TD L1 BST (Groupe 1) - 06/11/25

Calcul d'intégrales : CHANGEMENT de VARIABLE

- $\int e^y \, dy = e^y$
- $\int \cos(AAAH) \, d(AAAH) = \sin(AAAH)$

Introduction

$$\int e^{\underbrace{x+7}_{y}} \, dx = \int e^x \cdot \underbrace{e^7}_{\text{constante}} \, dx = e^7 \int e^x \, dx$$

$$= e^7 \cdot e^x$$

$$= e^{\underbrace{x+7}_{y}}$$

Soit $f = f(x)$ une fonction dérivable. On appelle « ÉLÉMENT DIFFERENTIEL (de f) »

$$df = f'(x) \, dx$$

Si on pose $y = x+7$, alors y est une fonction dérivable, avec $y'(x) = (x+7)' = 1$

$$\Rightarrow \quad dy = \underbrace{y'(x)}_{1} \, dx = dx$$

Si on a $y = x^2 + 2x$, alors

$$y' = 2x + 2$$

$$dy = y'\,dx = (2x+2)\,dx$$

Regle pour le changement de variable dans les intégrals:

Si $y = y(x)$

alors $$\int f(y)\,dy = \int f(y(x))\,y'(x)\,dx$$

Ex. 1 $$\int (x+1)^6\,dx$$

$$y = x+1$$

$$y' = 1$$

$$dy = \underbrace{y'}_{1}\,dx = dx$$

$$\int (\underbrace{x+1}_{y})^6\,\underbrace{dx}_{dy} = \int y^6\,dy = \frac{y^7}{7} = \frac{(x+1)^7}{7}$$

Ex. 2 $\left[\text{Rappel : } \int \frac{1}{z}\, dz = \ln(|z|) \right]$

$$\int \frac{1}{2x+3}\, dx$$

$z = 2x+3$

$z' = 2$

$dz = z'\, dx = 2\, dx$

$\Rightarrow dx = \frac{1}{2}\, dz$

$$\int \frac{1}{\underbrace{2x+3}_{z}} \underbrace{dx}_{\frac{1}{2}dz} = \int \frac{1}{z} \cdot \frac{1}{2}\, dz$$

$$= \frac{1}{2} \int \frac{1}{z}\, dz$$

$$= \frac{1}{2} \ln(|z|)$$

$$= \frac{1}{2} \ln(|2x+3|).$$

Ex. 3 $\displaystyle\int \frac{x}{x^2+1}\, dx$

$y = x^2+1$

$y' = 2x$

$dy = 2x \cdot dx$

$\Rightarrow dx = \frac{1}{2x}\, dy$

$$\Rightarrow \int \frac{x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\cdot x\,dx = \int \frac{1}{y}\cdot \cancel{x}\cdot\frac{1}{2\cancel{x}}\,dy = \int \frac{1}{2y}\,dy$$

$$= \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\ln(|y|) = \frac{1}{2}\ln(|x^2+1|)$$

$$= \frac{1}{2}\ln(x^2+1)$$

parce que $x^2+1>0$ pour tout $x\in\mathbb{R}$

Ex. 4 $\int x\, e^{-x^2}\, dx$

$y = -x^2$

$y' = -2x$

$dy = -2x\,dx$

$\Rightarrow dx = -\frac{1}{2x}\,dy$

$$= \int \cancel{x}\, e^{y}\left(-\frac{1}{2\cancel{x}}\right)dy = \int e^{y}\cdot\left(-\frac{1}{2}\right)dy$$

$$= -\frac{1}{2}\int e^{y}\,dy$$

$$= -\frac{1}{2}e^{y}$$

$$= -\frac{1}{2}e^{-x^2}$$

Ex. 4 $\int x\, e^{-x^2}\, dx$

$u = e^{-x^2}$

$u' = -2x\, e^{-x^2}$

Rappel

$(e^{f(x)})' = f'(x)\, e^{f(x)}$

$du = (-2x\, e^{-x^2})\, dx$

$\Rightarrow \quad dx = \dfrac{du}{-2x\, e^{-x^2}}$

$$= \int \cancel{x}\, \cancel{e^{-x^2}} \frac{du}{-2\cancel{x}\, \cancel{e^{-x^2}}} = \int \left(\frac{1}{-2}\right) du$$

$$= -\frac{1}{2} \int du = -\frac{1}{2} u$$

$$= -\frac{1}{2} e^{-x^2}$$

$\int f(u)\, u'\, du = F(u)$ avec F primitive de f

Ex. 5 $\int (\cos(x))^{12} \sin(x)\, dx$

$y = \cos(x)$

$y' = -\sin(x)$

$dy = -\sin(x)\, dx$

$\sin(x)\, dx = -dy$

$= \int y^{12} (-dy)$

$= -\int y^{12}\, dy = -\frac{y^{13}}{13} = -\frac{(\cos(x))^{13}}{13}$

Ex. 6 $\int \frac{x}{\sqrt[3]{2x^2+7}}\, dx$

$y = 2x^2 + 7$

$y' = 4x$

$dy = 4x\, dx$

$\frac{1}{4} dy = x\, dx$

$= \int \frac{1}{\sqrt[3]{y}} \frac{1}{4} dy = \frac{1}{4} \int \frac{1}{\sqrt[3]{y}} dy = \frac{1}{4} \int \frac{1}{y^{1/3}} dy$

$$= \frac{1}{4}\int y^{-1/3}\,dy = \frac{1}{4}\cdot\frac{y^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{1}{4}\cdot\frac{y^{2/3}}{2/3}$$

$$= \frac{3}{4\cdot 2}\,y^{2/3} = \frac{3}{8}\,y^{2/3} = \frac{3}{8}\,(2x^2+7)^{2/3}$$

$$\frac{1}{4}\cdot\frac{1}{2/3} = \frac{1}{4\cdot\frac{2}{3}} = \frac{1}{\frac{4\cdot 2}{3}} = \frac{3}{4\cdot 2}$$

$$= \frac{3}{8}\sqrt[3]{(2x^2+7)^2}$$

$$\circledast \quad dy = 4x\,dx \implies dx = \frac{1}{4x}\,dy$$

$$\implies \int \frac{x}{\sqrt[3]{2x^2+7}}\,dx = \int \frac{\cancel{x}}{\sqrt[3]{y}}\,\frac{1}{4\cancel{x}}\,dy$$

Exercice 33. A l'aide d'un changement de variable approprié, calculer les primitives suivantes, en précisant les intervalles où ce calcul est valable :

Ex.2

~~a.~~ $\displaystyle\int \frac{1}{2x+3}dx$ c. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}dx$ e. $\displaystyle\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx$ g. $\displaystyle\int xe^{x^2}\,dx$ i. $\displaystyle\int \frac{x\,dx}{(x^2+1)^3}$

~~b.~~ $\displaystyle\int \frac{x}{x^2+1}dx$ d. $\displaystyle\int \frac{\sin x}{\sqrt{2+\cos(x)}}dx$ f. $\displaystyle\int \frac{\ln(x)}{x}dx$ h. $\displaystyle\int \frac{\cos(x)}{(1+\sin x)^4}dx$ j. $\displaystyle\int \tan(x)dx$

Ex.3

Ind. Pour f. poser $y = \ln(x)$

Pour j. rappel $\tan(x) = \frac{\sin(x)}{\cos(x)}$

Sol.

a. et b. déjà faits.

c.)
$$\int \frac{x}{\sqrt{x^2+1}}\,dx \underset{\substack{y = x^2+1 \\ y' = 2x \\ dy = 2x\,dx}}{=} \int \frac{1}{\sqrt{y}} \cdot \underbrace{x\,dx}_{x\,dx = \frac{1}{2}dy}$$

$$= \int \frac{1}{\sqrt{y}} \cdot \frac{1}{2}\,dy$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{y}}\,dy = \frac{1}{2}\int y^{-1/2}\,dy = \frac{1}{2}\,\frac{y^{-1/2+1}}{-\frac{1}{2}+1}$$

$$= \cancel{\frac{1}{2}}\,\frac{y^{1/2}}{\cancel{1/2}} = y^{1/2} = \sqrt{y} = \sqrt{x^2+1}$$

d.) $\displaystyle\int \frac{\sin(x)}{\sqrt{2+\cos(x)}}\,dx \underset{\uparrow}{=} \int \frac{1}{\sqrt{y}} \sin(x)\,dx = \int \frac{1}{\sqrt{y}}(-dy)$

$y = 2+\cos(x)$

$y' = -\sin(x)$

$dy = -\sin(x)\,dx \longleftrightarrow \sin(x)\,dx = -dy$

$\displaystyle = -\int \frac{1}{\sqrt{y}}\,dy \underset{\uparrow}{=} -\frac{y^{1/2}}{1/2} = -2\sqrt{y}$

déjà vu

$= -2\sqrt{2+\cos(x)}$

e.) $\displaystyle\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx \underset{\uparrow}{=} \int \frac{1}{\sqrt{y}}\,dy = \frac{y^{1/2}}{1/2} = 2\sqrt{y}$

$y = x^2+x+1$

$y' = 2x+1$

$dy = (2x+1)\,dx$

$= 2\sqrt{x^2+x+1}$

f.) $\displaystyle\int \frac{\ln(x)}{x}\,dx \underset{\uparrow}{=} \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\ln(x)^2$

$u = \ln(x)$

$u' = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

peut-être écrit également

$\frac{1}{2}\ln^2(x)$

[NOTATION]

g.) $\displaystyle\int x\,e^{x^2}\,dx \underset{\uparrow}{=} \int e^y \cdot \frac{1}{2}\,dy = \frac{1}{2}\int e^y\,dy = \frac{1}{2}e^y$

$y = x^2$

$y' = 2x$

$dy = 2x\,dx \longleftrightarrow x\,dx = \frac{1}{2}dy$

$= \frac{1}{2}e^{x^2}$

Bsp:
$$y = e^{x^2}$$
$$y' = 2x\, e^{x^2}$$
$$dy = 2x\, e^{x^2}\, dx \implies x e^{x^2} dx = \frac{1}{2} dy$$
$$\implies \int x e^{x^2} dx = \int \frac{1}{2} dy = \frac{1}{2} y = \frac{1}{2} e^{x^2}.$$

h.) $$\int \frac{\cos(x)}{(1+\sin(x))^4} dx \overset{\uparrow}{=} \int \frac{dw}{w^4} = \int w^{-4} dw$$

$$w = 1+\sin(x)$$
$$w' = \cos(x)$$
$$dw = \cos(x)\, dx$$

$$= \frac{w^{-3}}{-3}$$
$$= -\frac{1}{3} (1+\sin(x))^{-3}$$
$$= -\frac{1}{3(1+\sin(x))^3}$$

i.) $$\int \frac{x}{(x^2+1)^3} dx \overset{\uparrow}{=} \int \frac{1}{u^3} \left(-\frac{1}{2}\right) du = -\frac{1}{2} \int u^{-3} du$$

$$u = x^2+1$$
$$u' = 2x$$
$$du = 2x\, dx \longleftrightarrow x\, dx = -\frac{1}{2} du$$

$$= -\frac{1}{2} \frac{u^{-2}}{-2} = \frac{1}{4u^2} = \frac{1}{4(x^2+1)^2}$$

1.) $\int \tan(x)\,dx = \int \frac{\sin(x)\,dx}{\cos(x)} = \int \frac{1}{u}(-du)$

$u = \cos(x)$

$u' = -\sin(x)$

$du = -\sin(x)dx$

$\sin(x)dx = -du$

$= -\int \frac{du}{u} = -\ln(|u|) = -\ln(|\cos(x)|).$