A REDUCTION TO THE COMPACT CASE FOR GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

ANNALISA CONVERSANO

Abstract. Let $\mathcal{N}(G)$ be the maximal normal definable torsion-free subgroup of a group G definable in an o-minimal structure M. We prove that the quotient $G/\mathcal{N}(G)$ has a maximal definable definably compact subgroup K, which is definably connected and unique up to conjugation. Moreover we show that K has a definable torsion-free complement, i.e. there is a definable torsion-free subgroup H such that $G/\mathcal{N}(G) = K \cdot H$ and $K \cap H = \{e\}$. It follows that G is definably homeomorphic to $K \times M^s$ (with $s = \dim G - \dim K$), and homotopy equivalent to K. This gives a (definably) topological reduction to the compact case, in analogy with Lie groups.

1. Introduction and preliminaries

Every n-dimensional group G definable in an o-minimal structure M admits a (unique) topology making it a topological group locally definably homeomorphic to M^n ([Pi1, 2.5]). It follows that when M is an o-minimal expansion of the reals, G is a n-dimensional real Lie group. The homotopy type of a real Lie group can be reduced to the compact case by the following classical theorem due to Iwasawa:

Fact 1.1. ([Iw]). *Let L be a connected real Lie group. Then L has a maximal compact subgroup K, which is connected and unique up to conjugation. Moreover L is homeomorphic to*

$$K \times \mathbb{R}^s$$

where $s = \dim L - \dim K$. In particular L is homotopy equivalent to K.

Topological properties of groups definable in o-minimal structures have been studied in many papers by several authors, in particular in the compact case in analogy with Lie groups (see [Ba, BaBe, Be1, Be2, BeMa, BeMaOt, BeOtPePi, EdEl, HrPePi1, HrPePi2, HrPi, Ma, Pi2]). But unlike the real case, definable groups may not have a maximal definably compact subgroup: one can see for instance the groups [PeSte, 5.6] or [Str, 5.3]. So it is not clear at first how (or even if) one can obtain a definable reduction to the compact case.

The main result of this paper is that the quotient of a definably connected group G by its maximal normal definable torsion-free subgroup $\mathcal{N}(G)$ always has a maximal definably compact subgroup, which is definably connected and unique up to conjugation (one can see [Co2] for a characterization of definable groups with maximal definably compact subgroups). Moreover for this canonical quotient we can show a definable compact·torsion-free decomposition, namely:

Theorem 1.2. *Let G be a definably connected group. Set $\bar{G} := G/\mathcal{N}(G)$. Then $\bar{G}$ has a maximal definably compact subgroup K, which is definably connected and unique up to conjugation. Moreover, K has a definable torsion-free complement H in $\bar{G}$, i.e.*

$$\bar{G} = K \cdot H \qquad \text{and} \qquad K \cap H = \{e\}.$$

Date: January 29, 2012.

The subgroup H can be decomposed as a product of definably connected subgroups $H = A \cdot N$, with A abelian and N nilpotent.

The previous theorem and the work on torsion-free definable groups made in [PeSta], yields a reduction to the compact case in analogy to the real case Fact 1.1:

Theorem 1.3. *Every definably connected group G definable in an o-minimal expansion of a real closed field M is definably homeomorphic to*

$$K \times M^s,$$

where K is the maximal definably compact subgroup of $G/\mathcal{N}(G)$, and $s = \dim G - \dim K$. Therefore G is definably homotopy equivalent to K.

This shows that in the o-minimal context the homotopy type of a definably connected group equals to the homotopy type of a definably compact group, as in the real case. The difference is that in the Lie case the compact group can be found within the group, while in the o-minimal context it is found in a canonical quotient.

The paper is organized as follows: the remain part of this section contains definitions and main results used later on. Section 2 starts with the analysis of definably simple groups, which is used together with a result about splitting extensions (Lemma 2.3) to give proofs of Theorem 1.2 and Theorem 1.3. From Theorem 1.2 we deduce that every group definable in an o-minimal structure has maximal definable torsion-free subgroups (Corollary 2.4). We conclude with a couple of consequences on contractibility (Proposition 2.5 and Corollary 2.6).

Preliminaries. We assume M is any o-minimal structure. A basic reference for o-minimality is [Dr]. See [Ot] for an overview about groups definable in o-minimal structures. By *definable* we mean "definable in M with parameters".

Let G be a definable group.

G is *definably connected* if it has no proper definable subgroup of finite index.

G is *definably compact* if every definable curve is completable in G. When the group topology on G agrees with the induced topology from the ambient space, G is definably compact if and only if it is *closed and bounded* ([PeSte, 1.1 and 2.1]).

G is *definably simple* if it is definably connected, non-abelian and with no proper non-trivial normal definable subgroup.

G is *semisimple* if it is definably connected and does not contain any infinite abelian normal (definable) subgroup. A fundamental theorem due to Peterzil, Pillay and Starchenko about semisimple definable groups is the following:

Fact 1.4. ([PePiSta1, 4.1], [PePiSta3, 5.1]). *Let G be an infinite semisimple definable group. Then there are definable real closed fields $\mathcal{R}_i$ such that $G/Z(G)$ is definably isomorphic to a direct product $H_1 \times \cdots \times H_s$, where for every $i = 1, \ldots, s$, H_i is a semialgebraic (over $\mathcal{R}_i$) definably simple subgroup of $\mathrm{GL}(n_i, \mathcal{R}_i)$.*

Corollary 1.5. *An infinite definable group is semisimple if and only if it has no infinite solvable normal definable subgroup.*

With the corollary above and an easy induction argument, one can prove the following well-known fact:

Fact 1.6. *Every definably connected group G has a unique normal solvable definably connected subgroup R such that G/R is semisimple. The subgroup R contains every normal solvable definably connected subgroup of G, so it is the unique maximal normal solvable definably connected subgroup of G. It is said to be* ***the solvable radical*** *of G.*

An important definably characteristic definable subgroup contained in the solvable radical is the maximal normal definable torsion-free subgroup:

Fact 1.7. ([CoPi]). *In every definable group G there is a normal definable torsion-free subgroup which contains every normal definable torsion-free subgroup of G. It is the unique normal definable torsion-free subgroup of G of maximal dimension. We will refer to it as* **the maximal normal definable torsion-free subgroup** *of G and we will denote it by $\mathcal{N}(G)$.*

We notice an interesting duality between definable torsion-free groups and definably compact groups:

Fact 1.8.

(a) *Every definable group which is not definably compact contains a definable 1-dimensional torsion-free subgroup* ([PeSte, 1.2]).

(b) *Every definably compact group has torsion* ([EdOt, HrPePi1, PePiSta1]).

Remark 1.9. It follows from 1.8(b) that a definable torsion-free group is not definably compact, and so definably compact groups have no definable torsion-free subgroup. This is because definable subgroups are closed by [Pi1, 2.8], so definable subgroups of definably compact groups are definably compact.

Therefore if $K, H < G$ are definable subgroups where K is definably compact and H is torsion-free, then $K \cap H = \{e\}$. If in addition $G = K \cdot H$, then K *is a maximal definably compact subgroup* and H *is a maximal torsion-free definable subgroup.*

Fact 1.10. (Iwasawa decomposition of semisimple Lie algebras and Lie groups) ([Iw], [Kn, Chapter 6]). *For every semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ there esists a basis $\{X_i\}$ of $\mathfrak{g}$ and subspaces $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ such that $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and the matrices representing $\mathrm{ad}(\mathfrak{g})$ with respect to $\{X_i\}$ have the following properties:*

- *the matrices of $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric,*
- *the matrices of $\mathrm{ad}(\mathfrak{a})$ are diagonal with real entries,*
- *the matrices of $\mathrm{ad}(\mathfrak{n})$ are upper triangular with 0's on the diagonal.*

Let G be a semisimple connected Lie group with finite center and Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as above. If K, A and N are connected analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ respectively, then:

(a) *the multiplication map*

$$\begin{array}{ccc} K \times A \times N & \longrightarrow & G \\ (\, k \;,\; a \;,\; n \,) & \mapsto & kan \end{array}$$

is a surjective diffeomorphism;

(b) *the group K is a maximal compact subgroup of G and any maximal compact subgroup of G is a conjugate of K;*

(c) *$A \cdot N = N \cdot A$, i.e. $A \cdot N$ is a subgroup.*

Notation 1.11. Let $m \in \mathbb{N}$.

- $O_m(\mathcal{R}) = \{[x_{ij}] \in \mathrm{GL}_m(\mathcal{R}) : [x_{ij}][x_{ji}] = I\}$ is the orthogonal group,
- $T_m^+(\mathcal{R}) = \{[x_{ij}] \in \mathrm{GL}_m(\mathcal{R}) : x_{ij} = 0 \;\forall\, i < j \text{ and } x_{ii} > 0 \;\forall\, i\}$ is the group of upper triangular matrices with positive elements on the diagonal,
- $UT_m(\mathcal{R}) = \{[x_{ij}] \in \mathrm{GL}_m(\mathcal{R}) : x_{ij} = 0 \;\forall\, i < j \text{ and } x_{ii} = 1 \;\forall\, i\}$ is the group of unipotent upper triangular matrices,
- $D_m^+(\mathcal{R}) = \{[x_{ij}] \in \mathrm{GL}_m(\mathcal{R}) : x_{ij} = 0 \;\forall\, i \neq j,\; x_{ii} > 0 \;\forall\, i\}$ is the group of diagonal matrices with positive elements on the diagonal.

2. Reducing to definably compact groups

Definably simple groups. Peterzil, Pillay and Starchenko thoroughly analyse definably simple groups in several papers: [PePiSta1, PePiSta2, PePiSta3]. The following theorem (which builds on their work) clarifies the structure of a definably simple group in terms of definably compact and definable torsion-free subgroups, and is the first step to prove Theorem 1.2. We refer to the notation given in 1.11.

Theorem 2.1. *Let G be a definably simple group. Then there is a definable real closed field $\mathcal{R}$ and some $m \in \mathbb{N}$ such that G is definably isomorphic to a group $G_1 < \mathrm{GL}_m(\mathcal{R})$ definable in $\mathcal{R}$, such that:*

- *$G_1 = K \cdot H$, with $K = G_1 \cap O_m(\mathcal{R})$ and $H = G_1 \cap T_m^+(\mathcal{R})$,*
- *$H = A \cdot N$, with $A = G_1 \cap D_m^+(\mathcal{R})$ and $N = G_1 \cap UT_m(\mathcal{R})$.*

Moreover K, H, A, N are all definably connected, and K is not trivial (so infinite).

Proof. By [PePiSta1, 4.1], there is a definable real closed field $\mathcal{R}$, such that we can suppose G definable in $\mathcal{R}$ and contained in $\mathrm{GL}_n(\mathcal{R})$, for some $n \in \mathbb{N}$.

Let $\mathfrak{g}$ be the Lie algebra of G. By Theorem 2.36 of [PePiSta1], $\mathfrak{g}$ is a simple Lie algebra over $\mathcal{R}$. As noticed in the proof of Theorem 5.1 in [PePiSta3], there is a first order formula φ which says that there are finitely many simple Lie subalgebras $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ of $M_n(\mathcal{R})$, such that any simple subalgebra of $M_n(\mathcal{R})$ is isomorphic to one of the $\mathfrak{g}_i$ (we suppose to know r). In addition in φ we can require that for every $i = 1, \dots, r$ there are subspaces $\mathfrak{k}_i, \mathfrak{a}_i, \mathfrak{n}_i$ of $\mathfrak{g}_i$, with $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$, such that the matrices representing $\mathrm{ad}(\mathfrak{g}_i)$ have the following properties:

- the matrices of $\mathrm{ad}(\mathfrak{k}_i)$ are skew-symmetric,
- the matrices of $\mathrm{ad}(\mathfrak{a}_i)$ are diagonal,
- the matrices of $\mathrm{ad}(\mathfrak{n}_i)$ are upper triangular with 0's on the diagonal.

All these properties are first order (see [PePiSta1, PePiSta3]).

By the Iwasawa decomposition of semisimple Lie algebras (1.10), φ is true in $\bar{\mathbb{R}} := \langle \mathbb{R}, <, +, \cdot \rangle$ and therefore it is true in $\mathcal{R}$ as well. Hence $\mathfrak{g}$ is isomorphic to a Lie algebra $\mathfrak{g}_i \in \{\mathfrak{g}_1, \dots, \mathfrak{g}_r\}$ with the above mentioned properties. Say $\mathfrak{g}_i = \mathfrak{g}_1$.

By the proof of Theorem 2.37 in [PePiSta1], G is definably isomorphic to $\mathrm{Aut}_{\mathcal{R}}(\mathfrak{g})^0$, the definably connected component of the identity in $\mathrm{Aut}_{\mathcal{R}}(\mathfrak{g})$. Therefore G is definably isomorphic also to $\mathrm{Aut}_{\mathcal{R}}(\mathfrak{g}_1)^0 = G_1(\mathcal{R})$, a semialgebraic linear group defined over $\mathcal{R}_{alg}$, the subfield of algebraic numbers. So it makes sense to consider the group $G_1(\mathbb{R})$ defined in $\bar{\mathbb{R}}$ by the same formula over $\mathbb{R}_{alg}$ defining $G_1(\mathcal{R})$ in $\mathcal{R}$. It is a simple Lie group equal to $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}_1)^0$. If K, A, N are connected subgroups of $G_1(\mathbb{R})$ corresponding to the Lie subalgebras $\mathfrak{k}_1$, $\mathfrak{a}_1$ and $\mathfrak{n}_1$, then by 1.10 we have $A \cdot N = N \cdot A$ and $G_1(\mathbb{R}) = K \cdot A \cdot N$. Moreover if $m = n^2$,

- the matrices of $\mathrm{ad}(\mathfrak{k}_1)$ are skew-symmetric $\Rightarrow$ $K \subseteq O_m(\mathbb{R})$,
- the matrices of $\mathrm{ad}(\mathfrak{a}_1)$ are diagonal $\Rightarrow$ $A \subseteq D_m^+(\mathbb{R})$,
- the matrices of $\mathrm{ad}(\mathfrak{n}_1)$ are upper triangular with 0's on the diagonal $\Rightarrow$ $N \subseteq UT_m(\mathbb{R})$.

Since $D_m^+(\mathbb{R}) \cap UT_m(\mathbb{R}) = \{I\} = O_m(\mathbb{R}) \cap T_m^+(\mathbb{R})$, we get that

- $K = G_1(\mathbb{R}) \cap O_m(\mathbb{R})$,
- $A = G_1(\mathbb{R}) \cap D_m^+(\mathbb{R})$,
- $N = G_1(\mathbb{R}) \cap UT_m(\mathbb{R})$.

Thus the first order formula in the language of ordered fields which says that every element $g \in G_1$ can be written in a unique way as a product $g = kan$, with

$k \in G_1 \cap O_m$, $a \in G_1 \cap D_m^+$, $n \in G_1 \cap UT_m$ and $an = na \; \forall\, a \in G_1 \cap D_m^+, \forall\, n \in G_1 \cap UT_m$ is true in $\bar{\mathbb{R}}$ and therefore it is true in $\mathcal{R}$ as well.

Since H, A, N are definable torsion-free groups, they are solvable and definably connected ([PeSta]). It follows that K cannot be trivial (because G is definably simple, so it is not solvable) and it is definably connected as well (because G and H are definably connected). □

Corollary 2.2. *Any definably simple group G has a maximal definably compact subgroup K, which is infinite, definably connected and unique up to conjugation. Moreover K has a definable torsion-free complement H (i.e. $G = K \cdot H$ and $K \cap H = \{e\}$).*

Proof. Let $G_1 < \mathrm{GL}_m(\mathcal{R})$ be a definable group definably isomorphic to G, as in Theorem 2.1. As we noticed in Remark 1.9, $K = G_1 \cap O_m(\mathcal{R})$ is a maximal definably compact subgroup of G_1.

We claim that K is unique up to conjugation. Therefore, if C is any definably compact subgroup of G_1, we want to show that C is contained in a conjugate of K. Since every definably compact subgroup of $\mathrm{GL}_m(\mathcal{R})$ is semialgebraic ([PePiSta3, 4.6]), the fact that C is a definably compact (i.e. closed and bounded by [PeSte]) definable subgroup of $\mathrm{GL}_m(\mathcal{R})$ can be expressed by a first order formula in the language of ordered fields. Suppose that now $C = C(y)$ is defined over a set of parameters $y = (y_1, \ldots, y_n)$. Since every compact (again closed and bounded) subgroup of $G_1(\mathbb{R})$ is contained in a conjugate of $K(\mathbb{R}) = G_1(\mathbb{R}) \cap O_m(\mathbb{R})$, the following formula

$$\forall y \; [C(y) \text{ is a closed and bounded subgroup of } G_1 \Rightarrow \exists\, x \in G_1 \; (C(y) \subset K^x)]$$

is true in $\bar{\mathbb{R}}$, so it is true in $\mathcal{R}$ as well.

By Theorem 2.1, $H = G_1 \cap T_m^+(\mathcal{R})$ is a definable torsion-free complement of K. □

A structure theorem. In order to prove Theorem 1.2, we need the following Lemma:

Lemma 2.3. *Let $\pi\colon G \to Q$ be a definable extension of a definable torsion-free group Q by a definably compact group $K \lhd G$. Then the definable exact sequence*

$$1 \longrightarrow K \xrightarrow{\;i\;} G \xrightarrow{\;\pi\;} Q \longrightarrow 1$$

splits definably in a direct product. Therefore $\mathcal{N}(G)$ is definably isomorphic to Q and

$$G \;=\; \mathcal{N}(G) \times K.$$

Proof. We proceed by induction on $n = \dim Q$. For $n = 1$, see [Ed, 5.1]. If $n > 1$, let $Q_1 \subset Q$ be a normal definable subgroup of Q of codimension 1 ([PeSta]) and let $G_1 = \pi^{-1}(Q_1)$. By induction, the definable exact sequence

$$1 \longrightarrow K \xrightarrow{\;i\;} G_1 \xrightarrow{\;\pi\;} Q_1 \longrightarrow 1$$

splits definably in a direct product. So there is a definable torsion-free subgroup H which is normal in G_1 and such that $G_1 = K \times H$. Since $G_1/H \cong K$ which is definably compact, it follows that $H = \mathcal{N}(G_1)$ and H is normal in G as well (because $\mathcal{N}(G_1)$ is definably characteristic in G_1, which is normal in G). Consider now the definable exact sequence

$$1 \longrightarrow G_1/H \longrightarrow G/H \longrightarrow (G/H)/(G_1/H) \longrightarrow 1$$

with the obvious maps. Because $G_1/H \cong K$ which is definably compact and $(G/H)/(G_1/H) \cong G/G_1 \cong Q/Q_1$ which is torsion-free, we can apply induction again and find a definable torsion-free subgroup S normal in G/H such that $G/H = G_1/H \times S$. Therefore the preimage of S in G is a normal definable torsion-free subgroup which is a direct complement to K in G. □

Proof of Theorem 1.2. We want to show that the definable group $G/\mathcal{N}(G)$ has a maximal definably compact subgroup K, and K has a definable torsion-free complement H. Denoted by R the solvable radical of G, note that the solvable radical of $G/\mathcal{N}(G)$ is $R/\mathcal{N}(G)$, which is definably compact by [Ed] (see also [Co1, 2.5.1] for an alternative proof).

So we assume G is a definably connected group with definably compact solvable radical R, and we prove our claim for G.

Moreover we suppose that the definably connected semisimple group G/R is centerless. Even if it is not, the center of G/R is finite, so its preimage in G is a normal solvable definably compact subgroup, which is the only assumption we will make on R. By [PePiSta1, 4.1], G/R is a direct product of definably simple groups G_i, $i = 1, \dots, s$. By the definably simple case (2.1), for every $i = 1, \dots, s$, $G_i = K_i \cdot H_i$, where K_i is definably compact and H_i is torsion-free, and every definably compact subgroup of G_i is contained in a conjugate (in G_i) of K_i (2.2). Let $\bar{K} = \{(k_1, \dots, k_s) : k_i \in K_i\}$ and $\bar{H} = \{(h_i, \dots, h_s) : h_i \in H_i\}$ in G/R. Again by the simple case, for every $i = 1, \dots, s$, $H_i = A_i \cdot N_i$, with A_i abelian and N_i nilpotent definable subgroups. Thus $\bar{H} = \bar{A} \cdot \bar{N}$, with $\bar{A}$ the product of the A_i's and $\bar{N}$ the product of the N_i's. Note that $\bar{K}$ is definably compact, $\bar{H}$ is torsion-free and $G/R = \bar{K} \cdot \bar{H}$.

Let $\pi\colon G \to G/R$ be the canonical projection. Since R is definably compact, $K = \pi^{-1}(\bar{K}) \subseteq G$ is definably compact. Define $\bar{H}_1 = \pi^{-1}(\bar{H}) \subset G$. The group $\bar{H}_1$ is a definable extension of the definable torsion-free group $\bar{H}$ by the solvable definably compact group R. Hence by 2.3, the definable exact sequence

$$1 \longrightarrow R \xrightarrow{i} \bar{H}_1 \xrightarrow{\pi} \bar{H} \longrightarrow 1$$

splits definably in a direct product $\bar{H}_1 = R \times H$, for some definable torsion-free subgroup $H < G$ definably isomorphic to $\bar{H}$. It is immediate that $G = K \cdot H$.

As we noticed in 1.9, K is a maximal definably compact subgroup of G and H is a maximal definable torsion-free subgroup of G. Therefore for every $g \in G$, K^g is a maximal definably compact subgroup of $G = K^g \cdot H^g$. We want to show that these are the only ones, which is equivalent to say that for every definably compact subgroup C of G, there is some $g \in G$ such that $C \subset K^g$.

For every $i = 1, \dots, s$, let $\pi_i\colon G \to G_i$ be the composition of the canonical projections, first on G/R and then from G/R to G_i. By the simple case (2.1) we know that $\pi_i(C) \subseteq K_i^{g_i}$ for some $g_i \in G_i$ and so $\pi(C) \subseteq \bar{K}^{\bar{g}}$ with $\bar{g} = (g_1, \dots, g_s) \in G/R$. Therefore $C \subseteq K^g$, for every $g \in \pi^{-1}(\bar{g})$.

Finally, because G and H are definably connected, also K (and every conjugate of it) is definably connected. □

Corollary 2.4. *Every definable group has maximal definable torsion-free subgroups.*

Proof. Let G be a definable group. We can assume G is definably connected.

Set $N = \mathcal{N}(G)$ and let $G/N = K \cdot H$ be a definable compact·torsion-free decomposition given by Theorem 1.2. Let P be the pre-image of H in G by the canonical projection $\pi\colon G \to G/N$. We claim that P is a maximal definable torsion-free subgroup of G. Note that H is definably isomorphic to P/N, so $|E(H)| = |E(P/N)| = |E(N)| = 1$, where $E(X)$ denotes the o-minimal Euler characteristic of a definable set X. It follows that $|E(P)| = |E(N)E(P/N)| = 1$ as well, and by [Str] P is torsion-free. Maximality of P follows by maximality of H (see Remark 1.9). $\square$

The homotopy type of a definable group. In this last subsection we assume $\mathcal{M}$ is an o-minimal expansion of a real closed field.

We recall from [Dr] the basics definitions about definable homotopy:

Let X, Y be definable sets and $f, g\colon X \to Y$ definable continuous maps. A *definable homotopy between* f and g is a definable continuous map $\mathcal{H}\colon X \times [0,1] \to Y$ such that $f(x) = \mathcal{H}(x,0)$ and $g(x) = \mathcal{H}(x,1)$ for every $x \in X$.
A definable set X is called *definably contractible* if there is a point $\bar{x} \in X$ and a definable homotopy $\mathcal{H}\colon X \times [0,1] \to X$ between the identity map on X and the map $X \to X$ taking the constant value $\bar{x}$.

Two definable sets X and Y are *definably homotopy equivalent* if there are definable continuous maps $f\colon X \to Y$, $g\colon Y \to X$ such that there exist definable homotopies $\mathcal{H}_X$ between $(f \circ g)\colon X \to X$ and the identical map on X, and $\mathcal{H}_Y$ between $(g \circ f)\colon Y \to Y$ and the identical map on Y.

Proof of Theorem 1.3. Suppose H is a n-dimensional definable torsion-free group. Peterzil and Starchenko proved in [PeSta] that H is definably homeomorphic to M^n (and so is definably contractible). Moreover they show that there is a definable continuous section $s\colon G/H \to G$ (we recall that by definition of a section, after composing the canonical projection $\pi\colon G \to G/H$ with $s\colon G/H \to G$ one obtains the identity on G). Therefore if $\dim \mathcal{N}(G) = n$, it follows that G is definably homeomorphic to $G/\mathcal{N}(G) \times M^n$.

By 1.2, $G/\mathcal{N}(G) = K \cdot H$, with K definably compact and H definable torsion-free subgroups. Thus $G/\mathcal{N}(G)$ is definably homeomorphic to $K \times H$ and G is definably homeomorphic to $K \times M^s$, with $s = \dim \mathcal{N}(G) + \dim H$. $\square$

We conclude with a couple of results about contractibility:

Proposition 2.5. *A definably connected group is definably contractible if and only if it is torsion-free.*

Proof. Suppose G is a definably connected group which is definably contractible. By Theorem 1.3, G is definably homotopy equivalent to the maximal definably compact subgroup K of $G/\mathcal{N}(G)$. By Theorem 3.7 in [BeMaOt], the o-minimal homotopy groups of K are isomorphic to the homotopy groups of the connected compact Lie group K/K^{00}. Therefore K/K^{00} must be trivial, otherwise some of its homotopy groups are not trivial ([GoOnVi]). By [HrPePi1], the dimension of K/K^{00} equals the dimension of K, which must be trivial too. Therefore G is torsion-free by Theorem 1.2.

The other implication is [PeSta, 5.7]. $\square$

Corollary 2.6. *Let G be a definably connected group, definable in an o-minimal expansion of $\langle \mathbb{R}, <, +, \cdot \rangle$. Then G is contractible (as a topological group) if and only if G is definably contractible.*

Proof. Assume G is contractible. Then G is diffeomorphic to $\mathbb{R}^n$ and has only the trivial compact subgroup (see [GoOnVi, 3.2] for a reference). Therefore G is torsion-free, and the above proposition applies. The other implication is obvious. $\square$

Acknowledgments. The results of this paper are part of my Ph.D. dissertation. I am deeply grateful to my thesis advisor Alessandro Berarducci for his guidance. Together with him I thank Marcello Mamino, Margarita Otero, Ludovico Pernazza, Ya'acov Peterzil and Anand Pillay for helpful conversations and explanations. I thank Marcello Mamino also for let me use his beautiful "o-minimal" typesetting and Margarita Otero for suggest me the problem to find an o-minimal analogue to the existence of maximal compact subgroups in Lie groups.

References

[Ba] E. Baro, On the o-minimal LS-category, *Israel J. Math.*, **185 No.1** (2011), pp. 61–76.

[BaBe] E. Baro and A. Berarducci, Topology of definable abelian groups in o-minimal structures, to appear in *Bull. London Math. Soc.*

[Be1] A. Berarducci, O-minimal spectra, infinitesimal subgroups and cohomology, *J. Symbolic Logic*, **72 No. 4** (2007), pp. 1177–1193.

[Be2] A. Berarducci, Cohomology of groups in o-minimal structures: acyclicity of the infinitesimal subgroup, *J. Symbolic Logic*, **74(3)** (2009), pp. 891–900.

[BeMa] A. Berarducci and M. Mamino, On the homotopy type of definable groups in an o-minimal structure, *J. London Math. Soc.*, **83** (2011), pp. 563–586.

[BeMaOt] A. Berarducci, M. Mamino and M. Otero, Higher homotopy of groups definable in o-minimal structures, *Israel J. Math.*, **180** (2010), pp. 143–161.

[BeOtPePi] A. Berarducci, M. Otero, Y. Peterzil and A. Pillay, A descending chain condition for groups definable in o-minimal structures, *Ann. Pure Appl. Logic*, **134** (2005), pp. 303–313.

[Co1] A. Conversano, On the connections between definable groups in o-minimal structures and real Lie groups: the non-compact case, Ph.D. thesis, University of Siena, 2009.

[Co2] A. Conversano, Maximal compact subgroups in the o-minimal setting, preprint 2012.

[CoPi] A. Conversano and A. Pillay, Connected components of definable groups, and o-minimality I, preprint 2011.

[Dr] L. van den Dries, **Tame Topology and o-minimal Structures**, LMS Lecture Notes Series, **248**, Cambridge University Press, 1998.

[Ed] M. Edmundo, Solvable groups definable in o-minimal structures, *J. Pure Appl. Algebra*, **185** (2003), pp. 103–145.

[EdEl] M. Edmundo and P. Eleftheriou, The universal covering homomorphism in o-minimal expansions of groups, *Math. Log. Quar.*, **53** (2007), pp. 571–582.

[EdOt] M. Edmundo and M. Otero, Definably compact abelian groups, *J. of Math. Logic*, **4** (2004), pp. 163–180.

[GoOnVi] V.V. Gorbatsevich, A.L. Onishchik and E.B. Vinberg, **Lie groups and Lie algebras, III**, Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin, 1994.

[HrPePi1] E. Hrushovski, Y. Peterzil and A. Pillay, Groups, measures, and the NIP, *J. Amer. Math. Soc.*, **21** (2008), pp. 563–596.

[HrPePi2] E. Hrushovski, Y. Peterzil and A. Pillay, On central extensions and definably compact groups in o-minimal structures, *J. Algebra*, **327** (2011), pp. 71–106.

[HrPi] E. Hrushovski and A. Pillay, On NIP and invariant measures, *Journal of the European Math. Soc.*, **13** (2011), pp. 1005–1061.

[Iw] K. Iwasawa, On some types of topological groups, *Ann. Math.*, **50(3)** (1949), pp. 507–558.

[Kn] A.W. Knapp, **Lie Groups: Beyond an Introduction**, Birkhäuser, 2002.

[Ma] M. Mamino, Splitting definably compact groups in o-minimal structures, *J. Symbolic Logic*, **76** (2011), pp. 973–986.

[Ot] M. Otero, A survey on groups definable in o-minimal structures, in **Model Theory with Applications to Algebra and Analysis, vol. 2**, Editors: Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie, LMS LNS 350, Cambridge University Press, 2008, pp. 177–206.

[PePiSta1] Y. Peterzil, A. Pillay and S. Starchenko, Definably simple groups in o-minimal structures, *Trans. Amer. Math. Soc.*, **352** (2000), pp. 4397–4419.

[PePiSta2] Y. Peterzil, A. Pillay and S. Starchenko, Simple algebraic and semialgebraic groups over real closed fields, *Trans. Amer. Math. Soc.*, **352** (2000), pp. 4421–4450.

[PePiSta3] Y. Peterzil, A. Pillay and S. Starchenko, Linear groups definable in o-minimal structures, *J. Algebra*, **247** (2002), pp. 1–23.

[PeSta] Y. Peterzil and S. Starchenko, On torsion-free groups in o-minimal structures, *Illinnois J. Math.*, **49(4)** (2005), pp. 1299–1321.

[PeSte] Y. Peterzil and C. Steinhorn, Definable compactness and definable subgroups of o-minimal groups, *J. London Math. Soc.*, **59** (1999), pp. 769–786.

[Pi1] A. Pillay, On groups and fields definable in o-minimal structures, *J. Pure and Appl. Algebra*, **53** (1988), pp. 239–255.

[Pi2] A. Pillay, Type-definability, compact Lie groups, and o-minimality , *J. of Math. Logic*, **Vol.4, No.2** (2004), pp. 147–162.

[Str] A. Strzebonski, Euler characteristic in semialgebric and other o-minimal groups, *J. Pure Appl. Algebra*, **86** (1994), pp. 173–201.