Lexicographic shellability

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15/07/2021

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Shelling

Definition

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The facets F_k such that $\left(\bigcup_{j=1}^{k-1} F_j\right) \cap F_k = \partial F_k$ are called **critical**.

Shellability

Theorem

Let Δ be a shellable simplicial complex and let Σ be the set of its critical facets. Then Δ is homotopy equivalent to a wedge of spheres

$$\Delta \sim \bigvee_{\sigma \in \Sigma} S^{\dim \sigma}.$$

Poset shellability

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Poset shellability

From now on P will be a countable, chain-finite poset and we set $\widehat{P} = P \cup \{\widehat{0}, \widehat{1}\}$. Notice that there is a bijection

$$\{\text{facets in } \Delta(P)\} \longleftrightarrow \{\text{maximal chains in } \widehat{P}.\}$$

Therefore our goal is to order the maximal chains of $\widehat{P}$.

Edge-labeling

Definition

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To any non refinable chain in $\widehat{P}$

$$\gamma : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$$

we associate

$$\lambda(\gamma) = (\lambda(x_0 \lessdot x_1), \ldots, \lambda(x_{k-1} \lessdot x_k))$$

as a word with letters in Λ.

Edge labeling

Definition

We consider on $\Gamma_{x,y} = \{\text{maximal chains in } [x, y]\}$ the induced **lexicographic order**: given $\gamma, \delta \in \Gamma_{x,y}$ with $\lambda(\gamma) = a_1 \dots a_k$, $\lambda(\delta) = b_1, \dots, b_h$, set

$$\gamma \prec \delta \iff \begin{cases} a_i = b_i \text{ for } i < r \text{ and} \\ a_r < b_r \text{ for some } r \leq \min(h, k). \end{cases}$$

Edge labeling

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1. for any $\gamma_1 \neq \gamma_2 \in \Gamma_{x,y}$, $\lambda(\gamma_1)$ is not a prefix of $\lambda(\gamma_2)$,
2. $\left(\Gamma_{\widehat{0},\widehat{1}}, \prec\right)$ is a well ordered set.

Under mildly stronger assumptions, we can require easily verifiable conditions that imply the second one:

- if P is ranked, we can require that $\Lambda_j = \{\lambda(x \lessdot y) \mid \rho(x) = j\}$ is a well ordered set;
- if all edges leaving $x \in \widehat{P}$ have different labels, we can require that $\Lambda_x = \{\lambda(x \lessdot y)\}$ is a well ordered set.

LEX-labeling

Definition

An edge labeling on $\widehat{P}$ is called a **LEX-labeling** if it satisfies the:

- **SBS-condition:** if $\gamma \in \Gamma_{x,t}$ is not lexicographically least then there are $y \lessdot q \lessdot z \in \gamma$ such that $\gamma|_{[y,z]}$ is not least in $\Gamma_{y,z}$.

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Such poset P is called **lexshellable**.

LEX-labeling

Proposition

Let P be a poset and fix an edge labeling of $\widehat{P}$. The following condition is equivalent to the SBS-condition:

- **(LEX-condition)**: for any $\gamma \in \Gamma_{x,t}$ and $y, z \in \gamma$ such that $x < y < z < t$, we have

$$\left.\begin{array}{r}\gamma|_{[x,z]} = \min \Gamma_{x,z} \\ \gamma|_{[y,t]} = \min \Gamma_{y,t}\end{array}\right\} \Longrightarrow \gamma = \min \Gamma_{x,t}.$$

LEX-labeling

SBS⇒LEX:
Let $\gamma \in \Gamma_{x,t}$ and $y < z \in \gamma$ with $\gamma|_{[y,t]}$ and $\gamma|_{[x,z]}$ lexicographically least in their intervals. By contradiction suppose γ is not minimal, then by (SBS) there are $p \lessdot q \lessdot r \in \gamma$ such that $\gamma|_{[p,r]}$ is not minimal.
WLOG $p \geq y$, then we can take $\delta \in \Gamma_{p,q}$ such that $\delta \prec \gamma|_{[p,r]}$.
Now the chain $\gamma|_{[y,p]} \circ \delta \circ \gamma|_{[r,t]}$ contradicts the minimality of $\gamma|_{[y,t]}$.

LEX-labeling

LEX⇒SBS:
We prove it by induction on the length n of a maximal chain $\gamma \in \Gamma_{x,t}$. If $n = 0, 1, 2$ is immediate. Let $n > 2$ and take $x \lessdot y \lessdot z \in \gamma$. Then (LEX) implies that at least one between $\gamma|_{[x,z]}$ and $\gamma_{[y,t]}$ is not lexicographically least and we conclude by inductive hypothesis.

Mediocre chains

Definition

Let P be a poset with an edge labeling. A non-refinable chain γ is called **mediocre** if for any $x \lessdot y \lessdot z \in \gamma$, the subchain $\gamma|_{[x,z]}$ is never lexicographically least.

Lexshellability

Theorem

Let P be a poset, then the following are equivalent:

1. the simplicial complex $\Delta(P)$ is shellable via a shelling order induced by an edge labeling
2. P is lexshellable.

Moreover the critical cells of a shelling induced by a LEX-labeling are exactly the mediocre maximal chains.

Lexshellability

(1)⇒(2) Take $\alpha \prec \beta \in \Gamma_{x,t}$ and extend them to maximal chains $\gamma_1 \prec \gamma_2$, different only on $[x, t]$. By the shelling condition there exists $\delta \prec \gamma_2$ maximal s.t. $\gamma_1 \cap \gamma_2 \subseteq \delta \cap \gamma_2 = \gamma_2 \setminus \{q\}$, with $q \in \gamma|_{[x,t]}$, $x < q < t$. Therefore we have $p \lessdot q \lessdot r \in \beta$, $p, r \in \delta$ s.t. $\delta_{[p,r]} \prec \beta_{[p,r]}$, which is exactly the SBS-condition.

Lexshellability

(2)⇒(1) Take $\gamma_1 \prec \gamma_2$ maximal and let $[x, t]$ be the first interval on which they differ. Set $\alpha = \gamma_1|_{[x,t]}$ and $\beta = \gamma_2|_{[x,t]}$. Since $\alpha \prec \beta$ by (SBS) there are $p \lessdot q \lessdot r \in \beta$ and $\delta \in \Gamma_{p,r}$ s.t. $\delta \prec \beta|_{[p,r]}$. Define $\widetilde{\delta} = \gamma_2|_{[\widehat{0},p]} \circ \delta \circ \gamma_2|_{[r,\widehat{1}]}$. by construction we have $\widetilde{\delta} \prec \gamma_2$ and $\gamma_1 \cap \gamma_2 \subseteq \gamma_2 \cap \widetilde{\delta} = \gamma_2 \setminus \{q\}$, that is the shelling condition is verified.

Lexshellability

Recall now that

$$\partial(\Delta(\gamma)) = \bigcup_{q \in \gamma} \Delta(\gamma \setminus \{q\}).$$

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Therefore

$$\gamma \in \Gamma_{\hat{0},\hat{1}} \text{ is critical}$$

$$\Updownarrow$$

$$\text{for all } p \lessdot q \lessdot r \in \gamma \text{ exists } \delta \prec \gamma \text{ s.t. } \delta \cap \gamma = \gamma \setminus \{q\}$$

$$\Updownarrow$$

$$\gamma \text{ is mediocre.}$$

Lexshellability

Corollary

Given a LEX-labeling on $\widehat{P}$, the facets corresponding to the mediocre chains give a basis for the cohomology $H^*(\Delta(P); \mathbb{Z})$.

Boolean poset

Example

Take $\overline{\mathcal{B}_n} = \mathcal{B}_n \setminus \{\emptyset, [n]\}$. For a covering relation $A \lessdot B \in \mathcal{B}_n$ we have $B \setminus A = \{x\}$, with $1 \leq x \leq n$. Set $\lambda(A \lessdot B) = x$. It is easy to check that this is an EL-labeling ($\mathcal{B}_n$ is a geometric lattice).

Lexshellability of rank selections

Definition

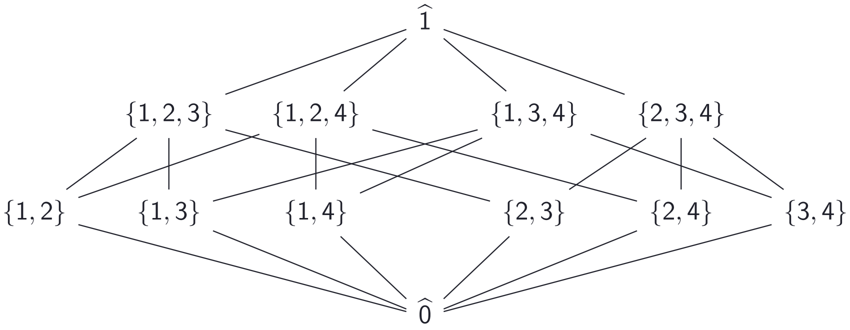
Let P be a poset with a rank function ρ and chain length n. For any $S \subseteq [n]$ we define the **rank selection**

$$P_S = \{x \in P \mid \rho(x) \in S\}.$$

Rank selection of Boolean poset

Example

Rank selection of $\mathcal{B}_4$ with $S = \{2, 3\}$.



Lexshellability of rank selections

Proposition

Any rank selection P_S of a lexshellable poset P is lexshellable.

Lexshellability of rank selections

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Fix a LEX-labeling λ on $\widehat{P}$ and label the covering relations in $\widehat{P_S}$ as follows:

$$\lambda_S(x \lessdot_S y) = \lambda(\gamma_{x,y}) \qquad \text{where } \gamma_{x,y} = \min \Gamma_{x,y}.$$

For every $\gamma \in \Gamma^S_{x,t}$ let $\widetilde{\gamma} \in \Gamma_{x,t}$ be its lexicographically least refinement. Observe that $\lambda_S(\gamma) = \lambda(\widetilde{\gamma})$, therefore

$$\gamma \prec_S \delta \iff \widetilde{\gamma} \prec \widetilde{\delta}.$$

Lexshellability of rank selections

Now take $\gamma \in \Gamma^{S}_{x,t}$ not lexicographically least. By (SBS) for $\widehat{P}$ there are $p \lessdot q \lessdot r \in \widetilde{\gamma}$ with $\widetilde{\gamma}|_{[p,r]}$ not least. Let

- $z = \min\{u \in \gamma \mid u \geq r\}$,
- $y = \max\{u \in \gamma \mid u \leq p\}$.

We have two cases to consider:

- $q \notin \widehat{P_S} \implies \lambda_S(y \lessdot_S z) = \lambda(\widetilde{\gamma}|_{[y,z]})$,
- $q \in \widehat{P_S} \implies y \lessdot_S q \lessdot_S z$ is a SBS.

Rank selection of Boolean poset

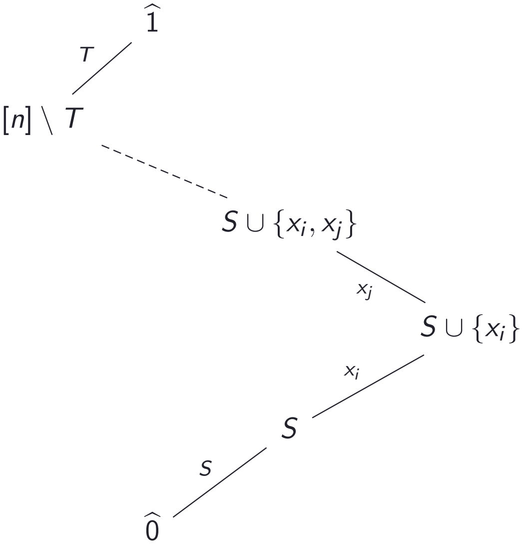
Example

Consider now the rank selection P of $\mathcal{B}_n$ given by $S = \{j \in [n] \mid k \leq j \leq m\}$, where $1 \leq k \leq m \leq n$.
Since P is ranked of length $m - k$, we know that

$$\Delta(P) \sim \bigvee_{i=1}^{\mu(P)} S^{m-k}.$$

We try to compute $\mu(P)$.

Rank selection of Boolean poset



Rank selection of Boolean poset

$$\mu(P) = \sum_{s=m}^{n} \sum_{t=1}^{k} \binom{s-t-1}{k-t,\, s-m-1,\, m-k}.$$

Bibliography

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