ONE LIE GROUP TO DEFINE THEM ALL

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Abstract. We prove that a first-order structure defines (resp. interprets) every connected Lie group if and only if it defines (resp. interprets) the real field expanded with a predicate for the integers.

1. Introduction

The purpose of this note is to exhibit a connected Lie group which is, from the point of view of model theory, as bad behaved as it possibly can. In particular, we produce a Lie group (which happens to be connected, solvable and non-compact) defining the real field expanded with a predicate for the integers – in itself a patently *wild* structure from the point of view of model theory – in turn, it is easy to see that this structure defines all Lie groups. Therefore our group must have maximal defining power in the class of Lie groups (connected or not), intended as model theoretic structures in the pure group language.

The immediate motivation for this note has been a question by Antongiulio Fornasiero, about the possibility to interpret every connected Lie group in a d-minimal structure (for results in this context see [5]), which is ruled out by our example. More in general, there is a growing number of classes of Lie groups known to enjoy nice model-theoretical properties, such as: Nash groups [14], algebraic groups [12], compact [10, 7] or semisimple [11] Lie groups, and covers of all the above [2, 8]. At the same time, model theoretic methods are being used to attack classical problems [3]. Our result provides a negative example, suggesting that there are non-compact Lie groups that are untractable for model theory. This contrasts with the compact case, in fact all compact Lie groups are isomorphic to groups definable in the real field, and, in turn, the real field can be recovered from any semisimple Lie group [11]. To study the class of Lie groups up to first order definability in more detail is a complicated and possibly interesting task; it is, however, beyond the scope of this note: all we intend to say is that this class has a maximal element.

Proviso: definable means definable over parameters, Lie group means real Lie group, the same symbol denotes a group and the underlying set.

Aknowledgments

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2. Construction of the group

Let $(\mathbb{R},+,\cdot,\mathbb{Z})$ denote the first order structure whose domain are the reals, with the field operations and a predicate for the subset of the integers. It is well-known that the definable sets in $(\mathbb{R},+,\cdot,\mathbb{Z})$ coincide with the projective sets (see, for instance, [9, exercise 37.6]). Therefore we get immediately the following fact.

Fact 2.1. *All Lie groups are definable in* $(\mathbb{R},+,\cdot,\mathbb{Z})$.

We show now that one needs the full power of $(\mathbb{R},+,\cdot,\mathbb{Z})$ to be able to define all (connected) Lie groups. Namely, there is a connected Lie group G such that $(\mathbb{R},+,\cdot,\mathbb{Z})$ itself is definable in the pure group structure $(G,\cdot)$.

Theorem 2.2. *There is a connected Lie group which is interdefinable with the real field expanded with a predicate for the integers.*

Proof. Recall that a structure A is interpretable in a structure B when the universe set of A is in bijection with a set definable in B modulo an equivalence relation, itself definable in B, in such a way that the relations of A map to relations definable in B. This notion is weaker than A being definable in B, in that, for definability, A must be in bijection with a set definable in B tout court, without equivalence relation.

First, we will construct a group G in which $(\mathbb{R},+,\cdot,\mathbb{Z})$ is interpretable, after that we will show how this group can be modified in order to define $(\mathbb{R},+,\cdot,\mathbb{Z})$. Consider the following group

$$G = H_3(\mathbb{R})/\Gamma$$

Where $H_3(\mathbb{R})$ is the Heisenberg group

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\} < GL_3(\mathbb{R})$$

and

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\} \triangleleft H_3(\mathbb{R})$$

is a discrete subgroup of the center of H_3 (we choose a particular Γ; however, up to isomorphism, G does not depend on this choice).

For ease of notation, write $[a,b,c]$ to denote the class of the element

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

of $H_3(\mathbb{R})$ in the quotient. One can check directly that $[a,b,c] = [a',b',c']$ if and only if $a = a'$, $b = b'$, and $c - c' \in \mathbb{Z}$. It follows that the centralizer of $[a,b,c]$ in G is

$$C([a,b,c]) = \left\{ [a',b',c'] : ab' - ba' \in \mathbb{Z} \right\}.$$

For any $a, b \in \mathbb{R}$ we can define the subgroup

$$L_{a,b} = C([a,b,0]) \cap C([\pi a, \pi b, 0])$$

where π denotes any irrational number, as this is enough to ensure that

$$L_{a,b} = \left\{ [a',b',c'] : ab' - ba' = 0 \right\}.$$

In particular, the following are definable subgroups in $(G, \cdot)$:

$$A \stackrel{def}{=} L_{0,1} = \{[0,b,c] \ : \ b,c \in \mathbb{R}\}$$

$$B \stackrel{def}{=} L_{0,1} \cap C([1,0,0]) = \{[0,b,c] \ : \ b \in \mathbb{Z}, \ c \in \mathbb{R}\}$$

Hence the following groups are interpretable over parameters in $(G, \cdot)$:

$$E \stackrel{def}{=} G/_{Z(G)} \qquad R \stackrel{def}{=} A/_{Z(G)} \qquad Z \stackrel{def}{=} B/_{Z(G)}$$

Clearly $E > R > Z$. Observe that two elements $[a,b,c]$ and $[a',b',c']$ of G are equivalent modulo $Z(G)$ if and only if $a = a'$ and $b = b'$. It follows that E, R, and Z are isomorphic respectively to $\mathbb{R}^2$, $\{0\} \times \mathbb{R}$, and $\{0\} \times \mathbb{Z}$ through the map $\iota \colon [a,b,c] \mapsto (a,b) \in \mathbb{R}^2$. These are the ingredients of our interpretation.

The group R is the interpretation of the domain $\mathbb{R}$ of $(\mathbb{R}, +, \cdot, \mathbb{Z})$, and Z is that of $\mathbb{Z}$. So we only have to define the field operations. It is a classical fact that the field operations on $\{0\} \times \mathbb{R}$ can be defined using only the incidence graph of the straight lines in $\mathbb{R}^2$. Now, let $\mathcal{L}$ denote the set $\left\{L_{a,b} \ : \ (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}\right\}$ of subgroups of G. Clearly the family $\{\iota(L)\}_{L \in \mathcal{L}}$ spans all the straight lines through $(0,0)$, so, if the set $\mathcal{L}$ happens to be definable, then we are done. Recall that the subgroups $L_{a,b}$ are intersections of pairs of centralizers, therefore, it suffices to find a definable predicate that tells whether, given $g_1, g_2 \in G \setminus Z(G)$, there are a, b such that $C(g_1) \cap C(g_2) = L_{a,b}$. Indeed, this happens if and only if the group $C(g_1) \cap C(g_2)$ is divisible by 2, i.e. for all $x \in C(g_1) \cap C(g_2)$ there is $y \in C(g_1) \cap C(g_2)$ such that $x = y \cdot y$.

Now, to get a group in which $(\mathbb{R}, +, \cdot, \mathbb{Z})$ is definable, as opposed to interpretable, we replace $H_3(\mathbb{R})$ with the group

$$H' = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & x & b \\ 0 & 0 & 1 \end{pmatrix} \ : \ a,b,c \in \mathbb{R} \wedge x \in \mathbb{R}^+ \right\} < GL_3(\mathbb{R})$$

Again Γ is central in H', and we can construct our replacement for G as

$$G' = H'/_{\Gamma}$$

Now, using the notation $[a,b,c,x]$ for elements of G', it is easy to recover the group G as the product of the centralizers of $[1,0,0,1]$ and $[0,1,0,1]$. Therefore we can carry on the construction of E, R, and Z as before and get an interpretation of $(\mathbb{R}, +, \cdot, \mathbb{Z})$. To turn this into an actual definition, we only need a choice of representatives for the elements of $R = A/_{Z(G)}$. To this aim, let O be the orbit of $[0,1,0,1]$ under conjugation by elements of $C([0,0,0,2])$. An easy computation shows that

$$C([0,0,0,2]) = \left\{[0,0,c,x] \ : \ c \in \mathbb{R} \wedge x \in \mathbb{R}^+\right\}$$

$$O = \left\{[0,b,0,1] \ : \ b \in \mathbb{R}^+\right\}$$

Hence $O \cup O^{-1} \cup \{[0,0,0,1]\}$ intersects each equivalence class of A modulo $Z(G)$ in a single point. $\square$

3. Additional remarks

The group G of Theorem 2.2 has minimal dimension, since all connected Lie groups of dimension up to 2 are definable in the real field: up to Lie isomorphism, connected 1-dimensional groups are $SO_2(\mathbb{R})$ and $\mathbb{R}$, 2-dimensional groups are $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}^{>0} \ltimes \mathbb{R}$, $SO_2(\mathbb{R}) \times \mathbb{R}$ and $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ (see for instance [1, p.36]). We don't know whether G' could be replaced by a group of dimension 3.

The group G, defined as the quotient of a connected real algebrac group by a discrete subgroup, could also be obtained as the quotient of a definably connected semialgebraic group by a definably connected $\bigvee$-definable subgroup (see [4, Example 5.9]).

The group G is not linear (for instance, by a Theorem of Gotô [6, Theorem 5] the derived subgroup of a connected solvable linear Lie group needs to intersect trivially any maximal compact subgroup, but the derived subgroup of G *coincides* with a maximal compact subgroup). In a private communication, Ya'acov Peterzil observed that, by a modification of [13, example on p. 5], there is a linear group interpreting $(\mathbb{C}, +, \cdot, \mathbb{Z})$—in fact, it suffices to repeat the same construction of the example with $\mathbb{R}$ replaced by $\mathbb{C}$. This raises the question of whether there is a linear group interpreting $(\mathbb{R}, +, \cdot, \mathbb{Z})$.

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