T^3 act transitively on the 7 structures.) The structure whose characteristic sublink is the entire link has $\mu = 8$. For a tricky proof, note that the fiber of the elliptic surface $E(1)$ is characteristic, so its complement is a spin manifold with signature -8 and bounded by T^3, implying that some spin structure on T^3 has $\mu = 8$. For a direct proof by Kaplan's algorithm, slide twice as in Figure 12.30 to convert the characteristic sublink into a 0-framed trefoil knot, then eliminate this as in the solution of Exercise 5.7.15(a). To do this, we blow up 2 $\mathbb{CP}^2$'s and 8 $\overline{\mathbb{CP}^2}$'s, then blow down 2 $\mathbb{CP}^2$'s, so the resulting spin manifold with T^3 boundary has signature -8. □

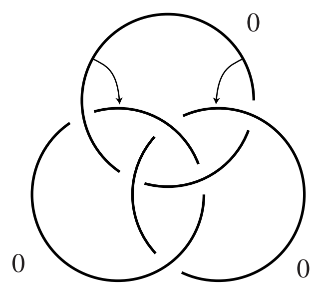

Figure 12.30

Solution of Exercise 5.7.17(d): We have $\sigma(X) - [S]^2 \equiv \mu(\partial X, s) \pmod{16}$. The proof is essentially the same as that of Exercise 5.7.7(b). To see that S continues to represent $w_2(X, s)$ after we blow up points on it, note that we have already shown this for $\partial X = \emptyset$, and the local picture of blowing up a point is the same in the general case. □

Solution of Exercise 5.7.21(b): Without loss of generality, we can assume that $p > |q| > 0$. As in the solution of Exercise 5.3.9(b), we can expand $-\frac{p}{q}$ as a continued fraction — by rounding to *even* integers, we obtain a_i even for $i \neq 1$ and $|a_i| \geq 2$ for all i $(= 1, \ldots, n)$. For p even, the Rohlin invariants of the two spin structures on $L(p, q)$ are then given by the mod 16 residues of

$$\sum_{i=1}^{n} \operatorname{sign}(a_i) \quad \text{and} \quad \sum_{i=1}^{n} \operatorname{sign}(a_i) - \sum_{i \equiv n (\operatorname{mod} 2)} a_i,$$

where $\operatorname{sign}(a_i) = \frac{a_i}{|a_i|}$. For p odd, the Rohlin invariant of the unique spin structure is given by the first formula if a_1 is even and the second if a_1 is odd. To prove this, first suppose that a_1 is even. Then the linear plumbing with coefficients a_i represents $L(p, q)$ as a spin boundary, and diagonalization shows that the signature is given by the first formula (since all $|a_i| \geq 2$). For p even, there must be a second spin structure. The only nonempty characteristic sublink of this plumbing is the union of the odd-indexed unknots. (Check, for example, that the second component cannot be in a characteristic sublink.) Thus, n must be odd in this case. Following Kaplan's algorithm, we slide one component of this sublink over the others, to obtain an unknot K with framing $\sum_{i \text{ odd}} a_i$. By blowing up meridians and blowing down K, we obtain a spin manifold with signature given by the second formula. In the case when a_1 is odd, the plumbing has a unique characteristic sublink, whose components are indexed by all $i \equiv n \pmod 2$, so p must be odd, and the previous computation yields the second formula. □

Solution of Exercise 6.1.1(a): Compute $F \cdot F$ by pushing F off of itself to obtain a surface F' transverse to F. Then $F \cdot F = F \cdot F'$. There are two kinds of intersections of F and F'. We obtain $e(\nu F)$ intersections (counted with sign) coming from the twisting of the normal bundle as in the embedded case. However, each self-intersection of F also contributes two intersections of the same sign, as in Figure 12.31, resulting in the additional term $2\,\operatorname{self}(F)$ in the formula. □

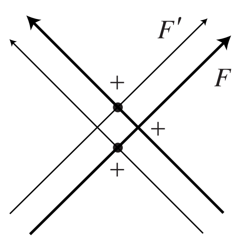

Figure 12.31

Solution of Exercise 6.1.4(a): If we add a 0-framed meridian to the top stage dotted circle of $X_{n,k}$, the 1- and 2-handles will clearly cancel in pairs. If we do the same cancellation without the 0-framed meridian, starting from the rightmost 2-handle, dragging along its dotted, doubled meridian and performing 1-handle slides when necessary, we will be left with a k-framed unknot and one dotted circle. The resulting 2-component link will be obtained from the Hopf link by doubling one component n times (taking untwisted, positive Whitehead doubles). Negative self-plumbings result in negative Whitehead doubles. A kinky handle with more than one self-plumbing results in a *ramified* Whitehead double — that is, we replace the relevant knot by parallel copies of itself using the 0-framing, and then double each copy. In short, an n-stage Casson tower is obtained from a 2-handle by removing a pushed-in family of embedded disks bounded by a link that forms an n-fold ramified Whitehead double of the belt circle, and the signs and ramification are determined by the signs of self-intersection and branching of the Casson tower. □

Solution of Exercise 6.1.4(c): To prove that R is simply connected, note that by compactness any loop γ in R is contained in some Y_n. As in the previous solution, we can cancel handles to draw Y_n as a 3-component link: the two left-most curves in Figure 6.16 together with a dotted circle forming the n-fold Whitehead double of the pictured circular dotted curve. We can now draw ∂Y_n as 0-surgery on a knot K in S^3, by cancelling the first two link components. (In fact, one obtains the n-fold double of the pretzel knot in Figure 6.24.) Since any loop in Y_n is clearly homotopic to one in ∂Y_n, we can write γ as a product of meridians of K (Exercise 5.2.2(b)). But Y_{n+1} is built from Y_n by adding a kinky handle to a meridian of K, so γ is nullhomotopic in Y_{n+1}. (To see this nullhomotopy directly in the picture, note that the attaching circle of the last 2-handle is nullhomotopic in $Y_n \cup$ 1-handle, cf. Figure 6.10.) Now we have $H_1(R;\mathbb{Z}) = \pi_1(R) = 0$. Since each 2-handle of R kills an element of infinite order in H_1, we have $H_2(R;\mathbb{Z}) = 0$, and since there are no handles of higher index, $H_i(R;\mathbb{Z}) = 0$ for $i \neq 0$. Thus, R is contractible, by standard CW-complex theory. To prove that R is simply connected at infinity, note that any compact $C \subset R$ lies in some Y_n, and set D equal to this Y_n (minus a collar of ∂Y_n). Then $R - D$ is obtained from 0-surgery on K (as above) by adding a Casson handle to a meridian. As before, any loop γ in $R - D$ lies in some $Y_m - D$, so we can homotope it into ∂Y_m, where it is nullhomotopic in $Y_{m+1} - D$. □

Solution of Exercise 6.2.2: Given an n-component link $L \subset S^3$ with rational coefficients, we obtain a Heegaard diagram of the corresponding surgered manifold M as follows. Connect each component of L to a base point p by an arc, and let X denote the complement of a regular neighborhood of L union the arcs. Then M is obtained from X by attaching a copy of $\natural n S^1 \times D^2$, which is n 2-handles and a 3-handle, and the 2-handles attach to

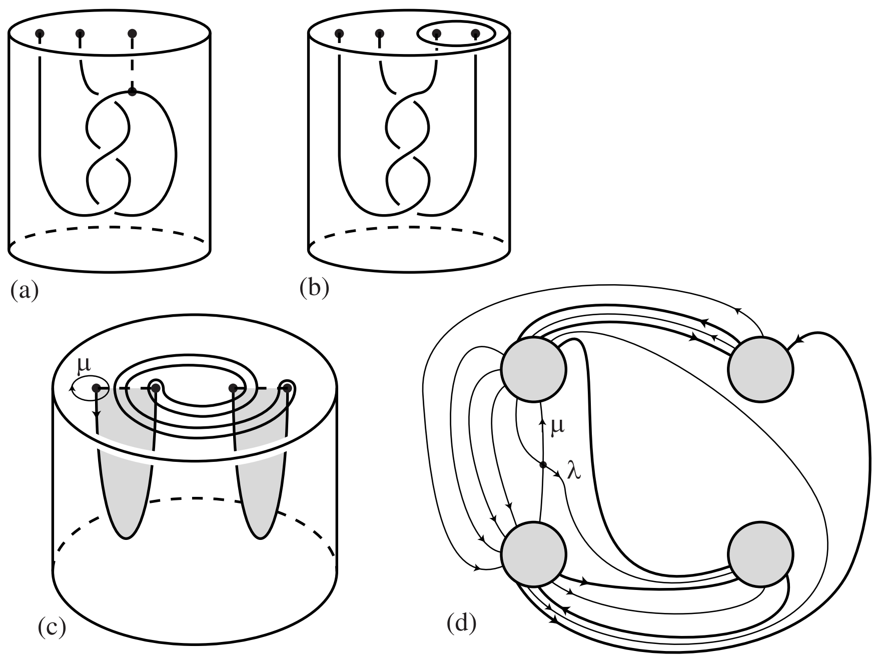

Figure 12.32. Trefoil knot complement.

∂X along the surgery curves given by the rational coefficients. To describe X as a handlebody, delete a ball containing p from S^3, realizing X as the complement in $D^3 = I \times D^2$ of a neighborhood of a proper embedding of n arcs. See Figure 12.32(a) for the trefoil. By Proposition 6.2.1, the corresponding handle diagram of X has one 0-handle, a 1-handle for each local minimum of an arc, and a 2-handle for each local maximum. To draw the union X_1 of 0- and 1-handles, we remove each 2-handle by drilling out a vertical arc as in (a) of the diagram, keeping track of the attaching circle of the 2-handle (b). We simplify the picture of X_1 by unwinding the arcs (c). It is then easy to convert to a planar diagram (d). (The 1-handles of (d) are produced by vertical identifications; the 2-handle is given by the heavy curve.) To obtain a diagram of M, we must add in the attaching curves of the remaining n 2-handles determined by the surgery coefficients. To do this, it suffices to identify the meridian μ and longitude λ of each component of L, as we have for the trefoil in (d). (Check that $\mu \cdot \lambda = +1$ in the boundary orientation of the tubular neighborhood of the knot, and that λ is nullhomologous in the knot complement, as required. These conditions determine λ.) □

Solution of Exercise 6.2.4(a): In Figure 6.19, the boundary of a regular neighborhood of the two circles union the band is a genus-2 surface; the

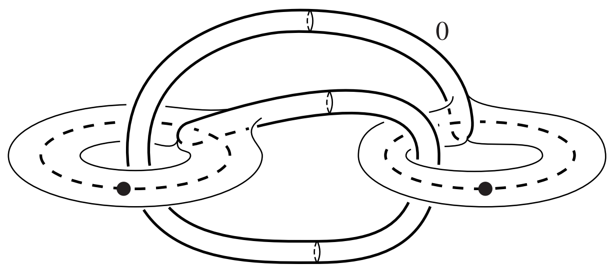

Figure 12.33

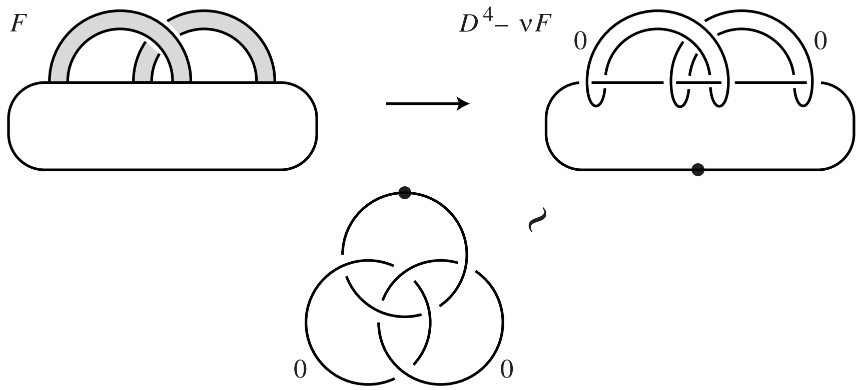

Figure 12.34. Unknotted punctured torus F and complement in D^4.

torus is obtained from this by surgery along the core disk of the 2-handle, Figure 12.33. □

Solution of Exercise 6.2.4(b): See Figure 12.34. The complement is the same as in Exercise 5.5.7(a) since the two embeddings $T^2 \subset S^4$ are the same. (Check this by explicitly drawing the torus in the top diagram of Figure 12.21 and then cancelling the handles.) □

Solution of Exercise 6.2.4(c): One way to draw both surfaces simultaneously is to start with two orthogonal Möbius bands in S^3 with the same core circle. Push one into D^4 and cap it with a 2-handle in the 4-handle of S^4; cap the other with a disk in D^4. □

Solution of Exercise 6.2.4(d): $\partial(I \times D^3) = S^3$, and $\partial(I \times K_0) = K\#\overline{K}$. (We have doubled the pair (D^3, K_0).) If we shrink D^3 to a point, the local maxima (of the radial function) on K_0 fall off of D^3, resulting in ribbon moves on $\partial(I \times K_0)$, and then the local minima of K_0 generate local minima of the ribbon disk. To see this in a diagram, imagine K_0 reflected across

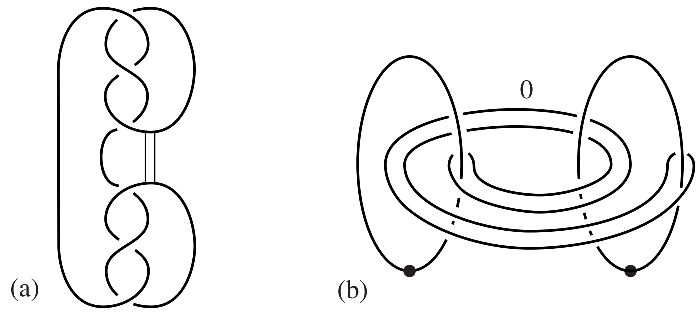

Figure 12.35. $I \times$ (trefoil knot complement).

a horizontal plane to obtain $K\#\overline{K}$ (Figure 12.35(a) for the trefoil). The ribbon moves are given by vertical bands connecting critical points, and the resulting circles form an unlink by the symmetry of the picture. (Imagine K being pushed into a mirror and disappearing.) For the resulting Kirby diagram for the trefoil, see Figure 12.35(b), and compare with Figure 12.32. □

Solution of Exercise 6.2.4(e): $S^2 \times D^2$ is built from $S^2 \times S^1$ by attaching a 2-handle along $\{p\} \times S^1$ with the product framing, together with a 4-handle. Thus, S^4 is built from X by attaching a 0-framed meridian to a dotted circle, after which the entire diagram cancels, and introducing a 4-handle. If we take $p \in S^2$ to be the north pole, then the Gluck twist fixes the attaching circle but changes its framing by 1 (or any odd number). Thus, the manifold resulting from the Gluck construction on S is obtained from X by adding a ± 1-framed (or any odd-framed) meridian to a dotted circle, and introducing a 4-handle. Cancelling the 1-2 pair is the same as turning the dotted circle into a ∓ 1-framed 2-handle and blowing it down. □

Solution of Exercise 6.2.5(b): For a knot complement $M = S^3 - \nu K$, $I \times M$ is given as the complement of the canonical ribbon disk for $K\#\overline{K}$ (Exercise 6.2.4(d)). For Dehn surgery on K, add a 2-handle and a 3-handle, where the 2-handle attaches along a circle in $\partial \nu K$ determined by the framing coefficient. (See Figure 12.36, which is from [**A6**], for the Poincaré homology sphere, -1-surgery on the left trefoil.) The procedure generalizes to surgery on a link L once we observe that the manifold $I \times X$, with X as given in the solution of Exercise 6.2.2, is obtained by summing corresponding components of L and $\overline{L}$ and deleting the corresponding ribbon disks from D^4. If we convert to ordinary dotted circle notation, we recover the diagram given by Example 4.6.8, since both are based on the handle decomposition of X given by Proposition 6.2.1. A description of $S^1 \times M$ can also be obtained from our diagram of $I \times M$ as in Example 4.6.8 (cf. [**A6**] and Figure 10.2). □

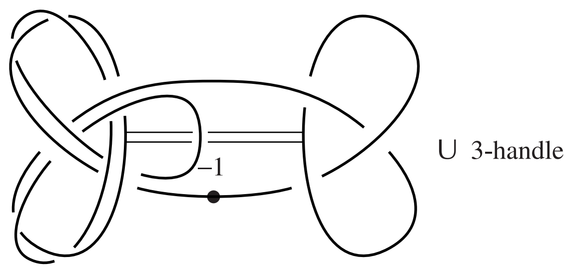

Figure 12.36. $I \times$ (Poincaré homology sphere).

Solution of Exercise 6.2.6(a): There is a genus-2 surface bounding a tubular neighborhood of the dotted circle(s) union a 3-ball surrounding Figure 6.27. The torus is obtained by surgering this along the core of the 2-handle (cf. Exercise 6.2.4(a)). See Figure 12.37. □

Solution of Exercise 6.2.11(b): The CW-complex corresponding to the given handle decomposition of F is a contractible 1-complex; after ambiently sliding 1-handles of F, we can assume it is homeomorphic to an interval. When we double (D^4, F), each 1-handle of F will generate an additional 1-handle in S (Figure 12.38(a)), and each 0-handle will generate a 2-handle in S. Thus, the handle decomposition of $S^4 - \nu S$ will have a 1- and 3-handle for each 0-handle of F, two 2-handles for each 1-handle of F (Figure 12.38(b)) and a 4-handle. We complete the Gluck construction by attaching a 1-framed meridian to the dotted circle corresponding to one endpoint of the linear graph, and a 4-handle (which cancels a 3-handle), as in Exercise 6.2.4(e). When we cancel the 1-handle, the 2-handle h leading to the next 1-handle

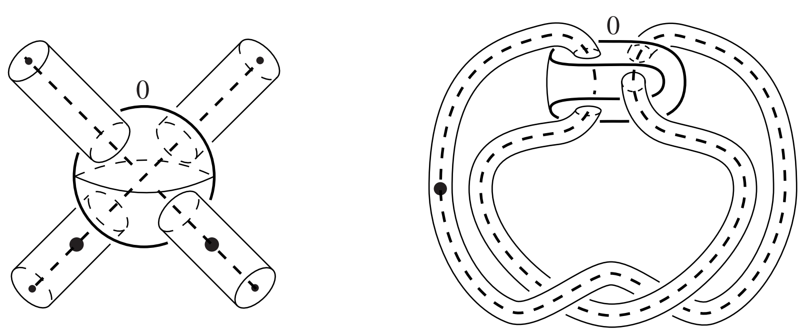

Figure 12.37

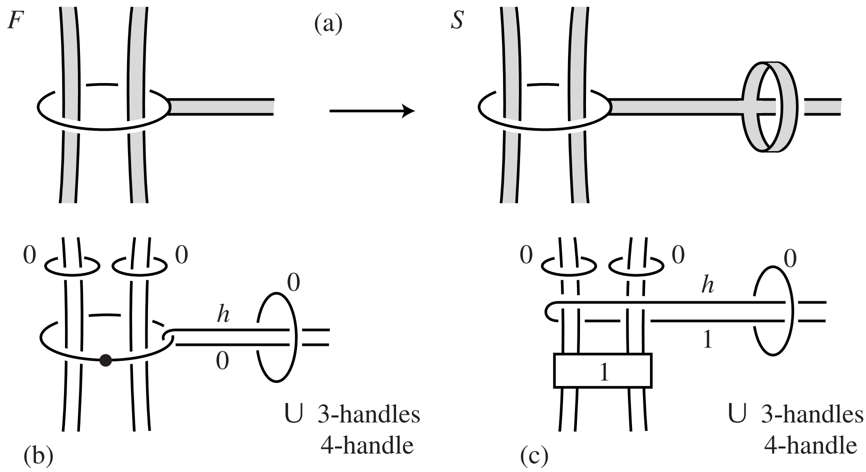

Figure 12.38

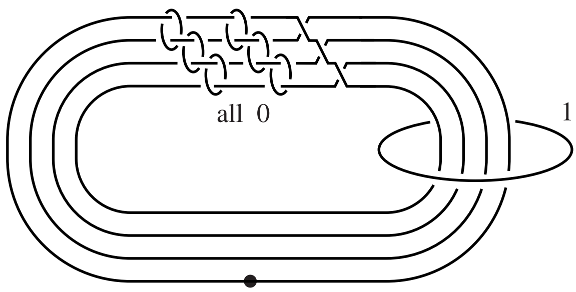

Figure 12.39. Complement of a quartic curve in $\mathbb{CP}^2$.

will become wrapped around anything linking the first dotted circle. (See Figure 12.38(c)). However, these additional strands occur in pairs with linking 0-framed circles, so we may untangle h from them by handle slides, after which it is attached to a 1-framed meridian of the second dotted circle. Thus, we can apply induction to cancel all 1-handles. We are left with a 0-framed unlink whose 2-handles correspond to the dual 1-handles of F; this cancels the 3-handles. □

Solution of Exercise 6.2.12(c): $\pi_1(X_d) \cong \mathbb{Z}_d$. For a picture with a unique 1-handle, use a column of bands in Figure 6.34 to cancel the extra 0-handles of F_d. See Figure 12.39 for $d = 4$. □

Solution of Exercise 6.3.3(a): See Figure 12.40. In the upper left corner, the knot has writhe 1, so the framing is (blackboard) + $e(Y) + 1$. In the double cover, both of the lifted knots have writhe 0, hence framing $e(Y) + 1$.

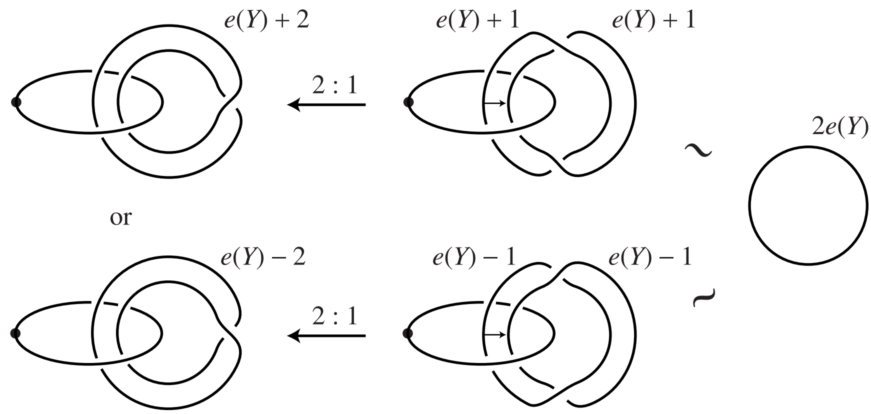

Figure 12.40. 2-fold cover of a D^2-bundle over $\mathbb{RP}^2$.

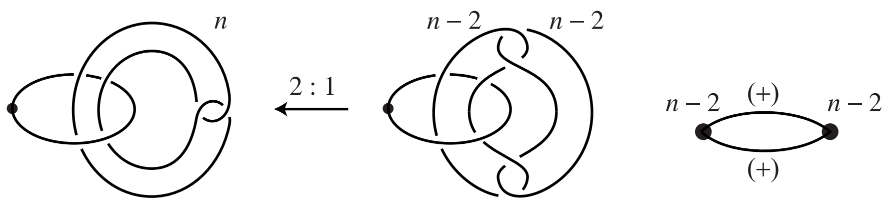

Figure 12.41. 2-fold cover of a self-plumbed D^2-bundle over S^2.

Sliding one 2-handle over the other to remove it from the 1-handle allows us to cancel the remaining handle pair, resulting in the required picture of X. □

Solution of Exercise 6.3.3(b): See Figure 12.41. The Whitehead curve has writhe 2, so the lifted curves (with writhe 0) have framing $n - 2$. The double cover is made from two D^2-bundles over S^2 with Euler number $n-2$ by plumbing them together twice (positively). This is easy to see directly, once we note that Y has Euler number $n-2$ by Exercise 6.1.1(a). Similarly, the d-fold cover of Y must be made from d bundles with Euler number $n-2$ by plumbing according to a circular graph. (Draw the Kirby diagram.) □

Solution of Exercise 6.3.5(a): Figure 12.42 shows the case $d = 4$; the general case is similar. □

Solution of Exercise 6.3.5(c): See Figure 12.43. The last blow-down produces $L(3, -1) = L(3, 2)$. □

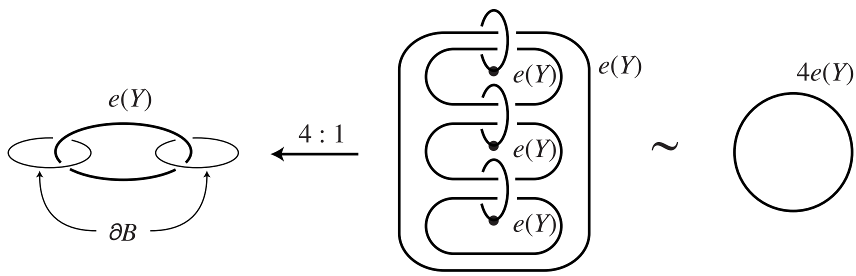

Figure 12.42. 4-fold cover of a D^2-bundle over S^2, branched along a pair of fibers.

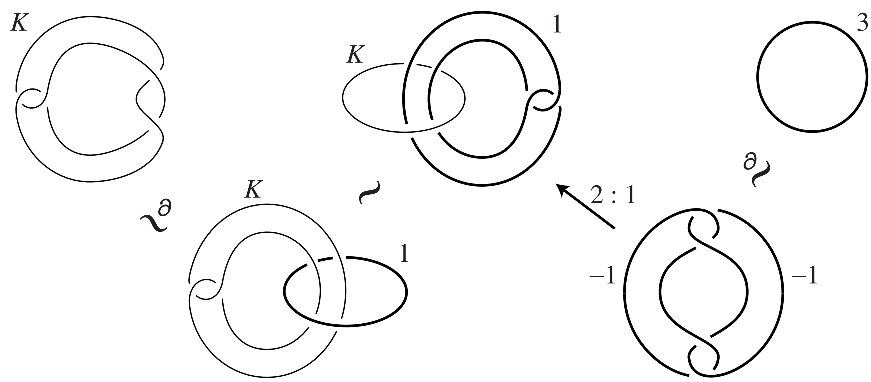

Figure 12.43. Double cover of S^3 branched along a trefoil knot K.

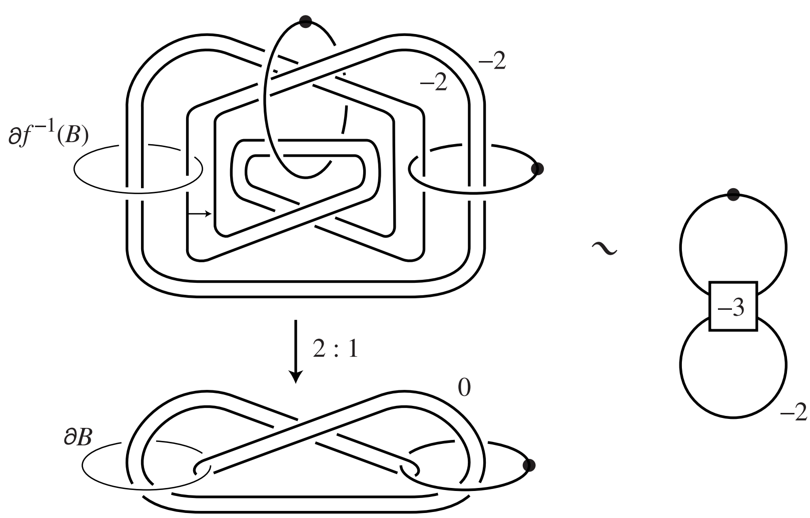

Figure 12.44. Double cover of D^4 branched along a ribbon disk.

Solution of Exercise 6.3.5(d): Take the double cover of Figure 6.20 (with both 1-handles mapping to $1 \in \mathbb{Z}_2$), then fill in $f^{-1}(B)$ by erasing one dotted circle, obtaining Figure 12.44. After the indicated slide (which is easy to visualize since all framings are given by the blackboard) we can cancel the lower 1-handle to obtain the simpler picture shown. For the square knot, note that Figure 12.35 differs from Figure 6.20 by a half-twist. Dragging this through the previous computation, we obtain the same simple figure with the coefficient changed from -2 to -3. By Example 4.6.8, this latter figure is $I \times (L(3,1) - \text{int } D^3)$. To explain the last observation, note that the branch locus in D^4 has the form $I \times K_0 \subset I \times D^3$ as in Exercise 6.2.4(d). The branched cover is $I \times X$, where X is the branched cover of D^3 along K_0. By (c) above, X is $\overline{L(3,1)} - \text{int } D^3$ (and $I \times \overline{X} = \overline{I} \times X = I \times X$). □

Solution of Exercise 6.3.9(b): By our algorithm, the double branched cover is the disk bundle over S^2 with $e(X) = k$. To describe the involution without Kirby diagrams, note that it covers reflection on S^2, and is a reflection on each D^2-fiber over the equator. There must be $\frac{k}{2}$ full twists in the fixed set (as we travel once around the equator), since it must appear the same in each of the two copies of $D^2 \times D^2$. □

Solution of Exercise 6.3.9(c): The Seifert surface is visible in Figure 6.43, so the double branched cover X is obtained by plumbing together two D^2-bundles over S^2 with Euler number -2. A slam-dunk shows that $\partial X \approx L(3,2)$ as required. □

Solution of Exercise 6.3.9(d): See Figure 12.45. For the untwisted double, the knotted band must be determined by the 0-framing on K, resulting in a 0-framing in the diagram of X. For the n-twisted double, this latter framing changes from 0 to $2n$. □

Solution of Exercise 6.3.9(e): It suffices to draw the plumbing with a $\mathbb{Z}_2$-symmetry reflecting each attaching circle. This can always be done; see Figure 12.46, for example. □

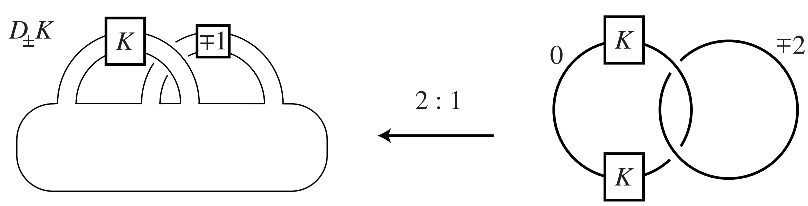

Figure 12.45. Double cover branched along a Seifert surface of a Whitehead double.

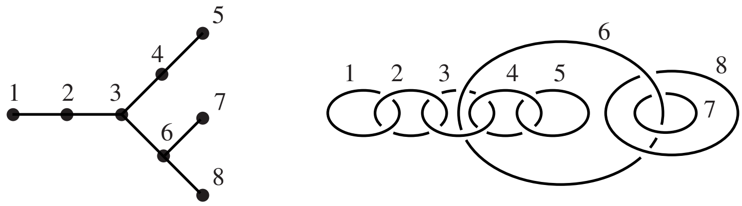

Figure 12.46. $\mathbb{Z}_2$-symmetry on a plumbing.

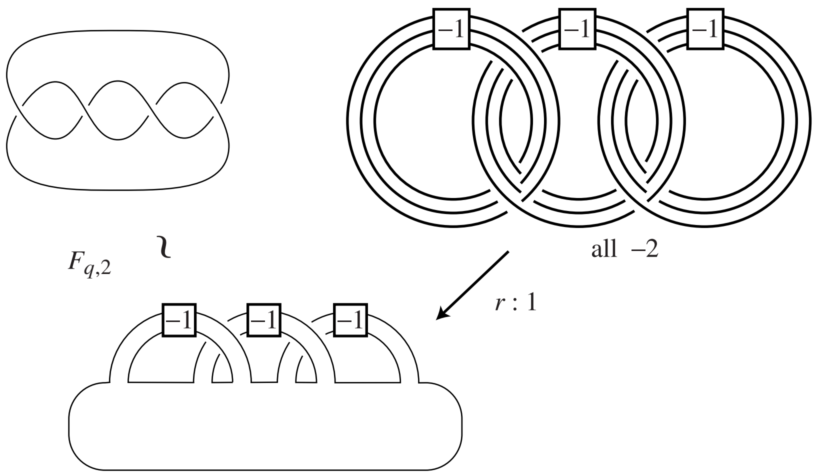

Figure 12.47. $M_c(r,q,2)$: $q-1$ rings of $r-1$ circles ($q=r=4$).

Solution of Exercise 6.3.13(a): See Figure 12.47. Note that we can pull a $-\frac{1}{2}$-twist out of each -1 and slide it clockwise to the bottom. The meshing with the adjacent ring flips over, and we obtain Figure 6.45. □

Solution of Exercise 6.3.16(a): The 2-fold branched covers are $S^2 \times S^2$ and $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}^2}$), respectively. (Recall that there are two different unknotted embeddings $\mathbb{RP}^2 \subset S^4$, related by a reflection — cf. Exercise 6.2.4(c).) For a general (Y, B), modify B by taking a pairwise connected sum with (S^4, T^2) or $(S^4, \mathbb{RP}^2)$. Since the double branched cover of a trivial disk pair (D^4, D^2) is again (D^4, D^2), the effect on the double branched cover of (Y, B) is to form its connected sum with $S^2 \times S^2$, $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$. Since $H_1(S^4 - \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}_2$, the above construction only generalizes to degree $d > 2$ in the case of T^2. In this case, summing B with a trivial torus changes the d-fold cyclic cover of Y by summing with $\#(d-1)S^2 \times S^2$, as is evident from Figure 12.48 (drawn for $d = 4$) and Proposition 5.1.4. □

Solution of Exercise 6.3.16(b): $S^2 \times S^2$ — See Figure 12.49. □

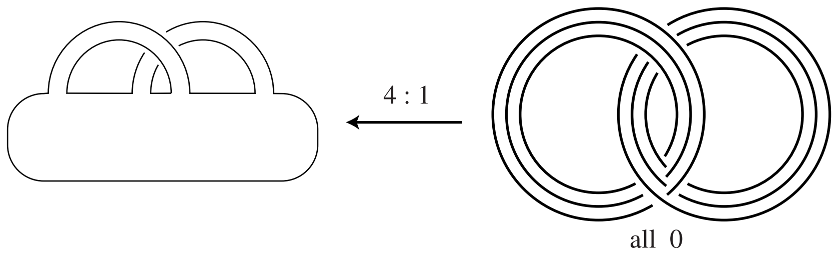

Figure 12.48. 4-fold cover of S^4 branched along T^2.

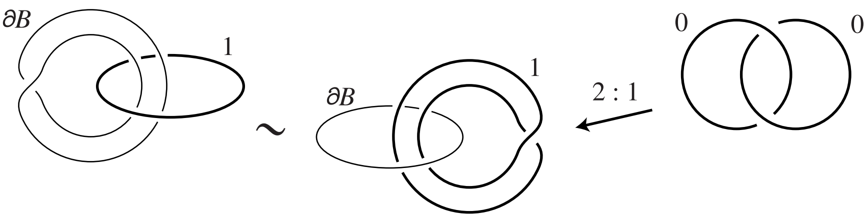

Figure 12.49. Double cover of $\mathbb{CP}^2$ branched along a quadric curve.

Solution of Exercise 6.3.18: The action is given by 180° rotation about the three coordinate axes in the right-hand diagram of Figure 12.49, with the given generators corresponding to the x- and y-axes, respectively. (Note that the former preserves either orientation of the positive Hopf link.) The quotients (corresponding to x, y, z and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, respectively) are $\mathbb{CP}^2$ and S^4 (cf. Exercises (b) and (a) above), $\overline{\mathbb{CP}^2}$ and S^4. □

12.3. Solutions of some exercises in Part 3

Solution of Exercise 7.1.10(b): Since $[C]\cdot[F_n] = 1$, the argument of the solution of Exercise 3.1.12(a) applies and proves that C is a section of $\mathbb{F}_n \to \mathbb{CP}^1$. Note that since $[C]^2 = n+2\beta$ and $\langle c_1(\mathbb{F}_n), [C]\rangle = n+2\beta+2$, the adjunction formula 1.4.17 implies that $g(C) = 0$, hence $C \approx \mathbb{CP}^1$. Consequently the complex curve C is an affine section iff $[C] = [S_n]+\beta[F_n]$ and $[C]\cdot[S_\infty] = 0$, i.e., iff $[C] = [S_n]$. □

Solution of Exercise 7.1.10(c): Suppose that $\mathbb{F}_n$ is nonminimal, so there is a rational curve $C \subset \mathbb{F}_n$ with $[C]^2 = -1$. If we set $[C] = a[S_n]+b[F_n]$, the facts that $[C]^2 = -1$ and $[C]\cdot[S_n] \geq 0$ imply $[C] = [S_n]-\frac{1+n}{2}[F_n]$. Now taking the product $[C]\cdot[S_\infty] = [C]([S_n]-n[F_n])$ we see that $[C]\cdot[S_\infty] = -\frac{n+1}{2}$. Since different complex curves intersect each other positively, the above equation shows that $C = S_\infty$, implying that $n = 1$, which concludes the solution. Note that for even n (i.e., when $\frac{1+n}{2}$ is not an integer) the manifold $\mathbb{F}_n$

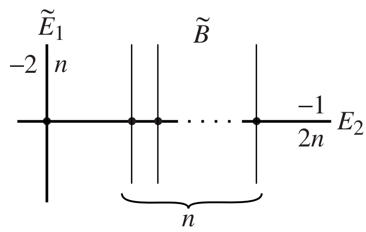

Figure 12.50. Resolution of the curve $f(x, y) = x^n + y^{2n}$.

has an even intersection form, hence does not contain any submanifold with self-intersection -1. □

Solution of Exercise 7.2.4(b): When we blow up the origin, the exceptional curve E_1 comes with multiplicity n, while the proper transform consists of n lines passing through the origin of one of the charts. Blowing up this point, we will get the desired configuration of a smooth proper transform with two exceptional curves $\widetilde{E}_1$ and E_2 ($m_1 = n,\ m_2 = 2n,\ e_1 = -2,\ e_2 = -1$), see Figure 12.50. □

Solution of Exercise 7.2.4(c): If we blow up the origin, then on the chart U_1 the equation of the total transform is $v^4(u^2 + v)(u^3 v + 1) = 0$, while on U_2 it is $(v')^4(1 + (u')^3 v')(v' + (u')^2) = 0$. This shows that the proper transform is tangent to the exceptional curve at the origins of both U_1 and U_2. Hence, to get a configuration with the desired properties we have to blow up both U_1 and U_2. (Details are left to the reader.) Note that for $g(x, y) = (x^2 + y^5)(x^5 + y^2)$ there are singularities on both charts. □

Solution of Exercise 7.2.5(a): First we solve the exercise for $k = 2$ and 3. Blowing up the curve $x^2 + y^2 = 0$ we get $v^2(u^2 + 1) = 0$ (on one chart); the corresponding diagram and dual graph are given by Figure 12.51(a). For $k = 3$ the first blow-up gives $v^2(u^2 + v) = 0$. Blowing up this configuration and considering the chart $U_{1,2}$, we have Figure 12.51(b); one further blow-up gives the final configuration, Figure 12.51(c). Now for arbitrary k, one blow-up reduces k by 2 (since the total transform is given by $v^2(u^2 + v^{k-2}) = 0$), so induction gives the solution and we get Figure 12.52(a) and (b) for odd and even k. (Note that in the last blow-up for k odd, we have to take the second chart instead of the first one.) □

Solution of Exercise 7.2.5(c): First we blow up the origin and take the curve in U_1, see Figure 12.53(a). (We have also indicated the u- and v-axes in U_1, although the latter does not lie in B'.) Now in the second blow-up we must take the chart $U_{1,2}$; on this chart the total transform is given by

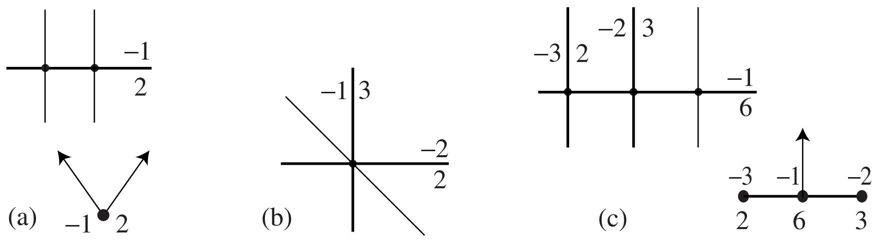

Figure 12.51. Resolution (a) of $f(x, y) = x^2 + y^2$ and (b), (c) of $f(x, y) = x^2 + y^3$.

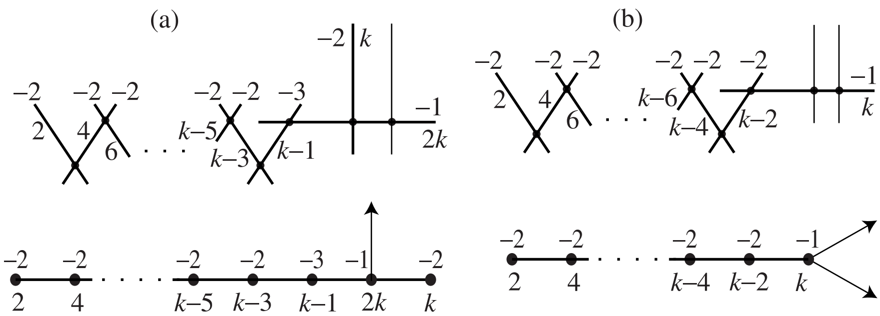

Figure 12.52. Diagram and dual graph of the resolution of $f(x, y) = x^2 + y^k$ with (a) k odd, and (b) k even.

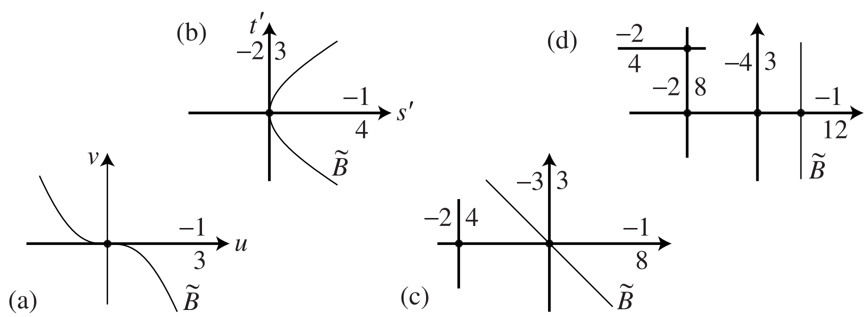

Figure 12.53. Resolution of $f(x, y) = x^3 + y^4$.

Figure 12.53(b). Blowing up the origin again, we get Figure 12.53(c), and a final blow-up gives Figure 12.53(d). The corresponding dual graph is given by Figure 12.54. □

Solution of Exercise 7.2.12(a): Following the algorithm described in Section 7.2 (and using the solutions of Exercises 7.2.5(a) and (c)), we get the diagrams shown in Figures 12.55 and 12.56. Note that for odd k the resulting desingularization of $z^2 = x^2 + y^k$ is not minimal: If we blow down

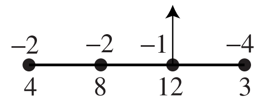

Figure 12.54. Dual graph of the resolution of $f(x, y) = x^3 + y^4$.

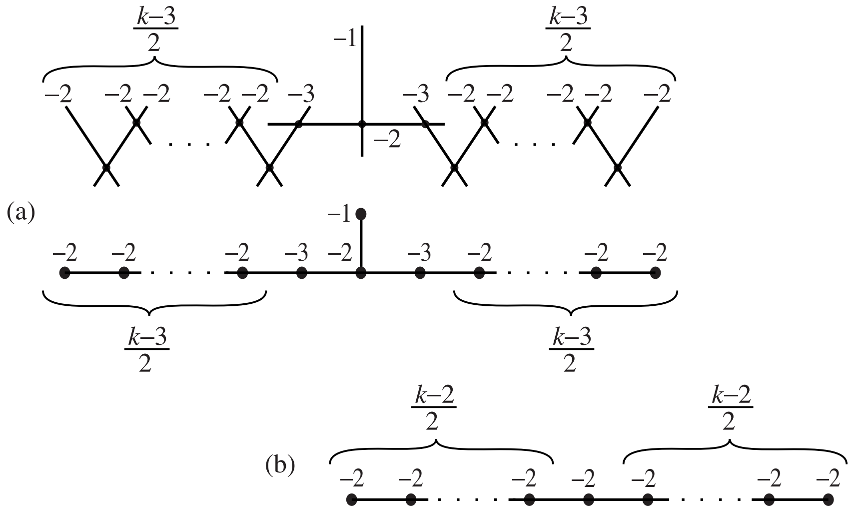

Figure 12.55. Resolution of the singularity $z^2 = x^2 + y^k$ for (a) k odd, and (b) k even.

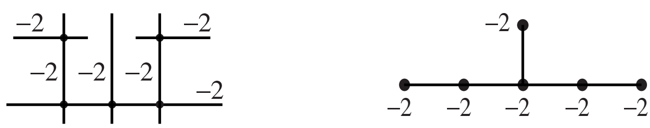

Figure 12.56. Diagram and dual graph of the resolution of $z^2 = x^3 + y^4$.

the rational -1-curve in the middle, the self-intersection of the -2-curve intersecting it will change to -1; after blowing this down we get a chain of $k-1$ copies of a -2-sphere, just as in the case with k even. For the final solution, see Figure 12.57. □

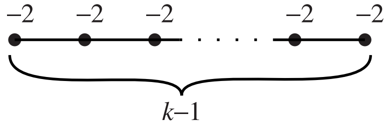

Figure 12.57. Dual graph (plumbing diagram) of the minimal resolution of $z^2 = x^2 + y^k$.

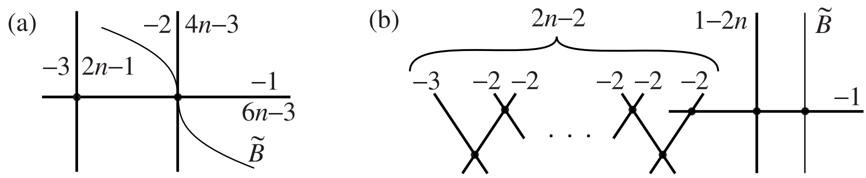

Figure 12.58. Resolution of the curve $f(x, y) = x^{2n-1} + y^{4n-3}$.

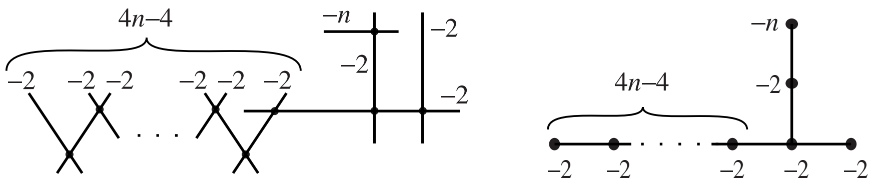

Figure 12.59. Diagram and dual graph of the resolution of the singularity $z^2 = x^{2n-1} + y^{4n-3}$.

Solution of Exercise 7.2.12(c): Following the algorithm described in Section 7.2, we must first desingularize the curve $\{x^{2n-1} + y^{4n-3} = 0\}$. In the following we will only highlight this process; details are left to the reader. After three blow-ups we get the configuration shown in Figure 12.58(a), and the equation we have to work with admits the form $u^{4n-3}v^{6n-3}(u + v^{2n-3}) = 0$. Each further blow-up will lower the exponent of v (in the parentheses) by 1; consequently $2n - 3$ further blow-ups will yield the required configuration. The end-result of this process is given in Figure 12.58(b); all multiplicities are odd, and all squares are equal to -2 except for a -1, a $1 - 2n$ and a -3 (the last one in the long chain). After separating odd multiplicities — which involves $2n$ further blow-ups — and taking the double branched cover, we end up with the configuration given by Figure 12.59. All but one of the $4n$ spheres in Figure 12.59 have self-intersection -2, and the remaining one has square $-n$. □

Solution of Exercises 7.2.15(a), (b) *and* (c): In Figure 12.60 we give the diagrams describing the desingularized curves, and the diagrams of the canonical and minimal resolutions. (For more examples of this type see [**La2**].) □

Solution of Exercise 7.3.8(a): Since a generic elliptic fiber $\widetilde{F}$ lifts from a fiber of pr_1, it will be visible in Figure 7.4 as in Figure 12.61. Blowing down as in the proof of Proposition 7.3.7, we obtain a smooth curve F in $\mathbb{CP}^2$ with degree $F \cdot H = 3$, where the hyperplane H is the image of S_1. Similarly, a pair of elliptic fibers blows down to a (nongeneric) pair of cubic curves.

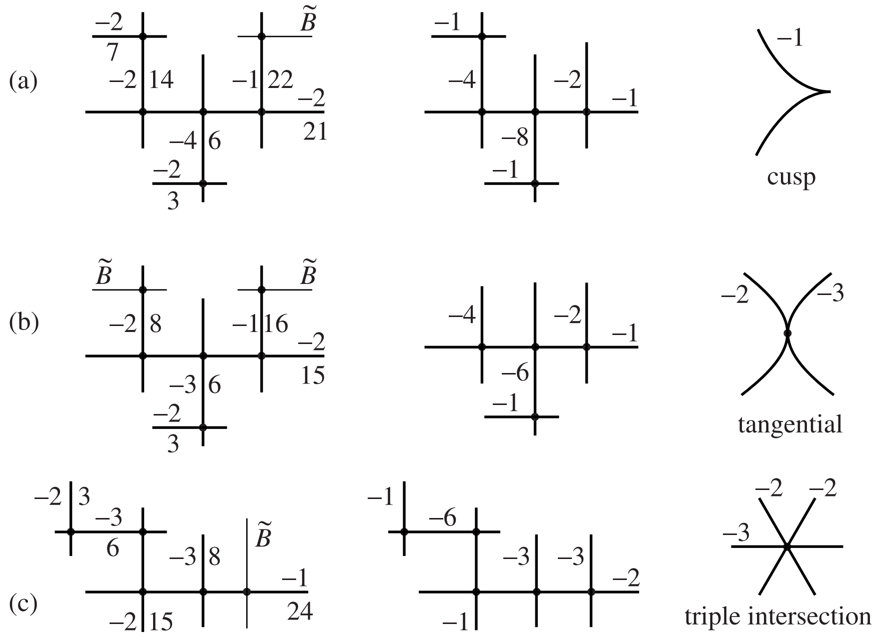

Figure 12.60. Examples of minimal resolutions involving singular curves, tangential and triple intersections.

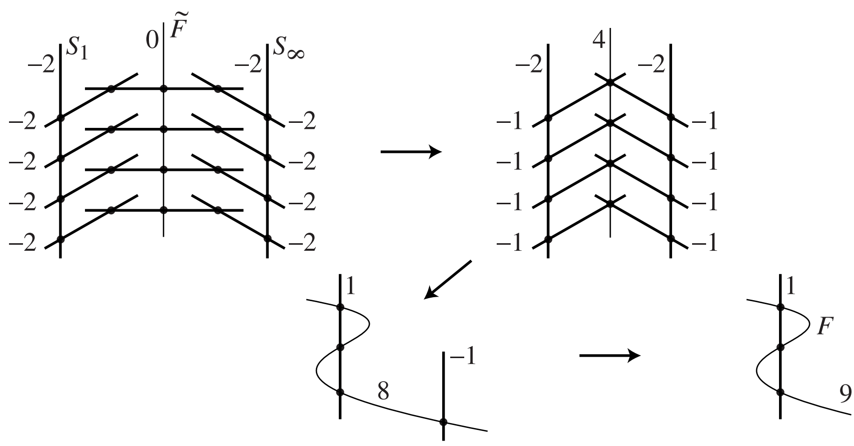

Figure 12.61. Keeping track of a fiber of $D'(1) \to \mathbb{CP}^1$ under blowing down.

After a perturbation, these curves generate an elliptic fibration on $E(1)$ as in Section 3.1. (Note that the normal framings induced on each fiber by the two elliptic fibrations must correspond, since these are determined over the 1-skeleton of F by the pencil on $\mathbb{CP}^2$.) □

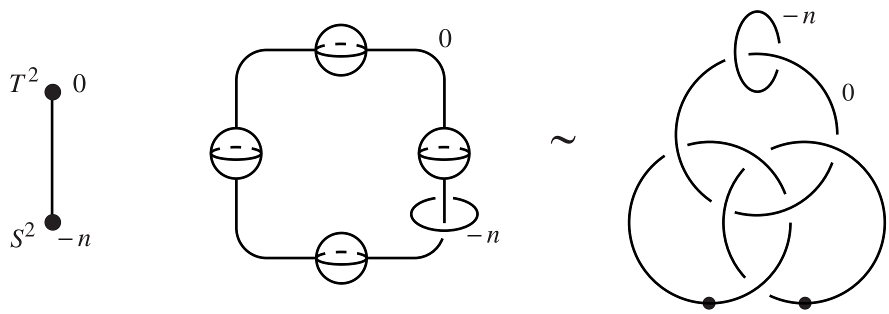

Figure 12.62. Plumbing and Kirby diagram for the neighborhood of a regular fiber and a section in $T(n)$.

Solution of Exercise 7.3.8(b): Recall that $X(n,1)$ is the desingularization of the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along $B_{n,1}$. Hence $X(n,1)$ is the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up in $4n$ points. Generalizing the proof of Proposition 7.3.7, we blow down the $2n$ rational -1-curves originating from the blow-ups of $\mathbb{CP}^1 \times \{p_i\}$ $(i = 1, \dots, 2n)$, and then further blow down $2n$ rational -1-curves to obtain $\mathbb{F}_1$, proving the claim. □

Solution of Exercise 7.3.16(a): Since $X(1,m) \approx \mathbb{CP}^2 \# (4m+1)\overline{\mathbb{CP}^2}$, the solution is obvious for $n = 1$. Now equip $X(n,m)$ with the genus-$(m-1)$ fibration over S^2 found earlier and apply induction on n. Since $X(n-1,m)$ and $X(1,m)$ both admit a section (i.e., a 2-sphere intersecting the fiber in a single point), we get that $\pi_1(X(1,m) - \nu F) = \pi_1(X(n-1,m) - \nu F) = 1$. Since $X(n,m) = X(n-1,m) \#_f X(1,m)$, the Seifert-Van Kampen Theorem completes the solution. (Since $X(n,m) \to S^2$ also admits a section, we conclude that $X(n,m) - \nu F$ is simply connected.) □

Solution of Exercise 7.3.17: Since it is a plumbing manifold (a torus of square 0 and a sphere of square $-n$ plumbed together), we have Figure 12.62. □

Solution of Exercise 7.3.21(a): The unique singular fiber in the fibration $M(2, 2m-1, 1) \to \mathbb{C}$ can be given as $C_s = \{(x,y) \in \mathbb{C}^2 \mid x^2 + y^{2m-1} = 0\}$. Hence a neighborhood of the singular fiber in the nucleus $N(m,n)$ can be given by the Kirby diagram consisting of the $(2, 2m-1)$-torus knot. (This is the knot in which the above singular curve intersects the boundary of a 4-ball neighborhood of its singular point in $\mathbb{C}^2$. We can regard the 4-ball as a 0-handle, and the rest of the singular fiber as the core of a 2-handle attached along the above knot.) Since the fiber has self-intersection 0, the framing of the knot is 0 as well. By plumbing a sphere of square $-n$ (corresponding to the section in $N(m,n)$) to the fiber, we get the Kirby diagram for $N(m,n)$ as it is shown by Figure 7.5. □

Solution of Exercise 7.3.27(b): Take two degree-d curves in general position, blow up the d^2 intersection points, and get the fibration $\mathbb{CP}^2\#d^2\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ described in Lemma 3.1.4. The fiber sum of two copies of this manifold is the desingularization of the double branched cover of $\mathbb{CP}^2$ along the two degree-d curves. Proposition 7.3.13 now concludes the argument. □

Solution of Exercise 7.4.1(a): Determine $c_1(\mathbb{F}_n)$; the rest is an obvious computation. (We identify second homology and cohomology using Poincaré duality.) If $c_1(\mathbb{F}_n) = x[S_n] + y[F_n]$, then the adjunction formula applied to the spheres S_n and F_n gives that $x = 2$ and $y = 2 - n$. Consequently $c_1(\mathbb{F}_n) = 2[S_n]+(2-n)[F_n]$; now we get $c_1^2(X) = 2a(a-2)n+4(a-2)(b-2)$ and $\chi_h(X) = \frac{1}{2}a(a-1)n + (a-1)(b-1) + 1$. Substituting $n = 1$, $a = 3$ and $b = m$ completes the solution. □

Solution of Exercise 7.4.9: Note that $y \leq 4x - 6$ implies $3(y - 2x + 6) \leq 2y - 2x + 12$, so $k = y - 2x + 6 \leq 2\frac{y-x+7-2a}{3} + 2\frac{2a-1}{3} = 2n + 2\frac{2a-1}{3}$. Since k is an integer, for $a = 0, 1$ or 2 this implies $k \leq 2n + 2[\frac{2a}{3}]$. □

Solution of Exercise 7.4.16: By definition, $X(n,m)$ is the desingularization of the double branched cover of $\mathbb{F}_0$ along $B_{n,m}$; consequently, $X(n,m)$ admits a holomorphic map $\pi\colon X(n,m) \to \mathbb{CP}^1$. Each singular fiber comes from a fiber of $\mathbb{F}_0$ lying in $B_{n,m}$. By the desingularization algorithm, a neighborhood of such a fiber is a plumbing on a star-shaped graph, where each point of the star is a sphere S_i of self-intersection -2 lifting an exceptional curve E_i in the resolution of $B_{n,m}$ (cf. the proof of Proposition 7.3.7). Clearly, such a fiber is simply connected. Since the map $S_i \to E_i$ is a covering map away from two points, it is easy to see that $d\pi$ is surjective at a generic point of S_i, so Lemma 7.4.15 yields the solution. □

Solution of Exercise 7.4.25: It is easy to see that for the Euler characteristic, $c_2(X) = \chi(X) = \chi(C_1)\chi(C_2) = (2 - 2g_1)(2 - 2g_2)$. Since $\pi_1(C_1 \times C_2) = \pi_1(C_1) \times \pi_1(C_2)$, we also know that $b_1(C_1 \times C_2) = 2g_1 + 2g_2$, hence $b_2(X) = 4g_1g_2 + 2$. A basis for $H_2(X;\mathbb{Z})$ can be given as follows: Suppose that $\{a_i, b_i \mid i = 1,\ldots, g_1\}$ (and $\{\alpha_j, \beta_j \mid i = 1,\ldots, g_2\}$ respectively) are the usual sets of circles in C_1 (C_2 resp.) representing a basis for $H_1(C_1;\mathbb{Z})$ (and $H_1(C_2;\mathbb{Z})$ resp.) as in the proof of Theorem 10.2.10; then the collection $\{[a_i \times \alpha_j], [a_i \times \beta_j], [b_i \times \alpha_j], [b_i \times \beta_j] \mid 1 \leq i \leq g_1,\ 1 \leq j \leq g_2\}$, together with $\{[C_1\times\{\text{pt.}\}], [\{\text{pt.}\}\times C_2]\}$, forms a basis of $H_2(X;\mathbb{Z})$. The generalized adjunction formula now implies that we have $\langle c_1(X), [a_i \times \alpha_j]\rangle = 0$

(and similarly for $a_i \times \beta_j$, $b_i \times \alpha_j$ and $b_i \times \beta_j$). The same argument shows that $c_1(X)$ is, in fact, Poincaré dual to $(2-2g_2)[C_1 \times \{\text{pt.}\}]+(2-2g_1)[\{\text{pt.}\} \times C_2]$. Consequently $c_1^2(X) = 2(2-2g_1)(2-2g_2)$, hence $c_1^2(X) = 2c_2(X)$ implying $\sigma(X) = 0$. Now the formula $\chi_h(X) = g_1 g_2 - g_1 - g_2 + 1$ is straightforward. Note that the fact that $\sigma(X) = 0$ can also be deduced by direct computation of the matrix representing Q_X in the basis described above, or by the existence of an orientation-reversing diffeomorphism, or using Lemma 9.1.5. (C_1 is the boundary of a 3-manifold M, hence $C_1 \times C_2 = \partial(M \times C_2)$.) The class $c_1(X)$ can also be computed using the identity $c_1(X) = c_1(\pi_1^* TC_1 \oplus \pi_2^* TC_2) = \pi_1^* c_1(C_1) + \pi_2^* c_1(C_2)$. □

Solution of Exercise 7.4.26: An easy computation (as discussed in Section 7.2) shows that $[G] = 5h - 3e_1 - 2e_2 \in H_2(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}; \mathbb{Z})$. Since $c_1(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}) = 3h - e_1 - e_2$, the adjunction formula gives $-\chi(G) = [G]^2 - c_1(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2})[G] = 12 - 10 = 2$. (Alternatively, the 5-fold branched covering shows that $\chi(G) = 5(\chi(\mathbb{CP}^1) - 3) + 3 = -2$.) Since G is connected, this implies that $g(G) = 2$. □

Solution of Exercise 8.1.1(a): The fiber has genus $\frac{1}{2}(d-1)(d-2)$, since it is a degree-d curve in the hyperplane $H_t \approx \mathbb{CP}^2$ cutting it out. For $d = 1$, we get the usual fibration of the Hirzebruch surface $\mathbb{F}_1 \approx \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ as constructed at the beginning of Section 3.1. For $d = 3$, we obtain an elliptic fibration on $E(1)$. (For $d = 2$, the fibration is obtained from the projection of either $\mathbb{F}_0$ or $\mathbb{F}_1$ into $\mathbb{CP}^1$ by blowing up twice; cf. Proposition 8.1.7.) □

Solution of Exercise 8.1.2(a): Perturb π to $\pi'(z_1, z_2) = p_1(z_1) + p_2(z_2)$ for p_i a generic polynomial of degree m_i. Then p_i has $m_i - 1$ quadratic critical points, so π' has $(m_1 - 1)(m_2 - 1)$ quadratic critical points. □

Solution of Exercise 8.1.6: Near B and the critical points, the charts given in Definition 8.1.4 define a complex structure. Elsewhere, we have a splitting of TX as a sum of the oriented 2-plane bundles tangent and normal to the fibers. Declare these real 2-plane bundles to be complex line bundles — we can do this extending the given structure near B and the critical points, since the fibers are already holomorphic there. □

Solution of Exercise 8.1.8(b): By Exercise 3.1.3, we are looking at all quadric curves through the four generic points $P_1, \dots, P_4$. There are three such singular curves (one for each way of splitting $\{P_1, \dots, P_4\}$ into two pairs), each a union of two complex lines. Since each of the lines contains two points P_i, blowing up the base locus gives a Lefschetz fibration whose three singular fibers are each a union of two exceptional spheres. To obtain a relatively minimal Lefschetz fibration, we must blow down one exceptional sphere in each of these three fibers. Looking at the combinatorics of the configuration of six exceptional spheres and four sections, we see that there are two ways to do this. We can blow down the three exceptional spheres intersecting one section, leaving a sphere bundle with a section of square +2 (the Hirzebruch surface $\mathbb{F}_2$ of Example 3.4.7), which is $S^2 \times S^2$ since its intersection form is even. Alternatively, we can blow down the three exceptional spheres disjoint from one section, leaving a bundle with a section of square -1, the Hirzebruch surface $\mathbb{F}_1 \approx S^2 \tilde{\times} S^2 \approx \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. In this case, the remaining three sections have square 1 and any two intersect once; these must be complex lines in $\mathbb{CP}^2$. □

Solution of Exercise 8.1.8(d): $3(d-1)^2$. Note that (after we blow up) each singular fiber contributes 1 to the Euler characteristic. □

Solution of Exercise 8.1.8(e): $\chi(X) = 3(d-1)^2 n + 2d(3-d)$, $\sigma(X) = (1-d^2)n$. X is simply connected because a section of $\mathbb{CP}^2 \# d^2\overline{\mathbb{CP}^2}$ provides a nullhomotopy for a meridian of the fiber. □

Solution of Exercise 8.1.10(a): The required local sections exist since each regular point of π lies on a disk in X mapping diffeomorphically to its image in Σ. Since π restricts to a fiber bundle on the complement of the singular fibers, it satisfies the Homotopy Lifting Property there [**Sp**]. In particular, given a path $\gamma\colon I \to X$ covering a loop $\pi \circ \gamma\colon I \to \Sigma$ avoiding the critical values, any homotopy rel $\{0,1\}$ of $\pi \circ \gamma$ avoiding critical values lifts to a homotopy rel $\{0,1\}$ of γ. It is now easy to prove the Homotopy Lifting Property for loops in Σ, that is, *any* (generic) homotopy rel $\{0,1\}$ of $\pi \circ \gamma$ in Σ lifts to γ fixing the endpoints: To push past a critical value $t \in \Sigma$, first homotope γ locally into a regular neighborhood νF_t, then push across F_t using a local section as above in the given component N of νF_t. (Note that each regular fiber near F_t intersects N in a connected surface.) To define the map $\pi_1(\Sigma) \to \pi_0(F)$, note that any loop representing $\alpha \in \pi_1(\Sigma)$ with interior avoiding critical values of π can be lifted to a path γ in X with $\gamma(0) \in F$ the base point of X (used in defining $\pi_1(X)$). The endpoint $\gamma(1)$ lies in a component of F that is independent of the choices of loops and lifts. (For a different path γ', the loop $\pi \circ (\gamma^{-1} * \gamma')$ in Σ is nullhomotopic; the corresponding homotopy in X changes $\gamma^{-1} * \gamma'$ to a path in F from $\gamma(1)$ to $\gamma'(1)$.) Exactness of the sequence at $\pi_1(X)$ follows immediately from lifting homotopies of loops in Σ; exactness at $\pi_1(\Sigma)$ and $\pi_0(F)$ is even easier. □

Solution of Exercise 8.1.10(b): We prove a version of the Homotopy Lifting Property for loops; the exact sequence then follows as in (a). As before,

π is a fiber bundle projection away from the singular fibers, so we only need to show how to push $\pi \circ \gamma$ past a critical value t. We homotope γ locally into a neighborhood of F_t as before. We no longer have local sections near t, but each Σ_i has a transverse disk D_i with ∂D_i mapping to Σ with winding number m_i around t. Componentwise surjectivity of f guarantees that we can access all disks D_i associated to one component of F_t by fiber-preserving homotopies of γ near F_t. Since this component has multiplicity $\gcd\{m_i\} = 1$, we can slide γ across a suitable linear combination of disks D_i to move $\pi \circ \gamma$ past t as required. Lemma 7.4.15 follows immediately from the proposition once we observe that the given map $\pi\colon X \to \mathbb{CP}^1$ satisfies the required hypotheses. (The fact that all fibers are connected, hence all connected components have multiplicity 1, follows from the existence of one such fiber with some $m_i = 1$, together with componentwise surjectivity of f. To verify the latter, let F' be a component of F_{t_0} and write $[F'] = \sum k_i[\Sigma_i]$, with each $k_i > 0$ and i ranging over suitable values. For each Σ_j not in the sum, we have $0 = F' \cdot \Sigma_j = \sum k_i \Sigma_i \cdot \Sigma_j$. Nonnegativity of holomorphic intersections implies $\Sigma_i \cdot \Sigma_j = 0$ for each $\Sigma_i \subset f(F')$. Thus, any Σ_j on which $[F']$ has coefficient 0 is completely disjoint from $f(F')$, as required.) □

Solution of Exercise 8.2.4: Before the $-\frac{1}{n}$-surgery we see the obvious T^2-fibration of $S^1 \times T^2$. The surgery circle C lies in a single fiber F. Let $A \subset F$ be an annulus neighborhood of C, and let T be a solid torus in $S^1 \times T^2$ with $T \cap F = A \subset \partial T$ and T lying behind A in the picture. To perform the $-\frac{1}{n}$-surgery, we cut out int T and glue T back in by a map that is the identity on $\partial T - A$. This map must be an n-fold Dehn twist on A, so the surgery is equivalent to changing the monodromy of the (trivial) T^2-bundle from id_{T^2} to ψ^n. (Check the orientation.) The case of a general Lefschetz singular fiber is similar (with $n = 1$). □

Solution of Exercise 8.2.6: To visualize the monodromy around ∂D in a Kirby diagram, start with a fiber F behind the vanishing cycles and pull it forward through the surgeries, accumulating Dehn twists as in the previous exercise. Pulling F through the point at infinity returns it to its original position, but modified by the self-diffeomorphism $\psi_1 * \cdots * \psi_n$. To realize a cyclic permutation, pull the fiber containing the front vanishing cycle C through infinity so that C moves to the back of the stack. To return F to its desired position behind all vanishing cycles, we must push F through C, acting on it by ψ_n^{-1}. (This is the required change in φ.) □

Solution of Exercise 8.2.7(a): Figure 8.7 is equivalent to sliding h_i over h_{i+1} as in Figure 12.63. Note that the attaching circle is changed by the

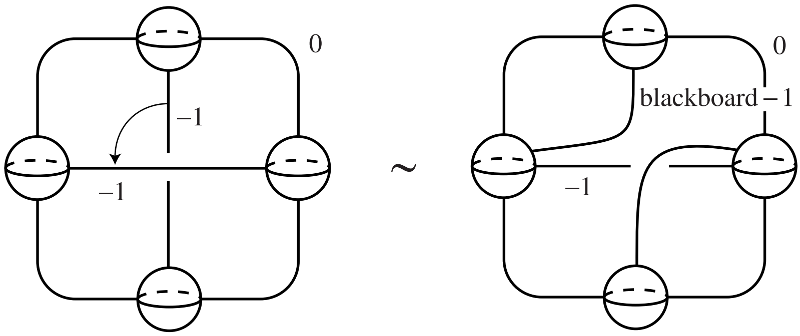

Figure 12.63. Elementary transformation.

monodromy as required. To understand this in general, recall the relation between the fibration and the surgeries, Exercise 8.2.4. □

Solution of Exercise 8.2.7(b): Move ψ_n into the first slot by $n-1$ elementary transformations (pushing it back one slot at a time), obtaining a cyclic permutation composed with the inner automorphism $\psi \mapsto \psi_n^{-1} * \psi * \psi_n$. Alternatively, push ψ_1 forward to the last slot, allowing it to conjugate at each stage, obtaining $(\psi_2, \ldots, \psi_n, (\psi_2 * \cdots * \psi_n)^{-1} * \psi_1 * (\psi_2 * \cdots * \psi_n))$. When $\psi_1 * \cdots * \psi_n = \mathrm{id}_F$, this is a cyclic permutation. □

Solution of Exercise 8.2.7(c): First note that we can realize cyclic permutations $A_1, \ldots, A_n$ as in the previous solution. Thus, we can assume the indices in Figure 8.7 are in $\mathbb{Z}_n$ (allowing $i = n$). Now apply the isotopy that turns the radial picture of $\{A_i'\}$ back into the original picture of it. Each time two endpoints t_i cross (as measured by radially projecting to ∂D^2), apply a move to keep the arcs radial. At the end, the endpoints are in their original positions (agreeing with those of $\{A_i\}$), but our chosen sequence of moves has kept the arcs radial, so the final picture is precisely $\{A_i\}$. □

Solution of Exercise 8.2.10: X splits as a nontrivial fiber sum if and only if the corresponding ordered collection of monodromies can be written (after elementary transformations) in the form $(\psi_1, \ldots, \psi_n)$ with $\psi_1 * \cdots * \psi_m$ isotopic to the identity for some positive $m < n$. Then $X = X_1 \#_f X_2$, where X_1 and X_2 are determined by $(\psi_1, \ldots, \psi_m)$ and $(\psi_{m+1}, \ldots, \psi_n)$, respectively, and the given fiber sum is formed using the identity map on a regular fiber. □

Solution of Exercise 8.3.4(a): By Lemma 8.3.3, $M_c(2,3,2m) \approx Y_0(2m, 0) \approx Y_0(4, m-2)$. Draw the latter, and as in the proof of that lemma (Figure 8.19) slide the long 1 over the outer left -1 and "blow down" the remaining -1 of the right ring. After cancelling the left 1-handle, "blow down" the remaining -1 to obtain Figure 8.16. □

Solution of Exercise 8.3.4(b): Follow the (alternate) solution of Exercise 5.1.12(a), replacing the $+1$-framed circle in Figure 12.9(a) by a dotted circle with a -1-framed meridian, and replacing blowing down by handle sliding as in the proof of Theorem 8.3.2. □

Solution of Exercise 8.3.4(c): Comparison of the two figures suggests that we try to slide the 0-framed 2-handle in Figure 8.16 so that it becomes a -1-framed meridian of the dotted circle. We can either find the slide by experimentation, using the linking matrix as a guide (Figure 12.64), or show that the two attaching circles are isotopic in the boundary of the union of the remaining handles. For the latter argument, it is routine to verify that the 3-manifold is -2-surgery on a left trefoil, and the attaching circles in question are -1-framed meridians. □

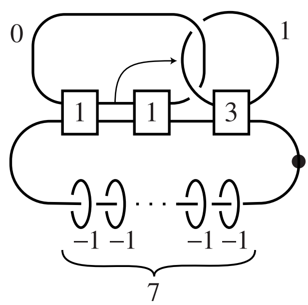

Figure 12.64

Solution of Exercise 8.3.16(a): In Figure 8.25(a), let λ and μ denote the homology classes in $\partial\nu T$ given by the bottom and middle arcs of H, respectively, oriented counterclockwise. The diffeomorphism $\varphi^{-1}\colon \partial\nu T \to \partial(T^2 \times D^2)$ given by Figure 8.26 sends $\mu \mapsto \lambda$, $\lambda \mapsto p\lambda - \mu$, so we have $\varphi_*(\mu) = p\mu - \lambda$. This is the class α from the definition of auxiliary multiplicity, and μ is given to be a positively oriented meridian of T, so (for $p \geq 0$) the auxiliary multiplicity is 1 and direction is $-\lambda$, as required. □

Solution of Exercise 8.3.16(b): The self-diffeomorphism of Q is given by 180° rotation in the plane of the paper on the left side of Figure 8.8 or about the z-axis in Figure 8.27(a). (In the latter case, look closely at the clasps to verify that the rotation preserves the link and orientation on T.) To see a diffeomorphism $Q_p \approx Q_{-p}$ in Figure 8.27(b), slide the $(p-1)$-framed 2-handle across the dotted circle parallel to it, then rotate the figure about the z-axis. (An easy way to deal with the p twists during the handle slide is to draw the parallel push-off of the dotted circle so that it runs through the box denoting p twists, then remove the extra twists between the dotted circle and its parallel by adding $-p$ twists elsewhere (Figure 12.65). When the 2-handle is band-summed into the push-off, it will immediately pull out of the p twists, leaving $-p$ twists between it and the dotted circle.) □

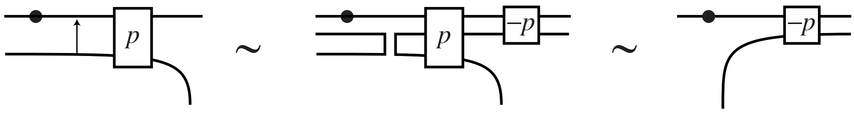

Figure 12.65. A trick for dealing with twists during handle slides.

Solution of Exercise 8.3.16(d): The $6n$ spheres with square -1 in $E(n)_0$ (Exercise 8.3.15) are visible in Figure 8.24 ($p = 0$) when we slide the $6n$ vertical -1-framed arcs over the 0-framed vertical arc. These spheres do not intersect int $\nu F'$, where $\nu F' = D^4 \cup 1\text{-handles} \cup 2\text{-handle}$ is the obvious neighborhood of a regular fiber of $E(n) - \text{int}\ \nu F \subset E(n)_0$. If we perform the remaining logarithmic transformations inside int $\nu F'$, the $\overline{\mathbb{CP}^2}$ summands will remain. Similarly, $N(n)_0$ (Figure 8.29, $p = 0$) has an obvious sphere with square -1. This is disjoint from the fibers on which we perform the remaining logarithmic transformations, since the latter are visible in (a) of the figure as spanning disks of the dotted circle, surgered to avoid the 0-framed 2-handle (cf. Figure 8.25). □

Solution of Exercise 8.4.2(c): See Figure 12.66. Cancelling 1-handles gives (b), which is isotopic to (c). (Grab the small writhes of the inner $-m$-framed circle and lift them toward the top of the picture, unwinding the -1-twists to obtain the top strand of (c).) Slide over each -2-framed handle as in (d) to obtain (e). The indicated slide yields Figure 8.35. The homology class of the fiber is given by the sum of the 0-framed 2-handles in (a), and is straightforward to follow (cf. Section 5.1). To construct $X(m,n)$ as a branched cover, we first blow up $2n$ fibers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ $2m$ times each. The total transforms of these fibers lift to the required singular fibers of $X(m,n)$ (cf. the solution of Exercise 7.4.16). □

Solution of Exercise 8.4.15(a): Examples 8.4.7 and 8.4.9 are the unique achiral Lefschetz fibration and pencil, respectively, on S^4, and no other homology 4-spheres X admit such structures: By Lemma 8.4.12, $q = 1$ for both choices of orientation of X. Thus, there are exactly two points of X that are not regular points of π, and their charts are oppositely oriented. If both

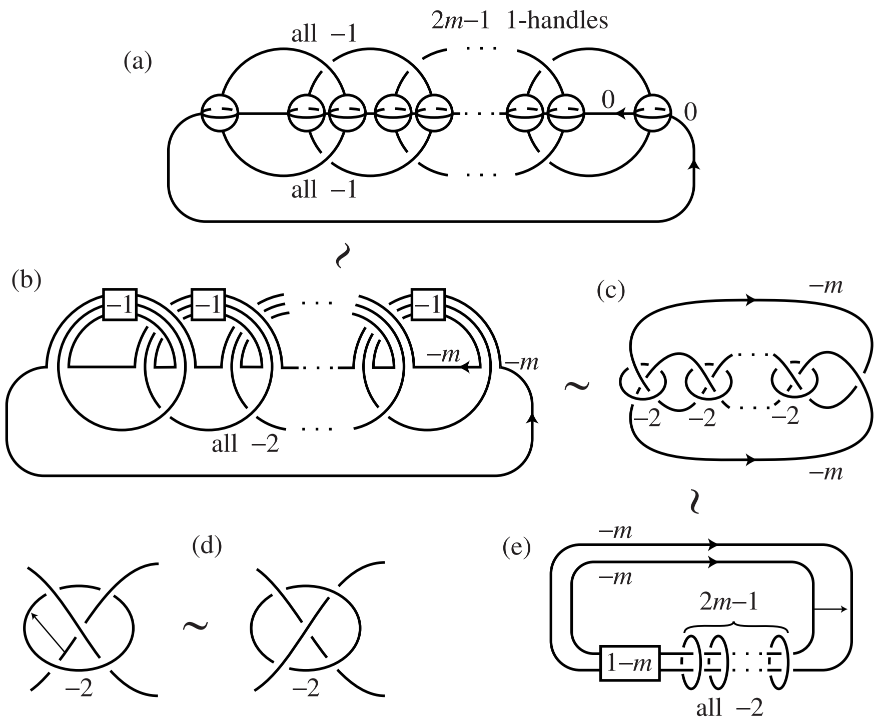

Figure 12.66. Singular fiber of $X(m,n)$.

points are in B, then blowing up gives an S^2-bundle $X\#\mathbb{CP}^2\#\overline{\mathbb{CP}^2} \to S^2$ which must be the usual S^2-bundle with $X = S^4$. If only one point is in B, we suitably orient X and blow up to obtain a Lefschetz fibration on $X\#\mathbb{CP}^2$ with exactly one critical point. Then $3 = \chi(X\#\mathbb{CP}^2) = 2\chi(F)+1$, contradicting orientability of the fibers. If $B = \emptyset$ then we have an achiral Lefschetz fibration on X with two oppositely oriented critical points. Since $2 = \chi(X) = 2\chi(F) + 2$, the genus must be 1. Since the monodromies must cancel, the vanishing cycles must be parallel, so the fibration differs from Example 8.4.7 by at most the choice of the multiplicity-1 logarithmic transformation. Its direction and one vanishing cycle must form a basis of $H_1(F)$ (where F is the regular fiber over the base point in X), since $H_1(X;\mathbb{Z}) = 0$. Identify F with $S^1 \times S^1$ so that this basis is standard. The result is Example 8.4.7. Note that there are achiral Lefschetz fibrations on rational homology spheres with $H_1(X;\mathbb{Z}) \cong \mathbb{Z}_p$ for any $p > 0$. □

Solution of Exercise 8.4.15(b): Since $T(X - P) \cong L \oplus \pi^*T\Sigma$, we have $c_1(X,J) = c_1(L) + \pi^*c_1(T\Sigma)$, and $c_1(T\Sigma) = \chi(\Sigma)[\Sigma]$. For a regular fiber F of an achiral Lefschetz fibration we have $L|F \cong TF$, so $\langle c_1(L), F\rangle = \langle c_1(TF), F\rangle = \chi(F)$. If F is nullhomologous, this must vanish. □

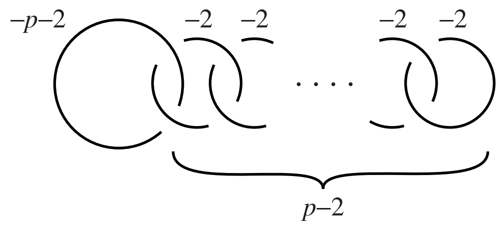

Figure 12.67. The plumbing C_p.

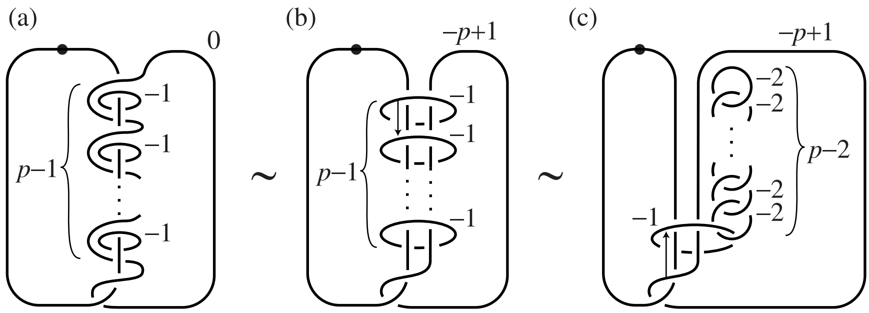

Figure 12.68. Alternative Kirby diagrams for C_p.

Solution of Exercise 8.5.1(a): Based on the plumbing diagram of Figure 8.39, the obvious Kirby diagram for C_p is given by Figure 12.67. To see that this is the same as Figure 8.40, draw the latter as in Figure 12.68(a), slide the 0-framed 2-handle over the $p-1$ meridians to get (b), then slide the −1-framed curves over each other as indicated (from the top down) to get (c). Cancelling the 1-handle yields Figure 12.67. □

Solution of Exercise 8.5.1(b): This is clear from Figure 8.40 by adding handles to cancel the 1-handle and 0-framed 2-handle. For a more algebraic geometric argument, fix a smooth quadric $s_1 \subset \overline{\mathbb{CP}^2}$ (with $[s_1]^2 = -4$) and a line $s_2 \subset \overline{\mathbb{CP}^2}$ intersecting it transversally twice. (Note that both s_1 and s_2 are spheres and $[s_2]^2 = -1$.) Orient s_1 and s_2 in such a way that the two intersections are positive. The tubular neighborhood of s_1 provides a copy of C_2 in $\overline{\mathbb{CP}^2}$. If we blow up one of the intersection points of $s_1 \cap s_2$, the proper transforms give rise to $C_3 \subset \#2\overline{\mathbb{CP}^2}$. The exceptional sphere of the blow-up can be oriented to give positive intersections with both proper transforms. Blowing up the intersection of the exceptional sphere and the proper transform of s_1, we get $C_4 \subset \#3\overline{\mathbb{CP}^2}$. Repeated blow-ups of the intersection of the last exceptional sphere with the proper transform of s_1 give the desired configuration of spheres in $\#(p-1)\overline{\mathbb{CP}^2}$. □

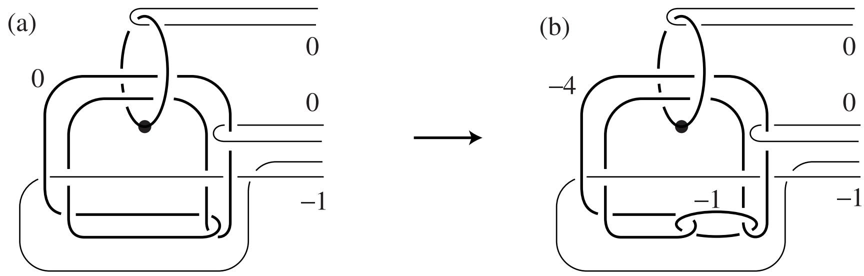

Figure 12.69. Blowing up a fishtail neighborhood.

Solution of Exercise 8.5.8(a): To exhibit the immersed sphere Σ in Figure 8.27(a), we cancel the lower 1-handle, obtaining (a) of Figure 12.69, where we have drawn the 2-handle with an extra half-twist to exhibit the self-intersection as a right clasp. Now blowing up the intersection yields (b) (note the sphere $\widetilde{\Sigma}$ with square -4), and additional blow-ups at the clasp on the left yield Figure 8.44(a). The construction exhibits the required embedding of C_p as the obvious sublink realizing Figure 12.67. □

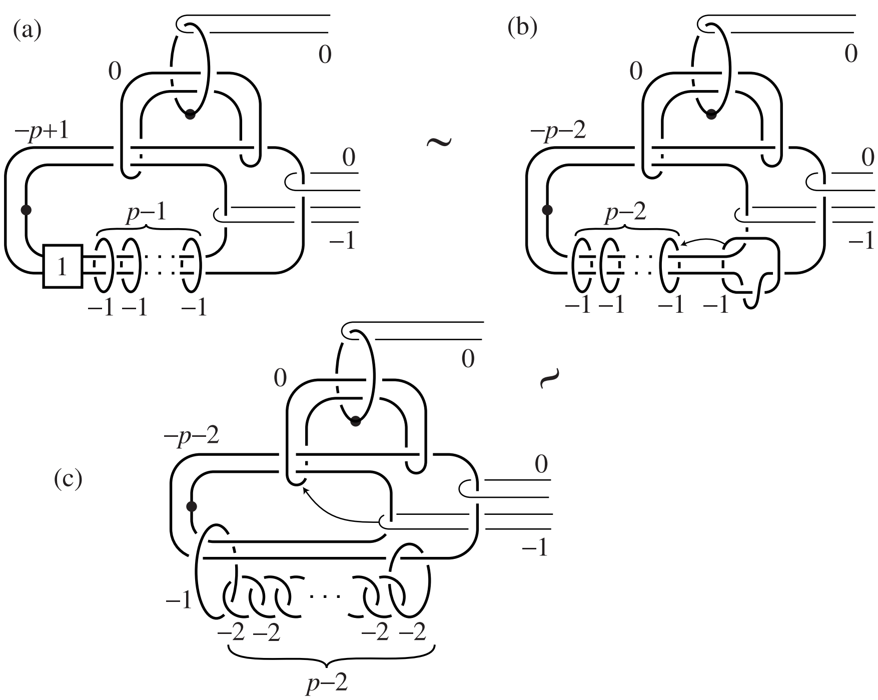

Figure 12.70. Identification of C_p inside the blown-up fishtail neighborhood $Q\#(p-1)\overline{\mathbb{CP}^2}$.

Solution of Exercise 8.5.8(b): Sliding over the -1-framed meridians as before changes Figure 8.44(b) to (a) of Figure 12.70; one more slide yields (b). Sliding -1-framed circles over each other as before results in (c). (If $p = 2$, interpret the chain of 2-handles at the bottom of (c) as a single -1-framed handle linking the $(-p-2)$-framed circle twice, and note that (b) and (c) are isotopic.) We slide the fine -1-framed curve as shown (which is easier if we pull the relevant 2-handle out of the upper dotted circle first, dragging the $(-p-2)$-framed circle). Cancelling the lower 1-handle against the -1-framed 2-handle results in Figure 8.44(a). Note that we never slid a handle across the upper 0-framed 2-handle or dotted circle. Thus, we can remove these two handles (and the fine curves) and verify that the remaining handlebody (which in Figure 8.44(b) is clearly equivalent to Figure 8.40) is sent to the copy of C_p in Figure 8.44(a) identified in the previous solution. □

Solution of Exercise 8.5.19(b): Recall that $\mathcal{B}as_{E(4)} = \{0, \pm PD(2f)\}$ where f denotes the homology class of the fiber of the elliptic fibration. Since C_2 is tautly embedded in $E(4)$, Theorem 8.5.18 implies $\mathcal{B}as_{P_1} = \{\pm PD(2f + \frac{\sigma_1}{2})\}$. (Here σ_1 stands for the homology class of the given section, which is primitive in $H_2(E(4);\mathbb{Z})$ but is divisible by 2 in $H_2(P_1;\mathbb{Z})$.) Since further copies of C_2 given by the sections $\sigma_2, \dots, \sigma_9$ are also tautly embedded, induction gives that $\mathcal{B}as_{P_j} = \{\pm PD(2f + \frac{1}{2}(\sigma_1 + \dots + \sigma_j))\}$. Since SW_{P_j} is not identically zero, P_j cannot be decomposed as $P_j = X_1 \# X_2$ with $b_2^+(X_i) > 0$, cf. Theorem 2.4.6. If $P_j = X_1 \# X_2$ with $b_2^+(X_2) = 0$, then by the blow-up formula, $b_2(X_2) = 1$ and $PD(2f + \frac{1}{2}(\sigma_1 + \dots + \sigma_j)) = E$ with $E^2 = -1$. Since $PD(2f + \frac{1}{2}(\sigma_1 + \dots + \sigma_j))^2 = j$, we get a contradiction, proving the irreducibility of P_j. □

Solution of Exercise 8.5.19(d): Apply the algorithm for rationally blowing down ($p = 2$) as in Figure 12.71, interchanging the 0-framed and dotted circles and blowing down the -1-framed meridian to replace C_2 by B_2. Cancelling the dotted circle yields the final picture. □

Solution of Exercise 9.1.20: If $\partial X \subset S^4$, then $S^4 = A \cup_{\partial X} B$, and since ∂X is a homology sphere, $Q_A = Q_B = Q_{S^4}$ is trivial. Hence $Y = X \cup_{\partial X} B$ is a closed 4-manifold with $Q_Y = Q_X$, contradicting Donaldson's Theorem 1.2.30. □

Solution of Exercise 9.3.5: Surgering out the sphere A corresponds to putting a dot on one of the heavy 0-framed circles in Figure 9.6; surgering out B puts a dot on the other circle instead. The resulting manifolds are obviously diffeomorphic to each other via 180° rotation. If we then cancel the two

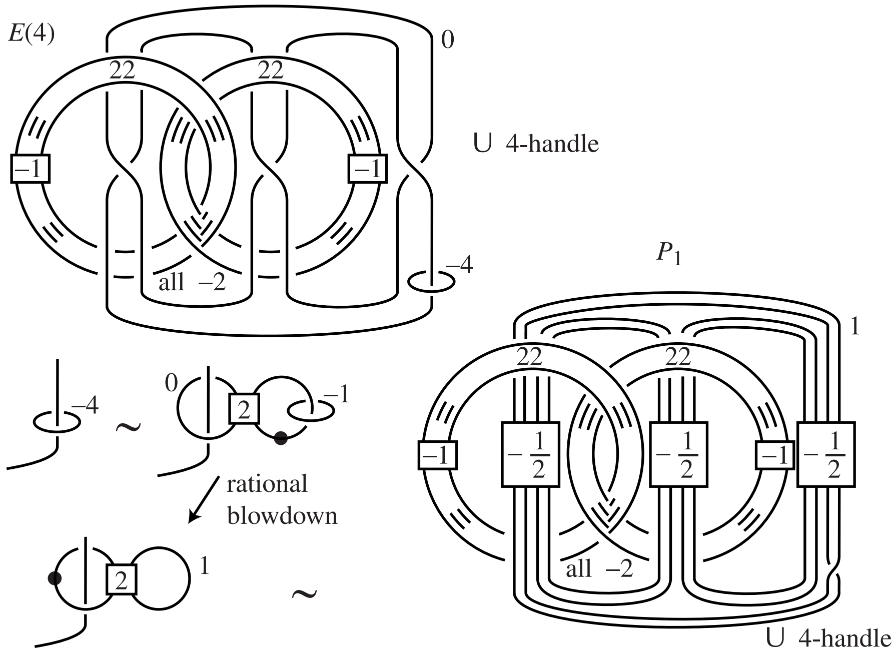

Figure 12.71. Rationally blowing down a section of $E(4)$ to obtain P_1, an irreducible, noncomplex, symplectic manifold.

1-handles in Figure 9.6 against the fine 2-handles, we obtain Figure 9.7. (An easy way to do this is to begin by sliding the lower dotted circle under the upper one so that it encircles the middle clasp. The fine 0-framed circle then cancels the upper dotted circle, and cancelling the -1-framed circle against the other dotted circle obviously gives Figure 9.7.) Now an isotopy (ignoring the curve δ) yields Figure 9.5(b). Clearly, 180° rotation in either Figure 9.6 or 9.7 is an involution of ∂Y that interchanges the roles of the two circles in Figure 9.5, so it realizes the required diffeomorphism φ. □

Solution of Exercise 9.4.1(a): The given compact handlebody X would be contractible. Since $\mathbb{R}^4$ is simply connected at infinity (as we see by taking D to be a topological ball in the definition, Exercise 6.1.4(c)), it is easy to see that ∂X would be simply connected. Thus, either ∂X would be a counterexample to the 3-dimensional Poincaré Conjecture or it would be diffeomorphic to S^3, in which case we could add a 4-handle to X, obtaining a homotopy 4-sphere X^* (which would be homeomorphic to S^4 by Freedman's Theorem). This could not be diffeomorphic to S^4, for if it were, then int $X = X^* - D^4 \approx S^4 - D^4$ would be diffeomorphic to $\mathbb{R}^4$. □

Solution of Exercise 9.4.1(b): Since any two orientation-preserving embeddings $D^4 \hookrightarrow$ int D^4 are isotopic, each $D_{i+1} -$ int D_i is diffeomorphic to

$I \times S^3$. The diffeomorphism $X \approx \mathbb{R}^4$ is now easy to construct by sending D_i to the round ball of radius i in $\mathbb{R}^4$. □

Solution of Exercise 9.4.8(a): First, consider $X_1 \natural X_2$. Remove all of each D_i from $X_1 \natural X_2$ except for the smooth neighborhoods of p_i in ∂D_i. The resulting complement Y is a smooth manifold with two boundary components, and is easily seen to be connected. Find an arc γ in Y connecting p_1 to p_2 and disjoint from ∂Y except for transverse intersections at the endpoints, and verify that $R_1 \cup R_2 \cup \nu\gamma \subset X_1 \natural X_2$ is diffeomorphic to $R_1 \natural R_2$. To see the difficulty with general embeddings $R_i \hookrightarrow X_i$, consider the 2-dimensional case for $X_1 = \mathbb{R}^2$ and R_1 given by the open unit disk minus a spiral converging toward the unit circle. To avoid such bad embeddings when proving $R_1 \natural R_2 \hookrightarrow X_1 \# X_2$, remove a closed tubular neighborhood of a ray (properly embedded in R_i) from each R_i. The diffeomorphism type of R_i is unaffected, but now each X_i has a coordinate chart in which R_i appears as the open upper half-space in $\mathbb{R}^4$. Form $X_1 \# X_2$ using a ball in the lower half-space of each chart. Then it is easy to connect R_1 and R_2 by an arc in $X_1 \# X_2$. □

Solution of Exercise 9.4.8(b): It is not hard to prove that for X noncompact and connected at infinity (e.g., X homeomorphic to $\mathbb{R}^4$), $X \# \infty \mathbb{CP}^2$ is well-defined. (There is a unique ambient isotopy class of proper embeddings $\mathbb{N} \hookrightarrow X$, hence, of proper, orientation-preserving embeddings $\coprod \infty D^4 \hookrightarrow X$.) Thus, we obtain a map called $\mathbb{CP}^2$-*stabilization* from $\mathcal{R}$ onto a smaller set, sending each X to $X \# \infty \mathbb{CP}^2$, and similarly a $\overline{\mathbb{CP}^2}$-stabilization map. Let $Y_k \subset R$ be the compact submanifold obtained from Figure 6.16 by cutting off all of the Casson handle CH_+ except its bottom k-stage tower T_k, and with the handles thinned so that $Y_k \subset \operatorname{int} Y_{k+1}$ and $\bigcup_{k=0}^{\infty} Y_k = R$. By blowing up $\mathbb{CP}^2$ in $Y_{k+1} - Y_k$, we create an embedded disk D attached to T_k that turns T_k into a 2-handle (cf. Figures 6.16 and 6.10). Thus, we find a 4-ball $D_k = Y_k \cup \nu D$ with $Y_k \subset D_k \subset Y_{k+1} \# \mathbb{CP}^2$. The method of Exercise 9.4.1(b) now shows that $R \# \infty \mathbb{CP}^2 \approx \mathbb{R}^4 \# \infty \mathbb{CP}^2$. On the other hand, $R \# \infty \overline{\mathbb{CP}^2}$ is not diffeomorphic to $\mathbb{R}^4 \# \infty \overline{\mathbb{CP}^2}$, as we can see by the proof of Theorem 9.3.8 modified to allow blow-ups whenever necessary. (Use the fact that $E(n) \# k \overline{\mathbb{CP}^2}$ has nontrivial Seiberg-Witten invariants for all $n \geq 2$.) Thus, $\mathbb{CP}^2$- and $\overline{\mathbb{CP}^2}$-stabilization together distinguish $\mathbb{R}^4$, R, $\overline{R}$ and $R \natural \overline{R}$. See [**BG**] Proposition 5.4 for more details. □

Solution of Exercise 9.4.8(c): Given such an embedding, remove both copies of $\operatorname{int} K$ from $L \subset \# 3 S^2 \times S^2$ and glue in two copies of X, to obtain a closed manifold with intersection form $-4E_8 \oplus 3H$, contradicting Furuta's Theorem. Then note that any infinite group of covering translations would contain an element moving the compact subset K' off of itself. □

Solution of Exercise 9.4.11(a): Such a diffeomorphism of ends would allow us to construct a periodic end as in the proof of Theorem 9.4.10, contradicting Taubes. Since $U \cup L = \mathbb{CP}^2$, we can define a radial family $U_t \subset U$ by $U_t = \mathbb{CP}^2 - c\ell(L_t)$, and the same argument applies. □

Solution of Exercise 9.4.11(b): There is no ψ turning U inside out; otherwise we could glue the region inside $\psi(U)$ to X (with end suitably trimmed) to construct a closed, smooth, negative definite manifold contradicting Donaldson's Theorem. We cannot have U and V' concentric by the proof of Theorem 9.4.10. Finally, U and V' cannot be disjoint and nonconcentric by the method of proof of Exercise 9.4.8(c). If there were a smooth isotopy of ∂L_q off of itself, then for suitable s, t' with $r \leq s < q < t'$ we could construct such a map ψ. □

Solution of Exercise 9.4.13(b): A diffeomorphism rel ∂ between the Casson handles of Theorem 9.4.12 would result in a diffeomorphism between the corresponding manifolds R_t. But there are only countably many self-diffeomorphisms of $\partial(CH) \approx S^1 \times \mathbb{R}^2$, since these are determined by their restriction to $S^1 \times D^2$. Alternatively, use the manifolds U_t of the solution of Exercise 9.4.11(a), which by Freedman theory can be assumed to have the form int $D^4 \cup CH$ for t in a Cantor set. □

Solution of Exercise 9.4.15(b): Any compact submanifold of $R_1 \natural R_3$ can be enlarged to one of the form $K_1 \natural K_3$ (boundary sum) with $K_i \subset R_i$ and ∂K_i connected. Embed $K_i \hookrightarrow R_{i+1} \subset R_2 \natural R_4$ and connect these by an arc. □

Solution of Exercise 9.4.20(a): Build R one handle at a time. After all 3-handles have been added, we have no way to kill H_2, so each subsequent handlebody K_n must have $b_2(K_n) = 0$ (since $b_2(R) = 0$). Now standard algebraic topology (e.g. the exact homology sequence of $(I \times K_n, \partial)$ and Poincaré duality) shows that $H_2(DK_n; \mathbb{Q}) = 0$, so we can set $X = DK_n$ in the definition of γ for each n. □

Solution of Exercise 9.4.20(b): Fix a diffeomorphism $\varphi\colon R_1 -$ int $K_1 \to R_2 -$ int K_2. To show that $\gamma(R_1) \leq \gamma(R_2)$, pick a compact $K_1' \subset R_1$. Let $K_2' \subset R_2$ be a compact submanifold containing $\varphi(K_1' -$ int $K_1)$ and a topological 4-ball D containing K_2. Now K_2' embeds in a spin 4-manifold X_2 with $\sigma(X_2) = 0$ and $\frac{1}{2}b_2(X_2) \leq \gamma(R_2)$. Cutting D out of X_2 and replacing it by a suitable ball in R_1 (glued in by φ), we obtain a spin manifold X_1 with $K_1' \subset X_1$, $\sigma(X_1) = 0$ and $\frac{1}{2}b_2(X_1) = \frac{1}{2}b_2(X_2) \leq \gamma(R_2)$. Thus $\gamma(R_1) \leq \gamma(R_2)$; interchanging the roles of R_1 and R_2 completes the solution. □

Solution of Exercise 9.4.20(c): The first inequality is because each $R_m \leq \natural_n R_n$. For the second, note that any compact submanifold of $\natural_n R_n$ is contained in one of the form $\natural_n K_n$, $K_n \subset R_n$ (and ∂K_n connected). By compactness, the boundary sum is finite. Embed $K_n \subset X_n$ with $\frac{1}{2}b_2(X_n) \leq \gamma(R_n)$. Then $\natural_n K_n \subset \#_n X_n$ and $\frac{1}{2}b_2(\#_n X_n) = \sum_n \frac{1}{2}b_2(X_n) \leq \sum_n \gamma(R_n)$. □

Solution of Exercise 9.4.20(d): Let L be as in Remark 9.4.5. By connecting the punctures of the associated $-2E_8$-manifold by smooth curves and deleting these, we obtain a smooth manifold with form $-2E_8$ and a connected end. Check that this end is diffeomorphic to that of $\natural 3L$. The proof of Corollary 9.4.7 shows that $\gamma(\natural nL)$ increases without bound as $n \to \infty$, so $\gamma(\natural \infty L) = \infty$. However, $\gamma(L) = 1$, so the previous exercise shows that $\gamma(\natural(n+1)L) = \gamma(\natural nL)$ or $\gamma(\natural nL) + 1$. Thus, $\gamma(\natural nL)$ must range over all nonnegative integers as $n \geq 0$ increases. □

Solution of Exercise 9.4.20(f): By (d) above, for $n \in \mathbb{N}$ we get a large exotic $\mathbb{R}^4$ with $\gamma(L) = n$. Let $L_t \subset L$ be a radial family. By (e) we must have $\gamma(L_t) = n$ for all sufficiently large t, so $\gamma^{-1}(n) \subset \mathcal{R}_\sim$ is uncountable by Theorem 9.4.10. (For $n = 1, 2$, construct a definite manifold whose end consists of three periodic components, and merge these as in (d).) Given $L' \subset \mathbb{CP}^2$ with $\gamma(L') = \infty$, let L_t be any radial family as in Theorem 9.4.16; that proof shows that the exotic $\mathbb{R}^4$'s $L' \natural \overline{L}_t$ realize uncountably many compact equivalence classes. But $\gamma(L' \natural \overline{L}_t) \geq \gamma(L') = \infty$. □

Solution of Exercise 9.4.20(g): If $R \subset X$ with X closed and spin, then there is a point $p \in X - R$. Since $X - \{p\} \approx X - D^4$ we can assume R is disjoint from a 4-ball in X. After summing with copies of the $K3$-surface we can assume $\sigma(X) = 0$; this contradicts the assumption $\gamma(R) = \infty$. If X is compact and spin with $\partial X \neq \emptyset$, then $R \subset DX = \partial(I \times X)$ and DX is closed and spin with $\sigma(DX) = 0$. If R embeds in any 4-manifold with $c\ell(R)$ a flat 4-ball, then a neighborhood U of $c\ell(R)$ is homeomorphic to $\mathbb{R}^4$, and as in the proof of Theorem 9.4.3 we can find a smooth, compact $K \subset U$ containing $c\ell(R)$. This K will be spin since U is. □

Solution of Exercise 9.4.23: For any $K \subset S^3$, X_K has a smooth (resp. flat topological) embedding in $\mathbb{R}^4$ iff K is smoothly (resp. topologically) slice. (Given an embedding $\varphi\colon X_K \hookrightarrow S^4$, the core of the 2-handle is a slice disk for K in $S^4 -$ int $\varphi(D^4) \approx D^4$.) For K topologically slice, fix φ and note that the topological manifold $\mathbb{R}^4 -$ int $\varphi(X_K)$ can be smoothed by Theorem 9.4.22. By uniqueness of smoothings on 3-manifolds, the homeomorphism $\varphi|\partial X_K$ is isotopic to a diffeomorphism, so the smooth structures on X_K and $\mathbb{R}^4 -$ int $\varphi(X_K)$ fit together to give a smoothing R of $\mathbb{R}^4$. If K

is also not smoothly slice, then X_K smoothly embeds in R but not in $\mathbb{R}^4$, so R is a large exotic $\mathbb{R}^4$. □

Solution of Exercise 10.1.20(a): Ambiently surgering Σ and applying the generalized adjunction formula 2.4.8 to the resulting -1-torus, we get $c_1(X,\omega)\cdot PD([\Sigma]) = \pm 1$. Hence the appropriate orientation of Σ provides the class e for which $c_1(X,\omega)\cdot PD(e) = 1$. Now the blow-up formula implies that $c_1(X,\omega) + 2PD(e) \in \mathcal{B}as_X$, concluding the solution. □

Solution of Exercise 10.1.20(b): Suppose that X is nonminimal, i.e., X is diffeomorphic to $Y\#\overline{\mathbb{CP}^2}$ for some 4-manifold Y. By the blow-up formula we know that there exists a basic class $L \in \mathcal{B}as_Y$ satisfying $c_1(X,\omega) = L - E$. Appealing to the blow-up formula again, we have $K = L + E$ is in $\mathcal{B}as_X$; now the relation $(c_1(X,\omega) - K)^2 = -4$ follows from the fact that $E^2 = -1$. For the converse direction, assume that there is a class $K \in \mathcal{B}as_X$ with $(c_1(X,\omega) - K)^2 = -4$. Taking $E = \frac{1}{2}(K - c_1(X,\omega))$, we have $E^2 = -1$; moreover the fact that $K^2 = c_1^2(X,\omega)$ (since a symplectic 4-manifold with $b_2^+ > 1$ has simple type) implies that $c_1(X,\omega)\cdot E = 1$. Now Remark 10.1.16(b) outlines the construction of a -1-sphere, implying that X is nonminimal. □

Solution of Exercise 10.1.20(c): Since $-c_1(X,\omega) \in \mathcal{B}as_X$, Theorem 10.1.15 shows that the Poincaré dual of $-c_1(X,\omega)$ can be represented by a pseudo-holomorphic curve $C = \bigcup C_i$. The adjunction formula now gives $2g(C_i)-2 = 2C_i^2$ for each component, i.e., $c_1^2(X,\omega) = \sum C_i^2 = \sum (g(C_i) - 1)$. Since the assumption $g(C_i) = 0$ would imply $C_i^2 = -1$ (hence the existence of a -1-sphere, contradicting the minimality of X), we conclude that $c_1^2(X,\omega)$ is a sum of nonnegative numbers, completing the solution. □

Solution of Exercise 10.1.20(e): First note that $b_2^+(N) = 0$, then apply Donaldson's Theorem 1.2.30 to diagonalize $H_2(N;\mathbb{Z})$. Using the appropriate form of the blow-up formula (Theorem 2.4.10) and (b) above, we conclude that X is nonminimal unless N is an integral homology sphere. Since $\pi_1(X) = 1$, the manifold N is simply connected; hence Freedman's Theorem 1.2.27 concludes the solution. □

Solution of Exercise 10.2.5(a): Remark 10.1.16(b) shows that if $-c_1(X,\omega)$ is nonzero (and $b_2^+(X) > 1$), then it can be represented by a pseudo-holomorphic (in particular, symplectic) submanifold, hence $-c_1(X,\omega)[\omega]$ is positive. Consequently, if $c_1(X,\omega)[\omega] = 0$ and $b_2^+(X) > 1$, then $c_1(X,\omega) = 0$. Now Theorem 10.1.11 shows that if $K \in \mathcal{B}as_X$, then $K\cdot[\omega] = 0$ since $|K\cdot[\omega]| \leq |c_1(X,\omega)\cdot[\omega]| = 0$, and since the latter is an equality, the same theorem implies that $K = \pm c_1(X, \omega) = 0$. Since $w_2(X)$ is the mod 2 reduction of $c_1(X, \omega)$, it follows that X is spin. □

Solution of Exercise 10.2.19: As we saw in Section 8.1, either a fiber F of the Lefschetz fibration $X \to \Sigma$ admits a decomposition into nonempty closed surfaces $F_0 \cup F_1$ (which happens if the vanishing cycle separates), or F cannot be decomposed in this manner — this is the case for generic fibers and for singular ones with a nonseparating vanishing cycle. (Recall that we assumed π is injective on the set of critical points.) Now by assumption $[F] \neq 0$ in $H_2(X; \mathbb{R})$, hence there is an element $a \in H^2_{dR}(X)$ with $\langle a, [F] \rangle > 0$. Suppose that for a closed surface F_0 contained in F we have $\langle a, [F_0] \rangle = r \leq 0$. Since $\langle a, [F] \rangle = \langle a, [F_0 \cup F_1] \rangle = s > 0$ and $[F_0] \cdot [F_1] = 1$, the cohomology class $a + (-r + \frac{1}{2}s)PD[F_1]$ evaluates positively on both F_0 and F_1. This modification does not affect the value of a on any other fiber, and there are only finitely many singular fibers. Consequently, after finitely many modifications we end up with a cohomology class which evaluates positively on every closed surface contained by a fiber. Representing this cohomology class by a closed 2-form completes the solution. □

Solution of Exercise 10.2.26: Since nondegeneracy is an open condition and since $H^2(X; \mathbb{Q})$ is dense in $H^2(X; \mathbb{R})$, we can find a closed and nondegenerate 2-form ω_1 close to ω such that ω_1 is in $H^2(X; \mathbb{Q})$. Now multiplying ω_1 by the appropriate denominator, we get a symplectic form ω' on X such that ω' is an integral form. □

Solution of Exercise 10.3.7(a): Suppose that X is a simply connected spin 4-manifold with $Q_X = 2kE_8 \oplus lH$. Take the decomposition of $X = X_1 \# \ldots \# X_n$ into irreducible pieces; assume that $Q_{X_i} = 2k_iE_8 \oplus l_iH$. Obviously $|k| \leq \sum |k_i|$ since $E_8 \oplus (-E_8) \cong 8H$; for similar reasons $\sum l_i \leq l$. Since $c_1^2(X_i) = 48k_i + 32|k_i| + 4l_i + 4$, the assumption $c_1^2(X_i) \geq 0$ implies that $12k_i + 8|k_i| + l_i + 1 \geq 0$. The irreducibility of X_i implies the same for $\overline{X}_i$, hence (by reversing the orientation of X_i if necessary) we can assume that $k_i \leq 0$. For $k_i \leq 0$, however, the above inequality gives $l_i + 1 \geq 4|k_i|$, which implies $l_i \geq 3|k_i|$. Putting all these together, we get $3|k| \leq 3\sum |k_i| \leq \sum l_i \leq l$, which shows that the $\frac{11}{8}$-Conjecture is true for X. □

Solution of Exercise 10.3.7(b): Since $\sigma(\overline{X}) = -\sigma(X)$ and $\chi(\overline{X}) = \chi(X)$, we have $c_1^2(\overline{X}) = 3\sigma(\overline{X}) + 2\chi(\overline{X}) = -3\sigma(X) + 2\chi(X) = -3\sigma(X) - 2\chi(X) + 4\chi(X) = 4c_2(X) - c_1^2(X)$. □

Solution of Exercise 10.3.8(a): Let $\Sigma \subset K3$ be the chosen sphere with $[\Sigma]^2 = -2$. Since the tubular neighborhood $\nu\Sigma$ is described by the diagram on the left of Figure 12.28, the 3-manifold $\partial(K3-\nu\Sigma)$ is diffeomorphic to $\mathbb{RP}^3$. Now Figure 12.28 shows the desired orientation-reversing diffeomorphism. Taking (for example) the description of the $K3$-surface as $X(2)$ (from Section 3.2), one can easily see that there is a sphere Σ' such that $\Sigma \cap \Sigma' = \{\text{pt.}\}$. Consequently $K3 - \nu\Sigma$ is simply connected, proving that $\pi_1(K3\#_2K3) = 1$. Now $\chi(K3\#_2K3) = 2\chi(K3 - \nu\Sigma) = 2\chi(K3) - 2\chi(\nu\Sigma) = 44$; since $\sigma(K3 - \nu\Sigma) = -15$, we have that $\sigma(K3\#_2K3) = -30$. This implies that $b_2^+(K3\#_2K3) = 6$, $b_2^-(K3\#_2K3) = 36$ and $c_1^2(K3\#_2K3) = -2$. □

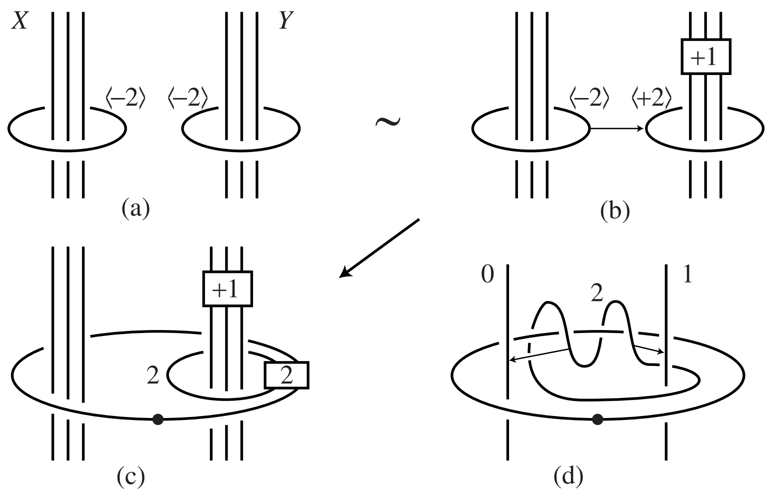

Figure 12.72. Construction of $X\#_2Y$ and $K3\#_2K3$.

Solution of Exercise 10.3.8(b): Let N denote a tubular neighborhood of a -2-framed sphere. One approach to drawing $X\#_2Y$ is to glue $X-\text{int } N$ and $Y-\text{int } N$ onto $I\times\partial N$, where the latter is drawn with $\partial_-(I\times\partial N) = \emptyset$ and the other manifolds have $\partial_+ = \emptyset$. This is equivalent to gluing $X\#Y-\text{int }(N\natural N)$ onto $I \times (\partial N - \text{int } D^3)$ as in Example 5.5.8. The first of these manifolds has a diagram obtained by drawing X and Y in the same picture (to get $X\#Y$) and putting $\langle -2\rangle$-coefficients on both unknots (Figure 12.72(a)). The other piece is $I \times (\mathbb{RP}^3 - \text{int } D^3)$, the D^2-bundle over $\mathbb{RP}^2$ with Euler number 0 (cf. Exercise 4.6.9(d)). To perform the gluing, change one coefficient in Figure 12.72(a) to $\langle +2\rangle$ by blowing up and down as in Figure 12.28 (Figure 12.72(b)), perform the indicated slide and attach the manifold to the D^2-bundle over $\mathbb{RP}^2$ as drawn in Figure 6.2 (Figure 12.72(c)). In the case of $K3\#_2K3$ (where we use the $-n$-framed meridian in Figure 8.15(c) with $n = 2$), the two handle slides shown in Figure 12.72(d), followed by a 1-handle cancellation, yield Figure 10.4. To verify that our gluing map φ was the correct one, it suffices to see that φ identifies the two 2-spheres S, S'

arising in the construction of $X\#Y - \text{int}\,(N\natural N)$ (separating $\partial N\#\partial N$) and $I\times(\partial N - \text{int}\,D^3)$ (namely, $\{\text{pt.}\}\times\partial D^3$). This can be checked explicitly in the diagrams, or we can invoke standard 3-manifold theory. (If $S'\cap\varphi(S)=\emptyset$, then the region between these spheres in $\partial N\#\partial N$ is diffeomorphic to $I\times S^2$ and we are done. Otherwise, some component of the complement in S' of the 1-manifold $S'\cap\varphi(S)$ must be a disk, and this splits $\varphi(S)$ into a pair of embedded spheres in $\partial N\#\partial N$, one of which bounds a copy of D^3. An isotopy of φ now reduces the number of components of $S'\cap\varphi(S)$.) □

Solution of Exercise 11.1.6: Given a generic link projection, the only difficulty is that some crossings may be opposite to the required configuration for a Legendrian link. This can be remedied by the isotopy in Figure 12.73.

□

Figure 12.73. Making a knot in (S^3,ξ_c) Legendrian.

Solution of Exercise 11.2.3(b): Since $\mathbb{CP}^N - H$ is biholomorphic to $\mathbb{C}^N$, the manifold $Y = X - X\cap H$ is a Stein manifold. The basic idea is to turn the handle structure of Y upside down. Since all handles now have index $\geq n$, they will not contribute to low-dimensional homology or homotopy. There are two technical difficulties, however: We do not know that the handle decomposition is finite, and we do not know how the handle structure fits together with a tubular neighborhood of $X\cap H$ in X. (In fact, Y is an affine *algebraic* manifold, so it is collared near infinity and has a finite handle decomposition, but we will not assume this.) To resolve these difficulties, we note that it suffices (via the long exact sequences) to prove $H_i(X, X\cap H) = 0 = \pi_i(X, X\cap H)$ for $i\leq n-1$. Turning Y upside down shows that any element of one of these groups is represented by a cycle (resp. map) in a preassigned tubular neighborhood of $X\cap H$ in X, hence it vanishes. (In fact, transversality of X and H is not really necessary, since even singular projective subvarieties have neighborhoods which deformation retract onto them.) □

Solution of Exercise 11.2.5(a): The case $F = S^2$ is easy. For $F = \#gT^2$, $g>1$, stack copies of Figure 11.7 vertically and band-sum them as in Figure 12.74. For $F=\mathbb{RP}^2$, Figure 4.38(b) gives Figure 12.75. (Note that Figure 4.38(a) is not optimal.) Band-sum these as in Figure 12.76 for the case $F = \#g\mathbb{RP}^2$. (Recall that the framing coefficient will be $e(X) - 2g$ (cf. Section 6.1), so each new summand increases $e(X)$ by 1 as required.) □

Solution of Exercise 11.2.5(b): See Figure 12.77. □

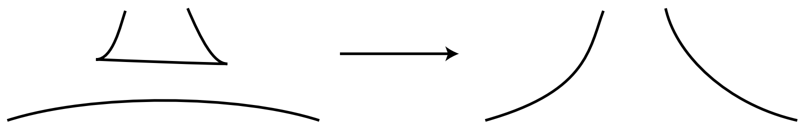

Figure 12.74

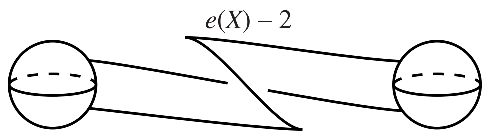

Figure 12.75. Stein structure on a D^2-bundle over $\mathbb{RP}^2$ with $e(X) \leq -1$.

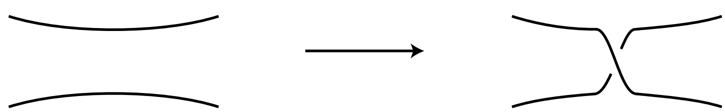

Figure 12.76. Band-summing Legendrian curves.

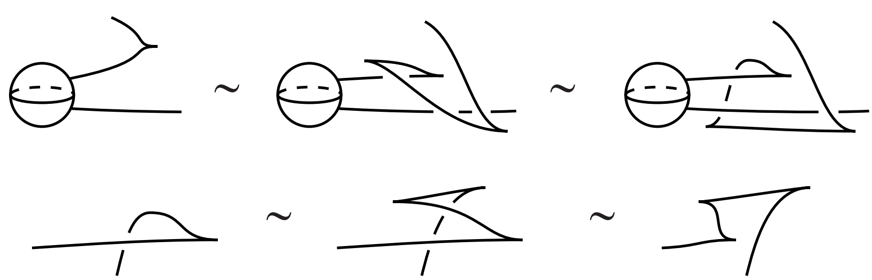

Figure 12.77. A Legendrian isotopy.

Solution of Exercise 11.2.5(c): Create the self-plumbing as in Figure 12.78 and verify that tb increases by ± 2. □

Solution of Exercise 11.2.9: Erase each component with coefficient $-\infty$. Expand each remaining coefficient r as a continued fraction as in the solution of Exercise 5.3.9(b), always rounding down. The resulting integers satisfy $a_n \leq r < tb(K)$ (so $a_n \leq tb(K) - 1$) and $a_i \leq -2$ for $i < n$. The proof is

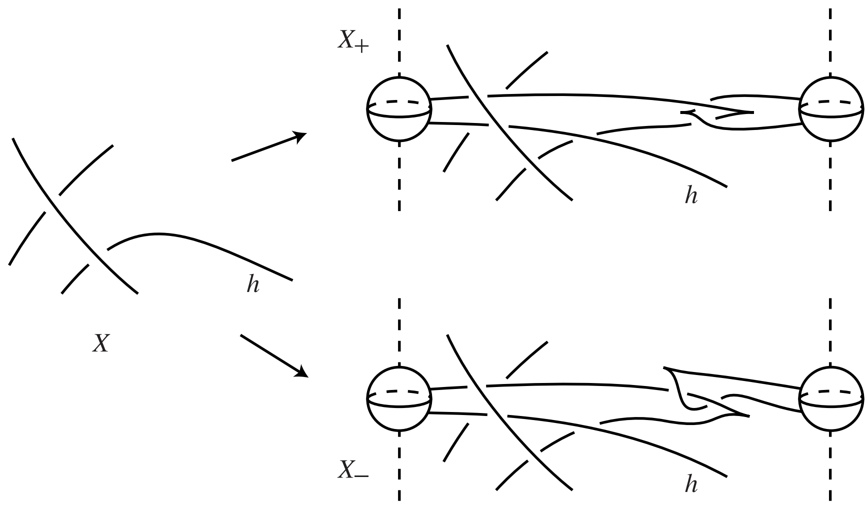

Figure 12.78. Adding a self-plumbing to a 2-handle.

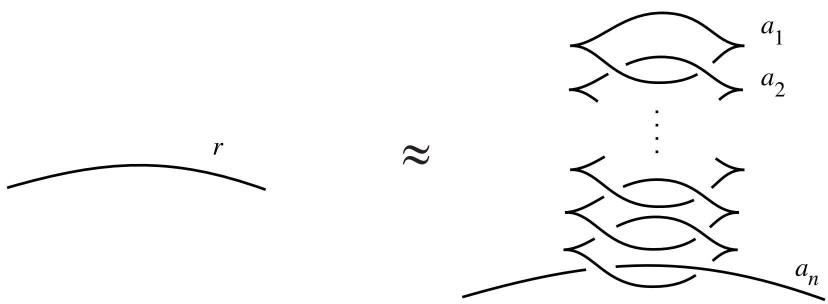

Figure 12.79. Realizing rational surgery diagrams as Stein boundaries.

completed by redrawing Figure 12.13 as in Figure 12.79, and adding zig-zags as necessary. □

Solution of Exercise 11.2.11(a): S^3 and $S^1 \times S^2$ bound D^4 and $D^4 \cup$1-handle, respectively. Any other lens space can be written as $L(p,q)$ with $p > q \geq 1$, or surgery on the unknot with coefficient $-\frac{p}{q} < -1 = tb(K)$, where K is the obvious Legendrian unknot in $\mathbb{R}^3$. □

Solution of Exercise 11.2.11(b): Figure 12.80 shows a Legendrian left trefoil with $tb = -6$, solving the problem for surgery coefficients $r < -6$. For $r = -1$, the manifold bounds the (negative) E_8-plumbing, which is easily described as a Stein surface in standard form. Thus, it seems reasonable to try to generalize the solution of Exercise 5.1.12(a), Figure 12.9. If we change the coefficient on the left trefoil in (h) of that figure to r, we can trace back

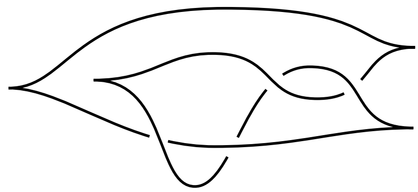

Figure 12.80. Legendrian left trefoil knot.

to (b), where the $+5$ is replaced by $r+6$. Slam-dunking this meridian, we obtain a linear chain of unknots with coefficients -2 and $-1-\frac{1}{r+6}$. This clearly represents a Stein surface provided that $r \geq -6$. □

Solution of Exercise 11.3.2(a): In Exercise 11.2.5(a), we constructed a Stein structure when $e(X) = -\chi(F)$. It is easy to check that the knot K in these diagrams has $r(K) = 0$. For a fixed orientation on K, adding a downward zig-zag increases $r(K)$ by 1, and an upward zig-zag decreases $r(K)$ by 1. Either operation decreases $e(X)$ by 1; the two operations together generate the general case. The second condition holds for all Stein structures since $\langle c_1(X), F\rangle|_2 = \langle w_2(X), F\rangle = e(X)|_2$. When F is nonorientable, the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2)$ is an isomorphism, so $c_1(X)$ is determined by the requirement that $c_1(X)|_2 = w_2(X)$. Thus, Exercise 11.2.5(a) realizes the unique possible Chern class in this case. □

Solution of Exercise 11.3.6(b): Let $n = tb(K) - 1$ be the framing coefficient. For $n \neq 0$, we have $\theta(\xi) = \frac{r(K)^2}{n} - 4 - 3\text{sign}(n)$, for $r(K) = n = 0$ we have $\theta(\xi) = -4$, and $\theta(\xi)$ is undefined otherwise. □

Solution of Exercise 11.3.6(c): Let r_i denote $r(K_i)$. Clearly, $|n_0| = |n_1|$ is the order of $H_1(M_i;\mathbb{Z})$. If $n_i = 0$ the group is $\mathbb{Z}$, $|r_i|$ is the divisibility of $c_1(\xi_i)$, and we are done. Otherwise, $\theta(\xi_0) = \theta(\xi_1)$, and these are given by the previous exercise. It now suffices to show that $n_0 = n_1$. If not, then $n_0 = -n_1$ and the formula $\theta(\xi_0) = \theta(\xi_1)$ shows that $r_0^2 + r_1^2 = 6|n_0|$. Now under the linking pairing, $[\mu_i]^2 \equiv -\frac{1}{n_i} \pmod 1$ (Exercise 5.3.13(g)), but $[\mu_0]^2 = (\varphi_*[\mu_0])^2 = k^2[\mu_1]^2 \pmod 1$, so $k^2 \equiv -1 \pmod{n_0}$. Reducing the equation $r_0^2 + r_1^2 = 6|n_0|$ modulo 3, we see that each r_i is divisible by 3 (since -1 is not a square mod 3), and so n_0 is divisible by 3. But the equation $k^2 \equiv -1 \pmod 3$ has no solutions. □

Solution of Exercise 11.3.14(a): Note that $(\hat{z} + z')^2 = \hat{z}^2 + 2\hat{z}\cdot z'$, and the last term is $2[\partial z]\cdot[z']$ in M, which can be arranged to be any multiple of $2d([\partial z])$ since $H_1(M;\mathbb{Z})/\text{torsion}$ is the dual space of $H_2(M;\mathbb{Z})$. But $[z+z'] = [z] \in H_2(X, M;\mathbb{Z})$, and by the long exact homology sequence of (X, M), any relative cycle z'' with $\partial z'' = \partial z$ and $[z''] = [z]$ is homologous in X to $z + z'$ for some z'. □

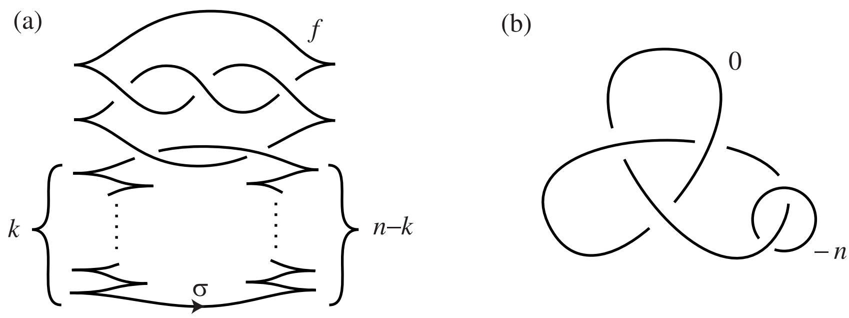

Figure 12.81. Stein structures on the nucleus $N(n)$.

Solution of Exercise 11.4.11(b): Take the Stein surface S_k corresponding to the link diagram of Figure 12.81(a). It is easy to see that $tb(f) = 1$, $r(f) = 0$, $tb(\sigma) = -n + 1$ and $r(\sigma) = 2k - n$. The underlying smooth 4-manifold is given by the Kirby diagram of Figure 12.81(b); consequently $S_k \approx N(n)$ for all $1 \leq k \leq n - 1$ $(n \geq 2)$. Now $\langle c_1(S_k), f\rangle = r(f) = 0$ and $\langle c_1(S_k), \sigma\rangle = r(\sigma) = 2k - n$, hence $c_1(S_k) = PD((2k - n)f)$. □

Solution of Exercise 11.4.11(c): Suppose that X admits a Stein structure and apply Theorem 11.4.7 to F. Suppose first that F is orientable. If $g(F) > 0$, then $-\chi(F) \geq [F]^2 + |c_1(X)[F]|$ and $[F]^2 = e(X)$, so $0 \leq |\langle c_1(X), [F]\rangle| \leq -e(X) - \chi(F)$. If F is a sphere, then $[F]^2 = e(X) \leq -2$ by Theorem 11.4.7, so $e(X)+\chi(F) = e(X)+2 \leq 0$. Summing F with a trivial torus and applying the theorem again, we obtain $|\langle c_1(X), [F]\rangle| \leq -e(X)$. If F is nonorientable, then pass to the double cover, which by Exercise 11.2.14 is also Stein. Since both e and χ double under the double cover, the required inequality follows from the previous case. For the congruence $\langle c_1(X), F\rangle \equiv e(X) \pmod 2$, see the solution of Exercise 11.3.2(a). □

Solution of Exercise 11.4.11(d): Since $H_2(X) \neq 0$ and there are no 3-handles, there are classes $a \in H^2(X;\mathbb{Z})$ and $b \in H_2(X;\mathbb{Z})$ with $\langle a, b\rangle \neq 0 \in \mathbb{Z}$. Let $c \in H^2(X;\mathbb{Z})$ be any integer lift of $w_2(X)$ (which exists since $H^3(X;\mathbb{Z}) = 0$, or by Remark 5.7.5). The classes $c + 2ka$ $(k \in \mathbb{Z})$ are all integer lifts of $w_2(X)$, so by Exercise 11.3.2(b) they all arise as Chern classes of Stein structures S_k on possibly different smooth structures on X. Since we can choose $|\langle c + 2ka, b\rangle|$ to be arbitrarily large, Theorem 11.4.7 implies that the minimum genus of a connected representative of b in S_k increases without bound as k increases. In particular, the underlying smooth structures must realize infinitely many isotopy classes. In the case of $\mathbb{R}^2$-bundles, these are clearly nondiffeomorphic. □

Notation, important figures

13.1. List of commonly used notation

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the ring of integers
$\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$	the fields of complex, real and rational numbers
$\mathbb{H}$	the field of quaternions
$\mathbb{Z}_n$	the ring of integers modulo n
$\mathbb{R}^n_+$	the closed upper half space of $\mathbb{R}^n$
$\gcd(p, q)$	greatest common divisor of p and q
$[X]$	the fundamental class of the manifold X
$\operatorname{int} X$	the interior of X
$c\ell(X)$	the closure of X
$\overline{X}$	the manifold X with the opposite orientation
∂X	boundary of the manifold X
$\partial_\pm X$	part of the boundary of X
$\cup_\partial$	gluing along a boundary
$\chi(X)$	the (topological) Euler characteristic of the manifold X
$\sigma(X)$	signature of the 4-manifold X
Q_X	intersection form of the 4-manifold X
$\mathcal{C}_X$	the set of characteristic elements in $H^2(X^4;\mathbb{Z})$

$b_2^+(X)$ ($b_2^-(X)$ resp.)	the dimension of the maximal positive (negative) definite subspace of $H_2(X;\mathbb{Z})$ with respect to the given intersection form Q_X
PD	Poincaré duality isomorphism
E_8, H	two important intersection forms
$\chi_h(S)$	holomorphic Euler characteristic of the complex surface S
$\kappa(S)$, $\kappa(X)$	the Kodaira dimension of the complex surface S (or symplectic 4-manifold X)
K_S	the canonical line bundle of the complex surface S
D^n	n-dimensional disk
S^n	n-dimensional sphere
T^n	n-dimensional torus
$\mathbb{RP}^n$	n-dimensional (real) projective space
$\mathbb{CP}^n$	n-dimensional (complex) projective space
$[z_0 : \ldots : z_n]$	homogeneous coordinates in $\mathbb{CP}^n$ or $\mathbb{RP}^n$
$E(n)$	the simply connected elliptic surface (with section) with $\chi_h(E(n)) = n$
$E(n)_{p_1,\ldots,p_k}$	the above elliptic surface after k logarithmic transformations
$M(p,q,r)$	Milnor fiber
Σ_g	Riemann surface of genus g
$g(\Sigma)$	genus of the Riemann surface Σ
$(\mathcal{M}_g, *)$	mapping class group of Σ_g, with multiplication $\varphi * \psi = \psi \circ \varphi$
$\nu\Sigma$	tubular neighborhood of the submanifold Σ
$\mathbb{F}_n$	Hirzebruch surface
$\mathbb{G}_{n,g}$	geometrically ruled surface over the Riemann surface Σ_g
$\approx$	orientation-preserving diffeomorphism of manifolds
$\sim$	orientation-preserving diffeomorphism of Kirby diagrams
$\overset{\partial}{\sim}$	orientation-preserving diffeomorphism of boundary 3-manifolds in a Kirby diagram
$\cong$	isomorphism of groups
$\#$	connected sum of manifolds
$\natural$	boundary sum, end sum

$\#_f$	fiber sum
$\amalg$	disjoint union
$\sim_c$	cobordant
$\langle n \rangle$	surgery coefficient of $\partial_- X$ (in Kirby diagrams); also used to denote the bilinear form on $\mathbb{Z}$ with matrix $[n]$
$P_G \to X$	principal G-bundle over X
$P_G \times_\rho F$	the associated fiber bundle (with fiber F) via the representation $\rho\colon G \to \mathrm{Aut}(F)$.
$\Gamma(X; E)$	the vector space of C^∞ sections of the vector bundle $E \to X$
Λ^i	the bundle of i-forms
$\Lambda^\pm$	the bundle of self-dual and anti-self-dual forms over a Riemannian 4-manifold
F_A	the curvature of the connection A
F_A^+	the self-dual part of the curvature of the connection A
$O(n),\ SO(n)$	n-dimensional orthogonal and special orthogonal group
$U(n),\ SU(n)$	n-dimensional unitary and special unitary group
$GL(n;R),\ SL(n;R)$	n-dimensional general and special linear group over the ring R
$Spin(n)$	n-dimensional spin group
$Spin^c(n)$	n-dimensional spin^c group
$Lie(G)$	Lie algebra of the Lie group G
$\mathcal{S}_X$	the set of spin structures on the manifold X
$\mathcal{S}^c_X$	the set of spin^c structures on the manifold X
$\mathcal{S}^c_{X,\xi}$	the set of spin^c structures on the manifold X inducing the contact structure ξ on ∂X
$S^\pm$	spinor bundles
$Met(X)$	the space of metrics on the manifold X
$\not\partial$	the Dirac operator on a spin Riemannian manifold
$\not\partial_A$	the twisted Dirac operator on a spin^c Riemannian manifold
$W^\pm$	spin^c spinor bundles
SW_X	Seiberg-Witten invariant of a closed 4-manifold X

$SW_{X,\xi}$	Seiberg-Witten invariant of a 4-manifold X with contact boundary $(\partial X, \xi)$
$\mathcal{P}ert(X)$	the space of perturbations on the 4-manifold X
$\mathcal{B}as_X$	the set of basic classes of a 4-manifold X
Cl_n (and $\mathbb{C}l_n$)	the n-dimensional real (and complexified) Clifford algebra
$\mathbb{C}l(X)$	the complex Clifford bundle over the spin manifold X
Ω_n	n-dimensional cobordism group
Ω_*	cobordism ring
(X, ω)	symplectic manifold with symplectic form ω
(M, ξ)	manifold with contact structure (or plane field) ξ
$\ell k(K_1, K_2)$	the linking number of the knots K_1, K_2
$w(K)$	writhe of a knot
$tb(K)$	the Thurston-Bennequin invariant of the Legendrian knot K
$r(K)$	the rotation number of the Legendrian knot K

13.2. Index of important diagrams

13.3. Index of Kirby moves and related operations

Preserving 4-manifold:

- Changing notation for 1-handles: Figure 5.35
- 1-handle/2-handle cancellation: Figures 5.12, 5.13, 5.38
- 2-handle/3-handle cancellation: Figure 5.15
- 1-handle slide: Figures 5.2, 5.39
- 2-handle slide: Figures 5.5, 5.8, 6.35
- Sliding under a 1-handle: Figure 5.36
- Ribbon disk slide: Figure 6.22
- Twisting a 1-handle: Figure 5.42
- Turning a handlebody upside down: Example 5.5.5
- Trick for following twists through handle slides: Figure 12.65

For 3-manifolds:

- Rolfsen twist: Figure 5.27
- Slam-dunk: Figure 5.30
- Changing rational surgery to integral surgery: Figure 12.13

Other:

- Blowing up/down: Figures 5.17-5.21
- Gluck twist: Figure 12.17
- Logarithmic transformation: Figure 8.26
- Rational blow-down: Figures 8.40, 8.41 and Definition 8.5.4
- Sliding a surface off of a 2-handle: Figures 6.36, 6.37
- Covers/branched covers: Section 6.3
- Doubling: Examples 4.6.3, 5.5.4

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For additional references on 4-manifolds see [**FQ**], [**DK**], [**K2**], [**K4**].

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Index

Since the early 1980s, there has been an explosive growth in 4-manifold theory, particularly due to the influx of interest and ideas from gauge theory and algebraic geometry. This book offers an exposition of the subject from the topological point of view. It bridges the gap to other disciplines and presents classical but important topological techniques that have not previously appeared in the literature.

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I greatly recommend this wonderful book to any researcher in 4-manifold topology for the novel ideas, techniques, constructions, and computations on the topic, presented in a very fascinating way. I think really that every student, mathematician, and researcher interested in 4-manifold topology, should own a copy of this beautiful book.

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