MDAL (A)

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Es. 1

$$V = \{ p \in \mathbb{R}[x] : \deg p \leq 3 \}$$

$$W \leq V, \quad W = \{ p \in V : p(1) = 0 \}$$

ESERCIZIO. W è sottosp. vettoriale

(a) Trovare una base W e dim W

$$V \xrightarrow{f} \mathbb{R}^4$$

$$p \mapsto (a_0, a_1, a_2, a_3)$$ → vettore dei coeff. di p

↓

polinomio $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

f isomorfismo (lineare + invertibile) $\Longrightarrow$ si può lavorare in $\mathbb{R}^4$ anziché in V

$$p(x) = \sum_{i=0}^{3} a_i x^i \in V$$

$$p \in W \Leftrightarrow p(1) = 0 \Leftrightarrow a_0 + a_1 + a_2 + a_3 = 0$$

$$\tilde{W} := f(W) = \left\{ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4 : a_0 + a_1 + a_2 + a_3 = 0 \right\}$$

INTUITIVAMENTE: $\dim \tilde{W} = 4 - 1 = 3$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ -a_0 - a_1 - a_2 \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

è una scrittura valida per ogni el. di $\tilde{W}$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ generano $\tilde{W}$

$$\tilde{W} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right)$$

Per l'indipendenza:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow{R_4 \to R_4+R_1+R_2+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Ci sono 3 pivot $\Rightarrow$ i tre vettori sono lin. indip.

$\Rightarrow$ formano una base di $\widetilde{W}$

$$\Rightarrow \dim \widetilde{W} = 3$$

$$\mathcal{B}_{\widetilde{W}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Passando a W:

$$\mathcal{B}_W = \{ 1-x^3, x-x^3, x^2-x^3 \}$$

Es. 2

$$f : \mathbb{R}^3 \to \mathbb{R}^3 , \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + 2y + 3z \\ y + z \end{pmatrix}$$

(a) Determinare $\operatorname{Ker}(f)$, $\operatorname{Imm}(f)$

$$\mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$$

matrice di f risp. alle basi canoniche $\mathcal{E}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

dipende dai primi due

$$\operatorname{Imm}(f) = \operatorname{Span}\left(\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}_{\text{indipendenti (non multipli)}}, \cancel{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}} \right)$$

$\Longrightarrow \dim \operatorname{Imm}(f) = 2$

$$\dim \ker(f) + \dim \operatorname{Imm}(f) = 3$$

$$\Rightarrow \dim \ker(f) = 3 - 2 = 1$$

Osservazione

$f: V \to W$ lineare, $v_1, v_2, v_3 \in V$ indip.

$$a f(v_1) + b f(v_2) + c f(v_3) = 0 \iff$$

$$\underset{f \text{ lin.}}{\iff} f(a v_1 + b v_2 + c v_3) = 0$$

$$\iff a v_1 + b v_2 + c v_3 \in \ker f.$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \ker f$$

(b) Trovare $g : \mathbb{R}^3 \to \mathbb{R}^3$ con $g \neq 0$ tale che $g \circ f = 0$.

$$\mathbb{R}^3 \xrightarrow{\ f\ } \mathbb{R}^3 \xrightarrow{\ g\ } \mathbb{R}^3$$

$$\begin{array}{ccccc}
\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & \longmapsto & \underline{0} & \longmapsto & \underline{0} \\[2ex]
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & \longmapsto & \underline{0} \\[2ex]
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} & \longmapsto & \underline{0} \\[2ex]
 & & \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} & \longmapsto & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{array}$$

base di Imm(f)

compl. a base di $\mathbb{R}^3$

In questo modo g è ben definita perché definita su una base di $\mathbb{R}^3$.
In più, $g \neq 0$ poiché $g\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ e
$g|_{\mathrm{Imm} f} = 0$ per costr. $\Longrightarrow g \circ f = 0$.

Es. 3

Cifra delle unità di 7^{2020}.

$n \geq 0$ intero $\Longrightarrow n = \sum_{i=0}^{r} c_i \cdot 10^i =$

$$= c_0 + c_1 \cdot 10 + \dots + c_r 10$$

cifra delle unità

$$\Longrightarrow \boxed{n \equiv c_0 \pmod{10}}$$

$$7^1 = 7, \quad 7^2 = 49 \equiv -1 \ (10)$$

$$\Longrightarrow 7^4 = (7^2)^2 \equiv 1 \ (10)$$

$$\Longrightarrow 7^{2020} = (7^4)^{505} \equiv 1^{505} \equiv 1 \ (10)$$

$\Longrightarrow$ cifra delle unità $= 1$.

ESERCIZIO. Qual è la cifra delle decine?

(pensare mod 100) [Sol. 0]

Es. 4

$$12x \equiv 5 \quad (29)$$

$(29, 12) = 1 \overset{(*)}{\Longrightarrow}$ ha sol.

$(*)$ $ax \equiv b \ (n)$ ha soluz. $\Updownarrow$ $(a,n) \mid b$

Cerchiamo l'inv. di 12 mod 29:

$$29 = 12 \cdot 2 + 5$$
$$12 = 5 \cdot 2 + 2$$
$$5 = 2 \cdot 2 + 1$$
$$2 = 1 \cdot 2 + 0$$

	29	12
29	1	0
12	0	1
5	1	−2
2	−2	5
1	5	−12

$$1 = 29 \cdot 5 + 12 \cdot (-12)$$

$$\Longrightarrow 1 \equiv 12 \cdot (-12) \quad (\text{mod } 29)$$

$$(-12) \cdot 12x \equiv (-12) \cdot 5 \quad (29)$$

$$x \equiv -60 \equiv -2 \quad (29)$$

Es. 5 $16x \equiv 10 \ (14)$

$(16,14) = 2 \mid 10 \Longrightarrow$ ha sol.

$$2x \equiv 10 \ (14)$$

$\underset{(*)}{\Longrightarrow}$ $x \equiv 5 \ (7)$

(*)

$$\boxed{\begin{array}{c} ax \equiv b \ (n) \\ d = (a,n) \\ \Downarrow \\ \frac{a}{d} x \equiv \frac{b}{d} \ \left(\frac{n}{d}\right) \end{array}}$$

primi tra loro

ESERCIZI

(i) $\begin{cases} 12x \equiv 5 \ (29) \\ 16x \equiv 10 \ (14) \end{cases}$

(ii) $\begin{cases} (6a-1)x \equiv 1 \ (21) \\ x \equiv a \ (35) \end{cases}$ con $a \in \mathbb{Z}$