TD L1 BST (Groupe 6) – 03/11/25

Intégration par parties

Rappels :

1) $\int e^x \, dx = e^x$

2) Si $a \neq 0$, $\int e^{ax} \, dx = \frac{1}{a} e^{ax}$

$$(e^{ax})' = a e^{ax}$$

$$\left(\frac{1}{a} e^{ax}\right)' = \frac{1}{a} (e^{ax})' = \frac{1}{a} \cdot a e^{ax} = e^{ax}$$

3)

$$\begin{tikzcd}
\sin(x) \arrow[d, "\text{dér}"] \\
\cos(x) \arrow[d, "\text{dér}"] \arrow[u, bend right, "\text{prim.}"'] \\
-\sin(x) \arrow[d, "\text{dér}"] \arrow[u, bend right, "\text{prim}"'] \\
-\cos(x) \arrow[u, bend right, "\text{prim}"'] \arrow[uuu, bend left=60, "\text{dér}"]
\end{tikzcd}$$

$\sin(x) \xrightarrow{\text{prim}} -\cos(x)$

$$\int \cos(x) \, dx = \sin(x)$$

4)
$$\int x^a \, dx = \begin{cases} \dfrac{x^{a+1}}{a+1}, & \text{si } a \neq -1 \\ \ln(|x|), & \text{si } a = -1 \end{cases}$$

Rappel : $x^{-1} = \frac{1}{x}$

Comment calculer $\int x\, e^x \, dx$?

On note que $\int f(x) \cdot g(x) \, dx \neq \int f(x) \, dx \cdot \int g(x) dx$

[En effet, on a $(f(x) \cdot g(x))' \neq f'(x) \cdot g'(x)$

parce que : $(f(x) \cdot g(x))' = f'(x) g(x) + f(x) g'(x)$]

Alors $\int (f(x) \cdot g(x))' \, dx = \int \left(f'(x) g(x) + f(x) g'(x)\right) dx$

$= \int f'(x) g(x) \, dx + \int f(x) g'(x) dx$

Mais $\int (f(x) \cdot g(x))' \, dx = f(x) \cdot g(x)$

déf de primitive

Donc on a trouvé la FORMULE D'INTÉGRATION PAR PARTIES

$$\int f'(x) \cdot g(x) \, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, dx$$

Ex. 1) $$\int \underbrace{x}_{g(x)} \underbrace{e^x}_{f'(x)} dx = \underbrace{e^x \cdot x}_{f(x) \cdot g(x)} - \int \underbrace{e^x \cdot 1}_{f(x) \cdot g'(x)} dx$$

$g'(x) = 1$

$f(x) = \int e^x dx = e^x$

$$= e^x \cdot x - \int e^x dx$$

$$= e^x \cdot x - e^x$$

$$\left(= e^x (x-1) \right)$$

Si on pose

$$\int \underbrace{x}_{f'(x)} \underbrace{e^x}_{g(x)} = \underbrace{\frac{x^2}{2} \cdot e^x}_{f(x) \cdot g(x)} - \int \underbrace{\frac{x^2}{2} e^x}_{f(x) \cdot g'(x)} dx$$

$f(x) = \int x\, dx = \frac{x^2}{2}$

$g'(x) = e^x$

mais c'est beaucoup plus difficile !

L'idée est de choisir $f'(x)$ et $g(x)$ de façon que le deuxième intégral soit plus facile à calculer !

2) $$\int \underbrace{x}_{g(x)} \underbrace{\sin(x)}_{f'(x)}\,dx = \underbrace{-\cos(x)\cdot x}_{f(x)\cdot g(x)} - \int \underbrace{(-\cos(x))\cdot 1}_{f(x)\cdot g'(x)}\,dx$$

$$f(x) = \int \sin(x)\,dx = -\cos(x)$$

$$g'(x) = 1$$

$$= -\cos(x)\cdot x + \int \cos(x)\,dx$$

$$= -\cos(x)\cdot x + \sin(x)$$

3) $$\int \underbrace{x}_{g(x)} \underbrace{\ln(x)}_{f'(x)}\,dx$$

$$f(x) = \int \ln(x)\,dx \quad \ldots \quad \text{NAHH}$$

$$\int \underbrace{x}_{f'(x)} \underbrace{\ln(x)}_{g(x)}\,dx = \underbrace{\frac{x^2}{2}}_{f(x)}\cdot \underbrace{\ln(x)}_{g(x)} - \int \underbrace{\frac{x^2}{2}}_{f(x)}\cdot \underbrace{\frac{1}{x}}_{g'(x)}\,dx$$

$$f(x) = \int x\,dx = \frac{x^2}{2}$$

$$g'(x) = \frac{1}{x}$$

$$= \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{2}\cdot\frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}.$$

4) Concaténation de la méthode

$$\int \underbrace{x^2}_{g(x)} \underbrace{\cos(x)}_{f'(x)}\, dx = \underbrace{\sin(x) \cdot x^2}_{f(x)\cdot g(x)} - \int \underbrace{\sin(x) \cdot 2x}_{f(x)\cdot g'(x)}\, dx$$

$g'(x) = 2x$

$f(x) = \int \cos(x)\, dx = \sin(x)$

$$= \sin(x) \cdot x^2 - 2 \underbrace{\int x \cdot \sin(x)\, dx}_{\text{déjà vu}}$$

$$\int \underbrace{x}_{g(x)} \cdot \underbrace{\sin(x)}_{f'(x)}\, dx = -\cos(x) \cdot x - \int (-\cos x) \cdot 1\, dx$$

$$= -\cos(x) \cdot x + \int \cos(x)\, dx$$

$$= \underbrace{-\cos(x) \cdot x + \sin(x)}$$

$$= \sin(x) \cdot x^2 - 2 \left(-\cos(x) \cdot x + \sin(x) \right)$$

$$= \sin(x) \cdot x^2 + 2\cos(x) \cdot x - 2 \sin(x) .$$

5) $\displaystyle\int \ln(x)\,dx = \int \underbrace{1}_{f'(x)} \cdot \underbrace{\ln(x)}_{g(x)}\,dx$

$f(x) = \int 1\,dx = x$

$g'(x) = \frac{1}{x}$

$$= \underbrace{x \cdot \ln(x)}_{f(x)\cdot g(x)} - \int \underbrace{x \cdot \frac{1}{x}}_{f(x)\cdot g'(x)}\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x\,.$$

Exercice 32. Calculer, grâce à le méthode d'intégration par partie, les primitives suivantes :

a. $\displaystyle\int xe^x\,dx$ ✓

b'. $\displaystyle\int x^2\ln(x)\,dx$

c. $\displaystyle\int (x+1)\sin(x)\,dx$

d. $\displaystyle\int e^x\sin(x)\,dx$

e. $\displaystyle\int \ln(x)\,dx$ ✓

f. $\displaystyle\int x^2\sin(x)\,dx$

g. $\displaystyle\int x^2e^x\,dx$

~~h.~~ $\displaystyle\cancel{\int \frac{x}{(\cos(x))^2}\,dx}$

i. $\displaystyle\int xe^{3x}\,dx$

(Ind. pour i. utiliser que, pour $a \in \mathbb{R}$ constant, $\int e^{ax}dx = \frac{1}{a}e^{ax} + \cancel{C}$).
et $a \neq 0$

c) , f) , g) , i) , b') , $\underbrace{\text{d)}}_{\text{Ind}}$

PAR PARTIES deux fois et après poser:
$I = \int e^x \sin(x)\,dx$

i) $\int \underbrace{x}_{g}\ \underbrace{e^{3x}}_{f'}\,dx = \frac{1}{3}e^{3x}\cdot x - \int \frac{1}{3}e^{3x}\,dx$

$f(x) = \int e^{3x}dx = \frac{1}{3}e^{3x}$

$g'(x) = 1$

$= \frac{1}{3}e^{3x}\cdot x - \frac{1}{3}\int e^{3x}\,dx$

$= \frac{1}{3}e^{3x}\cdot x - \frac{1}{3}\cdot\frac{1}{3}e^{3x}$

$= \frac{1}{3}e^{3x}\cdot x - \frac{1}{9}e^{3x}$.

Pour les autres : 23/10/25 Groupes 1 et 3
sur ma page personnelle

https://poisson.phc.dm.unipi.it/~dinunzio/2025_10_23-L1_BST_01