MAXIMAL COMPACT SUBGROUPS IN THE O-MINIMAL SETTING

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Abstract. A characterization of groups definable in o-minimal structures having maximal definable definably compact subgroups is given. This follows from a definable decomposition in analogy with Lie groups, where the role of maximal tori is played by maximal 0-subgroups. Along the way we give structural theorems for solvable groups, linear groups, and extensions of definably compact by torsion-free definable groups.

1. Introduction and Preliminaries

We consider groups G definable in an o-minimal structure $\mathcal{M}$ with underlying universe M. If $M = \mathbb{R}$, then G is a finite-dimensional real Lie group ([21]). It is well-known that a finite-dimensional real Lie group has maximal compact subgroups (all conjugate) whose homotopy type coincides to the homotopy type of the whole group. Moreover:

Fact 1.1. ([11]). *Let L be a connected real Lie group.*
For every maximal torus T of L there is a maximal compact subgroup K of G and 1-dimensional connected torsion-free closed subgroups $H_1, \dots, H_s$ such that

$$K = \bigcup_{x \in K} T^x \qquad L = K \cdot H_1 \cdots H_s \qquad K \cap H_i = \{e\}.$$

In the o-minimal context the corresponding notion of compactness is given in [20]: a definable set X is *definably compact* if every definable curve is completable in X. It turns out that a definable group G is definably compact if and only if G does not contain any infinite torsion-free definable subgroup ([8, 9, 16, 20]). As remarked in [19], definable torsion-free groups are definably connected, so one can note the analogy with Lie groups: a Lie group L is compact if and only if L does not contain any infinite connected torsion-free closed subgroup.

In general a definable group does not contain a maximal definable definably compact subgroup. However, if we restrict ourselves to the semisimple case, then we are able to prove the following (see Section 5):

Proposition 1.2. *Every definable group has a (unique up to conjugation) maximal definable semisimple definably compact subgroup.*

Moreover we will show how in the o-minimal setting the notion of "maximal torus" (i.e. maximal abelian connected compact subgroup) of a Lie group finds an analogue in the notion of "maximal 0-subgroups", i.e. 0-Sylow (see Section 2 for the definitions). Like in the Lie context, 0-Sylow are abelian and definably connected ([23]). However a 0-Sylow might not be definably compact, and its maximal definable torsion-free subgroup turns out to be an important invariant of

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the group, and the only obstruction to the existence of maximal definable definably compact subgroups:

Theorem 1.3. *Let T be a* 0*-Sylow of a definable group G. Then the maximal definable torsion-free subgroup $\mathcal{N}(T)$ of T does not depend on T. Moreover it is central and invariant by definable automorphisms of G. We denote it by $\mathcal{A}(G)$.*

Suppose G is definably connected. Then G has a maximal definable definably compact subgroup if and only if $\mathcal{A}(G)$ is trivial.

In the proof of Theorem 1.3 (see Proposition 5.5) we will actually show that $\mathcal{A}(G) = \{e\}$ and the property that G has a maximal definable definably compact subgroup are both equivalent to the fact that G admits a definable compact·torsion-free decomposition:

Definition 1.4. We say that a definable group G has a *definable compact·torsion-free decomposition* if there are definable subgroups K and H of G where

$$G = K \cdot H \qquad \text{and} \qquad K \cap H = \{e\}$$

such that K is definably compact and H is torsion-free.

Our main result is the following decomposition:

Theorem 1.5. *Let G be a definable definably connected group.*

For every 0*-Sylow T of G there is a maximal definable torsion-free subgroup H and a definable definably connected subgroup P such that*

$$(*) \qquad P = \bigcup_{x \in P} T^x \qquad G = P \cdot H \qquad P \cap H = \mathcal{A}(G).$$

Moreover for every cofactor C of $\mathcal{N}(T)$ in T there is an abstract subgroup $K \subset P$ such that

$$(**) \qquad K = \bigcup_{x \in K} C^x \qquad G = K \cdot H \qquad K \cap H = \{e\}.$$

The canonical projection $\pi\colon G \to G/\mathcal{N}(G)$ induces an isomorphism between K and the maximal definable definably compact subgroup of $G/\mathcal{N}(G)$.

Also $P = \mathcal{A}(G) \times K$ is the smallest definable subgroup of G containing K, and $[P, P] = [K, K]$ is a maximal definable semisimple definably compact subgroup of G.

Conversely, for every maximal definable semisimple definably compact subgroup S of G there is a 0*-Sylow T of G, a maximal definable torsion-free subgroup H, a definable definably connected subgroup P and an abstract subgroup K satisfying $(*)$ and $(**)$, and such that $S = [P, P] = [K, K]$.*

Corollary 1.6. *Let G be a definable definably connected group. Then $G/\mathcal{A}(G)$ has a definable compact·torsion-free decomposition, and it is the maximal definable quotient of G with a definable compact·torsion-free decomposition.*

It follows that every definable definably connected group is a central extension of a group with a definable compact·torsion-free decomposition by a definable definably characteristic torsion-free subgroup.

The paper is organized as follows: the remain part of Section 1 contains basic definitions and results. In Section 2 we introduce $\mathcal{A}(G)$. Section 3 is devoted to the analysis of solvable groups. In Section 4 it is proved that every definable linear group admits a definable compact·torsion-free decomposition. In Section 5 we give the proof of the main results in the general case.

We assume $\mathcal{M}$ is an o-minimal expansion of an ordered group. A basic reference for o-minimality is [5]. By "*definable*" we mean "definable in $\mathcal{M}$ with parameters". Many results about groups definable in o-minimal structures and some theorems about Lie groups are used. One can see [13] for an overview about groups definable in o-minimal structures, and [12] for an overview about Lie groups.

Let G, N, Q be definable groups. A *definable extension of Q by N* is a definable group G containing N, together with a definable homomorphism $\pi\colon G \to Q$ such that the sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

is exact, i.e. π is a surjective definable homomorphism and $i(N) = N$ is the kernel of π (where $i\colon N \to G$ is the inclusion map). We say that a definable extension $\pi\colon G \to Q$ *splits abstractly*, if there is a surjective homomorphism $s\colon Q \to G$ such that $\pi \circ s = id_Q$. We say that it *splits definably* if there is such a s which is moreover definable. Thus if N is a normal definable subgroup of a definable group G, we can always see G like a definable extension of G/N by N. The exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} G/N \longrightarrow 1$$

splits abstractly if and only if G contains a subgroup K isomorphic to G/N such that $N \cap K = \{e\}$ and $N \cdot K = G$, i.e. if and only if G is a *semidirect product* of N and K. In this case we will write $G = N \rtimes K$ or $G = K \ltimes N$ and we will say that K is a *cofactor* of N. The exact sequence above splits definably if and only if there is a *definable cofactor* K of N in G.

If $H, K < G$ are groups (no normality assumption here), we say that H is a *complement* of K in G whenever $G = K \cdot H$ and $K \cap H = \{e\}$. If all groups involved are definable we say that H is a *definable complement* of K in G.

We denote by G^0 the definably connected component of the identity in a definable group G. It is the smallest definable subgroup of finite index in G ([21]). We say that a group G is *definably connected* if it is definable and $G = G^0$, so if G has no proper definable subgroup of finite index.

A definably connected group G is said to be *semisimple* if does not contain any infinite abelian normal (definable) subgroup. It is *definably simple* if does not contain any proper non-trivial normal definable subgroup.

We summarize the main results which we will be using here:

Fact 1.7. *Let G be a definable group.*

1. *G has maximal definable torsion-free subgroups* ([2]).
2. *G has a maximal normal definable torsion-free subgroup $\mathcal{N}(G)$* ([3]).
3. *If G is solvable then $G/\mathcal{N}(G)$ is definably compact* ([6]), *and $G^0/\mathcal{N}(G)$ is abelian* ([18]).
4. *$G^0/\mathcal{N}(G) = K \cdot H$, where K is a maximal definable definably compact subgroup (and is definably connected and unique up to conjugation) and H is a maximal definable torsion-free subgroup* ([2]).
5. *If $G/\mathcal{N}(G)$ is definably compact, then G^0 has a (unique up to conjugation) definable Levi subgroup, i.e. a maximal definable semisimple subgroup S such that $G = \mathcal{N}(G) \cdot S$* ([4]).
6. *$G/Z(G)$ is a definable linear group* ([14]).
7. *Suppose G is definably compact and definably connected. Then the commutator subgroup $[G,G]$ is a definable semisimple group, and $G = Z(G)^0 \cdot [G,G]$* ([10]).

(8) *Suppose G is definably compact and definably connected. Then*

$$G = \bigcup_{x \in G} T^x$$

for every maximal definable torus T (i.e. T is a 0-Sylow) of G ([1], [7]).

2. 0-groups and 0-Sylow

In analogy with finite groups, Strzebonski develops in [23] a theory of definable p-groups, proving corresponding "Sylow's theorems", where the cardinality of finite p-groups is replaced by the o-minimal Euler characteristic of definable p-groups.

If $\mathcal{P}$ is a cell decomposition of a definable set X, *the o-minimal Euler characteristic* $E(X)$ is the integer defined as the number of even-dimensional cells in $\mathcal{P}$ minus the number of odd-dimensional cells in $\mathcal{P}$. This does not depend on $\mathcal{P}$ (see [5, Chapter 4]). When X is finite then $E(X) = \mathrm{card}(X)$. Since for every A, B definable sets, $E(A \times B) = E(A)E(B)$, the following holds:

Fact 2.1 ([23])**.** *Let $K < H < G$ be definable groups. Then*
- (a) $E(G) = E(H)E(G/H)$.
- (b) $E(G/K) = E(G/H)E(H/K)$.

Fact 2.2 ([23])**.** *Let G be a definable group. Then*

$$|E(G)| = 1 \;\Leftrightarrow\; G \text{ is torsion-free.}$$

In [23] Strzebonski also introduces 0-groups, which are further investigated by Berarducci in [1]:

Definition 2.3. ([23]).
- A 0**-group** is a definable group G such that for every proper definable subgroup H, $E(G/H) = 0$ (in particular, $E(G) = 0$).
- A 0**-subgroup** is a definable subgroup which is a 0-group.
- A 0**-Sylow** is a maximal 0-subgroup.
- A **definable torus** is a 0-group such that every definably connected subgroup of it is a 0-group.

Fact 2.4. ([1], [23]).
- (a) *Every definable group G with $E(G) = 0$ contains a 0-subgroup.*
- (b) *Every 0-group is abelian and definably connected.*
- (c) *Every 0-subgroup is contained in a 0-Sylow.*
- (d) *Any two 0-Sylow are conjugate.*
- (e) *If H is a 0-subgroup of a definable group G, then*

$$H \text{ is a 0-Sylow} \;\Leftrightarrow\; E(G/H) \neq 0.$$

- (f) *A definable group is a definable torus if and only if it is abelian definably connected and definably compact.*

Remark 2.5. If A is an abelian definable definably connected group and $H < A$ is a torsion-free definable subgroup, then they are both divisible ([23]) and therefore A contains a (possibly non-definable) direct cofactor of H (see for instance [22, 10.24]). Thus

$$A \;\cong\; H \;\times\; A/H.$$

In particular, if A is a 0-Sylow of a definable group G and $H = \mathcal{N}(A)$ is infinite, then every cofactor K of $\mathcal{N}(A)$ in A is not definable, but is abstractly isomorphic by the canonical projection $\pi\colon A \to A/\mathcal{N}(A)$ to a definable definably compact group.

Proposition 2.6. *Let T be a 0-Sylow of a definable group G. Then the maximal definable torsion-free subgroup $\mathcal{N}(T)$ of T does not depend on T. Moreover it is central and invariant by definable automorphisms of G. We denote it by $\mathcal{A}(G)$.*

Lemma 2.7. *Every linear 0-group is definably compact.*

Proof. Let G be a linear 0-group. By [17], every definable definably connected linear abelian group splits into the product of definable 1-dimensional subgroups. So G can be written as a product of definable groups $G = N \times K$ where N is torsion-free and K is definably compact. Since $E(G/K) = E(N) = \pm 1$ and G is a 0-group, it follows that $G = K$. □

Lemma 2.8. *Let T be a 0-Sylow of a definable group G. Then $\mathcal{N}(T) \subseteq Z(G)$.*

Proof. Since the quotient $G/Z(G)$ is a definable linear group, it follows that the group $T/(T \cap Z(G))$ is a linear 0-group, so it is definably compact (Lemma 2.7). It follows that $\mathcal{N}(T) \subseteq Z(G)$, otherwise $T/(T \cap Z(G))$ would contain an infinite definable torsion-free subgroup, in contradiction with the fact that is definably compact. □

Lemma 2.9. *If T_1 and T_2 are 0-Sylow of a definable group G, then $\mathcal{N}(T_1) = \mathcal{N}(T_2)$.*

Proof. Let $g \in G$ such that $T_1 = T_2^g$. As we observed in Remark 2.5, for $i = 1, 2$, $T_i = \mathcal{N}(T_i) \times K_i$, for some subgroups K_i which are abstractly isomorphic to definable definably compact groups. Thus $T_1 = \mathcal{N}(T_2)^g \times K_2^g$, and therefore $\mathcal{N}(T_1) = \mathcal{N}(T_2)^g$. Since $\mathcal{N}(T_2)$ is central in G (Lemma 2.8), the thesis follows. □

Proof of Proposition 2.6. By Lemma 2.8 and Lemma 2.9. Since a definable automorphism of G permutes its 0-Sylow, it follows that $\mathcal{A}(G)$ is definably characteristic, in particular it is a normal subgroup. Therefore $\mathcal{A}(G) \subseteq \mathcal{N}(G)$. □

3. Solvable definable groups

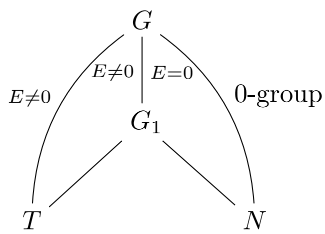
In this section we show structure results for solvable definably connected groups, giving the proof of Theorem 1.3 and Theorem 1.5 in the solvable case.

Proposition 3.1. *Let T be a 0-Sylow of a definable solvable definably connected group G. Then*

$$G = T \cdot \mathcal{N}(G) \quad \textit{and} \quad T \cap \mathcal{N}(G) = \mathcal{A}(G).$$

Moreover if $T_1 < G$ is a 0-subgroup such that $G = T_1 \cdot \mathcal{N}(G)$, then T_1 is a conjugate of T.

Proof. Set $N = \mathcal{N}(G)$. If G is torsion-free then $G = N$ and there is nothing to prove. So suppose $E(G) = 0$ and let T be a 0-Sylow of G. Set $G_1 = T \cdot N$ and consider the following diagram:



Since $E(G/G_1)E(G_1/T) = E(G/T) \neq 0$ (because T is a 0-Sylow: see Fact 2.4), it follows that $E(G/G_1) \neq 0$. On the other hand, G_1/N is a definable subgroup of the definable abelian, definably connected, definably compact group G/N, which is in particular a 0-group. Therefore $E(G/G_1) = E((G/N)/(G_1/N)) = 0$, unless $G/N = G_1/N$. It follows that $G = G_1$, as claimed.

Since G/N is definably compact, it follows that $T/(T \cap N)$ is definably compact as well, so $T \cap N = \mathcal{N}(T) = \mathcal{A}(G)$.

Let T_1 be a 0-subgroup of G such that $G = T_1 \cdot N$, and let S_1 be a 0-Sylow containing T_1. Thus S_1/T_1 is definably isomorphic to $(S_1 \cap N)/(T_1 \cap N)$, and then $E(S_1/T_1) = \pm 1$. But S_1 is a 0-group, so $S_1 = T_1$. It follows that T_1 is a 0-Sylow, and therefore a conjugate of T. □

Corollary 3.2. *Every definable solvable definably connected group G is abstractly isomorphic to a semidirect product $\mathcal{N}(G) \rtimes G/\mathcal{N}(G)$.*

Proof. Let T be a 0-Sylow of G. Then by 3.1 $G = T \cdot \mathcal{N}(G)$. If T is definably compact, then $T \cap \mathcal{N}(G) = \{e\}$ and T is a definable cofactor of $\mathcal{N}(G)$ in G. Otherwise, $T \cap \mathcal{N}(G) = \mathcal{N}(T)$ and it is infinite. If K is a cofactor of $\mathcal{N}(T)$ in T (see Remark 2.5), i.e. $T = \mathcal{N}(T) \times K$, then K is a cofactor of $\mathcal{N}(G)$ in G. □

Proposition 3.3. *Let G be a definable solvable definably connected group. Then the following are equivalent:*

(i) *G has maximal definable definably compact subgroups.*
(ii) *$G = K \cdot \mathcal{N}(G)$, for some definable definably compact subgroup $K < G$.*
(iii) *$\mathcal{A}(G) = \{e\}$.*

Proof. $(i) \Rightarrow (iii)$: Take K to be a maximal definable definably compact subgroup, and T a 0-Sylow of G. Suppose, for a contradiction, that $\mathcal{A}(G)$ is infinite. Then $T = \mathcal{A}(G) \times C$, for some abstract group C (see Remark 2.5). Note that K embeds in C because $K \cap \mathcal{N}(G) = \{e\}$ and C is abstractly isomorphic to $G/\mathcal{N}(G)$. Since C is not definable, it follows that $\dim K < \dim(G/\mathcal{N}(G))$, so there are arbitrarily large finite subgroups $F \subset C$ which are not in K (see [8] and [15] for the structure of the torsion subgroup of a definable definably compact abelian group). It follows that for any such an F, the definable group $F \cdot K$ is definably compact and contains K properly, contradiction.

$(iii) \Rightarrow (ii)$: By Proposition 3.1.

$(ii) \Rightarrow (i)$: Suppose $G = K \cdot \mathcal{N}(G)$, for some definable definably compact $K < G$. Then K is a maximal definable definably compact subgroup of G. □

4. Definable linear groups

Theorem 4.1. *Every definable definably connected linear group admits a definable compact·torsion-free decomposition.*

Proof. We refer to Definition 2.6. Suppose G is a definable definably connected linear group, set $N = \mathcal{N}(G)$ and let $\pi\colon G \to G/N$ be the canonical projection. Suppose $G/N = K_1 \cdot H_1$ is a definable compact·torsion-free decomposition (see Fact 1.7). Set $G_1 = \pi^{-1}(K_1)$. Note that G_1 is definably connected because N and K_1 are definably connected.

We claim that the following definable exact sequence

$$1 \longrightarrow N \xrightarrow{i} G_1 \xrightarrow{\pi} K_1 \longrightarrow 1$$

splits definably, i.e. N has a definable cofactor in G_1. Let us consider the possible cases for K_1:

If K_1 is *abelian*, then G_1 is solvable. Applying Proposition 3.1 and Lemma 2.7, we obtain that every 0-Sylow of G_1 is a definable cofactor of N.

If K_1 is *semisimple*, then N is the solvable radical of G_1 and by [17] G_1 contains a definable semisimple definably connected subgroup S such that $G_1 = N \cdot S$ and $N \cap S$ is finite. Since N is torsion-free, it follows that $N \cap S = \{e\}$ and therefore S is a definable cofactor of N in G_1.

Finally suppose K_1 is *neither abelian nor semisimple*, and set $G_2 = \pi^{-1}([K_1, K_1])$. Note that G_2 is definably connected because $[K_1, K_1]$ and N are definably connected. By the semisimple case, the definable exact sequence

$$1 \longrightarrow N \xrightarrow{i} G_2 \xrightarrow{\pi} [K_1, K_1] \longrightarrow 1$$

splits definably. Let S be a definable cofactor of N in G_2, i.e. a definable subgroup of G_2 definably isomorphic to $[K_1, K_1]$.

Claim. There is a 0-Sylow T of G_1 which is contained in $N_{G_1}(S)$, the normalizer of S in G_1.

Proof of the Claim. Suppose T is a 0-Sylow of $N_{G_1}(S)$. We want to show that T is a 0-Sylow of G as well. Take $g \in G_1$ and consider S^g. Since G_2 is a normal subgroup of G_1 (because $[K_1, K_1]$ is normal in K_1), it follows that $S^g \subset G_2$, and it is a definable Levi subgroup of G_2. Therefore there is some $x \in N$ such that $S^g = S^x$. This gives a well-defined bijective definable map

$$G/N_{G_1}(S) \longleftrightarrow N/N_N(S).$$

Since N is torsion-free, it follows that $E(N/N_N(K)) = \pm 1 = E(G/N_{G_1}(S))$. Then $E(G_1/T) = E(G_1/N_{G_1}(S))E(N_{G_1}(S)/T) \neq 0$, and T is a 0-Sylow of G as well (Fact 2.4). □

Since $T \subset N_{G_1}(S)$, it follows that $T \cdot S$ is a subgroup. Call it K. The fact that T and S are definably compact (Lemma 2.7) implies that K is definably compact as well. We know that $\pi(S) = [K_1, K_1]$, so in order to conclude that K is a definable cofactor of N in G_1, it is enough to show that $\pi(T)$ contains $Z(K_1)^0$.

Because T is definably compact (Lemma 2.7), the restriction of π to T is a definable isomorphism between T and a maximal definable torus T_1 of K_1. Since $Z(K_1)^0$ is a normal definable torus and all maximal tori are conjugate, it follows that T_1 (and any other maximal torus) contains $Z(K_1)^0$.

We proved that $G_1 = K \ltimes N$, for some definable definably compact K.

Define $H = \pi^{-1}(H_1) < G$. Then H is torsion-free and $G = K \cdot H$ is a definable compact·torsion-free decomposition of G. □

5. The general case

The maximal definable semisimple definably compact subgroup. We show now that every definable group G has a maximal definable semisimple definably compact subgroup S, and it is unique up to conjugation. It turns out that S is definably isomorphic by the canonical projection $\pi\colon G \to G/\mathcal{N}(G)$ to the commutator subgroup of the maximal definable definably compact subgroup of $G/\mathcal{N}(G)$.

Proof of Proposition 1.2. Let G be a definable group. Since semisimple definable groups are definably connected by definition, we can suppose G is definably connected. Let K_1 be the maximal definable definably compact subgroup of $G/\mathcal{N}(G)$, and note that $[K_1, K_1]$ is the unique maximal definable semisimple definably compact subgroup of K_1.

Let $\pi\colon G \to G/\mathcal{N}(G)$ be the canonical projection and define $G_1 = \pi^{-1}([K_1, K_1])$. By [4] (see Fact 1.7), G_1 has a definable Levi subgroup S_1, and it is unique up to conjugation. We claim that S_1 is a maximal definable semisimple definably compact subgroup of G. If not, let S be a definable semisimple definably compact subgroup containing S_1 properly. Since $S \cap \mathcal{N}(G) = \{e\}$, it follows that $\pi(S)$ is a definable semisimple definably compact subgroup containing $[K_1, K_1]$ properly, contradiction.

Let S_2 be another maximal definable semisimple definably compact subgroup of G. We want to show that S_2 is a conjugate of S_1. Again, since $S_2 \cap \mathcal{N}(G) = \{e\}$ it follows that $\pi(S_2)$ is a maximal definable semisimple definably compact subgroup of $G/\mathcal{N}(G)$. Let K_2 be a maximal definable definably compact subgroup of $G/\mathcal{N}(G)$ containing $\pi(S_2)$. Then $\pi(S_2) = [K_2, K_2]$. Define $G_2 = \pi^{-1}([K_2, K_2])$.

Since K_2 is a conjugate of K_1, let $\bar{g} \in G/\mathcal{N}(G)$ such that $[K_2, K_2] = [K_1, K_1]^{\bar{g}}$. Set $G_2 = \mathcal{N}(G) \rtimes S_2$. Then for every $g \in \pi^{-1}(\bar{g})$, denoted by $a(g)$ the conjugation map given by $x \mapsto gxg^{-1}$, the following diagram commutes:

$$\begin{tikzcd}
S_1 \arrow[r, "a(g)"] \arrow[d, "\pi_{|S_1}"] & S_1^g \arrow[r, "i"] \arrow[d, "\pi_{|S_1^g}"] & G_2 \arrow[d, "\pi_{|G_2}"] \\
{[K_1, K_1]} \arrow[r, "a(\bar{g})"] & {[K_1, K_1]^{\bar{g}}} \arrow[r, leftrightarrow, "\mathrm{id}"] & {[K_2, K_2]}
\end{tikzcd}$$

Thus S_1^g is a Levi subgroup of G_2 and therefore a conjugate of S_2 in G_2. This proves that S_1 and S_2 are conjugate in G. □

Remark 5.1. Suppose G is a definable group and $\pi \colon G \to G/\mathcal{N}(G)$ is the canonical projection. The proof above shows that:

(a) if S is a maximal definable semisimple definably compact subgroup of G, then $\pi(S) = [K, K]$, where K is a maximal definable definably compact subgroup of $G^0/\mathcal{N}(G)$ containing $\pi(S)$;

(b) conversely, for every maximal definable definably compact subgroup K of $G^0/\mathcal{N}(G)$ there is a maximal definable semisimple definably compact subgroup S of G such that $\pi(S) = [K, K]$.

Extensions of a definably compact group by a torsion-free group. In [2] it is shown that every definable extension of a definable torsion-free group by a definable definably compact group splits definably in a direct product. We consider here the specular case of a definable extension of a definably compact group by a torsion-free definable group, and we will make use of the following:

Lemma 5.2. *Let G be a definable group. Then:*

(i) *for every maximal definable semisimple definably compact subgroup S of G, there is a 0-Sylow T of G such that $T \subset N_G(S)$;*

(ii) *for every 0-Sylow T of G there is a maximal definable semisimple definably compact subgroup S of G such that $T \subset N_G(S)$.*

Proof. We can suppose G is definably connected.

Note that $(i) \Leftrightarrow (ii)$ since all 0-Sylow are conjugate (Fact 2.4) and all maximal definable semisimple definably compact subgroups are conjugate as well (Proposition 1.2). Indeed for every $g \in G$,

$$T \subset N_G(S) \ \Leftrightarrow \ T^g \subset N_G(S^g).$$

We show (i). Let S be a maximal definable semisimple definably compact subgroup of G and consider the canonical projection $\pi \colon G \to G/\mathcal{N}(G)$. Suppose K is a maximal definable definably compact subgroup of $G/\mathcal{N}(G)$ containing $\pi(S)$. Define $G_1 := \pi^{-1}(K)$ and $G_2 := [K, K]$. Note that S is a definable cofactor of $\mathcal{N}(G)$ in G_2, so we can apply the same argument used in the proof of Claim I of Theorem 4.1 to find a 0-Sylow T of G_1 in $N_G(S)$. Note that $E(G/G_1) = E(H) = \pm 1$ (where H is a torsion-free definable complement of K in $G/\mathcal{N}(G)$), so T is a 0-Sylow of G as well. □

Proposition 5.3. *Let G be a definable extension of a definable definably connected definably compact group K by a definable torsion-free group N. Then the exact sequence*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} K \longrightarrow 1$$

splits abstractly, and it splits definably if and only if $\mathcal{A}(G)$ is trivial.

Moreover for every direct complement C of $\mathcal{A}(G)$ in a 0-Sylow T of G (see Remark 2.5), there is a cofactor K_C of N in G such that

$$K_C = \bigcup_{x \in K} C^x.$$

The commutator subgroup $[K_C, K_C]$ is definable, and is a maximal definable semisimple definably compact subgroup of G.

Proof. Note that $N = \mathcal{N}(G)$. Let us consider the possible cases for K:

If K is *abelian*, then G is solvable and the thesis follows by Corollary 3.2 and Proposition 3.3. In this case $K_C = C$ and $[K_C, K_C]$ is trivial.

If K is *semisimple*, then every definable Levi subgroup S of G (see Fact 1.7) is a definable cofactor of N and the extension above splits definably. Every definable maximal torus T of S is a 0-Sylow of G (Fact 2.4) and then $\mathcal{A}(G) = \{e\}$. Note that definable Levi subgroups of G are also maximal definable definably compact subgroups.

Suppose finally K is *neither abelian nor semisimple*. Let T be a 0-Sylow of G, and C a direct complement of $\mathcal{N}(T) = \mathcal{A}(G)$ in T. By 1.2, 5.1 and 5.2 there is a maximal definable semisimple definably compact subgroup S of G which is definably isomorphic to $[K, K]$ by $\pi_{|S} : S \to [K, K]$, and such that $T \subset N_G(S)$. In particular $C \subset N_G(S)$ and therefore $K_C := C \cdot S$ is a subgroup. The map $\pi_{|K_C} : K_C \to K$ is an isomorphism, $[K_C, K_C] = S$, and $\pi(T) = \pi(C)$ is a maximal definable torus of K. It follows that

$$K = \bigcup_{x \in K} \pi(C)^x,$$

and therefore

$$K_C = \bigcup_{x \in K_C} C^x.$$

If T is definably compact (and therefore $\mathcal{A}(G)$ is trivial), then K_C is definable.

Otherwise, suppose for a contradiction that there is a definable cofactor K_1 of N in G. Then every definable maximal torus of K_1 would be a definably compact 0-Sylow of G, in contradiction with the fact that T is not. □

A definable Lie-like decomposition. We give now the proof of Theorem 1.5.

Lemma 5.4. *Let A be a 0-group. Suppose C is a cofactor of $\mathcal{N}(A)$ in A. Then there are no proper definable subgroup of A containing C.*

Proof. If not, suppose B is a proper definable subgroup of A containing C. Set $N = \mathcal{N}(A)$. Since A/B is definably isomorphic to $(A \cap N)/(B \cap N)$, it follows that $E(A/B) = E((A \cap N)/(B \cap N)) = \pm 1$ (Fact 2.2), in contradiction with the fact that A is a 0-group. □

Proof of Theorem 1.5. If $\mathcal{N}(G)$ is trivial, then $\mathcal{A}(G)$ is trivial too and the thesis follows by Fact 1.7. So suppose $\mathcal{N}(G)$ is infinite and consider the canonical projection $\pi\colon G \to G/\mathcal{N}(G)$.

Fix T a 0-Sylow of G and C a cofactor of $\mathcal{N}(T) = \mathcal{A}(G)$ in T. Set $T_1 := \pi(T)$ and let K_1 be a maximal definable definably compact subgroup of $G/\mathcal{N}(G)$ containing T_1. Let H_1 be a definable torsion-free complement of K_1 in $G/\mathcal{N}(G)$. Define $G_1 := \pi^{-1}(K_1)$ and $H = \pi^{-1}(H_1)$. By Proposition 5.3 applied to G_1 there is a subgroup $K = K_C$ of G_1 such that K and H satisfy $(**)$. Moreover $[K,K]$ is definable and is a maximal definable semisimple definably compact subgroup of G_1 (and therefore of G) and $K = C \cdot [K,K]$.

Let P be the smallest definable subgroup of G containing K. In order to show $(*)$ we prove now some claims:

Claim I. $P = T \cdot [K,K]$.

Proof of Claim I. Define $S := [K,K]$. Since $S \lhd K$ and $C \subset K$, of course $C \subset N_G(S)$. Moreover $T = \mathcal{A}(G) \times C$ and $\mathcal{A}(G)$ is central in G, so $T \subset N_G(S)$. It follows that $T \cdot S$ is a definable subgroup containing K, so $P \subseteq T \cdot S$. On the other hand, by Lemma 5.4, $T \subset P$, so $T \cdot S \subseteq P$, and Claim I is proved. □

Since $T = \mathcal{A}(G) \times C$, $K = C \cdot [K,K]$ and $\mathcal{A}(G) \subseteq Z(G)$ (Proposition 2.6), it follows that $P = \mathcal{A}(G) \times K$.

Claim II. $(T \cap S)^0$ is a 0-Sylow of S.

Proof of Claim II. Being abelian (because T is abelian), definably connected and definably compact (because S is definably compact), $(T \cap S)^0$ is a 0-group (Fact 2.4). Moreover by Claim I the definable set $S/(T \cap S)$ is in definable bijection with P/T, therefore

$$E(S/(T\cap S)^0) = E(P/T)E((T\cap S)/(T\cap S)^0) \neq 0,$$

and by Fact 2.4, Claim II is proved. □

By Claim II and Fact 1.7, it follows that $S = \bigcup_{x \in S}(T \cap S)^x$ and $S \subset \bigcup_{x\in P} T^x$. For the other inclusion, note that

$$T \subset N_G(S) \quad \Rightarrow \quad T^y \subseteq T \cdot S = P \quad \forall\, y \in S \quad \Rightarrow \quad T^x \subseteq P \;\forall\, x \in P.$$

Therefore $P = \bigcup_{x \in P} T^x$.

Conversely, assume S is a maximal definable semisimple definably compact subgroup of G, and take T to be a 0-Sylow of G such that $T \subset N_G(S)$ (Lemma 5.2). Define $P := T \cdot S$ and apply Proposition 5.3 to P to find K. Given $K_1 = \pi(K) \subset G/\mathcal{N}(G)$, take H_1 to be a definable torsion-free complement of K_1 and set $H = \pi^{-1}(H_1)$. □

Maximal definably compact subgroups and $\mathcal{A}(G)$. In this last subsection we complete the proof of Theorem 1.3.

Proposition 5.5. *Let G be a definable definably connected group. The following are equivalent:*

- (i) *G has a maximal definable definably compact subgroup.*
- (ii) *G admits a definable compact-torsion-free decomposition.*
- (iii) *Every 0-subgroup of G is definably compact (i.e. $\mathcal{A}(G) = \{e\}$).*

Proof. If G is solvable, then see Proposition 3.3. Assume G is not solvable.

Suppose $G = K \cdot H$ is a definable compact·torsion-free decomposition. Then by Theorem 1.5, $\mathcal{A}(G) = \{e\}$. Note that K is a maximal definable definably compact subgroup of G. If not, let K_1 be a maximal definable definable definably compact subgroup of G containing K. Then $K_1 \cap H$ is torsion-free (because H is torsion-free) and definably compact (because every definable subgroup is closed by [21]). So $K_1 \cap H = \{e\}$ and $K_1 = K$.

Assume now G does not admit a definable compact·torsion-free decomposition. Then by Theorem 1.5, $\mathcal{A}(G)$ is infinite. Suppose, for a contradiction, K_1 is a maximal definable definably compact subgroup of G. We can assume K_1 is definably connected. Let T_1 be a 0-Sylow of K_1, and T be a 0-Sylow of G containing T_1. Note that $K_1 = \bigcup_{x \in K_1} T_1^x$ (Fact 1.7), and $T = \mathcal{A}(G) \times T_1 \times C_1$ for some infinite abelian (abstract) group C_1 (see Remark 2.5). Define $C = T_1 \times C_1$, and apply Theorem 1.5 to find an abstract subgroup K satisfying $(**)$. Then for every finite subgroup $F \subset C_1$, we get $K_1 \subsetneq F \cdot K_1 \subsetneq K$, in contradiction with the fact that K is a maximal definable definably compact subgroup of G. □

Proof of Theorem 1.3. The first part is Proposition 2.6. The second part follows by Proposition 5.5. □

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