ON THE CONNECTIONS BETWEEN DEFINABLE GROUPS IN O-MINIMAL STRUCTURES AND REAL LIE GROUPS: THE NON-COMPACT CASE

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In the last twenty years many authors have investigated the analogies between definable groups in o-minimal structures and real Lie groups (see [Ot:08] for a survey). By a theorem of Pillay ([Pi:88]), every definable group G in an o-minimal structure M can be equipped with a topology τ which makes it a topological group. Moreover, if $n \in \mathbb{N}$ is the o-minimal dimension of $G \subset M^k$, then G with the topology τ is locally definably homeomorphic to M^n, just like a real Lie group of dimension n is locally homeomorphic to $\mathbb{R}^n$. Thus, if G is a definable group in an o-minimal structure with universe $\mathbb{R}$, then it is a Lie group. Moreover every definably compact definable group G has torsion ([BeOt:03, BeEdOt:07]), and in the definably connected abelian case, the torsion subgroup of G is isomorphic to the torsion subgroup of a compact connected abelian Lie group (a torus) ([EdOt:04, Pe:08]).

In the non-compact case some outstanding differences come out. Every non-compact connected real Lie group is homotopically equivalent to any of its maximal compact subgroups. In fact, if C is a maximal compact subgroup of a connected real Lie group L, then L is homeomorphic to $C \times \mathbb{R}^l$, for some $l \in \mathbb{N}$ ([Iwa:49, 6]). On the contrary, not every definable group in an o-minimal structure has maximal definably compact definable subgroups (see for example [PeSte:99, 5.6] and [Str:94a, 5.3]). While one can find maximal definably compact definably connected definable subgroups, there are examples such that no one of them is definably homotopically equivalent to the whole group.

Nevertheless, as we prove in the **first part** of this dissertation (Chapter 1-3), for every definably connected definable group G there is a canonical quotient of G (the quotient G/N by the maximal normal torsion-free definable subgroup N) which always contains maximal definably compact definable subgroups K (all definably connected and conjugate). If the structure expands a real closed field M, it turns out that G is definably homeomorphic to $K \times M^l$, for some $l \in \mathbb{N}$, and G is definably homotopically equivalent to K. Therefore the study of the o-minimal homotopy invariants of G is reduced to the study of those of K.

In the **second part** (Chapter 4-5) we study the correspondence between definable groups and compact Lie groups arising from the work on the "Pillay's conjecture" ([Pi:04]). That is, every definable group G in a saturated o-minimal structure contains a smallest type-definable subgroup of bounded index G^{00}, and the quotient G/G^{00} equipped with the logic topology is a compact Lie group ([BeOtPePi:05]). When G is definably compact, the connection between G and G/G^{00} is very strong:

- The o-minimal dimension of G equals the Lie dimension of G/G^{00} ([ElSta:07, HrPePi:08a, Pe:08]).

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- The functor $\Phi\colon G \mapsto G/G^{00}$ is exact ([Be:07]) and preserves cohomology ([Be:08]) and homotopy ([BeMaOt:08]).
- There is an elementary embedding (with respect to the group structure) $\sigma\colon G/G^{00} \to G$ which is a section for the canonical projection $\pi\colon G \to G/G^{00}$ ([HrPePi:08b]), and therefore

$$\langle G, \cdot\rangle \equiv \langle G/G^{00}, \cdot\rangle.$$

When G is not definably compact, the situation is more complicated. For instance, we prove that the Lie dimension of G/G^{00} is smaller than the o-minimal dimension of G, and that the functor $G \mapsto G/G^{00}$ extended to the category of all definable groups is not exact anymore.

We obtain that G/G^{00} is always a (explicitly determined) quotient of K/K^{00}, where K is any maximal definably compact definable subgroup of G/N, N the maximal normal torsion-free definable subgroup of G. Moreover we characterize the cases when this quotient is not proper and G/G^{00} is Lie isomorphic to K/K^{00}, giving several equivalent conditions to this fact.

We give below a more detailed summary of this work:

In **Chapter 1** we prove that every definable group G contains a unique maximal normal torsion-free definable subgroup N. This is always trivial if G is definably compact, but it can be trivial even if G is not definably compact: take $G = SL_2(\mathbb{R})$. One can observe that if G is defined in an o-minimal expansion of a real closed field M, then G/N is definably homotopically equivalent to G: by results of Y. Peterzil and S. Starchenko ([PeSta:05]), every n-dimensional torsion-free definable group N is definably diffeomorphic to M^n and there is a definable continuous global section $s\colon G/N \to G$. Thus one can deduce that G is definably homeomorphic to $G/N \times M^n$.

Chapter 2 is dedicated to the analysis of solvable groups. The main result is that if G is a solvable definable group, then the quotient G/N by the maximal normal torsion-free definable subgroup N is definably compact. Along the way we show that every solvable extension of a torsion-free definable group H by a definably compact definable group K is definably isomorphic to their direct product $K \times H$. The main ingredients of its proof are the fact that every definable group which is not definably compact contains a 1-dimensional torsion-free definable subgroup ([PeSte:99]), and that there are no infinite definable families of definable homomorphisms between abelian definably compact definable groups ([PeSta:00]).

In addition, given a definably connected solvable definable group G with maximal normal torsion-free definable subgroup N, we prove that $G = AN$ for every 0-Sylow A of G, where a 0-Sylow is a maximal 0-group (A is a 0-group if for every proper definable subgroup Q of A, $E(A/Q) = 0$, E the o-minimal Euler characteristic). In general, $N \cap A$ can be infinite (it is the maximal torsion-free definable subgroup of A), but if G is a definable subgroup of the general linear group $GL_n(\mathcal{M})$, then $N \cap A = \{e\}$. The reason is that by a theorem of Y. Peterzil, A. Pillay and S. Starchenko ([PePiSta:02]), every 0-group in $GL_n(\mathcal{M})$ is definably compact. As a consequence we get that every solvable definably connected definable subgroup G of $GL_n(\mathcal{M})$ is a definable semidirect product $\mathbf{T}^d \ltimes N$, where $\mathbf{T} = SO_2(\mathcal{M})$, just like a connected solvable Lie subgroup of $GL_n(\mathbb{R})$ is a semidirect product $\mathbb{T}^d \ltimes H$, with $\mathbb{T} = SO_2(\mathbb{R})$ and H a Lie subgroup diffeomorphic to $\mathbb{R}^{\dim H}$.

In **Chapter 3** we apply the results of Chapter 2 to find, for every definable group G in an o-minimal expansion of a real closed field, a definably compact definable group K definably homotopically equivalent to it.

Using the study of definably simple definable groups made by Y. Peterzil, A. Pillay and S. Starchenko in [PePiSta:00a] and [PePiSta:02], we prove that every definably connected definable group G with definably compact solvable radical (= the maximal normal solvable definably connected definable subgroup) contains maximal definably compact definable subgroups K, all definably connected and conjugate to each other. Furthermore, for every such K, there is a torsion-free definable subgroup H with $K \cap H = \{e\}$ and $G = KH$. If the structure expands a real closed field M, again H is definably diffeomorphic to $M^{\dim H}$, G is definably homeomorphic to $K \times H$ and thus G is definably homotopically equivalent to K.

If now G is any definably connected definable group in an o-minimal expansion of a real closed field, and N is its maximal normal torsion-free definable subgroup, then G/N has definably compact solvable radical (if R is the solvable radical of G, then R/N is the solvable radical of G/N and it is definably compact by the results about solvable groups), and therefore, as we said above, it is definably homotopically equivalent to every maximal definably compact definable subgroup in it. Recalling that G is definably homotopically equivalent to G/N, we get that G is definably homotopically equivalent to every maximal definably compact definable subgroup K of G/N. Actually G is definably homeomorphic to $K \times M^l$, with $l \in \mathbb{N}$ the dimension of any maximal torsion-free definable subgroup of G.

In the first section, in analogy with connected real Lie groups, we discuss the cases where a definably connected definable group G is an almost semidirect product of the solvable radical R and a semisimple definable subgroup S (Levi decomposition). While definably connected definable linear groups always admit such a decomposition ([PePiSta:02]), we show that this is not the case for every definable group. Indeed, we give an example of a definably connected semialgebraic group G which does not admit a definable Levi decomposition, and whose commutator subgroup $[G, G]$ is not semialgebraic (i.e. $[G, G]$ is not definable).

In **Chapter 4 and 5** we examine the smallest type-definable subgroup of bounded index G^{00} and the compact Lie group G/G^{00}, without assuming that G is definably compact. We give two characterizations of G/G^{00}: one in terms of K_G, the maximal normal definably compact definably connected definable subgroup of G/N (where N is the maximal normal torsion-free definable subgroup of G), and the other in terms of any maximal definably compact definable subgroup K of G/N. We prove that if G belongs to the class of definably connected *definable groups with the exactness property* (i.e. for every $H \lhd G$, $H^{00} = G^{00} \cap H$), which includes for instance definably compact groups ([HrPePi:08a, Be:07]), solvable groups and centerless semisimple groups, then G/G^{00} is Lie isomorphic to K_G/K_G^{00}. In general,

$$G/G^{00} \;\cong\; K_G/(K_G \cap (G/N)^{00})$$

and therefore

$$\dim_{\mathbb{R}} G/G^{00} = \dim_{\mathcal{M}} K_G - \dim_{\mathbb{R}}(K_G \cap (G/N)^{00}/K_G^{00}).$$

On the other hand, G/G^{00} can be viewed as a quotient of K/K^{00}, for every maximal definably compact definable subgroup K of G/N. More precisely,

$$G/G^{00} \;\cong\; (K/K^{00})/(K \cap (G/N)^{00}/K^{00}).$$

It is natural to ask in which cases the quotient is not proper. It turns out that the following conditions are equivalent:

(1) G/G^{00} is Lie isomorphic to K/K^{00}.
(2) G/N is definably compact.
(3) G^{00} is torsion-free.
(4) *G has the strong exactness property*: for every definable subgroup $S < G$, $S^{00} = G^{00} \cap S$.

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