TD L1 BST (Groupes 1 et 3, LAS) – 20/11/25

Séance d'exercices : SOLUTIONS

Exercice 1. Déterminer les réels a et b tels que

$$\frac{x+1}{x^2+2x-3} = \frac{a}{x+3} + \frac{b}{x-1},$$

puis calculer

$$\int \frac{x+1}{x^2+2x-3}\,\mathrm{d}x.$$

Solution.

$$\frac{a}{x+3} + \frac{b}{x-1} = \frac{(x-1)a + (x+3)b}{(x+3)(x-1)} = \frac{ax - a + bx + 3b}{(x+3)(x-1)}$$

$$= \frac{(a+b)x - a + 3b}{x^2+2x-3} = \frac{x+1}{x^2-2x+3}$$

$$(a+b)x - a + 3b = x+1 = 1\cdot x + 1$$

$$\begin{cases} a+b = 1 \\ -a+3b = 1 \end{cases} \iff \begin{cases} a = 1-b \\ b-1+3b = 1 \end{cases} \iff \begin{cases} a = 1-b \\ 4b = 2 \end{cases} \iff \begin{cases} a = 1-1/2 = 1/2 \\ b = 1/2 \end{cases}$$

$$\int \frac{x+1}{(x+3)(x-1)}\,dx = \int \frac{1/2}{x+3}\,dx + \int \frac{1/2}{x-1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x+3}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$u = x-1 \Rightarrow dx = du$

$y = x+3$, $y' = 1$, $dy = dx$

$$= \frac{1}{2}\int \frac{1}{y}\,dy + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|y| + \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|x+3| + \frac{1}{2}\ln|x-1| = \overbrace{\frac{1}{2}\ln|(x+3)(x-1)|}^{\text{pas nécessaire}}$$

Exercice 2. Déterminer les réels a et b tels que

$$\frac{2x+5}{x^2+5x} = \frac{a}{x} + \frac{b}{x+5},$$

puis calculer

$$\int_1^2 \frac{2x+5}{x^2+5x}\,\mathrm{d}x.$$

Solution.

$\frac{2x+5}{x^2+5x}$ doit être égal à $\frac{a}{x} + \frac{b}{x+5} = \frac{(x+5)a + bx}{x(x+5)} = \frac{ax+5a+bx}{x^2+5x}$

$$= \frac{(a+b)x + 5a}{x^2+5x}$$

Donc $2x + 5 = (a+b)x + 5a$

$$\begin{cases} a+b = 2 \\ 5a = 5 \end{cases} \iff \begin{cases} b = 2-a = 2-1 = 1 \\ a = 1 \end{cases}$$

Alors

comme avant $y = x+5$ ...

$$\int_1^2 \frac{2x+5}{x^2+5x}\,dx = \int_1^2 \left(\frac{1}{x} + \frac{1}{x+5}\right) dx = \Big[\ln|x| + \ln|x+5|\Big]_1^2$$

$$= \ln(2) + \ln(7) - (\underbrace{\ln(1)}_{=0} + \ln(6))$$

$$= \ln(2\cdot 7) - \ln(6)$$

$$= \ln\left(\frac{2\cdot 7}{6}\right) = \ln\left(\frac{7}{3}\right).$$

« Par curiosité »

"avec un chang. de variable c'est plus facile"
(mais j'ai vous demandé de le faire comme avant)

Rmq. Si on pose $z = x^2+5x$, on a $z' = 2x+5$ et donc

$dz = (2x+5)dx \Rightarrow \int \frac{2x+5}{x^2+5x}\,dx = \int \frac{dz}{z} = \ln|z| = \ln|x^2+5x| = \ln|x| + \ln|x+5|$

Exercice 3. En utilisant la méthode d'intégration par parties, calculer

$$a)\ \int (x+2)e^x\,dx \qquad b)\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-x)\sin(x)\,dx \qquad c)\ \int x^4\ln(x)\,dx \qquad d)\ \int_0^{\pi} \sin^2(x)\,dx$$

plus compliqué

[Indication : pour d), écrire $\sin^2(x) = \sin(x)\cdot\sin(x)$ et utiliser la formule $\sin^2(x)+\cos^2(x) = 1$.]

Solution.

a) $$\int \underbrace{(x+2)}_{g}\underbrace{e^x}_{f'}dx = \underbrace{(x+2)}_{g}\underbrace{e^x}_{f} - \int \underbrace{1}_{g'}\cdot\underbrace{e^x}_{f}dx = (x+2)e^x - e^x$$

$$= (x+2-1)e^x$$

$$= (x+1)e^x.$$

$f' = e^x \Rightarrow f = e^x$

$g = (x+2) \Rightarrow g' = 1$

b) $$\int_{-\frac{\pi}{2}}^{\pi/2} (1-x)\sin(x)\,dx$$

Deux méthodes (équivalentes)

1. Méthode STANDARD : • on calcule $F(x) = \int (1-x)\sin(x)\,dx$;

• on calcule $[F(x)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = F(\frac{\pi}{2}) - F(-\frac{\pi}{2})$.

$$F(x) = \int \underbrace{(1-x)}_{g}\underbrace{\sin(x)}_{f'}dx = \underbrace{-\cos(x)}_{f}\underbrace{(1-x)}_{g} - \int \underbrace{(-\cos(x))}_{f}\underbrace{(-1)}_{g'}dx$$

$f' = \sin(x) \Rightarrow f = -\cos(x)$

$g = 1-x \Rightarrow g' = -1$

$$= (x-1)\cos(x) - \int \cos(x)\,dx$$

$$= (x-1)\cos(x) - \sin(x)$$

$[F(x)]_{-\pi/2}^{\pi/2}$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-x)\sin(x)\,dx = \Big[(x-1)\cos(x) - \sin(x)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \cancel{(\tfrac{\pi}{2}-1)\underbrace{\cos\tfrac{\pi}{2}}_{=0}} - \underbrace{\sin\tfrac{\pi}{2}}_{=1} - \Big(\cancel{(-\tfrac{\pi}{2}-1)\underbrace{\cos(-\tfrac{\pi}{2})}_{=0}} - \underbrace{\sin(-\tfrac{\pi}{2})}_{=-1}\Big)$$

$$= -1 - (-(-1)) = -1 - 1 = -2.$$

2. Méthode 2 : « laisser les extrema »

il faut faire

$$\int_{-\pi/2}^{\pi/2} \underbrace{(1-x)}_{g}\underbrace{\sin(x)}_{f'}\,dx = \Big[\underbrace{-\cos(x)}_{f}\cdot\underbrace{(1-x)}_{g}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\pi/2}\underbrace{(-\cos(x))}_{f}\cdot\underbrace{(-1)}_{g'}\,dx$$

$$= \left(-\cancel{\cos\left(\tfrac{\pi}{2}\right)}\left(1-\tfrac{\pi}{2}\right) - \left(-\cancel{\cos\left(-\tfrac{\pi}{2}\right)}\left(1+\tfrac{\pi}{2}\right)\right)\right) - \int_{-\frac{\pi}{2}}^{\pi/2}\cos(x)\,dx$$

$$= -\int_{-\pi/2}^{\pi/2}\cos(x)\,dx = -\big[\sin(x)\big]_{-\frac{\pi}{2}}^{\pi/2} = -(1-(-1)) = -2.$$

c) $$\int \underbrace{x^4}_{f'}\underbrace{\ln(x)}_{g}\,dx = \underbrace{\frac{x^5}{5}}_{f}\underbrace{\ln(x)}_{g} - \int \underbrace{\frac{x^5}{5}}_{f}\cdot\underbrace{\frac{1}{x}}_{g'}\,dx = \frac{x^5}{5}\ln(x) - \frac{1}{5}\int x^4\,dx$$

$f' = x^4 \Rightarrow f = \frac{x^5}{5}$

$g = \ln(x) \Rightarrow g' = \frac{1}{x}$

$$= \frac{x^5}{5}\ln(x) - \frac{1}{5}\cdot\frac{x^5}{5}$$

$$= \frac{x^5}{5}\left(\ln(x) - \frac{1}{5}\right).$$

d) On appelle $I = \int_0^\pi \sin^2(x)\,dx$. Alors

On utilise la Méthode 2 précédente

$$I = \int_0^\pi \underbrace{\sin(x)}_{f'}\cdot\underbrace{\sin(x)}_{g}\,dx = \Big[\underbrace{-\cos(x)}_{f}\,\underbrace{\sin(x)}_{g}\Big]_0^\pi - \int_0^\pi \underbrace{(-\cos(x))}_{f}\,\underbrace{\cos(x)}_{g'}\,dx$$

$f(x) = -\cos(x)$

$g'(x) = \cos(x)$

$$= -\cos(\pi)\underbrace{\sin(\pi)}_{=0} + \cos(0)\underbrace{\sin(0)}_{=0} + \int_0^\pi \cos^2(x)\,dx$$

$$= \int_0^\pi \cos^2(x)\,dx = \int_0^\pi (1-\sin^2(x))\,dx = \int_0^\pi 1\cdot dx - \int_0^\pi \sin^2(x)\,dx$$

$\sin^2(x) + \cos^2(x) = 1$

$$= [x]_0^\pi - I$$

$$= \pi - I.$$

Donc $I = \pi - I$, alors $2I = \pi$ d'où $\boxed{I = \frac{\pi}{2}}$.

Exercice 4. En utilisant un changement de variable approprié, calculer

$$a)\ \int 2x(x^2+2)^3\,\mathrm{d}x \qquad b)\ \int_{\sqrt[3]{9}}^{2} \frac{x^2}{(x^3-7)^4}\,\mathrm{d}x \qquad c)\ \int_0^{(\ln(2))^2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

[Indication : pour c), utiliser $y=\sqrt{x}$ et rappeler que $\sqrt{x}=x^{\frac{1}{2}}$.]

Solution.

a) On pose $u = x^2+2$. Alors $u' = 2x$ et donc $du = 2x\,dx$.

$$\int 2x(x^2+2)^3\,dx = \int u^3\,du = \frac{u^4}{4} = \frac{(x^2+2)^4}{4}.$$

b) $\int_{\sqrt[3]{9}}^{2} \frac{x^2}{(x^3-7)^4}\,dx$.

Deux méthodes

1. Méthode STANDARD

- on calcule $F(x)=\int \frac{x^2}{(x^3-7)^4}\,dx$;
- on calcule $[F(x)]_{\sqrt[3]{9}}^{2} = F(2)-F(\sqrt[3]{9})$.

$$F(x) = \int \frac{x^2}{(x^3-7)^4}\,dx = \int \frac{1}{u^4}\,\frac{1}{3}\,du = \frac{1}{3}\int u^{-4}\,du$$

$u = x^3-7$
$u' = 3x^2$
$du = 3x^2\,dx$
$\Rightarrow x^2\,dx = \frac{1}{3}du$

$$= \frac{1}{3}\cdot\frac{u^{-3}}{-3}$$

$$= -\frac{1}{9u^3}$$

$$= -\underbrace{\frac{1}{9(x^3-7)^3}}$$

$[F(x)]_{\sqrt[3]{9}}^{2}$

$$\int_{\sqrt[3]{9}}^{2} \frac{x^2}{(x^3-7)^4}\,dx = \left[-\frac{1}{9(x^3-7)^3}\right]_{\sqrt[3]{9}}^{2} = -\frac{1}{9(8-7)^3} - \left(-\frac{1}{9}\,\frac{1}{(9-7)^3}\right)$$

$$= -\frac{1}{9} + \frac{1}{9}\cdot\frac{1}{8} = -\frac{7}{72}.$$

2. Méthode 2 « laisser les extrema »

$$\int_{\sqrt[3]{9}}^{2} \frac{x^2}{(x^3-7)^4}\,dx = \int_2^1 \frac{1}{u^4}\cdot\frac{1}{3}\,du = \frac{1}{3}\int_2^1 u^{-4}\,du$$

comme avant $\begin{cases} u = x^3-7 \\ x^2dx = \frac{1}{3}du \end{cases}$

mais : $x = \sqrt[3]{9} \Rightarrow u = (\sqrt[3]{9})^3 - 7 = 2$
$x = 2 \Rightarrow u = 2^3 - 7 = 1$

$$= \frac{1}{3}\left[\frac{u^{-3}}{-3}\right]_2^1$$

$$= \frac{1}{3}\left(\frac{1^{-3}}{-3} - \frac{2^{-3}}{-3}\right)$$

$$= \frac{1}{3}\left(-\frac{1}{3} + \frac{1}{24}\right)$$

$$= \frac{1}{3}\left(-\frac{7}{24}\right) = -\frac{7}{72}.$$

c) On utilise la méthode 2 précédente.

- Élément différentiel :

$y = \sqrt{x} = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$

$\Rightarrow \frac{1}{\sqrt{x}}\,dx = 2\,dy$

- Extrema :

$x = 0 \Rightarrow y = \sqrt{0} = 0$

$x = \ln^2(2) \Rightarrow y = \sqrt{\ln^2(2)} = |\ln(2)| = \ln(2)$ Parce que $\ln(2) > 0$

$\ln(x) > 0$ pour $x > 1$

$$\int_0^{\ln^2(2)} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^{\ln(2)} e^y\, 2\,dy = 2\int_0^{\ln(2)} e^y\,dy$$

$$= 2\left[e^y\right]_0^{\ln(2)}$$

$$= 2(\underbrace{e^{\ln(2)}}_{=2} - \underbrace{e^0}_{=1}) = 2.$$

Rappel : $\forall x \in]0,+\infty[,\ e^{\ln(x)} = x$
$\forall x \in \mathbb{R},\ \ln(e^x) = x$

Exercice 5. Calculer
$$\int_0^{\ln(3)} \frac{e^x}{e^{2x}-16}\, dx$$
en utilisant le changement de variable $y = e^x$ et la méthode de décomposition en éléments simples.

Solution.

→ Rappel : $e^{2x} = (e^x)^2$.

On utilise (encore) la méthode 2 pour le changement de variable.

- Élément différentiel

$y = e^x \implies y' = e^x \implies dy = e^x\, dx$

- Extrema

$x = 0 \implies y = e^0 = 1$

$x = \ln(3) \implies y = e^{\ln(3)} = 3$ (voir au-dessus)

Donc $$\int_0^{\ln(3)} \frac{e^x}{e^{2x}-16}\, dx = \int_1^3 \frac{1}{y^2-16}\, dy$$

$e^{2x} = (e^x)^2 = y^2$

Maintenant, on utilise la méthode de décomposition en éléments simples pour $\frac{1}{y^2-16} = \frac{1}{(y-4)(y+4)}$.

On cherche $a, b \in \mathbb{R}$ tels que $$\frac{1}{y^2-16} = \frac{a}{y-4} + \frac{b}{y+4}$$

On a $$\frac{a}{y-4} + \frac{b}{y+4} = \frac{a(y+4)+b(y-4)}{(y-4)(y+4)} = \frac{(a+b)y + 4a - 4b}{y^2-16}$$

$$\implies (a+b)y + 4a - 4b = 1 = 0 \cdot y + 1$$

Alors $\begin{cases} a+b=0 \\ 4a-4b=1 \end{cases} \iff \begin{cases} b=-a \\ 4a+4a=1 \end{cases} \iff \begin{cases} b=-a=-1/8 \\ a=1/8 \end{cases}$

et donc $\dfrac{1}{y^2-16} = \dfrac{1/8}{y-4} - \dfrac{1/8}{y+4}$,

d'où $\displaystyle\int_1^3 \frac{1}{y^2-16}\,dy = \int_1^3 \left(\frac{1/8}{y-4} - \frac{1/8}{y+4}\right) dy$

$$= \frac{1}{8}\int_1^3 \frac{1}{y-4}\,dy - \frac{1}{8}\int_1^3 \frac{1}{y+4}\,dy$$

$u = y-4$ $\quad$ $v = y+4$

$du = dy \ldots$ $\quad$ $dv = dy \ldots$

$$= \frac{1}{8}\Big[\ln|y-4|\Big]_1^3 - \frac{1}{8}\Big[\ln|y+4|\Big]_1^3$$

$$= \frac{1}{8}\big(\ln|-1| - \ln|-3|\big) - \frac{1}{8}\big(\ln|7| - \ln|5|\big)$$

$$= \frac{1}{8}\big(\underbrace{\cancel{\ln(1)}}_{=0} - \ln(3) - \ln(7) + \ln(5)\big)$$

$$= \frac{1}{8}\big(\ln(5) - \ln(3) - \ln(7)\big)$$

Pas nécessaire $\displaystyle = \frac{1}{8}\ln\left(\frac{5}{3\cdot 7}\right) = \frac{1}{8}\ln\left(\frac{5}{21}\right).$