Lecture Notes

Elements of Complex Analysis

Course held by

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Disclaimer

These are the notes I have written during the *Elements of Complex Analysis* course, held by Professor Marco Abate in the second semester of the academic year 2017/2018.

To report any mistakes, misprints, or if you have any question, feel free to send me an email to **francescopaolo (dot) maiale (at) gmail (dot) com**.

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Chapter 1

Complex Analysis in One Variable

In this chapter, we investigate the theory of functions of one complex variable following (mainly) the book by Narasimhan and Nievergelt [1], and the book [2] due to Rudin.

1.1 Holomorphic Functions

In this first section, we recollect some of the basic notions and results in complex analysis that should be already familiar to the reader.

Definition 1.1 (Holomorphic Function)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \longrightarrow \mathbb{C}$ be a function. We say that f is *holomorphic* on Ω if and only if it is $\mathbb{C}$-differentiable, that is, for all $a \in \Omega$ the limit of the increment

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and we indicate it by $f'(a)$. Furthermore, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions $f : \Omega \longrightarrow \mathbb{C}$.

Proposition 1.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \longrightarrow \mathbb{C}$ be a function. The following assertions are equivalent:*

(1) *The function f is holomorphic on Ω.*

(2) *For all $a \in \Omega$ there exist a neighborhood $U_a \subseteq \Omega$ of a and a sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ such that*

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n \quad \text{for all } z \in U_a.$$

Additionally, the series converges absolutely for all $z \in U_a$, and we refer to a function f satisfying this property as analytic.

(3) *The function f is continuous, the partial derivatives $\partial_x f$ and $\partial_y f$ exist and satisfy the Cauchy-Riemann equations on Ω, that is,*

$$\frac{\partial f}{\partial x}(z) = -\imath \frac{\partial f}{\partial y}(z) \tag{1.1}$$

for all $z \in \Omega$*, where we consider the decomposition* $z = x + \imath y$ *for* $x, y \in \mathbb{R}$*.*

This characterization of holomorphic functions is one of the fundamental results in complex analysis since it connects the $\mathbb{C}$-differentiability with the fact that f is locally equal to its Taylor series.

The third condition, the Cauchy-Riemann equations (1.1), are usually rewritten using the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right).$$

Indeed, it is easy to see that

$$\frac{\partial f}{\partial x}(z) = \frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(z),$$

and

$$\frac{\partial f}{\partial y}(z) = \imath \left[\frac{\partial f}{\partial z}(z) - \frac{\partial f}{\partial \bar{z}}(z)\right],$$

which means that (1.1) can be replaced by the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}}(z) = 0. \tag{1.2}$$

In particular, a function $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic if and only if it is continuous, its directional derivatives exist, and f does not depend on the complex conjugate variable $\bar{z}$.

Remark 1.1. The formulas defining the operators ∂_z and $\partial_{\bar{z}}$ do not come out of nowhere. In fact, the decomposition $z = x + \imath y$ yields to

$$\mathrm{d}z = \mathrm{d}x + \imath \mathrm{d}y \quad \text{and} \quad \mathrm{d}\bar{z} = \mathrm{d}x - \imath \mathrm{d}y,$$

and it is easy to check that $\{\partial_z, \partial_{\bar{z}}\}$ is the dual basis of $\{\mathrm{d}z, \mathrm{d}\bar{z}\}$.

Theorem 1.3. *Let* $\Omega \subseteq \mathbb{C}$ *be an open set, and let* $f : \Omega \longrightarrow \mathbb{C}$ *be a function.*

(i) *Let* $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ *be a sequence of complex numbers, and let*

$$R := \limsup_{n\to+\infty} |c_n|^{1/n} \in [0, \infty].$$

Then $\sum_{n=0}^{\infty} c_n z^n$ *converges absolutely for all* $z \in \mathbb{C}$ *such that* $|z| < R$*, and diverges for all* $z \in \mathbb{C}$ *such that* $|z| > R$*. Furthermore, the convergence is uniform on the open disk* $B(\rho)$ *of radius* $\rho < R$ *strictly, where*

$$B(\rho) := \{z \in \mathbb{C} \, : \, |z| < \rho\}$$

We refer to R *as the radius of convergence of the series* $\sum_{n=0}^{\infty} c_n z^n$*.*

(ii) *The series* $\sum_{n=1}^{\infty} n c_n z^{n-1}$ *has the same radius of convergence of* ***(i)****. In particular, the derivative of the function*

$$f(z) := \sum_{n=0}^{\infty} c_n z^n$$

is given by

$$f'(z) := \sum_{n=1}^{\infty} n c_n z^{n-1}$$

for all $z \in \mathbb{C}$ *for which both functions are well-defined.*

(iii) *Let $f \in \mathcal{O}(\Omega)$, and let $a \in \Omega$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(z-a)^n,$$

and this series converges on the maximal disk $B(a, r)$, centered in a, contained in Ω.

Theorem 1.4 (Cauchy-Goursat and Morera)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \longrightarrow \mathbb{C}$ be a function. Then f is holomorphic if and only if for all γ piecewise C^1 curve homotopic to a point in Ω it turns out that*

$$\oint_{\gamma} f(z)\, \mathrm{d}z = 0.$$

Theorem 1.5 (Cauchy Formula)**.** *Let $f \in \mathcal{O}(\Omega)$, and let $\overline{D} \subseteq \Omega$ be a closed rectangle/disk contained in Ω. Then*

$$f(a) = \frac{1}{2\pi\imath} \int_{\partial D} \frac{f(\xi)}{\xi - a}\, \mathrm{d}\xi \tag{1.3}$$

and, more generally,

$$f^{(n)}(a) = \frac{n!}{2\pi\imath} \int_{\partial D} \frac{f(\xi)}{(\xi - a)^{n+1}}\, \mathrm{d}\xi \tag{1.4}$$

where $f^{(n)}(z)$ denotes the nth derivative of f evaluated at the point ξ, for all $a \in D$.

A straightforward consequence of the Cauchy formula (1.4) is the following results, which estimates the value of the nth derivative at a certain point.

Theorem 1.6. *Let $f \in \mathcal{O}(D(a, r))$, and let $0 < \rho < r$ and*

$$M(\rho) := \max_{|z-a|=\rho} |f(z)|.$$

Then the following estimate holds for all $n \in \mathbb{N}$:

$$|f^{(n)}(a)| \leq \frac{n!}{\rho^n} M(\rho). \tag{1.5}$$

Corollary 1.7 (Liouville Theorem)**.** *A bounded function $f \in \mathcal{O}(\mathbb{C})$ holomorphic on the whole complex plane is constant.*

Exercise 1.1. Let $f \in \mathcal{O}(D(a, r))$, and let $0 < \rho < \rho' < r$ and

$$M(\rho) := \max_{|z-a|=\rho} |f(z)|.$$

Then the following estimate holds for all $n \in \mathbb{N}$ and $z \in D(a, \rho)$:

$$|f^{(n)}(z)| \leq \frac{n!\rho'}{(\rho' - \rho)^n} M(\rho'). \tag{1.6}$$

1.2 Compact-Open Topology and Convergence Results

In this section, we first introduce suitable topologies on $\mathcal{O}(\Omega)$ and compare them. Next, we prove that holomorphic functions are stable under the uniform convergence on compact sets (Weierstrass), and also that an Ascoli-Arzelà theorem (Stieltjes-Osgood-Montel) holds true.

1.2.1 Pointwise Convergence and Uniform Convergence

Denote by $\mathbb{C}^{\mathbb{C}}$ the set of all functions $f : \mathbb{C} \longrightarrow \mathbb{C}$, and endow the codomain $\mathbb{C}$ with its usual metric. Then the Tychonov topology τ_p on $\mathbb{C}^{\mathbb{C}}$ is the coarser topology such that makes all the projections p_x, defined by

$$p_x : \mathbb{C}^{\mathbb{C}} \longrightarrow \mathbb{C}_x, \qquad p_x(f) = f(x),$$

continuous. It is easy to show that a sequence of function $(f_n)_{n\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{C}}$ converges to $f \in \mathbb{C}^{\mathbb{C}}$ with respect to the topology τ_p if and only if

$$f_n(x) \xrightarrow{n\to+\infty} f(x)$$

with respect to the metric space $\mathbb{C}$. The topology τ_p is, for this reason, usually referred to as *topology of pointwise convergence*.

Remark 1.2. The topology τ_p is well-defined on $\mathcal{O}(\Omega)$, but we need to introduce a finer topology (uniform convergence on compact sets) because $(\mathcal{O}(\Omega),\, \tau_p)$ is not closed.

Idea of the Proof. We know that if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ is a sequence of holomorphic functions, converging pointwise to some f on Ω, then (**Osgood's theorem**) we can find an open, dense set $D \subseteq \Omega$ such that f is holomorphic on D.

The main issue is that, in general, D is a proper subset of Ω and thus f does not belong to $\mathcal{O}(\Omega)$ - see here for a concrete counterexample -. □

We now denote by $C^0(\Omega)$ the set of all continuous functions $f : \Omega \longrightarrow \mathbb{C}$, and we notice that $\mathcal{O}(\Omega)$ is a proper subset. The *topology of uniform convergence on compact sets*, denoted by τ_{uc}, can be easily defined on $C^0(\Omega)$, and it is easy to see that

$$\left(C^0(\Omega),\, \tau_{uc}\right)$$

is a Fréchet space. We will soon define τ_{uc} exactly, but we first need to recall what a Fréchet space and a seminorm are.

Definition 1.8 (Seminorm)**.** Let X be a $\mathbb{C}$-vector space. We say that $p : X \longrightarrow \mathbb{R}$ is a *seminorm* if the following properties holds true:

(a) **Subadditivity.** For every $x,\, y \in X$, it turns out that

$$p(x+y) \le p(x) + p(y).$$

(b) **Positive Homogeneity.** For every $\alpha \in \mathbb{C}$ and $x \in X$, it turns out that

$$p(\alpha x) = |\alpha|\, p(x).$$

Definition 1.9 (Topological Vector Space)**.** A *topological vector space* over $\mathbb{C}$ (or $\mathbb{R}$) is a pair $(X,\, \tau)$ satisfying the following properties:

(a) X is a complex (or real) vector space.

(b) $(X,\, \tau)$ is a topological space.

(c) The vector sum and the scalar product are continuous with respect to the topology τ.

(d) The singlet $\{0\}$ is closed in τ, that is, X is T_1.

Definition 1.10 (Locally Convex)**.** A topological vector space (X, τ) is *locally convex* if the origin has a local basis of absolutely convex[1] absorbent sets.

Definition 1.11 (F-space)**.** A topological vector space (X, τ) is a *F-space* if the following properties are satisfied:

(i) The topology τ is induced by a translation-invariant metric d.

(ii) The metric space (X, d) is complete.

Definition 1.12 (Fréchet space)**.** A topological vector space (X, τ) is a *Fréchet space* if the following properties are satisfied:

(i) (X, τ) is locally convex.

(ii) (X, τ) is a F-space.

The key idea is that the topology of a Fréchet space (X, τ) can also be introduced via a countable family of seminorms $(p_n)_{n\in\mathbb{N}}$ such that

$$p_n(x) = 0 \quad \text{for all } n \in \mathbb{N} \implies x = 0.$$

The topology τ that makes X a topological vector space is nothing but the metric topology induced by the distance (check!) given by

$$d(x, y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{p_n(x-y)+1},$$

and therefore X is Fréchet if and only if d is a complete distance. It is not hard to see that a sequence of points $(x_k)_{k\in\mathbb{N}} \subset X$ converges to some $x \in X$ if and only if

$$p_n(x_k) \xrightarrow{k\to+\infty} p_n(x)$$

for all $n \in \mathbb{N}$. In our case, we can consider on $C^0(\Omega)$ the seminorms

$$p_K(f) := \sup_{x\in K} |f(x)|$$

for all $K \subset \Omega$ compact. Since Ω is a subset of $\mathbb{C}$, we can always cover it with an exhaustion by compact sets, that is, a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets such that $K_j \subsetneq \mathring{K}_{j+1}$ and

$$\Omega \subseteq \bigcup_{n\in\mathbb{N}} K_n.$$

It follows that we can simply consider on $C^0(\Omega)$ the countable (we stress again that it is fundamental to have a metric d that the seminorms are separated and countably many!) collection of seminorms

$$\mathcal{P} := \left\{ p_n(f) := \sup_{x\in K_n} |f(x)| \, : \, n \in \mathbb{N} \right\}.$$

[1] We say that a set is absolutely convex if and only if it is convex and balanced.

It turns out that $(f_n)_{n\in\mathbb{N}} \subset C^0(\Omega)$ converges with respect to this topology τ_{uc} to some $f \in C^0(\Omega)$ if and only if it converges uniformly to f on every compact set in Ω (or, equivalently, on every compact set of an exhaustion.)

We will see soon enough that the $\mathcal{O}(\Omega)$ is closed with respect to the uniform convergence (**Weierstrass theorem**), which means that, if a sequence of holomorphic functions converges uniformly on all compact sets to some f, then f is holomorphic on Ω as well.

Notation. We will often refer to the seminorm $p_K(\cdot)$ defined as the supremum over the compact set K with the symbol $\|\cdot\|_K$.

1.2.2 Compact-Open Topology

Let X and Y be topological spaces. The *compact-open topology*, denoted by τ_{co}, of Y^X is the topology whose prebasis is given by

$$W(K,\, U) := \left\{f \in Y^X \ : \ f(K) \subseteq U\right\}$$

where $K \subseteq X$ is a compact set and $U \subseteq Y$ is an open set.

Remark 1.3. Note that, in general, the topology τ_p is coarser than τ_{co}. Furthermore, if Y is a metric space, then τ_{uc} is coarser than τ_{co}.

The compact-open topology is interesting because it coincides with the topology of uniform convergence on compact sets on $C^0(\Omega)$ and, also, on $\mathcal{O}(\Omega)$.

Lemma 1.13. *Let X be a topological space and Y be a metric space. Then τ_{uc} and τ_{co} coincide on $C^0(X,\, Y)$.*

Idea of the Proof. Fix $f \in C^0(X,\, Y)$. We first need to prove that for every prebasis element $W(K,\, U)$ in $C^0(X,\, Y)$, there exists a positive $\epsilon > 0$ such that

$$W_{K,\,\epsilon}(f) \subseteq W(K,\, U),$$

where $W_{K,\,\epsilon}(f)$ is an element of the basis that generates the topology τ_{uc}, that is,

$$W_{K,\,\epsilon}(f) := \left\{ g \in C^0(X,\, Y) \ : \ \sup_{x\in K} d(g(x),\, f(x)) < \epsilon \right\}.$$

The key idea here is that $f(K)$ is compact in Y and contained in U; it follows that $B_\delta(f(K)) \subseteq U$, and therefore

$$W_{K,\,\delta/2}(f) \subseteq W(K,\, U).$$

We need to be a little bit more careful for the opposite inclusion. Given K compact subset of X and $\epsilon > 0$, we need to find $K_i \subseteq X$ compact sets and $U_i \subseteq Y$ open sets such that $f(K_i) \subseteq U_i$, and

$$W_{K,\,\epsilon}(f) \supseteq \bigcap_{i=1}^{n} W(K_i,\, U_i).$$

The reader might try to fill in the details by themselves as an exercise (only basic topology is required here.) □

1.2.3 Weierstrass Theorem

In the previous sections, we proved that τ_{uc} coincides with the compact-open topology on $\mathcal{O}(\Omega)$. We will now endow it with the mentioned topology and investigate its main properties (closure and compactness.)

Theorem 1.14 (Weierstrass)**.** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence of functions τ_{uc}-converging to some f. Then f belongs to $\mathcal{O}(\omega)$, and the sequence of the derivatives $(f_n')_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ τ_{uc}-converges to f'.*

Proof. Let $\overline{D}$ be a closed disk contained in Ω. The Cauchy formula (1.3) asserts that

$$f_n(z) = \frac{1}{2\pi\imath}\int_{\partial D} \frac{f_n(\xi)}{\xi - z}\,\mathrm{d}\xi$$

for all $n \in \mathbb{N}$ and for all $z \in D$. Taking the limit as $n \to +\infty$ yields to the formula

$$f(z) = \frac{1}{2\pi\imath}\int_{\partial D} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi,$$

and this is enough to infer that f is holomorphic since it holds for an exhaustion of Ω with compact disks.

For the derivative, it suffices to apply the generalized Cauchy formula (1.3) which asserts that for all $n \in \mathbb{N}$ and $z \in D$ we have

$$f_n'(z) = \frac{1}{2\pi\imath}\int_{\partial D} \frac{f_n(\xi)}{(\xi - z)^2}\,\mathrm{d}\xi.$$

Taking the limit as $n \to +\infty$ yields to the same conclusion as above for f' and, clearly, we can iterate the same for all derivatives of f. □

Corollary 1.15. *The set $\mathcal{O}(\Omega)$ is closed in $C^0(\Omega)$, and it is thus a Fréchet space.*

1.2.4 Montel's Theorem and Vitali's Theorem

We will now show that a Ascoli-Arzelà compactness result holds for holomorphic functions under the unique assumption that the family is uniformly bounded on every compact set.

Theorem 1.16 (Montel)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a family of functions such that for all compact sets $K \Subset \Omega$ there exists a constant $M_K > 0$ satisfying*

$$\|f\|_K < M_K \quad \text{for all } f \in \mathcal{F}.$$

Then $\mathcal{F}$ is relatively compact in $\mathcal{O}(\Omega)$, that is, every sequence in $\mathcal{F}$ admits a converging subsequence.

Proof. The proof is rather long, so we will divide it into four steps in order to ease the notations and make it more clear.

Step 1. Fix $a \in \Omega$, and let $0 < r < d(a, \partial\Omega)$ so that $\overline{D(a, r)} \subset \Omega$. Set $c_n(f) := \frac{f^{(n)}(a)}{n!}$ and notice that the series expansion of f is equal to

$$f(z) = \sum_{n=0}^{+\infty} c_n(f)(z-a)^n \quad \text{for all } z \in \overline{D(a, r)}.$$

Recall that the Cauchy inequality (1.5) for the nth derivative of f shows that

$$|c_n(f)| \leq \frac{\|f\|_{\overline{D(a, r)}}}{r^n}. \tag{1.7}$$

Step 2. Let us consider a sequence $\{f_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{F}$, and denote by M the positive constant such that

$$\|f_\nu\|_{\overline{D(a, r)}} < M \quad \text{for all } \nu \in \mathbb{N}.$$

It follows directly from this uniform estimate that there is an increasing subsequence $(\nu_j^1)_{j\in\mathbb{N}}$ such that there is pointwise convergence

$$c_0(f_{\nu_j^1}) = f_{\nu_j^1}(a) \xrightarrow{j\to+\infty} c_0 \in \mathbb{C}.$$

Furthermore, the estimate (1.7) is also uniform since $\|f\|_{\overline{D(a, r)}}$ is bounded by the constant M, and thus we can find for all $n \in \mathbb{N}$ a subsequence $(\nu_j^n)_{j\in\mathbb{N}} \subset (\nu_j^{n-1})_{j\in\mathbb{N}}$ such that

$$c_n(f_{\nu_j^n}) \xrightarrow{j\to+\infty} c_n \in \mathbb{C}.$$

We now exploit the well-known Cantor diagonal argument, letting $\nu_j := \nu_j^{j+1}$ and noticing that for all $n \in \mathbb{N}$ we have $c_n(f_{\nu_j}) \to c_n$ for $j \to +\infty$.

Step 3. We now want to show that the subsequence $\{f_{\nu_j}\}_{j\in\mathbb{N}}$ converges uniformly in a smaller (open) disk $D_{a,r}$, given for example by

$$D_{a,r} := D\left(a, \frac{r}{2}\right),$$

to the function defined by the power expansion $\sum_{n=0}^{+\infty} c_n(z-a)^n$, where c_n is the sequence defined by the limits above. For all $z \in D_{a,r}$ and all $N \in \mathbb{N}$, we have the estimate

$$|f_{\nu_k}(z) - f_{\nu_h}(z)| \leq \sum_{n=0}^{N} |c_n(f_{\nu_k}) - c_n(f_{\nu_h})||z-a|^n + \sum_{n>N} \frac{2M}{r^n}|z-a|^n$$

as a consequence of (1.7). The point z belongs to the disk $D_{a,r}$, so the distance from a cannot be more than $r/2$, so that

$$|f_{\nu_k}(z) - f_{\nu_h}(z)| \leq \max\left\{1, \frac{r^N}{2^N}\right\} \sum_{n=0}^{N} |c_n(f_{\nu_k}) - c_n(f_{\nu_h})| + 2M \sum_{n>N} 2^{-n} \leq$$

$$\leq \underbrace{\max\left\{1, \frac{r^N}{2^N}\right\}}_{:=K_N} \sum_{n=0}^{N} |c_n(f_{\nu_k}) - c_n(f_{\nu_h})| + 2^{1-N}M.$$

Fix $\epsilon > 0$ and let $N \geq 1$ in such a way that $2^{1-N}M < \frac{\epsilon}{2}$. Let $n_0 \in \mathbb{N}$ be such that for all natural numbers h, $k \geq n_0$ we have the estimate

$$|c_n(f_{\nu_k}) - c_n(f_{\nu_h})| < \frac{\epsilon}{2NK_N} \quad \text{for all } n \in \{0, \ldots, N\},$$

and this is possible because $(c_n(f_{\nu_k}))_{k\in\mathbb{N}}$ is a Cauchy sequence and N is finite. We conclude that $(f_{\nu_k}(z))_{k\in\mathbb{N}}$ is a Cauchy sequence uniformly with respect to $z \in D_{a,r}$.

Step 4. The argument above proves the statement only on the disk $D_{a,r}$, and the subsequence depends on it, so we need to find a way to extend it to the whole Ω. To do it, consider a sequence of points $\{a_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{C}$ such that

$$\Omega = \bigcup_{\nu\in\mathbb{N}} D_{a_\nu, r}.$$

Now a Cantor diagonal argument, together with the fact that a subsequence exists on each disk, allows us to extract a subsequence $(f_{\nu_j})_{j\in\mathbb{N}}$ that converges uniformly on all disks $D_{a_\nu,r}$. Since compact subsets $K \Subset \Omega$ can be covered by a finite number of $D_{a_\nu,r}$, it is easy to see that $(f_{\nu_j})_{j\in\mathbb{N}}$ converge uniformly on all compact sets, and thus we can apply the Weierstrass Theorem to infer the thesis. □

Theorem 1.17 (Vitali)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $\{f_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence of holomorphic functions such that for all compact sets $K \Subset \Omega$ there exists a constant $M_K > 0$ satisfying the estimate*

$$\|f_\nu\|_K < M_K \quad \textit{for all } \nu \in \mathbb{N}.$$

Suppose also that there is a set $A \subseteq \Omega$, with at least an accumulation point in Ω, such that the sequence $\{f_\nu(a)\}_{\nu\in\mathbb{N}}$ converges in $\mathbb{C}$ for all $a \in A$. Then $\{f_\nu\}_{\nu\in\mathbb{N}}$ converges uniformly on all compact sets $K \Subset \Omega$ to a holomorphic function $f \in \mathcal{O}(\Omega)$.

The fundamental idea here is to apply the Montel's theorem to the family $\mathcal{F} := \{f_\nu\}_{\nu\in\mathbb{N}}$, but we first need a result concerning accumulation points in metric spaces, whose proof is left to the reader as an easy exercise.

Exercise 1.2. Let (X, d) be a metric space, and let $\{x_\nu\}_{\nu\in\mathbb{N}}$ be a sequence with compact closure that admits a unique accumulation point x_∞. Prove that

$$d(x_\nu, x_\infty) \xrightarrow{\nu\to+\infty} 0.$$

Proof of Vitali's Theorem. We argue by contradiction. Suppose that there exists a compact set $K \Subset \Omega$, $\delta > 0$, and sequences $\{n_k\}_{k\in\mathbb{N}}$, $\{m_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}} \subset K$ such that

$$|f_{n_k}(z_k) - f_{m_k}(z_k)| \geq \delta \quad \text{for all } k \in \mathbb{N}. \tag{1.8}$$

Now Montel's Theorem asserts that, up to subsequences, we can assume that

$$f_{n_k} \xrightarrow{k\to+\infty} f \in \mathcal{O}(\Omega),$$

$$f_{m_k} \xrightarrow{k\to+\infty} g \in \mathcal{O}(\Omega),$$

$$z_k \xrightarrow{k\to+\infty} z \in K.$$

It immediately follows from the uniform convergence and (1.8) that

$$|f(z) - g(z)| \geq \delta \implies f(z) \neq g(z).$$

On the other hand, by assumption $f\big|_A \equiv g\big|_A$ as functions, and thus the *identity principle*[2] for holomorphic functions asserts that

$$f \equiv g$$

on all Ω, which is a contradiction with $f(z) \neq g(z)$. □

1.3 Meromorphic Functions and Laurent Series

In this section, we recall some notions the reader should be already familiar with, such as meromorphic functions, Laurent series, zeros, poles, essential singularities, etc.

Definition 1.18 (Biholomorphism)**.** A function $f : \Omega_1 \longrightarrow \Omega_2$ is a *biholomorphism* if and only if f is holomorphic, invertible, and its inverse is also holomorphic.

Caution!

In some books, the term *conformal mapping* is also used to refer to a biholomorphism. To us, a conformal map will be a map that preserves angles.

Definition 1.19 (Local Biholomorphism)**.** A function $f : \Omega \longrightarrow \mathbb{C}$ is a *local biholomorphism* if and only if $f \in \mathcal{O}(\Omega)$ and for all $a \in \Omega$ there exists a neighborhood $U \subseteq \Omega$ of a such that

$$f\big|_U : U \longrightarrow f(U)$$

is a biholomorphism.

Theorem 1.20. *Let $0 \leq r_1 < r_2 \leq \infty$, and let $A(r_1, r_2)$ denote the annulus given by*

$$A(r_1, r_2) := \{\zeta \in \mathbb{C} \,:\, r_1 < |\zeta| < r_2\}\,.$$

Let $f \in \mathcal{O}(A(r_1, r_2))$. Then there exists a unique sequence $\{c_n\}_{n\in\mathbb{Z}}$ such that

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n \quad \textit{for all } z \in A(r_1, r_2),$$

usually called Laurent expansion of f, and the convergence is absolute and uniform on all compact subsets of $A(r_1, r_2)$.

In particular, given $\Omega \subseteq \mathbb{C}$ open subset and $a \in \Omega$, a function $f \in \mathcal{O}(\Omega \setminus \{a\})$ admits a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n \quad \textit{for all } z \in \overline{D(a, r)} \setminus \{a\},$$

where $0 < r < d(a, \partial\Omega)$.

[2]**Theorem.** Let f and g be holomorphic functions defined on a connected open set Ω. If f equals g on some nonempty open subset $A \subset \Omega$, then f equals g on all Ω.

We will not prove this theorem, as it should be already known to the reader, but we will show that the coefficient c_n is given by the expression

$$c_n = \frac{1}{2\pi\imath}\int_{\partial D(a,\, r)} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, \mathrm{d}\zeta.$$

To do it, set $a := 0$ and denote $D(0,\, r)$ simply by D_r. The function f admits a Laurent expansion near the origin, so we write

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n,$$

and we notice that

$$\frac{1}{2\pi\imath}\int_{\partial D_r} \frac{f(\zeta)}{\zeta^{n+1}}\, \mathrm{d}\zeta = \frac{1}{2\pi\imath}\int_{\partial D_r} \frac{\sum_{k=-\infty}^{+\infty} c_k \zeta^k}{\zeta^{n+1}}\, \mathrm{d}\zeta.$$

The converge (of the series) is uniform, so we can apply a monotone-type result to interchange sum with integral

$$\frac{1}{2\pi\imath}\int_{\partial D_r} \frac{\sum_{k=-\infty}^{+\infty} c_k \zeta^k}{\zeta^{n+1}}\, \mathrm{d}\zeta = \frac{1}{2\pi\imath}\sum_{k=-\infty}^{+\infty} c_k \int_{\partial D_r} \zeta^{k-n-1}\, \mathrm{d}\zeta.$$

We now apply the change of variable $\zeta \longmapsto \zeta := r\mathrm{e}^{\imath\theta}$, which is true because ζ belongs to the boundary of D_r, to obtain

$$\frac{1}{2\pi\imath}\sum_{k=-\infty}^{+\infty} c_k \int_{\partial D_r} \zeta^{k-n-1}\, \mathrm{d}\zeta = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} c_k \int_0^{2\pi} r^{k-n-1}\mathrm{e}^{\imath(k-n-1)\theta} r\mathrm{e}^{\imath\theta}\, \mathrm{d}\theta =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} c_k r^{k-n} \int_0^{2\pi} \mathrm{e}^{\imath(k-n)\theta}\, \mathrm{d}\theta.$$

We now exploit the orthogonality relation between the exponentials to infer that

$$\int_0^{2\pi} \mathrm{e}^{\imath(k-n)\theta}\, \mathrm{d}\theta \neq 0 \iff k = n,$$

and this implies that

$$\frac{1}{2\pi\imath}\int_{\partial D_r} \frac{f(\zeta)}{\zeta^{n+1}}\, \mathrm{d}\zeta = c_k.$$

Definition 1.21 (Residue)**.** Let f be as above. The *residue* of f at $a \in \Omega$ is defined as the coefficient c_{-1} of its Laurent expansion, that is,

$$\mathrm{res}_f(a) := \frac{1}{2\pi\imath}\int_{\partial D(a,\, r)} f(\zeta)\, \mathrm{d}\zeta.$$

Theorem 1.22 (Riemann)**.** *Let $f \in \mathcal{O}(D(a,\, r) \setminus \{a\})$ be a function such that*

$$\lim_{z \to a}(z-a)f(z) = 0. \tag{1.9}$$

Then f can be holomorphically extended to the whole disk $D(a,\, r)$.

Proof. The previous theorem asserts that f admits a Laurent expansion in such a way that

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n \quad \text{for all } z \in D(a,\, r) \setminus \{a\},$$

and therefore it turns out that

$$(z-a)f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^{n+1} \quad \text{for all } z \in D(a,\, r) \setminus \{a\}.$$

The assumption (1.9) implies that c_n must be equal to zero for all $n < -1$. Additionally, it is easy to see that the residue of f at a is zero, which means that also $c_{-1} = 0$. Then

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n \quad \text{for all } z \in D(a,\, r) \setminus \{a\},$$

and this means that f extends holomorphically to a. □

Definition 1.23 (Order)**.** Let $f \in \mathcal{O}(D(a,\, r) \setminus \{a\})$. The *order* of f at a is defined as

$$\operatorname{ord}_f(a) := \inf \{n \in \mathbb{N} \,:\, c_n \neq 0\} \in \mathbb{Z} \cup \{-\infty\}.$$

Furthermore, we say that a is

- an **essential singularity** if its order is minus infinity, that is, $\operatorname{ord}_f(a) = -\infty$;
- a **pole** if its order is strictly negative, that is, $\operatorname{ord}_f(a) < 0$;
- a **regular point** if its order is nonnegative, that is, $\operatorname{ord}_f(a) \geq 0$;
- a **zero** if its order is strictly positive, that is, $\operatorname{ord}_f(a) > 0$.

Theorem 1.24 (Casorati-Weierstrass)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f \in \mathcal{O}(\Omega \setminus \{a\})$ for some $a \in \Omega$. If a is an essential singularity, then for all $r > 0$ it turns out that*

$$f(D(a,\, r) \setminus \{a\}) \subseteq \mathbb{C} \text{ is a dense inclusion.}$$

Definition 1.25 (Meromorphic Function)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $E \subset \Omega$ be a discrete set. A function $f \in \mathcal{O}(\Omega \setminus E)$ is *meromorphic* on Ω, and we denote it by $f \in \mathcal{M}(\Omega)$, if and only if it is locally the quotient of holomorphic functions.

A function $f \in \mathcal{O}(\Omega \setminus E)$ is meromorphic if and only if for all $a \in E$ there are a neighborhood $U_a \subset \Omega$ of a and holomorphic functions $f,\, g \in \mathcal{O}(U)$ such that

$$f\,\big|_{U \setminus \{a\}} \equiv \frac{g}{h}\,\big|_{U \setminus \{a\}}.$$

To better understand the behavior of the function f near a point $a \in E$, we consider the power expansions of g and h, given by

$$g(z) = (z-a)^{k_0} \sum_{n \geq k_0} a_n (z-a)^{n-k_0} \quad \text{and} \quad h(z) = (z-a)^{k_1} \sum_{n \geq k_1} b_n (z-a)^{n-k_1}.$$

It follows that the local behavior of f is given by the ratio between these two power expansions, that is,

$$f(z) = (z-a)^{k_0-k_1} \frac{g_1(z)}{h_1(z)},$$

where $g_1(a) \neq 0$ and $h_1(a) \neq 0$. The quotient g_1/h_1 is holomorphic on the whole U, and thus it admits a power expansion around a; it follows that

$$f(z) = (z-a)^{k_0-k_1} \sum_{n \geq 0} c_n (z-a)^n = \sum_{n \geq 0} c_n (z-a)^{n+k_0-k_1}.$$

This is the Laurent expansion of the meromorphic function f, and it is easy to see that $k_0 - k_1$ is finite, which means that

$$f \in \mathcal{M}(\Omega) \implies \text{for all } a \in E,\ a \text{ is } \textbf{not} \text{ an essential singularity.}$$

Corollary 1.26. *Let E be a discrete subset of Ω. Then:*

(1) A function $f \in \mathcal{O}(\Omega \backslash E)$ is meromorphic on Ω if and only if for all a is not an essential singularity for any $a \in E$.

(2) A function $f \in \mathcal{O}(\Omega \backslash E)$ is meromorphic on Ω if and only if either the limit $\lim_{z \to a} f(z)$ *exists (regular point) or* $\lim_{z \to a} |f(z)| = +\infty$ *(pole).*

1.3.1 Riemann Sphere

Denote by $\hat{\mathbb{C}}$ the *Alexandroff one-point compactification* of $\mathbb{C}$, that is, the set $\mathbb{C} \cup \{\infty\}$ endowed with the topology τ_c defined as follows:

- The topology τ_c coincides with the euclidean topology on $\mathbb{C}$, that is, the neighborhoods of the points $a \in \mathbb{C}$ are the usual ones.
- The open neighborhoods of ∞ are the sets $(\mathbb{C} \setminus K) \cup \{\infty\}$, where K ranges among all compact subsets $K \Subset \mathbb{C}$.

It is easy to see that the stereographic projection induces a topological spaces homeomorphism between $\hat{\mathbb{C}}$ and the sphere S^2, which is usually referred to as *Riemann sphere*.

Remark 1.4. Let $f \in \mathcal{M}(\Omega)$. Then f can be extended to a continuous function

$$f : \Omega \longrightarrow \hat{\mathbb{C}}$$

by setting $f(a) := \infty$ for all $a \in E$. We will soon see that a sort of vice versa holds.

To define the notion of holomorphic with values in $\hat{\mathbb{C}}$, the idea is to change chart and identify ∞ with the origin in the following way. The mapping

$$\tau : \hat{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{C}, \qquad w \longmapsto \begin{cases} \frac{1}{w} & \text{if } w \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{if } w = \infty \end{cases}$$

is continuous and invertible, and its inverse is given by

$$\tau^{-1} : \mathbb{C} \longrightarrow \hat{\mathbb{C}} \setminus \{0\}, \qquad z \longmapsto \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ \infty & \text{if } z = 0. \end{cases}$$

Definition 1.27 (Holomorphic Function)**.** Let $z_0 \in \Omega$ be such that $f(z_0) = \infty$. A function $f : \Omega \longrightarrow \hat{\mathbb{C}}$ is holomorphic in a neighborhood of z_0 if and only if $\tau \circ f$ is holomorphic at z_0, which means that

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0 \quad \text{and} \quad \frac{1}{f} \text{ is holomorphic on } U \setminus \{z_0\}.$$

Definition 1.28. We say that $f \in \mathcal{O}(\Omega, \hat{\mathbb{C}})$ if and only if f is holomorphic at all points with image contained in $\mathbb{C}$ and holomorphic in a neighborhood of all points with image ∞.

Corollary 1.29. *A function $f \in \mathcal{O}(\Omega \setminus E)$ belongs to $\mathcal{M}(\Omega)$ if and only if its extension to the Riemann sphere $\hat{\mathbb{C}}$ - that sends E to ∞ - belongs to $\mathcal{O}(\Omega, \hat{\mathbb{C}})$.*

Definition 1.30 (Holomorphic Function)**.** A function $f : \hat{\mathbb{C}} \longrightarrow \mathbb{C}$ is holomorphic if and only if $f\big|_{\mathbb{C}}$ is holomorphic and $f \circ \tau^{-1}$ is holomorphic in a neighborhood of the origin.

Putting together the definition of $\mathcal{O}(\Omega, \hat{\mathbb{C}})$ and the definition of $\mathcal{O}(\hat{\mathbb{C}}, \mathbb{C})$, we get for free the notion of holomorphic function between Riemann spheres.

Corollary 1.31. *A function f belongs to $\mathcal{O}(\hat{\mathbb{C}}, \mathbb{C})$ if and only if f is constant.*

Corollary 1.32. *Any function f that belongs to $\mathcal{O}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ is necessarily surjective.*

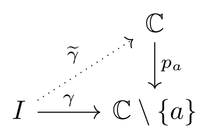
1.4 Residue Theorem

In this section, we recall the residue theorem and we use it to deduce properties of the zeros of meromorphic functions (argument principle, Rouché, etc.)

Let $p : \mathbb{C} \longrightarrow \mathbb{C}^*$ be the universal covering defined by the complex exponential, and consider for all $a \in \mathbb{C}$ the translation

$$p_a : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{a\}, \qquad p_a(z) = \mathrm{e}^z + a.$$

Let $\gamma : [0, 1] \longrightarrow \mathbb{C} \setminus \{a\}$ be a closed curve that can be lifted (via the universal covering property) to a curve $\widetilde{\gamma}$ such that the following diagram commutes



that is, we have $\widetilde{\gamma} \circ p = \gamma$, and thus $\widetilde{\gamma}(t) = \log(\gamma(t) - a)$. It follows that $\widetilde{\gamma}$ is not necessarily a closed curve, but rather we have

$$\widetilde{\gamma}(1) - \widetilde{\gamma}(0) = 2k\pi\imath,$$

where $k \in \mathbb{Z}$ corresponds to the number of turns around the point. We denote it by

$$n(\gamma, a) := \frac{1}{2\pi\imath} \left[\widetilde{\gamma}(1) - \widetilde{\gamma}(0)\right] \in \mathbb{Z}.$$

Proposition 1.33. *Under the assumptions above, we have the identity*

$$n(\gamma, a) = \frac{1}{2\pi\imath}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z.$$

Theorem 1.34 (Residue Theorem)**.** *Let $E \subset \Omega$ be a discrete set, and let $\gamma : [0, 1] \longrightarrow \Omega \setminus E$ be a closed curve homotopic to a constant in Ω. Then*

$$\frac{1}{2\pi\imath}\int_\gamma f(z)\,\mathrm{d}z = \sum_{a\in E} \mathrm{res}_f(a) n(\gamma, a), \tag{1.10}$$

holds for all $f \in \mathcal{O}(\Omega \setminus E)$, and the right-hand series is finite.

Theorem 1.35 (Argument Principle)**.** *Let $f \in \mathcal{M}(\Omega)$ be a meromorphic function, and denote by Z_f and P_f the set of zeros and the set of poles of f respectively. Let*

$$\gamma : [0, 1] \longrightarrow \Omega \setminus (Z_f \cup P_f)$$

be a closed curve homotopic to a constant path on Ω. Then

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{a\in Z_f\cup P_f} n(\gamma, a)\cdot \mathrm{ord}_f(a). \tag{1.11}$$

In particular, if γ is a oriented parametrization of a closed disc $D \Subset \Omega$ with $\partial D \cap (Z_f \cup P_f) = \varnothing$, then

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{a\in Z_f\cup P_f} \mathrm{ord}_f(a).$$

Proof. It suffices to apply (1.10) with $f := \frac{f'}{f}$. □

Proposition 1.36 (Rouché)**.** *Let f and g be holomorphic functions on the interior of the disk D. Assume that*

$$|f(z) - g(z)| < |g(z)| \quad \text{for all } z \in \partial D. \tag{1.12}$$

Then f and g have the same number of zeros, counted with their own multiplicity, in D.

Proof. Consider the function $f_t(z) := g(z) + t(f(z) - g(z))$ for $t \in [0, 1]$. The reverse triangular inequality shows that

$$|f_t(z)| \geq |g(z)| - t|f(z) - g(z)| \geq |g(z)| - |f(z) - g(z)| > 0$$

for all $z \in \partial D$, which means that $f_t\big|_{\partial D}$ is always different from zero. It follows from (1.11) that

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f_t'(\zeta)}{f_t(\zeta)}\,\mathrm{d}\zeta = \sum_{a\in Z_f\cup P_f} \mathrm{ord}_{f_t}(a),$$

and this is enough to conclude since the left-hand side depends continuously on t, while the right-hand side takes value in a discrete set ($\mathbb{N}$). Therefore, the function

$$t \longmapsto \frac{1}{2\pi\imath}\int_\gamma \frac{f_t'(\zeta)}{f_t(\zeta)}\,\mathrm{d}\zeta = \sum_{a\in Z_f\cup P_f} \mathrm{ord}_{f_t}(a)$$

takes the same value at $t = 0$ and $t = 1$, that is,

$$\sum_{a \in Z_f \cup P_f} \operatorname{ord}_f(a) = \sum_{a \in Z_f \cup P_f} \operatorname{ord}_g(a).$$

□

Corollary 1.37. *Let $\Delta \subset \mathbb{C}$ denote the unit disk. Assume that $f \in \mathcal{O}(\Delta, \Delta)$ is a function whose image $f(\Delta)$ is relatively compact in Δ. Then f admits a unique fixed point.*

Proof. Let $r < 1$ be a real number such that $|f(z)| < r$ for all $z \in \Delta$. It follows that

$$|z - (z - f(z))| = |f(z)| < r = |z|$$

for all $z \in \partial\Delta_r$. By Rouché id_Δ and $\mathrm{id}_\Delta - f$ have the same number of zeros (with multiplicity) in Δ, which means that there exists a unique $z^* \in \Delta$ such that

$$\mathrm{id}_\Delta(z^*) - f(z^*) = 0 \implies z^* = f(z^*).$$

□

1.5 Hurwitz Theorems

The fundamental result by Hurwitz asserts that a sequence of holomorphic functions, τ_{uc}-converging to a holomorphic nonconstant function f, is definitively equal to $f(z)$ for all z.

Theorem 1.38 (Hurwitz)**.** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence converging uniformly on all compact sets to $f \in \mathcal{O}(\Omega)$. Suppose that f is not constant. Then for all $z \in \Omega$ there are a sequence of points $(z_n)_{n\in\mathbb{N}} \subset \Omega$ converging to z and an index $N := N(z) \in \mathbb{N}$ such that*

$$f_n(z_n) = f(z) \quad \text{for all } n \geq N.$$

Proof. Set $w := f(z)$. Since f is nonconstant, the preimage $f^{-1}(w)$ is a discrete subset of Ω, and thus there is a positive constant $\delta > 0$ such that

$$f^{-1}(w) \cap D(z, \delta) = \{z\} \quad \text{and} \quad D(z, \delta) \subseteq \Omega.$$

For all $k \geq 1$ denote by γ_k the boundary of the rescaled disk $D(z, \frac{\delta}{k})$, and notice that $w \notin f(\gamma_k)$. Consider now the distance

$$\delta_k := \min\{|f(\zeta) - w| \,:\, \zeta \in \gamma_k\} = d(w, \gamma_k),$$

and take $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ increasing sequence in such a way that for all $n \geq n_k$ and all $k \geq 1$ it turns out that

$$\max_{\zeta \in \gamma_k} |f(\zeta) - w| < \frac{\delta_k}{2}.$$

Fix $k \geq 1$. For all $n \geq n_k$ and $\zeta \in \gamma_k$ we have the inequality

$$|(f_n(\zeta) - w) + (f(\zeta) - w)| = |f_n(\zeta) - f(\zeta)| < \frac{\delta_k}{2} < \delta_k \leq |f(\zeta) - w|,$$

and thus it follows from the Rouché Theorem 1.36 that the functions $f_n(\zeta) - w$ and $f(\zeta) - w$ have the same number of zeros in $D_k := D(z, \frac{\delta}{k})$, which means at least one.

In particular, there exists a sequence $z_{n,k} \in D_k$ such that $f_n(z_{n,k}) = w = f(z)$. To conclude the proof we extract a subsequence $(z_n)_{n\in\mathbb{N}} \subset \Omega$ as follows: set $z_n := z_{n,k}$, where k is the unique integer such that $n_k \le n \le n_{k+1}$. □

Corollary 1.39 (First Hurwitz Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence converging uniformly on all compact sets to $f \in \mathcal{O}(\Omega)$. Suppose that each f_n has no zeros. Then f is either nonzero at all points or constant.*

More in general, if there exists $w \in \mathbb{C}$ such that $w \notin f_n(\Omega)$ for all[3] $n \in \mathbb{N}$, then f is either constant or w does not belong to $f(\Omega)$.

Proof. Suppose that f is nonconstant. If $w \in f(\Omega)$, then by Hurwitz Theorem 1.38 it also belongs (definitively) to $f_n(\Omega)$, which is a contradiction. □

Example 1.1. The sequence of functions $f_n(z) := \frac{1}{n}e^z$ satisfies the assumptions of the previous corollary, and its limit is the constant value zero.

Corollary 1.40 (Second Hurwitz Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence converging uniformly on all compact sets to $f \in \mathcal{O}(\Omega)$. Suppose that each f_n is injective. Then f is either constant or injective.*

Proof. We argue by contradiction. Suppose that f is not injective (and not constant) and let $z_1 \neq z_2 \in \Omega$ be two points such that $f(z_1) = f(z_2)$. Set

$$h_n(z) := f_n(z) - f_n(z_2),$$

$$h(z) := f(z) - f(z_2).$$

Then h_n converges uniformly on all compact sets to h and $h(z_1) = 0$, so that h is nonconstant and admits a zero in Ω. By Hurwitz Theorem 1.38 we can find a sequence of points $(z_n)_{n\in\mathbb{N}}$ such that $h_n(z_n) = 0$ and $z_n \to z_1$. Thus

$$f_n(z_n) = f_n(z_2),$$

and since z_n converges to $z_1 \neq z_2$, we have that $z_n \neq z_2$ definitively, which is the desired contradiction. □

Example 1.2. The sequence of functions $f_n(z) := \frac{1}{n}z$ satisfies the assumptions of the previous corollary, and its limit is once again the constant value zero.

Exercise 1.3. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ be a sequence converging uniformly on all compact sets to $f \in \mathcal{O}(\Omega)$.

(a) Prove that, if f_n is a local biholomorphism for all $n \in \mathbb{N}$, then f is also a local biholomorphism.

(b) Prove that, if f_n is injective for all $n \in \mathbb{N}$ and f is not constant, then f_n^{-1} converges uniformly on all compact sets to f^{-1}.

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[3] It is clearly enough to require this to happen definitively w.r.t. n. The same is true for all the results presented in this section.

Chapter 2

Automorphism Groups

In this chapter, we will investigate the automorphism groups of several subsets of the complex plane, focusing mainly on the unit disk (which is a model for every simply connected bounded domain), the complex plane and the Riemann sphere.

2.1 Automorphism Group of the Unit Disk Δ

In this section, we first introduce a fundamental result known as Schwarz's Lemma. Next, we use Möbius transformations, together with rotations, to describe the automorphism group of the unit disk.

Lemma 2.1 (Schwarz). *Let $f \in \mathcal{O}(\Delta, \Delta)$ be a function such that $f(0) = 0$. Then the following assertions hold:*

(i) *For all $z \in \Delta$ we have the estimate*

$$|f(z)| \leq |z|. \tag{2.1}$$

(ii) *The derivative of f is bounded at the origin, and*

$$|f'(0)| \leq 1. \tag{2.2}$$

(iii) *The equality holds in **(ii)** or in **(i)** for some $z \in \Delta \setminus \{0\}$ if and only if the equalities holds at all points if and only if f is a rotation, that is,*

$$f(z) = \mathrm{e}^{\imath\theta} z.$$

To prove this result the key is to apply the maximum principle to a suitable function that is strictly related to f. We first recall the statement of the maximum modulus principle.

Proposition 2.2. *Let f be a function holomorphic on some connected open set $\Omega \subset \mathbb{C}$. If z_0 is a point in Ω such that*

$$|f(z_0)| \geq |f(z)|$$

for all z in a neighborhood of z_0, then the function f is constant on the whole Ω.

Proof. The function $g(z) := \frac{f(z)}{z}$ is holomorphic in Δ since

$$g(z) = \frac{\sum_{n\geq 1} a_n z^n}{z} = \sum_{n\geq 0} a_{n+1} z^n,$$

and it is thus clear that $g(0) = f'(0)$. Let $z \in \Delta$ and let $|z| < r < 1$. The maximum modulus principle asserts that

$$|g(z)| \leq \max_{|\zeta|=r} |g(\zeta)| = \max_{|\zeta|=r} \frac{|f(z)|}{r} \leq \frac{1}{r}.$$

Taking the limit as $r \to 1^-$, we conclude that $|g(z)| \leq 1$ for all $z \in \Delta$, which is equivalent to both **(i)** and **(ii)**. To prove **(iii)** we simply notice that, if there exists $z_0 \in \Delta$ such that $|g(z_0)| = 1$, then the maximum modulus principle implies that g is constant and of modulus one, which means that

$$g(z) \equiv e^{\imath\theta} \implies f(z) = e^{\imath\theta} z.$$

□

Definition 2.3. The automorphism group of an open set $\Omega \subset \mathbb{C}$ is defined as

$$\mathrm{Aut}(\Omega) := \{f \in \mathcal{O}(\Omega) \, : \, f(\Omega) = \Omega \text{ and } f \text{ is injective}\}.$$

It is not hard to prove that the automorphism group can also be rewritten using the notion of biholomorphism, that is, we have

$$\mathrm{Aut}(\Omega) = \{f : \Omega \longrightarrow \Omega \, : \, f \text{ is biholomorphic}\}.$$

Definition 2.4 (Möbius Transformation)**.** Let $a \in \Delta$. The a-Möbius transformation, denoted by γ_a, is defined by

$$\gamma_a(z) := \frac{z-a}{1-\bar{a}z}.$$

Exercise 2.1. Let $a \in \Delta$.

(1) The preimage of the origin via γ_a is $\{a\}$, and thus $\gamma_a(a) = 0$.

(2) For all $z \in \Delta$ we have

$$1 - |\gamma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

(3) The function γ_a can be extended by continuity to the boundary $\partial\Delta$.

(4) The function γ_a is an automorphism of the unit disk Δ and its inverse is given by

$$\gamma_a^{-1}(z) = \frac{z+a}{1+\bar{a}z} = \gamma_{-a}(z).$$

It follows from these properties that $\mathrm{Aut}(\Delta)$ acts transitively on Δ, which means that for all a, $b \in \Delta$ there exists $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(a) = b$. The reader may check as an exercise that γ is nothing but the composition $\gamma_b^{-1} \circ \gamma_a$.

Corollary 2.5. *The automorphism group of* Δ *is given by*

$$\mathrm{Aut}(\Delta) = \{\rho_\theta \circ \gamma_a \, : \, \theta \in \mathbb{R}, \; a \in \Delta\},$$

where ρ_θ *denotes the rotation of angle* θ*.*

Proof. Let $\gamma \in \mathrm{Aut}(\Delta)$ be an automorphism such that $\gamma(0) = 0$. Then γ^{-1} also belongs to $\mathrm{Aut}(\Delta)$ and maps 0 to 0. Now (2.2) implies that

$$|\gamma'(0)| \leq 1 \quad \text{and} \quad \frac{1}{|\gamma'(0)|} = |(\gamma^{-1})'(0)| \leq 1,$$

and this is possible if and only if $|\gamma'(0)| = 1$. The third point of the Schwarz's Lemma implies that $\gamma(z)$ must be a rotation, that is,

$$\gamma(z) = \rho_\theta(z).$$

Suppose now that γ is an automorphism such that $\gamma^{-1}(0) = a$. Then $\gamma \circ \gamma_a^{-1}$ maps the origin to the origin, and therefore

$$\gamma \circ \gamma_a^{-1}(z) = \rho_\theta(z) \implies \gamma(z) = \rho_\theta \circ \gamma_a(z).$$

□

Exercise 2.2. Show that the properties **(2)** and **(3)** of the previous exercise hold true for all automorphisms, and not only for Möbius transformations.

Exercise 2.3. Prove the following assertions.

(1) If $f \in \mathcal{O}(\Delta, \Delta)$ has two different fixed points $z_1, z_2 \in \Delta$, then f is the identity map.

(2) Let $\gamma \in \mathrm{Aut}(\Delta) \setminus \{\mathrm{id}\}$. Then one and only one of the following hold:

   (a) **Elliptic.** γ has a unique fixed point in Δ and no fixed points in $\partial\Delta$.

   (b) **Parabolic.** γ has no fixed points in Δ and a unique fixed point in $\partial\Delta$.

   (c) **Hyperbolic.** γ has no fixed points in Δ and two fixed points in $\partial\Delta$.

(3) The automorphism group $\mathrm{Aut}(\Delta)$ is doubly transitive on the boundary, that is, for all[1] $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \partial\Delta \times \partial\Delta$ there exists an automorphism γ whose boundary extension maps (ξ_1, ξ_2) to (η_1, η_2).

Lemma 2.6 (Schwarz-Pick)**.** *Let* $f \in \mathcal{O}(\Delta, \Delta)$*. Then the following assertions hold:*

(i) *For all* $z, w \in \Delta$ *we have the estimate*

$$\left| \frac{f(z) - f(w)}{1 - \bar{f(w)} f(z)} \right| \leq \left| \frac{z - w}{1 - \bar{w} z} \right|. \tag{2.3}$$

(ii) *For all* $z \in \Delta$ *we have*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{2.4}$$

[1]We assume that the trivially false cases are not to be considered here.

(iii) *The equality holds in* ***(ii)*** *for some* $z \in \Delta$ *or in* ***(i)*** *for some* $(z, w) \in \Delta^2$ *if and only if* f *is an automorphism, that is,* $f \in \mathrm{Aut}(\Delta)$.

Proof. It follows from Schwarz's Lemma applied to $\gamma_{f(w)} \circ f \circ \gamma_w^{-1}$. □

There is another way to rewrite this result, introducing a function that will also be useful from a topological point of view. Let $\omega : \Delta^2 \longrightarrow \mathbb{R}_+$ be the function

$$(z,\, w) \longmapsto \operatorname{atanh}\left|\frac{z - w}{1 - \bar{w}z}\right|.$$

Notice that $[0,\, 1)$ is mapped into $[0,\, \infty)$, and the function is strictly increasing.

Corollary 2.7. *Let* $f \in \mathcal{O}(\Delta,\, \Delta)$. *For all* $z,\, w \in \Delta$ *we have the estimate*

$$\omega(f(z),\, f(w)) \le \omega(z,\, w), \tag{2.5}$$

and the equality holds if and only if $f \in \mathrm{Aut}(\Delta)$.

It is not hard to see that ω is a distance, known as *Kobayashi distance*, which was first introduced to give a manifold structure to the so-called hyperbolic space.

Exercise 2.4 (Kobayashi Distance Properties)**.** Prove the following assertions:

(1) For all $w,\, z \in \Delta$ it turns out that

$$\omega(w,\, z) = \omega(0,\, \gamma_w(z)).$$

(2) The group of isometries of ω is given by the automorphisms of the unit disk Δ and their conjugates, that is,

$$\mathrm{Iso}(\omega) = \mathrm{Aut}(\Delta) \cup \overline{\mathrm{Aut}(\Delta)}.$$

(3) For all $r \in (0,\, 1]$ and $z_0 \in \Delta$ it turns out that

$$B_\omega(z_0,\, r) = B_{\mathbb{C}}\left(\frac{(1 - \tanh(r))^2}{(1 - \tanh^2(r))|z_0|^2} z_0,\; \frac{(1 - |z_0|^2)\tanh(r)}{(1 - \tanh^2(r))|z_0|^2}\right),$$

where $B_{\mathbb{C}}$ denotes the euclidean ball.

We now give the definition of *geodesic*, but the reader should be aware of the fact that there is a much more general definition (on Riemannian manifold) that is usually introduced in a differential geometry course.

Definition 2.8 (Geodesic)**.** A curve $\gamma : \mathbb{R} \longrightarrow \Delta$ is a *geodesic* with respect to ω, or ω-geodesic, if for all $t_1,\, t_2 \in \mathbb{R}$ it turns out that

$$\omega(\gamma(t_1),\, \gamma(t_2)) = |t_2 - t_1|.$$

Exercise 2.5. Prove that the curve

$$\mathbb{R} \ni t \longmapsto \tanh(t)\frac{z_0}{|z_0|}$$

is a ω-geodesic for all $z_0 \in \Delta \setminus \{0\}$.

Exercise 2.6. A reasonable parametrization of a circumference arc, orthogonal to the boundary of the disk, is a geodesic.

Hint. It suffices to show that all ω-geodesics are images of diameters via the automorphism group of the unit disk. □

The reader without a solid background in differential geometry may skip the following paragraph. The Kobayashi distance ω is induced by the Riemannian metric

$$k_\delta^2 : \Delta \times T\Delta \longmapsto \mathbb{C}, \qquad (z,\, w) \longmapsto \frac{|v|^2}{(1-|z|^2)^2},$$

which is usually known as Poincaré metric and gives a manifold structure to the hyperbolic plane. The metric allows us to compute the length $L(\cdot)$ of a curve, and thus we can define a distance as

$$d(x,\, y) := \inf\left\{L(\gamma) \,:\, \gamma \text{ regular curve in } \Delta \text{ between } x \text{ and } y\right\}.$$

The length $L(\cdot)$ can be defined as the integral of the norm of the tangent vectors, and it is not hard to see that d coincides with ω.

Remark 2.1. The Gaussian curvature of the hyperbolic plane with the Poincaré metric is constant and negative, a phenomenon that cannot happen in $\mathbb{R}^3$ as a consequence of a result due to Hilbert (which means that the unit disk cannot be embedded in $\mathbb{R}^3$.)

2.2 Automorphism Group of the Hyperbolic Plane

There is another model of the hyperbolic plane, which is completely equivalent to $(\Delta,\, \omega)$. We consider the upper half-plane

$$H^+ := \{z \in \mathbb{C} \,:\, \mathfrak{Im}(z) > 0\},$$

with boundary (seen in the Riemann sphere) given by

$$\partial H^+ := \{z \in \mathbb{C} \,:\, \mathfrak{Im}(z) = 0\} \cup \{\infty\}.$$

The Caley transform sends the unit disk Δ into the upper half-plane H^+ as follows:

$$\Psi(z) = \imath \frac{1+z}{1-z}.$$

We can easily check that Ψ is a well-defined map since for all $|z| < 1$ it turns out that

$$\mathfrak{Im}(\Psi(z)) = \mathfrak{Re}\left(\frac{1+z}{1-z}\right) > 0 \iff |z| < 1.$$

The inverse of the Caley transform can be computed explicitly, and equals

$$\Psi^{-1}(w) = \frac{w-\imath}{w+\imath}.$$

We have $\Psi(0) = \imath$, $\Psi(1) = \infty$ and $\Psi(-1) = 0$, so that the diameter of the unit disk Δ is mapped into the half-line $[0, +\infty]$. The reader should also check that Ψ is a biholomorphism between Δ and H^+, and therefore

$$f \in \mathcal{O}(H^+, H^+) \iff \Psi^{-1} \circ f \circ \Psi \in \mathcal{O}(\Delta, \Delta),$$

which means that we can investigate the leading properties of f both in Δ and H^+ with no significant difference.

Proposition 2.9. *The mapping*

$$\mathrm{SL}(2, \mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(w \longmapsto \frac{aw + b}{cw + d} \right) \in \mathrm{Aut}(H^+)$$

induces an isomorphism of groups

$$\mathrm{PSL}(2, \mathbb{R}) := {}^{\mathrm{SL}(2, \mathbb{R})}\big/_{\{\pm \mathrm{Id}_{2\times 2}\}} \cong \mathrm{Aut}(H^+) \cong \mathrm{Aut}(\Delta).$$

Proof. We know that $\gamma \in \mathrm{Aut}(H^+)$ if and only if $\Psi^{-1} \circ \gamma \circ \Psi \in \mathrm{Aut}(\Delta)$, and the automorphisms of the unit disk have a nice explicit formula. It turns out that

$$\Psi^{-1} \circ \gamma \circ \Psi(z) = \rho_\theta \circ \gamma_a(z),$$

and the thesis follows immediately from a simple computation. □

In the previous section, we classified the automorphisms of the unit disk in elliptic, parabolic and hyperbolic. Now we are finally ready to write them explicitly through the biholomorphism with the half-plane model.

A **elliptic** automorphism has a unique fixed point in Δ. We assume without loss of generality that the fixed point is the origin, and it is easy to check that

$$\rho_\theta(z) = z \iff z = 0.$$

In the general case, let $a \in \Delta$ be the fixed point. Then $\gamma_a(a) = 0$, and therefore

$$\gamma_a^{-1} \circ \rho_\theta \gamma_a \in \mathrm{Aut}(\Delta)$$

is the unique automorphism, for a fixed angle $\theta \in \mathbb{R} \setminus \{0\}$, that fixes only the point $a \in \Delta$.

Suppose now that $\gamma \in \mathrm{Aut}(H^+)$ is a **parabolic** automorphism. As before, we may assume without loss of generality that the unique fixed point in the boundary $\partial\Delta$ is ∞. It follows that

$$\gamma(\infty) = \infty \iff \frac{a}{c} = \infty \iff c = 0,$$

which means that

$$\gamma(w) = \frac{a}{d} w + \frac{b}{d}.$$

We now require that γ does not have any fixed point inside Δ. It is easy to check that

$$\left(\frac{a}{d} - 1 \right) w_0 = -\frac{b}{d} \quad \text{for all } w_0 \in \Delta \iff a = d,$$

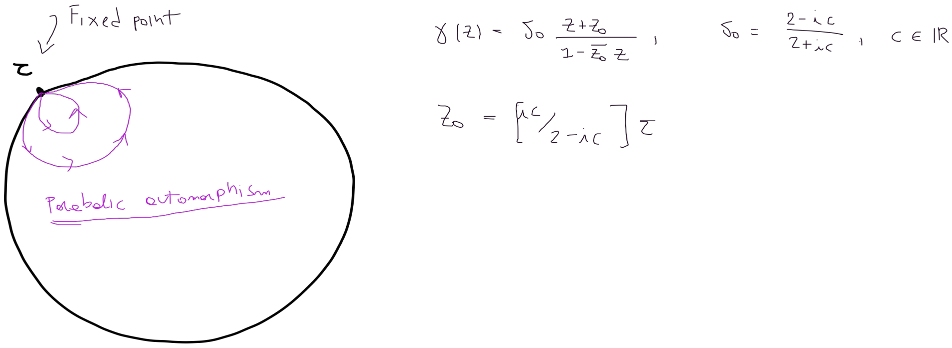


Figure 2.1: A parabolic automorphism.

which means that the desired parabolic automorphism is given by

$$\gamma(w) = w + \beta$$

for some $\beta \in \mathbb{R}$. In Figure 2.1 we show the graph of a parabolic automorphism in the half-plane H^+ and how difficult it is in the unit disk Δ.

Suppose now that $\gamma \in \mathrm{Aut}(H^+)$ is a **hyperbolic** automorphism. As before, we may assume without loss of generality that the fixed points in the boundary $\partial\Delta$ are 0 and ∞. It follows that

$$\gamma(\infty) = \infty \iff \frac{a}{c} = \infty \iff c = 0,$$

and

$$\gamma(0) = 0 \iff b = 0,$$

which means that the desired parabolic automorphism is given by

$$\gamma(w) = \lambda w$$

for some $\lambda \in \mathbb{R}_{>0}$. In Figure 2.2 we illustrate the graph of a hyperbolic automorphism in the half-plane H^+ and how different it is in the unit disk Δ.

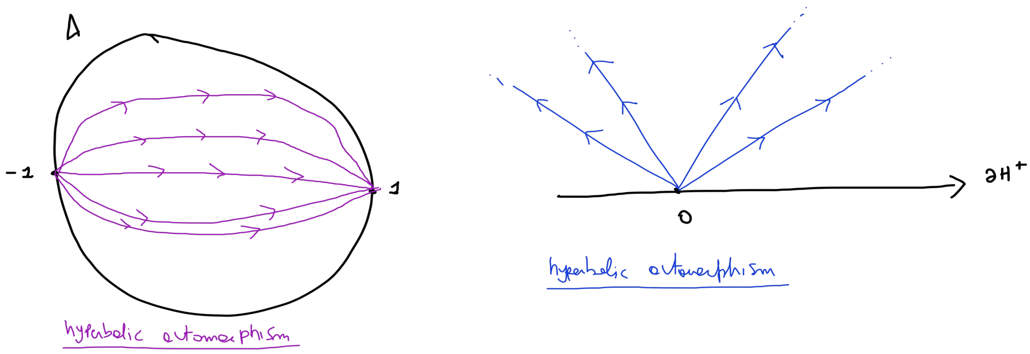


Figure 2.2: A hyperbolic automorphism.

2.3 Automorphism Group of the Complex Plane

We will see that simply connected domains Ω are biholomorphic to the unit disk, except for the complex plane $\mathbb{C}$. Therefore, it makes sense to study its automorphism group $\mathrm{Aut}(\mathbb{C})$.

Proposition 2.10. *Let $f \in \mathcal{O}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ be a holomorphic function. Then there exist $p, q \in \mathbb{C}[z]$ coprime polynomials ($(p, q) = 1$) such that*

$$f(z) = \frac{p(z)}{q(z)} \quad \text{for all } z \in \hat{\mathbb{C}}.$$

Proof. Let Z_f and P_f be respectively the set of zeros and the set of poles of f in $\mathbb{C}$, and notice that these are discrete subset in a compact space $\hat{\mathbb{C}}$, and thus are finite. Set

$$Z_f := \{z_1, \dots, z_r\},$$

and

$$P_f := \{w_1, \dots, w_s\}.$$

The function

$$g(z) = \frac{\prod_{i=1}^{s}(z - w_i)}{\prod_{j=1}^{r}(z - z_j)} f(z)$$

has no zeros nor poles in $\mathbb{C}$, so we only need to address what happens at ∞. If $g(\infty) \in \mathbb{C}$, then g maps $\hat{\mathbb{C}}$ into $\mathbb{C}$, and this implies that g is a constant function, i.e.,

$$\lambda = \frac{\prod_{i=1}^{s}(z - w_i)}{\prod_{j=1}^{r}(z - z_j)} f(z) \implies f(z) = \lambda \frac{\prod_{j=1}^{r}(z - z_j)}{\prod_{i=1}^{s}(z - w_i)}.$$

If $g(\infty) = \infty$, then we simply consider the function $\frac{1}{g}$ and apply the same argument to it; since $\frac{1}{g}(\infty) = 0$, we infer that

$$\frac{1}{\lambda} = \frac{\prod_{i=1}^{s}(z - w_i)}{\prod_{j=1}^{r}(z - z_j)} f(z) \implies f(z) = \lambda' \frac{\prod_{j=1}^{r}(z - z_j)}{\prod_{i=1}^{s}(z - w_i)}.$$

□

Remark 2.2. To compute $f(\infty)$ we simply compute $f(0)$ in the opposite chart (with transition map $\frac{1}{z}$). In particular, we have

$$f(\infty) = \lim_{w \to 0} \frac{p(\frac{1}{w})}{q(\frac{1}{w})} = \frac{a_m}{b_n} w^{n-m},$$

and therefore the value of $f(\infty)$ depends on $n - m$, where a_m and b_n are the leading coefficients of the polynomials p and q respectively.

Definition 2.11 (Order)**.** Let $f \in \mathcal{O}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ be a holomorphic function. The order of f at ∞, denoted by $\mathrm{ord}_f(\infty)$ is given by

$$\mathrm{ord}_f(\infty) := \deg q - \deg p.$$

Definition 2.12 (Degree)**.** Let $f \in \mathcal{O}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ be a holomorphic function. The degree of f is defined as the maximum between the degrees of the polynomials, that is,

$$\deg f := \max\{\deg p, \deg q\}.$$

Definition 2.13 (Multiplicity)**.** Let f be a holomorphic function, and let $f(z_0) = w_0$. The *multiplicity* of f at z_0 is thus given by

$$\partial_f(z_0) := \begin{cases} \operatorname{ord}_{f-w_0}(z_0) & \text{if } w_0 \in \mathbb{C}, \\ -\operatorname{ord}_f(z_0) & \text{if } w_0 = \infty. \end{cases}$$

Theorem 2.14. *Let $f \in \mathcal{O}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ be a holomorphic nonconstant function. Then for all $w_0 \in \hat{\mathbb{C}}$ we have the identity*

$$\sum_{f(z_0)=w_0} \delta_f(z_0) = \deg f. \tag{2.6}$$

Proof. We first prove it for $w_0 = 0$. The key idea is to split the sum between the z_0s in $\mathbb{C}$ and ∞, that is,

$$\sum_{f(z_0)=0} \delta_f(z_0) = \sum_{\substack{f(z_0)=0 \\ z_0\in\mathbb{C}}} \delta_f(z_0) + \sum_{f(\infty)=0} \delta_f(\infty) = \deg p + \max\{0, \deg q - \deg p\} = \deg f.$$

We now prove it for $w_0 = \infty$. Again we have

$$\sum_{f(z_0)=\infty} \delta_f(z_0) = \sum_{\substack{f(z_0)=\infty \\ z_0\in\mathbb{C}}} \delta_f(z_0) + \sum_{f(\infty)=\infty} \delta_f(\infty) = \deg q + \max\{0, \deg p - \deg q\} = \deg f.$$

Finally, if $w_0 \in \mathbb{C} \setminus \{0\}$, then it is easy to see that

$$f(z) - w_0 = \frac{p(z) - w_0 q(z)}{q(z)},$$

and the degree of $f - w_0$ is clearly equal to the degree of f, so the previous case applies. □

Corollary 2.15. *The group of automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ is isomorphic to* $\mathrm{PSL}(2, \mathbb{C})$*.*

Proof. It suffices to apply (2.6) and note that γ has necessarily degree one. □

Exercise 2.7. The group of automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ is 3-transitive.

Exercise 2.8. The group of automorphisms of the complex plane $\mathbb{C}$ is isomorphic to

$$\{az + b \,:\, a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

2.4 Wolff-Denjoy Theorem

Let f be a holomorphic map from the unit disk Δ to the unit disk Δ, and assume that there exists $z_0 \in \Delta$ such that $f(z_0) = z_0$. It follows from (2.3) that

$$\left| \frac{f(z) - z_0}{1 - \bar{z}_0 f(z)} \right| \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|,$$

and this is equivalent to the inequality

$$1 - \left| \frac{f(z) - z_0}{1 - \bar{z}_0 f(z)} \right|^2 \geq 1 - \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|^2.$$

We now employ the formula

$$1 - |\gamma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a} z|^2}$$

to infer that

$$\frac{|1 - \bar{z}_0 f(z)|^2}{1 - |f(z)|^2} \leq \frac{|1 - \bar{z}_0 z|^2}{1 - |z|^2} \tag{2.7}$$

holds for all $z \in \Delta$, provided that f admits a fixed point z_0. Unfortunately, this argument does not work for a function without fixed points inside the unit disk.

The next goal is to show that it is possible to find a point $\tau \in \partial\Delta$ such that (2.7) holds, even if $f(\tau)$ is not well-defined.

Definition 2.16 (Horocycle)**.** Let $\tau \in \partial\Delta$ and $R > 0$. The *horocycle* of "center" τ and radius Δ is the set given by

$$E(\tau, R) := \left\{ z \in \Delta \,:\, \frac{|1 - \bar{\tau} z|^2}{1 - |z|^2} < R \right\}.$$

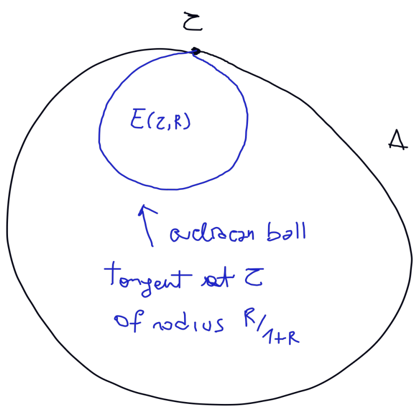


Figure 2.3: The horocycle of center τ and radius R.

Remark 2.3. Let ω be the Kobayashi distance. Then

$$\lim_{\zeta\to\tau}\left[\omega(z,\,\zeta)-\omega(0,\,\zeta)\right]=\frac{1}{2}\log\frac{|1-\bar{\tau}z|^2}{1-|z|^2}$$

holds for all $z\in\Delta$, and therefore the horocycle $E(\tau,\,R)$ is nothing but the open ω-ball of radius $\frac{1}{2}\log R$.

Theorem 2.17 (Wolff's Lemma)**.** *Let $f\in\mathrm{Hom}(\Delta,\,\Delta)$ be a holomorphic function with no fixed point in Δ. Then there exists a unique $\tau\in\partial\Delta$ such that*

$$\frac{|1-\bar{\tau}f(z)|^2}{1-|f(z)|^2}\le\frac{|1-\bar{\tau}z|^2}{1-|z|^2}\tag{2.8}$$

for all $z\in\Delta$. Furthermore, the equality in (2.8) holds at some $z_0\in\Delta$ if and only if at all points of the unit disk Δ and f is a parabolic automorphism with fixed point τ.

Remark 2.4. The condition (2.8) is equivalent to the fact that

$$f(E(\tau,\,R))\subseteq E(\tau,\,R)$$

for all $R>0$.

Proof. We first prove the uniqueness since it follows from an easy geometric argument, and then we prove the existence using a simple trick.

Uniqueness. Suppose that $\tau_1\neq\tau_2\in\partial\Delta$ both satisfy these properties. We can always find - see Figure 2.4 - $R_1>0$ and $R_2>0$ such that the horocycles $E(\tau_1,\,R_1)$ and $E(\tau_2,\,R_2)$ are tangent at some $z_0\in\Delta$, that is,

$$\overline{E(\tau_1,\,R_1)}\cap\overline{E(\tau_2,\,R_2)}=\{z_0\}.$$

Then $f(z_0)$ is contained in both $E(\tau_i,\,R_i)$, and thus it must be equal to z_0, which is absurd since we assumed f to be a map without fixed points in Δ.

Existence. Let $r_\nu\nearrow 1$ and set $f_\nu:=r_\nu\cdot f$. The inclusion $f_\nu(\Delta)\Subset\Delta$ is relatively compact, and hence there exists a unique $w_\nu\in\Delta$ fixed point such that $f_\nu(w_\nu)=w_\nu$. Up to subsequences, we may assume that

$$w_\nu\xrightarrow{\nu\to\infty}\tau\in\bar{\Delta}.$$

If τ were a point of Δ, then the continuity of f in Δ would imply that

$$f(w_\nu)\xrightarrow{\nu\to\infty}f(\tau)\quad\text{and}\quad f(w_\nu)=\frac{1}{r_\nu}w_\nu\xrightarrow{\nu\to\infty}\tau,$$

and, by assumption, f does not admit any fixed point in Δ. Thus $\tau\in\partial\Delta$. Now the inequality (2.7) applies to f_ν, and we have that

$$\frac{|1-\bar{w}_\nu f_\nu(z)|^2}{1-|f_\nu(z)|^2}\le\frac{|1-\bar{w}_\nu z|^2}{1-|z|^2}$$

passes to the limit for $\nu\to\infty$, and we conclude that (2.8) holds.

Equality. Suppose that (2.8) holds with the equal at some point $z_0 \in \Delta$. We have

$$\mathfrak{Re}\left(\frac{z+f(z)}{z-f(z)} - \frac{\tau+z}{\tau-z}\right) \leq 0$$

at all points $z \in \Delta$, and

$$\mathfrak{Re}\left(\frac{z+f(z)}{z-f(z)} - \frac{\tau+z}{\tau-z}\right)\Big|_{z=z_0} = 0.$$

The maximum principle for the real part of a holomorphic function immediately implies that it must be equal to a constant imaginary function, which means that

$$\frac{z+f(z)}{z-f(z)} - \frac{\tau+z}{\tau-z} = \imath c$$

for some $c \in \mathbb{R}$. We solve the equation for $f(z)$ and we easily infer that f is a parabolic automorphism. □

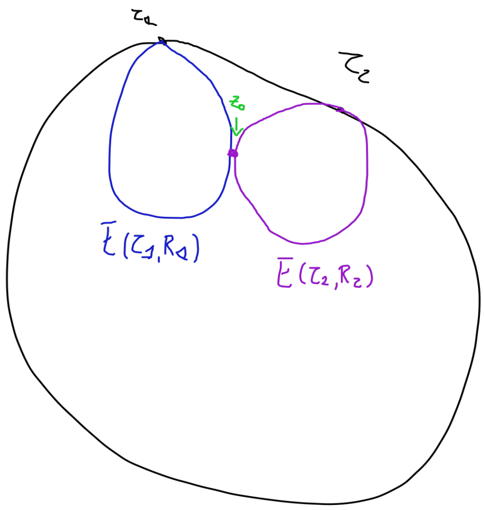


Figure 2.4: The idea behind the proof of the uniqueness in Wolff's Lemma.

Definition 2.18. Let $f \in \mathrm{Hom}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$ be a holomorphic function. The *Wolff point* of f is the fixed point in Δ, if any, or the $\tau \in \partial\Delta$ given by the previous result.

2.4.1 Discrete Dynamical Systems

In this section, we shall denote by f^n the function obtained iterating n times f, that is,

$$f^n(z) = f \circ \cdots \circ f(z) = f^{n-1} \circ f(z).$$

Let $f \in \mathrm{Hom}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$. The *orbit* of a point $z_0 \in \Delta$ is given by

$$\mathcal{O}_f^+(z_0) := \{f^n(z_0) \ : \ n \in \mathbb{N}\}.$$

We are interested in the behavior of $\mathcal{O}_f^+(z_0)$ as z ranges among the unit disk and, also, as f ranges among Hom(Δ, Δ).

The stability of a discrete dynamical system is strictly related to the differences between $\mathcal{O}_f^+(z_0)$ and $\mathcal{O}_f^+(z_1)$ whenever z_0 is near (in some sense) to z_1; the structural stability, on the other hand, concerns the dynamics associated to f_1 and f_2 whenever these are near.

Theorem 2.19 (Wolff-Denjoy). *Let $f \in \mathrm{Hom}(\Delta, \Delta)$ be a holomorphic map that is not an elliptic automorphism. Then f^k converges uniformly on all compact sets to τ, the Wolff point of f.*

Proof. We divide the proof into two cases: when τ is an interior point, and when τ is the point on the boundary.

Case 1. Suppose that $\tau \in \Delta$. Up to automorphisms[2], we may assume that τ is the origin since we can easily show that

$$g = \gamma^{-1} \circ f \circ \gamma \implies g^k = \gamma^{-1} \circ f^k \circ \gamma,$$

so that the dynamic associated to f and the dynamic associated to g coincide. Now for all $z \in \Delta$ we have $|f(z)| < |z|$ - as f is not elliptic -, and therefore for all $r < 1$ there exists $\lambda_r \in (0, 1)$ such that

$$|f(z)| \leq \lambda_r |z|$$

for all $z \in \Delta_r$. It follows that $f(\bar{\Delta}_r) \subseteq \Delta_r$, and therefore for all $n \in \mathbb{N}$ we have

$$|f(z)| \leq \lambda_r^n |z| \leq \lambda_r^n r \xrightarrow{n \to \infty} 0.$$

Case 2. Suppose that $\tau \in \partial\Delta$, and suppose that for some $z_0 \in \Delta$ we have

$$f^n(z_0) \xrightarrow{n \to \infty} x \in \bar{\Delta}.$$

If x were a point of Δ, then we would be able to conclude that

$$f(x) = \lim_{n \to \infty} f(f^n(z_0)) = \lim_{n \to \infty} f^{n+1}(z_0) = x,$$

which is absurd since f has no fixed points in Δ. Thus

$$f^n(z_0) \xrightarrow{n \to \infty} x \in \bar{\Delta} \implies x \in \partial\Delta.$$

It is easy to see that, if we can show that $f^n(z_0)$ converges to τ for all $z_0 \in \Delta$, then we can apply Vitali's theorem to infer the thesis. Now notice that

$$\{f^n(z_0)\}_{n \in \mathbb{N}} \subseteq \bar{\Delta}$$

and $\bar{\Delta}$ is compact, which means that everything will come along if we can prove that τ is the unique accumulation point.

[2] Let γ be an automorphism satisfying $\gamma(0) = \tau$ in such a way that $g(0) = 0$.

(a) Suppose that $f(\partial E(\tau, R))$ intersects $\partial E(\tau, R)$ in a point $\sigma \neq \tau$. Then f is a parabolic automorphism, and therefore in the upper half-plane model

$$f(z) = z + c \implies f^n(z) = z + n \cdot c \xrightarrow{n\to\infty} \infty,$$

which is the Wolff point of f and the unique fixed point.

(b) Suppose that $f(\partial E(\tau, R))$ intersects $\partial E(\tau, R)$ only in τ. Then f is not parabolic, and it is easy to see that there exists $\lambda_R \in (0, 1)$ such that

$$\frac{|1 - \bar{\tau} f^n(z)|^2}{1 - |f^n(z)|^2} \leq \lambda_R^n \frac{|1 - \bar{\tau} z|^2}{1 - |z|^2} \leq \lambda_R^n \cdot R$$

for all $z \in \overline{E(\tau, R)}$ and all $n \in \mathbb{N}$. If $f^{n_k}(z_0)$ converges to $\sigma \neq \tau$ for $k \to \infty$, then $f^{n_k}(z_0)$ does not belong to $\overline{E(\tau, r)}$ for all n big enough and r sufficiently small. On the other hand, the inequality above implies that

$$f^n(z_0) \in \overline{E(\tau, \lambda_R^n \cdot R)},$$

and therefore $f^n(z_0)$ belongs to $\overline{E(\tau, \rho)}$ for some $\rho > 0$ and all $n \geq N$.

□

Chapter 3

Sheaf Theory

3.1 Holomorphic Functions Sheaf

Let $a \in \mathbb{C}$ be an arbitrary point, and let us consider the set of couples

$$\{(U,\, f)\ :\ U \text{ neighbourhood of } a \text{ and } f \in \mathrm{Hol}(U,\, \mathbb{C})\}$$

equipped with the equivalence relation $(U,\, f) \sim (W,\, g)$ if and only if there exists V, neighbourhood of a, such that $V \subseteq U \cap W$ and

$$f\,\big|_U \equiv g\,\big|_W.$$

The set of all equivalence classes, known as *stalk*, is usually denoted by $\mathcal{O}_a$, and we refer to the class $[(U,\, f)]$ as the *germ* of f at a.

Definition 3.1 (Sheaf)**.** The union of all stalks as a ranges in $\mathbb{C}$, denoted by

$$\mathcal{O} := \bigcup_{a\in\mathbb{C}} \mathcal{O}_a,$$

is called *germs sheaf* of holomorphic functions, and it is endowed with the projection

$$p : \mathcal{O} \longrightarrow \mathbb{C}, \qquad \mathcal{O}_a \longmapsto a.$$

Let $[(U,\, f)] =: f_a$ be an element of the stalk $\mathcal{O}_a$, and let $(U,\, f)$ be a representative chosen in such a way that U is an open neighbourhood of p. We define

$$N(U,\, f) := \{f_z\ :\ z \in U,\, f_z \in \mathcal{O}_z\}\,.$$

Exercise 3.1. There exists a unique topology τ such that the family $N(U,\, f)$, as $(U,\, f)$ ranges in f_a, is a complete system of neighbourhoods of f_a for all germs of $\mathcal{O}$.

Proposition 3.2. *The topology τ defined in the exercise is Hausdorff*[1].

Proof. Let $f_a \neq g_b \in \mathcal{O}$ be distinct germs. Then either $a \neq b$ or $a = b$.

[1] Recall that a topological space X is Hausdorff if and only if for all $x \neq y \in X$ we can find neighbourhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$

Case 1. Take $(U,\, f)$ and $(W,\, g)$ representatives such that $U \ni a$ and $W \ni b$ are disjoint neighbourhoods (which is possible because $\mathbb{C}$ is Hausdorff). Then

$$N(U,\, f) \cap N(W,\, g) = \varnothing.$$

Case 2. Take $(U,\, f) \in f_a$ and $(W,\, g) \in g_a$, and let $D \ni a$ be a disk contained in the intersection $U \cap V$. We now claim that

$$f_a \neq g_a \implies N(D,\, f) \cap N(D,\, g) = \varnothing. \tag{3.1}$$

We argue by contradiction. Assume that there exists $h_z \in N(D,\, f) \cap N(D,\, g)$. Then $z \in D$ and, by definition, we have

$$h_z = f_z \quad \text{and} \quad h_z = g_z \implies f_z = g_z.$$

It follows that we can find a subset $W \subset D$ such that $f\big|_W \equiv g\big|_W$, and therefore $f\big|_D \equiv g\big|_D$, which yields to a contradiction with the assumption $f_a \neq g_a$. □

Remark 3.1. In the argument by contradiction, we employed the well-known identity principle for holomorphic functions to infer that

$$f\big|_W \equiv g\big|_W \implies f\big|_D \equiv g\big|_D.$$

Corollary 3.3. *The topology τ induces (with the inclusion topology) the discrete topology*[2] *on all the stalks $\mathcal{O}_a$, for $a \in \mathbb{C}$.*

Proposition 3.4. *The projection $p : \mathcal{O} \longrightarrow \mathbb{C}$ is continuous, open and a local homeomorphism. In particular, τ is not the discrete topology on $\mathcal{O}$.*

Proof. Let $V \subseteq \mathbb{C}$ be an open set. The preimage of V via p is given by

$$p^{-1}(V) = \bigcup \left\{ N(U,\, f) \;:\; U \subseteq V \text{ open and } f \in \mathrm{Hol}(U,\, \mathbb{C}) \right\},$$

and thus $p^{-1}(V)$ is open ($\Longrightarrow p$ is continuous) because it is equal to the arbitrary union of open sets. Furthermore, we have that

$$p\left(N(U,\, f)\right) = V$$

for all $U \subseteq V$ open and $f \in \mathrm{Hol}(U,\, \mathbb{C})$, which implies that p is open. In conclusion, notice that the map defined by

$$V \ni z \longmapsto f_z \in N(U,\, f)$$

is a local inverse of p, which is also continuous and equal to $\left(p\big|_{N(U,\, f)}\right)^{-1}$. □

Definition 3.5 (Analytic Extension)**.** Let $f_a \in \mathcal{O}$, and let $\gamma : [0,\, 1] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(0) = a$. An *analytic extension* of f_a along γ is a continuous lift

$$\widetilde{\gamma} : [0,\, 1] \longrightarrow \mathcal{O}$$

such that $\widetilde{\gamma}(0) = f_a$ and the diagram commutes, that is, $p \circ \widetilde{\gamma} = \gamma$.

[2] This is true in the general framework as well, as no property of holomorphic functions is necessary.

Remark 3.2. The projection p is not a covering.

Proof. Set $a := 1 \in \mathbb{C}$, $f_a := (z^{-1})_a$ and $\gamma(t) := 1 - t$. The reader should check that f_1 does not admit an analytic extension along γ, and conclude that p is not a covering (otherwise such a lift would exist always.) We recommend to first solve the following general exercise:

Exercise 3.2. Let $\gamma : [0,\, 1] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(0) = a$, and let $f_a,\, g_a \in \mathcal{O}$. Assume that $p \in \mathbb{C}[X,\, Y]$ is a polynomial such that

$$P(f_a,\, g_a) = 0.$$

Assume also that both f_a and g_a admit analytic extensions $f_{\gamma(t)}$ and $g_{\gamma(t)}$ along γ. Then

$$P(f_{\gamma(t)},\, g_{\gamma(t)}) = 0 \quad \text{for all } t \in [0,\, 1].$$

□

Definition 3.6. Fix $a \in \mathbb{C}$. We can endow the stalk $\mathcal{O}_a$ with a sum given by

$$f_a + g_a := \left[\left(U \cap V,\, (f + g)\left.\right|_{U \cap V}\right)\right],$$

where $f_a = [(U,\, f)]$ and $g_a = [(V,\, g)]$, and a product

$$f_a \cdot g_a := \left[\left(U \cap V,\, (f \cdot g)\left.\right|_{U \cap V}\right)\right],$$

that make $\mathcal{O}_a$ a $\mathbb{C}$-algebra.

Exercise 3.3. Prove that the operations defined above are well-defined.

Exercise 3.4. Prove that the $\mathbb{C}$-algebra $(\mathcal{O}_a,\, +,\, \cdot)$ has a unique maximal ideal, which is explicitly given by

$$\mathfrak{M}_a := \{f_a \in \mathcal{O}_a \;:\; f_a(a) = 0\}.$$

Remark 3.3. The local nature of the germ f_a proves that $f_a(a)$ is well-defined, and equal to $f(a)$ for any representative $(U,\, f)$ of f_a.

Remark 3.4. Let $k \in \mathbb{N}$. The argument above works also for the kth derivative, $f_a^{(k)}(a)$, as it depends only on the local behaviour around a, and thus

$$f_a^{(k)}(a) := f^{(k)}(a) \quad \text{for any } f \text{ such that } (U,\, f) \in f_a$$

is also well-defined.

Remark 3.5. If $\Omega \subseteq \mathbb{C}$ is an open subset, then the preimage $p^{-1}(\Omega)$ is given by $\mathcal{O}_\Omega$.

Definition 3.7 (Section)**.** A section s of the projection p on some $\Omega \subseteq \mathbb{C}$ is a continuous[3] map $s : \Omega \longrightarrow \mathcal{O}$ such that

$$p \circ s = \mathrm{id}_\Omega.$$

Exercise 3.5. The set of all sections defined on Ω, denoted by $\mathcal{O}(\Omega)$, is in a 1-1 correspondence with holomorphic maps defined on Ω, that is, $\mathrm{Hol}(\Omega,\, \mathbb{C})$.

[3] We usually define sections without any regularity requirement, but, since we will only be interested with continuous ones, we add it to the definition.

We now define a derivative map $d : \mathcal{O} \longrightarrow \mathcal{O}$ that maps a germ f_a to its "derivative", that is, we set

$$d(f_a) := [(U,\, f')] \quad \text{for some } (U,\, f) \in f_a.$$

Theorem 3.8. *The map $d : \mathcal{O} \longrightarrow \mathcal{O}$ is a covering.*

Lemma 3.9. *Let $D \subset \mathbb{C}$ be an open disk. Then any holomorphic function $f \in \mathcal{O}(D)$ admits a unique, up to additive constants, primitive $F \in \mathcal{O}(D)$.*

Proof. We have that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-a)^n,$$

and the series converges uniformly on all compact subsets of D. Then

$$F(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} (z-a)^{n+1}$$

is a primitive of f in D, and it is clearly unique up to additive constant. □

Proof of Theorem 3.8. Let f_a be a germ in $\mathcal{O}_a$, and take any $(U,\, f) \in f_a$. Let $D := D(a,\, r)$ be an open disk centered at a so that

$$(D,\, f\big|_D) \in f_a,$$

and let $F \in \mathcal{O}(D)$ be a primitive of f. We now consider the family of neighbourhoods

$$\mathcal{F} := \{N(D,\, F+c)\}_{c\in\mathbb{C}}\,,$$

and we claim that

$$d^{-1}(D) = \bigsqcup_{c\in\mathbb{C}} N(D,\, F+c). \tag{3.2}$$

To prove this claim, we first notice that the derivative map d maps each element of $\mathcal{F}$ onto D, that is,

$$d(N(D,\, F+c)) = D \quad \text{for all } c \in \mathbb{C}.$$

Vice versa, let $g_z \in d^{-1}(D)$ be any germ with $z \in D$ and $d(g_z) = f_z$. It follows easily that there exists $V \subseteq D$ open connected neighbourhood of z such that

$$g'\big|_V \equiv f\big|_V.$$

Since $F'\big|_V$ coincides with $f\big|_V$ as well and V is a connected open set, we infer that

$$g(x) = F(x) + c \quad \text{for all } x \in V \text{ and some } c \in \mathbb{C},$$

and therefore we conclude that $g_z \in N(D,\, F+c)$. In particular, the claim (3.2) holds true, and thus it remains only to show that

$$d\big|_{N(D,\, F+c)} : N(D,\, F+c) \longrightarrow D$$

is a homeomorphism for all $c \in \mathbb{C}$. The map $d\big|_{N(D,\, F+c)}$ is obviously continuous (as d itself is), surjective and open since

$$d(N(U,\, g)) = N(U,\, g').$$

The injectivity of the restriction $d\big|_{N(D,\, F+c)}$ is also obvious since it preserves the stalks, and therefore $d\big|_{N(D,\, F+c)}$ is a homeomorphism and d a covering. □

Corollary 3.10. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected subset, and let $f \in \mathcal{O}(\Omega)$. Then f admits a primitive $F \in \mathcal{O}(\Omega)$ which is unique up to an additive constant.*

Proof. Consider the diagram

$$\begin{array}{ccc} & & \mathcal{O}_\Omega \\ & \overset{F}{\nearrow} & \big\downarrow d \\ \Omega & \xrightarrow[f]{} & \mathcal{O}_\Omega \end{array}$$

Since d is a covering and Ω is simply connected, we can always find a lift F of f such that the diagram is commutative, that is, $d \circ F = f$. Furthermore, we have that

$$\mathrm{id}_\Omega = p \circ f = p \circ d \circ F = p \circ F$$

since $p \circ d = p$, and hence F is a section and belongs to $\mathcal{O}(\Omega)$. $\square$

Corollary 3.11. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected subset, and let $f \in \mathcal{O}(\Omega)$. Assume also that $f(z) \neq 0$ for all $z \in \Omega$. Then the following properties hold:*

(1) *There exists $g \in \mathcal{O}(\Omega)$ such that $f = \mathrm{e}^g$, and g is unique up to additive constants of the form $2k\pi\imath$ for $k \in \mathbb{Z}$.*

(2) *For all $n \in \mathbb{N}$ there exists $h_n \in \mathcal{O}(\Omega)$ such that $f(z) = h_n(z)z^n$, and h_n is unique up to the roots of 1.*

Proof. The idea is to take g equal to the primitive of $\frac{f'}{f}$, and $h_n := \mathrm{e}^{\frac{g}{n}}$. $\square$

3.2 Riemann Uniformization Theorem

The primary goal of this section is to give a proof of the Riemann uniformisation theorem, which gives a complete characterisation of Riemann surfaces and the universal coverings.

Theorem 3.12 (Riemann)**.** *Let X be a Riemann surface, and let $\pi : \widetilde{X} \longrightarrow X$ be its universal covering. Then of the following possibilities hold:*

(i) Elliptic Case. *There are biholomorphisms $X \cong \hat{\mathbb{C}}$ and $\widetilde{X} \cong \hat{\mathbb{C}}$, with universal covering given by the identity map.*

(ii) Parabolic Case. *The universal covering $\widetilde{X}$ is biholomorphic to $\mathbb{C}$, while X is either biholomorphic to $\mathbb{C}$, $\mathbb{C}^*$, or homeomorphic[4] to a torus $X \cong \mathbb{C}/\mathbb{Z}^2$.*

(iii) Hyperbolic Case. *The universal covering $\widetilde{X}$ is biholomorphic to Δ and X is none of the previous ones.*

In this course, we will not prove the Riemann uniformisation theorem in all its generality, but we will settle with a slightly weaker result.

[4]The reason is that we can find Riemann surfaces holomorphic to torii that are not biholomorphic.

Theorem 3.13. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then Ω is biholomorphic to Δ.*

Theorem 3.14. *Let $\Omega \Subset \mathbb{C}$ be a bounded domain, and let $z_0 \in \Omega$. Then there is a unique holomorphic covering $\varphi : \Delta \longrightarrow \Omega$ such that*

$$\varphi(0) = z_0 \quad \text{and} \quad \varphi'(0) > 0.$$

Proposition 3.15. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then Ω is biholomorphic to a bounded domain.*

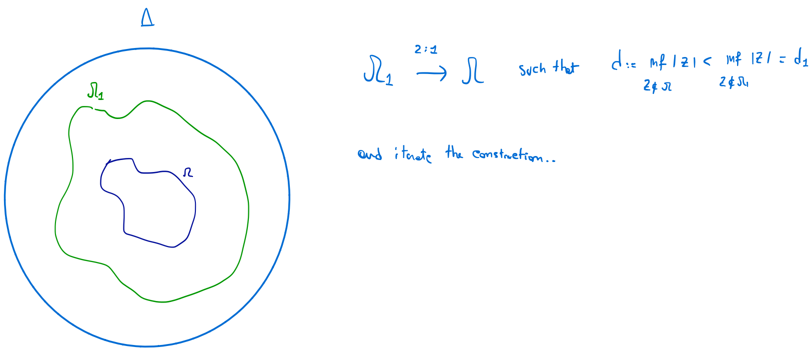


Figure 3.1: Idea of the argument used to prove Proposition 3.15.

Lemma 3.16. *Let $\Omega \Subset \mathbb{C}$ be a bounded domain strictly contained in Δ and such that $0 \in \Omega$. Then there exists a holomorphic map $f \in \mathrm{Hol}(\Delta, \Delta)$ such that the following properties are satisfied:*

(1) We have $f(0) = 0$, $f'(0) > 0$, and $f(\Delta) \supset \Omega$.

(2) If Ω_1 is the connected component of $f^{-1}(\Omega)$ containing the origin, then $f\big|_{\Omega_1}$ is a covering of degree 2.

(3) We have

$$\inf_{z \notin \Omega_1} |z| =: d_1 > d := \inf_{z \notin \Omega} |z|. \tag{3.3}$$

Proof. Let $a \in \Delta \setminus \Omega$, and let $b \in \Omega$ be a square root of $-a$, that is, $b^2 = -a$. Let us consider the automorphisms Φ and Ψ of Δ given by

$$\Phi(z) := \frac{z+a}{1+\bar{a}z} \quad \text{and} \Psi(z) := \frac{z+a}{1+\bar{b}z}$$

Now let $f : \Delta \longrightarrow \Delta$ be given by

$$f(z) := \frac{\bar{b}}{|b|}\Phi(\Psi^2(z)),$$

and recall that $z \longmapsto z^2$ is a covering $\Delta^* \longrightarrow \Delta^*$ of degree 2.

(1) A straightforward computation shows that

$$f(\Delta) = \Delta \quad \text{and} \quad f(0) = 0,$$

and thus f is surjective. Furthermore, we have that

$$f'(0) = 2|b| \frac{1 - |b|^2}{1 - |a|^2} > 0.$$

(2) The preimage via Φ of Ω is a subset of Δ^* because $\Phi(0) = a$ does not belong to Ω by assumption. The reader might check as an exercise that

$$f\,\Big|_{\Omega_1} : \Omega_1 \longrightarrow \Omega$$

is a covering of degree 2.

(3) If $d_1 = 1$, then (3.3) is trivially satisfied and $\Omega_1 = \Delta$. Suppose now that $d_1 < 1$ strictly, and let $z_1 \in \partial\Omega_1$ be a point such that $d_1 = |z_1|$. Then

$$f(z_1) \notin \Omega \implies |f(z_1)| \geq d,$$

and this is enough to conclude since by Schwarz (2.1) we have

$$d_1 = |z_1| > |f(z_1) \geq d.$$

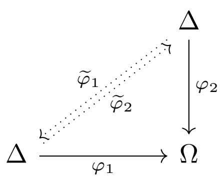
□

Proof of Theorem 3.14. We divide the proof into two steps.

Uniqueness. Suppose that there are two holomorphic coverings φ_1 and φ_2 satisfying

$$\varphi_i(0) = z_0 \quad \text{and} \quad \varphi_i'(0) > 0.$$

We consider the diagram



Then we can find a lift $\widetilde{\varphi}_i$ of φ_i, for $i = 1, 2$, such that

$$\begin{cases} \varphi_2 \circ \widetilde{\varphi}_1 = \varphi_1, \\ \varphi_1 \circ \widetilde{\varphi}_2 = \varphi_2, \end{cases}$$

where $\widetilde{\varphi}_i : \Delta \longrightarrow \Delta$ is a holomorphic covering satisfying

$$\widetilde{\varphi}_i(0) = 0 \quad \text{and} \quad \widetilde{\varphi}'(0) > 0.$$

The unique automorphism satisfying these two properties is the identity, and thus $\varphi_2 \equiv \varphi_1$, which gives us the uniqueness of the holomorphic covering.

Existence. Let us consider the family

$$\mathcal{F} := \{f \in \mathrm{Hol}(\Delta, \Delta) \ : \ f \text{ satisfies (1) and (2)}\}.$$

We will denote by Ω_f the connected component given in (2) rather than Ω_1. The previous result asserts that $\mathcal{F}$ is a nonempty family, and

$$d_f := \inf_{z \notin \Omega_f} |z| \leq 1$$

is equal to one if and only if Ω_f coincides with Δ. Let

$$d := \sup_{f \in \mathcal{F}} d_f \leq 1,$$

and let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of functions such that

$$d_{f_n} \xrightarrow{n\to\infty} d.$$

By Montel's Theorem we conclude[5] that, up to subsequences, f_n converges to a function $f_0 \in \mathrm{Hol}(\Delta, \Delta)$; it remains to show that f_0 belongs to $\mathcal{F}$.

Existence - (1). Let $r > 0$ be a positive number such that $\Delta_r \Subset \Omega$, and let h_n be the restriction (see Figure 3.2) of f_n to Δ_r satisfying $h_n(0) = 0$. Then

$$h_n \in \mathrm{Hol}(\Delta_r, \Delta) \quad \text{and} \quad f_n \circ h_n = \mathrm{id}_{\Delta_r}.$$

It turns out that, up to subsequences, h_n converges to some $h_0 \in \mathrm{Hol}(\Delta_r, \Delta)$ satisfying $f_0 \circ h_0 = \mathrm{id}_{\Delta_r}$, which gives

$$f_0(0) = f_0(h_0(0)) = 0 \quad \text{and} \quad f_0'(0) > 0.$$

Existence - (2.1). Let Ω_{f_0} be the connected component containing the origin. We will first show that

$$f_0(\Omega_{f_0}) = \Omega.$$

Let $z_0 \in \Omega$ and $\gamma : [0, 1] \longrightarrow \Omega$ continuous curve such that $\gamma(0) = 0$ and $\gamma(1) = z_0$. We can always (see Figure 3.3) cover $\gamma([0, 1])$ with a finite number of closed disks $D_0, \ldots, D_J \Subset \Omega$ such that

$$0 \in D_0, \quad z_0 \in D_J \quad \text{and} \quad D_j \cap D_{j+1} \neq \varnothing.$$

Denote by $h_{n,0}$ the inverse of $f_n\big|_{D_0}$ satisfying $h_{n,0}(0) = 0$, and denote by $h_{n,j}$ the inverse of $f_n\big|_{D_j}$, $j = 1, \ldots, J$, satisfying

$$h_{n,j}\big|_{D_j \cap D_{j-1}} \equiv h_{n,j-1}\big|_{D_j \cap D_{j-1}}.$$

By Vitali's Theorem, up to subsequences, for all $j = 1, \ldots, J$ we have

$$h_{n,j} \xrightarrow{n\to\infty} h_{0,j} \in \mathrm{Hol}(D_j, \Delta)$$

[5] Note that Montel's theorem was proved for sequences in $\mathcal{O}(\Delta)$, but it is easy to see that we can restrict the codomain to Δ.

since the disks intersect in a nonempty region, and it turns out that

$$f_0 \circ h_{0,\,j} = \mathrm{id}_{D_j}$$

for all j. Note that for $j = J$ we obtain $f_0 \circ h_{0,\,J} = \mathrm{id}_{D_J}$, and thus

$$f_0(h_{0,\,J}(z_0)) = z_0 \implies z_0 \in f_0(\Delta),$$

but this is not enough. We define the curve

$$\widetilde{\gamma}(t) := h_{0,\,J} \circ \gamma(t),$$

and we notice that it is a continuous lift of γ, that is, $f_0 \circ \widetilde{\gamma} = \gamma$. Furthermore, $\widetilde{\gamma}$ connects 0 and $h_{0,\,J}(z_0)$, which means that

$$h_{0,\,J}(z_0) \in \Omega_{f_0} \implies z_0 \in f_0(\Omega_{f_0}),$$

and this is exactly what we wanted to prove.

Existence - (2.2). We will now prove that $f_0\big|_{\Omega_{f_0}}$ is a covering of degree two. Let $z_0 \in \Omega$ and let D be a disk, centered at z_0, compactly embedded in Ω. We claim that for all $w_0 \in f_0^{-1}(z_0) \cap \Omega_{f_0}$, we can find a neighbourhoods $U_{w_0} \ni w_0 \subset \Omega_{f_0}$ such that

$$f\big|_{U_{w_0}} : U_{w_0} \longrightarrow D$$

is a biholomorphism, and $U_{w_i} \cap U_{w_j} = \varnothing$ for all i and j. By Hurwitz's Theorem there exists $n_1 \geq 1$ such that for all $n \geq n_1$ there is $w_n \in \Delta$ with the following properties:

$$f_n(w_n) = z_0 \quad \text{and} \quad w_n \xrightarrow{n\to\infty} w_0.$$

It follows that $w_n \in \Omega_{f_0}$ for n sufficiently big. Let $h_n : D \longrightarrow \Delta$ be the local inverse of f_n satisfying $h_n(z_0) = w_n$. Up to subsequences, we have

$$h_n \xrightarrow{n\to\infty} h_0 \in \mathrm{Hol}(D,\,\Delta),$$

$h_0(z_0) = w_0$, and $f_0 \circ h_0 = \mathrm{id}_D$. Then we can take $U_{w_0} := h_0(D)$, and it is easy to see that the restriction

$$f\big|_{U_{w_0}} : U_{w_0} \longrightarrow D$$

is a biholomorphism. To prove that $U_{w_i} \cap U_{w_j} = \varnothing$, we consider a point $\widetilde{w}_0 \neq w_0$ in $f_0^{-1}(z_0) \cap \Omega_{f_0}$ and we denote by $\widetilde{h}_0$ the local inverse above. The idea is to argue by contradiction assuming that

$$\exists\, w \in U_{w_0} \cap U_{\widetilde{w}_0}.$$

Then $w = h_0(z_1) = \widetilde{h}_0(\widetilde{z}_1)$, and thus

$$z_1 = f_0 \circ h_0(z_1) = f_0 \circ \widetilde{h}_0(\widetilde{z}_1) = \widetilde{z}_1 \implies z_1 = \widetilde{z}_1.$$

Therefore, the point w is a zero of the function $h_0 - \widetilde{h}_0$. On the other hand, the sequence $h_n - \widetilde{h}_n$ has no zeroes for n sufficiently big because $w_0 \neq \widetilde{w}_0$.

Applying Hurwitz's Theorem again, we conclude that the limit function $h_0 - \widetilde{h}_0$ must be identically equal to zero, which means that

$$h_0 \equiv \widetilde{h}_0 \implies w_0 = \widetilde{w}_0,$$

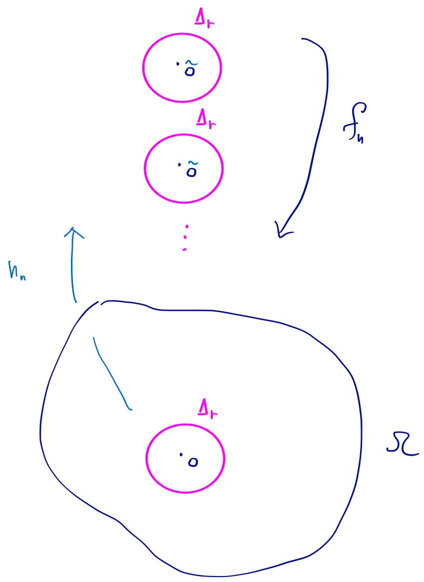


Figure 3.2: The mapping h_n as the local inverse of f_n.

and this is the desired contradiction. It follows that $f_0 \in \mathcal{F}$ and $d_{f_0} = d$. If $d = 1$, then there is nothing else left to do; if $d < 1$, then we apply the result above to Ω_{f_0} and obtain $\hat{f} \in \mathcal{F}$ such that $d_{\hat{f}} > d$ and so on - stopping when d becomes 1. $\square$

Proof of Proposition 3.15. Let $a \in \mathbb{C} \setminus \Omega$. The function $z - a$ does not vanish on Ω, and therefore there exists $h \in \mathcal{O}(\Omega)$ such that

$$h^2(z) = z - a.$$

Let $z_1, z_2 \in \Omega$. If $h(z_1) = \pm h(z_2)$, then

$$h^2(z_1) = h^2(z_2) \implies z_1 = z_2,$$

and therefore h is injective and $h(\Omega) \cap (-h(\Omega)) = \varnothing$. Now fix $z_0 \in \Omega$ and let $r > 0$ be small enough to have the compact inclusion

$$D := D_{h(z_0),\, r} \Subset h(\Omega).$$

The property showed above implies that $-D \cap h(\Omega)$ is empty, and thus

$$|h(z) + h(z_0)| \geq r \quad \text{for all } z \in \Omega \implies 2|h(z_0)| \geq r.$$

The function

$$f(z) := \frac{r}{4} \frac{1}{|h(z_0)|} \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

is obviously injective, and also such that

$$\left| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right| = |h(z_0)| \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \leq 4 \frac{|h(z_0)|}{r},$$

which in turn implies that $f(\Omega) \subseteq \Delta$, and thus Ω is biholomorphic to the bounded domain $f(\Omega)$. $\square$

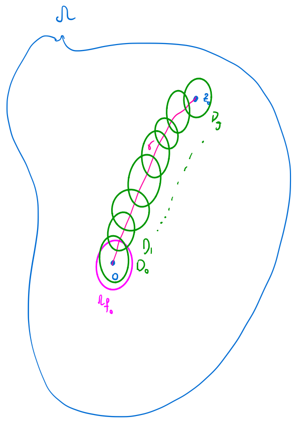


Figure 3.3: Cover the image of the curve γ with a finite number of closed disks compactly embedded in Ω.

Part II

Complex Analysis in Several Variables

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