Lecture Notes

Calculus of Variations A

Course held by

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Disclaimer

I wrote these notes to summarise the content of the course *Calculus of Variations A*, held by Professor Giovanni Alberti at the University of Pisa in 2018/2019.

I tried to include all the topics that were discussed in class and combine it with some additional information from several other courses to produce a self-contained document.

I will try to review them periodically, but I am sure that at the end there will be a large number of mistakes and oversights. To report them, feel free to send me an email at **francesco (dot) maiale (at) sns (dot) it**.

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CHAPTER 1

Introduction to Calculus of Variations

In this chapter, we will briefly describe how and why the field of *calculus of variations* became so crucial in modern mathematics and introduce the basic notions such as the first variation of a functional.

Since this is more of a descriptive part, we shall assume that everything is as much regular as we need to carry out all the computations and come back to deal with it at a later time.

1. Introduction

Let $\mathfrak{X}$ be a space of functions and F a real-valued functional defined on $\mathfrak{X}$. The "goal" of the course is to find minimisers of F on $\mathfrak{X}$, namely

$$\inf_{u\in\mathfrak{X}} F(u) = \sup_{u\in\mathfrak{X}}\{-F(u)\}.$$

We are interested in the existence and regularity of solutions, which is usually possible if F is "nice" enough. However, sometimes even finding a way to approximate a solution via numerical methods will be crucial.

Classical approach. If $\mathfrak{X} = \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$, then $\bar{x} \in \mathbb{R}^n$ minimiser of F satisfies the equation

$$\nabla F(\bar{x}) = 0.$$

If, in addition, F is *convex*, then the converse is also true. In other words, minimisers are completely characterised by the following relation:

$$\nabla F(\bar{x}) = 0 \iff F(\bar{x}) = \inf_{x\in\mathbb{R}^n} F(x).$$

In the general framework, if u is a minimiser of F, then u satisfies the *Euler-Lagrange equation* associated with F. We will also see that, if F is convex, then the vice versa holds.

Abstract setting. In general, $\mathfrak{X}$ is an open set in a Banach space or a Banach manifold (i.e., a C^1-manifold without boundary modeled on a Banach space.) The functional

$$F : \mathfrak{X} \longrightarrow \mathbb{R}$$

is Gateaux-differentiable (G-differentiable) at every point $u \in \mathfrak{X}$ with differential $\mathrm{d}f(u) : \mathrm{Tan}(X,\, u) \to \mathbb{R}$. We expect that something along the lines of the following holds:

THEOREM 1.1. *If u is a local maximum (or minimum) of F, then $\langle \mathrm{d}f(u),\, v\rangle = 0$ for all $v \in \mathrm{Tan}(\mathfrak{X},\, u)$.*

This framework can be developed (e.g., in the *Variational Methods* course), but we will not do it because we are more interested in the existence of minimisers. Furthermore, some problems arising in applications do not fit in this framework.

2. First variation of functionals and Euler-Lagrange equation

DEFINITION 1.2. Let $\mathfrak{X}$ be a vector space (of functions). Fix a point $u \in \mathfrak{X}$ and fix a *direction* $v \in \mathfrak{X}$. The *directional derivative* of F at u in the direction v is defined by setting

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u + tv).$$

We can either indicate it with the symbol $\frac{\partial F}{\partial v}(u)$ or, with some abuse of notation, $\langle \mathrm{d}F(u),\, v\rangle$.

REMARK 1.3. We will try to avoid the second notation, $\langle \mathrm{d}F(u),\, v\rangle$, because when $\mathfrak{X}$ is only a vector space there is no guarantee that there exists such a linear mapping.

DEFINITION 1.4. Let $\mathfrak{X}$ be a vector space. The map

$$\mathfrak{X} \ni v \longmapsto \frac{\partial F}{\partial v}(u) \in \mathbb{R}$$

is called *first variation* of F at u.

REMARK 1.5. If u is a minimiser (or local minimiser) of F, then the first variation of F at u is equal to zero, that is,

$$\frac{\partial F}{\partial v}(u) = 0 \quad \text{for all } v \in \mathfrak{X}. \tag{1.1}$$

Note that this is the Euler-Lagrange equation in its simplest form. Furthermore, if F is convex, then the vice versa holds; in other words, the relation (1.1) implies that u is a minimiser.

The relation (1.1) is often useless because using the Euler-Lagrange equation to find minimisers requires a way to compute the directional derivative rather than the abstract theory behind it.

We will now present several examples of simple minimisation problems which, we hope, will give a satisfying understanding of the following phenomena:

- **(a)** Weak and strong formulation of the Euler-Lagrange equation.
- **(b)** Dirichlet boundary conditions and the appearance of Neumann ones.
- **(c)** Obstacle-type problems.
- **(d)** Higher-order problems; namely, the functional depends on derivatives of order bigger than or equal to two.
- **(e)** Vector-valued problems, e.g., u maps $\mathbb{R}^n$ into $\mathbb{R}^m$.
- **(f)** The space $\mathfrak{X}$ is not linear but, for example, affine.

2.1. Technical result. To formally compute the Euler-Lagrange equation, we first need to recall a few technical results such as the divergence theorem and the fundamental lemma in the calculus of variations.

LEMMA 1.6 (Divergence theorem). *Let U be a vector field defined on Ω of class C^1. Then*

$$\int_{\partial\Omega} U \cdot \nu \, \mathrm{d}\sigma = \int_{\Omega} \mathrm{div} U \, \mathrm{d}x.$$

COROLLARY 1.7. *Let U be a vector field defined on Ω of class C^1 and let $v \in C_0^1(\Omega)$. Then*

$$\int_{\partial\Omega} (vU) \cdot \nu \, \mathrm{d}\sigma = \int_{\Omega} \mathrm{div}(vU) \, \mathrm{d}x = \int_{\Omega} \nabla v \cdot U \, \mathrm{d}x + \int_{\Omega} v \, \mathrm{div} U \, \mathrm{d}x.$$

We now give the statement of the *fundamental lemma in calculus of variations*. In (1.4) we do not need such generality, but we take the opportunity to give it once and for all.

LEMMA 1.8. *Let $f : \Omega \to \mathbb{R}$ be a $L^1_{\mathrm{loc}}(\Omega)$ function. If*

$$\int_{\Omega} f(x)v(x) \, \mathrm{d}x = 0 \quad \textit{for all } v \in C_c^\infty(\Omega), \tag{1.2}$$

then f is identically equal to zero almost everywhere.

PROOF. The condition (1.6) implies that

$$\int_{\Omega} f(x)v(x) \, \mathrm{d}x = 0 \quad \text{for all } v \in C_c(\Omega)$$

by means of an approximation argument. We now claim that we can further require that

$$\int_{\Omega} f(x)v(x) \, \mathrm{d}x = 0 \quad \text{for all } v \in L_c^\infty(\Omega).$$

The reason is that $L_c^\infty(\Omega)$ can be obtained as the "pointwise limit" of uniformly bounded[1] continuous functions. Assume now that f is not a.e. zero and let E be a set of positive finite measure such that

$$f\big|_E > 0.$$

Then we can choose $v := \chi_E$ as a test function and notice that $\int_\Omega f(x)\chi_E(x) \, \mathrm{d}x > 0$. This gives a contradiction with the assumption and concludes the proof. □

2.2. Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider the vector space of continuously differentiable functions equal to zero at the boundary:

$$\mathfrak{X} := \left\{ u : \bar{\Omega} \to \mathbb{R} \,:\, u \in C^1(\Omega),\ u\big|_{\partial\Omega} \equiv 0 \right\} = C_0^1(\Omega).$$

We are interested in the minimisation of the following functional[2]

$$F(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x + \int_{\Omega} f(x,\, u(x)) \, \mathrm{d}x.$$

[1] This is important because we need to use Lebesgue's dominated convergence theorem.

[2] The first term is usually referred to in the literature as *Dirichlet energy*.

Let $t \in \mathbb{R}$. To compute the first variation, we first evaluate F at $u + tv \in \mathfrak{X}$,

$$F(u+tv) = \frac{1}{2}\int_\Omega |\nabla u + t\nabla v|^2 \, \mathrm{d}x + \int_\Omega f(x,\, u+tv)\, \mathrm{d}x,$$

and then we take (assuming that f is regular enough) the derivative with respect to t at $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u+tv) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega f_u(x,\, u) v \, \mathrm{d}x, \tag{1.3}$$

where $f_u(x,\, u)$ is the derivative of f with respect to the second variable. We will refer to (1.3) as *weak formulation* of the first variation. To go further, we can apply Corollary 1.7 to infer that

$$\frac{\partial F}{\partial v}(u) = \int_{\partial\Omega} (\nabla u \cdot \nu) v \, \mathrm{d}\sigma - \int_\Omega \Delta u v \, \mathrm{d}x + \int_\Omega f_u(x,\, u) v \, \mathrm{d}x,$$

and, since $v \in \mathfrak{X}$ is zero at the boundary, we finally get

$$\frac{\partial F}{\partial v}(u) = \int_\Omega [f_u(x,\, u) - \Delta u]\, v \, \mathrm{d}x. \tag{1.4}$$

We will refer to (1.4) as *strong formulation* of the first variation since it requires more regularity on u (e.g., C^2) and Ω (e.g., Lipschitz) than (1.3). Now, if u is a minimiser, then

$$\int_\Omega [f_u(x,\, u) - \Delta u]\, v \, \mathrm{d}x = 0 \quad \text{for all } v \in \mathfrak{X},$$

from which it follows (see Lemma 1.8) that u satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta u(x) = f_u(x,\, u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{1.5}$$

REMARK 1.9. If $f(x,\, u) = \rho(x)u(x)$, then (1.5) becomes the so-called *Poisson equation*, which is the PDE describing the shape of the electric potential u inside a non-conductive domain Ω (with conductive boundary) under the effect of a distributional charge ρ.

2.3. Neumann boundary conditions. Let $\mathfrak{X} = C^1(\bar{\Omega})$ so that there is no boundary condition. As before, we find that the directional derivative is given by

$$\frac{\partial F}{\partial v}(u) = \int_{\partial\Omega} (\nabla u \cdot \nu) v \, \mathrm{d}\sigma + \int_\Omega [f_u(x,\, u) - \Delta u]\, v \, \mathrm{d}x,$$

but the first term is not zero anymore. We now claim that

$$\frac{\partial F}{\partial v}(u) = 0 \quad \text{for all } v \in C^1(\bar{\Omega})$$

if and only if u solves

$$\begin{cases} -\Delta u(x) = f_u(x,\, u(x)) & \text{if } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{1.6}$$

The condition $\frac{\partial u}{\partial \nu} = 0$ is referred to as *Neumann boundary condition*. One implication is obvious, so we only have to prove that u is a minimiser implies u solution of (1.6). But now we know that

$$0 = \int_\Omega [f_u(x,\, u) - \Delta u]\, v \,\mathrm{d}x \quad \text{for all } v \in C_c^\infty(\Omega),$$

which implies (by Lemma 1.8) that $-\Delta u(x) = f_u(x,\, u(x))$. Now plug this back into the directional derivative to find that

$$\int_{\partial\Omega} (\nabla u \cdot \nu) v \,\mathrm{d}\sigma = 0$$

for all $v \in C^1(\bar{\Omega})$. However, the space of compactly supported smooth functions is a subset of $\mathfrak{X}$, and hence

$$\int_{\partial\Omega} (\nabla u \cdot \nu) v \,\mathrm{d}\sigma = 0 \quad \text{for all } v \in C_c^\infty(\Omega) \implies \nabla u \cdot \nu(x) = \frac{\partial u}{\partial \nu}(x) = 0.$$

2.4. Mixed boundary conditions. We now present a couple of one-dimensional minimisation problems in which the Lagrangian f is entirely general.

EXAMPLE 1.10. *Let $\mathfrak{X} = C^1([a,\, b])$ and consider the general functional*

$$F(u) = \int_a^b f(x,\, u,\, \dot{u}) \,\mathrm{d}x.$$

Then the first variation is given by

$$\langle \mathrm{d}F(u),\, v\rangle = \int_a^b [f_u(x,\, u,\, \dot{u})v + f_\xi(x,\, u,\, \dot{u})\dot{v}] \,\mathrm{d}x,$$

where ξ denotes the third variable of F, i.e., $\dot{u}$. Integrating by parts, we find that

$$\langle \mathrm{d}F(u),\, v\rangle = [f_\xi(x,\, u,\, \dot{u})v]_{x=a}^b + \int_a^b \left[-\frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, u,\, \dot{u}) + f_u(x,\, u,\, \dot{u})\right] v \,\mathrm{d}x.$$

The Euler-Lagrange equation associated to the problem is given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, u,\, \dot{u}) = f_u(x,\, u,\, \dot{u}) & \text{if } x \in [a,\, b], \\ f_\xi(x,\, u,\, \dot{u}) = 0 & \text{if } x = a \text{ or } x = b, \end{cases}$$

where the Neumann boundary conditions give the uniqueness of the solution. It is worth remarking that, if we consider

$$\mathfrak{X} = \{u \in C^1([a,\, b]) \,:\, u(a) = 0\},$$

then a straightforward computation shows that the Euler-Lagrange equation associated to the problem is given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, u,\, \dot{u}) = f_u(x,\, u,\, \dot{u}) & \textit{if } x \in [a,\, b], \\ u(x) = 0 & \textit{if } x = a, \\ f_\xi(x,\, u,\, \dot{u}) = 0 & \textit{if } x = b. \end{cases}$$

In particular, the presence of a Dirichlet condition at some point of the boundary prevents the appearance of a Neumann-type condition there.

REMARK 1.11. Notice that, if we consider $\mathfrak{X} = C^1([a,\, b];\; \mathbb{R}^m)$, then we need to be careful carrying out these computations because the Lagrangian is given by

$$f : [a,\, b] \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R},$$

which means that f_u has to be replaced with $\nabla_u f$, the gradient with respect to u of f. It is easy to check that the first variation is given by

$$\langle \mathrm{d}F(u),\, v\rangle = \int_a^b \left[\nabla_u f(x,\, u,\, \dot{u}) \cdot v + \nabla_\xi f(x,\, u,\, \dot{u}) \cdot \nabla v\right] \mathrm{d}x,$$

which, integrating by parts, leads to

$$\langle \mathrm{d}F(u),\, v\rangle = \left[\nabla_\xi f(x,\, u,\, \dot{u}) \cdot v\right]_{x=a}^b + \int_a^b \left[-\frac{\mathrm{d}}{\mathrm{d}x}\nabla_\xi f(x,\, u,\, \dot{u}) + \nabla_u f(x,\, u,\, \dot{u})\right] \cdot v \,\mathrm{d}x.$$

The Euler-Lagrange system of equations is obtained using Lemma 1.8 component by component. It turns out that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} \nabla_\xi f(x,\, u,\, \dot{u}) = \nabla_u f(x,\, u,\, \dot{u}) & \text{if } x \in [a,\, b], \\ \nabla_\xi f(x,\, u,\, \dot{u}) = 0 & \text{if } x = a \text{ or } x = b, \end{cases}$$

which corresponds to the system (as $j = 1,\, \dots,\, m$)

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} \partial_{\xi_j} f(x,\, u,\, \dot{u}) = \partial_{u_j} f(x,\, u,\, \dot{u}) & \text{if } x \in [a,\, b], \\ \nabla_{\xi_j} f(x,\, u,\, \dot{u}) = 0 & \text{if } x = a \text{ or } x = b. \end{cases}$$

The same computation with $\mathfrak{X}$ replaced by $C^1_{u_0}(\bar{\Omega})$ leads to a similar system in which the Neumann conditions are replaced by the Dirichlet ones

$$u_j(x) = (u_0)_j(x) \quad \text{for all } x \in \partial\Omega \text{ and all } j \in \{1,\, \dots,\, n\}.$$

At this point, one might wonder why we only talk about Dirichlet and Neumann boundary conditions since, from a mathematical point of view, there is no reason why we cannot require something different. It is important to notice, however, that most problems arising from Physics (or other disciplines) are naturally endowed with such conditions. The next example will show why the problem is more delicate than it seems.

EXAMPLE 1.12. *Let* $F(u) = \int_a^b f(x,\, \dot{u})\,\mathrm{d}x$ *and let*

$$\mathfrak{X} = \{u \in C^1([a,\, b]) \ :\ u(a) = u_0(a),\, u(b) = u_0(b),\, \dot{u}(a) = v_0(a),\, \dot{u}(b) = v_0(b)\}.$$

Clearly $\mathfrak{X}$ *is an affine space whose underlying vector space is*

$$\mathfrak{V} = \{v \in C^1([a,\, b]) \ :\ v\big|_{\{a,\, b\}} = \dot{v}\big|_{\{a,\, b\}} = 0\},$$

so the first variation is given by

$$\langle \mathrm{d}F(u),\, v\rangle = [f_\xi(x,\, \dot{u})v]_{x=a}^b + \int_a^b \left[-\frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, \dot{u}) +\right] v\,\mathrm{d}x.$$

The Euler-Lagrange equation associated to this variational problem is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, u,\, \dot{u}) = 0 & \textit{if } x \in [a,\, b], \\ u(x) = u_0(x),\, \dot{u}(x) = v_0(x) & \textit{if } x = a \textit{ or } x = b, \end{cases}$$

and, unless we are very lucky, the boundary conditions are incompatible because fixing the value of u *on* $\{a,\, b\}$ *is already enough to infer existence and* **uniqueness**. *On the other hand, if we consider*

$$\mathfrak{X} = \{u \in C^1([a,\, b]) \ :\ \dot{u}(a) = v_0(a),\, \dot{u}(b) = v_0(b)\},$$

then it is easy to verify that the Euler-Lagrange equation is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x,\, u,\, \dot{u}) = 0 & \textit{if } x \in [a,\, b], \\ \dot{u}(x) = v_0(x) & \textit{if } x = a \textit{ or } x = b, \\ f_\xi(x,\, \dot{u}) = 0 & \textit{if } x = a \textit{ or } x = b. \end{cases}$$

The latter condition is obtained because v *needs not to be zero at the boundary anymore, and this leads to the same problem as above: the equation is over-determined.*

The issue about what boundary condition is appropriate for a specific problem is strictly related to the weak formulation in Sobolev spaces, which will be investigated later on in the course.

2.5. Affine vector spaces. We now show that we can make the same computations in a functional space $\mathfrak{X}$ which is **not** a vector space, but rather an affine space. The main issue to deal with is that

$$u,\, v \in \mathfrak{X} \implies\!\!\!\!\!\!/\;\; u + tv \in \mathfrak{X}.$$

Fix $u_0 : \partial\Omega \to \mathbb{R}$ function defined on the boundary and define

$$\mathfrak{X} := \{u : \bar{\Omega} \to \mathbb{R} \ :\ u \in C^1(\Omega),\ u\big|_{\partial\Omega} \equiv u_0\} = C^1_{u_0}(\Omega).$$

The space $\mathfrak{X}$ is not a vector space, but an affine one. We compute the directional derivative as before and obtain

$$\frac{\partial F}{\partial v}(u) = \int_{\partial\Omega} \nabla u \cdot \nabla v\,\mathrm{d}x + \int_\Omega f_u(x,\, u)v\,\mathrm{d}x.$$

Unfortunately, we cannot take anymore $v \in \mathfrak{X}$ because the element $u + tv$ will not necessarily belong to $\mathfrak{X}$ because of the affine structure. That said, it is easy to see that

$$v \in C_0^1(\bar{\Omega}) \implies u + tv \in \mathfrak{X} \quad \text{for all } x \in \mathfrak{X},$$

which, in turn, implies that

$$\frac{\partial F}{\partial v}(u) = \int_\Omega [f_u(x,\, u) - \Delta u]\, v \,\mathrm{d}x.$$

This means that u minimiser implies u solution of the boundary-value PDE

$$\begin{cases} -\Delta u(x) = f_u(x,\, u(x)) & \text{if } x \in \Omega, \\ u(x) = u_0(x) & \text{if } x \in \partial\Omega. \end{cases} \tag{1.7}$$

2.6. Higher-order derivatives. The Lagrangian usually depends on the triple $(x,\, u,\, \dot{u})$, but this is by no means a rule. Indeed, there are a lot of problems - arising from the real world - for which

$$f = f(x,\, u,\, \dot{u},\, \ldots,\, u^{(m)}).$$

On the other hand, if no derivative appears in the Lagrangian f, then the problem is trivial from a theoretical point of view (formally we have order-zero Euler-Lagrange equation), but it can be not-so-easy from a computational stand.

EXERCISE 1.1 (Higher derivatives). Let $\ddot{u}$ be the second-order derivative of u. Compute first variation and Euler-Lagrange equation for the functional

$$F(u) = \int_a^b f(x,\, \ddot{u}) \,\mathrm{d}x$$

in the following cases:

(i) $\mathfrak{X} = C^2([a,\, b])$;

(ii) $\mathfrak{X} = C^2_{u_0}([a,\, b])$;

(iii) $\mathfrak{X} = C^2_{u_0,\, v_0}([a,\, b]) := \{u \in C^2_{u_0}([a,\, b]) \,:\, \dot{u}(a) = v_0(a),\ \dot{u}(b) = v_0(b)\}$.

EXERCISE 1.2 (Higher derivatives). Let Δ denote the Laplace operator (sum of second derivatives). Compute first variation and Euler-Lagrange equation for the functional

$$F(u) = \int_\Omega |\Delta u|^2(x) \,\mathrm{d}x$$

in the following cases:

(i) $\mathfrak{X} = C^2(\bar{\Omega})$;

(ii) $\mathfrak{X} = C^2_{u_0}(\bar{\Omega})$;

(iii) $\mathfrak{X} = C^2_{u_0,\, v_0}(\bar{\Omega})$.

2.7. Obstacles. A significant problem in the *Calculus of Variations* is to find an equilibrium profile of an elastic membrane Ω with fixed boundary and constrained to lie above an obstacle. Applications include the study of fluid filtration in porous media, constrained heating and elasto-plasticity - see [**?**].

In this brief section, we will show a simple example of an obstacle-type problem, and we will try to point out the main difficulties which derive from the fact that $u + tv$ may fall **below** the obstacle.

EXERCISE 1.3 (Obstacle). Let $\mathfrak{X}$ be the subset of $C_0^1(\bar{\Omega})$ which elements satisfies the obstacle inequality

$$u(x) \geq u_0(x) \quad \text{for all } x \in E,$$

where E is a compact subset of Ω and $u_0 : E \to \mathbb{R}$ is given. Compute the first variation and the Euler-Lagrange equation for the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

SOLUTION. First, notice that when $u(x_0) = u_0(x_0)$ for some $x_0 \in E$, it suffices to pick a $v \in C_0^1(\Omega)$ satisfying $v(x_0) < 0$ to conclude that

$$\exists t_0 \in (-\delta,\, \delta) \ : \ u + t_0 v \notin \mathfrak{X}$$

no matter how small $\delta > 0$ is. Indeed, it is easy to verify that

$$u + tv \in \mathfrak{X} \text{ for all } t \in (-\delta,\, \delta) \implies v\Big|_{E_u} \equiv 0,$$

where $E_u := \{x \in E \ : \ u(x) = u_0(x)\}$. However, if we work with this class of test functions, then making the usual computations leads to an incomplete Euler-Lagrange equation since

$$-\int_\Omega v \Delta u \, \mathrm{d}x = 0 \quad \text{for all } v \in E_u$$

implies

$$\begin{cases} -\Delta u = 0 & \text{if } x \in \Omega \setminus E_u, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.8}$$

but we have no information whatsoever on the behaviour of u on the set E_u. We thus go back to the choice of v and notice that requiring something less (on t) leads to a larger class of test functions:

$$u + tv \in \mathfrak{X} \text{ for all } t \in (0,\, \delta) \implies v \in C_0^1(\Omega) \text{ and } v\Big|_{E_u} \geq 0\,.$$

Recall that for a function $p : [a,\, b] \to \mathbb{R}$ the fact that a is a critical point only implies that $p'(a) \geq 0$ and, similarly, b critical point implies $p'(b) \leq 0$. We now compute the first variation of the functional taking into account that $t \mapsto F(u + tv)$ is only defined on the $(0,\, \delta)$; it turns out that

$$-\int_\Omega v \Delta u \, \mathrm{d}x \geq 0 \quad \text{for all } v \in C_0^1(\Omega) \text{ and } v\Big|_{E_u} \geq 0.$$

Using a variation of the *fundamental lemma of calculus of variations*, Lemma 1.13, we infer that

$$-\Delta u \geq 0 \quad \text{if } x \in \Omega.$$

This, together with the information obtained in (1.8), leads to the full Euler-Lagrange equation

$$\begin{cases} -\Delta u(x) = 0 & \text{if } x \in \Omega \setminus E_u, \\ -\Delta u(x) \geq 0 & \text{if } x \in E_u, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

□

LEMMA 1.13. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a function such that*

$$\int_\Omega f\varphi \,\mathrm{d}x \geq 0 \quad \textit{for all } \varphi \geq 0,\, \varphi \in C_c^\infty(\Omega), \tag{1.9}$$

then $f(x) \geq 0$ for a.e. $x \in \Omega$.

PROOF. First, notice that given a nonnegative continuous function $\varphi \in C_c(\Omega)$, we can approximate it via nonnegative smooth functions. Indeed, it suffices to take the sequence

$$\varphi_\epsilon(x) := \varphi * \rho_\epsilon(x),$$

where $\{\rho_\epsilon\}_{\epsilon>0}$ is an *approximate identity*. It is easy to see that φ_ϵ converges uniformly to φ and $\varphi_\epsilon \geq 0$ if we choose ρ nonnegative. We now claim that

$$\int_\Omega f\varphi \,\mathrm{d}x \geq 0 \quad \text{for all } \varphi \in L_c^\infty(\Omega)$$

satisfying $\varphi(x) \geq 0$ almost everywhere. This is also achieved by approximating $f \in L_c^\infty(\Omega)$ via a sequence of continuous functions f_n such that

$$f_n(x) \xrightarrow{n\to\infty} f(x) \text{ at a.e. } x \in \Omega \text{ and } |f_n(x)| \leq C.$$

Since these functions are uniformly bounded, we can apply Lebesgue's dominated convergence theorem to prove the claim. Notice that we can pick an approximating sequence of nonnegative by replacing f_n with

$$g_n(x) := \max\{0,\, f_n(x)\}.$$

Finally, we argue by contradiction. If $f < 0$ on E of positive finite measure, then take $\varphi = \chi_E$ (or replace it with a subset if it is not compactly supported) and notice that

$$\int_E f(x)\,\mathrm{d}x < 0,$$

which contradict the assumption (1.9) with $L_c^\infty(\Omega)$ in place of $C_c^\infty(\Omega)$. □

Abstract framework. To deal with obstacle-type problems, it is natural to require $\mathfrak{X}$ to be either the closure of an open set in a Banach space with C^1-regular boundary or a Banach manifold **with** boundary.

REMARK 1.14. It is possible to give a notion of manifold with boundary in the infinite-dimensional setting, although the definition is not topological and relies on the differential structure - see [**?**] and the reference therein.

If we consider a Gateaux-differentiable function $F : \mathfrak{X} \to \mathbb{R}$ at all $u \in \mathfrak{X}$, then it is not so immediate to find an equivalent of Theorem 1.1 and some additional regularity is required.

EXERCISE 1.4. Under which assumptions on E and u_0 a solution u of the Euler-Lagrange equation

$$\begin{cases} -\Delta u(x) = 0 & \text{if } x \in \Omega \setminus E_u, \\ -\Delta u(x) \geq 0 & \text{if } x \in E_u, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

is also a minimizer of F on $\mathfrak{X}$ (as defined in the previous exercise)?

2.8. Further examples.

EXAMPLE 1.15. *Let $\mathfrak{X} = C^1(\bar{\Omega})$, Ω open bounded subset of $\mathbb{R}^n$, and*

$$F(u) = \int_\Omega f(x,\, u,\, \nabla u)\, \mathrm{d}x.$$

Then the first variation is given by

$$\langle \mathrm{d}F(u),\, v\rangle = \int_\Omega [f_u(x,\, u,\, \nabla u)\cdot v + f_\xi(x,\, u,\, \nabla u)\cdot \nabla v]\, \mathrm{d}x,$$

and integrating by parts yields

$$\langle \mathrm{d}F(u),\, v\rangle = \int_{\partial\Omega} (f_\xi(x,\, u,\, \nabla u)\cdot \eta) v\, \mathrm{d}\sigma + \int_\Omega [-\mathrm{div}_x f_\xi(x,\, u,\, \nabla u) + f_u(x,\, u,\, \nabla u)]\cdot v\, \mathrm{d}x.$$

The Euler-Lagrange equation associated to the problem is given by

$$\begin{cases} \mathrm{div}_x f_\xi(x,\, u,\, \nabla u) = f_u(x,\, u,\, \dot{u}) & \text{if } x \in [a,\, b] \\ f_\xi(x,\, u,\, \nabla u)\cdot \nu = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

with Neumann boundary conditions if we do not require anything on $\mathfrak{X}$. Consider the slightly modified ambient space

$$\mathfrak{X}_\Gamma = \{u \in C^1(\bar{\Omega}) \,:\, u\big|_\Gamma \equiv u_0,\ \Gamma \subset \Omega\},$$

where Γ is closed and $u_0 : \Gamma \to \mathbb{R}$ is given. Then the first variation is the same, but the admissible directions v are the ones such that $v \in C^1(\bar{\Omega})$ and $v\big|_\Gamma \equiv 0$. Then

$$\begin{cases} \mathrm{div}_x f_\xi(x,\, u,\, \nabla u) = f_u(x,\, u,\, \dot{u}) & \textit{if } \Omega, \\ f_\xi(x,\, u,\, \nabla u) \cdot \nu = 0 & \textit{if } x \in \partial\Omega \setminus \Gamma, \\ u = u_0 & \textit{if } x \in \Gamma, \end{cases}$$

is the Euler-Lagrange equation, which boundary conditions are obtained by applying Lemma 1.8 to the integral on $\partial\Omega \setminus \Gamma$ only.

EXAMPLE 1.16. *Let $\Omega \subset \mathbb{R}^n$, $\mathfrak{X} = C^1_{u_0}(\bar{\Omega},\, \mathbb{R}^m)$ and let*

$$F(u) = \int_\Omega f(\nabla u)\, \mathrm{d}x.$$

Notice that, in this case, ∇u is a $m \times n$ matrix so one would need first to introduce matrices scalar product; to avoid this issue, we only specify all components. Let v be an element of the underlying vector space and compute the t-derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u + tv) = \int_\Omega \left[\sum_{i,\,j} f_{\xi_{ij}}(\nabla u) \frac{\partial v_i}{\partial_{x_j}} \right] \mathrm{d}x =$$

$$\int_\Omega \left[\sum_{i=1}^m f_{\xi_i}(\nabla u) \cdot \nabla v_i \right] \mathrm{d}x.$$

We can now apply Corollary 1.7 to infer that

$$\frac{\partial F}{\partial v}(u) = \int_{\partial\Omega} \left[\sum_{i=1}^m f_{\xi_i}(\nabla u)\nu \cdot v_i \right] \mathrm{d}\sigma + \int_\Omega \left[\sum_{i=1}^m \mathrm{div}(f_{\xi_i}(\nabla u)) \cdot v_i \right] \mathrm{d}x =$$

$$= \int_\Omega \left[\sum_{i=1}^m \mathrm{div}(f_{\xi_i}(\nabla u)) \cdot v_i \right] \mathrm{d}x,$$

employing the boundary condition ($v = 0$ on $\partial\Omega$). Then Lemma 1.8 applies to each term of the sum separately which, in turns, shows that the Euler-Lagrange system is given by

$$\begin{cases} \mathrm{div}(f_{\xi_i}(\nabla u)) = 0 & \textit{if } x \in \Omega, \\ u_i(x) = (u_0)_i(x) & \textit{if } x \in \partial\Omega, \end{cases}$$

for all $i \in \{1,\, \dots,\, m\}$.

EXAMPLE 1.17 (Laplace operator). *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then its gradient ∇u is a $m \times n$ matrix. Consider the functional*

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

and endow the space of matrices with the norm $\|M\|_2^2 = \sum_{i,j} |M_{i,j}|^2$. It follows that

$$F(u) = \sum_i \frac{1}{2} \int_\Omega |\nabla u_i|^2 \, \mathrm{d}x.$$

In this case, the decoupled structure of the problem shows that the Euler-Lagrange equations are the Laplace ones for each component u_i; namely, we have

$$\Delta u_i = 0 \quad \textit{for all } i = 1, \ldots, n.$$

This is, unfortunately, a very lucky case. Indeed, if we replace the gradient with the symmetric gradient, then the Euler-Lagrange system is not so simple. For example, the functional

$$F(u) = \frac{1}{2} \int_\Omega \left| \frac{\nabla u + \nabla^T u}{2} \right|^2 \, \mathrm{d}x$$

represents the energy in linearized elasticity. *The Euler-Lagrange system of equations, in this case, is not made up of decoupled equations.*

EXERCISE 1.5. Compute first variation and Euler-Lagrange equation for the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x$$

where $\mathfrak{X}$ is $C_0^1(\bar{\Omega})$ with prescribed L^2-norm $\int_\Omega |u|^2 \, \mathrm{d}x = 1$.

SOLUTION. Notice that we cannot employ the methods we applied to all the previous problems because the constraint on the L^2-norm is hard to deal with. However, we can use *Lagrange multipliers* and consider

$$F_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \frac{\lambda}{2} G(u),$$

where $G(u) = \int_\Omega |u|^2 \, \mathrm{d}x - 1$. Then

$$F_\lambda(u + tv) = \frac{1}{2} \left[\int_\Omega |\nabla(u + tv)|^2 \, \mathrm{d}x - \lambda G(u + tv) \right],$$

and hence the first variation is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F_\lambda(u+tv) &= \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x - \lambda\int_\Omega uv\,\mathrm{d}x = \\ &= -\int_\Omega(\Delta u+\lambda u)v\,\mathrm{d}x + \int_{\partial\Omega}(\nabla u\cdot\nu)v\,\mathrm{d}x = \\ &= -\int_\Omega(\Delta u+\lambda u)v\,\mathrm{d}x.\end{aligned}$$

This leads to the Euler-Lagrange equation

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{if } x\in\Omega, \\ u(x) = 0 & \text{if } x\in\partial\Omega, \end{cases}$$

is nothing but the *eigenvalue problem* of the Laplace operator with Dirichlet boundary conditions. □

EXERCISE 1.6. Compute first variation and Euler-Lagrange equation for the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_{\partial\Omega} gu\,\mathrm{d}\sigma,$$

where $g:\partial\Omega\to\mathbb{R}$ is given and $\mathfrak{X} = C^1(\bar\Omega)$ so that $\int_{\partial\Omega} gu\,\mathrm{d}\sigma$ is nontrivial. Compare it with the case $g = u$, that is,

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2(x)\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma.$$

SOLUTION. Let $v\in\mathfrak{X}$ and $t\in\mathbb{R}$. Then

$$F(u+tv) = \frac{1}{2}\int_\Omega |\nabla u + t\nabla v|^2\,\mathrm{d}x + \int_{\partial\Omega} g(u+tv)\,\mathrm{d}\sigma,$$

and thus the first variation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u+tv) = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\partial\Omega} gv\,\mathrm{d}\sigma.$$

We can apply Corollary 1.7 to the first term on the right-hand side. It turns out that u is a minimiser if

$$\int_{\partial\Omega}(\nabla u\cdot\nu)v\mathrm{d}\sigma - \int_\Omega \Delta uv\,\mathrm{d}x + \int_{\partial\Omega} gv\,\mathrm{d}\sigma = 0.$$

Choosing $v\in C_c^\infty(\Omega)\subset\mathfrak{X}$, we can use Lemma 1.8 to infer that

$$\Delta u = 0,$$

while the Neumann boundary condition changes accordingly to the presence of g:

$$\nabla u\cdot\nu(x) = -g(x) \quad \text{for all } x\in\partial\Omega.$$

If we consider the functional

$$F(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2(x)\,\mathrm{d}x+\frac{1}{2}\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma,$$

then there are differences even at the level of first variation. Indeed, it turns out that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}F(u+tv)=\int_{\Omega}\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\partial\Omega}uv\,\mathrm{d}\sigma=$$

$$=-\int_{\Omega}\Delta uv\,\mathrm{d}x+\int_{\partial\Omega}(\nabla u\cdot\nu+u)v\,\mathrm{d}\sigma$$

The Euler-Lagrange equation corresponding to the problem is

$$\begin{cases}-\Delta u(x)=0 & \text{if } x\in\Omega,\\ \nabla u\cdot\nu(x)+u(x)=0 & \text{if } x\in\partial\Omega.\end{cases}$$

□

3. Inner variation and free boundary problem

It is often useful to consider minimisation problems for functionals whose natural domains are sets of functions which admit a finite number of discontinuities. These can either be fixed or be an unknown themselves and for this reason, the latter will be called *free-discontinuity problems*.

EXAMPLE 1.18. *Let $c\in[a,\,b]$ be a* **fixed point**. *Our goal is to find the first variation and the Euler-Lagrange equation of the functional*

$$F(u)=\int_a^b f(x,\,\dot u)\,\mathrm{d}x$$

when u belongs to the class

$$\mathfrak{X}_c:=\left\{u:[a,\,b]\to\mathbb{R}\ :\ u\in C^1([a,\,b])\ \textit{possibly discontinuous at } c\right\}.$$

Since c is fixed we can simply split $\int_a^b$ to avoid the discontinuity point c. It turns out that

$$F(u)=\int_a^c f(x,\,\dot u)\,\mathrm{d}x+\int_c^b f(x,\,\dot u)\,\mathrm{d}x=:F_1(u)+F_2(u),$$

and it is easy to verify that minimising F is equivalent to minimising F_1 and F_2 on $C^1([a,\,c])$ and $C^1([c,\,b])$ respectively. The Euler-Lagrange equation is

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}x}(f_\xi(x,\,\dot u))=0 & \textit{if } x\in[a,\,c)\cup(c,\,b],\\ f_\xi(x,\,\dot u)=0 & \textit{if } x=a,\ x=b \textit{ and } x=c^{\pm},\end{cases}\tag{1.10}$$

where $x=c^-$ and $x=c^+$ denote, respectively, the following limits:

$$\lim_{x\to c^-}f_\xi(x,\,\dot u)=0\quad\textit{and}\quad\lim_{x\to c^+}f_\xi(x,\,\dot u)=0.$$

In this example, we decided to keep the discontinuity point fixed, but it makes sense to wonder if moving it around $[a, b]$ might lead to a better result in terms of minimising the functional. Note that computing

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(u + tv)$$

is not enough anymore because it does not take into account the fact that c moves. For this reason, we now introduce a different type of variation which is usually referred to as *inner variation*.

Inner variation. Let us consider the functional

$$F(u) = \int_a^b f(x, u, \dot{u})\,\mathrm{d}x$$

and suppose that u belongs to a class of functions with enough regularity to justify the following computations. For $\eta \in C_c^1([a, b])$, define

$$\Phi_t(x) := x + t\eta(x) \quad \text{for } |t| < \delta.$$

The idea is to use Φ_t to obtain a new function through the composition. In particular, we introduce the function

$$u_t(y) := u \circ \Phi_t^{-1}(y)$$

and call it the *inner variation* of u with respect to η.

REMARK 1.19. We are no longer deriving along straight lines, but we are considering paths in the graph of the function u which means that our options are highly dependent on the shape of u.

The map $t \mapsto F(u_t)$ is well-defined for $|t| < \delta$ and maps an open interval to the real line. It follows that if u is a local minimum/maximum, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(u_t) = 0 \quad \text{for every } \eta \in C_c^1(a, b).$$

Before we compute the derivative explicitly, we would like to point out two important properties of Φ_t. Indeed, if we take $\delta > 0$ small enough[3], then it is easy to verify that

$$\Phi_t'(x) = 1 + t\eta'(x) > 0,$$

[3] For example, smaller than the uniform norm of η'.

which implies that Φ_t is increasing, C^1 with C^1 inverse and mapping $[a,\, b]$ into $[a,\, b]$. We are thus allowed to use the change of variables formula and find that

$$\begin{aligned} F(u_t) &= \int_a^b f(y,\, u_t(y),\, \dot u_t(y))\,\mathrm{d}y \\ &= \int_a^b f\left(y,\, u(\Phi_t^{-1}(y)),\, \dot u(\Phi_t^{-1}(y))\frac{1}{\Phi_t'(\Phi_t^{-1}(y))}\right)\,\mathrm{d}y \\ &= \int_a^b f\left(\Phi_t(x),\, u(x),\, \dot u(x)\frac{1}{\Phi_t'(x)}\right)\Phi_t'(x)\,\mathrm{d}x. \end{aligned}$$

Now derive the function with respect to t. A simple computation shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}F(u_t) = &\int_a^b f\left(\Phi_t(x),\, u(x),\, \dot u(x)\frac{1}{\Phi_t'(x)}\right)\frac{\partial}{\partial t}\Phi_t'(x)\,\mathrm{d}x + \cdots \\ &\cdots + \int_a^b \Bigg[f_x\left(\Phi_t(x),\, u(x),\, \dot u(x)\frac{1}{\Phi_t'(x)}\right)\frac{\partial}{\partial t}\Phi_t(x) + \cdots \\ &\cdots + \int_a^b f_\xi\left(\Phi_t(x),\, u(x),\, \dot u(x)\frac{1}{\Phi_t'(x)}\right)\dot u(x)\frac{-\frac{\partial}{\partial t}\Phi_t'(x)}{(\Phi_t'(x))^2}\Bigg]\Phi_t'(x)\,\mathrm{d}x. \end{aligned}$$

The expression for the derivative is rather complicated, but when we evaluate it at $t = 0$ everything simplifies immensely because the following identities hold:

$$\Phi_0'(x) = 1, \quad \frac{\partial}{\partial t}\Phi_0(x) = \eta(x), \quad \frac{\partial}{\partial t}\Phi_0'(x) = \dot\eta(x).$$

It turns out that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}F(u_t) &= \int_a^b f(x,\, u,\, \dot u)\dot\eta\,\mathrm{d}x + \int_a^b [f_x(x,\, u,\, \dot u)\eta - f_\xi(x,\, u,\, \dot u)\dot u\dot\eta]\,\mathrm{d}x = \\ &= \int_a^b [f(x,\, u,\, \dot u) - f_\xi(x,\, u,\, \dot u)\dot u]\,\dot\eta\,\mathrm{d}x + \int_a^b f_x(x,\, u,\, \dot u)\eta\,\mathrm{d}x = \\ &= \int_a^b \left[f_x - \frac{\mathrm{d}}{\mathrm{d}x}(f - f_\xi\dot u)\right]\eta\,\mathrm{d}x = 0. \end{aligned}$$

Since this is zero for all $v \in C_c^1([a,\, b])$, we can apply Lemma 1.8 to obtain the identity

$$\begin{aligned} f_x(x,\, u,\, \dot{u}) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x,\, u,\, \dot{u}) - f_\xi(x,\, u,\, \dot{u})\dot{u}\right) \\ &= f_x + f_u\dot{u} + f_\xi\ddot{u} - f_\xi\ddot{u} - f_{\xi x}\dot{u} - f_{\xi u}\dot{u}^2 - f_{\xi\xi}\ddot{u}\dot{u} = \\ &= f_x + f_u\dot{u} - f_{\xi x}\dot{u} - f_{\xi u}\dot{u}^2 - f_{\xi\xi}\dot{u}\ddot{u}, \end{aligned}$$

which ultimately leads to a slightly weaker type of Euler-Lagrange equation:

$$\left[f_u - \frac{\mathrm{d}}{\mathrm{d}x} f_\xi\right]\dot{u} = 0. \tag{1.11}$$

This result is slightly weaker than the one obtained via the straight lines variation because directional derivatives are strictly related to the differentiability of F, while inner variation may not give all possible directions.

EXAMPLE 1.20 (Free-boundary problem). *Now consider the problem solved in Exercise 1.18, but let the discontinuity point c move; namely, consider the class of functions*

$$\mathfrak{X} = \left\{u : [a,\, b] \to \mathbb{R} \,:\, u \in C^1([a,\, b]) \textit{ discontinuous at most at a single point}\right\}.$$

It is easy to compute the first variation with $u + tv \in \mathfrak{X}$, choosing v to be discontinuous at most at the point where u is. However, since we are not exploiting the fact that the discontinuity point can move*, we find once again the equation* (1.10) *although we expect that more freedom yields a better result.*

The idea is to use the inner variation discussed above. For $\eta \in C_c^1([a,\, b])$, define $\Phi_t(x) := x + t\eta(x)$ and denote by u_t the function

$$u_t(y) := u \circ \Phi_t^{-1}(y).$$

The computations carried out above are still valid until we get to

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(u_t) = \int_a^b \left[f(x,\, u,\, \dot{u}) - f_\xi(x,\, u,\, \dot{u})\dot{u}\right]\dot{\eta}\,\mathrm{d}x + \int_a^b f_x(x,\, u,\, \dot{u})\eta\,\mathrm{d}x.$$

However, at this point we cannot integrate by parts because of the discontinuity c. Splitting the integral in the usual way (to avoid c) yields to

$$\int_a^b \left(f_x - \frac{\mathrm{d}}{\mathrm{d}x}(f - f_\xi\dot{u})\right)\eta + \left[(f - f_\xi\dot{u})\eta\right]_{c^-}^{c^+} = 0$$

for all $\eta \in C_c^1([a,\, b])$. The integral is zero as a consequence of the Euler-Lagrange equation (1.10) *so the boundary term must also be zero. This means that the function*

$$f - f_\xi\dot{u}$$

is continuous at c (i.e., the limit for x to c^- and c^+ coincide). Notice that this is an extra condition since it does not follow from any other in (1.10).

The inner variation leads to the Euler-Lagrange equation whenever $\dot{u}(x) \neq 0$. Intuitively, this makes sense because we are moving along the graph of u and hence we do not expect to find any kind of information where u is **flat**. The next remark shows how limited inner variation can be in some situations.

REMARK 1.21. In the vector-valued case, namely $u : [a, b] \to \mathbb{R}^m$, the same computation shows that

$$f_x = \frac{\mathrm{d}}{\mathrm{d}x}\left(f - f_\xi \cdot \dot{u}\right),$$

where $\cdot$ denotes the scalar product. Differently from the variation along straight lines here we only get one equation (instead of m) because we are just moving in the one-dimensional set $u([a, b])$.

REMARK 1.22. If $m = 1$ and $f(u, \dot{u})$ does not depend on x directly, then (1.11) reduces to

$$f - f_\xi \dot{u} = \text{constant}.$$

This means that we get a first-order equation which holds even if u belongs to C^1 since we do not need to integrate by parts. Indeed, recall that

$$0 = \int_a^b (f(u, \dot{u}) - f_\xi(u, \dot{u})\dot{u})\dot{\eta}\,\mathrm{d}x$$

for all $\eta \in C^1_c([a, b])$; however, we cannot apply Lemma 1.8 because we have $\dot{\eta}$. The following result is a slightly different version and allows us to conclude that $f - f_\xi \dot{u}$ is constant even if u is C^1 only.

LEMMA 1.23 (Du Bois-Reymond). *If $g \in L^1_{\mathrm{loc}}((a, b))$ and*

$$\int_a^b g\dot{\varphi}\,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in C^\infty_c([a, b]),$$

then g is equal to a constant at a.e. $x \in (a, b)$.

REMARK 1.24. Observe that with the language of distributions this lemma merely asserts that a function whose distributional derivative is zero must be constant almost everywhere.

PROOF. Suppose that $(a, b) = \mathbb{R}$. Take an approximation identity ρ_ϵ and notice that $\varphi * \epsilon$ is still an admissible test function, that is,

$$\int_a^b g(\varphi * \rho_\epsilon)'\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C^\infty_c(a, b).$$

The regularity properties of the convolution implies that

$$\int_a^b (g * \rho_\epsilon)\varphi'\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C^\infty_c(a, b)$$

and, since $g * \rho_\epsilon$ belongs to C^1, we can integrate by parts and deduce that

$$g * \rho_\epsilon = c(\epsilon).$$

Finally the pointwise convergence a.e. of the convolution implies that g is constant almost everywhere and the constant does not depend on ϵ.

If $(a,\, b)$ is a proper subinterval of $\mathbb{R}$, one can fix a smaller interval $(a-\delta,\, b+\delta)$, $\delta > 0$, and use the fact that the convolution is defined there. The reader might fill in the details as an exercise. $\square$

EXERCISE 1.7. Let $\Omega \subseteq \mathbb{R}^n$ be a connected subset. Prove that if $g \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_a^b g \operatorname{div}(\varphi)\, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

then g is equal to a constant at a.e. $x \in \Omega$.

EXERCISE 1.8. Let $\mathfrak{X} := C^1_0(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, with the additional constraint that $\frac{1}{2}\|u\|_{L^2(\Omega)} = 1$. Consider the Dirichlet energy

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

Compute the Euler-Lagrange equation of this functional using the Lagrange multiplier method, namely

$$\langle \mathrm{d}F(u),\, v\rangle = \lambda \langle \mathrm{d}G(u),\, v\rangle,$$

and try to justify the use of this kind of theorem in such an abstract framework.

SOLUTION. If u is a local minimum/maximum for F, then the Lagrange multiplier theorem implies that there exists $\lambda \in \mathbb{R}$ such that

$$\langle \mathrm{d}F(u),\, v\rangle = \lambda \langle \mathrm{d}G(u),\, v\rangle \quad \text{for all } v \in C^1_0(\Omega).$$

The left-hand side gives $\int_\Omega (-\Delta u) v \, \mathrm{d}x$, while the right-hand side $\lambda \int_\Omega uv \, \mathrm{d}x$. Using Lemma 1.8 we can deduce the equation

$$\begin{cases} -\Delta u = \lambda u & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

which means that u is an eigenfunction of the Laplace operator. It can be proved that it corresponds to the *first eigenvalue* for any elliptic operator $\operatorname{div} L$, that is,

$$\lambda_1(\Omega,\, L) := \min_{u \in H^1_0(\Omega)} \frac{\sum_{i,\,j=1}^N a_{i,\,j}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u \, \mathrm{d}x + \int_\Omega a_0(x)|u|^2 \, \mathrm{d}x}{\int_\Omega |u|^2 \, \mathrm{d}x}.$$

$\square$

We will now provide a proof that shows why we can do all those computations (note that we require the functionals F and G to be actually of class C^1). Since

$$\langle \mathrm{d}G(u),\, v\rangle = \int_\Omega uv \, \mathrm{d}x \implies \mathrm{d}G(u) \neq 0$$

as a linear map, for otherwise it would be necessarily $u \equiv 0$ which is impossible because the L^2-norm of u is equal to one. We say that v is *tangent* to $\mathfrak{X}$ at u if

$$\int_\Omega uv \,\mathrm{d}x = 0,$$

which is to say that $v \perp u$ in $L^2(\Omega)$. We now claim that for all $v \perp u$ there exists a path $(-\delta,\, \delta) \ni t \mapsto u_t \in \mathfrak{X}$ for $\delta > 0$ sufficiently small such that $u_0 \equiv u$ and

$$\frac{\partial}{\partial_t}\Big|_{t=0} u_t(x) \equiv v(x).$$

The idea is to take $u_t := u + tv$ and normalize it in such a way that its L^2-norm becomes equal to one. One also needs to check that the derivative of $F(u_t)$ is equal to the derivative of F along the direction v or, in other words, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u_t) \stackrel{?}{=} \langle \mathrm{d}F(u),\, v\rangle.$$

If we give this for granted and take a local minimum/maximum u, then it is easy to check that

$$-\int_\Omega v\Delta u \,\mathrm{d}x = 0 \quad \text{for all } v \in\perp u \text{ in } L^2(\Omega),$$

and using the inclusion $\ker(F) \supset \ker(G)$, by a standard linear algebra result we deduce that these functionals are proportional. Namely, there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega (-\Delta u - \lambda u)v \,\mathrm{d}x = 0,$$

and this holds for **all** $v \in \mathfrak{X}$. We can now conclude as we did before using Lemma 1.8. We will come back to formalise these computations once we introduce Sobolev spaces which arethe right framework.

REMARK 1.25. Notice that $F(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$ is a *positive-definite* quadratic form. Using the divergence theorem, we easily deduce that

$$F(u) = \int_\Omega (-\Delta u)u \,\mathrm{d}x = \langle -\Delta u,\, u\rangle_{L^2(\Omega)}$$

is also a positive-definite quadratic form and, more precisely, the one associated to the Laplace operator $-\Delta$ with respect to the scalar product in $L^2(\Omega)$. Furthermore, it follows from the identity

$$\langle -\Delta u,\, u\rangle_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \langle u,\, -\Delta u\rangle_{L^2(\Omega)}$$

that $-\Delta$ is a self-adjoint operator.

4. Geodesics

Let $\mathcal{M}$ be a k-dimensional surface[4] of class C^2 embedded in $\mathbb{R}^n$.

DEFINITION 1.26. Let $x_0,\, x_1 \in \mathcal{M}$. A *geodesic* from x_0 to x_1 is a curve $\gamma : [0,\, 1] \to \mathcal{M}$ connecting these two points and such that it minimizes the length functional

$$L(\gamma) := \int_0^1 |\dot{\gamma}(s)|\, \mathrm{d}s.$$

Denote by $\mathfrak{X}$ the set of all curves connecting x_0 and x_1. The reader might notice that this is slightly different from the one usually introduced in *differential geometry* to indicate the length of geodesics; namely,

$$E(\gamma) := \left[\int_0^1 |\dot{\gamma}(s)|^2\, \mathrm{d}s\right]^{\frac{1}{2}}.$$

The reason is that we can use either Jensen's inequality or Hölder's inequality to infer that

$$L(u) \leq \left[\int_0^1 |\dot{\gamma}(s)|^2\, \mathrm{d}s\right]^{\frac{1}{2}},$$

with the equality that holds if and only if $|\dot{\gamma}|$ is constant. However[5], for all γ there exists a reparametrization $\tilde{\gamma} := \gamma \circ \sigma$ such that its speed is constant.

COROLLARY 1.27. *If γ_0 minimizes E in $\mathfrak{X}$, then it minimizes L. Moreover, the speed $|\dot{\gamma}_0|$ is constant.*

PROOF. Let $\gamma \in \mathfrak{X}$ be any curve joining x_0 and x_1, and let $\tilde{\gamma}$ be the reparametrization with constant speed. Then the inequalities above show that

$$L(\gamma) = L(\tilde{\gamma}) = E(\tilde{\gamma}) \geq E(\gamma_0) \geq L(\gamma_0),$$

so γ_0 minimizes the length L. Also $L(\gamma) \geq E(\gamma_0) \geq L(\gamma_0)$ at $\gamma = \gamma_0$ gives $E(\gamma_0) = L(\gamma_0)$, which is possible only when γ_0 has constant speed. $\square$

REMARK 1.28. Notice that L has a significant group of invariants such as reparametrizations, but this is not true for $E(\gamma)$ and it actually **selects** the parametrisation which gives constant speed. This is why it makes more sense to minimise E rather than L.

PROPOSITION 1.29. *If γ_0 minimizes E on $\mathfrak{X}$, then the Euler-Lagrange equation is*

$$\ddot{\gamma} \perp \mathcal{M} \text{ at every } s \in [0,\, 1]. \tag{1.12}$$

In other words, the curvature $\vec{\kappa}$ is orthogonal to $\mathcal{M}$ at all points $\gamma(s)$.

---

[4]We might develop the same theory in the general framework of Riemannian manifold with a few changes. The definition of geodesic and Proposition 1.29 need to be fixed, but otherwise everything else holds.

[5]Assuming that $\dot{\gamma}$ is different from zero at all points, for example.

PROOF. Start by considering a path $t \mapsto \gamma_t$ in $\mathfrak{X}$ that originates from γ and is defined in such a way that $\gamma_t(s) \in \mathcal{M}$ for all $t,\, s \in [0,\, 1]$. Then $s \mapsto \gamma_t(s)$ is a path in $\mathcal{M}$ and hence

$$\frac{\partial}{\partial t}\Big|_{t=0} \gamma_t(s) \in \mathrm{Tan}_{\gamma(s)}\mathcal{M}$$

for all s. We define the set

$$\mathrm{Tan}_\gamma \mathfrak{X} := \{v : [0,\, 1] \to \mathbb{R}^n \ :\ v(s) \in \mathrm{Tan}_{\gamma(s)}\mathcal{M},\, v(0) = v(1) = 0\}$$

as the space on which we will move the possible variations of γ. We claim that there exists $t \mapsto \gamma_t$ path in $\mathfrak{X}$ such that $\gamma_0 \equiv \gamma$ and, as before, satisfying

$$\frac{\partial}{\partial_t}\Big|_{t=0} \gamma_t(x) \equiv v(x).$$

We can choose $\gamma_t(s) = \gamma(s) + tv(s) + \omega(t,\, s)$, where the remainder ω belongs to $\mathcal{O}(t^2)$ uniformly with respect to s and its derivative

$$\frac{\partial \omega}{\partial s}(t,\, s) = \mathcal{O}(t^2),$$

once again uniformly with respect to s. Using the existence of a *tubular neighbourhood* we can define a projection from $\mathbb{R}^n$ to $\mathcal{M}$ (at least locally) in such a way that $\gamma_t(s)$ for $|t| < \delta$ and $s \in [0,\, 1]$ remains inside of $\mathcal{M}$. Finally, one simply needs to verify that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(\gamma_t) = \langle \mathrm{d}E(\gamma),\, v\rangle,$$

and then use the explicit formula for E to deduce that

$$\int_0^1 \ddot{\gamma} \cdot v \,\mathrm{d}x = 0$$

for all $v \in \mathrm{Tan}_\gamma \mathfrak{X}$. This implies $\ddot{\gamma}$ orthogonal to all v's in such a tangent space and hence using an appropriate variation of Lemma 1.8 we conclude. ☐

5. Minimal surfaces

PROPOSITION 1.30. *If Σ_0 is a d-dimensional surface of class C^2 in $\mathbb{R}^{d+1}$ which minimizes the area functional $A(\Sigma)$ among all Σ d-dimensional surfaces in $\mathbb{R}^{d+1}$ with prescribed boundary Γ, then*

$$H_{\Sigma_0}(x) \equiv 0, \tag{1.13}$$

where H_Σ is the mean curvature *of Σ.*

PROOF. The set of competitors in this case is given by all d-dimensional surfaces with the same prescribed boundary and to avoid technicalities we define

$$\mathfrak{X} := \{\Sigma \subset \mathbb{R}^d \ :\ \Sigma \text{ d-dim. surface s.t. } \partial\Sigma = \Gamma\}.$$

Let η_0 be a *unit normal* to Σ_0 and let $v : \Sigma_0 \to \mathbb{R}$ be a regular function which is zero[6] on a neighbourhood of the boundary $\partial\Sigma_0$. Define

$$\Phi_t(x) := x + tv(x) \cdot \eta_0(x)$$

and notice that for $|t| < \delta$, δ small enough, this is a diffeomorphism so that $\Sigma_t = \Phi_t(\Sigma_0) \cong \Sigma_0$ and $\Sigma_t \in \mathfrak{X}$ since points on the boundary are not moving. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A(\Sigma_t) = 0,$$

but we first need to give a meaning to the area functional and evaluate it at Σ_t.

(1) **Step 1.** The differential of $\Phi_t : \Sigma_0 \to \Sigma_t$ is a linear map between the respective tangent spaces

$$\mathrm{d}\Phi_t(x) : \mathrm{Tan}_x\Sigma_0 \to \mathrm{Tan}_x\Sigma_t.$$

We can choose two orthonormal bases on these tangent spaces and find a matrix that represent $\mathrm{d}\Phi_t(x)$; this is usually denoted by $\widetilde{\nabla}_\tau\Phi_t(x)$. The Jacobian is hence given by

$$J\Phi_t(x) := \left|\det(\widetilde{\nabla}_\tau\Phi_t(x))\right|,$$

so we can use the change of variables formula to infer that

$$A(\Sigma_t) = \int_{\Sigma_0} J\Phi_t(x)\,\mathrm{d}x.$$

However, the choice of the basis on Σ_t depends on t and hence the formula above is quite hard to deal with. Consider the differential

$$\mathrm{d}\Phi_t(x) : \mathrm{Tan}_x\Sigma_0 \to \mathbb{R}^{d+1}$$

and complete a basis of $\mathrm{Tan}_x\Sigma_0$ to a basis of $\mathbb{R}^{d+1}$ by adding $\eta_0(x)$. The basis on the codomain vector space is now independent of t and

$$\nabla_\tau^T\Phi_t(x)$$

is the $(d+1) \times d$ matrix associated to the differential by this choice. One can prove that

$$J\Phi_t(x) = \sqrt{\det(\nabla_\tau^T\Phi_t(x)\nabla_\tau\Phi_t(x))},$$

which only makes sense because $\nabla_\tau^T\Phi_t(x)\nabla_\tau\Phi_t(x)$ is a square matrix.

(2) **Step 2.** We can now compute the differential explicitly as

$$\mathrm{d}\Phi_t(x)[h] := h + t\left(\eta_0(x)\langle \mathrm{d}v(x),\, h\rangle + v(x)\langle \mathrm{d}\eta_0(x),\, h\rangle\right).$$

Then the matrix associated to this linear map is

$$\nabla_\tau\Phi_t(x) = \left(\begin{array}{c} \mathrm{Id}_{d\times d} + tv(x)\nabla_\tau\eta(x) \\ \hline t\nabla_\tau v(x) \end{array}\right)$$

[6] This is a technical assumption which is useful to define variations of Σ_0. Below it will be more clear why we need it.

and hence

$$\nabla_\tau^T \Phi_t(x) \nabla_\tau \Phi_t(x) = \mathrm{Id}_{d\times d} + tv(x)(\nabla\eta(x) + \nabla^T\eta(x)) + \mathcal{O}(t^2).$$

The determinant is thus given (using Taylor's expansion) by

$$\det(\nabla_\tau^T \Phi_t(x) \nabla_\tau \Phi_t(x)) = 1 + 2tv(x)\mathrm{Tr}(\nabla\eta(x)) + \mathcal{O}(t^2),$$

and $\mathrm{Tr}(\nabla\eta(x))$ is the mean curvature because $\nabla\eta(x)$ is the second fundamental form. Then

$$J\Phi_t(x) = \sqrt{\det(\nabla_\tau^T \Phi_t(x) \nabla_\tau \Phi_t(x))} \simeq 1 + tv(x)H(x) + \mathcal{O}(t^2),$$

exploiting the Taylor's expansion of the square root.

(3) **Step 3.** A simple computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A(\Sigma_t) = \int_{\Sigma_0} v(x)H(x)\,\mathrm{d}x = 0 \quad \text{for all } v \text{ as above},$$

and we can apply Lemma 1.8 since v can be chosen among the ones with compact support in Σ and we deduce that $H \equiv 0$.

□

CHAPTER 2

Existence Results via Direct Method

In this chapter, we discuss the existence of solutions to minimisation/maximisation problems via the so-called *direct method*, which consists in working in a weaker framework where properties such as compactness and lower semicontinuity are straightforward.

1. Introduction to the direct method

Let $F : \mathfrak{X} \to [-\infty, \infty]$ be a functional. The idea behind the direct method is to investigate the problem $\inf_{u\in\mathfrak{Y}} F(u)$, where $\mathfrak{Y}$ may be a space that includes $\mathfrak{X}$ endowed with a topology that makes F lower semicontinuous and sublevels of F compact.

DEFINITION 2.1. Let (X, d) be a metric space. A functional $F : X \to [-\infty, \infty]$ is *lower semicontinuous* at a point $x_0 \in X$ if

$$f(x_0) \le \liminf_{x\to x_0} f(x).$$

The definition of lower semicontinuity can be stated in topological spaces using the preimages of half-open sets (r, ∞), but in this course we will not need this degree of generality.

THEOREM 2.2. *If K is a compact metric space and $F : K \to [-\infty, \infty]$ is lower semicontinuous, then there exists $\bar{u} \in K$ minimum point for F.*

PROOF. Let u_n be a minimizing sequence and denote by m the infimum of F. By compactness, there exists a subsequence $n(k)$ such that

$$u_{n(k)} \xrightarrow{k\to\infty} \bar{u}$$

and since F is lower semicontinuous, we also have that

$$m = \liminf_{k\to\infty} F(u_{n(k)}) \ge F(\bar{u}) \implies F(\bar{u}) = m.$$

□

The assumption K *compact metric space* for the applications is usually not verified. Indeed, we can work in a general metric space and only require that F satisfies an additional property.

DEFINITION 2.3 (Coercivity). Let (X, d) be a metric space. A functional $F : X \to [-\infty, \infty]$ is *coercive* if for all sequences $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$F(u_n) \le c < \infty,$$

there is a converging subsequence.

THEOREM 2.4. *Let X be a metric space and let $F : X \to [-\infty, \infty]$ be a lower semicontinuous, coercive functional. Then there exists $\bar{u} \in X$ minimum point for F.*

PROOF. This is left as an exercise. A possible way is to observe that $F : X \to [-\infty, \infty]$ is lower semicontinuous and coercive if and only if all sublevels

$$\Lambda_M(F) := \{u \in X \;:\; F(u) \leq M\}$$

are compact with respect to the topology given by the metric structure of X. □

REMARK 2.5. Since we are interested in the minimization problem $\inf_{u \in X} F(u)$, it makes sense to weaken the coercivity assumption and merely ask $\Lambda_M(F)$ compact for some $C > \inf_{u \in X} F(u)$.

REMARK 2.6. In $X = \mathbb{R}^n$, the definition of coercive functional is simpler. Indeed, one can prove that

$$\lim_{|x| \to \infty} F(x) = \infty$$

is **equivalent** to F coercive since bounded sets are relatively compact in $\mathbb{R}^n$. Therefore, the only sequences with no converging subsequences are the unbounded ones.

REMARK 2.7. If X is an open subset of $\mathbb{R}^n$ and $F : X \to [-\infty, \infty]$ a functional, then F is coercive if and only if $F(x_n) \to \infty$ when $|x_n| \to \infty$ or x_n converges to the boundary of Ω. Note that

$$\lim_{|x| \to \infty} F(x) = \infty$$

is enough if we replace Ω with its one-point compactifiation (the boundary is identified with ∞).

We conclude this section by describing an example of an infinite-dimensional metric space X in which we can actually apply the stronger result Theorem 2.2.

THEOREM 2.8. *Let (K, d_K) be a compact metric space, $x_0, x_1 \in K$ and assume that there exists a curve of finite length $\gamma : [0, 1] \to K$ such that*

$$\gamma(0) = x - 0 \quad \textit{and} \quad \gamma(1) = x_1.$$

Then there exists $\bar{\gamma}$ geodesic connecting x_0 and x_1.

DEFINITION 2.9. Let (X, d) be a metric space. We say that $f : X \to \mathbb{R}$ is L-Lipschitz if

$$d(f(x), f(y)) \leq L|x - y| \quad \text{for all } x, y \in X.$$

The sharpest constant in the inequality is called Lipschitz constant of f and it is usually denoted by $\mathrm{Lip}(f)$.

PROOF. Let $X := \{\gamma : [0, 1] \to K \; : \; \gamma$ is L-Lipschitz and $\gamma(0) = x_0, \gamma(1) = x_1\}$ endowed with the distance defined by setting

$$d_X(\gamma, \gamma') := \sup_{0<t<1} d_K(\gamma(t), \gamma'(t)).$$

The space X is compact (by Ascoli-Arzelà) and the functional F that associated γ to its length is lower semicontinuous because it is given by the supremum of continuous functionals; namely,

$$F(\gamma) = \sup_{0\le t_1<\cdots<t_n\le 1} \left\{ \sum_{i=2}^{n} d_K(\gamma(t_i), \gamma(t_{i-1})) \right\}.$$

Then Theorem 2.2 shows that there exists a minimiser $\bar{\gamma} \in X$, but a priori we do not know that it is the optimal path also among the ones that are not L-Lipschitz. The following result concludes:

LEMMA 2.10. *If $\gamma : [0, 1] \to X$ is continuous and has length $L(\gamma) = \ell < \infty$, then there exists $\sigma : [0, 1] \to [0, 1]$ reparametrization such that $\tilde{\gamma} := \gamma \circ \sigma$ satisfies:*

(i) $d_K(\tilde{\gamma}(t), \tilde{\gamma}(t') \le \ell |t - t'|$;

(ii) $\tilde{\gamma}$ *is ℓ-Lipschitz.*

□

2. Compactness in Banach spaces

The notion of compactness in Banach spaces is not as simple as in Euclidean spaces, but there is a theorem that achieves it under mild assumptions.

DEFINITION 2.11. Let X be a topological vector space, and let $S \subset X$. The *polar* set associated to S is the set defined by

$$S^\circ := \{f \in X^* \mid |f(x)| \le 1 \text{ for all } x \in S\}$$

THEOREM 2.12 (Banach-Alaoglu). *Let X be a topological vector space, and let V be a neighborhood of the origin. The polar V° is convex, weakly-$*$ closed and weakly-$*$ compact.*

COROLLARY 2.13. *Let $\mathfrak{X}$ be a Banach space. The closed unit ball $\overline{B_{\mathfrak{X}^*}}$ in $\mathfrak{X}^*$ is weakly-$*$ compact.*

The version of the Banach-Alaoglu we are interested in requires $\mathfrak{X}$ to be a separable Banach space. The reason is that it yields a metric structure on the space.

THEOREM 2.14 (Banach-Alaoglu). *Let $\mathfrak{X}$ be a separable Banach space. Then the closed ball $\overline{B_{\mathfrak{X}^*}(0, 1)} \subset \mathfrak{X}^*$ is weakly-$*$ metrizable and it is sequentially weakly-$*$ compact.*

REMARK 2.15. If $\mathfrak{X}$ is a reflexive separable Banach space and $F : \mathfrak{X} \to [-\infty, \infty]$ a functional, then F is coercive with respect to the weak topology of $\mathfrak{X}$ if and only if

$$\lim_{\|x_n\|_{\mathfrak{X}} \to \infty} F(x_n) = \infty.$$

DEFINITION 2.16. Let $1 \leq p \leq \infty$. A function f belongs to $W^{1,p}(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n)$ and its distributional derivative $Df \in L^p(\mathbb{R}^n)$.

LEMMA 2.17. *The space $W^{1,p}(\mathbb{R}^n)$ is reflexive for all $1 < p < \infty$ and separable for all $1 \leq p < \infty$.*

THEOREM 2.18. *Let Ω be a regular*[1] *open set in $\mathbb{R}^n$, $1 < p < \infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ be a bounded sequence. Then there exists a subsequence $n(k)$ such that*

$$\begin{cases} \|u_{n(k)} - u\|_{L^p(\Omega)} \to 0, \\ \nabla u_{n(k)} \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega). \end{cases}$$

PROOF. The sequence u_n is uniformly bounded in the $W^{1,p}$-norm so there exists a positive constant c such that

$$\|u_n\|_{L^p(\Omega)} + \|\nabla u_n\|_{L^p(\Omega)} \leq c < \infty.$$

It follows from Theorem 2.14 that (up to subsequences) u_n and ∇u_n converge weakly in $L^p(\Omega)$ to u and v respectively. It is easy to verify that in a distributional sense we have

$$v = \nabla u,$$

and therefore $u \in W^{1,p}(\Omega)$. Finally, we can apply the *Sobolev (compact) embedding theorem* that holds for the inclusion

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

to infer that u_n converges **strongly** to u in $L^p(\Omega)$. ☐

REMARK 2.19. If $F : W^{1,p}(\Omega) \to [-\infty, \infty]$, then F coercive with respect to the *weak topology* is equivalent to $F(x_n) \to \infty$ whenever $\|x_n\|_{W^{1,p}(\Omega)} \to \infty$. In other words, the following are equivalent:

(a) For all sequences u_n such that $F(u_n) \leq c < \infty$ there exists a subsequence such that $u_{n(k)} \to u$ strongly in $L^p(\Omega)$ and $\nabla u_{n(k)} \to \nabla u$ weakly in $L^p(\Omega)$.

(b) Whenever $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$ there holds

$$F(u_n) \to \infty.$$

PROPOSITION 2.20. *Let $\mathfrak{X} := W^{1,2}(\Omega)$, where Ω is a regular open subset of $\mathbb{R}^n$. Let*

$$F(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + g(x, u)\right] \mathrm{d}x$$

and assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

[1] As an exercise, the reader might try to find the minimal assumptions on Ω for which the result holds. Observe that the regularity is only required for the compact embedding.

(a) *It is Borel-measurable in both variables.*

(b) *The map $u \mapsto g(x,\, u)$ is lower semicontinuous at almost every $x \in \Omega$.*

(c) *It has a quadratic growth, that is,*

$$g(x,\, u) \geq c|u|^2 \quad \textit{for all } u \in \mathbb{R} \textit{ and almost every } x \in \Omega.$$

Then F is (weakly) lower semicontinuous and coercive on $\mathfrak{X}$. In particular, F admits a minimum point $\bar{u} \in \mathfrak{X}$.

PROOF. The coercivity of F is an easy consequence of $(\mathbf{c})$ since

$$F(u_n) \geq \frac{1}{2}\|\nabla u_n\|^2_{L^2(\Omega)} + c\|u_n\|^2_{L^2(\Omega)} \geq c'\|u_n\|^2_{W^{1,\,2}(\Omega)}$$

so $F(u_n)$ goes to ∞ whenever $\|u_n\|_{W^{1,\,2}(\Omega)} \to \infty$. For the lower semicontinuity simply split F as the sum of $F_1 + F_2$ and observe that the Dirichlet energy

$$F_1(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

is lower semicontinuous since this is always the case for a norm on a Banach space (with respect to the weak topology). Now let $u_n \rightharpoonup u$ and use *Fatou's lemma* to deduce that

$$\liminf_{n\to\infty} F_2(u_n) = \liminf_{n\to\infty}\int_\Omega g(x,\, u_n)\,\mathrm{d}x \geq \int_\Omega \liminf_{n\to\infty} g(x,\, u_n)\,\mathrm{d}x.$$

Since u_n converges strongly to u in $L^2(\Omega)$, up to a subsequence we know that u_n converges pointwise to u almost everywhere and hence by $(\mathbf{b})$ we infer that

$$\int_\Omega \liminf_{n\to\infty} g(x,\, u_n)\,\mathrm{d}x \geq \int_\Omega g(x,\, u)\,\mathrm{d}x,$$

and this concludes the proof. □

REMARK 2.21. The almost everywhere pointwise convergence forces us to pass through a subsequence. However, in the last step the $\liminf$ is with respect to n. This is justified because we can use the usual sub-subsequence trick.

EXERCISE 2.1. The space $\mathfrak{X}$ is usually unknown for a given functional. Why in Proposition 2.20 we work in the Sobolev space $W^{1,\,2}(\Omega)$? What happens if we try to do the same in $W^{1,p}(\Omega)$ for some $p \neq 2$?

HINT. The functional is coercive on $W^{1,p}(\Omega)$ when $1 \leq p \leq 2$, but it is not when $p > 2$. The semicontinuity, on the other hand, is harder in $W^{1,p}(\Omega)$ because we have the $W^{1,2}(\Omega)$-norm and requires the characterization of closed convex sets that follows. □

LEMMA 2.2. *Let $\mathfrak{X}$ be a Banach space.*

(a) If $C \subset \mathfrak{X}$ is a convex set, then C is closed in norm if and only if C is weakly closed.

(b) If $C \subset \mathfrak{X}^*$ is a convex set, then C is closed in norm if it is weakly-$*$ closed.

PROOF. One implication is clear in both the assertions: if C is closed in the weak (or weak-$*$) topology, then it is bounded in norm.

(a) Assume that C is closed in the strong topology: we want to prove that the complement $A := C^c$ is weakly open. Let $x_0 \notin C$ and apply Hahn-Banach theorem to find a linear continuous functional $f \in X^*$ and a real number $\alpha \in \mathbb{R}$ such that

$$\langle f,\, x_0 \rangle < \alpha < \langle f,\, x \rangle \quad \text{for all } x \in C.$$

It follows that $V := \{x \in X \mid \langle f,\, x \rangle < \alpha\}$ is an open neighbourhood of x_0, strictly contained in A, and this is enough to infer that A is weakly open.

(b) It follows easily from the definitions, but it is interesting to see a counterexample for the opposite implication. For example, a non-reflexive Banach space X is closed in norm (as a subset of the bi-dual X^{**}), but it is weakly-$*$ dense and thus it cannot be weakly-$*$ closed.

□

EXERCISE 2.3. Prove that, if we replace the assumption (**c**) in Proposition 2.20 with

$$g(x,\, u) \geq \omega(|u|), \text{ where } \omega(t) \to \infty \text{ as } t \to \infty,$$

then the same conclusion holds.

Before we can solve Exercise 2.3, we need a characterization result for the weak (weak-$*$ in $W^{1,\infty}(\Omega)$) convergence in Sobolev spaces in terms of uniform boundedness.

PROPOSITION 2.22. *Let $1 \leq p \leq \infty$, Ω open subset of $\mathbb{R}^n$ and u_n a sequence in $W^{1,p}(\Omega)$. Then the following assertions are equivalent:*

(i) u_n converges weakly to u in $W^{1,p}(\Omega)$ (resp. weakly-$\star$ in $W^{1,\infty}(\Omega)$).

(ii) u_n and ∇u_n converge weakly to u and ∇u respectively in $L^p(\Omega)$ (weakly-$*$ if $p = \infty$).

(iii) If $p > 1$, then u_n converges weakly to u in $L^p(\Omega)$ and ∇u_n is uniformly bounded.

(iv) If $p > 1$ and Ω is regular (i.e., with Lipschitz boundary), then u_n converges strongly to u in $L^p(\Omega)$ and ∇u_n is uniformly bounded.

REMARK 2.23. In (iii) we need $p > 1$ because the weak limit of a sequence in $W^{1,1}(\Omega)$ can be a L^1-function with weak differential ∇u in the class of measures (more precisely, $u \in \mathrm{BV}(\Omega)$).

REMARK 2.24. In (iv) we need, in addition, a regularity assumption on Ω because the equivalence requires a setting in which the Sobolev embedding theorem works.

PROOF. The first equivalence follows from the fact that the embedding $W^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ is an isomorphism of Banach spaces. □

PROPOSITION 2.25. *If $1 < p < \infty$, a weakly closed subset E of $W^{1,p}(\Omega)$ is weakly-compact if and only if it is bounded.*

HINT. The Banach space $W^{1,p}(\Omega)$ is both reflexive and separable for $1 < p < \infty$. □

COROLLARY 2.26. *The topology is metrizable on E. In particular, a lower semicontinuous coercive functional is a sequentially lower semicontinuous coercive functionals and vice versa.*

REMARK 2.27. If $p = \infty$, the results above are true if we replace the weak topology with the weak-$*$ topology.

LEMMA 2.28. *Let X be a weakly closed subset of $W^{1,p}(\Omega)$ for $1 < p < \infty$. A functional $F : X \to \mathbb{R} \cup \{\pm\infty\}$ is coercive if and only if*

$$\|u_n\|_{W^{1,p}(\Omega)} \to \infty \implies F(u_n) \to \infty.$$

SOLUTION OF EXERCISE 2.3. To prove that F is coercive, let $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ be a sequence such that $\|u_n\|_{W^{1,2}(\Omega)} \to \infty$. Thanks to the previous remark, we only need to prove that

$$\lim_{n\to\infty} F(u_n) = \infty.$$

Notice that $\|u_n\|_{W^{1,2}(\Omega)} \to \infty$ is equivalent to saying that at least one between $\|u_n\|_{L^2(\Omega)} \to \infty$ and $\|\nabla u_n\|_{L^2(\Omega)}$ goes to infinity. If

$$\|\nabla u_n\|_{L^2(\Omega)} \to \infty,$$

then it is easy to verify that $F(u_n) \geq \frac{1}{2}\|\nabla u_n\|_{L^2(\Omega)} \to \infty$, so we can assume without loss of generality that

$$\|u_n\|_{L^2(\Omega)} \to \infty \quad \text{and} \quad \|\nabla u_n\|_{L^2(\Omega)} \leq C < \infty.$$

By Poincaré's inequality (see Theorem 2.35) we have

$$\int_\Omega |u_n - (u_n)_\Omega| \,\mathrm{d}x \lesssim \|\nabla u_n\|^2_{L^2(\Omega)} \leq c',$$

where

$$v_\omega := \fint_\omega v(x)\,\mathrm{d}x$$

is the *mean value of* v *on* $\omega \subset \mathbb{R}^n$. We now claim that for each $M > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$|\{|u_n| \geq M\}| \geq \frac{|\Omega|}{2} \quad \text{for } n \geq n_0.$$

This would be enough to conclude since we have the estimate

$$\int_\Omega f(x,\, u_n)\,\mathrm{d}x \geq \frac{|\Omega|}{2}\omega(M) \xrightarrow{M\to\infty} \infty,$$

which gives that F is coercive. To prove the claim, consider the set

$$A^c = \{x \in \Omega \,:\, |u_n(x)| \leq M\}.$$

We remarked above that by Poincaré's inequality we have

$$\begin{aligned}
(|(u_n)_\Omega| - M)\,|A^c| &= |(u_n)_\Omega||A^c| - M|A^c| \\
&\leq \int_{A^c} (-|u_n| + |(u_n)_\Omega|)\,\mathrm{d}x \\
&\leq \int_{A^c} |u_n - (u_n)_\Omega|\mathrm{d}x \leq C,
\end{aligned}$$

which ultimately means that $|A^c| \to 0$ since $(|(u_n)_\Omega| - M) \to \infty$. □

EXERCISE 2.4. Let $\mathfrak{X} = W^{1,2}(\Omega)$ and consider the functional

$$F(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,\mathrm{d}x + \int_\Omega \rho u\,\mathrm{d}x,$$

where $\rho : \Omega \to \mathbb{R}$ is a function with zero mean, that is,

$$\int_\Omega \rho\,\mathrm{d}x = 0.$$

Discuss the existence of a solution $\bar{u}$ and show that, if we assume $\int_\Omega \rho\,\mathrm{d} \neq 0$, then there is no minimizer and $\inf_{u\in\mathfrak{X}} F(u) = -\infty$.

SOLUTION. If $\rho_\Omega \neq 0$, then it is easy to see that $\inf_{u\in\mathfrak{X}} F(u) = -\infty$ since

$$u(x) \equiv t \implies F(u) = t\rho_\Omega,$$

which goes to minus infinity when $t \to \infty$ (or $t \to -\infty$, depending on the sign of ρ_Ω). On the other hand, if $\rho_\Omega = 0$, then

$$F(u) = F(u + t) \quad \text{for all } t \in \mathbb{R}$$

so it is not restrictive to assume that $u_\Omega = 0$. Let

$$\mathfrak{X}' := \{u \in \mathfrak{X} \;:\; u_\Omega = 0\},$$

and notice that it is closed (and thus weakly closed) in $\mathfrak{X}$. The functional F is coercive on $\mathfrak{X}'$ because the Sobolev norm,

$$\|\nabla u\|_{L^2(\Omega)} + |u_\Omega|,$$

is equivalent (see Section 2.2) to the usual one, namely $\|u\|_{W^{1,2}(\Omega)}$. Furthermore, on $\mathfrak{X}'$ the L^2-norm of the gradient is all that is left since $u_\Omega = 0$ and hence

$$F(u) \geq \frac{1}{2}\|u\|^2_{\mathfrak{X}'} - \|\rho\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - C\|\rho\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)},$$

and this goes to infinity as soon as $\|\nabla u\|_{L^2(\Omega)} \to \infty$. □

2.1. Trace operator. We now would like to consider a minimization problem for a function F defined on a Sobolev space with the additional constraint that $u\big|_{\partial\Omega} \equiv u_0$. We begin with a fundamental lemma.

LEMMA 2.29. *Let $\Omega = \mathbb{R}^n_+$ and $1 \leq p < \infty$. Then there exists a constant C such that*

$$\left[\int_{\partial\Omega} |u(x',\,0)|^p \,\mathrm{d}x'\right]^{\frac{1}{p}} \leq C\|u\|_{W^{1,\,p}(\Omega)} \quad \textit{for all } u \in C^1_c(\Omega).$$

An immediate consequence of this lemma is that the map $u \mapsto u\big|_\Gamma$ with $\Gamma = \partial\Omega \cong \mathbb{R}^{n-1} \times \{0\}$ defined from $C^1_c(\Omega)$ into $L^p(\Gamma)$ extends, by density, to a bounded linear operator $W^{1,p}(\Omega) \to L^p(\Omega)$. This operator is, by definition, the *trace of* u *on* Γ and corresponds to the restriction to Γ when u is regular.

There is a fundamental difference between $W^{1,p}(\Omega)$ and $L^p(\Omega)$: the functions in the latter do not have a trace on Ω. Anyway, it is easy to extend the argument above to Ω regular open set of $\mathbb{R}^n$ (for example, of class C^1 with Γ bounded) via local charts.

REMARK 2.30. The Sobolev space $W^{1,\,p}_0(\Omega)$ is the kernel of the trace operator, that is,

$$W^{1,\,p}_0(\Omega) = \left\{u \in W^{1,\,p}(\Omega) \,:\, u\big|_{\partial\Omega} = 0\right\}.$$

EXAMPLE 2.31. *Let $\mathfrak{X} := W^{1,2}_{u_0}(\Omega) = \{u \in W^{1,2} \;:\; u\big|_{\partial\Omega} \equiv u_0\}$ and consider the functional*

$$F(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x.$$

It is easy to verify that $W^{1,\,2}_{u_0}(\Omega)$ is a closed affine subspace of $W^{1,\,2}(\Omega)$, and hence weakly closed. The functional F is lower semicontinuous as usual and coercive on $W^{1,\,2}_{u_0}(\Omega)$ since

$$\Phi(u) := \|u\|_{L^2(\partial\Omega)} + \|\nabla u\|_{L^2(\Omega)}$$

is an equivalent norm on $W^{1,\,2}(\Omega)$. Since u must coincide with u_0 at the boundary, the term $\|u\|_{L^2(\partial\Omega)}$ is fixed and finite so

$$\|u_n\|_{W^{1,\,2}(\Omega)} \to \infty \implies \|\nabla u\|_{L^2(\Omega)} \to \infty,$$

which means that F is coercive.

2.2. Equivalent norms on Sobolev spaces. We now want to prove that the norms we introduced earlier are actually equivalent to the Sobole norm. We prove a much more general result from which the Poincarè inequality will follow as well.

PROPOSITION 2.32. *Let Ω be a connected subset of $\mathbb{R}^n$, k, p positive integers and $\mathbb{E}$ a Banach space. Assume that the operator*

$$T : W^{k,\,p}(\Omega) \to \mathbb{E},$$

is linear, bounded and satisfies $\ker(T) \cap \mathbb{P}_{<k}[t] = \{0\}$. Then

$$\Phi(u) = \|Tu\|_{\mathbb{E}} + \|\nabla^k u\|_{L^p(\Omega)}$$

is an equivalent norm to $\|\cdot\|_{W^{k,\,p}(\Omega)}$, where ∇^k is the kth derivative.

PROOF. It is easy to verify that Φ is a seminorm. We only need to check that

$$\Phi(u) = 0 \implies u = 0,$$

but this is an immediate consequence of the kernel assumption. Indeed, we know that $\Phi(u) = 0$ implies that both $T(u) = 0$ and $\nabla^k u = 0$. A standard result asserts that the latter (on a connected set) implies that u is a polynomial of degree $k-1$ a.e. so

$$u \in \ker(T) \cap \mathbb{P}_{<k}[t] = \{0\} \implies u = 0.$$

The operator T is bounded, and hence one inequality is for free:

$$\Phi(u) \leq C\|u\|_{W^{k,p}(\Omega)}.$$

For the opposite one, we argue by contradiction. Let u_n be a sequence in $W^{1,p}(\Omega)$ such that

$$\Phi(u_n) \leq \frac{1}{n} \quad \text{and} \quad \|u_n\|_{W^{k,p}(\Omega)} = 1.$$

Since $\Phi(u_n) \to 0$, we also have $Tu_n \to 0$ in $\mathbb{E}$ and $\nabla^k u_n \to 0$ in $L^p(\Omega)$. Then u_n converges (up to subsequences) weakly to some u and

$$\nabla^k u = 0 \quad \text{almost everywhere in } \Omega \implies u \text{ is a polynomial of degree at most } k-1.$$

Furthermore, we have the convergence

$$Tu_n \to Tu$$

by continuity of T, which immediately tells us that $Tu = 0$. Hence u must be zero since no polynomial of degree $k-1$ belongs to $\ker(T)$. By Sobolev embedding u_n converges strongly (with all its derivatives) to zero and this leads to a contradiction since

$$1 = \liminf_{n\to\infty} \|u_n\|_{W^{k,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)} = 0.$$

□

REMARK 2.33. Notice that we can remove the assumption Ω connected replacing $\ker(T) \cap \mathbb{P}_{<k}[t] = \{0\}$ with $\ker(T) \cap \ker(\nabla^k) = \{0\}$ but we must require that $\ker(\nabla^k)$ is a finite-dimensional vector space.

COROLLARY 2.34.

(i) If Ω is regular (i.e., C^1 and bounded), then the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ is equivalent to

$$\|Tu\|_{L^p(\partial\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

(ii) If Ω is connected (otherwise the kernels might intersect), the Sobolev norm is equivalent to

$$|u_\Omega| + \|\nabla u\|_{L^p(\Omega)}$$

for $1 \leq p < \infty$.

(iii) If Ω has finitely many connected components (e.g., it is regular), then

$$\|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

is equivalent for all $q \leq p^*$, the critical Sobolev exponent.

COROLLARY 2.35 (Poincarè inequality). *Let Ω be connected. If $\Lambda : W^{k,p}(\Omega) \to \mathbb{E}$ is a bounded operator, $\mathbb{E}$ Banach, and $\ker(\Lambda)$ contains all polynomials in P_{k-1}, then there exists C such that*

$$\|\Lambda(u)\|_{\mathbb{E}} \leq C\|\nabla^k u\|_{L^p(\Omega)}. \tag{2.1}$$

PROOF. Assume for simplicity $k = 1$. Let $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ be a bounded linear projection on constant functions, for example

$$T(u) = \fint_A u \,\mathrm{d}x$$

for some $A \subset \Omega$ with finite measure. Then

$$\Phi(u) := \|Tu\|_{W^{1,p}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

is an equivalent norm by Proposition 2.32, which means that

$$\|\Lambda u\|_{\mathbb{E}} \leq C\Phi(u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

But $\Lambda(u) = \Lambda(u - Tu)$ because of the assumption on $\ker(\Lambda)$. It follows that

$$\|\Lambda u\|_{\mathbb{E}} \leq C\Phi(u - Tu) = \underbrace{\|T(u - Tu)\|_{W^{1,p}(\Omega)}}_{=0} + C\|\nabla u\|_{L^p(\Omega)},$$

and this concludes the proof. □

REMARK 2.36. The same proof works for $k > 1$, but finding a projection T on polynomials of degree at most $k - 1$ is not as trivial. The reader might try to find it out as an exercise.

COROLLARY 2.37. *Let Ω be a bounded regular connected set and let $1 \leq q \leq p^*$. Then*

$$\|u - u_A\|_{L^q(A)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where u_A is the average of u over a set of positive and finite measure A.

PROOF. The operator $\Lambda : W^{1,p}(\Omega) \to L^q(A)$ defined by setting

$$\Lambda(u) := u - u_A$$

is linear and bounded. We can apply Corollary 2.35 to conclude. □

EXERCISE 2.5. The constant C in Corollary 2.37 depends on p, q, Ω and A. If we let $A = \Omega$, how does C behaves with respect to scaling of Ω?

3. General theory of existence for integral functionals on Sobolev spaces

In this section, our goal is to discuss the existence under general assumptions of critical points for functionals of the form

$$F(u) = \int_\Omega f(x, u, \nabla u) \,\mathrm{d}x, \tag{2.2}$$

where[2] $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \to [0, \infty]$ is a Borel (measurable would not be enough) function. In the sequel, we will always assume that $W^{1,p}(\Omega)$ is endowed with the **weak topology** for $1 < p < \infty$ because we characterized coercivity already in a simple way.

3.1. Coercivity. We prove a first criterion for coercivity when f does not depend on u. In this case, coercivity can only be achieved in subspaces such as $W_0^{1,p}(\Omega)$ because

$$F(u) = F(u + c)$$

so we can take a sequence of constants $c_n \to \infty$ and $u \in W^{1,p}(\Omega)$ to conclude that $u_n := u + c_n$ converges to infinite in norm, but $F(u_n) = F(u)$ does not.

PROPOSITION 2.38. *Let Ω be connected. If $F(u) = \int_\Omega f(x, \nabla u)\,\mathrm{d}x$, then F is coercive in $W_0^{1,p}(\Omega)$ if there exists $\delta > 0$ such that the following p-growth condition holds:*

$$f(x, \xi) \geq \delta|\xi|^p$$

PROOF. By Corollary 2.34, on $W_0^{1,p}(\Omega)$ the norm $\|\nabla u\|_{L^p(\Omega)}$ is equivalent to the Sobolev norm (since u at the boundary is equal to zero). Therefore

$$\|u_n\|_{W^{1,p}(\Omega)} \to \infty \implies \|\nabla u\|_{L^p(\Omega)} \to \infty,$$

and hence using the growth condition we conclude that

$$F(u_n) \geq \delta\|\nabla u_n\|_{L^p(\Omega)} \xrightarrow{n\to\infty} \infty.$$

□

EXERCISE 2.6. Is it possible that δ in the p-growth condition can depend on x without an uniform lower bound?

We now prove a criterion in the general case, that is, for f that depends on $(x, u, \nabla u)$.

PROPOSITION 2.39. *Let Ω be connected. If $F(u) = \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x$, then F is coercive in $W^{1,p}(\Omega)$ if there exists $\delta > 0$ such that the following p-growth condition holds:*

$$f(x, u, \xi) \geq \delta(|u|^p + |\xi|^p).$$

The proof of this result is similar to the previous one, but it is actually possible to require much less. For example, we can require that

$$f(x, u, \xi) \geq \delta|\xi|^p + \omega(|u|)\chi_A(x), \tag{2.3}$$

where $0 < |A| < \infty$ and $\omega(t) \to \infty$ as $t \to \infty$.

PROOF. By Corollary 2.34, on $W^{1,p}(\Omega)$ the norm $Phi(u) := \|\nabla u\|_{L^p(\Omega)} + |\int_A u\,\mathrm{d}x|$ is equivalent to the Sobolev norm since Ω is connected. Let u_n be a sequence such that

$$\Phi(u_n) \to \infty.$$

[2]In general f may take values in $(-\infty, \infty]$, but in that case one needs additional assumptions on f because (2.2) may not be defined.

If $\|\nabla u\|_{L^p(\Omega)} \to \infty$, then coercivity is trivial using (3.1). So we can assume that $\|\nabla u_n\|_{L^p(\Omega)}$ is uniformly bounded by a constant C and $|\int_A u\,\mathrm{d}x| \to \infty$. Then by Poincaré's inequality

$$\int_A |u_n - (u_n)_A|\,\mathrm{d}x \le c\|\nabla u_n\|_{L^p(\Omega)} \le C',$$

and if we take $M > 0$ arbitrarily big, the set $A_n := \{x \in A \,:\, |u_n|(x) \le M\}$ has measure that goes to zero using the same argument as in Exercise 2.3. Then

$$F(u_n) \ge \int_{A\setminus A_n} f(x,\, u_n,\, \nabla u_n)\,\mathrm{d}x \ge \omega(M)|A \setminus A_n| \to \omega(M)|A|$$

using the second part of (3.1), and the conclusion follows by taking the limit as M goes to infinity. □

EXERCISE 2.7. If we assume that f takes values in $(-\infty,\, \infty]$, what additional conditions are sufficient for a similar result to hold? Note that we not only need to prove coercivity, but also the well-definition of F.

3.2. Weak lower semicontinuity. We now prove a criterion for the weak lower semicontinuity of $F : W^{1,p}(\Omega) \to (-\infty,\, \infty]$ in terms of strong lower semicontinuity and convexity.

THEOREM 2.40. *If F is lower semicontinuous and convex, then F is weakly lower semicontinuous provided that $1 \le p < \infty$.*

This result follows from the following more general statement since being lower semicontinuous is equivalent to the closeness of sublevels $C_t := \{u \in W^{1,p}(\Omega) \,:\, F(u) \le t\}$.

LEMMA 2.41. *If $\mathbb{E}$ is a Banach space and $C \subset \mathbb{E}$ is convex and closed, then C is weakly closed.*

REMARK 2.42. The problem with $p = \infty$ is that the weak topology is not metrizable on compact sets and hence we cannot use the equivalence between compactness and sequential compactness.

COROLLARY 2.43. *If $F(u) = \int_\Omega f(x,\, \nabla u)\,\mathrm{d}x$, f Borel, and $(x,\, \cdot)$ is lower semicontinuous and convex with respect to ξ for a.e. $x \in \Omega$, then F is lower semicontinuous and convex on $W^{1,p}(\Omega)$.*

PROOF. Lower semicontinuity is a consequence of Fatou's lemma (as we did in the proof of Proposition 2.20) while the convexity is trivial. □

We conclude this section with a result that generalises the previous one to integral functionals whose Lagrangian depends on $(x,\, u,\, \nabla u)$.

THEOREM 2.44. *Let $F(u) = \int_\Omega f(x,\, u,\, \nabla u)\,\mathrm{d}x$ and suppose that the Lagrangian*

$$f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m^2} \to [0,\, \infty]$$

is positive[3] *and Borel*[4]*. Then the following assertions hold:*

(i) If for a.e. $x \in \Omega$ *the function* $(s, \xi) \mapsto f(x, s, \xi)$ *is lower semicontinuous, then* F *is strongly lower semicontinuous on* $W^{1,p}(\Omega)$*.*

(ii) If for a.e. $x \in \Omega$ *the function* $(s, \xi) \mapsto f(x, s, \xi)$ *is convex and lower semicontinuous, then* F *is convex and weakly lower semicontinuous on* $W^{1,p}(\Omega)$*.*

PROOF. We first prove (i). Let u_n be a sequence converges strongly in $W^{1,p}(\Omega)$ to some u and recall that this is equivalent to

$$u_n \to u \quad \text{and} \quad \nabla u_n \to \nabla u,$$

strongly in $L^p(\Omega)$. Up to subsequences, we can assume that u_n converges almost everywhere to u and similarly ∇u_n to ∇u. By lower semicontinuity of f we have

$$f(x, u(x), \nabla u(x)) \leq \liminf_{n\to\infty} f(x, u_n(x), \nabla u_n(x)),$$

and this concludes the proof since

$$\begin{aligned} F(u) &\leq \int_\Omega \liminf_{n\to\infty} f(x, u_n, \nabla u_n)\,\mathrm{d}x \\ &= \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)\,\mathrm{d}x = \liminf_{n\to\infty} F(u_n). \end{aligned}$$

The assertion (ii) now follows easily since in Banach spaces it is always true that strongly lower semicontinuous plus convex is equivalent to weakly lower semicontinuous. For this, simply apply Proposition 2.2 with a nonempty sublevel of F in place of C. □

REMARK 2.45. A Lebesgue-measurable function coincide with a Borel one up to a set of zero measure, so we can always assume that functions in $L^p(\Omega)$ are Borel (choosing the right representative).

4. Is convexity always a necessary condition?

In the general setting of Banach spaces, convexity is not always a necessary condition for a functional to be **weakly** lower semicontinuous. In this section, we will discuss a few particular instances in which convexity is needed and see how the scalar-valued and the vector-valued cases are different from each other.

THEOREM 2.46. *Let* $F(u) = \int_\Omega f(\nabla u)\,\mathrm{d}x$*, where* u *is a* **scalar** *function, and assume that* $f : \mathbb{R}^n \to [0, \infty]$ *is Borel. Then the following assertions hold:*

(i) If F *is strongly lower semicontinuous on* $W^{1,p}(\Omega)$*, then* f *is lower semicontinuous.*

[3] As we mentioned above, this assumption can be replaced with something else that makes the integral of f well-defined.

[4] This assumption is necessary because the composition of Lebesgue-measurable functions is not necessarily Lebesgue-measurable.

(ii) If F is weakly lower semicontinuous on $W^{1,p}(\Omega)$, then f is convex.

PROOF. For simplicity, we prove (i) under the additional assumption "Ω bounded", but with little effort it can be removed. Take $\xi_n \to \xi$ in $\mathbb{R}^n$ and consider

$$u_n(x) := \xi_n x \quad \text{and} \quad u(x) := \xi x.$$

Then $u_n \to u$ in $C^1(\Omega)$ and strongly in $W^{1,p}(\Omega)$ for all $p \geq 1$. By Fatou's lemma we have

$$|\Omega| \liminf_{n\to\infty} f(\xi_n) = \liminf_{n\to\infty} F(u_n) \geq F(u) = |\Omega| f(\xi),$$

and this shows f that lower semicontinuous with respect to ξ. To prove (ii) let $\xi_0,\ \xi_1 \in \mathbb{R}^n$, $\lambda \in [0,\ 1]$ and consider the convex combination

$$\xi = \lambda\xi_0 + (1-\lambda)\xi_1.$$

The goal is to construct a sequence u_ϵ that converges weakly in $W^{1,p}(\Omega)$ to some u such that $\nabla u = \xi$ everywhere and, for all $\epsilon > 0$,

$$\nabla u_\epsilon(x) = \begin{cases} \xi_0 & \text{on a set } A_\epsilon \text{ with } |A_\epsilon| \to \lambda|\Omega|, \\ \xi_1 & \text{on the set } \Omega \setminus A_\epsilon \text{ with } |\Omega \setminus A_\epsilon| \to (1-\lambda)|\Omega|. \end{cases}$$

Suppose we have a sequence satisfying these properties. Then

$$F(u_\epsilon) = f(\xi_0)|A_\epsilon| + f(\xi_1)|\Omega \setminus A_\epsilon|,$$

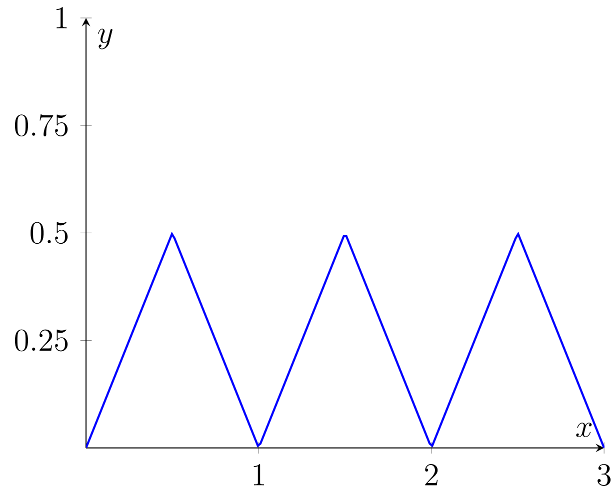
and by Fatou's lemma we infer the thesis:

$$|\Omega|\left[f(\xi_0)\lambda + f(\xi_1)(1-\lambda)\right] = \lim_{\epsilon\to 0^+} F(u_\epsilon) \geq F(u) = |\Omega| f(\xi).$$

To construct the sequence, let $u(x) = \xi x$ and define $v : \mathbb{R} \to \mathbb{R}$ to be a 1-periodic function with slope $\dot{v}$ that satisfies

$$\dot{v}(x) = \begin{cases} \lambda & \text{if } x \in \mathbb{Z} + [0,\ 1-\lambda], \\ \lambda - 1 & \text{if } x \in \mathbb{Z} + [1-\lambda,\ 1], \end{cases}$$

as in the following picture:



Set $u_\epsilon(x) := \epsilon v\left(\frac{(\xi_1-\xi_0)x}{\epsilon}\right) + \xi x$. Then u_ϵ converges uniformly to u as $\epsilon \to 0$ and hence in $L^p(\Omega)$ for all $1 \leq p \leq \infty$. Furthermore, the gradient is given by

$$\nabla u_\epsilon(x) = \xi + \dot{v}\left(\frac{(\xi_1-\xi_0)x}{\epsilon}\right)(\xi_1-\xi_0) = \begin{cases} \xi_1 & \text{if } x \in \Omega \setminus A_\epsilon, \\ \xi_0 & \text{if } x \in A_\epsilon, \end{cases}$$

where A_ϵ is the union of stripes of tickness $\lambda\epsilon$ and $(1-\lambda)\epsilon$ orthogonal to $\xi_1 - \xi_0$. Then ∇u_ϵ is uniformly bounded in $L^\infty(\Omega)$ and hence in every $L^p(\Omega)$, which means that

$$u_\epsilon \rightharpoonup u \quad \text{in } W^{1,p}(\Omega)$$

weakly for all $1 \leq p < \infty$ and weakly-$*$ for $p = \infty$. Since[5] $|A_\epsilon| \to \lambda|\Omega|$ as $\epsilon \to 0^+$ and similarly $|\Omega \setminus A_\epsilon|$ converges to $(1-\lambda)|\Omega|$, this concludes the proof. $\square$

The same result can be proved when $W^{1,p}(\Omega)$ is replaced with $W^{1,p}_{u_0}(\Omega)$, the Sobolev space with prescribed value at the boundary. The construction of u_ϵ, in this case, requires extra care as we will see soon.

THEOREM 2.47. *Let $F(u) = \int_\Omega f(\nabla u)\,\mathrm{d}x$, where u is a* **scalar** *function, and assume that $f : \mathbb{R}^n \to [0, \infty]$ is Borel. Then the following assertions hold:*

(i) If F is strongly lower semicontinuous on $W^{1,p}_{u_0}(\Omega)$, then f is lower semicontinuous.

(ii) If F is weakly lower semicontinuous on $W^{1,p}_{u_0}(\Omega)$, then f is convex.

[5]This implication is left as an exercise for the reader. A possible approach would be to take $\varphi : \mathbb{R} \to \mathbb{R}$ which is 1-periodic and equal to zero for $0 \leq t < 1-\lambda$ and 1 for $1-\lambda \leq t \leq 1$. Then $\varphi(\epsilon^{-1}t)$ converges weakly (or weakly-$*$) to λ.

PROOF. We only prove (ii) when $u_0 = 0$. Let $\xi_0,\ \xi_1 \in \mathbb{R}^n$, $\lambda \in [0,\ 1]$ and consider the convex combination

$$\xi = \lambda\xi_0 + (1-\lambda)\xi_1.$$

Our goal is to construct $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$, $1 \le p < \infty$, such that $\nabla u(x) = \xi$ on some $\Omega' \subset \Omega$ compactly contained and ∇u_ϵ as in the previous theorem but inside Ω', and

$$\nabla u_\epsilon = \nabla u \quad \text{in } \Omega \setminus (\Omega' \cup B_\epsilon)$$

with $|\nabla u_\epsilon| \le C$ on B_ϵ, a set whose volume satisfies $|B_\epsilon| \to 0$. Suppose for the time being that we are able to construct such a sequence. Then

$$F(u_\epsilon) = \int_{\Omega\setminus(\Omega'\cup B_\epsilon)} f(\nabla u)\,\mathrm{d}x + \int_{B_\epsilon} f(\nabla u_\epsilon)\,\mathrm{d}x + f(\xi_0)|A_\epsilon| + f(\xi_1)|\Omega' \setminus A_\epsilon|,$$

and taking the limit as $\epsilon \to 0^+$ the right-hand side converges to

$$\lim_{\epsilon\to 0^+} F(u_\epsilon) = \int_{\Omega\setminus\Omega'} f(\nabla u)\,\mathrm{d}x + {\color{red}0} + |\Omega'|\left[\lambda f(\xi_0) + (1-\lambda)f(\xi_2)\right].$$

Notice that the red part is not true, unless we put an additional assumption on f, but we come back to this later. Assuming that the convergence above holds, we have

$$\int_{\Omega\setminus\Omega'} f(\nabla u)\,\mathrm{d}x + |\Omega'|\left[\lambda f(\xi_0) + (1-\lambda)f(\xi_2)\right] = \liminf_{\epsilon\to 0} F(u_\epsilon)$$

$$\ge F(u) = \int_{\Omega\setminus\Omega'} f(\nabla u)\,\mathrm{d}x + |\Omega| f(\xi),$$

and this immediately leads (the additional term vanishes) to

$$\lambda f(\xi_0) + (1-\lambda)f(\xi_2) \ge f(\xi) \implies f \text{ is convex.}$$

Notice that $\int_{\Omega\setminus\Omega'} f(\nabla u)\,\mathrm{d}x$ can be removed if the value of the integral is finite. A possible assumption that makes this true together with red is

$$f(\xi) \le \omega(|\xi|),$$

where ω is increasing and maps $[0,\ \infty)$ to $[0,\ \infty)$. To construct u_ϵ, let Ω' be a ball with closure strictly contained in Ω and take $u \in C_c^\infty(\Omega)$ in such a way that

$$u(x) = \xi x \quad \text{in } \Omega'$$

using a smoothing argument. Then

$$u_\epsilon(x) = u(x) + \epsilon v\left(\frac{(\xi_1 - \xi_0)}{\epsilon}\right)\sigma_\epsilon(x),$$

where $\sigma_\epsilon(x)$ is a cutoff function equal to 1 on Ω' and 0 on $\Omega \setminus (\Omega')_{r_\epsilon}$, where $(\Omega')_{r_\epsilon}$ denotes a r_ϵ open neighborhood of Ω'. Then it makes sense to choose

$$B_\epsilon = (\Omega')_{r_\epsilon} \setminus \Omega',$$

and all it is left is to balance $r(\epsilon)$ properly, i.e. in such a way that the gradient of u_ϵ is uniformly bounded as required on the set B_ϵ. A simple computation shows that

$$\nabla u_\epsilon = \nabla u + \sigma_\epsilon \nabla v_\epsilon + \nabla \sigma_\epsilon v_\epsilon,$$

where

$$v_\epsilon(x) = \epsilon v \left(\frac{(\xi_1 - \xi_0)}{\epsilon} \right).$$

The first term is uniformly bounded and, similarly, the second term is uniformly bounded because ∇v_ϵ only takes two values. It follows that

$$\|\nabla u_\epsilon\|_\infty \leq \|\nabla u\|_\infty + \|\nabla v_\epsilon\|_\infty + \|\nabla \sigma_\epsilon\|_\infty \|v_\epsilon\|_\infty \leq C_1 + \|\nabla \sigma_\epsilon\|_\infty \|v_\epsilon\|_\infty.$$

By definition, $\|v_\epsilon\|_\infty$ has order ϵ while $\|\nabla \sigma_\epsilon\|_\infty$ goes to infinity as r_ϵ. Thus, to equilibrate these two quantities, we simply choose r_ϵ in such a way that

$$\epsilon r_\epsilon \xrightarrow{\epsilon \to 0^+} 0.$$

For example, one can take $r(\epsilon) = \sqrt{\epsilon}$. The construction works as the reader can verify and this concludes the proof. □

4.1. General theory. We now give two statements relating lower semicontinuity of the functional and convexity of the Lagrangian under general assumptions, completing the scalar and the one-dimensional cases.

THEOREM 2.48. *Let* $F(u) = \int_\Omega f(x, u, \nabla u)\, \mathrm{d}x$ *with* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to [0, \infty]$ *Borel and assume that the following holds:*

(i) $f(x, \cdot, \cdot)$ *is lower semicontinuous for almost every* $x \in \Omega$*;*

(ii) $f(x, u, \cdot)$ *is convex for*[6] *almost every* $x \in \Omega$ *and every* u*.*

Then F *is weakly lower semicontinuous on* $W^{1,p}(\Omega)$ *for any* p*.*

This theorem follows immediately from a much more general result that holds even if the gradient ∇u is replaced by an independent function $v \in L^p(\Omega)$.

THEOREM 2.49. *Let* $F(u, v) = \int_\Omega f(x, u, v)\, \mathrm{d}x$ *with* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^M \to [0, \infty]$ *Borel and assume that the following holds:*

(i) $f(x, \cdot, \cdot)$ *is lower semicontinuous for almost every* $x \in \Omega$*;*

(ii) $f(x, u, \cdot)$ *is convex for almost every* $x \in \Omega$ *and every* u*.*

Then F *is lower semicontinuous on* $L^p_{\mathrm{s}}(\Omega, \mathbb{R}^m) \times L^p_{\mathrm{w}}(\Omega, \mathbb{R}^M)$*, where* s *and* w *denote, respectively, strong and weak topology.*

We prove this theorem under the additional assumptions that $f(x, \cdot, \xi)$ is continuous uniformly with respect to (x, u, ξ) and $|\Omega|$ is finite.

[6] Notice that you cannot interchange the quantifiers in this assertion as the meaning would be entirely different.

PROOF. Take $u_n \to u$ strongly in $L^p(\Omega)$ and $v_n \rightharpoonup v$ weakly in $L^p(\Omega)$ and suppose[7] that u_n converges to u uniformly. Fix η and take δ such that

$$\forall (x,\, u,\, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^M \ :\ |u - u'| \le \delta \implies |f(x,\, u,\, \xi) - f(x,\, u',\, \xi)| \le \eta.$$

Then there exists $\bar{n}$ such that for all $n \ge \bar{n}$ it turns out that $|u_n(x) - u(x)| \le \delta$ for all $x \in \Omega$, and hence by uniform continuity we have

$$|f(x,\, u_n(x),\, \xi) - f(x,\, u(x),\, \xi)| \le \eta.$$

Integrate both sides with respect to x to obtain the estimate

$$|F(u_n,\, v_n) - F(u,\, v_n)| \le \eta|\Omega|,$$

which holds for all $n \ge \bar{n}$. However, we know that $v \mapsto F(u,\, v)$ is weakly lower semicontinuous on $L^p(\Omega,\, \mathbb{R}^M)$ by assumption using Theorem 2.44, and hence

$$\liminf_{n\to\infty} F(u_n,\, v_n) \ge \liminf_{n\to\infty} F(u,\, v_n) - \eta|\Omega| \ge \liminf_{n\to\infty} F(u,\, v) - \eta|\Omega|.$$

Since $|\Omega|$ has finite measure, we conclude the proof using the fact that $\eta > 0$ can be chosen arbitrarily. $\square$

We now briefly explain how to remove the additional assumptions, both via some kind of approximation argument. First, to remove the uniform continuity one notices that:

(1) If $F(u) = \sup_{n\in\mathbb{N}} F_n(u)$ for every u and F_n is lower semicontinuous, then F is also lower semicontinuous.

(2) If $g : X \to [0,\, \infty]$ is lower semicontinuous, where X is a metric space, then

$$g_\epsilon(x) := \inf_{y\in X} \{g(y) + \frac{1}{\epsilon} d(x,\, y)\}$$

is the pointwise infimum of $\frac{1}{\epsilon}$-Lipschitz functions so it is also $\frac{1}{\epsilon}$-Lipschitz. Moreover, g_ϵ is monotonically convergent to g and given an explicit formula for the approximation, so it is not hard to introduce more variables.

For what it concerns the measure of Ω, the idea is to take a sequence $\Omega'_n \subset \Omega$ of sets with finite measure and use them to approximate in the "right" way Ω.

THEOREM 2.50. *Let* $F(u) = \int_\Omega f(x,\, u,\, \nabla u)\, \mathrm{d}x$ *with* u *either scalar* ($m = 1$) *or one-dimensional*[8] *and assume that*

(i) f *is continuous with respect to all variables;*

(ii) F *is weakly lower semicontinuous on* $W^{1,p}_{u_0}(\Omega)$ *for some* p *and some* u_0.

Then $f(x,\, u,\, \cdot)$ *is convex for every* x *and every* u.

[7] Here you can, for example, use the usual subsequences argument together with Egoroff's theorem.

[8] In other words, u is defined on $[a,\, b]$ and hence $n = 1$.

PROOF OF THE SCALAR CASE. Fix $\bar{x} \in \Omega$, $\bar{u} \in \Omega$ and consider the convex combination $\bar{\xi} := \lambda\xi_0 + (1-\lambda)\xi_1$. The thesis is equivalent to proving that

$$f(\bar{x},\, \bar{u},\, \bar{\xi}) \leq \lambda f(\bar{x},\, \bar{u},\, \xi_1) + (1-\lambda) f(\bar{x},\, \bar{u},\, \xi_1).$$

Fix $\eta > 0$. We claim that there exists Ω' neighborhood of $\bar{x}$ and $\delta > 0$ such that

$$|f(x,\, u,\, \xi) - f(\bar{x},\, \bar{u},\, \xi)| \leq \eta \quad \text{for all } x \in \Omega',\, |u-\bar{u}| \leq \delta \text{ and } \xi \in B(\bar{\xi},\, R)$$

with R big enough for $\xi_0,\, \xi_1$ to belong to the ball. As in the proof of Theorem 2.46, we can choose $u(x) = \bar{u} + \bar{\xi}x$ for all $x \in \Omega'$ and

$$v_\epsilon(x) := \epsilon\varphi\left(\frac{x \cdot (\xi_1 - \xi_0)}{\epsilon}\right).$$

Let σ_ϵ be a cutoff function such that $\sigma_\epsilon \equiv 1$ on Ω' and $\sigma_\epsilon(x) = 0$ if $d(x,\, \Omega') \geq \sqrt{\epsilon}$. The approximating sequence is, once again, given by

$$u_\epsilon(x) = u + \sigma_\epsilon v_\epsilon,$$

and therefore

$$\begin{aligned} F(u) &= \int_{\Omega\setminus\Omega'} f(x,\, u,\, \nabla u)\,\mathrm{d}x + \int_{\Omega'} f(x,\, u,\, \bar{\xi})\,\mathrm{d}x \\ &\geq \int_{\Omega\setminus\Omega'} f(x,\, u,\, \nabla u)\,\mathrm{d}x + f(\bar{x},\, \bar{u},\, \bar{\xi})|\Omega'| - \eta|\Omega'|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F(u_\epsilon) &= \int_{\Omega\setminus(\Omega'\cup B_\epsilon)} f(x,\, u,\, \nabla u)\,\mathrm{d}x + \int_{B_\epsilon} f(x,\, u,\, \bar{\xi})\,\mathrm{d}x \\ &\quad + \int_{A_\epsilon} f(x,\, u,\, \xi_0)\,\mathrm{d}x + \int_{\Omega'\setminus A_\epsilon} f(x,\, u,\, \xi_1)\,\mathrm{d}x \\ &\leq \int_{\Omega\setminus(\Omega'\cup B_\epsilon)} f(x,\, u,\, \nabla u)\,\mathrm{d}x + \int_{B_\epsilon} f(x,\, u,\, \bar{\xi})\,\mathrm{d}x \\ &\quad + f(\bar{x},\, \bar{u},\, \xi_0)|A_\epsilon| + f(\bar{x},\, \bar{u},\, \xi_1)|\Omega' \setminus A_\epsilon| + \eta|\Omega'|. \end{aligned}$$

Taking the limit as $\epsilon \to 0^+$, we obtain the estimate

$$\begin{aligned} f(\bar{x},\, \bar{u},\, \bar{\eta}|\Omega'| - \eta|\Omega'| &\leq \lim_{\epsilon\to 0^+} F(u_\epsilon) \\ &\leq \lambda f(\bar{x},\, \bar{u},\, \xi_0)|\Omega'| + (1-\lambda) f(\bar{x},\, \bar{u},\, \xi_1)|\Omega'| + \eta|\Omega'|, \end{aligned}$$

and this concludes the proof because η can be chosen arbitrarily small. $\square$

EXERCISE 2.8. Prove the existence of minimizers, for $1 < p < \infty$, of

$$F(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega g(x,\, u) \,\mathrm{d}x,$$

where $u \in W^{1,p}_{u_0}(\Omega)$ and $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is lower semicontinuous in u for almost every $x \in \Omega$ under the following sets of assumptions:

(i) If $g(x,\, u)$ is nonnegative.

(ii) If $g(x,\, u) \geq -c_0$ for some positive constant $c_0 \in \mathbb{R}$.

(iii) If $g(x,\, u) \geq -c_0 - c_1|u|^q$ for $1 < q < p$. In this case, prove non-existence for

$$F(u) = \int_\Omega |\nabla u|^p - \lambda |u|^q \,\mathrm{d}x$$

on $W^{1,p}_0(\Omega)$ if $q > p$ and $\lambda > 0$. Show also that in the case $p = q$ the existence/non-existence depends on the value of the parameter λ.

EXERCISE 2.9. Study the existence of minimizer of

$$F(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega g(x,\, u) \,\mathrm{d}x,$$

where $u \in W^{1,p}(\Omega)$ and $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is lower semicontinuous in u for almost every $x \in \Omega$ under the following sets of assumptions:

(i) If $g(x,\, u)$ is nonnegative, show that existence is not guaranteed. Use, for example, $g = \mathrm{e}^u$.

(ii) If $g(x,\, u) \geq \omega(|u|)$, where $\omega(t) \to \infty$ as $t \to \infty$, show that we can recover existence.

5. Young measures

In this section, we show that Theorem 2.48 (which we restate below for the reader's convenience) can also be proved via a general tool, known as *Young measure*, which will also be helpful later on in the course.

THEOREM 2.51. *Let $F(u) = \int_\Omega f(x, u, \nabla u) \,\mathrm{d}x$ with $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0,\, \infty]$ Borel and assume that the following holds:*

(i) $f(x, \cdot, \cdot)$ is lower semicontinuous for almost every $x \in \Omega$;

(ii) $f(x, u, \cdot)$ is convex for almost every $x \in \Omega$ and for every $u \in \mathbb{R}^m$.

Then F is weakly lower semicontinuous on $W^{1,p}(\Omega)$ for any p.

REMARK 2.52. The standard proof of this result follows from the following two reductions to simpler cases:

(a) The function $f(x, u, \xi)$ is continuous in all variables and convex with respect to ξ.

(b) The function f depends only on ξ and is convex.

To follow this strategy, it is first useful to prove the following Lusin-type result which works, for example, for functions of two variables.

LEMMA 2.53. *Let $g : A \times B \to \mathbb{R}$ be Borel and continuous with respect to the second variable b. Assume that B is separable. Then for all $\epsilon > 0$ there exists $\tilde{g} : A \times B \to \mathbb{R}$ continuous with respect to both variables such that*

$$g(a, b) = \tilde{g}(a, b)$$

for all $a \notin E$ and all $b \in B$, where $E \subset A$ is a set of measure $|E| < \epsilon$.

HINT. To prove this lemma it is useful to consider g as a function of one variable, namely

$$A \ni x \longmapsto g(x, \cdot) \in C(B, \mathbb{R}),$$

and apply a Lusin-type theorem. □

5.1. Young measures. Let $u_n : \Omega \to \mathbb{R}$ be a sequence of functions converging in measure (pointwise a.e.) to some u. Then, for all $g : \mathbb{R} \to \mathbb{R}$ continuous, we have

$$g \circ u_n \to g \circ u.$$

But, if u_n converges weakly (in some L^p-space) to u, one might wonder what happens to the limit of the composition. It is well-known that if g is affine, then

$$g \circ u_n \rightharpoonup g \circ u,$$

but, if g is general, this is not true anymore as the next example shows.

EXAMPLE 2.54. *Let $g(s) = s^2$ and let u_n be a sequence of oscillating functions taking values in $\{\pm 1\}$ that converges weakly to zero. Then*

$$g \circ u_n(x) = 1,$$

which means that it converges to the constant function 1, but $g \circ u$ is the function identically equal to zero and hence the convergence fails.

5.2. Setting. Let Ω be an open set in $\mathbb{R}^n$, K a compact metric[9] space and let $u_n : \Omega \to K$ be a sequence of functions. Let $\mathcal{P}(K)$ be the space of probability measures on K.

DEFINITION 2.55. A function $\mu : \Omega \to \mathcal{P}(K)$, $x \mapsto \mu_x$, is *weak$\star$-Borel* if it is Borel with respect to the weak$\star$ topology on $\mathcal{P}(K)$. In other words, the mapping

$$x \longmapsto \int_K g(y)\, \mathrm{d}\mu_x(y)$$

is Borel for all $g \in C(K)$.

The following result, known as the *fundamental theorem for Young measures*, shows that each sequence (actually, a subsequence) of maps as above generates a weak$\star$-Borel function with specific properties.

[9]Separable would be enough here, but the metric structure simplifies the notations.

THEOREM 2.56. *Let $u_n : \Omega \to K$ be a sequence of maps. Then there are a subsequence u_{n_k} and a weak$\star$-Borel map $\mu : \Omega \to \mathcal{P}(K)$ such that the following hold:*

(i) For every $g : \Omega \times K \to \mathbb{R}$ such that $g(x, \cdot)$ continuous at a.e. $x \in \Omega$ and $\int_\Omega \sup_{y \in K} |g(x, y)| \,\mathrm{d}x < \infty$, we have

$$\int_\Omega g(x, u_{n_k}(x)) \,\mathrm{d}x \to \int_\Omega \left(\int_K g(x, y) \,\mathrm{d}\mu_x(y) \right) \mathrm{d}x.$$

(ii) For every continuous map $g : K \to \mathbb{R}$

$$g(u_{n_k}(x)) \overset{\star}{\rightharpoonup} \int g(y) \,\mathrm{d}\mu_x(y) \quad \text{in } L^\infty(\Omega, \mathbb{R}).$$

(iii) For all $g : \Omega \times K \to \mathbb{R}$ Borel and bounded such that $g(x, \cdot)$ is continuous for a.e. $x \in \Omega$. Then it turns out that

$$g(x, u_{n_k}(x)) \overset{\star}{\rightharpoonup} \int_K g(x, y) \,\mathrm{d}\mu_x(y) \quad \text{in } L^\infty(\Omega, \mathbb{R}).$$

(iv) If $K \subset \mathbb{R}^m$, then $u_{n_k} \overset{\star}{\rightharpoonup} u_\infty$ in $L^\infty(\Omega, \mathbb{R}^m)$, where

$$u_\infty(x) = \int_K y \,\mathrm{d}\mu_x(y).$$

(v) The measure μ_x equals $\delta_{u_\infty(x)}$ for a.e. $x \in \Omega$ if and only if u_{n_k} converges in measure to u_∞.

The map $x \mapsto \mu_x$ is called the **Young measure** *generated by the family of functions u_n (by the subsequence, to be more precise).*

EXAMPLE 2.57. *Let $K = [-1, 1]$, $\Omega = \mathbb{R}$ and $u_n(x) = f(nx)$ where f is the 1-periodic function defined by setting*

$$f(x) = \begin{cases} y_1 & \text{if } 0 \le x < \lambda, \\ y_2 & \text{if } \lambda \le x \le 1. \end{cases}$$

In this case, $\mu_x = \lambda\delta_{y_1} + (1 - \lambda)\delta_{y_2}$ is the map given by the Theorem 2.56. Note that

$$g(u_n(x)) \to \lambda g(y_1) + (1 - \lambda) g(y_2) \quad \text{for a.e. } x \in \Omega$$

follows immediately from (iii), but it can be proved by hands even without relying on such a powerful theorem.

Before giving the proof of Theorem 2.56, we recall a few basic facts from functional analysis regarding dual spaces.

THEOREM 2.58. *If E separable Banach space, then the dual of $L^1(\Omega, E)$ is $L^\infty_w(\Omega, E^\star)$.*

REMARK 2.59. Note that the subscript w in $L^\infty_w(\Omega, E^\star)$ indicates that we consider Borel functions with respect to the weak$\star$ topology on $E^\star$.

REMARK 2.60. The separability of E allows us to avoid measurability issues on $L^1(\Omega, E)$ because the σ-algebras generated by strong and weak topology coincide. We need to specify the topology on $L^\infty_w(\Omega, E^\star)$ because

$$E \text{ separable } \not\Longrightarrow E^\star \text{ separable.}$$

PROOF OF THEOREM 2.56. For each $n \in \mathbb{N}$, consider the map defined by setting

$$\mu^n : \Omega \ni x \mapsto \delta_{u_n(x)} \in \mathcal{P}(K).$$

Then $(\mu^n)_{n\in\mathbb{N}}$ is a sequence of maps from Ω to the probability measure space on K, which is a subset of $\mathcal{M}(K)$, the space of signed measures on K. A well-known duality theorem asserts that

$$\mathcal{M}(K) = (C(K))^\star$$

which means that $(\mu^n)_{n\in\mathbb{N}}$ can also be considered as a sequence of elements that belong to $L^\infty_w(\Omega, (C(K))^\star)$. This space is the dual (by Theorem 2.58) of

$$L^1(\Omega, C(K))$$

and, since $\|\mu^n\|_\infty = 1$ by definition, we can apply Banach-Alaoglu to find a measure μ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\mu^{n_k} \rightharpoonup \mu \quad \text{in } L^\infty_w(\Omega, \mathcal{M}(K)). \tag{2.4}$$

Now let $G \in L^1(\Omega, C(K))$. We can see G as a function $g : \Omega \times K \to \mathbb{R}$ satisfying

$$\int \sup_{y\in K} |g(x, y)| \,\mathrm{d}x < \infty.$$

The convergence (2.4) can also be rewritten as

$$\int_\Omega \langle \mu^{n_k}(x),\, G(x)\rangle \,\mathrm{d}x \to \int_\Omega \langle \mu(x),\, G(x)\rangle \,\mathrm{d}x,$$

for all $G \in L^1(\Omega, C(K))$, which is equivalent to requiring that

$$\int_\Omega g(x, u_{n_k}(x)) \,\mathrm{d}x \to \int_\Omega \left(\int_K g(x, y) \,\mathrm{d}\mu_x(y) \right) \mathrm{d}x, \tag{2.5}$$

where g is the function associated to each G. This proves (i). To verify that μ_x belong to $\mathcal{P}(K)$ for all $x \in \Omega$, plug the function

$$g(x, y) := \alpha(x) \in L^1(\Omega)$$

into (2.5). It turns out that

$$\int_\Omega \alpha(x) \,\mathrm{d}x = \int_\Omega \alpha(x)\mu_x(K)\mathrm{d}x,$$

which implies $\mu_x(K) = 1$ for a.e. $x \in \Omega$ since α is arbitrary in $L^1(\Omega)$. To prove that μ_x is not a signed measure, we simply notice that

$$\|\mu^{n_k}\|_\infty = 1 \implies \|\mu\|_\infty \leq 1,$$

and this is enough to infer that μ_x is positive since we just showed that it has mass equal to K at a.e. $x \in \Omega$. Now take

$$g(x,y) := \alpha(x)\beta(y)$$

with $\alpha \in L^1(\Omega)$. We use (2.5) once again and find that

$$\int_\Omega \alpha(x)\beta(u_{n_k}(x))\,\mathrm{d}x \to \int_\Omega \alpha(x)\left(\int_K \beta(y)\,\mathrm{d}\mu_x(y)\right)\mathrm{d}x,$$

which proves (ii). The assertion (iii) follows in a similar way so we leave it as an exercise for the reader. To prove (iv), consider the function

$$g(x,y) := \alpha(x)y$$

and use (2.5) to infer that

$$\int_\Omega \alpha(x)u_{n_k}(x)\,\mathrm{d}x \to \int_\Omega \alpha(x)\left(\int_K y\,\mathrm{d}\mu_x(y)\right)\,\mathrm{d}x$$

holds for all $\alpha \in L^1(\Omega)$. Finally, to prove (v), we consider the function

$$f(x,y) = d_K(y, u_\infty(x)),$$

where d_K is the distance on K (which is a metric space). It follows that

$$\int_\Omega d_K(u_{n_k}(x), u_\infty(x))\,\mathrm{d}x \to \int_\Omega \left(\int_K d_K(y, u_\infty(x))\,\mathrm{d}\mu_x\right)\mathrm{d}x$$

and, since u_{n_k} converges in measure to u_∞, we can use the Lebesgue's dominated convergence theorem (using the boundedness of the space) and infer that

$$0 = \int_\Omega \left(\int_K d_K(y, u_\infty(x))\,\mathrm{d}\mu_x\right)\mathrm{d}x,$$

which is equivalent to $\int_K d_K(y, u_\infty(x))\,\mathrm{d}\mu_x(y)$ for a.e. $x \in \Omega$ or, in other words, μ_x is supported on $\{u_\infty(x)\}$. This concludes the proof of the theorem. $\square$

REMARK 2.61. The measure μ_x is the Young measure generated by the subsequence u_{n_k} and it depends on the choice of such subsequence.

REMARK 2.62. If K is replaced by $\mathbb{R}^m$ we lose compactness; thus we consider the one-point compactification of $\mathbb{R}^m$ and use it to produce a μ in such a way that

$$\mu_x \in \mathcal{P}(\mathbb{R}^n \cup \{\infty\}).$$

The Young measure is usually the restriction of μ_x to $\mathbb{R}^n$ (removing the ∞ point), that is,

$$\bar{\mu}_x := \mu_x \llcorner \mathbb{R}^n.$$

In general, $\bar{\mu}_x$ is a sub-probability measure. However, if one puts additional assumptions on the sequence u_n, then it is possible to show that there is no mass at infinity and hence $\bar{\mu}_x$ is a probability measure.

EXERCISE 2.10. Find sufficient assumptions on u_n for $\bar{\mu}_x$ to be a probability measure.

Note that identifying μ for a given sequence is usually complicated, but there are a few examples in which this is possible with few efforts.

EXAMPLE 2.63. *Let $\Omega = [0,1]$, $K = [-L, L]$ and $u_n(x) = u(nx)$ and $u : \mathbb{R} \to \mathbb{R}$ is a 1-periodic function that takes values in $\{y_1, y_2\}$ with proportion λ and $(1-\lambda)$. Then*

$$\mu_x = \lambda\delta_{y_1} + (1-\lambda)\delta_{y_2},$$

and it is interesting to notice that it does not depend on x.

EXAMPLE 2.64. *Let u_n be as above, but consider the 1-periodic function $u(x) = \sin(2\pi x)$. Then μ_x does not depend on x and*

$$\mu_x = u_{\#}(\mathcal{L}^1 \llcorner [0,1]) = \rho(y)\mathcal{L}^1 \llcorner [-1,1],$$

where ρ is a density function that can be computed explicitly. This holds more in general for 1-periodic functions (although it is not always easy to compute ρ).

There is an alternative way to construct μ. Consider $\Omega \times K$ and let λ^n positive measures on $\Omega \times K$ defined by

$$\lambda^n := \int_\Omega \delta_{x,u_n(x)}\,\mathrm{d}x.$$

Then $\|\lambda^n\| = |\Omega|$ and, up to subsequences, $\lambda^n \rightharpoonup^\star \lambda$ on $\Omega \times K$ in $C_0(\Omega, K)$. It is easy to see that

$$(\pi_x)^{\#}\lambda = \mathcal{L}^n \llcorner \Omega,$$

where π_x is the projection on Ω, and by a well-known result we can disintegrate λ as

$$\int_\Omega \lambda_x\,\mathrm{d}x$$

with $\lambda_x \in \mathcal{P}(\{x\} \times K)$ for all $x \in \Omega$. So $\lambda_x = \delta_x \otimes \mu_x$, and $\mu_x \in \mathcal{P}(K)$ is the image of $x \in \Omega$ via μ, i.e., the Young measure.

6. Relaxation with semicontinuity

Let $u_n : \Omega \to K$ be a sequence and let $x \mapsto \mu_x$ the Young measure associated to a specific subsequence (we will ignore the subscript n_k in this section) of u_n.

LEMMA 2.65. *Let $f : \Omega \times K \to [0, \infty]$ be a Borel function such that $f(x, \cdot)$ is lower semicontinuous for almost every $x \in \Omega$ and set*

$$F(u) := \int_\Omega f(x,u)\,\mathrm{d}x.$$

Then

$$\liminf_{n\to\infty} F(u_n) \geq \int_\Omega \left(\int_K f(x,y)\,\mathrm{d}\mu_x(y) \right) \mathrm{d}x.$$

IDEA OF THE PROOF. Write f as the pointwise supremum of an increasing sequence $\phi_i : \Omega \times K \to [0,\infty)$ of Borel functions such that $\phi_i(x,\cdot)$ is continuous for almost every $x \in \Omega$. We can thus apply (i) of Theorem 2.56, find that

$$\int_\Omega f(x, u_n(x))\,\mathrm{d}x \geq \int_\Omega \phi_i(x, u_n(x))\,\mathrm{d}x \to \int_\Omega \left(\int_K \phi_i(x,y)\,\mathrm{d}\mu_x(y)\right)\mathrm{d}x,$$

and use the standard tricks we have seen already to conclude. □

REMARK 2.66. If we assume moreover that $K \subset \mathbb{R}^m$ and $f(x,\cdot)$ is convex for all $x \in \Omega$. Then Lemma 2.65, together with Jensen's inequality, allows us to infer that

$$\liminf_{n\to\infty} F(u_n) \geq \int_\Omega f\left(x, \int_K y\,\mathrm{d}\mu_x(y)\right)\,\mathrm{d}x = \int_\Omega f(x, u_\infty(x))\,\mathrm{d}x,$$

where u_∞ is the baricenter given in (iv) of Theorem 2.56. Notice that Jensen inequality is also necessary, which means that the convexity of $f(x,\cdot)$ here is necessary to obtain the weakly lower semicontinuity of F.

THEOREM 2.67. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to [0,\infty]$ be a Borel function such that $f(x,\cdot,\cdot)$ is lower semicontinuous for a.e. $x \in \Omega$ and $f(x,u,\cdot)$ is convex for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^n$. Then the functional*

$$F(u) = \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x$$

is weak⋆ lower semicontinuous on $W^{1,\infty}(\Omega,\mathbb{R}^n)$.

PROOF. Let $u_n \overset{\star}{\rightharpoonup} u_\infty$ in $W^{1,\infty}(\Omega)$ and consider the Young measure $x \mapsto \lambda_x \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^{m\times n})$ generated by a subsequence of

$$v_n := (u_n, \nabla u_n).$$

By Sobolev embedding theorem

$$u_n \to u_\infty \quad \text{strongly (i.e., pointwise convergence)}$$

so λ_x is a Dirac mass on the first variable, that is, $\lambda_x = \delta_{u_\infty(x)} \otimes \mu_x$. Then

$$\begin{aligned}
\liminf_{n\to\infty} F(u_n) &\geq \int_\Omega \left(\int_{\mathbb{R}^n\times\mathbb{R}^{m\times n}} f(x,u,\xi)\,\mathrm{d}\lambda(u,\xi)\right)\mathrm{d}x \\
&= \int_\Omega \left(\int_{\mathbb{R}^{m\times n}} f(x,u_\infty(x),\xi)\,\mathrm{d}\mu_x(\xi)\right)\mathrm{d}x \\
&\geq \int_\Omega f(x, u_\infty(x), \nabla u_\infty(x))\,\mathrm{d}x,
\end{aligned}$$

where the last inequality follows once again from Jensen and the fact that the baricenter of μ_x is exactly given by $u_\infty(x)$. □

The weak$\star$ lower semicontinuity of $F(u) = \int_\Omega f(x, u, \nabla u)\, \mathrm{d}x$ proved above is a consequence of the following more general result which can be proved with minor changes.

THEOREM 2.68. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to [0, \infty]$ be a Borel function such that $f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$ and $f(x, u, \cdot)$ is convex for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^n$. Then the functional*

$$F(u, v) = \int_\Omega f(x, u, v)\, \mathrm{d}x$$

is lower semicontinuous with respect to strong convergence of u in some L^p-space and weak convergence of v in some L^q-space.

REMARK 2.69. As usual, when $q = \infty$ in the statement the weak convergence should be replaced by the weak$\star$ convergence.

PROOF. Let $u_n \to u_\infty$ strongly in $L^p(\Omega)$, $v_n \rightharpoonup v_\infty$ in $L^q(\Omega)$ and consider the Young measure $x \mapsto \lambda_x \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^{m\times n})$ generated by a subsequence of

$$w_n := (u_n, v_n).$$

The strong convergence in the first variable implies that λ_x is a Dirac mass on the first variable, that is, $\lambda_x = \delta_{u_\infty(x)} \otimes \mu_x$. Then

$$\begin{aligned}
\liminf_{n\to\infty} F(u_n, v_n) &\geq \int_\Omega \left(\int_{\mathbb{R}^n \times \mathbb{R}^{m\times n}} f(x, u, \xi)\, \mathrm{d}\lambda(u, \xi) \right) \mathrm{d}x \\
&= \int_\Omega \left(\int_{\mathbb{R}^{m\times n}} f(x, u_\infty(x), \xi)\, \mathrm{d}\mu_x(\xi) \right) \mathrm{d}x \\
&\geq \int_\Omega f(x, u_\infty(x), v_\infty(x))\, \mathrm{d}x,
\end{aligned}$$

where the last inequality follows once again from Jensen and the fact that the baricenter of μ_x is exactly given by $v_\infty(x)$. $\square$

REMARK 2.70. Notice that we tacitly assumed that u_n and v_n are uniformly bounded sequences for otherwise the proof above would fail. The idea is to consider

$$K := (\mathbb{R}^n \cup \{\infty\}) \times (\mathbb{R}^N \cup \{\infty\}),$$

the one-point compactification of $\mathbb{R}^n \times \mathbb{R}^N$, and the corresponding Young measure λ_x. It is not hard to verify that

$$\int_\Omega g(x, u_n, v_n)\, \mathrm{d}x \to \int_\Omega \left(\int_K g(x, y)\, \mathrm{d}\lambda_x(y) \right) \mathrm{d}x$$

holds for all functions $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ which are continuous in (u, v) and, for example, tend to zero as u and v tend to zero.

This additional requirement makes g continuous on K as well. Since it is still possible to approximate f with an increasing sequence of functions of this kind, we can prove the weak lower semicontinuity using a similar (but much more technical) strategy as in Theorem 2.68.

CHAPTER 3

Convexity and Lower Semicontinuity

I will write the introduction after the chapter is complete!

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ and consider the functional

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x,$$

where $f : \mathbb{R}^{m\times n} \to [0,\infty]$ is a lower semicontinuous function. We proved in Theorem 2.46 the following chain of implications

$$f \text{ convex} \implies F \text{ convex} \implies F \text{ weakly lower semicontinuous on } W^{1,p}(\Omega,\mathbb{R}^m)$$

for all $1 \le p \le \infty$ - replacing weakly by weakly$\star$ when $p = \infty$ -.

REMARK 3.1. Notice that F weakly lower semicontinuous on $W^{1,p}(\Omega,\mathbb{R}^m)$ implies that the Lagrangian f is convex if either $m = 1$ or $n = 1$ (see Theorem 2.50), so in these two special cases we can close the circle of implications and we get the equivalence

$$f \text{ convex} \iff F \text{ weakly lower semicontinuous on } W^{1,p}(\Omega,\mathbb{R}^m).$$

REMARK 3.2. Recall that in Theorem 2.50 we defined a sequence of functions u_ϵ in such a way that the gradients are given by

$$\nabla u_\epsilon(x) = \begin{cases} \xi_1 & \text{if } x \in \Omega \setminus A_\epsilon, \\ \xi_0 & \text{if } x \in A_\epsilon, \end{cases}$$

where A_ϵ is the union of stripes of tickness $\lambda\epsilon$ and $(1-\lambda)\epsilon$ orthogonal to $\xi_1 - \xi_0$. This means that, when ξ_1 and ξ_0 are vectors, there exists $\vec{e}$ such that

$$\xi_1 - \xi_0 \perp \vec{e}.$$

However, when ξ_1 and ξ_2 are matrices (namely, m and n are both different from 1), the condition on the stripes translates to

$$\xi_1 - \xi_0 = \vec{a} \otimes \vec{e},$$

which means that the matrix $\xi_1 - \xi_0$ is equal to the matrix given by the tensor product of two vectors $\vec{a}$ and $\vec{e}$. This should give a hint on why convexity is hard to achieve as soon as $\min\{m,n\} > 1$, but the next result gives a precise explanation.

DEFINITION 3.3. A function $f : \mathbb{R}^{m\times n} \to [0,\infty]$ is *rank-one convex* if

$$f(\lambda\xi_0 + (1-\lambda)\xi_1) \le \lambda f(\xi_0) + (1-\lambda)f(\xi_1)$$

for all $\lambda \in [0,1]$ and all $\xi_0, \xi_1 \in \mathbb{R}^{m\times n}$ such that

$$\operatorname{rank}(\xi_1 - \xi_0) = 1.$$

PROPOSITION 3.4. *Let* $\min\{m,n\} > 1$*. If* F *weakly lower semicontinuous on* $W^{1,p}(\Omega,\mathbb{R}^m)$*, then* f *is rank-one convex.*

We will be more precise in the next section, but there is an additional notion, *polyconvexity*, which is easily seen to be different from convexity such that

$$f \text{ polyconvex } \implies F \text{ weakly lower semicontinuous on } W^{1,p}(\Omega,\mathbb{R}^m),$$

with the same replacement as above if $p=\infty$. In particular, we can summarise everything we said until now in the following scheme of implications:

$$f \text{ polyconvex } \implies F \text{ weakly lsc on } W^{1,p}(\Omega,\mathbb{R}^m) \implies f \text{ rank-one convex}.$$

This means that there must be a kind of "middle ground notion of convexity", different from the others, which gives equivalence with weakly lower semicontinuity. This notion is known in the literature as *quasiconvexity* and it can be proved that

$$f \text{ convex } \implies f \text{ polyconvex } \implies f \text{ quasiconvex } \implies f \text{ rank-one convex}.$$

In particular, if $m=1$ or $n=1$, then these classes coincide with one another (because the smallest and the largest do) and it is known that for $n\ge 2$ and $m \ge 3$ they are all different.

REMARK 3.5. [Open Problem] If $n=m=2$ it is still an open problem (with many important consequences) to determine whether or not

$$f \text{ quasiconvex } \iff f \text{ rank-one convex}.$$

On the other hand, it is well-known that convexity, polyconvexity and quasiconvexity are all different classes.

2. Polyconvexity

The goal of this section is to give a formal definition of *policonvexity* and show that it is indeed enough to obtain weak lower semicontinuity of the functional - although some restrictions on p appear due to technical reasons -.

DEFINITION 3.6. A lower semicontinuous function $f : \mathbb{R}^{m\times n} \to [0,\infty]$ is *polyconvex* if it can be written as

$$f(\xi) = g(M(\xi))$$

where $M(\xi)$ is the vector of all minors of the matrix ξ and g is a convex function.

EXAMPLE 3.7. *If* $m=n=2$*,* f *is polyconvex if* f *can be written as*

$$f(\xi) = g(\xi, \det\xi)$$

with $g : \mathbb{R}^{2\times 2}\times\mathbb{R} \to [0,\infty]$ *convex.*

THEOREM 3.8. *If f is polyconvex, then $F(u) = \int_\Omega f(\nabla u)\,\mathrm{d}x$ is weakly lower semicontinuous on $W^{1,p}(\Omega)$ for $p \geq \min\{m,n\}$.*

Before we can deal with this theorem, we first need to state and prove a couple of technical results concerning the convergence of minors.

LEMMA 3.9. *Let $m = n = 2$. If $u_n \rightharpoonup u_\infty$ in $W^{1,p}(\Omega,\mathbb{R}^2)$ and $p > 2$, then*

$$\det(\nabla u_n) \rightharpoonup \det(\nabla u_\infty) \quad \textit{in } L^{\frac{p}{2}}(\mathbb{R}^2).$$

PROOF. First, assume that $u \in C^2(\Omega,\mathbb{R}^2)$. It is easy to show that the determinant of the gradient, which is equal to

$$\det(\nabla u) = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_1},$$

can also be rewritten as

$$\det(\nabla u) = \frac{\partial}{\partial x_1}\left(u^1\frac{\partial u^2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u^1\frac{\partial u^2}{\partial x_1}\right), \tag{3.1}$$

where u^1 and u^2 are the components of u. We now claim that (3.1) holds even if there is less regularity, namely $u \in W^{1,p}(\Omega,\mathbb{R}^2)$ with $p > 2$, but the equal sign has to be intended in the distributional sense.

This follows from a standard approximation argument via smooth functions ν_n (converging strongly in $W^{1,p}$ to ν) so it suffices to show that the left-hand and the right-hand sides of (3.1) are both continuous in the distributional topology. By Hölder's inequality,

$$\nu_n^1 \xrightarrow{n\to\infty} \nu^1 \text{ and } \frac{\partial \nu_n^2}{\partial x_2} \xrightarrow{n\to\infty} \frac{\partial \nu_n^2}{\partial x_2} \text{ in } L^p \implies \nu_n^1\frac{\partial \nu_n^2}{\partial x_2} \xrightarrow{n\to\infty} \nu^1\frac{\partial u^2}{\partial x_2} \text{ in } L^{\frac{p}{2}},$$

which gives the continuity in the distributional topology of the right-hand side of (3.1). A similar argument works for the left-hand side. Now, by Sobolev embedding,

$$u_n^i \text{ converges strongly to } u^i \text{ for } i = 1,2 \text{ in } L^p,$$

and, since the sequence of the derivatives converges weakly in L^p, a well-known result in functional analysis asserts that the product $u^1\frac{\partial u^2}{\partial x_2}$ converges weakly in $L^{\frac{p}{2}}$. Therefore

$$\det(\nabla u_n)$$

converges in the sense of distributions to $\det(\nabla u_\infty)$ and, since, $\det(\nabla u_n)$ is uniformly bounded in $L^{\frac{p}{2}}$ we easily infer that

$$\det(\nabla u_n) \rightharpoonup \det(\nabla u_\infty) \quad \text{in } L^{\frac{p}{2}}.$$

□

REMARK 3.10. If $p = 2$, the same result holds true but we have to replaced the weak convergence with the weak$\star$ one. The reader might try to prove it as an exercise.

Note that in the proof of this lemma the dimension of the ambient space, $\Omega \subset \mathbb{R}^m$, does not matter and hence m is arbitrary. We can thus replace the gradient with any 2×2 minor $M(\nabla u_n)$, and the proof only requires small changes.

LEMMA 3.11. *Let $m = n$. If $u_k \rightharpoonup u_\infty$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ and $p > n$, then*

$$\det(\nabla u_n) \rightharpoonup \det(\nabla u_\infty) \quad \textit{in } L^{\frac{p}{n}}(\mathbb{R}^n).$$

PROOF. The argument is similar but it is necessary to use a different identity. More precisely, if $u \in C^2(\Omega, \mathbb{R}^n)$, then we can use multilinear algebra to write

$$\begin{aligned}\det(\nabla u)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n &= \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n \\ &= \mathrm{d}(u_1 \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n).\end{aligned}$$

This identity holds, as above, even for functions with lower regularity (namely in $W^{1,p}$). Furthermore, it allows us to conclude by induction because

$$u^1 \text{ converges strongly by Sobolev embedding in } L^p,$$

and $\mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n$ converges weakly in $L^{\frac{p}{n-1}}$ by inductive hypothesis. Then the product converges weakly in $L^{\frac{p}{n}}$ as explained before, and hence

$$\det(\nabla u_n)$$

converges in the sense of distributions to $\det(\nabla u_\infty)$. Since $\det(\nabla u_n)$ is uniformly bounded in $L^{\frac{p}{n}}$, we easily infer that

$$\det(\nabla u_n) \rightharpoonup \det(\nabla u_\infty) \quad \text{in } L^{\frac{p}{n}}.$$

□

PROOF OF THEOREM 3.8. We put the additional assumption $p > \min\{m, n\}$. Now, if we write

$$F(u_n) = \int_\Omega g(M(\nabla u_n))\,\mathrm{d}x,$$

then we can use Lemma 3.11 together with the convexity of g to infer that F is weakly lower semicontinuous (as in Theorem 2.46). □

3. Quasiconvexity

Let $f : \mathbb{R}^{m\times n} \to [0, \infty)$ be lower semicontinuous and locally bounded. Before we give the definition of quasiconvexity, we first recall a useful characterisation of convexity.

PROPOSITION 3.12. *Let B be the unit ball of $\mathbb{R}^n$. The function f is convex if and only if for every $\xi \in \mathbb{R}^{m\times n}$ and every $v : B \to \mathbb{R}^{m\times n}$ with zero average,*

$$\int_B v\,\mathrm{d}x = 0,$$

it turns out that

$$⨍_B f(\xi + v(x))\,\mathrm{d}x \geq f(\xi).$$

The following definition of quasiconvexity is due to Morrey. See, for example, the paper [**1**].

DEFINITION 3.13. The function f is *quasiconvex* if for every $\xi \in \mathbb{R}^{m\times n}$ and every $u \in W_0^{1,\infty}(B, \mathbb{R}^m)$ it turns out that

$$\fint_B f(\xi + \nabla u(x))\,\mathrm{d}x \geq f(\xi). \tag{3.2}$$

PROPOSITION 3.14. *If f is convex, then f is quasiconvex.*

PROOF. It is enough to show that functions $u \in W_0^{1,\infty}(B, \mathbb{R}^m)$ have gradient with zero average over B. Equivalently, we can prove that

$$\int_B \frac{\partial u^1}{\partial x_1}\,\mathrm{d}x = 0,$$

where u^1 is the first component of u. This is fairly easy because we can use the identity

$$\int_B \frac{\partial u^1}{\partial x_1}\,\mathrm{d}x = \int_B \mathrm{div}((u^1, 0, \ldots, 0))\,\mathrm{d}x,$$

and the right-hand side is, by the divergence's theorem, equal to

$$\int_{\partial B} u^1 \cdot \nu_1\,\mathrm{d}x.$$

Since u vanishes on the boundary (as it belongs to $W_0^{1,\infty}(B, \mathbb{R}^m)$), we infer that the last integral is zero and thus $\int_B \frac{\partial u^1}{\partial x_1}\,\mathrm{d}x = 0$ as well. □

We are now ready to give the main result concerning quasiconvex functions, which is the equivalence between quasiconvexity and weak⋆ lower semicontinuity on $W^{1,\infty}(\Omega, \mathbb{R}^m)$.

THEOREM 3.15. *Let f be as above, $\Omega \subset \mathbb{R}^n$, and consider the functional*

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x.$$

Then the following assertions hold:

- *(i) If f is quasiconvex, then F is weakly⋆ lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^m)$.*
- *(ii) If F is weakly⋆ lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^m)$, then f is quasiconvex.*

REMARK 3.16. With extra (growth) assumptions on f, one can prove that f quasiconvex implies F weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$ for all $1 < p < \infty$.

We will now only prove the point (ii) since it is easier and come back to (i) after we studied specific properties of quasiconvex functions.

PROOF OF (II). Let $\xi \in \mathbb{R}^{m\times n}$ and $u \in W_0^{1,\infty}(B)$. Our goal is to prove (3.2), that is,

$$\fint_B f(\xi + \nabla u(x))\,\mathrm{d}x \geq f(\xi).$$

Assume $\xi = f(0) = 0$ for simplicity[1] and suppose that there exists $r > 0$ such that

$$\Omega \supseteq Q := \mathrm{Cube}(r),$$

namely Ω contains a cube with side-length r. For $\ell \in \mathbb{N}$ fix a partition of the cube in ℓ^n sub-cubes with sides of length $\frac{r}{\ell}$ and in each small cube C_i of this partition fit a ball

$$B(x_i, r_i) := B\left(x_i, \frac{r}{2\ell}\right).$$

Let us consider the sequence of functions

$$u_\ell(x) := \begin{cases} r_i\, u\left(\frac{x-x_i}{r_i}\right) & \text{if } x \in B(x_i, r_i) \text{ for some } i, \\ 0 & \text{if } x \text{ does not belong to any ball } B(x_i, r_i). \end{cases}$$

Then u_ℓ converges uniformly to the function identically zero because

$$\|u_\ell\|_\infty = \frac{c}{2\ell}\|u\|_\infty \xrightarrow{\ell \to \infty} 0,$$

and it is easy to verify that $\|\nabla u_\ell\|_\infty \leq C$, with C uniform constant, since

$$\nabla u_\ell(x) = \begin{cases} \nabla u\left(\frac{x-x_i}{r_i}\right) & \text{if } x \in B(x_i, r_i) \text{ for some } i, \\ 0 & \text{if } x \text{ does not belong to any ball } B(x_i, r_i). \end{cases}$$

It can be proved (using, for example, Proposition 3.17) that $u_\ell \in W_0^{1,\infty}(\Omega)$. If we give this for granted, then it is easy to show that

$$\begin{aligned} F(u_\ell) &= \int_\Omega f(\nabla u_\ell)\,\mathrm{d}x = \\ &= \sum_i \int_{B_i} f\left(\nabla u\left(\frac{x-x_i}{r_i}\right)\right)\mathrm{d}x = \\ &\overset{(*)}{=} \sum_i \left[\int_B f(\nabla u(x))\,\mathrm{d}x\right] r_i^n = \\ &= \left(\frac{r}{2\ell}\right)^n \ell^n \int_B f(\nabla u(x))\,\mathrm{d}x = \\ &= \left(\frac{r}{2}\right)^n \int_B f(\nabla u(x)\,\mathrm{d}x), \end{aligned}$$

[1]The proof for a generic ξ is more or less the same, while the condition $f(0) = 0$ can always be achieved by adding or subtracting a suitable constant to f.

where the equality $(*)$ follows from the change of variables formula. By weak$\star$ lower semicontinuity of F we infer that

$$\fint_B f(\nabla u(x))\,\mathrm{d}x = \left(\frac{r}{2}\right)^n \int_B f(\nabla u(x)\,\mathrm{d}x) \geq F(0) = 0 = f(0),$$

and this concludes the proof. $\square$

Concerning the technical point which we avoided, namely the fact that u_ℓ belongs to $W_0^{1,\infty}(\Omega)$, one could try to use the following characterization.

PROPOSITION 3.17. *A function u belongs to $W_0^{1,\infty}(\Omega) \cap C(\Omega)$ if and only if u belongs to $u \in \mathrm{Lip}(\mathbb{R}^n)$ with $u = 0$ on $\mathbb{R}^n \setminus \Omega$.*

REMARK 3.18. We say that $u \in W_0^{1,\infty}(\Omega) \cap C(\Omega)$ because we require u to be the **continuous representative** of its class of equivalence (to avoid troubles in negligible sets).

PROOF FOR $n = 1$. Let $u \in \mathrm{Lip}(\mathbb{R})$ and denote by Du its weak gradient. It is easy to see that

$$\frac{u(x+h) - u(x)}{h} \to Du(x)$$

in the sense of distributions. This is an easy consequence of how distributions are defined since we have the identity

$$\int_{\mathbb{R}} \frac{u(x+h) - u(x)}{h} \Phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} u(x) \frac{\Phi(x) - \Phi(x+h)}{h}$$

for any smooth function Φ with compact support and the incremental ratio of a smooth function converges to its derivative. Finally, using the fact that

$$\frac{u(x+h) - u(x)}{h}$$

is uniformly bounded (because u is Lipschitz), it must converge (by Banach-Alaoglu) to a bounded function which means that $Du \in L^\infty(\mathbb{R})$ and hence $u \in W_0^{1,\infty}(\Omega)$. $\square$

REMARK 3.19. If $u \in W^{1,p}(\Omega) \cap C(\Omega)$, $p > n$, then u is differential at a.e. $x \in \Omega$ with distributional gradient that coincide with the gradient $\nabla u(x)$.

We will come back to the almost everywhere differentiability at a later time, but for the time being it is useful to keep it in mind when $p = \infty$.

PROPOSITION 3.20. *In the definition of quasiconvexity, we can replace the unit ball B by any bounded open set $A \subset \mathbb{R}^n$.*

PROOF. To prove this result, we first introduce a "definition" which will at the end be equivalent to quasiconvexity. Namely, we say that f is A-quasiconvex if

$$\fint_A f(\xi + \nabla u(x))\,\mathrm{d}x \geq f(\xi) \quad \text{for all } \xi \in \mathbb{R}^{m\times n} \text{ and } u \in W_0^{1,\infty}(A).$$

A simple observation tells us that

$$f \text{ quasiconvex} \iff f \text{ is } B(x,r)\text{-quasiconvex for any ball } B(x,r)$$

and now it is easy to conclude since if $B(x,r) \subset A$ we get

$$f \text{ is } A\text{-quasiconvex} \implies f \text{ is quasiconvex}$$

and, if $B(x,R) \supset A$, we get the opposite implication

$$f \text{ is } A\text{-quasiconvex} \impliedby f \text{ is quasiconvex.}$$

□

We now want to take a closer look to the chain of implications between the different notions of convexity, that is,

$$f \text{ convex} \implies f \text{ polyconvex} \implies f \text{ quasiconvex} \implies f \text{ rank-one convex}$$

in the nontrivial case (=when both m and n are bigger than or equal to two).

PROPOSITION 3.21. *The function f is polyconvex if f is convex.*

PROOF. This follows trivially from the definition of polyconvexity because we can choose g to be exactly equal to f. □

The next implication, f polyconvex $\implies$ f quasiconvex, follows from Theorem 3.15 together with Theorem 3.8, but there is also a direct proof which we will see immediately.

REMARK 3.22. Suppose $\min\{m,n\} > 1$. We saw already an example of a polyconvex function which is not convex, while for

$$f \text{ quasiconvex} \not\Longrightarrow f \text{ polyconvex}$$

there is some work to do. Finally, showing that

$$f \text{ rank-one convex} \not\Longrightarrow f \text{ quasiconvex}$$

is even harder and there are no known counterexamples in the case $n = m = 2$. If $n \geq 2$ and $m \geq 3$, the first work in that direction can be found in the paper [**2**].

Now, before we can give the mentioned direct proof, we need to present a few technical results.

LEMMA 3.23. *If $u \in W_0^{1,\infty}(B)$ and $\xi \in \mathbb{R}^{m\times n}$, then*

$$\fint_B M(\xi + \nabla u(x))\,\mathrm{d}x = M(\xi).$$

To prove this result, it is sufficient to replace M with the determinant (we leave to the reader the task to figure out why). Before that, recall a well-known property:

LEMMA 3.24. *For every $v, v' \in W^{1,\infty}(\Omega,\mathbb{R}^n)$ such that $v \equiv \tilde{v}$ on $\partial\Omega$ we have*

$$\int_\Omega \det(\nabla v)\,\mathrm{d}x = \int_\Omega \det(\nabla \tilde{v})\,\mathrm{d}x.$$

PROOF. Suppose that v is smooth. Then

$$\begin{aligned}\int_\Omega \det(\nabla v(x))\,\mathrm{d}x &= \int_\Omega \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n \\ &= \int_\Omega \mathrm{d}(v^1 \mathrm{d}v^2 \wedge \cdots \wedge \mathrm{d}v^n) \\ &= \int_{\partial\Omega} v^1 \mathrm{d}v^2 \wedge \cdots \wedge \mathrm{d}v^n,\end{aligned}$$

and the last equality follows from Stoke's theorem. Since we can replace the differential form with its restriction to the tangent bundle, the integrand depends on v^1 and on the tangential derivatives of the other components, which means that they ultimately depend only on the value of v at the boundary. □

LEMMA 3.25. *If $u \in W_0^{1,\infty}(B, \mathbb{R}^n)$ and $\xi \in \mathbb{R}^{n\times n}$, then*

$$\fint_B \det(\xi + \nabla u(x))\,\mathrm{d}x = \det(\xi).$$

PROOF. We use Lemma 3.24 with $v(x) = \xi x + u(x)$ and $\tilde{v}(x) = \xi x$ since u is equal to zero at the boundary so v and $\tilde{v}$ coincide there. □

PROPOSITION 3.26. *The function f is quasiconvex if f is polyconvex.*

PROOF. Suppose that f is polyconvex and let g be the convex function such that $f(\xi) = g(M(\xi))$. Then

$$\fint_B f(\xi + \nabla u(x))\,\mathrm{d}x = \fint g(M(\xi + \nabla u(x)))\,\mathrm{d}x,$$

and, using Lemma 3.23 together with Jensen's inequality, we obtain

$$\fint_B f(\xi + \nabla u(x))\,\mathrm{d}x \geq g(M(\xi)) = f(\xi),$$

which is exactly the definition of quasiconvexity (3.2). □

PROPOSITION 3.27. *The function f is rank-one convex if f is quasiconvex.*

PROOF. Suppose f quasiconvex and fix $\xi_0, \xi_1 \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(\xi_1 - \xi_0) = 1$, i.e., $\xi_1 - \xi_0 = a \otimes e$. Fix $\lambda \in [0,1]$ and denote by ξ the convex combination:

$$\xi := \lambda\xi_0 + (1-\lambda)\xi_1.$$

Our goal is to prove that

$$f(\xi) \leq \lambda f(\xi_0) + (1-\lambda) f(\xi_1).$$

The idea is the same one we used in the 1-dimensional case in Theorem 2.46. Namely, we wish to construct a sequence $u_\epsilon : \Omega \to \mathbb{R}^m$ such that

$$u_\epsilon \to u(x) := \xi x \quad \text{uniformly}$$

and ∇u_ϵ is equal to ξ_0 on a set A^0_ϵ with constant measure λM, equal to ξ_1 on a set A^1_ϵ with constant measure $(1-\lambda)M$, bounded on the "bad set" B_ϵ which measure satisfies $|B_\epsilon| \to 0$ and equal to ξ on the rest. It is then easy to verify that

$$\fint_B f(\nabla u_\epsilon)\,\mathrm{d}x \geq f(\xi)$$

as a consequence of the quasiconvexity of f. By passing this inequality to the limit as $\epsilon \to 0^+$ we finally get the desired inequality:

$$f(\xi) \leq \lambda f(\xi_0) + (1-\lambda) f(\xi_1).$$

□

REMARK 3.28. We tacitly assumed f to be locally bounded. Indeed, in the bad part B_ϵ we cannot estimate the gradient and hence we need $\int_{B_\epsilon} f(\nabla u_\epsilon)\,\mathrm{d}x$ to be finite.

We now want to show that rank-one convex functions are locally Lipschitz and, as a consequence, the same is true for polyconvex and quasiconvex functions.

LEMMA 3.29. *Let $\Omega \subseteq \mathbb{R}^n$ be a convex set. If $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is convex, then g is locally Lipschitz on the interior part of Ω.*

THEOREM 3.30. *Let $f : \mathbb{R}^{m\times n} \to [0,\infty)$ be lower semicontinuous and locally bounded. If f is rank-one convex, then f is locally Lipschitz.*

IDEA OF THE PROOF. Since f is rank-one convex, by definition f is convex on rank-one lines in the space of matrices $\mathbb{R}^{m\times n}$, which means that

$$\operatorname{rank}(\xi_1 - \xi_0) = 1 \implies f \text{ convex on the line of direction } \xi_1 - \xi_0.$$

By Lemma 3.29, the function f is locally Lipschitz on rank-one lines. This is enough to conclude because it can be proved that **any** two matrices can be connected via rank-one segments in such a way that the total length is bounded. □

REMARK 3.31. The spirit of this result is similar to the assertion "$f : \mathbb{R}^2 \to \mathbb{R}$ locally Lipschitz on both horizontal and vertical lines implies f locally Lipschitz on $\mathbb{R}^2$."

We are now ready to prove the main result concerning quasiconvex Lagrangians (Theorem 3.15), but for the sake of clarity, we give the accurate statement here again.

THEOREM 3.32. *Let $f : \mathbb{R}^{m\times n} \to [0,\infty)$ be lower semicontinuous and finite and let $\Omega \subset \mathbb{R}^n$. Consider the functional*

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x.$$

Then the following assertions hold:

(i) If f is quasiconvex, then F is weakly$\star$ lower semicontinuous on $W^{1,\infty}(\Omega,\mathbb{R}^m)$.

(ii) If F is weakly$\star$ lower semicontinuous on $W^{1,\infty}(\Omega,\mathbb{R}^m)$, then f is quasiconvex.

(iii) If, in addition, f satisfies the p-growth condition

$$f(\xi) \leq C(1 + |\xi|^p)$$

for some $1 < p < \infty$, then F is weakly lowersemicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$.

We are now ready to give the proof of the assertion (i) since (ii) was proved earlier and (iii) is more complicated and left to the interested reader.

PROOF OF (I). Let $u_n \overset{\star}{\rightharpoonup} u_\infty$ in $W^{1,\infty}(\Omega, \mathbb{R}^m)$. The idea is to first prove the theorem under restrictive assumptions on both Ω and u_∞, and then pass to the general case using some tools from geometric measure theory.

Step 1. Assume $\Omega = B$, where B is any ball, and u_∞ identically equal to 0. Note that, if each u_n is identically zero at the boundary of B, then

$$F(u_n) = \int_B f(\nabla u_n)\,\mathrm{d}x \geq |B| f(0) = F(u_\infty),$$

where the inequality $\geq$ follows from the quasiconvexity (3.2). If u_n does not vanish at the boundary of B, then the idea is to exploit the fact that u_n converges uniformly to zero and modify u_n to some $\tilde{u}_n$ satisfying, more or less, the following "properties":

$$\tilde{u}_n \Big|_{\partial B} \equiv 0 \quad \text{and} \quad F(\tilde{u}_n) \simeq F(u_n).$$

To make this argument formal, start by introducing the distance function $d(x) := \operatorname{dist}(x, \partial B)$ and the function g, defined on $B \times \mathbb{R}^m$ and given by

$$g(x, u) := u \left(1 \wedge \frac{d(x)}{|u|}\right).$$

We can introduce the *truncation operator* T which is defined as

$$Tu(x) := g(x, u(x))$$

for all $u \in W^{1,\infty}(B, \mathbb{R}^m)$. Now notice that g is Lipschitz as a function of (x, u) and hence we can use the equivalence between $W^{1,\infty}$ and Lip discussed earlier to conclude that

$$\begin{aligned} u \in W^{1,\infty}(B, \mathbb{R}^m) &\implies u \in \mathrm{Lip}(B, \mathbb{R}^m) \implies \cdots \\ \cdots &\implies Tu \in \mathrm{Lip}(B, \mathbb{R}^m) \implies Tu \in W^{1,\infty}(B, \mathbb{R}^m). \end{aligned}$$

Furthermore, it is not hard to verify that the following properties hold:

(1) $Tu(x) = 0$ whenever $x \in \partial B$ since $d(\cdot)$ vanishes at the boundary of B;

(2) $Tu(x) = u(x)$ on $B \setminus A_u$, where $A_u = \{x \in B \,:\, |u(x)| > d(x)\}$;

(3) $\nabla(Tu)(x) = \nabla u(x)$ at **almost every** $x \in B \setminus A_u$ because they are Lipschitz (so a.e. differentiable and a.e. point is a density point);

(4) $|\nabla(Tu)(x)| \leq \mathrm{Lip}(Tu) \leq c \cdot \mathrm{Lip}(u)$ at a.e. point and, in particular, at a.e. $x \in A_u$.

Now recall that (by Theorem 3.30) f is finite and quasiconvex if and only if f is finite and rank-one convex, which means that f is locally bounded. Let

$$\omega(t) := \sup_{|M| \leq t} f(M) < \infty$$

be the modulus of growth for f, which is finite for $t > 0$ sufficiently small. It follows that

$$\begin{aligned} F(Tu) &= \int_B f(\nabla(Tu))\,\mathrm{d}x \\ &\leq \int_{B \setminus A_u} f(\nabla u)\,\mathrm{d}x + \int_{A_u} f(\nabla(Tu))\,\mathrm{d}x \\ &\leq F(u) + \omega(c \cdot \mathrm{Lip}(u))|A_u|. \end{aligned}$$

In particular, we can reverse the inequality to obtain

$$F(u) \geq F(Tu) - \omega(c \cdot \mathrm{Lip}(u))|A_u|$$

and, given that Tu is zero at the boundary of B, we use that f is quasiconvex (3.2) to infer that

$$F(u) \geq |B| f(0) - \omega(c \cdot \mathrm{Lip}(u))|A_u|.$$

Now notice that u_n converges to zero almost everywhere, so we can apply this inequality to $u := u_n$ and conclude that

$$F(u_n) \geq |B| f(0) - o(1) = F(u_\infty) - o(1)$$

because the maps u_n are uniformly Lipschitz and - as the reader might try to check this by themselves - we have $|A_{u_n}| \to 0$. Taking the limit as $n \to \infty$ yields

$$\lim_{n \to \infty} F(u_n) \geq F(u_\infty),$$

and this concludes the proof of the first step.

Step 2. Assume $\Omega = B$ and $u_\infty =$ affine function. Then u_∞ has gradient equal to some fixed ξ and we can define a more efficient truncation by setting

$$\widetilde{T}u := u_\infty + T(u - u_\infty),$$

where T is the truncation operator defined in the first step. Arguing as above yields

$$F(u) \geq |B| f(\xi) - \omega(c \cdot \mathrm{Lip}(u))|\widetilde{A}_u|,$$

where $\widetilde{A}_u := \{x \in B \ : \ |u(x) - u_\infty(x)| > d(x)\}$. Since u_n converges to u_∞ almost everywhere, we follow the proof above and obtain the desired inequality:

$$\lim_{n \to \infty} F(u_n) \geq F(u).$$

Step 3. Suppose Ω and u_∞ arbitrary and fix $\epsilon > 0$. Since u_∞ is differentiable a.e. in Ω, we can always find finitely many disjoint balls $B_i := B(x_i, r_i) \subset \Omega$ such that

$$\left| \Omega \setminus \bigcup_{i=1}^{\kappa} B_i \right| \leq \epsilon$$

and, if we set

$$u^i(x) := u_\infty(x_i) + \nabla u_\infty(x_i)(x - x_i),$$

the following properties hold:

(1) $|u_\infty(x) - u^i(x)| \le \epsilon \cdot r_i$ for all $x \in B_i$;

(2) $|\nabla u_\infty(x) - \nabla u_\infty(x_i)| \le \epsilon$ for all $x \in B_i \setminus E_i$, where $|E_i| \le \epsilon |B_i|$.

We will come back to this fact at the end of the proof. Now take u and let $\widetilde{T}_i$ be the truncation operator on B_i constructed in the previous step; namely,

$$\widetilde{T}_i u(x) := u^i(x) + T_i(u(x) - u^i(x))$$

where, setting $d_i(x) := \operatorname{dist}(x, \partial B_i)$, we have

$$T_i u(x) := u(x)\left(1 \wedge \frac{d_i(x)}{|u(x)|}\right).$$

Arguing as in the previous steps, we can prove that

$$\int_{B_i} f(\nabla u)\,\mathrm{d}x \ge |B_i| f(\xi_i) - \omega(c \cdot \operatorname{Lip}(u)) |\widetilde{A}^i_u|$$

where $\widetilde{A}^i_u := \{x \in B_i \,:\, |u(x) - u^i(x)| > d_i(x)\}$. Then

$$\begin{aligned}
F(u) &\ge \sum_{i=1}^{\kappa} \int_{B_i} f(\nabla u)\,\mathrm{d}x \\
&\ge \sum_{i=1}^{\kappa} |B_i| f(\xi_i) - \omega(c \cdot \operatorname{Lip}(u)) \sum_{i=1}^{\kappa} |\widetilde{A}^i_u| \\
&\ge \sum_{i=1}^{\kappa} \int_{B_i} f(\nabla u_\infty)\,\mathrm{d}x - c' \sum_{i=1}^{\kappa} |\widetilde{A}^i_\infty(u)| - \mathcal{O}(\epsilon) \\
&\ge F(u_\infty) - c' \left[\epsilon + \sum_{i=1}^{\kappa} |\widetilde{A}^i_\infty(u)|\right],
\end{aligned}$$

where $\widetilde{A}^i_\infty(u) := \{x \in B_i \,:\, |u(x) - u_\infty(x)| > d_i(x)\}$. Now notice that u_n is a sequence converging to u_∞ almost everywhere, so it is easy to verify that

$$|A^i_\infty(u_n)| \xrightarrow{n\to\infty} 0,$$

and this leads to the desired inequality up to a big-O of ϵ, namely

$$\lim_{n\to\infty} F(u_n) \ge F(u_\infty) - c'\epsilon,$$

but this is enough to conclude because ϵ was arbitrary. $\square$

As promised, we will now make the geometric argument more precise and "show" that such a decomposition in balls exists. We first need a couple of technical ingredients:

THEOREM 3.33. *Let μ be locally finite measures defined on a metric space X satisfying certain technical assumptions*[2]*, and let $u \in L^p_{\text{loc}}(X,\mu)$. Then*

$$\fint_{B(x,r)} |u(y)-u(x)|^p \, \mathrm{d}\mu(y) \xrightarrow{r\to 0^+} 0, \tag{3.3}$$

*for μ-a.e. $x \in X$. In other words, the function u is L^p-**approximately continuous** in X.*

COROLLARY 3.34. *The gradient of the function u_∞, given in Theorem 3.32, is L^p-approximately continuous for all $1 \le p < \infty$. In other words,*

$$|\nabla u_\infty(x) - \nabla u_\infty(\bar{x})| \le \epsilon \quad \text{for all } x \in B(\bar{x},r)\setminus E,$$

where $E = E(\epsilon)$ is called exceptional set *and satisfies $|E| \le \epsilon |B(\bar{x},r)|$.*

THEOREM 3.35. *Let $\Omega \subset \mathbb{R}^n$ and $p > n$. Any function $u \in W^{1,p} \cap C^0(\Omega,\mathbb{R}^m)$ is differentiable at a.e. $x \in \Omega$.*

PROOF. Fix a ball $B := B(\bar{x},r)$ and notice that by the generalised Poincaré inequality

$$\sup_{x\in B} |u(x)-u(\bar{x})| \le c \left(\fint_B |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

It is not hard to verify (use, for example, a dimensional argument) that c is a constant that depends linearly on r. It follows that

$$\sup_{x\in B} |u(x)-u(\bar{x})| \le c' r \left(\fint_B |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

for some universal constant c'. We can subtract any linear function to u, so we choose $\nabla u(\bar{x})$ and assume that $\bar{x}$ is a point of L^p-approximate continuity. It turns out that

$$\sup_{x\in B} \|u(x)-u(\bar{x}) - \nabla u(\bar{x})(x-\bar{x})\| \le c' r \left(\fint_B |\nabla u(x) - \nabla u(\bar{x})|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

which concludes the proof because the right-hand side is $\mathcal{O}(r)$ using (3.3) together with the fact that a.e. point of Ω is a density point for Ω. ☐

COROLLARY 3.36. *The function u_∞ in Theorem 3.32 is differentiable at a.e. $x \in \Omega$.*

We can now go back to the existence of the balls for u_∞. Fix $\epsilon > 0$ and use the results above to infer the existence of some $r(\epsilon) > 0$ such that for all $r \le r(\epsilon)$ we have

$$|u_\infty(x) - u_\infty(\bar{x}) - \nabla u_\infty(\bar{x})(x-\bar{x})| \le \epsilon |x-\bar{x}| \le \epsilon r \quad \text{for every } x \in B(\bar{x},r),$$

and, by approximate continuity, also

$$|\nabla u(x) - \nabla u(\bar{x})| \le \epsilon \quad \text{for every } x \in B(\bar{x},r)\setminus E,$$

where E is a set depending on ϵ satisfying $|E| \le \epsilon |B(\bar{x},r)|$. Let $\mathcal{F}$ be the family of such balls for which, given any $x \in \Omega$, one can find $B(\bar{x},r_n) =: B_n \in \mathcal{F}$ with $r_n \to 0^+$.

[2] In this point, we are purposely vague because we do not need this result in its full generality. The interested reader might find more information in any standard GMT book.

LEMMA 3.37. *Let X be a metric space, let $\mathcal{F} := \{B(x_i, r_i)\}_{i\in I}$ be a family of closed balls with uniformly bounded radii that covers a set $E \subseteq X$. Then there exists a disjoint subfamily $\mathcal{F}'$ such that the rescaling*

$$\widehat{\mathcal{F}}' := \{\widehat{B}(x_i, r_i) \mid B(x_i, r_i) \in \mathcal{F}'\}$$

is a covering of E.

This result is well-known in GMT and usually referred to as *Vitali's covering lemma*. A simple consequence of this result is that from our family $\mathcal{F}$ we can extract finitely many balls in such a way that

$$|\Omega \setminus \bigcup_{i=1}^{\kappa} B_i| \leq \epsilon$$

and the required properties are satisfied.

4. A few remarks on Theorem 3.32

In Theorem 3.32, we proved the equivalence between quasiconvexity of the Lagrangian f and weak$\star$ lower semicontinuity on $W^{1,\infty}(\Omega, \mathbb{R}^m)$ of the corresponding functional

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x.$$

At this point, it makes sense to ask what happens when f does not depend only on ξ. More precisely, we now consider a Borel function

$$f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0, \infty]$$

such that $f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$ and $f(x, u, \cdot)$ is quasiconvex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^m$. To investigate the weak$\star$ lower semicontinuity of

$$F(u) := \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x$$

on $W^{1,\infty}(\Omega)$, Young measures play a fundamental role but, as expected, there are a few details that needs to be discussed.

REMARK 3.38. If g is a convex function, then the inequality

$$\fint g(\xi + y)\,\mathrm{d}\mu(y) \geq g(\xi)$$

holds for all μ probability measures with zero barycenter, that is,

$$\int y\,\mathrm{d}\mu(y) = 0.$$

This means that a quasiconvex function does not satisfy Jensen's inequality with any probability measure μ, and thus we need to obtain additional information on Young measures.

The result that follows asserts that Young measures generated by gradients belong to a particular class of which has, among several others properties, barycenter zero.

PROPOSITION 3.39. *Let u_n be a sequence of uniformly bounded functions in $W^{1,p}(\Omega,\mathbb{R}^m)$ and let $(\mu_x)_{x\in\Omega}$ be the Young (probability) measure on $\mathbb{R}^{m\times n}$ generated by $(\nabla u_n)_{n\in\mathbb{N}}$. Then*

$$\mu_x \in \left\{\tau_\xi(\nabla v)_{\#}\mathcal{L} \,:\, v\in W_0^{1,\infty}(B,\mathbb{R}^m)\right\}$$

for a.e. $x\in\Omega$, where τ_ξ is the translation operator on $\mathbb{R}^{m\times n}$ and $\mathcal{L}$ is the renormalized Lebesgue measure on the ball B.

REMARK 3.40. Since $v\big|_{\partial B}\equiv 0$, it is easy to verify that

$$\int_B \nabla v\,\mathrm{d}x = 0,$$

which means that Young measures generated by sequences of the type $(\nabla u_n)_{n\in\mathbb{N}}$ always have barycenter equal to zero.

REMARK 3.41. The main consequence of Proposition 3.39 is that a quasiconvex function satisfy a Jensen-type inequality. Namely, we have

$$\int_B f(\xi+M)\,\mathrm{d}(\nabla v)_{\#}\mathcal{L}(x) \geq f(\xi)$$

for all $v\in W_0^{1,\infty}(B,\mathbb{R}^m)$.

CHAPTER 4

Relaxation

I will write the introduction after the chapter is complete!

1. Introduction

Let $\mathfrak{X}$ be a topological space and let $F : \mathfrak{X} \to [-\infty, \infty]$ be a functional which is **not** necessarily lower semicontinuous.

DEFINITION 4.1. The *relaxation* of F on $\mathfrak{X}$, denoted by $\bar{F}$, is the lower semicontinuous envelope of F, that is,

$$\bar{F}(u) := \liminf_{v \to u} F(v) = \inf \left\{ \liminf_{n \to \infty} F(v_n) \mid v_n \to_{\mathfrak{X}} u \right\}$$

REMARK 4.2. Notice that $\bar{F}$ is the largest lower semicontinuous functional on $\mathfrak{X}$ (by definition) which is $\leq F$ (using the constant sequence $u_n \equiv u$).

PROPOSITION 4.3. *Let F be a coercive functional on $\mathfrak{X}$. Then $\bar{F}$ is coercive.*

PROOF. Recall that F coercive is equivalent to saying that the sublevel $\{u \in \mathfrak{X} : F(u) \leq M\}$ is relatively compact in $\mathfrak{X}$ for all $M \in \mathbb{R}$. Since

$$\{u \in \mathfrak{X} : \bar{F}(u) \leq M\} \subset \overline{\{u \in \mathfrak{X} : F(u) \leq M\}},$$

it is easy to conclude that the sublevels of $\bar{F}$ are relatively compact in $\mathfrak{X}$, and hence $\bar{F}$ is coercive. $\square$

REMARK 4.4. The relaxation $\bar{F}$ is lower semicontinuous by definition. Therefore, if F is coercive, then $\bar{F}$ admits a minimizer. Namely, if u_n is a minimizing sequence for F,

$$F(u_n) \xrightarrow{n \to \infty} \inf_{v \in \mathfrak{X}} F(v),$$

then one can show that u_n converges (up to subsequences) to some $u \in \mathfrak{X}$ and

$$\bar{F}(u) = \min_{v \in \mathfrak{X}} \bar{F}(v) = \inf_{v \in \mathfrak{X}} F(v).$$

PROOF. Let u_n be a minimizing sequence for F and let $u \in \mathfrak{X}$ be the limit of the converging subsequence. By lower semicontinuity and $\bar{F} \leq F$ we have

$$\bar{F}(u) \leq \lim_{n \to \infty} F(u_{n_k}) = \inf_{v \in \mathfrak{X}} F(v),$$

while the opposite inequality follows from the definition of $\bar{F}$. $\square$

Before we start developing the theory of relaxation even further, a couple of comments/properties are in order:

(1) Minimizers of $\bar{F}$ "track" the behavior of minimizing sequences for F.

(2) The functional F has a minimizer if and only if there exists $u \in \mathfrak{X}$ minimizer of $\bar{F}$ such that
$$\bar{F}(u) = F(u).$$

Note that the property (2) is, in practice, more useful to prove that **there are no** minimizers for a specific functional F rather than the opposite.

LEMMA 4.5. *If g is continuous, then the relaxation of $F + g$ is equal to $\bar{F} + g$. In other words, the following identity holds:*
$$\overline{F + g} = \bar{F} + g.$$

At this point, one might wonder how to determine whether or not a specific functional G is the relaxation of F. The following results answer this question.

PROPOSITION 4.6. *Let F be as above. A functional G is the relaxation of F (that is, $G \equiv \bar{F}$) if the following two properties hold:*

(i) G is lower semicontinuous on $\mathfrak{X}$ and $G(u) \leq F(u)$ for all $u \in \mathfrak{X}$.

(ii) For all $u \in \mathfrak{X}$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathfrak{X}$ converging to u such that
$$G(u) = \lim_{n\to\infty} F(u_n).$$

The condition (ii) can be replaced by an seemingly weaker one which is, in practice, often much easier to verify. First, we need to introduce the notion of *dense in energy*:

DEFINITION 4.7. Let $G : \mathfrak{X} \to [-\infty, \infty]$ be a functional. A set $\mathcal{D} \subset \mathfrak{X}$ is G-dense (or dense in energy for G) if for all $u \in \mathfrak{X}$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that
$$u_n \xrightarrow{n\to\infty} u \text{ in } \mathfrak{X} \quad \text{and} \quad G(u) \lim_{n\to\infty} G(u_n).$$

We can now give the weaker characterization of the relaxation $\bar{F}$ replacing $\mathfrak{X}$ in (ii) with a smaller set $\mathcal{D}$ which might be easier to deal with.

PROPOSITION 4.8. *Let F be as above. A functional G is the relaxation of F (that is, $G \equiv \bar{F}$) if the following two properties hold:*

(i) G is lower semicontinuous on $\mathfrak{X}$ and $G(u) \leq F(u)$ for all $u \in \mathfrak{X}$.

(ii) For all $u \in \mathcal{D}$, where $\mathcal{D}$ is G-dense, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathfrak{X}$ converging to u such that
$$G(u) = \lim_{n\to\infty} F(u_n).$$

REMARK 4.9. Let $\mathfrak{X}$ be a standard Sobolev space. It often happens that F is continuous in the strong topology and lower semicontinuous in the weak one. In this case, it can be proved that $\mathcal{D} = C^\infty$ is dense in energy for F.

2. Examples of relaxation theorems

In this section, we investigate the relaxations of a few functionals which are of common occurrence such as the Dirichlet energy. We start with a general result:

THEOREM 4.10. *Let $\Omega \subset \mathbb{R}^n$ and consider the functional*

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x,$$

where $u : \Omega \to \mathbb{R}$ and $f : \mathbb{R}^n \to [0,\infty]$ is a lower semicontinuous function. Suppose that F is coercive. Then the relaxation with respect to the $W^{1,p}(\Omega)$-weak topology is given by

$$\bar{F}(u) = \int_\Omega g(\nabla u)\,\mathrm{d}x, \tag{4.1}$$

where g is the convex envelope of f.

REMARK 4.11. The coercivity of F can be obtained under suitable growth assumptions on the Lagrangian f because the weak topology on $W^{1,p}(\Omega)$ is metrizable so

$$F \text{ coercive} \iff \lim_{|u|\to\infty} F(u) = \infty.$$

REMARK 4.12. Guessing what functional could be the relaxation of F is much easier if one knows that $\bar{F}$ has to be of the form

$$\bar{F}(u) = \int_\Omega g(\nabla u)\,\mathrm{d}x.$$

However, this is not easy at all to prove and it requires some integral representation theory.

In the remainder of the section, we will investigate the relaxations of some specific functionals without relying on the result above. However, we first try to motivate the importance of computing explicitly the relaxation through a simple example.

REMARK 4.13. We proved that the minimum of the problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x \;:\; u\,\big|_{\partial\Omega} \equiv u_0\right\}$$

is a function u that satisfies the Dirichlet's boundary condition

$$\begin{cases} -\Delta u = 0 & \text{if } x \in \Omega,\\ u = u_0 & \text{if } x \in \partial\Omega. \end{cases}$$

Notice that we tacitly assumed that u was smooth enough to integrate by parts twice taking the second derivative. However, how can we be sure that such a u exists?

The idea is to extend $F(u)$ to $W^{1,2}_{u_0}(\Omega)$, prove the existence there and finally show the regularity of the minimizer(s) to obtain a smooth solution.

We will describe this procedure (usually referred to as *direct method*) in the most general case, explain what the strategy is and what are the problems.

2.1. Direct Method. Let F be a functional defined on some space $\mathfrak{X}'$ of "regular" functions and suppose that $\mathfrak{X}'$ has no good compactness properties.

STEP 1. The functional F is not coercive. Find $\mathfrak{X} \supset \mathfrak{X}'$ such that the extension of F, denoted by F_{ext}, is coercive and lower semicontinuous on $\mathfrak{X}$.

STEP 2. Find a function $\bar{u} \in \mathfrak{X}$ that solves the minimizes the functional F_{ext}.

STEP 3. Use regularity theory to conclude that $\bar{u} \in \mathfrak{X}'$ so that

$$F(\bar{u}) = \min_{v \in \mathfrak{X}'} F(v).$$

The procedure looks simple, but there are a couple of issues regarding the last step that we need to consider (at least in the general case). More precisely:

(1) Regularity theory is, in general, quite hard.

(2) Regularity might fail. In other words, the minimizer $\bar{u}$ of F_{ext} may not be an element of the starting space $\mathfrak{X}'$.

The second issue is quite delicate to deal with because, if $\bar{u}$ does not belong to $\mathfrak{X}'$, then it might not even be a meaningful solution to our problem.

2.2. Relaxation. Let $\mathfrak{X}' \subset \mathfrak{X}$ be two functional spaces and F_{ext} the extension that is coercive and lower semicontinuous on $\mathfrak{X}$. If u_n is a minimizing sequence for F and $\bar{u} \in \mathfrak{X}$ a minimum of F_{ext}, that is,

$$F_{\text{ext}}(\bar{u}) = \min_{u \in \mathfrak{X}} F_{\text{ext}}(u),$$

then we expect u_n to converge to $\bar{u}$. Therefore, it makes sense to wonder whether or not F_{ext} and the relaxation $\bar{F}$ of F are in some ways connected to each other.

DEFINITION 4.14. Let $F : \mathfrak{X}' \subset \mathfrak{X} \to [-\infty, \infty]$ be a functional. The *relaxation* of F on $\mathfrak{X}$ is defined as the relaxation possibly extended to ∞. More precisely, we have

$$\bar{F}(u) = \begin{cases} \inf \left\{ \liminf_{n \to \infty} F(v_n) \mid (v_n) \subset \mathfrak{X}' \text{ s.t. } v_n \to u \right\} & \text{if } u \in \overline{\mathfrak{X}'}, \\ \infty & \text{if } u \notin \overline{\mathfrak{X}'}. \end{cases}$$

REMARK 4.15. The functional $\bar{F}$ is the largest lower semicontinuous functional on $\mathfrak{X}$ that satisfies the inequality $\bar{F} \le F$ on $\mathfrak{X}'$.

EXAMPLE 4.16. *Let $\Omega \subset \mathbb{R}^n$ and consider the functional*

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega g(x, u) \, \mathrm{d}x,$$

where $g : \Omega \times \mathbb{R} \to [0, \infty]$ is a Borel function such that $g(x, \cdot)$ is continuous for a.e. $x \in \Omega$ and satisfies the estimate

$$|g(x, u)| \le C(1 + |u|^2).$$

Let $\mathfrak{X}' := C^\infty(\bar{\Omega})$. Then the relaxation of F on $\mathfrak{X} := W^{1,2}(\Omega)$ endowed with the weak topology is

$$\bar{F}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega g(x, u) \, \mathrm{d}x.$$

HINT OF THE PROOF. It is enough to show that $\mathfrak{X}'$ is dense in energy in $W^{1,2}(\Omega)$. Indeed, the functional F is continuous with respect to the strong convergence and

$$W^{1,2}(\Omega) = \overline{C^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}.$$

$\square$

REMARK 4.17. What happens if $g(x,\cdot)$ is lower semicontinuous? The proof above fails, but, if $\mathfrak{X}' = C^1(\bar\Omega)$, we could use a Lusin-type theorem (for Sobolev spaces) and obtain the same conclusion.

EXAMPLE 4.18. *Let $\mathfrak{X}' := C^\infty_{u_0}(\bar\Omega)$, where u_0 is a smooth function, and suppose that $\partial\Omega$ is smooth. The relaxation of the functional*

$$F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

on $\mathfrak{X} := W^{1,2}_{u_0}(\Omega)$ is given by

$$G(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

PROOF. Since G is lower semicontinuous on $W^{1,2}_{u_0}(\Omega)$ with respect to the weak topology and $G(v) = F(v)$ for $v \in \mathfrak{X}'$, it is sufficient to show that for all $u \in W^{1,2}_{u_0}(\Omega)$ there exists a smooth sequence $(u_k)_{k\in\mathbb{N}} \subset \mathfrak{X}'$ such that

$$u_k \rightharpoonup u \text{ in } W^{1,2}_{u_0}(\Omega) \quad \text{and} \quad F(u_k) \to G(u).$$

Since $F(u_k) = G(u_k)$ and G is continuous with respect to the strong topology, it is enough to find a sequence such that $u_k \to u$ strongly in $W^{1,2}(\Omega)$. In other words, we need to prove that $\mathfrak{X}'$ is dense in energy in $W^{1,2}_{u_0}(\Omega)$, but this is hard to do directly so we introduce an auxiliary space which is easier to deal with, namely

$$\mathfrak{X}'' := \{u \in W^{1,2}(\Omega) \,:\, u(x) = u_0(x) \text{ for all } x \text{ in a neighborhood of } \partial\Omega\}.$$

Step 1. We claim that $\mathfrak{X}'$ is dense in energy in $\mathfrak{X}''$. Let $u \in \mathfrak{X}''$, extend[1] it to $W^{1,2}(\mathbb{R}^n)$, fix $\epsilon > 0$ and take an approximating sequence $(v_\epsilon)_{\epsilon>0} \subset C^\infty_c(\mathbb{R}^n)$ such that

$$\|u - v_\epsilon\|_{W^{1,2}(\mathbb{R}^n)} \xrightarrow{\epsilon\to\infty} 0.$$

Let U_ϵ be a 2ϵ-neighbourhood of $\partial\Omega$. We can always find a smooth function

$$\lambda_\epsilon : \bar\Omega \to [0,1]$$

such that $\lambda_\epsilon \equiv 1$ on a ϵ-neighborhood of $\partial\Omega$, $\lambda_\epsilon \equiv 0$ outside of U_ϵ and $|\nabla\lambda_\epsilon| \le \epsilon^{-1}$. The function defined by setting

$$u_\epsilon(x) := \lambda_\epsilon(x)u_0(x) + (1-\lambda_\epsilon(x))v_\epsilon(x),$$

[1]Here you can use any extension operator defined on $\mathfrak{X}'$ as it does not really matter for the proof.

is smooth and, actually, belongs to $\mathfrak{X}'$ for all $\epsilon > 0$. We wish to estimate the L^2-norm of $u_\epsilon - u$ so we start off by noticing that

$$u_\epsilon - u = \lambda_\epsilon(u_0 - u) + (1 - \lambda_\epsilon)(v_\epsilon - u).$$

The L^2-norm of the first addendum is easy to estimate since

$$\|\lambda_\epsilon(u_0 - u)\|^2_{L^2(\Omega)} = \int_\Omega \lambda_\epsilon^2 (u_0 - u)^2 \,\mathrm{d}x \le \int_{U_\epsilon} |u_0 - u|^2 \,\mathrm{d}x \xrightarrow{\epsilon \to 0^+} 0$$

using the dominated convergence theorem as $|u_0 - u|^2$ is integrable and U_ϵ tends to the empty set. The second addendum is even easier because

$$\|(1 - \lambda_\epsilon)(v_\epsilon - u)\|_{L^2(\Omega)} \le \|v_\epsilon - u\|_{L^2(\Omega)},$$

and the right-hand side goes to zero by construction of v_ϵ. In particular, u_ϵ converges strongly to u in the L^2 strong topology. Now notice that

$$\nabla(u_\epsilon - u) = \nabla(u_0 - u)\lambda_\epsilon + (1 - \lambda_\epsilon)\nabla(v_\epsilon - u) + \nabla\lambda_\epsilon((u_0 - u) + (u - v_\epsilon)),$$

and the first two terms can be estimated with no problems as above. The third addendum can also be rewritten as follows

$$\nabla\lambda_\epsilon((u_0 - u) + (u - v_\epsilon)) = \nabla\lambda_\epsilon(u - v_\epsilon)$$

because u coincides with u_0 on a neighborhood of $\partial\Omega$. It follows that

$$\|\nabla\lambda_\epsilon(u - v_\epsilon)\|_{L^2(\Omega)} \le \|\nabla\lambda_\epsilon\|_\infty \|v_\epsilon - u\|_{L^2(\Omega)} \le \epsilon^{-1}\|v_\epsilon - u\|_{L^2(\Omega)}$$

and, since we can decide how fast v_ϵ tends to u in $L^2(\Omega)$, we infer that $u_\epsilon \in \mathfrak{X}'$ tends to $u \in \mathfrak{X}''$ strongly in $W^{1,2}(\Omega)$.

Step 2. We claim that $\mathfrak{X}''$ is dense in energy in $W^{1,2}_{u_0}(\Omega)$. Let $u \in W^{1,2}_{u_0}(\Omega)$. The goal is to find a sequence $u_\epsilon \in \mathfrak{X}''$ that converges to u strongly in $W^{1,2}(\Omega)$.

Step 2.1. Assume that, given $x \in \partial\Omega$ and η the external unit normal, it turns out that

$$u_0(x + t\eta(x)) = u_0(x) \quad \text{for } |t| \le \delta \text{ small enough.}$$

Let Ω_δ be the δ-neighborhood of Ω given by

$$\Omega_\delta := \{x \in \mathbb{R}^n \,:\, d(x, \Omega) < \delta\},$$

and extend u to Ω_δ by setting it identically equal to u_0 outside of Ω so that[2] the result is still an element of $W^{1,2}(\Omega_\delta)$.

Step 2.2. Take $\Phi_\epsilon : \Omega \to \Omega_\epsilon$ family of diffeomorphisms (we give the existence for granted momentarily) satisfying the following three properties:

(1) For all $\epsilon > 0$, $|\Phi_\epsilon(x) - x| = \mathcal{O}(\epsilon)$.

(2) For all $\epsilon > 0$, $|\nabla\Phi_\epsilon(x) - \mathrm{Id}| = \mathcal{O}(\epsilon)$.

(3) For $x \in \partial\Omega$ and $0 \le t \le \epsilon$, $\Phi(x - t\eta(x)) = x + (\epsilon - t)\eta(x)$.

[2]If we extend u with trace u_0 out of Ω with a function having u_0 as trace, then the extension stays in the Sobolev space.

The conclusion now follows easily because we can take the sequence $u_\epsilon := u \circ \Phi_\epsilon \in \mathfrak{X}''$ and show that it converges strongly in $W^{1,2}(\Omega)$ to $u \in W^{1,2}_{u_0}(\Omega)$.

Step 2.3. The existence of the family of diffeomorphisms Φ_ϵ is a consequence of the *tubular neighborhood theorem* using that

$$\Omega \times [-\delta, \delta] \ni (x,t) \longmapsto x + t\eta(x) \in U_\delta(\partial\Omega),$$

for $\delta > 0$ sufficiently small, is a diffeomorphism. □

EXAMPLE 4.19. *Consider the Dirichlet energy functional*

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

defined on the space of smooth functions with mixed boundary conditions, namely

$$\mathfrak{X}' = \left\{ u \in C^\infty(\bar{\Omega}) \;:\; u\big|_{\partial\Omega} \equiv u_0,\, \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} \equiv 0 \right\}.$$

Then the Neumann boundary condition does **not** *make sense on the Sobolev space* $W^{1,2}(\Omega)$ *because* $\frac{\partial u}{\partial \eta} \in L^2(\Omega)$ *has no trace on* $\partial\Omega$*. Nonetheless, the relaxation of* F *on* $W^{1,2}_{u_0}(\Omega)$ *is*

$$\bar{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

This means that, if $\bar{u}$ *is a minimizer of* $\bar{F}$ *on* $W^{1,2}_{u_0}(\Omega)$ *there are two possibilities:*

(a) Either $\frac{\partial \bar{u}}{\partial \eta}\Big|_{\partial\Omega} \equiv 0$ *and* $\bar{u}$ *is a minimizer for the original problem; or*

(b) $\frac{\partial \bar{u}}{\partial \eta}\Big|_{\partial\Omega} \not\equiv 0$ *and the original problem does not admit any minimizer.*

PROOF. The same proof given above works because of the construction along the normal unit vector on the boundary. □

EXAMPLE 4.20. *Let* $\Omega \subseteq \mathbb{R}^n$*,* $n \geq 2$*, and consider the Dirichlet energy functional* F *defined on the space*

$$\mathfrak{X}' = \{ u \in C^\infty(\bar{\Omega}) \;:\; u(x_0) = 0,\, u\big|_{\partial\Omega} \equiv u_0 \},$$

where $x_0 \in \Omega$ *is fixed. Then the relaxation of* F *on* $W^{1,2}_{u_0}(\Omega)$ *is given by*

$$\bar{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

REMARK 4.21. The main issue here is that the Sobolev space $W^{1,2}(\Omega)$ does not embed into $C(\Omega)$ for these values of n.

Before we can prove that the relaxation of F on the Sobolev space $W^{1,2}_{u_0}(\Omega)$ is the one given above, we need a technical result.

LEMMA 4.22. *Let* $n \geq 2$*. Then there exists a sequence* $v_k \in C^\infty_c(\mathbb{R}^n)$ *satisfying the following properties:*

(a) $v_k(x) \in [0,1]$ *for all* $x \in \mathbb{R}^n$*;*

(b) $v_k(0) = 1$*;*

(c) $\operatorname{supp}(v_k) \subset B(0, \frac{1}{k})$*, which means that* $\|v_k\|_{L^p(\mathbb{R}^n)} \to 0$ *for all* $p \geq 1$*;*

(d) $\|\nabla v_k\|_{L^2(\mathbb{R}^n)} \to 0$.

PROOF. We first assume that $n > 2$. Let $v : \mathbb{R}^n \to [0,1]$ be a C^∞ function supported in the ball $B(0,1)$ such that $v(0) = 1$, and set

$$v_k(x) := v(kx).$$

Since $\nabla v_k(x) = k\nabla v(kx)$ we have

$$\int_{\mathbb{R}^n} |\nabla v_k|^2 \,\mathrm{d}x = k^2 \int_{\mathbb{R}^n} |\nabla v(kx)|^2 \,\mathrm{d}x = Ck^{2-n} \to 0$$

because $2 - n < 0$. If $n = 2$ this proof does not work anymore and one needs a different construction using, for example, non self-similar radial functions. □

PROOF OF EXAMPLE 4.20. Let $\mathfrak{X}'' = C^\infty_{u_0}(\bar{\Omega})$. Since $\mathfrak{X}''$ is dense in energy in $W^{1,2}_{u_0}(\Omega)$, it suffices to show that $\mathfrak{X}'$ is dense in energy in $\mathfrak{X}''$. Take $u \in \mathfrak{X}''$ and set

$$u_k(x) := u(x) - u(x_0)v_k(x - x_0),$$

where v_k is the sequence given in Lemma 4.22. It is easy to check that $u_k(x_0) = 0$ and that $u_k \to u$ strongly in $W^{1,2}(\Omega)$. □

REMARK 4.23. The embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow C_0(\mathbb{R}^2)$ fails because

$$I : C^\infty_c(\mathbb{R}^2) \to C_0(\mathbb{R}^2)$$

is **not** bounded, for otherwise it could be extended to $W^{1,2}(\mathbb{R}^2)$. The counterexample here gives a good sequence of functions to prove Lemma 4.22 when $n = 2$.

EXAMPLE 4.24. *Let* $\Omega \subset \mathbb{R}^n$ *and consider the* p*-Dirichlet energy*

$$F(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

for all $u \in \mathfrak{X}'$*, where*

$$\mathfrak{X}' = \{u \in C^\infty(\bar{\Omega}) \ : \ u\big|_{\partial\Omega} \equiv u_0,\ u(x_0) = 0\}$$

and u_0 *is a smooth function defined on* $\partial\Omega$*. The relaxation of* F *on* $W^{1,p}_{u_0}(\Omega)$ *is*

$$\bar{F}(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

if $p \leq n$*, and*

$$\bar{F}(u) = \begin{cases} \int_\Omega |\nabla u|^p \,\mathrm{d}x & \text{if } u \in W^{1,p}_{u_0}(\Omega) \text{ and } u(x_0) = 0, \\ \infty & \text{otherwise}, \end{cases}$$

if $p > n$. The reason is that the condition $u(x_0) = 0$ survives under relaxation and this can be shown using the continuous representative.

EXAMPLE 4.25. *If $\mathfrak{X}' = C^\infty_{u_0}(\bar{\Omega})$ and $F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$, then the relaxation of F on $W^{1,2}(\Omega)$ is given by*

$$\bar{F}(u) = \begin{cases} \int_\Omega |\nabla u|^2 \,\mathrm{d}x & \textit{if } u \in W^{1,2}_{u_0}(\Omega) \textit{ and } u \equiv u_0 \textit{ on } \partial\Omega, \\ \infty & \textit{otherwise}. \end{cases}$$

This is, once again, due to the continuity of the trace operator that survives under relaxation.

3. Relaxation and p-capacity

Let F be the Dirichlet energy,

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x,$$

and consider the functional space $\mathfrak{X}' := \{C^\infty_{u_0}(\bar{\Omega}) \;:\; u(x_0) = 0\}$ for some $x_0 \in \Omega$. We proved that the relaxation on $W^{1,2}_{u_0}(\Omega)$ endowed with the weak topology is

$$\bar{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

However, the condition $u(x_0) = 0$ is not preserved under relaxation, and therefore a minimizer $\bar{u}$ of $\bar{F}$ is also a minimizer of F if and only if it satisfies $\bar{u}(x_0) = 0$.

3.1. p-Capacity.

Let F be as above and consider the functional space

$$\mathfrak{X}' := \{C^\infty_{u_0}(\bar{\Omega}) \;:\; u\big|_K \equiv 0\},$$

where $K \subset \Omega$ is a compact set. At this point, it makes sense to wonder for which K the relaxation of F on $W^{1,2}_{u_0}(\Omega)$ is (once again) given by

$$\bar{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Let $1 < p < \infty$. To deal with this question, we first need to introduce the definition of p-capacity.

DEFINITION 4.26. The *p-capacity* of a subset $A \subseteq \mathbb{R}^n$ is defined as

$$\mathrm{Cap}_p(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p \,\mathrm{d}x \;:\; u \in C^\infty_c(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on some } U \in \mathcal{N}(A)\right\},$$

where $\mathcal{N}(A)$ denotes the set of all open neighborhoods of A.

The following theorem asserts that the value of the p-capacity of K determines how the relaxation of the Dirichlet energy looks like.

THEOREM 4.27. *Let F be the p-Dirichlet energy,*

$$F(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x,$$

defined on $\mathfrak{X}' := \{C^\infty_{u_0}(\bar\Omega) \,:\, u\big|_K \equiv 0\}$,, and let $\bar F$ be the relaxation on $W^{1,p}_{u_0}(\Omega)$. Then the following assertions hold:

(i) *If $\mathrm{Cap}_p(K) = 0$, then*

$$\bar F(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x.$$

(ii) *Let us consider the functional space*

$$\bar{\mathfrak{X}} := \{u \in W^{1,p}_{u_0}(\Omega) \,:\, \mathrm{Cap}_p(\{x \in K \,:\, u(x) \neq 0\}) = 0\},$$

where u is the representative which is approximately p-continuous at every $x \in \Omega$ except in a set of null p-capacity. Then

$$\bar F(u) = \begin{cases} \int_\Omega |\nabla u|^p \, \mathrm{d}x & \textit{if } u \in \bar{\mathfrak{X}}, \\ \infty & \textit{otherwise}. \end{cases}$$

REMARK 4.28. Let $u \in W^{1,p}_{u_0}(\Omega)$ be a Sobolev function. If we set

$$\widetilde{u}(x) := \lim_{r \to 0^+} \fint_{B(x,r)} u(y) \, \mathrm{d}y$$

then the Lebesgue theorem asserts that $\widetilde{u}$ is p-continuous at every $x \in \Omega$ except in a set of null p-capacity. It is usually referred to as *precise representative* of u.

PROOF OF (I). We must prove that for each $u \in W^{1,p}_{u_0}(\Omega)$ there exists a sequence of smooth functions $u_k \in C^\infty_c(\bar\Omega)$ satisfying the constraint $u_k\big|_K \equiv 0$ such that

$$u_k \rightharpoonup u \text{ (weakly) in } W^{1,p}(\Omega) \quad \text{and} \quad F(u_k) \xrightarrow{k \to \infty} F(u).$$

Step 1. The space $C^\infty_{u_0}(\bar\Omega)$ is dense in energy for F in $W^{1,p}_{u_0}(\Omega)$ because it is strongly dense and F is strongly continuous. Therefore, it suffices to consider $u \in C^\infty_{u_0}(\bar\Omega)$.

Step 2. Since K has p-capacity equal to zero, we can find a sequence of smooth functions $v_k \in C^\infty_c(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |\nabla v_k|^p \, \mathrm{d}x \xrightarrow{k \to \infty} 0$$

and $v_k \geq 1$ on an open neighborhood of K. We would like to consider

$$u_k(x) := u(x)(1 - v_k(x)),$$

but we first need to modify v_k in such a way that it satisfies the following two properties:

(a) v_k is identically equal to 1 on K;

(b) v_k has compact support contained in Ω.

Assume for the time being that we can modify v_k to satisfy these properties. Then

$$u_k(x) = u(x)(1 - v_k(x)) = 0 \quad \text{for all } x \in K$$

and

$$\|u - u_k\|_{L^p(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \|v_k\|_{L^p(\Omega)}.$$

Since v_k has compact support in Ω, we know that there exists a dimensional constant $c > 0$ such that the following holds:

$$\|v_k\|_{L^p(\Omega)} \leq c\|\nabla v_k\|_{L^p(\Omega)}.$$

We plug this inequality into the one above and find that

$$\|u - u_k\|_{L^p(\Omega)} \leq c\|u\|_{L^\infty(\Omega)} \|\nabla v_k\|_{L^p(\Omega)} \xrightarrow{k\to\infty} 0,$$

which means that u_k converges strongly to u in $L^p(\Omega)$. In a similar fashion, we have

$$\|\nabla u - \nabla u_k\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)} \|v_k\|_{L^p(\Omega)} + \|\nabla v_k\|_{L^p(\Omega)} \|u\|_{L^\infty(\Omega)} \xrightarrow{n\to\infty} 0,$$

and this concludes the proof of the theorem.

Step 3. To modify v_k and achieve the property (b), namely that v_k is compactly supported in Ω, it suffices to replace v_k with $v_k \cdot \Theta$, where Θ is a smooth cutoff function.

To obtain the property (a) we can either truncate (and show that $C_c^\infty(\mathbb{R}^n)$ in the definition of p-capacity is too much), or compose v_k with a suitable function. □

We now list a few properties of p-capacity (without proofs), which will be quite useful in the investigation of the relaxation of specific functionals.

REMARK 4.29.

(a) In the definition of p-capacity, the space $C_c^\infty(\mathbb{R}^n)$ can be replaced by Lipschitz functions in $\mathrm{Lip}_c(\mathbb{R}^n)$. In this case, we can assume u to be identically equal to one on a neighborhood of A because truncation works in the Lipschitz class.

(b) If K is a compact set, the capacity is also given by

$$\mathrm{Cap}_p(K) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p \,\mathrm{d}x \ : \ u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } K \right\}.$$

PROPOSITION 4.30. *Let $1 < p < \infty$. The capacity $\mathrm{Cap}_p(\cdot)$ is an outer measure.*

REMARK 4.31. If $p = 1$, then the p-capacity coincides with the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$.

REMARK 4.32. If $p > n$, then $\mathrm{Cap}_p(A) > 0$ for every nonempty set A. In other words, the p-capacity of points is positive, that is,

$$\mathrm{Cap}_p(\{x\}) > 0.$$

This is strictly related to the fact that for these values of p we have the embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n).$$

PROPOSITION 4.33. *Let $1 < p \leq n$. Then there exists a constant $c > 0$ such that*

$$\mathrm{Cap}_p(B(x,r)) = cr^{n-p}.$$

PROOF. The scaling with r is easy to verify, while the existence of the constant c is due to the fact that sets of positive measure cannot have p-capacity zero when $n - p \geq 0$. $\square$

PROPOSITION 4.34. *Let $1 < p \leq n$. If $\mathcal{H}^{n-p}(A) = 0$, then $\mathrm{Cap}_p(A) = 0$.*

PROOF. Fix $\epsilon > 0$. Since $\mathcal{H}^{n-p}(A) = 0$ we can find a family of balls B_i such that

$$A \subseteq \bigcup_{i \in \mathbb{N}} B_i \quad \text{and} \quad \sum_{i \in \mathbb{N}} r_i^{n-p} \leq \epsilon.$$

The p-capacity is subadditive (because it is an outer measure) so

$$\mathrm{Cap}_p(A) \leq \sum_{i \in \mathbb{N}} \mathrm{Cap}(B_i) = c \sum_{i \in \mathbb{N}} r_i^{n-p} \leq c\epsilon,$$

and we conclude using the arbitrariness of ϵ. $\square$

EXAMPLE 4.35 (Electrostatic potential). *A function $\bar{u}$ that achieves the infimum*

$$\mathrm{Cap}_2(K) = \inf \left\{ \int_\Omega |\nabla u|^2 \,\mathrm{d}x \;:\; u \in C_c^\infty(\mathbb{R}^n),\, u\big|_K \equiv 1 \right\}$$

is also a solution to the PDE

$$\begin{cases} \Delta \bar{u}(x) = 0 & \text{if } x \in \mathbb{R}^d \setminus K, \\ u(x) = 1 & \text{if all } x \in K, \\ \lim_{|x| \to \infty} \bar{u}(x) = 0. \end{cases}$$

Integrating by parts and using the equation satisfied by $\bar{u}$ yields

$$\mathrm{Cap}_2(K) = -\int_\Omega \bar{u} \Delta \bar{u} \,\mathrm{d}x = -\int_K \Delta \bar{u} \,\mathrm{d}x$$

because $\Delta \bar{u}$ vanishes outside of K and $\bar{u}$ is identically one on K.

4. Relaxation that does not coincide with the initial functional

Let us consider the functional

$$F(u) = \int_\Omega f(x, u, \nabla u) \,\mathrm{d}x,$$

where f is continuous in all three variables and such that $f(x, u, \cdot)$ is convex and consider the restriction on smooth functions $\mathfrak{X}' = C^\infty(\bar{\Omega})$. We would like to know if the relaxation $\bar{F}$ on the Sobolev space $W^{1,p}(\bar{\Omega})$ coincide with F or not.

REMARK 4.36. If $\bar{F}(u) = F(u)$ for $u \in \mathfrak{X}'$, studying minimizers in the Sobolev space gives important information on minimizing sequence in the space of smooth functions.

REMARK 4.37. If F is continuous with respect to the strong topology on $W^{1,p}(\Omega)$, then the answer is affirmative and one can prove that $\bar{F}$ coincide with F.

The next example, which is based on the pioneer work by **Lavrentiev**, gives an example of a functional for which the relaxation fails to coincide with F.

EXAMPLE 4.38 (???, Ball-James). *Let $p < \infty$ and consider the functional*

$$F(u) = \int_0^1 |2x^\alpha - u|^\beta |\dot{u}|^\gamma \,\mathrm{d}x.$$

Choose $\alpha < 1$ in such a way that $2x^\alpha \in W^{1,p}_{u_0}([0,1])$, where u_0 is a function satisfying the following boundary constraint: $u_0(0) = 0$ and $u_0(1) = 2$. Then

$$F(2x^\alpha) = \min_{u \in W^{1,p}_{u_0}([0,1])} F(u) = 0,$$

and it is the unique minimizer with such boundary values. However, it can be proved that

$$\min_{u \in W^{1,\infty}_{u_0}([0,1])} F(u) \geq c > 0$$

by choosing properly β and γ. This implies that the relaxation on $W^{1,p}_{u_0}([0,1])$ of the restriction to $W^{1,\infty}_{u_0}([0,1])$ is not equal to F.

REMARK 4.39. The minimizer of F on $W^{1,p}([0,1])$ is not regular, that is, it is not of class C^1.

We now show how to choose β and γ to prove the upper bound on the minimum of F among all functions in the class $W^{1,\infty}_{u_0}([0,1])$.

PROOF. Let $u \in W^{1,\infty}_{u_0}([0,1])$. Then there exists $0 < a < 1$ such that $u(x) \leq x^\alpha$ for all $x \in [0,a]$. It turns out that

$$F(u) \geq \int_0^a x^{\alpha\beta} |\dot{u}|^\gamma \,\mathrm{d}x.$$

Let $v(x^\delta) := u(x)$ for some $\delta > 0$ and notice that $\dot{u}(x) = \delta x^{\delta-1} \dot{v}(x^\delta)$. Then

$$F(u) \geq \int_0^a x^{\alpha\beta + (\delta-1)\gamma} \delta^\gamma |\dot{v}(x^\delta)|^\gamma \,\mathrm{d}x,$$

and the change of variables $t = x^\delta$ yields the inequality

$$F(u) \geq \int_0^{a^\delta} |\dot{v}(t)|^\gamma \mathrm{d}t \geq a^{\delta(1-\gamma)+\alpha\gamma},$$

provided that $\alpha\beta + (\delta - 1)\gamma = \delta - 1$, which can be rewritten as

$$\delta = \frac{\alpha\beta + 1 - \gamma}{1 - \gamma}.$$

Finally choose γ and β in such a way that the exponent of a does not allow the right-hand side $a^{\delta(1-\gamma)+\alpha\gamma}$ to go to zero as $a \to 0^+$. $\square$

5. Lavrentiev phenomenon

In the previous section, we provided an example of the so-called *Lavrentiev phenomenon*, which can be "defined" in the following way:

DEFINITION 4.40. Let F be a weakly lower semicontinuos functional on a Sobolev space $\mathfrak{X}$. We say that F exhibits the *Lavrentiev phenomenon* if the relaxation on $\mathfrak{X}$ of the restriction of F to a space $\mathfrak{X}'$ of "regular" functions does not coincide with F.

REMARK 4.41. If $\mathfrak{X}'$ is (strongly) dense and F is (strongly) continuous, then the relaxation of F coincide with F itself.

We will now show another (rare) example of the Lavrentiev phenomenon, which is particularly important because we consider vector-valued functions.

EXAMPLE 4.42. *Let $B := B(0,1)$ be the disk in $\mathbb{R}^2$ and take a function $\varphi : \mathbb{R}^2 \to [0,\infty)$ such that $\varphi(x) = 0$ if and only if $|x| = 1$. Let us consider the functional*

$$F(u) := \int_B \varphi(u)|\nabla u|^2 \,\mathrm{d}x,$$

where $u : B \to \mathbb{R}^2$. In addition, set $u_0(x) := x$ to be the constant function defined on the boundary ∂B.

THEOREM 4.43. *Let $p < 2$. The minimum of F over $W^{1,p}_{u_0}(B,\mathbb{R}^2)$ is*

$$\min_{u \in W^{1,p}_{u_0}(B,\mathbb{R}^2)} F(u) = 0,$$

and it is achieved by the function $\bar{u}(x) = \frac{x}{|x|}$. Furthermore, there exists a constant $c > 0$ independent of u such that

$$F(u) \geq c \quad \text{for every } u \in C^1(\bar{B},\mathbb{R}^2) \text{ such that } u\big|_{\partial B} \equiv u_0. \tag{4.2}$$

REMARK 4.44. In (4.2) we can replace C^1 functions with Lipschitz ones. More precisely, it can be proved that there exists a positive constant c independent of u such that

$$F(u) \geq c \quad \text{for every } u \in \mathrm{Lip}(\bar{B},\mathbb{R}^2) \text{ such that } u\big|_{\partial B} \equiv u_0.$$

PROOF. First, we use the simple algebraic inequality $a^2 + b^2 \geq 2ab$ to infer a similar estimate between $|\nabla u|^2$ and the determinant of the gradient; namely, we have

$$|\nabla u|^2 = \sum_{i,j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}\right)^2 \geq 2\det(\nabla u).$$

It follows that

$$F(u) \geq 2\int_B \varphi(u)|\det(\nabla u)| \,\mathrm{d}x \geq 2\int_{u(B)} \varphi(y) \,\mathrm{d}y$$

applying the change of variables formula with $y = u(x)$. Notice that we do not have the equality because u might not be injective and hence the multiplicity at each y be greater than one. A standard result in topology tells us that $B \subseteq u(B)$, and thus

$$F(u) \geq 2 \int_B \varphi(y)\,\mathrm{d}y =: c > 0.$$

It is easy to verify that, as claimed, the constant does not depend on u. □

To conclude the section, a few remarks and exercises concerning this result are in order.

REMARK 4.45.

(a) The condition $p < 2$ is necessary to ensure that the function $\bar{u}(x) = \frac{x}{|x|}$ actually belongs to $W^{1,p}_{u_0}(B)$.

EXERCISE 4.1. If $p < 2$, it is easy to see that $\nabla \bar{u} \in L^p(B)$. However, this only proves that $u \in W^{1,p}(B \setminus \{0\})$. How can one obtain $u \in W^{1,p}(B)$?

(b) The functional F is weakly lower semicontinuous on $W^{1,p}(B, \mathbb{R}^2)$. However, the relaxation of the restriction of F to smooth functions is not equal to F itself.

(c) If $\bar{F}$ is the aforementioned relaxation and c the constant in (4.2), one might wonder if it is true that the following equality holds:

$$\bar{F}\left(\frac{x}{|x|}\right) = c.$$

This is possible only if all the inequalities in the proof of Theorem 4.43 are, in fact, equalities.

EXERCISE 4.2. Verify whether or not $\bar{F}\left(\frac{x}{|x|}\right) = c$ holds. The second and third inequality are easy to fix, while the first one requires some care.

(d) As mentioned before, (4.2) is true even if we replace $C^1_{u_0}(\bar{B})$ with Lipschitz functions because the coarea formula used in the proof of Theorem 4.43 holds without any changes.

(e) In (4.2) we can even replace $C^1_{u_0}(\bar{B})$ with $W^{1,q}_{u_0}(B)$ with $q > 2$, but the coarea formula is not obtained as easily as above.

(f) The functional F is strongly continuous on $W^{1,p}_{u_0}(B)$ for $p > 2$, so it does not exhibit the Lavrentiev phenomenon. Not sure about $p = 2$?

(g) The functional F is not coercive on $W^{1,p}_{u_0}(B)$, but the perturbation

$$F_\epsilon(u) := F(u) + \epsilon \int |\nabla u|^p \,\mathrm{d}x$$

is coercive for every $\epsilon > 0$. Furthermore, if ϵ is small enough F_ϵ still exhibits the Lavrentiev phenomenon (and the non-regularity of minimizers).

CHAPTER 5

First Variation on Sobolev Spaces

I will write the introduction after the chapter is complete!

1. Introduction

Let $I \subset \mathbb{R}$ be a time interval and $\Omega \subset \mathbb{R}^n$ and let $g : I \times \Omega \to \mathbb{R}^m$ be a function satisfying the following properties:

(i) For every $x \in \Omega$, $g(\cdot, x) \in C^1(I)$;

(ii) there exists $t_0 \in I$ such that $g(t_0, \cdot) \in L^1(\Omega)$;

(iii) if $g_t(s, x)$ denotes the derivative with respect to t of g, then

$$\alpha(x) := \sup_{s \in I} |g_t(s, x)| \in L^1(\Omega).$$

PROPOSITION 5.1. *The function defined by setting*

$$G(t) := \int_\Omega g(t, x) \, \mathrm{d}x$$

is well-defined for every $t \in I$ and belongs to $C^1(I)$. Furthermore, we have

$$\dot{G}(s) = \int_\Omega g_t(s, x) \, \mathrm{d}x. \tag{5.1}$$

PROOF. Using the property (iii) we find that for all $t \in I$ we have

$$|g(t, x)| \leq |g(t_0, x)| + \alpha(x)|t - t_0|,$$

which, exploiting the property (ii), gives $g(t, \cdot) \in L^1(\Omega)$ for every $t \in I$. It follows that $G(t)$ is well-defined and finite for all $t \in I$. In a similar fashion, we have

$$g_t(s, \cdot) \in L^1(\Omega)$$

for all $s \in I$ because it is bounded by a $L^1(\Omega)$-function, α, by assumption (iii). To prove (5.1) we start by defining the function

$$F(s) := \int_\Omega g_t(s, x) \, \mathrm{d}x.$$

The dominated convergence theorem (using α) shows that F is continuous. Therefore, to prove that $\dot{G}(s) = F(s)$, by the fundamental theorem in calculus it is enough to show

$$G(s_2) - G(s_1) = \int_{s_1}^{s_2} F(s) \, \mathrm{d}s$$

for all $s_1 < s_2 \in I$. This can be proved easily using, for example, Fubini's theorem. We leave to the reader the task to fill in the details here. $\square$

2. First variation in weak form and regularity theory

Let us consider the functional

$$F(u) = \frac{1}{2}\int_a^b \dot{u}^2 \,\mathrm{d}x + \int_a^b g(x,u)\,\mathrm{d}x,$$

where $u \in W^{1,2}_{u_0}((a,b))$ and $g : (a,b) \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(i) g is Borel;

(ii) $g(x,\cdot) \in C^1(\mathbb{R})$ for every $x \in (a,b)$ and there are $c \in L^1((a,b))$ and a modulus of continuity ω such that

$$|g_u(x,u)| \leq c(x)\omega(|u|),$$

where g_u denotes the derivative of g with respect to the second variable.

We are now ready to compute the first variation of the functional F. We first take test functions $v \in W^{1,\infty}_0((a,b))$ because we need $\dot{v}$ to be bounded. It turns out that

$$\begin{aligned}\langle \mathrm{d}F(u), v\rangle &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(u+tv) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\frac{1}{2}\int_a^b \dot{u}^2\,\mathrm{d}x + \int_a^b g(x,u)\,\mathrm{d}x\right] \\ &= \int_a^b (\dot{u}\dot{v} + g_u(x,u)v)\,\mathrm{d}x\end{aligned}$$

by applying the integral-derivative swap of Proposition 5.1. We shall refer to the last term as *first variation in weak form of F*.

REMARK 5.2. The functional F is **not** differentiable on $W^{1,2}_{u_0}((a,b))$ because we can only compute a smaller set of directional derivatives with directions $v \in W^{1,\infty}_0((a,b))$. If we want F to be differentiable, then we should assume $c(x) \in L^2((a,b))$.

Assume now that u is a minimizer for F on $W^{1,2}_{u_0}((a,b))$. Then for all $v \in C^\infty_c([a,b]) \subset W^{1,\infty}_0(a,b)$ we have

$$\langle \mathrm{d}F(u), v\rangle = \int_a^b (\dot{u}\dot{v} + g_u(x,u)v)\,\mathrm{d}x = 0,$$

which can also be rewritten as

$$-\int_a^b \dot{u}\dot{v}\,\mathrm{d}x = \int_a^b g_u(x,u)v\,\mathrm{d}x.$$

If we denote by $D(\dot{u})$ the distributional derivative of $\dot{u}$, then this identity can be rewritten (in the language of distributions) as

$$D(\dot{u}) = g_u(x,u).$$

This shows that $u \in W^{2,1}\left((a,b)\right)$ and, if $c \in L^2\left((a,b)\right)$, then we can go as far as to say that $u \in W^{2,2}\left((a,b)\right)$. In particular, we conclude that

$$\ddot{u} = g_u(x,u).$$

REMARK 5.3. If, in addition, the derivative g_u is continuous, then the minimizer u is regular and belongs to $C^2([a,b])$.

If we now assume that u is a minimizer for F on $W^{1,2}\left((a,b)\right)$ with no boundary conditions, then we find that

$$-\int_a^b \dot{u}\dot{v}\,\mathrm{d}x = \int_a^b g_u(x,u)v\,\mathrm{d}x.$$

holds for all $v \in C^\infty([a,b]) \subset W^{1,\infty}\left((a,b)\right)$. As above we infer that

$$\ddot{u} = g_u(x,u)$$

using as test functions the subspace $C_c^\infty([a,b]) \subset C^\infty([a,b])$. Furthermore, using the inclusion

$$u \in W^{2,1}\left((a,b)\right) \subset C^1\left([a,b]\right)$$

we can integrate by parts and obtain Neumann boundary conditions (which make sense only because the function belongs to C^1):

$$\dot{u}(a) = \dot{u}(b) = 0.$$

REMARK 5.4. If we replace $\dot{u}^2$ by $\dot{u}^p$, then the same computation leads to

$$\frac{p}{2}\int_a^b \dot{u}^{p-1}\dot{v}\,\mathrm{d}x + \int_a^b g_u(x,u)v\,\mathrm{d}x = 0$$

for all appropriate v. However, integrating by parts (assuming that we can do it) yields

$$\int \dot{u}^{p-2}\ddot{u}v\,\mathrm{d}x = \int_a^b g_u(x,u)v\,\mathrm{d}x,$$

but it does not give any information on the summability of $\ddot{u}$ because of the term $\dot{u}^{p-2}$ with $p > 2$ (the situation is even worse if $p < 2$). There are other ways around in higher-dimensions, but here we cannot deduce anything on the regularity of $\ddot{u}$ from

$$\dot{u}^{p-2}\ddot{u} = g_u(x,u).$$

2.1. Vector-valued computation. Now let us consider the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega g(x, u) \, \mathrm{d}x,$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Borel function such that $g(x, \cdot) \in C^1(\mathbb{R})$ for every $x \in \Omega$ and satisfies the estimates

$$|g(x, u)|, \; |g_u(x, u)| \leq c(1 + |u|^2). \tag{5.2}$$

If we make the same computation as above (assuming that we can swap integral and derivative without any problems), we find that

$$\begin{aligned}
\langle \mathrm{d}F(u), v \rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(u + tv) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left[\frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega g(x, u) \, \mathrm{d}x \right] \\
&= \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega g_u(x, u) v \, \mathrm{d}x.
\end{aligned}$$

If u is a minimizer for F in $W^{1,2}_{u_0}(\Omega)$, then it is easy that thanks to the estimates (5.3) we can directly take as test functions all $v \in W^{1,2}_0(\Omega)$. It turns out that

$$- \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega g_u(x, u) v \, \mathrm{d}x$$

holds, in particular, for all $v \in C^\infty_c(\Omega)$. In the language of distributions, this is as to say that

$$-\Delta u = g_u(x, u),$$

but it is not anymore trivial (and, in fact, it might require some additional assumptions?) to show that from $\Delta u \in L^2(\Omega)$ we can conclude $u \in W^{2,2}(\Omega)$.

2.2. Vector-valued computation. Consider the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega g(x,u)\,\mathrm{d}x,$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Borel function such that $g(x,\cdot) \in C^1(\mathbb{R})$ for every $x \in \Omega$ and it satisfies the following estimates:

$$\begin{aligned} |g(x,u)| &\le c(1+|u|^2), \\ |g_u(x,u)| &\le c(1+|u|^2). \end{aligned} \tag{5.3}$$

If we make the same computation as above (assuming that we can swap integral and derivative without any issues), we find that

$$\begin{aligned} \langle \mathrm{d}F(u), v\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(u+tv) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left[\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega g(x,u)\,\mathrm{d}x\right] \\ &= \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega g_u(x,u)v\,\mathrm{d}x. \end{aligned}$$

If u is a minimizer for F in $W^{1,2}_{u_0}(\Omega)$, then it is easy that thanks to the estimates (5.3) we can directly take as test functions all $v \in W^{1,2}_0(\Omega)$. It turns out that

$$-\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x = \int_\Omega g_u(x,u)v\,\mathrm{d}x$$

holds, in particular, for all $v \in C^\infty_c(\Omega)$. If we denote by Δu the distributional divergence of ∇u, we find the Euler-Lagrange equation

$$-\mathrm{div}(\nabla u) = -\Delta u = g_u(x,u),$$

where the equality has to be considered in a distributional sense. However, we need the minimizer u to be more regular to talk about the Laplace operator in a pointwise sense.

REMARK 5.5. As in the one-dimensional case, we can also study minimizers $u \in W^{1,2}(\Omega)$ of F with no prescribed boundary conditions.

At this point, we are looking for results saying that $u \in W^{1,2}(\Omega)$ and $\Delta u \in L^2(\Omega)$ is enough to conclude that $u \in W^{2,2}(\Omega)$.

We will see soon that the regularity of g plays an important role, but without extra conditions we can only go as far as to say that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

REMARK 5.6. Obtaining more regularity (namely, $u \in W^{2,2}_{\mathrm{loc}}$) is crucial because we want to integrate by parts the weak form of the first variation, namely

$$-\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x = \int_\Omega g_u(x,u)v\,\mathrm{d}x.$$

The following result, which we give for granted, says that a Sobolev function $u \in W^{1,p}(\Omega)$ is a.e. *approximately differentiable* when $p \leq n$. For $p > n$ we know already that Sobolev functions are a.e. differentiable so for all values $1 \leq p \leq \infty$ it will make sense to talk about the "differential" as a pointwise function.

THEOREM 5.7. *Let $1 \leq p \leq \infty$ and let $u \in W^{1,p}(\Omega)$. Then for almost every $\bar{x} \in \Omega$ it turns out that*

$$\fint_{B(\bar{x},r)} \left| \frac{v(x) - v(\bar{x}) - \nabla v(\bar{x})(x - \bar{x})}{|x - \bar{x}|} \right|^{p^*} \mathrm{d}x \to 0$$

as r goes to zero.

REMARK 5.8. Consequently, if we can prove that a minimizer u of F belongs to $W^{2,2}(\Omega)$, then it correct to talk about its Laplacian. Moreover, the Euler-Lagrange equation

$$-\Delta u = g_u(x, u)$$

will not be anymore an equality in the distributional sense.

PROPOSITION 5.9. *If $u \in L^2(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$, where Δu is the distributional gradient, then $u \in W^{2,2}(\mathbb{R}^n)$ and there exists a constant $c > 0$ such that*

$$\|u\|_{W^{2,2}(\mathbb{R}^n)} \leq c(\|u\|_{L^2(\mathbb{R}^n)} + \|\Delta u\|_{L^2(\mathbb{R}^n)}). \tag{5.4}$$

We will now give a proof of the estimate (5.4) which is "almost" correct because, as we will see at the end, it works only under an additional assumption.

PROOF. For the reader's convenience, we divide the proof into three steps.

Step 1. Assume $u \in C_c^\infty(\mathbb{R}^n)$. We define the Fourier transform of u by setting

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) \mathrm{e}^{-\imath \xi \cdot x} \, \mathrm{d}x,$$

so that, using a well-known property of the Fourier transform, there results

$$\widehat{\nabla u}(\xi) = \imath \xi \hat{u}(\xi).$$

In a similar fashion, if we denote by $\nabla^2 u$ the Hessian of u, we find that

$$\widehat{\nabla^2 u}(\xi) = -(\xi \otimes \xi)\hat{u}(\xi),$$

where $\xi \otimes \xi$ is the matrix with entries $(\xi \otimes \xi)_{i,j} := \xi_i \xi_j$. The trace is given by

$$\mathrm{Tr}\left(\widehat{\nabla^2 u}(\xi)\right) = -|\xi|^2 \hat{u}(\xi),$$

and by **Plancherel's identity** we have

$$\|\Delta u\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^4 |\hat{u}|^2 \, \mathrm{d}\xi = \frac{1}{(2\pi)^n} \| |\xi|^2 \hat{u} \|_{L^2(\mathbb{R}^n)}^2.$$

Now notice that

$$\sum_{i,j=1}^n |(\xi \otimes \xi)_{i,j}|^2 = \sum_{i,j=1}^n |\xi_i \xi_j|^2 = |\xi|^2 \sum_{j=1}^n |\xi_j|^2 = |\xi|^4,$$

from which (using Plancherel's identity again) it follows that

$$\|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\||\xi|^2\hat{u}\|^2_{L^2(\mathbb{R}^n)}.$$

In particular, we have the quite surprising equality

$$\|\Delta u\|^2_{L^2(\mathbb{R}^n)} = \|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)}.$$

Next, we can use the trivial inequality $ab \le \frac{1}{2}(a^2+b^2)$ with $b=1$ to infer that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n}\int |\xi|^2|\hat{u}|^2\,\mathrm{d}\xi \\ &\le \frac{1}{(2\pi)^n}\int \frac{1}{2}(1+|\xi|^4)|\hat{u}|^2\,\mathrm{d}\xi \\ &\le \frac{\|u\|^2_{L^2(\mathbb{R}^n)} + \|\Delta u\|^2_{L^2(\mathbb{R}^n)}}{2}.\end{aligned}$$

In conclusion, we use the definition of the $W^{2,2}(\mathbb{R}^n)$-norm to achieve (5.4) for smooth functions:

$$\begin{aligned}\|u\|^2_{W^{2,2}(\mathbb{R}^n)} &= \|u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)} \\ &\le \frac{3}{2}(\|u\|^2_{L^2(\mathbb{R}^n)} + \|\Delta u\|^2_{L^2(\mathbb{R}^n)}).\end{aligned}$$

REMARK 5.10. To prove that $\|\Delta u\|^2_{L^2(\mathbb{R}^n)} = \|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)}$ we do not need the use the Fourier transform. Let $n=2$ (for simplicity) and notice that

$$\int_{\mathbb{R}^2} |\nabla^2 u|^2\,\mathrm{d}x = \int_{\mathbb{R}^2}(u_{11}^2 + u_{22}^2 + u_{12}^2 + u_{21}^2)\,\mathrm{d}x,$$

where u_{ij} is equal to $\partial_{x_i}\partial_{x_j}u$. A double integration by parts shows that

$$\int_{\mathbb{R}^2} u_{12}^2\,\mathrm{d}x = -\int_{\mathbb{R}^2} u_1 u_{122}\,\mathrm{d}x = \int_{\mathbb{R}^2} u_{11}u_{22}\,\mathrm{d}x,$$

and we conclude since

$$\begin{aligned}\int_{\mathbb{R}^2}(u_{11}^2 + u_{22}^2 + u_{12}^2 + u_{21}^2)\,\mathrm{d}x &= \int_{\mathbb{R}^2}(u_{11}^2 + u_{22}^2 + 2u_{11}u_{22})\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}(u_{11}+u_{22})^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}|\Delta u|^2\,\mathrm{d}x.\end{aligned}$$

Step 2. Now assume that $u \in L^2(\mathbb{R}^n)$ has compact support. Take $u_\epsilon := u * \rho_\epsilon$ where ρ_ϵ is the usual regularizing kernel with compact support. Then $u_\epsilon \in C_c(\mathbb{R}^n)$ so

$$\|u_\epsilon\|^2_{W^{2,2}(\mathbb{R}^n)} \le \frac{3}{2}(\|u_\epsilon\|^2_{L^2(\mathbb{R}^n)} + \|\Delta u_\epsilon\|^2_{L^2(\mathbb{R}^n)}).$$

However, we know that $\Delta u_\epsilon = (\Delta u) * \rho_\epsilon$ so these estimates can be rewritten as

$$\begin{aligned}\|u_\epsilon\|^2_{W^{2,2}(\mathbb{R}^n)} &\le \frac{3}{2}\left(\|u_\epsilon\|^2_{L^2(\mathbb{R}^n)} + \|(\Delta u) * \rho_\epsilon\|^2_{L^2(\mathbb{R}^n)}\right) \\ &\le \frac{3}{2}(\|u\|^2_{L^2(\mathbb{R}^n)} + \|\Delta u\|^2_{L^2(\mathbb{R}^n)})\end{aligned}$$

because the convolution decreases the norms. It follows that $u_\epsilon \in W^{2,2}(\mathbb{R}^n)$, and this is enough to conclude that $u \in W^{2,2}(\mathbb{R}^n)$ by semicontinuity of the norm as $\epsilon \to 0$.

Step 3. Let Θ be a smooth cutoff function from $\mathbb{R}^n$ to $[0,1]$ such that $\Theta(x) = 1$ for all $|x| \le 1$ and $\Theta(x) = 0$ for all $|x| \ge 2$. Let $\Theta_\epsilon(x) := \Theta(\epsilon x)$ and consider the sequence

$$u_\epsilon(x) := u(x)\Theta_\epsilon(x).$$

The functions u_ϵ are compactly supported so by the previous step we have

$$\|u_\epsilon\|^2_{W^{2,2}(\mathbb{R}^n)} \le \frac{3}{2}\left(\|u_\epsilon\|^2_{L^2(\mathbb{R}^n)} + \|\Delta u_\epsilon\|^2_{L^2(\mathbb{R}^n)}\right),$$

but, this time, things are not as easy because the Laplacian is more complicated:

$$\Delta u_\epsilon = \Theta_\epsilon \Delta u + u\Delta\Theta_\epsilon + 2\,\nabla u \cdot \nabla\Theta_\epsilon.$$

We now estimate the L^2-norm of the left-hand side as follows:

$$\|\Delta u_\epsilon\|^2_{L^2(\mathbb{R}^n)} \le \|\Delta u\|^2_{L^2(\mathbb{R}^n)} \underbrace{\|\Theta_\epsilon\|^2_\infty}_{\le 1} + \|u\|^2_{L^2(\mathbb{R}^n)}\mathcal{O}(\epsilon^4) + \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\mathcal{O}(\epsilon^2)$$

since it is easy to verify that

$$\|\Delta\Theta_\epsilon\|_{L^\infty(\mathbb{R}^n)} = \epsilon^2\|\Theta_\epsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\epsilon^2).$$

However, the right-hand side of the estimate is a problem because the term $\|\nabla u\|^2_{L^2(\mathbb{R}^n)}\mathcal{O}(\epsilon^2)$ is fine only if we know already that $\nabla u \in L^2(\mathbb{R}^n)$. Under this **extra assumption** we conclude that

$$\|u\|_{W^{2,2}(\mathbb{R}^n)} \le c(\|u\|_{L^2(\mathbb{R}^n)} + \|\Delta u\|_{L^2(\mathbb{R}^n)}).$$

□

REMARK 5.11. To prove (5.4) without assuming $\nabla u \in L^2(\mathbb{R}^n)$ use the fact that every $u \in L^2(\mathbb{R}^n)$ is a tempered distribution. Then everything works as in **Step 1** because the Fourier transform is well-defined on the space of tempered distributions.

COROLLARY 5.12. *If $u \in W^{1,2}(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in W^{2,2}_{\text{loc}}(\Omega)$.*

HINT OF THE PROOF. Let Ω' be a relatively compact subset of Ω and Θ a cutoff function. Apply Proposition 5.9 to the function $\tilde{u} := u\Theta$. □

REMARK 5.13. We cannot improve this result. Indeed, for every $\Omega \subset \mathbb{R}^n$ there exists $u \in W^{1,2}(\Omega) \cap C^\infty(\Omega)$ harmonic ($\Delta u = 0$) that does not belong to $W^{2,2}(\Omega)$.

PROOF. Consider $\Omega = B(0,1)$ and let u_n be the real part of z^n, that is,

$$u_n = \mathfrak{Re}(z^n) = \rho^n \cos(n\theta).$$

It is easy to show that $\|u_n\|^2_{L^2(\Omega)} \simeq n^{-1}$, $\|\nabla u_n\|^2_{L^2(\Omega)} \simeq n$ and $\|\nabla^2 u_n\|^2_{L^2(\Omega)} \simeq n^3$. Now consider the function defined by setting

$$u := \sum_{n\in\mathbb{N}} \alpha_n u_n,$$

where α_n are coefficients that need to be chosen in such a way that:

(i) The series $\sum_{n\in\mathbb{N}} \alpha_n u_n$ converges in $W^{1,2}(\Omega)$ and, by completeness, we obtain $u \in W^{1,2}(\Omega)$.

(ii) The series $\sum_{n\in\mathbb{N}} \alpha_n \|\nabla^2 u_n\|_{L^2(\Omega)}$ diverges to infinity.

(iii) The function u does not belong to $W^{2,2}(\Omega)$ - which is not guaranteed by the previous requirement.

This is not trivial so we propose another way to get the same conclusion. Let $\mathfrak{X}$ be the space of harmonic functions in $W^{1,2}(\mathrm{m})$ and consider the operator

$$T : \mathfrak{X} \ni u \longmapsto T(u) = \nabla^2 u \in L^2(\Omega).$$

It is not hard to see that T is unbounded. If ρ_ϵ is a regularizing convolution kernel, then we can define the family of operators $\{T_\epsilon\}_{\epsilon>0}$ by setting

$$T_\epsilon(u) = \nabla^2 u * \rho_\epsilon.$$

These are all well-defined and continuous, but

$$\sup_{\epsilon>0} \|T_\epsilon\| = \infty$$

so by Banach-Steinhaus theorem we can find u such that

$$\|T_\epsilon(u)\|_{L^2(\Omega)} \xrightarrow{\epsilon\to 0^+} \infty$$

and, if one is careful enough, it can be proved that Δu does not belong to $L^2(\Omega)$. □

PROPOSITION 5.14. *If $u \in W^{1,2}_0(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in W^{2,2}_0(\Omega)$ provided that Ω is "sufficiently regular".*

REMARK 5.15. With sufficiently regular we mean that Ω must have a boundary of class at least C^2. Note that piecewise polygonal is not good enough in this case.

COROLLARY 5.16. *Let $u_0 \in W^{2,2}(\Omega)$. If $u \in W^{1,2}_{u_0}(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in W^{2,2}_{u_0}(\Omega)$ provided that Ω is "sufficiently regular".*

We now give two **wrong** proofs of Proposition 5.14 to get a better understand of what could go wrong when we try to extend the function u from Ω to $\mathbb{R}^n$.

WRONG PROOF 1. Approximate $u \in W_0^{1,2}(\Omega)$ with a sequence of smooth functions $u_k \in C_c^\infty(\mathbb{R}^n)$ and apply Proposition 5.9 to each u_k. We find the estimate

$$\|u_k\|_{W^{2,2}(\mathbb{R}^n)} \le c(\|u_k\|_{L^2(\mathbb{R}^n)} + \|\Delta u_k\|_{L^2(\mathbb{R}^n)}).$$

for each $k \in \mathbb{N}$, but the problem is that Δu_k has no reason at all to converge to Δu. For example, the function

$$u(x) = \sin(x)$$

belongs to $W_0^{1,2}((0,\pi))$ and its distributional second-order derivative is given by

$$\delta_0 + \delta_\pi - \sin(x)$$

since $-\sin(x)$ is the second derivative inside the interval $(0,\pi)$, but we also have to keep in mind the contributes from the boundary which are Dirac masses. However, if

$$u_k''(x) \xrightarrow{k\to\infty} \delta_0 + \delta_\pi - \sin(x) \quad \text{in } L^2((0,\pi)),$$

then we are not approximating u via smooth functions since $u_k'' \notin L^2((0,\pi))$. □

WRONG PROOF 2. Let $u \in W_0^{1,2}(\Omega)$. We can extend it to $\tilde{u}$ zero outside of Ω because the trace is zero, but it might happen that

$$\Delta u \neq \Delta \bar{u}$$

because the normal derivative of the extension could have jumps at the boundary created by the extension itself. □

Before we can give the correct proof of Proposition 5.14, a remark about odd extensions of Sobolev functions is in order.

REMARK 5.17. If $f \in W_0^{2,2}((0,\infty))$, then the odd extension

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x > 0, \\ -f(-x) & \text{if } x < 0 \end{cases}$$

belongs to $W^{2,2}(\mathbb{R})$. Indeed, it is easy to verify $\tilde{f} \in W^{1,2}(\mathbb{R})$ since we are "gluing" together two functions with the same trace so we only need to prove that

$$g(x) = \begin{cases} f''(x) & \text{if } x > 0, \\ -f''(-x) & \text{if } x < 0, \end{cases}$$

is the second-order distributional derivative of the function $\tilde{f}$. A simple computation shows that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{f}(x)\varphi''(x)\,\mathrm{d}x &= \int_0^\infty f(x)(\varphi''(x) - \varphi''(-x))\,\mathrm{d}x \\ &= \int_0^\infty f''(x)(\varphi(x) - \varphi(-x))\,\mathrm{d}x = \\ &= \int_{\mathbb{R}} g(x)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

We integrated by parts twice and no boundary terms appear, even if $\varphi(x) - \varphi(-x)$ has derivative which is not compactly supported. The remaining boundary term would be

$$[f(x)(\varphi'(x) - \varphi'(-x)]_{x=0}^{\infty},$$

but $f \in W_0^{2,2}(0,\infty)$ so it vanishes.

CORRECT PROOF. Suppose Ω is the half-space, namely $\Omega = \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Take $u \in W_0^{1,2}(\Omega)$ with $\Delta u \in L^2(\Omega)$ and define the odd extension as

$$\tilde{u}(x) = \tilde{u}(x_1, \dots, x_n) = \begin{cases} u(x_1, \dots, x_n) & \text{if } x_1 > 0, \\ -u(-x_1, x_2, \dots, x_n) & \text{if } x_1 < 0. \end{cases}$$

As above, it is easy to verify that $\tilde{u} \in W^{1,2}(\mathbb{R}^n)$ and $\Delta\tilde{u}$ is the odd extension of Δu so it must be an element $L^2(\mathbb{R}^n)$. To conclude apply Proposition 5.9 to $\tilde{u}$. □

REMARK 5.18. If Ω is a general subset with C^2 boundary, the idea is to take a partition of unity and decompose u accordingly. Then each "piece" can be reduced to the half-plane by a change of variables - which is why we have to assume the boundary to be the graph of a C^2 function -.

In the previous lecture, we proved that $u \in W^{1,2}(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$, where Δu is the distributional Laplacian, is enough to conclude $u \in W^{2,2}(\mathbb{R}^n)$ and

$$\|u\|_{W^{2,2}(\mathbb{R}^n)} \le c\left(\|u\|_{L^2(\mathbb{R}^n)} + \|\Delta u\|_{L^2(\mathbb{R}^n)}\right).$$

We also proved that the same is not true if $\mathbb{R}^n$ is replaced by Ω, unless we add additional conditions as in the next proposition:

PROPOSITION 5.19. *Let $\Omega \subset \mathbb{R}^n$ be a set with boundary of class C^2. If $u \in W_0^{1,2}(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in W_0^{2,2}(\Omega)$ and there is a constant $c := c(\Omega) > 0$ such that*

$$\|u\|_{W^{2,2}(\Omega)} \le c(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}).$$

REMARK 5.20. The same conclusion is true with $u \in W_{u_0}^{1,2}(\Omega)$, but we must require the boundary datum u_0 to belong to the class $W^{2,2}(\Omega)$.

PROPOSITION 5.21. *Let $\Omega \subset \mathbb{R}^n$ be a set with boundary of class C^2. If $u \in W^{1,2}(\Omega)$, $\Delta u = f \in L^2(\Omega)$ and $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$, then $u \in W^{2,2}(\Omega)$ and*

$$\|u\|_{W^{2,2}(\Omega)} \le c\left(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right).$$

for some positive constant c that depends on Ω.

REMARK 5.22. Since u, a priori, only belongs to $W^{1,2}(\Omega)$, the normal derivative does not have a trace and hence the condition

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

is not a pointwise equality. However, it makes perfect sense in the "world" of distributions and it is easy to verify that requiring

$$\int_\Omega (f\varphi + \nabla u \cdot \nabla\varphi)\, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n)$$

is an **equivalent** condition to both $\Delta u = f$ and $\frac{\partial u}{\partial n}\big|_{\partial\Omega} \equiv 0$.

Before we can prove Proposition 5.21, we need to briefly discuss how to extend a function and preserve the condition on the normal derivative.

REMARK 5.23. Let $v \in W^{2,2}((0,\infty))$ be such that $\dot v(0) = 0$. The odd extension does not work in this situation, but the even extension

$$v_p(x) := \begin{cases} v(x) & \text{if } x > 0, \\ v(-x) & \text{if } x < 0, \end{cases}$$

does the trick, namely $\dot v_p(x) = 0$.

EXERCISE 5.1. Let $v \in W^{2,2}((0,\infty))$. Show that $v_p \in W^{2,2}(\mathbb{R})$ and the second derivative $\ddot v_p$ is the even extension of $\ddot v$.

PROOF OF PROPOSITION 5.21. Assume that $\Omega = (0,\infty) \times \mathbb{R}^{n-1}$. The idea is similar to Proposition 5.14, but given $u \in W_{1,2}(\Omega)$ with $\nabla u \in L^2(\Omega)$ we define

$$u_p(x) = \tilde u(x_1, \dots, x_n) = \begin{cases} u(x) & \text{if } x_1 > 0, \\ u(-x_1, x_2, \dots, x_n) & \text{if } x_1 < 0, \end{cases}$$

and apply Proposition 5.9 to u_p. The case in which Ω is not the half-plane can be dealt with in the exact same way (partition of unity + change of variables). □

REMARK 5.24. After one shows that $u \in W^{2,2}(\Omega)$, the trace of the normal derivative exists and hence the condition

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$$

is a pointwise equality. Similarly, the Laplacian Δu is not anymore a distribution but an actual $L^2(\Omega)$-function.

THEOREM 5.25. *If* $1 < p < \infty$, $u \in W^{1,p}(\mathbb{R}^n)$ *and* $\Delta u \in L^p(\mathbb{R}^n)$, *then* $u \in W^{2,p}(\mathbb{R}^n)$.

REMARK 5.26. The proof of this statement, unlike the case $p = 2$, is nontrivial. The idea is to use once again the Fourier transform and show the key estimate

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^n)} \le c\|\Delta u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. If $f := \Delta u$, then

$$\hat f(\xi) = -|\xi|^2 \hat u(\xi).$$

We can recover $\hat u$ by writing (at least formally) that

$$\hat u(\xi) = -\frac{1}{|\xi|^2}\hat f(\xi).$$

It is well-known that there exists K such that $\hat{K} = -\frac{1}{|\xi|^2}$. Therefore, using standard properties of the Fourier transform we find that

$$\hat{u}(\xi) = \hat{K}(\xi)\hat{f}(\xi) = \mathcal{F}(K * f),$$

where $\mathcal{F}$ denotes the Fourier transform. It follows that

$$u = K * f,$$

where K is (up to a constant) the fundamental solution of the Laplacian ($|x|^{2-n}$ if $n \geq 3$ and $\log|x|$ if $n = 2$). Next, we apply the following fundamental result:

THEOREM 5.27 (Calderón–Zygmund). *Suppose that the operator T given by*

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \,\mathrm{d}y$$

has kernel K that satisfies the following conditions:

(a) $|K(x)| \leq c|x|^{-n}$ *for all* $x \in \mathbb{R}^n$*;*

(b) $\int_{r<|x|<R} K(x) \,\mathrm{d}x = 0$ *for all* $0 < r < R < \infty$*;*

(c) $\int_{|x|>2|y|} |K(x-y) - K(x)| \,\mathrm{d}x \leq c'$ *for a positive constant* c' *when* $|y| > 0$.

Then for $1 < p < \infty$ *we have that* $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is a bounded operator.*

At this point notice that $K * f$ and $\nabla K * f$ both make sense as distributions, but $\nabla^2 K * f$ only makes sense as a singular integral (as in the CZ theorem).

CHAPTER 6

Γ-convergence and Applications

I will write the introduction after the chapter is complete!

1. Introduction and first examples

The Γ-convergence is, roughly speaking, a notion of convergence of functionals designed to have the convergence of minimizers. Namely, if $\mathfrak{X}$ is a metric space and

$$F_n : \mathfrak{X} \to [0, \infty]$$

a sequence of lower semicontinuous functions, then $F_n \xrightarrow{\Gamma} F$ should imply "that a sequence x_n of minimizers for F_n converges to a minimum point x of F".

We will now give the formal definition of Γ-convergence and show that, under certain conditions, the assertion in quotes holds.

DEFINITION 6.1. [Γ-convergence] Let $\mathfrak{X}$ be a metric space and $F_n : \mathfrak{X} \to [0, \infty]$. We say that F_n Γ-converges to $F : \mathfrak{X} \to [0, \infty]$ if the following properties hold:

(i) For all $x \in \mathfrak{X}$ and all sequences $(x_n)_{n\in\mathbb{N}} \subset \mathfrak{X}$ such that $x_n \to x$ it turns out that

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x). \tag{6.1}$$

(ii) For all $x \in \mathfrak{X}$ there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathfrak{X}$ such that $x_n \to x$ and

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x). \tag{6.2}$$

The sequence x_n is usually referred to in the literature as *recovery sequence*.

REMARK 6.2. The condition (6.1) is usually called $\Gamma - \liminf$ inequality, while (6.2) is known as $\Gamma - \limsup$ inequality.

PROPOSITION 6.3. *Let $\bar{x}_n$ be a minimizer for F_n and suppose that F is the Γ-limit of the sequence F_n. If $\bar{x}_n \to \bar{x} \in \mathfrak{X}$, then $\bar{x}$ is a minimizer for F.*

PROOF. It suffices to prove that given any $x \in \mathfrak{X}$ we have

$$F(x) \geq F(\bar{x}).$$

Fix $x \in \mathfrak{X}$ and take a recovery sequence x_n. Then

$$F_n(\bar{x}_n) \leq F_n(x_n) \implies \liminf_{n\to\infty} F_n(\bar{x}_n) \leq \lim_{n\to\infty} F_n(x_n) = F(x)$$

follows from (6.2), while the opposite inequality

$$\liminf_{n\to\infty} F_n(\bar{x}_n) \geq F(\bar{x})$$

follows immediately from (6.1). $\square$

EXAMPLE 6.4. *Let $\mathfrak{X} = \mathbb{R}$ and $F_n(x) := x^2 + \sin(nx)$. We claim that*

$$\Gamma - \lim_{n\to\infty} F_n(x) = x^2 - 1.$$

The $\Gamma - \liminf$ inequality is easy to check because -1 is the minimum of $\sin(nx)$. Similarly, given $x \in \mathbb{R}$ the recovery sequence is $x_n \in \mathbb{R}$ such that

$$\sin(nx_n) = -1 \quad \textit{for all } n \in \mathbb{N}.$$

An immediate consequence is that, if $\bar{x}_n$ is a minimizer for $F_n(x)$, then $\bar{x}_n \to 0$ as the function $x^2 + 1$ has its unique minimum at $\bar{x} = 0$.

It is worth noticing that Proposition 6.3 requires $\bar{x}_n$ to be a sequence that converges to some element of $\mathfrak{X}$. Thus, it makes sense to introduce the following notion:

DEFINITION 6.5. A sequence of functionals $F_n : \mathfrak{X} \to [0, \infty]$ is said to be *equicoercive* if $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathfrak{X}$ whenever

$$F_n(x_n) \le C < \infty$$

for some uniform constant C that does not depend on n.

PROPOSITION 6.6. *Let $\bar{x}_n$ be a minimizer for F_n and suppose that F is the Γ-limit of the sequence F_n. If F_n is a equicoercive, then $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathfrak{X}$ and each accumulation point is a minimizer for F.*

REMARK 6.7. In the applications, we need to choose the “right” topology on the space $\mathfrak{X}$. In fact, we need to find a good ”balance” between these two points:

(i) If the topology is too weak, then the $\Gamma - \liminf$ inequality has to be tested on more sequences and thus it might fail.

(ii) If the topology is too strong, then (6.1) is easier but the equicoerciveness (strictly related to compactness) might fail.

REMARK 6.8. The statement of Proposition 6.6 concerns the convergence of minimizers, but it does not say anything about local minimizers.

REMARK 6.9. From a numerical point of view Γ-converge is rather useless because it gives no information about the rate of convergence of minimizers (so, given $\bar{x}_n$, we do not know where to ”stop” to find a good approximation of $\bar{x}$).

EXAMPLE 6.10. *Let $\mathfrak{X} = \mathbb{R}$ and $F_n(x) = \frac{x^2}{n} + \sin(nx)$. Then*

$$F_n \xrightarrow{\Gamma} F \equiv -1,$$

where F is the function identically -1. Each $x \in \mathbb{R}$ minimizes F so the Γ-convergence here gives no information whatsoever on $\bar{x}_n \in \operatorname{argmin}(F_n)$. However, the functional

$$G_n(x) := n(F_n(x) + 1) = x^2 + n(\sin(nx) + 1)$$

is well-defined and satisfies $\operatorname{argmin}(F_n) = \operatorname{argmin}(G_n)$. *Furthermore, one can prove that*

$$\Gamma - \lim_{n\to\infty} G_n = x^2,$$

which admits a unique minimum point. Therefore, any sequence $\bar{x}_n$ *of minimizers for* F_n *must converge to* $x = 0$.

REMARK 6.11.

(i) To prove that F is the Γ-limit of F_n, it suffices to show the $\Gamma - \liminf$ inequality and to find a recovery sequence for all $x \in D$, with D a F-dense subset of $\mathfrak{X}$.

(ii) $F_n \xrightarrow{\Gamma} F$ if and only if the epigraphs of F_n converge (in the sense of **Kuratowski**) to the epigraph of F.

(iii) The Γ-limit does not always coincide with the pointwise limit. For example, let $\mathfrak{X} = \mathbb{R}$ and consider the sequence

$$F_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \geq \frac{2}{n}, \\ f_n(x) & \text{if } 0 < x < \frac{2}{n}, \end{cases}$$

where f_n is a continuous function satisfying $f_n\left(\frac{1}{n}\right) = -1$. The pointwise limit of F_n as $n \to \infty$ is the identically zero function, but

$$F_n \xrightarrow{\Gamma} F(x) := \begin{cases} 0 & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

However, if F_n is pointwise increasing and lower semicontinuous for all $n \in \mathbb{N}$, then the Γ-limit and the pointwise limit coincide.

2. Ginzburg-Landau scalar model

Suppose that we have a two-phases (say A^+ and A^-) fluid inside a container. Our goal is to study the equilibrium configuration of the fluid through the so-called *Ginzburg-Landau theory* which was introduced in a more general situation to describe phases transitions.

2.1. Mathematical model. Let Ω be a bounded smooth domain in $\mathbb{R}^d$ ($d = 3$ in the applications) and let $u : \Omega \to [-1, 1]$ be the *order parameter*, which describes the configuration of the fluid and it is defined in such a way that

$$\begin{aligned} u(x) = 1 &\implies \text{pure phase } A^+, \\ u(x) = -1 &\implies \text{pure phase } A^-, \\ u(x) \in (-1, 1) &\implies \text{mixed phase.} \end{aligned}$$

At the equilibrium configuration, u minimizes the functional

$$F_\epsilon(u) = \int_\Omega \left(W(u) + \epsilon^2 |\nabla u|^2\right) \, \mathrm{d}x$$

with the constraint

$$⨍_\Omega u \, \mathrm{d}x = m \in (-1, 1), \tag{6.3}$$

m fixed, where $W : \mathbb{R} \to \mathbb{R}$ is a *double-well potential*, that is, W is everywhere positive and $W(1) = W(-1) = 0$. The most natural example of such a potential is

$$W(u) := (u^2 - 1)^2.$$

REMARK 6.12. Notice that W favors phases separation, while the gradient term favors the homogeneity of the fluid (because sudden transitions make $|\nabla u|$ bigger).

QUESTION 1. *If u_ϵ denotes a minimizer of the energy F_ϵ, what can we say about its asymptotic behavior as $\epsilon \to 0$?*

To answer this question thoroughly, we plan (in the next couple of lectures) to carry out the following strategy:

(1) Use a heuristic argument (from physics) to determine the behavior of F_ϵ and u_ϵ.
(2) Exploit the heuristic result to guess a possible Γ-limit.
(3) Introduce tools (such as finite perimeter sets) to find the right setting for which the Γ-limit exists and makes sense.
(4) Prove a relevant Γ-convergence result and derive information about the asymptotic behavior of u_ϵ as $\epsilon \to 0$.

2.2. Heuristic argument, I. The first observation is that, since we only care about the limit behavior as $\epsilon \to 0$, we can restrict to small values of ϵ. In this case, the potential W "dominates" the integral, and therefore we can expect u_ϵ to take the values 1 and -1 respectively in two sides of Ω which are separated by an "interface" (**not** a surface in general) which we **guess** might be a δ-neighborhood of some surface Σ as in Figure **??**.

The energy should consequently be concentrated in the phases transition δ-neighborhood and should be given by

$$F_\epsilon(u) \simeq \left(1 + \epsilon^2 \frac{1}{\delta^2}\right) \delta|\Sigma| = \left(\delta + \frac{\epsilon^2}{\delta}\right) |\Sigma|, \tag{6.4}$$

where $\simeq$ means "equal up to a constant". The term δ^{-2} comes out of $|\nabla u|^2$ because the neighborhood has thickness equal to δ.

REMARK 6.13. The term $\delta|\Sigma|$ that multiplies everything above is a good approximation of the volume of the δ-neighborhood of Σ.

It is worth remarking that we cannot choose $\delta > 0$ to be too small or too large because the right-hand side of (6.4) contains both δ and δ^{-1}. Thus, we consider

$$\min_{\delta>0} \left\{\delta + \frac{\epsilon^2}{\delta}\right\}$$

and notice that the minimum is achieved at $\delta = \epsilon$. This "proves" that the thickness of the interface should be of order ϵ and F_ϵ more or less equal to $\epsilon|\Sigma|$, but this is not yet an asymptotic estimate.

2.3. Heuristic argument, II. To find (heuristically) a more precise asymptotic estimate, we use the following ansatz:

$$u(x) = v\left(\frac{d_\Sigma(x)}{\epsilon}\right),$$

where d_Σ is the *oriented distance* from Σ. Since it is arbitrary, we choose d_Σ to be positive on the side where $u = 1$ and negative on the side where $u = -1$. It follows that

$$|\nabla u(x)| = \frac{1}{\epsilon}\left|\dot{v}\left(\frac{d_\Sigma(x)}{\epsilon}\right)\right|$$

because (well-known fact) the modulus of the gradient of the distance d_Σ is always equal to one. Using this ansatz, the energy can be rewritten as

$$F_\epsilon(u) = \int_\Omega \left[W\left(v\left(\frac{d_\Sigma(x)}{\epsilon}\right)\right) + \left|\dot{v}\left(\frac{d_\Sigma(x)}{\epsilon}\right)\right|^2\right] \mathrm{d}x.$$

Let $\Sigma_t := \{x \in \Omega \,:\, d(x,\Sigma) = t\}$ for $t > 0$ (in such a way that $\Sigma = \Sigma_0$). Using the *coarea formula* (which we will discuss briefly at another time) we find that

$$\begin{aligned} F_\epsilon(u) &= \int_{-\infty}^{\infty} \left[W\left(v\left(\frac{t}{\epsilon}\right)\right) + \left|\dot{v}\left(\frac{t}{\epsilon}\right)\right|^2\right] |\Sigma_t| \,\mathrm{d}t \sim \\ &\sim \int_{-\infty}^{\infty} \left[W\left(v\left(\frac{t}{\epsilon}\right)\right) + \left|\dot{v}\left(\frac{t}{\epsilon}\right)\right|^2\right] |\Sigma| \,\mathrm{d}t = \\ &= \epsilon|\Sigma| \int_{-\infty}^{\infty} \left[W\left(v\left(t\right)\right) + \left|\dot{v}\left(t\right)\right|^2\right] \,\mathrm{d}t \end{aligned}$$

where the symbol $\sim$ means asymptotically equivalent. Notice that we can replace $|\Sigma_t|$ with $|\Sigma|$ because the ϵ-neighborhoods (as $\epsilon \to 0$) are so close to Σ that the surface should not change substantially. The equality, on the other hand, simply follows from the change of variables $t' := \frac{t}{\epsilon}$. Therefore, the "right" (heuristically) expansion of the energy should be

$$F_\epsilon(u) \sim \sigma_0 \epsilon |\Sigma|,$$

where σ_0 is the *surface tension* and it is obtained by minimizing the functional

$$\int_{-\infty}^{\infty} \left[W(v(t)) + |\dot{v}(t)|^2\right] \,\mathrm{d}t$$

among all $v : \mathbb{R} \to [-1, 1]$ such that $v(\pm\infty) = \pm 1$. We will come back to compute σ_0 explicitly in the next point of the strategy.

Heuristic conclusion. The function u_ϵ takes the values 1 and -1 on two sides of Ω separated by an "interface" of thickness of order ϵ around a surface Σ. More precisely,

$$u_\epsilon(x) = v\left(\frac{d_\Sigma(x)}{\epsilon}\right),$$

where v is a (quasi)minimizer of $\int_{-\infty}^{\infty}[W(v) + |\dot{v}|^2]\,\mathrm{d}t$ and Σ minimizes the area among all surfaces that divide Ω in two sets A^+ and A^- such that the volume constraint (6.3) is satisfied, that is,

$$\frac{|A^+| - |A^-|}{\Omega} = m.$$

2.4. Heuristic on Γ-convergence. At this point, we would like to know whether or not we can derive the asymptotic behavior of u_ϵ formally using Γ-convergence. This was proved in '77 by Modica and, a few years later, an even stronger result was obtained via a more "direct" proof, but we will not discuss it in this course. In any case, we have

$$F_\epsilon \xrightarrow{\Gamma} F(u) := \int_\Omega W(u)\,\mathrm{d}x,$$

but this Γ-limit is rather useless because **any** function taking values in $\{\pm 1\}$ is a good minimizer for F. The heuristic argument above suggests that it could be better to consider

$$G_\epsilon(u) := \frac{F_\epsilon(u)}{\epsilon}$$

and study the corresponding Γ-limit, which will be denoted by G.

REMARK 6.14. If the Γ-limit G exists, then a natural guess would be a functional equal to $\sigma_0|\Sigma|$ if $u(x) = \pm 1$ at a.e. $x \in \Omega$, where Σ is the "interface" between the two phases A^+ and A^-, and equal to ∞ otherwise.

The functional F_ϵ is well-defined on $W^{1,2}(\Omega)$, but this is not the right space to study the problem because functions in $W^{1,2}(\Omega)$ cannot possibly take only two values (jumps are not allowed). Therefore, we need to find a way to solve the following issues:

(1) What functional space should replace the Sobolev one $W^{1,2}(\Omega)$?

(2) How does the Γ-limit G looks like? The idea mentioned in Remark 6.14 is not good enough because functions taking only two values form a class that is not broad enough for equicoerciveness, etc.

(3) What kind of “interfaces” should we consider? Surely, it cannot be too regular for otherwise, we would have issues similar to (2).

To conclude, we would like to compute σ_0 explicitly. Recall that it is given by the minimum value of

$$\int_{-\infty}^{\infty} \left[W(v(t)) + |\dot{v}(t)|^2\right] \,\mathrm{d}t,$$

where $v : \mathbb{R} \to [-1, 1]$ ranges in some class and satisfies $v(\pm\infty) = \pm 1$. Notice that we need $\dot{v} \in L^2(\mathbb{R})$, but the condition $v(\pm\infty) = \pm 1$ implies that v cannot possibly be in $L^2(\mathbb{R})$ so the Sobolev space $W^{1,2}(\mathbb{R})$ is not good enough. There are further issues:

(a) Translation-invariance.

(b) A function with derivative in $L^2(\mathbb{R})$ admits a continuous representative. However, the values at $\pm\infty$ might not be well-defined. In particular, if v_n is a minimizing sequence converging in $W^{1,2}(\Omega)$ to some v, there is no way to guarantee that v will also satisfy the condition at infinity.

The idea is to prove existence of minimizers in a completely different way, namely we provide a lower bound on the functional and show that it is achieved by some v.

REMARK 6.15. Assume that $v \in C^1(\mathbb{R})$. The algebraic inequality $(a + b)^2 \geq 2ab$ and a simple change of variables $s = v(t)$ allow us to conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \left[W(v(t)) + |\dot{v}(t)|^2\right] \,\mathrm{d}t, &\geq 2 \int_{-\infty}^{\infty} \sqrt{W(v(t))}|\dot{v(t)}| \,\mathrm{d}t \\ &\geq \int_{-\infty}^{\infty} 2\sqrt{W(v(t))}\dot{v(t)} \,\mathrm{d}t \\ &= \int_{-1}^{1} 2\sqrt{W(s)} \,\mathrm{d}s = \sigma_0. \end{aligned}$$

Now notice that the inequality is $(a + b)^2 \geq 2ab$ is an equality if and only if $a = b$, while $|\dot{v}| = \dot{v}$ is true provided that $\dot{v} \geq 0$. It follows that the optimal v is the solution of

$$\dot{v}(t) = \sqrt{W(v(t))}$$

with the additional constraint that $v(\pm\infty) = \pm 1$, and it can be proved that it is of class C^1 and, depending on the regularity of W, even more regular.

2.5. Tools: finite perimeter sets. The first step is to introduce functions with *bounded variation*. Let Ω be a regular bounded domain in $\mathbb{R}^d$ and define

$$\mathrm{BV}(\Omega) := \{u \in L^1(\Omega) \ :\ Du \text{ is a vector-valued measure}\},$$

where Du is the distributional derivative of u. In other words, if $u \in \mathrm{BV}(\Omega)$, then there are μ positive finite Borel measure and f unit Borel vector field such that

$$Du = f\mu,$$

as distributions, which can immediately be translated to the condition

$$\int_\Omega u \operatorname{div}(\phi) \,\mathrm{d}x = -\int_\Omega \phi \, Du = -\int_\Omega \phi \cdot f \,\mathrm{d}\mu \quad \text{for every } \phi \in C_c^\infty(\Omega). \tag{6.5}$$

A standard argument in functional analysis allows us to extend the class of test functions to $C_0^1(\Omega)$. The space $\mathrm{BV}(\Omega)$ is a non-separable Banach space with norm

$$\|u\|_{\mathrm{BV}} := \|u\|_{L^1(\Omega)} + \|Du\| = \|u\|_{L^1(\Omega)} + \mu(\Omega).$$

PROPOSITION 6.16. *For every $1 \le p \le \infty$, the Sobolev space $W^{1,p}(\Omega)$ is contained in* $\mathrm{BV}(\Omega)$.

EXAMPLE 6.17. *Piecewise-C^1 functions are contained in $\mathrm{BV}(\Omega)$. Let $u : \Omega \to \mathbb{R}$ be such a function, let Ω_i be disjoint open subsets of Ω such that*

$$\bigcup_{i=1}^M \overline{\Omega_i} = \bar{\Omega},$$

with boundary $\partial\Omega_i$ of class piecewise-C^1, and let $u_i \in C^1(\overline{\Omega_i})$ be functions such that $u \equiv u_i$ in the interior of Ω_i for all $i = 1, \dots, m$. We claim that

$$Du = \sum_{i=1}^M \nabla u_i \cdot \mathcal{L}^d \llcorner \Omega_i + \sum_{i<j} (u_j - u_i) n_{i,j} \cdot \mathcal{H}^{d-1} \llcorner (\partial\Omega_i \cap \partial\Omega_j),$$

where $\mathcal{L}^d$ is the d-dimensional Lebsgue measure, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $n_{i,j}$ is the normal pointing from Ω_i to Ω_j.

REMARK 6.18. The Sobolev space $W^{1,1}(\Omega)$ is a closed subspace of $\mathrm{BV}(\Omega)$.

We now list a few well-known results concerning bounded variation functions. We need some regularity on Ω so we will tacitly assume that $\partial\Omega$ is Lipschitz.

THEOREM 6.19 (Embedding). *The immersion $\mathrm{BV}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for every $1 \le p \le p^*$ and compact for every $1 \le p < p^*$.*

REMARK 6.20. The Poincaré inequality for $\mathrm{BV}(\Omega)$ functions works in the exact same way as for Sobolev functions in $W^{1,1}(\Omega)$.

THEOREM 6.21 (Compactness). *Let u_n be a sequence of functions such that $\|u_n\|_{\mathrm{BV}(\Omega)} \le C < \infty$. Then (up to subsequences) u_n converges in $L^1(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$ and*

$$Du_n \rightharpoonup Du$$

in the sense of measures, that is, if $Du_n = f_n\mu_n$ and $Du = f\mu$, then

$$\int_\Omega \phi f_n \,\mathrm{d}\mu_n \xrightarrow{n\to\infty} \int_\Omega \phi f \,\mathrm{d}\mu \quad \textit{for all } \phi \in C_0(\Omega, \mathbb{R}^d).$$

Furthermore, we have

$$\liminf_{n\to\infty} \mu_n(\Omega) \geq \mu(\Omega).$$

THEOREM 6.22 (Approximation). *Let* $u \in \mathrm{BV}(\Omega)$. *Then there exists a sequence* $(u_n)_{n\in\mathbb{N}} \in C^\infty(\bar{\Omega})$ *such that the following properties hold:*

(i) u_n *converges to* u *strongly in* $L^1(\Omega)$*;*

(ii) $Du_n \rightharpoonup Du$ *in the sense of measures;*

(iii) $\|\nabla u_n\|_{L^1(\Omega)} \to \|Du\|$.

DEFINITION 6.23. A measurable set $E \subseteq \Omega$ *has finite perimeter in* Ω if the characteristic function χ_E belongs to $\mathrm{BV}(\Omega)$. The perimeter of E inside of Ω is defined by

$$\mathrm{Per}(E,\Omega) := \|D\chi_E\| = \sup\left\{\int_E \mathrm{div}(\phi)\,\mathrm{d}x \;:\; \phi \in C_c^\infty(\Omega), \|\phi\|_\infty \leq 1\right\}. \tag{6.6}$$

EXAMPLE 6.24. *If* $E \subset \Omega$ *and* $\partial E \cap \Omega$ *is a* C^1*-hypersurface, then* E *has finite perimeter and its distributional derivative is given by*

$$D\chi_E = -\eta \cdot \mathcal{H}^{d-1} \llcorner (\partial E \cap \Omega),$$

where η *is the external unit normal of* E. *Furthermore, the perimeter is given by*

$$\mathrm{Per}(E,\Omega) = \mathcal{H}^{d-1}(\partial E \cap \Omega).$$

REMARK 6.25. It is worth noticing that the definition of perimeter **in** Ω does not take into account the portion of the perimeter of E which is on the boundary of Ω.

REMARK 6.26. The compactness and the approximation results given for $\mathrm{BV}(\Omega)$ functions can be easily specialized to hold for finite perimeter sets.

2.6. Tools: finite perimeter sets. Recall that, given $E \subset \Omega$, we denote by χ_E the characteristic function of E.

DEFINITION 6.27. A measurable set $E \subseteq \Omega$ *has finite perimeter in* Ω if the characteristic function χ_E belongs to $\mathrm{BV}(\Omega)$. The perimeter of E inside of Ω is defined by

$$\mathrm{Per}_\Omega(E) := \|D\chi_E\| = \sup\left\{\int_E \mathrm{div}(\phi)\,\mathrm{d}x \;:\; \phi \in C_c^\infty(\Omega), \|\phi\|_\infty \leq 1\right\}. \tag{6.7}$$

EXAMPLE 6.28. *If $E \subset \Omega$ and $\partial E \cap \Omega$ is a C^1 surface, then E has finite perimeter and its distributional derivative is given by*

$$D\chi_E = -\eta \cdot \mathcal{H}^{d-1} \llcorner (\partial E \cap \Omega),$$

where η is the external unit normal of E. Furthermore, the perimeter is given by

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{d-1}(\partial E \cap \Omega).$$

PROPOSITION 6.29. *If $\mathcal{H}^{d-1}(\partial E \cap \Omega)$ is finite, then E has finite perimeter and*

$$\mathrm{Per}_\Omega(E) \leq \mathcal{H}^{d-1}(\partial E \cap \Omega) \tag{6.8}$$

The inequality in (6.8), as the next example shows, can be strict as a consequence of the fact that the distributional derivative "does not take into account what happens if we modify a negligible set in E."

EXAMPLE 6.30. *Let E be the unit disk $B(0,1) \subset \Omega$. If we remove the diameter, we obtain a new set $\tilde{E} \subset \Omega$ such that*

$$\mathcal{H}^{d-1}(\partial E \cap \Omega) < \mathcal{H}^{d-1}(\partial \tilde{E} \cap \Omega).$$

On the other hand, it is easy to prove that $D\chi_E = D\chi_{\tilde{E}}$ so using (6.8) *we conclude that*

$$\mathrm{Per}_\Omega(\tilde{E}) = \mathrm{Per}_\Omega(E) \leq \mathcal{H}^{d-1}(\partial E \cap \Omega) < \mathcal{H}^{d-1}(\partial \tilde{E} \cap \Omega).$$

REMARK 6.31. If $E \subset \Omega$ and ∂E is $\mathcal{H}^{d-1}$-finite, then E has finite perimeter in Ω but it is not trivial to determine the distributional derivative.

In the next example, we show that finite perimeter sets can be rather "wild" since they might have boundary which is not negligible w.r.t. the Lebesgue measure $\mathcal{L}^d$.

EXAMPLE 6.32. *Let $B_n := B(x_n, r_n) \subset \Omega$ be a sequence of balls satisfying*

$$\sum_{n\in\mathbb{N}} r_n^{d-1} < \infty.$$

The set $E := \bigcup_{n\in\mathbb{N}} B_n$ has finite perimeter as a consequence of Proposition 6.29, but depending on how we choose x_n, several weird behaviors can arise:

(1) If $(x_n)_{n\in\mathbb{N}}$ is dense in Ω and $|E| < |\Omega|$, then E is dense in Ω, its topological boundary is given by $\Omega \setminus E$, and

$$|\partial E| = |\Omega \setminus E| = |\Omega| - |E| > 0.$$

(2) *There exists a set E of finite perimeter in Ω such that the topological boundary of E is equal to $\bar{\Omega}$.*

The following compactness result follows immediately from the compactness of $\mathrm{BV}(\Omega)$-functions. Is $|\cdot|$ denotes the Lebesgue measure, notice that

$$\|\chi_A - \chi_B\|_{L^1(\Omega)} = |A \triangle B|,$$

where $A \triangle B$ denotes the *symmetric difference* of the sets $A, B \subset \Omega$.

THEOREM 6.33 (Compactness). *Let $E_n \subset \Omega$ be a sequence of sets with uniformly bounded perimeter, that is,*

$$\mathrm{Per}_\Omega(E) \leq C < \infty.$$

Then there exists a finite perimeter set $E \subset \Omega$ such that up to subsequences

$$|E \triangle E_n| \xrightarrow{n\to\infty} 0.$$

Furthermore, the perimeter is lower semicontinuous:

$$\mathrm{Per}_\Omega(E) \leq \liminf_{n\to\infty} \mathrm{Per}_\Omega(E_n).$$

EXERCISE 6.1. Find a Borel set $A \subset \Omega$ such that $\mathrm{Per}_\Omega(A) = \infty$.

THEOREM 6.34 (Approximation). *Let $E \subset \Omega$ be a set with finite perimeter. Then there exists a sequence of sets $E_n \subset \Omega$ with smooth boundary in Ω such that*

$$|E_n \triangle E| \xrightarrow{n\to\infty} 0$$

and

$$\mathrm{Per}_\Omega(E) = \lim_{n\to\infty} \mathrm{Per}_\Omega(E_n).$$

The next result, known as *structure theorem*, is highly nontrivial and the proof requires a great deal of geometric measure theory.

REMARK 6.35. For $d = 1$, any finite perimeter set is, up to a Lebesgue-null set, equal to a finite unions of intervals.

DEFINITION 6.36. The *d-dimensional density* of E at x is defined by

$$\Theta_d(E, x) := \lim_{r\to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

wherever this limit exists.

DEFINITION 6.37. The *measure theoretic boundary* of E, denoted by $\partial_* E$, is the set

$$\partial_* E := \{x \in E \ : \ \text{either } \Theta_d(E,x) \text{ does not exists or it belongs to } (0,1)\}.$$

THEOREM 6.38 (Structure theorem). *Let $E \subset \Omega$ be a set with finite perimeter. Then the following assertions hold true:*

(i) *The measure theoretic boundary $\partial_* E$ is $\mathcal{H}^{d-1}$-finite and rectifiable (=it can be covered, up to a $\mathcal{H}^{d-1}$-null set, by countably many hypersurfaces Σ_i of class C^1).*

(ii) For almost every $\bar{x} \in \partial_* E$, there exists $\nu(\bar{x})$ such that

$$H(\bar{x}) := \{x \in \mathbb{R}^d \;:\; (x - \bar{x}) \cdot \nu(\bar{x}) < 0\}$$

satisfies the property

$$|(H(\bar{x}) \triangle E) \cap B(\bar{x}, r)| \ll r^d.$$

In other words, the density of $H(\bar{x}) \triangle E$ at $\bar{x}$ is equal to zero and, consequently,

$$\Theta_d(E, \bar{x}) = \frac{1}{2}.$$

The vector $\nu(\bar{x})$ is usually called measure-theoretic outer normal *of E at $\bar{x}$.*

(iii) The distributional derivative of χ_E is given by

$$D\chi_E = -\nu \cdot \mathcal{H}^{d-1} \llcorner (\partial_* E \cap \Omega).$$

2.7. Tools: coarea formula. Let $u : \Omega \to \mathbb{R}$ be a function of class C^1 such that $\nabla u(x) \neq 0$ for all $x \in \Omega$. For every $t \in \mathbb{R}$, we can consider the C^1 hypersurface

$$\Sigma_t := \{x \in \Omega \;:\; u(x) = t\}.$$

THEOREM 6.39 (Coarea). *Let $h : \Omega \to [0, \infty]$ be a positive*[1] *Borel function. Then*

$$\int_{-\infty}^{\infty} \left(\int_{\Sigma_t} h(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) \right) \mathrm{d}t = \int_\Omega h(x) |\nabla u(x)| \, \mathrm{d}x. \tag{6.9}$$

The proof of this result is left to the reader as an exercise. Notice that it can be easily proved under these assumptions using a change of variables to reduce to Fubini's theorem.

REMARK 6.40. The assumption $\nabla u(x) \neq 0$ that makes Σ_t a regular surface is not necessary, but proving the identity (6.9) is harder because we cannot reduce to Fubini.

REMARK 6.41. The assumption $u \in C^1(\Omega)$ is not necessary and, in fact, we can replace it with Lipschitz. On the other hand, being differentiable a.e. is not enough - dealing with less regularity is no easy task (nut it should be true if $W^{1,p}(\Omega)$ for $p > d$).

REMARK 6.42. The coarea formula for bounded variation functions holds, but it is necessary to change the definition of Σ_t accordingly.

2.8. Γ-convergence result. Let us go back to the Ginzburg-Landau scalar model but, for the sake of simplicity, we will assume that the double-well potential

$$W : \mathbb{R} \to [0, \infty)$$

is a continuous function and satisfies $W(u) = 0$ when $u = 0$ (instead of $u = -1$) and $u = 1$. For $\epsilon > 0$, recall that the energy is given by

$$F_\epsilon(u) = \int_\Omega \left[W(u) + \epsilon^2 |\nabla u|^2 \right] \mathrm{d}x.$$

[1] As usual we require the integrand to be positive for the integral to be well-defined and taking values in $[0, \infty]$.

We mentioned earlier that the Γ-limit of F_ϵ is not interesting so we consider the rescaled functional

$$E_\epsilon(u) := \frac{1}{\epsilon} F_\epsilon(u) = \int_\Omega \left[\frac{1}{\epsilon} W(u) + \epsilon |\nabla u|^2 \right] \mathrm{d}x.$$

Recall that we consider the constraint $\fint_\Omega u \, \mathrm{d}x = m$, where m is a fixed number in $(0,1)$ and notice that moving the well from $u = -1$ to $u = 0$ changes the value of σ_0 to

$$\sigma_0 = 2 \int_0^1 \sqrt{W(s)} \, \mathrm{d}s.$$

The following theorem was proved by Modica in 1987, but the result is usually referred to as Modica-Mortola theorem because it is primarily based on the method introduced by them in a paper in 1978.

THEOREM 6.43 (Modica-Mortola). *Let us consider the functional space*

$$\mathfrak{X} := \{u : \Omega \to [0,1] \ : \ \fint_\Omega u \, \mathrm{d}x = m\} \subseteq L^1(\Omega)$$

equipped with distance induced by the $L^1(\Omega)$ one. For $\epsilon > 0$ and $u \in \mathfrak{X}$ define

$$E_\epsilon(u) = \begin{cases} \int_\Omega \left[\frac{1}{\epsilon} W(u) + \epsilon |\nabla u|^2 \right] \mathrm{d}x & \textit{if } u \in W^{1,2}(\Omega), \\ \infty & \textit{otherwise} \end{cases}$$

and

$$E(u) = \begin{cases} \sigma_0 \mathrm{Per}_\Omega(A) & \textit{if } u = \chi_A \in \mathrm{BV}(\Omega), \\ \infty & \textit{otherwise}. \end{cases}$$

Then E_ϵ is a equicoercive family of functionals and $E_\epsilon \xrightarrow{\Gamma} E$ as $\epsilon \to 0^+$.

COROLLARY 6.44. *Minimizers $\bar{u}_\epsilon$ of F_ϵ converge in $L^1(\Omega)$ to minimizers $\bar{u}$ of E.*

Notice that the functional E admits a minimizer because we can apply lower semicontinuity and compactness results for finite perimeter sets.

PROOF OF THEOREM 6.43. We can assume $u_\epsilon \in C^\infty(\bar{\Omega})$ because smooth functions are strongly dense and E_ϵ is continuous with respect to the strong topology.

Step 1. We shall now prove equicoerciveness and the $\Gamma - \liminf$ inequality contemporaneously. Let $u_\epsilon \in \mathfrak{X}$ be a sequence of functions such that

$$E_\epsilon(u_\epsilon) \leq C < \infty$$

for a uniform constant $C > 0$. Then (up to subsequences) u_ϵ converges in $L^1(\Omega)$ to some $u \in \mathfrak{X}$ and we have the inequality

$$\liminf_{\epsilon \to 0} E_\epsilon(u_\epsilon) \geq E(u).$$

Using the algebraic inequality $a^2 + b^2 \geq 2ab$ we find that

$$E_\epsilon(u_\epsilon) \geq \int_\Omega 2\sqrt{W(u_\epsilon)}|\nabla u_\epsilon| \, \mathrm{d}x.$$

If $H : \mathbb{R} \to \mathbb{R}$ is a function satisfying $H(0) = 0$ and $\dot{H}(u) = \sqrt{W(u)}$, then

$$E_\epsilon(u_\epsilon) \geq 2\int_\Omega \dot{H}(u_\epsilon)|\nabla u_\epsilon| \, \mathrm{d}x = \int_\Omega |\nabla(H \circ u_\epsilon)| \, \mathrm{d}x.$$

Now denote $H \circ u_\epsilon$ by v_ϵ and notice that the inequality above implies

$$\|\nabla v_\epsilon\|_{L^1(\Omega)} \leq C < \infty,$$

where C is a uniform constant. Since $\mathrm{BV}(\Omega)$ has good compactness properties (see Theorem 6.21) we have

$$\|v_\epsilon - v\|_{L^1(\Omega)} \xrightarrow{\epsilon \to 0} 0 \quad \text{and} \quad \liminf_{\epsilon \to 0} \|\nabla u_\epsilon\|_{L^1(\Omega)} \geq \|Dv\|.$$

The limit function $v \in \mathrm{BV}(\Omega)$ may not be in $W^{1,1}(\Omega)$ so we need to use the distributional derivative Dv. In any case, this implies that

$$u_\epsilon = H^{-1}(v_\epsilon) \xrightarrow{L^1(\Omega)} u = H^{-1}(v)$$

and, consequently, we conclude that

$$\int_\Omega W(u_\epsilon(x)) \, \mathrm{d}x \to 0 \implies \int_\Omega W(u(x)) \, \mathrm{d}x = 0 \implies u \equiv \chi_A.$$

Since $v = H \circ u$ takes the values $H(0) = 0$ and $H(1) = \sigma_0$, we necessarily have

$$v = \sigma_0 \chi_A$$

and, given that $v \in \mathrm{BV}(\Omega)$, the set A must have finite perimeter. The $\Gamma - \liminf$ inequality is now easy since

$$\liminf_{n \to \infty} E_\epsilon(u_\epsilon) \geq \liminf_{n \to \infty} \int_\Omega |\nabla v_\epsilon|^2 \, \mathrm{d}x \geq \|Dv\|$$

and $\|Dv\| = \sigma_0 \mathrm{Per}_\Omega(A)$.

Step 2. We can find a recovery sequence for functions of the form $u = \chi_A$ with ∂A smooth and transversal to $\partial\Omega$ as a consequence of Theorem 6.34. If

$$u_\epsilon(x) := v\left(\frac{d_{\partial A}(x)}{\epsilon}\right),$$

where v is the solution of $\dot{v}(s) = \sqrt{W(v(s))}$ with boundary conditions $v(-\infty) = 0$ and $v(\infty) = 1$, then it is easy to show that

$$\lim_{\epsilon \to 0} E_\epsilon(u_\epsilon) = E(\chi_A),$$

and this concludes the proof. □

CHAPTER 7

Rearrangement inequalities

I will write the introduction after the chapter is complete!

1. Introduction

Let $A \subset \mathbb{R}^d$. Throughout this section we will always denote by $|A|$ the volume (w.r.t. the Lebesgue measure) of the set A.

DEFINITION 7.1. The *symmetric (or radial) rearrangement* of A, denoted by A^*, is the open ball with center zero and volume $|A|$. In other words,

$$A^* := \{x \in \mathbb{R}^d \,:\, \omega_d |x|^d < |A|\},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

DEFINITION 7.2. Let $u : \mathbb{R}^d \to [0, \infty]$ be a measurable function. The *symmetric decreasing rearrangement* of u is the function $u^* : \mathbb{R}^d \to [0, \infty]$ such that

$$\{x \in \mathbb{R}^d \,:\, u^*(x) > t\} = \big(\{x \in \mathbb{R}^d \,:\, u(x) > t\}\big)^* \tag{7.1}$$

holds for all $t > 0$.

REMARK 7.3. If we require u to take values in $[-\infty, 0]$, we can define the symmetric increasing rearrangement by simply replacing superlevels with sublevels in (7.1).

REMARK 7.4. It is possible to define the *decreasing rearrangement* of a measurable $u : \mathbb{R}^d \to [-\infty, \infty]$ that is finite a.e. using the distribution

$$d_u(s) : [0, \infty] \to [0, \infty]$$

defined as $d_u(s) := |\{x \in \mathbb{R}^d \,:\, f(x) > s\}|$. The decreasing rearrangement (which is not symmetric anymore) is the function

$$\tilde{u}(t) := \inf\{s \in [0, \infty] \,:\, d_u(s) \leq t\}.$$

It can be proved that it shares several properties with the symmetric rearrangement introduced above, but we will not discuss it any further in this course.

REMARK 7.5. Let $u^*(x)$ be the symmetric rearrangement of $u : \mathbb{R}^d \to [0, \infty]$. Then there exists a radially symmetric function $\rho : [0, \infty) \to [0, \infty]$ such that

$$u^*(x) = \rho(|x|).$$

Furthermore, the function ρ is **decreasing** and left-continuous.

We conclude this section with a final remark.

REMARK 7.6. The definition can be extended to any $\Omega \subseteq \mathbb{R}^d$ and $u : \Omega \to [0, \infty]$, but we usually introduce u^* to compare it with u so we would like to have

$$\Omega = \Omega^*.$$

This means that either Ω is $\mathbb{R}^d$ or Ω is an open ball centered at the origin 0.

2. Main properties of the symmetric rearrangement

The goal of this section is to study the main properties of the symmetric rearrangement which will later be useful in a couple of applications.

REMARK 7.7. We denote by $E_t(u)$ the superlevel of u, that is,

$$E_t(u) := \{x \in \Omega \,:\, u(x) > t\}$$

in such a way that the following identity holds by definition:

$$(E_t(u))^* = E_t(u^*).$$

In what follows, we will always assume that u and v are measurable functions defined on $\mathbb{R}^d$ taking values in $[0, \infty]$ so that the symmetric rearrangement is well-defined.

PROPOSITION 7.8. *If $u \geq v$, then $u^* \geq v^*$.*

PROOF. If $u \geq v$, then $E_t(v) \subseteq E_t(u)$ for all $t > 0$. This immediately implies that

$$E_t(v^*) = (E_t(v))^* \subseteq (E_t(u))^* = E_t(u^*),$$

which, in turn, gives $u^* \geq v^*$ as claimed. □

THEOREM 7.9. *Let u be as above and let $g : [0, \infty) \to [0, \infty]$ be a lower semicontinuous function with $g(0) = 0$. Then*

$$\int_{\mathbb{R}^d} g(u(x))\,\mathrm{d}x = \int_{\mathbb{R}^d} g(u^*(x))\,\mathrm{d}x. \tag{7.2}$$

PROOF. We divide the proof into two steps: we first prove (7.2) under additional regularity assumptions and then we reduce the general case to it by approximation.

Step 1. Assume that $g \in C^\infty([0, \infty))$ with compact support $\mathrm{spt}(g) \subset (0, \infty)$ so that it stays away from origin and ∞. For every $s \geq 0$ we have

$$g(s) = \int_0^s \dot{g}(t)\,\mathrm{d}t,$$

so taking $s = u(x)$ yields

$$g(u(x)) = \int_0^{u(x)} \dot{g}(t)\,\mathrm{d}t \implies \int_{\mathbb{R}^d} g(u(x))\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}x \int_0^{u(x)} \dot{g}(t)\,\mathrm{d}t.$$

By Fubini's theorem (which we can apply thanks to the extra assumptions on g) we find that

$$\int_{\mathbb{R}^d} g(u(x))\,\mathrm{d}x = \int_0^\infty \dot{g}(t)|E_t(u)|\,\mathrm{d}t = \int_0^\infty \dot{g}(t)|E_t(u^*)|\,\mathrm{d}t = \int_{\mathbb{R}^d} g(u^*(x))\,\mathrm{d}x$$

and this concludes the proof.

Step 2. We use the usual approximation via an increasing sequence of functions g_n and use the monotone convergence theorem to conclude that

$$\int_{\mathbb{R}^d} g_n(u(x))\,\mathrm{d}x \xrightarrow{n\to\infty} \int_{\mathbb{R}^d} g(u(x))\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^d} g_n(u^*(x))\,\mathrm{d}x \xrightarrow{n\to\infty} \int_{\mathbb{R}^d} g(u^*(x))\,\mathrm{d}x.$$

□

COROLLARY 7.10. *For all $1 \le p \le \infty$ the symmetric decreasing rearrangement preserves the L^p-norm, that is,*

$$\|u\|_{L^p(\mathbb{R}^d)} = \|u^*\|_{L^p(\mathbb{R}^d)}. \tag{7.3}$$

THEOREM 7.11. *Let u and v be as above and let $h : \mathbb{R} \to [0,\infty]$ be a lower semicontinuous, even and convex function with $h(0) = 0$. Then*

$$\int_{\mathbb{R}^d} h(u(x) - v(x))\,\mathrm{d}x \ge \int_{\mathbb{R}^d} h(u^*(x) - v^*(x))\,\mathrm{d}x. \tag{7.4}$$

PROOF. We divide, once again, the proof into two steps assuming at first more regularity and then using an approximation argument to conclude.

Step 1. Assume that $h : \mathbb{R} \to [0,\infty)$ is even, convex, belongs to the class C^2 and $h(0) = 0$, and notice that $\dot{h}(0) = 0$ as well. Given $s' > s \ge 0$ we have

$$h(s' - s) = \int_s^{s'} \dot{h}(t' - s)\,\mathrm{d}t' = \int_s^{s'} \mathrm{d}t' \int_s^{t'} \ddot{h}(t' - t)\,\mathrm{d}t.$$

Taking $s' = u(x)$ and $s = v(x)$ leads to

$$h(u(x) - v(x)) = \int_{v(x)}^{u(x)} \mathrm{d}t' \int_{v(x)}^{t'} \ddot{h}(t' - t)\,\mathrm{d}t.$$

Let $A^+ = \{x \in \mathbb{R}^d \;:\; u(x) \ge v(x)\}$ so that the identity above holds for all $x \in A^+$. Then

$$\int_{A^+} h(u(x) - v(x))\,\mathrm{d}x = \int_{A^+} \left[\int_{v(x)}^{u(x)} \mathrm{d}t' \int_{v(x)}^{t'} \ddot{h}(t' - t)\,\mathrm{d}t\right] \mathrm{d}x$$

and it is rather easy to see that in the right-hand side x ranges in $E_{t'}(u) \setminus E_t(v)$. We now apply Fubini's theorem and find that

$$\int_{A^+} h(u(x) - v(x))\,\mathrm{d}x = \iint_{0 \le t \le t' < \infty} \ddot{h}(t'-t)|E_{t'}(u) \setminus E_t(v)|\,\mathrm{d}t'\mathrm{d}t.$$

If we use $A^- = \{x \in \mathbb{R}^d \,:\, u(x) < v(x)\}$ instead of A^+, we obtain a similar identity:

$$\int_{A^-} h(u(x) - v(x))\,\mathrm{d}x = \iint_{0 \le t \le t' < \infty} \ddot{h}(t'-t)|E_{t'}(v) \setminus E_t(u)|\,\mathrm{d}t'\mathrm{d}t.$$

The decomposition $\mathbb{R}^d = A^+ \cup A^-$ immediately implies that

$$\int_{\mathbb{R}^d} h(u(x) - v(x))\,\mathrm{d}x = \iint_{0 \le t \le t' < \infty} \ddot{h}(t'-t)\left(|E_{t'}(u) \setminus E_t(v)| + |E_{t'}(v) \setminus E_t(u)|\right)\,\mathrm{d}t'\mathrm{d}t.$$

We now notice that, given A and B sets in $\mathbb{R}^d$, it is always true that

$$|A \setminus B| \ge (|A| - |B|)^+,$$

where $(\cdot)^+$ denotes the positive part of the quantity between brackets. On the other hand, the rearrangements A^* and B^* are balls centered at the origin so

$$(|A^*| - |B^*|)^+ = |A^* \setminus B^*|$$

because they are contained one inside the other. In particular, it turns out that

$$\begin{aligned} |E_{t'}(u) \setminus E_t(v)| &\ge (|E_{t'}(u)| - |E_t(v)|)^+ \\ &= (|E_{t'}(u^*)| - |E_t(v^*)|)^+ \\ &= |E_{t'}(u^*) \setminus E_t(v^*)|, \end{aligned}$$

and, similarly,

$$|E_{t'}(v) \setminus E_t(u)| \ge |E_{t'}(v^*) \setminus E_t(u^*)|.$$

Since we proved already that

$$\int_{\mathbb{R}^d} h(u^*(x) - v^*(x))\,\mathrm{d}x = \iint_{0 \le t \le t' < \infty} \ddot{h}(t'-t)\left(|E_{t'}(u^*) \setminus E_t(v^*)| + |E_{t'}(v^*) \setminus E_t(u^*)|\right)\,\mathrm{d}t'\mathrm{d}t,$$

we simply put these last two estimates together to obtain (7.4).

Step 2. We use the usual approximation via an increasing sequence of functions h_n and use the monotone convergence theorem as above. $\square$

COROLLARY 7.12. *The symmetric decreasing rearrangement $*$ is a contraction from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1 \le p \le \infty$, that is,*

$$\|u - v\|_{L^p(\mathbb{R}^d)} \ge \|u^* - v^*\|_{L^p(\mathbb{R}^d)}.$$

THEOREM 7.13. *If $u \in W^{1,p}(\mathbb{R}^d)$ with $p \ge 1$, then $u^* \in W^{1,p}(\mathbb{R}^d)$. Similarly, if $u \in \mathrm{BV}(\mathbb{R}^d)$, then u^* also belongs to $\mathrm{BV}(\mathbb{R}^d)$.*

THEOREM 7.14. *Let $1 \le p \le \infty$ and let $u \in W^{1,p}(\mathbb{R}^d)$, $1 \le p \le \infty$. Suppose that $f : [0,\infty) \times [0,\infty) \to [0,\infty]$ is a function satisfying the following properties:*

(1) f is lower semicontinuous in both variables;

(2) $f(u,\cdot)$ is convex for all $u \in [0,\infty)$;

(3) $f(u,0)=0$ for all $u \in [0,\infty)$.

Then

$$\int_{\mathbb{R}^d} f(u,|\nabla u|)\,\mathrm{d}x \geq \int_{\mathbb{R}^d} f(u^*,|\nabla u^*|)\,\mathrm{d}x. \tag{7.5}$$

PROOF. Assume enough regularity for u and f. Since $f(u,\cdot)$ is convex we can find coefficients a, b depending on both u and $\bar{\xi}$ such that

$$f(u,\xi) \geq a\xi - b \quad \text{for every } \xi \in [0,\infty) \text{ and } f(u,\bar{\xi}) = a\bar{\xi} - b.$$

Let $u = u^*(x)$ and $\bar{\xi} = |\nabla u^*(x)|$. Then

$$f(u^*(x),|\nabla u^*(x)|) = a(u^*(x),|\nabla u^*(x)|)|\nabla u^*(x)| - b(u^*(x),|\nabla u^*(x)|),$$

but it can be readily checked that a and b only depend on u^* because the gradient of radial functions has modulus which is radial as well. Then

$$\int_{\mathbb{R}^d} f(u^*(x),|\nabla u^*(x)|)\,\mathrm{d}x = \int_{\mathbb{R}^d} a(u^*(x))|\nabla u^*(x)|\,\mathrm{d}x - \int_{\mathbb{R}^d} b(u^*(x))\,\mathrm{d}x,$$

and we can apply (7.2) to conclude that

$$\int_{\mathbb{R}^d} b(u^*(x))\,\mathrm{d}x = \int_{\mathbb{R}^d} b(u(x))\,\mathrm{d}x.$$

To deal with the other integral we use the coarea formula and write

$$\begin{aligned}\int_{\mathbb{R}^d} a(u^*(x))|\nabla u^*(x)|\,\mathrm{d}x &= \int_0^\infty a(t)\,\mathrm{d}t \int_{\Sigma_t(u^*)} \mathrm{d}\mathcal{H}^{d-1} \\ &= \int_0^\infty a(t)\mathcal{H}^{d-1}(\Sigma_t(u^*))\,\mathrm{d}t,\end{aligned}$$

where $\Sigma_t(u^*)$ is the level surface of u^*, namely

$$\Sigma_t(u^*) = \{x \in \mathbb{R}^d \ : \ u^*(x) = t\}.$$

Since $\Sigma_t(u^*)$ is the boundary of $E_t(u^*)$, we can apply the *isoperimetric inequality* to assert that

$$|E_t(u^*)| = |E_t(u)| \implies \mathcal{H}^{d-1}(\Sigma_t(u^*)) \leq \mathcal{H}^{d-1}(\Sigma_t(u))$$

because $E_t(u^*)$ is a ball so its perimeter is the smallest one for fixed volume. This means that

$$\int_0^\infty a(t)\mathcal{H}^{d-1}(\Sigma_t(u^*))\,\mathrm{d}t \leq \int_0^\infty a(t)\mathcal{H}^{d-1}(\Sigma_t(u))\,\mathrm{d}t$$

and applying coarea formula again to the right-hand side we find that

$$\int_0^\infty a(t)\mathcal{H}^{d-1}(\Sigma_t(u))\,\mathrm{d}t = \int_{\mathbb{R}^d} a(u(x))|\nabla u(x)|\,\mathrm{d}x.$$

Putting everything we proved so far together leads to (7.5). □

COROLLARY 7.15 (Pólya-Szego). *For* $1 \leq p \leq \infty$ *and* $u \in W^{1,p}(\mathbb{R}^d)$ *we have*

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^d)}. \tag{7.6}$$

2.1. Application: best constant for Sobolev embedding. It is well-known that

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d)}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, so we can also extend it to the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the seminorm $\|\nabla u\|_{L^p(\mathbb{R}^d)}$. The sharp constant $C > 0$ is given by

$$\sup_{u \in C_c^\infty(\mathbb{R}^d)} \frac{\|u\|_{L^{p^*}(\mathbb{R}^d)}}{\|\nabla u\|_{L^p(\mathbb{R}^d)}}.$$

However, applying (7.6) and (7.3) immediately implies that we can take the supremum among radial functions. Determining the right u and the value of C is particularly important in several fields of mathematics such as analysis of PDEs.

2.2. Application: first eigenvalue of $-\Delta$. Let Ω be a bounded set in $\mathbb{R}^d$. Then

$$\lambda_1(\Omega) := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$$

is the first eigenvalue of the Laplacian in Ω with Dirichlet boundary conditions. It is easy to see that

$$\lambda(\Omega^*) \leq \lambda(\Omega)$$

so for a fixed volume m of Ω the optimal eigenvalue (i.e., the smallest possible) is achieved by the ball.

CHAPTER 8

Isoperimetrical Inequalities (by Aldo Pratelli)

I will write the introduction after the chapter is complete!

1. Introduction to finite perimeter sets

Given $m > 0$, we would like to solve the problem of minimizing the perimeter for a given area, that is,

$$J(m) := \min_{|E|=m} \operatorname{Per}(E), \tag{8.1}$$

where E ranges among all sets in $\mathbb{R}^N$. A priori there is no reason why a minimum should exist, but we will soon see that any ball of volume m works.

REMARK 8.1. The minimization problem (8.1) is translation invariant. In other words, it is easy to see that

$$J(m) = \operatorname{Per}(E) \implies J(m) = \operatorname{Per}(E + x)$$

for all $x \in \mathbb{R}^N$.

Before we can go any further, we need to establish what we mean exactly by "perimeter". It could seem a good idea to define

$$\operatorname{Per}(E) = \mathcal{H}^{N-1}(\partial E),$$

where ∂E is the topological boundary of E but, as the next example shows, this is a good definition only for sets E with boundary smooth enough.

EXAMPLE 8.2. *Let E be the unit disk $B(0, 1) \subset \mathbb{R}^N$. If we remove the diameter, we obtain a new set $\tilde{E} \subset \mathbb{R}^N$ such that*

$$\mathcal{H}^{N-1}(\partial E) < \mathcal{H}^{N-1}(\partial \tilde{E}).$$

This is a situation that we would like to avoid, so the idea is to use bounded variation functions as "the characteristic will not be able to see that the diameter was removed."

DEFINITION 8.3. A set $E \subset \mathbb{R}^N$ has finite perimeter if $\chi_E \in \mathrm{BV}(\mathbb{R}^N)$. The perimeter of E is given by the total variation of the distributional derivative, that is,

$$\operatorname{Per}(E) := |D\chi_E|(\mathbb{R}^N). \tag{8.2}$$

REMARK 8.4. If E has finite perimeter and Lipschitz boundary, then the distributional derivative is given by

$$D\chi_E = -\nu_E \cdot \mathcal{H}^{N-1}(\partial E),$$

and therefore the total variation coincides with the naïve notion of perimeter:

$$|D\chi_E|(\mathbb{R}^N) = \mathcal{H}^{N-1}(\partial E).$$

EXAMPLE 8.5. *If E is the unit disk $B(0,1) \subset \mathbb{R}^N$ and $\tilde{E}$ the set obtained by removing the diameter, then it is easy to see that*

$$|D\chi_E|(\mathbb{R}^N) = |D\chi_{\tilde{E}}|(\mathbb{R}^N) \implies \mathrm{Per}(E) = \mathrm{Per}(\tilde{E}).$$

REMARK 8.6. The situation can be much more complex. If E is a ball and $\tilde{E}$ the set obtained from E by adding all hyper-planes with a rational coordinate, then

$$\mathrm{Per}(E) = \mathrm{Per}(\tilde{E}),$$

but the topological boundary changes drastically ∂E is the sphere and $\partial\tilde{E}$ the whole $\mathbb{R}^N$.

The notion of topological boundary is therefore not good in this setting and must be replaced by a different one. The idea is to find it in such a way that

$$\mathrm{Per}(E) = \mathcal{H}^{N-1}(\partial^* E)$$

holds even if E is not regular, where $\partial^* E$ is the "new boundary".

DEFINITION 8.7. Let E be a set of finite perimeter. We say that x belongs to the *reduced boundary* of E, denoted by $x \in \partial^* E$, if there exists $\nu \in \mathbb{S}^{N-1}$ such that

$$\lim_{\epsilon\to 0} \frac{|(B^+(x,\epsilon)\cap E)|}{|B^+(x,\epsilon)|} = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{|(B^-(x,\epsilon)\cap E)|}{|B^-(x,\epsilon)|} = 1,$$

where B^+ and B^- are respectively defined as

$$B^+(x,\epsilon) = \{y \in B(x,\epsilon) \,:\, (y-x)\cdot\nu > 0\},$$
$$B^-(x,\epsilon) = \{y \in B(x,\epsilon) \,:\, (y-x)\cdot\nu < 0\}.$$

In this case, the unique ν achieving this will be called *outer normal to $\partial^* E$ at x*.

THEOREM 8.8. *Let E be a set of finite perimeter. Then*

$$\mathrm{Per}(E) = \mathcal{H}^{N-1}(\partial^* E).$$

DEFINITION 8.9. The *N-dimensional density* of E at x is defined by

$$\Theta_N(E,x) := \lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{|B(x,r)|}$$

wherever this limit exists. We shall denote by E^d the set $\{p \in E \,:\, \Theta_N(E,p) = d\}$.

REMARK 8.10. If E has finite perimeter, then the characteristic function χ_E belongs to $L^1(\mathbb{R}^N)$. It follows that $\mathcal{H}^N$-a.e. $p \in E$ either belongs to E^0 or E^1.

COROLLARY 8.11. *Let E be a set of finite perimeter. If $p \in \partial^* E$, then $p \in E^{\frac{1}{2}}$.*

For a general set $E \subset \mathbb{R}^N$ we have the chain of inclusions (that would be equalities with some regularity assumptions):

$$\partial^* E \subseteq E^{\frac{1}{2}} \subseteq \mathbb{R}^N \setminus (E^0 \cup E^1).$$

It is not completely trivial, but it can be proved that these are at most different up to $\mathcal{H}^{N-1}$-negligible sets of points.

2. Existence of minimizers

Let us take a step back in the previous section and consider the functional we are interested in minimizing, that is,

$$J(m) := \inf\{\mathrm{Per}(E) \ : \ |E| = m, \ E \subset \mathbb{R}^N\}.$$

Notice that $J(m) \geq 0$ but, a priori, it may only be an infimum. Furthermore, for all $x \in \mathbb{R}^N$ and $\lambda > 0$ we have

$$|E + x| = |E| \quad \text{and} \quad |\lambda E + x| = \lambda^N |E|.$$

The perimeter is translation invariant but scales with a different factor, that is,

$$\mathrm{Per}(\lambda E + x) = \lambda^{N-1} \mathrm{Per}(E),$$

so it is enough to solve the isoperimetric problem for a fixed value of $m = 1$ and find the value of the minimum for all $m > 0$ using the identity

$$J(m) = m^{\frac{N-1}{N}} J(1).$$

2.1. Existence, Part I. Let E_j be a minimizing sequence for $J(m)$, namely they satisfy the volume constraint $|E_j| = m$ and

$$\mathrm{Per}(E_j) \searrow J(m).$$

The sequence χ_{E_j} is contained in $\mathrm{BV}(\mathbb{R}^N)$ and it is uniformly bounded because

$$\|\chi_{E_j}\|_{L^1(\mathbb{R}^N)} = m \quad \text{and} \quad \mathrm{Per}(E_j) \leq C < \infty$$

as the sequence of perimeters converges to $J(m)$ as real numbers. By compactness

$$\chi_{E_j} \rightharpoonup^\star f$$

for some $f \in \mathrm{BV}(\mathbb{R}^N)$ in the weak sense of measures (up to subsequences). However, this implies the strong convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and hence

$$\chi_{E_j} \xrightarrow{L^1_{\mathrm{loc}}} f.$$

This gives pointwise convergence a.e. starting from functions that only take values in $\{0, 1\}$ so f is also the characteristic function of some set E. Furthermore, we have

$$|D\chi_E|(\mathbb{R}^N) \leq \liminf_{j \to \infty} |D\chi_{E_j}|(\mathbb{R}^N) = \liminf_{j \to \infty} \mathrm{Per}(E_j) = J(m)$$

so E has finite perimeter which is less than or equal to $J(m)$ so, if E has volume m (which is not true in general), then E is the desired minimum point.

Before we can go on with the proof of the existence, we need to find a reasonable way to modify the sequence E_j in such a way that the limit has volume exactly equal to m.

REMARK 8.12. Let $\epsilon > 0$ be fixed, $F \subset \mathbb{R}^N$ and consider a section $\{(x_1, \ldots, x_n) \in F \,:\, x_1 = t\}$ satisfying

$$\mathcal{H}^{N-1}(F_t) \leq \epsilon.$$

The section divides F in two parts, the left one F_ℓ given by $\{x \in F \,:\, x_1 < t\}$ and the right part F_r given by $\{x \in F \,:\, x_1 > t\}$. It is easy to verify that

$$\mathrm{Per}(F) \geq \mathrm{Per}(F_\ell) + \mathrm{Per}(F_r) - 2\epsilon \geq J(1)\left[|F_\ell|^{\frac{N-1}{N}} + |F_r|^{\frac{N-1}{N}}\right] - 2\epsilon.$$

If we assume that $\mathrm{Per}(F) \leq J(|F|) + 1$, then it follows that

$$J(1)\left[|F|^{\frac{N-1}{N}} - |F_\ell|^{\frac{N-1}{N}} - |F_r|^{\frac{N-1}{N}}\right] \geq -3\epsilon$$

Using the sub-additivity of the function $f(x) = x^{\frac{N-1}{N}}$ we find that

$$|F|^{\frac{N-1}{N}} \leq |F_\ell|^{\frac{N-1}{N}} + |F_r|^{\frac{N-1}{N}},$$

which in turn implies that there must be some positive $\delta = \delta(\epsilon)$ such that

$$\min\{|F_\ell|, |F_r|\} < \delta.$$

In other words, the volume of one of the two parts must be close to zero, that is, less than δ where $\delta \to 0$ as $\epsilon \to 0$.

2.2. Existence, Part II. Up to a translation we can always assume that

$$|E_j \cap \{x_1 > 0\}| = \frac{m}{2}.$$

If $\epsilon > 0$ is given, then for all j big enough it turns out that

$$\mathrm{Per}(E_j) < J(m) + \epsilon.$$

By Fubini's theorem we can always find $0 < t < \frac{m}{2\epsilon}$ such that

$$\mathcal{H}^{N-1}(E_j \cap \{x_1 = t\}) < \epsilon,$$

and by Remark 8.12 there is $\delta > 0$ such that

$$|E_j \cap \{x_1 > t\}| < \delta.$$

Notice that it cannot be the left part because t is positive and we already know that there is $m/2$ mass in the interval $t < 0$. Therefore

$$\left|E_j \cap \left\{x_1 \geq \frac{m}{\epsilon}\right\}\right| \leq \delta$$

and, in a similar fashion, one can prove that

$$\left|E_j \cap \left\{x_1 \leq -\frac{m}{\epsilon}\right\}\right| \leq \delta.$$

In both cases, the number δ does not depend on j. If we do the same for all directions in $\mathbb{R}^N$, then we easily find that

$$\left| E_j \cap \left[-\frac{m}{\epsilon}, \frac{m}{\epsilon} \right]^N \right| > m - 2N\delta.$$

We have strong convergence in $L^1_{\text{loc}}(\mathbb{R}^N)$ so inside these boxes the convergence is strong, and therefore it suffices to notice that

$$|E| \geq \left| E \cap \left[-\frac{m}{\epsilon}, \frac{m}{\epsilon} \right]^N \right| = \lim_{j\to\infty} \left| E_j \cap \left[-\frac{m}{\epsilon}, \frac{m}{\epsilon} \right]^N \right| \geq m - 2N\delta.$$

The arbitrariness of ϵ allows one to conclude that $|E| \geq m - 2N\delta$ and, by taking the limit as $\epsilon \to 0^+$, we easily obtain

$$|E| \geq m \implies |E| = m.$$

3. Symmetries in minimizing the perimeter

Let $E \subset \mathbb{R}^N$ be a set of finite perimeter, let Π be an affine $(N-1)$-dimensional hyperplane and define the two half-spaces $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$ in such a way that

$$\mathbb{R}^N \setminus \Pi = \mathbb{R}^N_+ \cup \mathbb{R}^N_-.$$

Assume that

$$|E \cap \mathbb{R}^N_+| = |E \cap \mathbb{R}^N_-|$$

and define E^- and E^+ as the two set symmetric sets with respect to Π, that is,

$$E^+ := (\mathbb{R}^N_+ \cap E) \cup \pi(\mathbb{R}^N_+ \cap E)$$

where π is the symmetry reflects $(\mathbb{R}^N_+ \cap E)$ with respect to the hyperplane Π, and in a similar way we can define E^-. Notice that

$$|E| = |E^+| = |E^-|$$

holds by construction.

PROPOSITION 8.13. *Let E be as above. Then*

$$\text{Per}(E) \geq \frac{\text{Per}(E^+) + \text{Per}(E^-)}{2}.$$

PROOF. First, notice that the reduced boundary of E^+ satisfies

$$\partial^* E^+ \subseteq (\mathbb{R}^N_+ \cap \partial^* E) \cup \pi(\mathbb{R}^N_+ \cap \partial^* E) \cup (\Pi \cap \partial^* E)$$

where π is the symmetry defined earlier. Replacing $\mathbb{R}^N_+$ with $\mathbb{R}^N_-$ we find a similar inclusion for the reduced boundary of E^-, which leads to

$$\begin{aligned} \text{Per}(E^+) + \text{Per}(E^-) &\leq 2\left[\mathcal{H}^{N-1}(\mathbb{R}^N_+ \cap \partial^* E) + \mathcal{H}^{N-1}(\mathbb{R}^N_- \cap \partial^* E) + \mathcal{H}^{N-1}(\Pi \cap \partial^* E) \right] \\ &= 2\text{Per}(E), \end{aligned}$$

and this concludes the proof. $\square$

REMARK 8.14. Suppose that E^+ has smaller perimeter. Since E and E^+ have the same volume, the previous proposition tells us that E^+ is a better candidate for

$$\min\left\{\mathrm{Per}(A)\ :\ A \subset \mathbb{R}^N,\ |A| = m\right\}.$$

We can use any bisecting hyperplane and, in particular, the coordinate ones, to infer that E can always be replaced by its N-symmetrization.

4. Steiner symmetrization

Let $E \subset \mathbb{R}^N$ be a set and for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^{N-1}$ define

$$E_x := \{t \in \mathbb{R}\ :\ (x,t) \in E\} \subseteq \mathbb{R}.$$

The set E_x is well-defined $\mathcal{H}^1$-almost everywhere (thanks to Fubini's theorem), and therefore we have a function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}^+$ such that

$$\varphi(x) = \mathcal{H}^1(E_x).$$

DEFINITION 8.15. The *Steiner symmetrization* of E is the set

$$\tilde{E} := \left\{(x',t) \in \mathbb{R}^N\ :\ -\frac{\varphi(x')}{2} < t < \frac{\varphi(x')}{2}\right\}.$$

We will now study the main properties of Steiner symmetrization. It is worth remarking that it is useful in the minimization problem because it does not increase the perimeter.

PROPOSITION 8.16. *Let $\tilde{E}$ be the Steiner symmetrization of E. Then*

$$\mathrm{Per}(\tilde{E}) \leq \mathrm{Per}(E).$$

PROOF. Suppose $|E| < \infty$.

Step 1. Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of smooth sets such that $\chi_{E_j} \to \chi_E$ in $L^1(\mathbb{R}^N)$, $D\chi_{E_j} \rightharpoonup^* D\chi_E$ in the weak sense of measures and

$$\mathrm{Per}(E_j) \xrightarrow{j\to\infty} \mathrm{Per}(E).$$

Now recall that the perimeter is lower semicontinuous and $\chi_{\tilde{E}_j} \to \chi_{\tilde{E}}$ in $L^1(\mathbb{R}^N)$ as a consequence of $\chi_{E_j} \to \chi_E$. It turns out that

$$\begin{aligned}\mathrm{Per}(E) &= \lim_{j\to\infty} \mathrm{Per}(E_j)\\ &\geq \liminf_{j\to\infty} \mathrm{Per}(E_j)\\ &\geq \liminf_{j\to\infty} \mathrm{Per}(\tilde{E}_j) \geq \mathrm{Per}(\tilde{E})\end{aligned}$$

so we only need to prove the thesis for smooth sets.

Step 2. We now assume that E is a smooth set of finite perimeter with at most a finite number of points with vertical tangent at the boundary, that is,

$$\{x \in \partial^* E\ :\ \nu(x) \in \mathbb{R}^{N-1}\} = \{x_1, \dots, x_q\}.$$

We can consequently find finitely many disjoint open sets $A_\ell \subset \mathbb{R}^{N-1}$ and associate to each one a number $k(\ell)$ defined as the number of "sections" in which the set

$$(A_\ell \times \mathbb{R}) \cap E$$

is divided as a consequence of the fact that some of the points with vertical tangent will also be in A_ℓ. Thus for all ℓ we can find smooth functions $g^+_{i,\ell}, g^-_{i,\ell} : A_\ell \to \mathbb{R}$ such that

$$g^-_{1,\ell} < g^+_{1,\ell} < g^-_{2,\ell} < \cdots < g^+_{k(\ell),\ell}$$

and $(A_\ell \times \mathbb{R}) \cap E$ is contained in the following union of graphs:

$$\bigcup_{i=1}^{k(\ell)} \left(\operatorname{graph}(g^+_{i,\ell}) \cup \operatorname{graph}(g^-_{i,\ell})\right).$$

In other words, the reduced boundary $\partial^* E$ is locally the graph of a smooth function and the perimeter of E can be written as

$$\operatorname{Per}(E) = \mathcal{H}^{N-1}\left(\partial^* E \cap \left(\bigcup_\ell A_\ell \times \mathbb{R}\right)\right).$$

We now notice that

$$\mathcal{H}^{N-1}\left(\{(x', g(x')) \ : \ x \in A\}\right) = \int_A (1 + |\nabla g|^2)\, \mathrm{d}\mathcal{H}^{N-1}(x')$$

and the right-hand side integrand $(1+|\nabla g|^2)$ is a strictly convex function of x' so it satisfies the following inequality:

$$\frac{\sqrt{1+|\nabla g^+|^2} + \sqrt{1+|\nabla g^-|^2}}{2} \geq \sqrt{1 + \left(\frac{|\nabla g^+| + |\nabla g^-|}{2}\right)^2}.$$

We can now use the perimeter formula above with $\tilde{E}$ in place of E. In particular, it is easy to prove that

$$\begin{aligned}
\mathcal{H}^{N-1}\left(\partial^* E \cap (A_\ell \times \mathbb{R})\right) &= 2\int_{A_\ell} \sqrt{1 + \left(\frac{\nabla\varphi}{2}\right)^2}\, \mathrm{d}\mathcal{H}^{N-1}(x') \\
&\leq 2\int_{A_\ell} \sqrt{1 + \left(\frac{|\nabla g^+| + |\nabla g^-|}{2}\right)^2}\, \mathrm{d}\mathcal{H}^{N-1}(x') \\
&\leq \int_{A_\ell} \left[\sqrt{1+|\nabla g^+|^2} + \sqrt{1+|\nabla g^-|^2}\right] \mathrm{d}\mathcal{H}^{N-1}(x'),
\end{aligned}$$

assuming that $k(\ell) = 1$ and noticing that $\varphi(x') = g^+(x') - g^-(x')$ so that $\nabla\varphi(x') = \nabla g^+(x') - \nabla g^-(x')$ and

$$|\nabla\varphi(x')| \leq |\nabla g^+(x')| + |\nabla g^-(x')|.$$

A similar argument works for $k(\ell) > 1$. Since there are only finitely many A_ℓ we obtain similar estimates for all ℓ and put all together to conclude that

$$\mathrm{Per}(\tilde{E}) \leq \mathrm{Per}(E),$$

proving the theorem. □

REMARK 8.17. The functions used above are strictly increasing and strictly convex.

REMARK 8.18. Suppose that Q is a smooth cube (i.e., a cube with smoothened vertexes) and add a "small" cubic perturbation on one of the sides obtaining Q_p. Then

$$\mathrm{Per}(Q_p) = \mathrm{Per}(\tilde{Q}_p),$$

but after the Steiner symmetrization the set $\tilde{Q}_p$ only moves the perturbation to look "more symmetric" but it does not get absorbed to become once again a smooth cube.

At this point, we would like to say that all the sections are segments, but this is only true in the smooth case as we show in the next lemma.

LEMMA 8.19. *Let E be a smooth set with no vertical points at the boundary and let*

$$\Gamma := \{x' \in \mathbb{R}^{N-1} \;:\; E_{x'} \text{ consists of at least } 2 \text{ segments}\},$$

where "consists of at least 2 segments" is a well-defined notion up to a $\mathcal{H}^{N-2}$-negligible set and can be written as

$$\#(\partial^* E_{x'}) \geq 2.$$

Then there exists a continuous and increasing function ψ such that $\psi(0) = 0$ and $\psi(x) > 0$ for all $x > 0$ for which the following perimeter estimate holds:

$$\mathrm{Per}(E) \geq \mathrm{Per}(\tilde{E}) + \mathrm{Per}(E) \cdot \psi\left(\frac{\mathcal{H}^{N-1}(\Gamma)}{\mathrm{Per}(E)}\right). \tag{8.3}$$

COROLLARY 8.20. *If $E \subset \mathbb{R}^N$ is a set of finite perimeter, then*

$$\mathrm{Per}(E) > \mathrm{Per}(\tilde{E}).$$

PROOF. Let E_j be a sequence of smooth sets with no vertical points at the boundary such that $\chi_{E_j} \to \chi_E$ in $L^1(\mathbb{R}^N)$, $D\chi_{E_j} \rightharpoonup^* D\chi_E$ in the weak sense of measures and

$$\mathrm{Per}(E_j) \xrightarrow{j\to\infty} \mathrm{Per}(E).$$

Notice that

$$\liminf_{j\to\infty} \mathcal{H}^{N-1}(\Gamma_j) \geq \mathcal{H}^{N-1}(\Gamma),$$

but the right-hand side is actually equal to the limit the sequence $\mathcal{H}^{N-1}(\Gamma_j)$. In any case, using (8.3) and the properties of ψ we find that

$$\begin{aligned}
\operatorname{Per}(E) &= \lim_{j\to\infty} \operatorname{Per}(E_j) \\
&\geq \lim_{j\to\infty} \left[\operatorname{Per}(\tilde{E}_j) + \operatorname{Per}(E_j)\cdot\psi\left(\frac{\mathcal{H}^{N-1}(\Gamma_j)}{\operatorname{Per}(E_j)}\right)\right] \\
&\geq \liminf_{j\to\infty} \operatorname{Per}(\tilde{E}_j) + \liminf_{j\to\infty}\left[\operatorname{Per}(E_j)\cdot\psi\left(\frac{\mathcal{H}^{N-1}(\Gamma_j)}{\operatorname{Per}(E_j)}\right)\right] \\
&\geq \operatorname{Per}(\tilde{E}) + \operatorname{Per}(E)\cdot\liminf_{j\to\infty}\psi\left(\frac{\mathcal{H}^{N-1}(\Gamma_j)}{\operatorname{Per}(E_j)}\right) \\
&\geq \operatorname{Per}(\tilde{E}) + \operatorname{Per}(E)\cdot\psi\left(\frac{\liminf_{j\to\infty}\mathcal{H}^{N-1}(\Gamma_j)}{\operatorname{Per}(E)}\right) \\
&\geq \operatorname{Per}(\tilde{E}) + \operatorname{Per}(E)\cdot\psi\left(\frac{\mathcal{H}^{N-1}(\Gamma)}{\operatorname{Per}(E)}\right),
\end{aligned}$$

where the last inequality follows from ψ increasing. Since $\psi(x) > 0$ for all $x > 0$ we easily conclude from this estimate that

$$\operatorname{Per}(E) > \operatorname{Per}(\tilde{E}).$$

$\square$

COROLLARY 8.21. *Isoperimetric sets are convex. In other words, if E is a minimum point, then the set of points with density 1 (denoted by E^1) is convex.*

In the previous lecture, we stated (without proof) the following lemma, which gives a precise estimate of the difference between the perimeter of E and the perimeter of $\tilde{E}$.

LEMMA 8.22. *Let E be a smooth set with no vertical points at the boundary and let*

$$\Gamma := \{x' \in \mathbb{R}^{N-1} \ : \ E_{x'} \text{ consists of at least } 2 \text{ segments}\},$$

where "consists of at least 2 segments" is a well-defined notion up to a $\mathcal{H}^{N-2}$-negligible set and can be written as

$$\#(\partial^* E_{x'}) \geq 2.$$

Then there exists a continuous and increasing function ψ such that $\psi(0) = 0$ and $\psi(x) > 0$ for all $x > 0$ for which the following perimeter estimate holds:

$$\mathrm{Per}(E) \geq \mathrm{Per}(\tilde{E}) + \mathrm{Per}(E) \cdot \psi\left(\frac{\mathcal{H}^{N-1}(\Gamma)}{\mathrm{Per}(E)}\right). \tag{8.4}$$

We also stated and proved a corollary that holds for set of finite perimeter so it now only remains to show that (8.4) holds for smooth sets.

PROOF. Let $\tau(t) := \sqrt{1+t^2}$ and recall that the perimeter of $\tilde{E}$ can also be written as

$$\mathrm{Per}(\tilde{E}) = 2\int_{\mathbb{R}^{N-1}\cap\{\varphi(x')>0\}} \tau\left(\frac{|\nabla\varphi(x')|}{2}\right) \, \mathrm{d}\mathcal{H}^{N-1}(x'),$$

while the perimeter of E is slightly more complicated and equal to

$$\mathrm{Per}(E) = \int_{\mathbb{R}^{N-1}\cap\{\varphi(x')>0\}} \sum_{j=1}^{q}\sum_{i=1}^{k(j)} \left[\tau(|\nabla g^+_{i,j}(x')|) + \tau(|\nabla g^-_{i,j}(x')|)\right] \, \mathrm{d}\mathcal{H}^{N-1}(x'),$$

where q is the number of disjoint sets A_ℓ and $k(j)$ is the number defined in the previous lecture. Remember also that

$$\varphi(x') = \sum_{j=1}^{q}\sum_{i=1}^{k(j)} \left[g^+_{i,j}(x') + g^-_{i,j}(x')\right],$$

so we have an easy estimate on the absolute value of the gradient of φ:

$$|\nabla\varphi(x')| \leq \sum_{j=1}^{q}\sum_{i=1}^{k(j)} \left[|\nabla g^+_{i,j}(x')| + |\nabla g^-_{i,j}(x')|\right].$$

Using the strict convexity of φ for a fixed $1 \leq j \leq q$ we find that

$$\begin{aligned}\sum_{i=1}^{k(j)} \left[\tau(|\nabla g^+_{i,j}(x')|) + \tau(|\nabla g^-_{i,j}(x')|\right] &\geq 2k(j)\tau\left(\frac{\sum_{i=1}^{k(j)}\left[|\nabla g^+_{i,j}(x')| + |\nabla g^-_{i,j}(x')|\right]}{2k(j)}\right) \\ &\geq 2k(j)\tau\left(\frac{|\nabla\varphi(x')|}{2k(j)}\right).\end{aligned}$$

Summing over all possible j's, we find the following inequality:

$$\sum_{j=1}^{q}\sum_{i=1}^{k(j)}\left[\tau(|\nabla g_{i,j}^{+}(x')|)+\tau(|\nabla g_{i,j}^{-}(x')|\right]\geq\sum_{j=1}^{q}2k(j)\tau\left(\frac{|\nabla\varphi(x')|}{2k(j)}\right).$$

There are now two possible scenarios. If $k(j)=1$ for all j, then we simply obtain

$$\mathrm{Per}(E)\geq\mathrm{Per}(\tilde{E}),$$

which was already known. On the other hand, if $k(j)\geq 2$ for some (at least one) $j\in\{1,\dots,q\}$, then we find a more precise perimeter inequality:

$$\mathrm{Per}(E)-\mathrm{Per}(\tilde{E})\geq\int_{\Gamma}\left[4\tau\left(\frac{|\nabla\varphi(x')|}{4}\right)-2\tau\left(\frac{|\nabla\varphi(x')|}{2}\right)\right]\,\mathrm{d}\mathcal{H}^{N-1}(x')$$

We now introduce a positive and strictly decreasing function f in such a way that the right-hand side can be rewritten in a more compact way. More precisely, let f be such that

$$\int_{\Gamma}\left[4\tau\left(\frac{|\nabla\varphi(x')|}{4}\right)-2\tau\left(\frac{|\nabla\varphi(x')|}{2}\right)\right]\,\mathrm{d}\mathcal{H}^{N-1}(x')=\int_{\Gamma}f(|\nabla\varphi(x')|)\,\mathrm{d}\mathcal{H}^{N-1}(x').$$

Notice that $f(0)=2$ and $\lim_{s\to\infty}f(s)=0$. Consider the decomposition $\Gamma=\Gamma^{+}\cup\Gamma^{-}$ in such a way that the measures are equal, that is,

$$\mathcal{H}^{N-1}(\Gamma^{+})=\mathcal{H}^{N-1}(\Gamma^{-})=\frac{\mathcal{H}^{N-1}(\Gamma)}{2}.$$

It is easy to verify that $|\nabla\varphi(x')|\geq|\nabla\varphi(y')|$ for all $x'\in\Gamma^{+}$ and all $y'\in\Gamma^{-}$. Therefore, we can find a positive constant M such that

$$|\nabla\varphi(x')|\geq M\geq|\nabla\varphi(y')|$$

for all $x'\in\Gamma^{+}$ and all $y'\in\Gamma^{-}$. This means that

$$\mathrm{Per}(E)\geq\mathrm{Per}(\tilde{E})\geq 2\int_{\Gamma^{+}}\tau\left(\frac{|\nabla\varphi(x')|}{2}\right)\,\mathrm{d}\mathcal{H}^{N-1}(x')\geq\tau\left(\frac{M}{2}\right)\mathcal{H}^{N-1}(\Gamma)$$

and, given that $\tau\left(\frac{M}{2}\right)\geq\frac{M}{2}$, we obtain an upper bound for M:

$$M\leq\frac{2\mathrm{Per}(E)}{\mathcal{H}^{N-1}(\Gamma)}.$$

Since f is strictly decreasing, we can use this bound to say that

$$f(M)\geq f\left(\frac{2\mathrm{Per}(E)}{\mathcal{H}^{N-1}(\Gamma)}\right).$$

To conclude, we put this together with the estimate on the difference between the perimeters obtain above to find that

$$\begin{aligned}\mathrm{Per}(E)-\mathrm{Per}(\tilde{E}) &\geq \int_\Gamma f(|\nabla\varphi(x')|)\,\mathrm{d}\mathcal{H}^{N-1}(x')\\ &\geq f(M)\frac{\mathcal{H}^{N-1}(\Gamma)}{2}\\ &\geq f\left(\frac{2\mathrm{Per}(E)}{\mathcal{H}^{N-1}(\Gamma)}\right)\frac{\mathcal{H}^{N-1}(\Gamma)}{2\mathrm{Per}(E)}\mathrm{Per}(E).\end{aligned}$$

In particular, the desired inequality (8.4) is obtained by choosing ψ as

$$\psi(t) := \frac{t}{2}f\left(\frac{2}{t}\right).$$

The function ψ satisfies all the required properties and hence Lemma 8.22 is proved. ☐

REMARK 8.23. The same formula (8.4) can be proved for finite perimeter sets using the approximation by smooth sets with no vertical points at the boundary.

5. The isoperimetric set is a ball

The final goal of this section is to prove that the set with the minimal perimeter for a fixed volume is the ball. First, we need a technical result:

LEMMA 8.24. *Let E be a isoperimetric set which is symmetric with respect to the origin (any point would be fine). Then E is a ball.*

PROOF. Throughout this proof, we will always use the symbol E to denote E^1, the set of points with density one. In particular, we can assume that E is convex.

Step 1. We first notice that (due to our assumptions) there exists a function $\ell : \mathbb{S}^{N-1} \to \mathbb{R}$ such that for all $\theta \in \mathbb{S}^{N-1}$ we can write

$$E^1 \cap \mathbb{R}\cdot\theta = [-\ell(\theta), \ell(\theta)]\cdot\theta.$$

In other words, ℓ describes the segment in E^1 with direction given by θ. Fix $\theta, \nu \in \mathbb{S}^{N-1}$ with $\theta\cdot\nu > 0$ and let Π be the hyperplane passing through ν such that

$$(\mathrm{Span}(\theta,\nu))^\perp \subseteq \Pi.$$

Then Π bisects E (i.e., it divides E in two halves with same volume) which is symmetric w.r.t. the origin, so we can divide it as in the previous lecture:

$$E = E^+ \cup E^-.$$

Now let $P = \ell(0)\cdot\theta$ and Q be its symmetric with respect to Π. Denote by E' the set obtained by symmetrization (w.r.t. Π) starting from the half-space that contains P, that is,

$$P \in \mathbb{R}^N_+ \implies E' = E^+ \cup \rho_\Pi(E^+),$$

where ρ_Π is the reflection with respect to Π (and similarly if $P \in \mathbb{R}^N_-$). Then E' is isoperimetric and both P and Q belong to the boundary $\partial E'$. Moreover, for $\lambda \in (0,1)$ we have

$$\lambda P, \lambda Q \in (E')^1$$

and, more precisely, the whole segment with extrema λP and λQ does:

$$[\lambda P, \lambda Q] \subseteq (E')^1.$$

Now let $R := \frac{P+Q}{2} \in \Pi$ and notice that $\lambda R \in (E')^1$. Since E' coincides with E on one half of the space, it also turns out that $\lambda R \in E$ has density one. It follows that

$$\ell(\nu) \geq |R| = \ell(\theta)\theta \cdot \nu \tag{8.5}$$

for all $\theta, \nu \in \mathbb{S}^{N-1}$ such that $\theta \cdot \nu > 0$.

Step 2. Let us consider a finite number of directions θ_i, $1 \leq i \leq M$, in such a way that $\theta_0 = \theta$, $\theta_M = \nu$ and for all i we have

$$\theta_i \in \mathrm{Span}\langle \nu, \theta \rangle.$$

Let $\alpha \in \mathbb{S}^1$ be the angle between θ and ν in such a way that

$$\theta \cdot \nu = \cos\alpha.$$

It is easy to verify that the scalar product between any two consecutive directions is

$$\theta_i \cdot \theta_{i+1} = \cos\left(\frac{\alpha}{M}\right),$$

so using (8.5) a finite number of times yields

$$\ell(\nu) = \ell(\theta_M) \geq \ell(\theta_{M-1}) \cos\left(\frac{\alpha}{M}\right) \geq \ell(\theta) \cos\left(\frac{\alpha}{M}\right)^M.$$

If we send $M \to \infty$, we find that

$$\ell(\nu) \geq \ell(\theta) \implies \ell(\nu) = \ell(\theta),$$

which means that ℓ is a constant function and therefore E is a ball. □

THEOREM 8.25. *Let E be an isoperimetrical set. Then E is a ball.*

PROOF. Up to translations we can always assume that

$$|E \cap \{x_1 > 0, \dots, x_j > 0\}| = 2^{-j}|E| \quad \text{for any } 1 \leq j \leq N.$$

Now E^1 is isoperimetric because $\{x_1 = 0\}$ bisects E, where E^1 is the symmetrization of the right part, that is,

$$\{x \in \mathbb{R}^N \ : \ (|x_1|, x_2, \dots, x_N) \in E\}.$$

Similarly, the second hyperplane $\{x_2 = 0\}$ bisects E_1 (although it may not bisect E), and hence the set

$$E^2 := \{x \in \mathbb{R}^N \ : \ (|x_1|, |x_2|, x_3, \dots, x_N) \in E\}$$

is isoperimetric as well. We can iterate the argument to find that

$$E^N = \{x \in \mathbb{R}^N \ : \ (|x_1|, \dots, |x_N|) \in E\}$$

is isoperimetric. Furthermore, by construction it is symmetric w.r.t. the origin (actually, something more), so we can use Lemma 8.24 to infer that E^N is a ball. Then

$$E \cap \{x_1 > 0, \dots, x_N > 0\}$$

is also a ball and, since we can do the same for all possible quadrants of the space $\mathbb{R}^N$, we immediately conclude that E itself is a ball. ☐

6. Generalization to more than one set: clusters

Let $m \in (\mathbb{R}^+)^M$ be a M-tuple of real positive numbers. The goal of this section is to minimize the functional

$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^{M} \partial^* E_i\right)$$

among all sets $E_1, \dots, E_M \subset \mathbb{R}^N$ with empty intersections ($E_i \cap E_j = \emptyset$) and satisfying the volume constraint $|E_1| = m_1, \dots, |E_M| = m_M$.

REMARK 8.26. The problem is nontrivial. Suppose that $M = 2$ and take two disjoint balls that satisfy, singularly, the isoperimetrical problem with masses m_1 and m_2. If we move them closer and closer, we notice that the parts of the boundaries around points which are almost "in contact" look flatter and flatter. Therefore, we are essentially counting the perimeter of these flat parts twice (once for each ball), and this is obviously not so convenient. In particular, the solution to the clustering problem is not a disjoint union of balls but rather (?) a figure obtained by merging them so that the issue above does not arise.

6.1. Existence of a minimizer. Fix $m \in (\mathbb{R}^+)^M$. We shall denote clusters by italic letters as follows

$$\mathcal{E} = \{E_1, \dots, E_M\}.$$

We will also say that $\mathcal{E}$ has mass m and write $|\mathcal{E}| = m$ if and only if

$$|E_i| = m_i \quad \text{for all } 1 \le i \le M.$$

We would like to find an optimal cluster $\bar{\mathcal{E}} := \{\bar{E}_1, \dots, \bar{E}_M\}$ that achieves the minimum of the variational problem

$$J(m) := \inf_{|\mathcal{E}|=m} \operatorname{Per}(\mathcal{E}), \tag{8.6}$$

where the perimeter of the cluster is defined by setting

$$\operatorname{Per}(\mathcal{E}) := \mathcal{H}^{N-1}\left(\bigcup_{i=1}^{M} \partial^* E_i\right).$$

6.2. Proof, Part I. Let $\{\mathcal{E}^j\}_{j\in\mathbb{N}}$ be a minimizing sequence of clusters for the functional J, and denote by E_ℓ^j the sets inside of each $\mathcal{E}$. Fix $1 \le \ell \le M$ and notice that

$$\|\chi_{E_\ell^j}\|_{\mathrm{BV}(\mathbb{R}^N)} \le C_\ell < \infty$$

is uniformly bounded in BV and hence, up to subsequences, we have that $\chi_{E_\ell^j}$ converges in BV to a characteristic function $\chi_{\bar{E}_\ell}$ for all $1 \le \ell \le M$. Furthermore,

$$\chi_{E_\ell^j} \cdot \chi_{E_{\ell'}^j} = 0$$

holds for all $\ell \neq \ell'$ so we can send $j \to \infty$ (as both converge strongly locally in L^1 and in other any L^p) to find that the limit sets are also disjoint:

$$\chi_{\bar{E}_\ell} \cdot \chi_{\bar{E}_{\ell'}} = 0 \implies \bar{E}_\ell \cap \bar{E}_{\ell'} = \emptyset.$$

It is easy to see that

$$\mathrm{Per}(\bar{\mathcal{E}}) = \mathcal{H}^{N-1}\left(\cup_{\ell=1}^M \partial^* \bar{E}_\ell\right) \le \liminf_{j\to\infty} \mathrm{Per}(\mathcal{E}^j) = J(m)$$

by lower semicontinuity, so we only have to prove that the limit cluster $\bar{\mathcal{E}}$ satisfies the volume constraint.

REMARK 8.27. Notice that the perimeter of any cluster can be rewritten as

$$\mathrm{Per}(\mathcal{E}) = \frac{1}{2}\left[\sum_{j=1}^M \mathrm{Per}(E_j) + \mathrm{Per}\left(\bigcup_{j=1}^M E_j\right)\right],$$

and this is exactly what we need to prove the lower semicontinuity since each perimeter in the right-hand side is computed at a single set rather than a cluster.

The goal of today's lecture is to prove that we can find an optimal cluster $\bar{\mathcal{E}}$ that satisfies the mass constraint and, not surprisingly, we will follow what we did for a single set.

REMARK 8.28. Recall that, in the isoperimetrical problem with a single set, a fundamental step was proving the inequality

$$J(m' + m'') < J(m') + J(m'')$$

for all $m', m'' > 0$. This was an immediate consequence of the scaling property

$$J(m) = m^{\frac{N-1}{N}} J(1),$$

which holds for all $m > 0$. However, for clusters we do not have such a scaling property because two vectors m and m' in $(\mathbb{R}^+)^M$ are not necessarily proportional.

Nevertheless, we can assume (for the time being) that for all $m', m'' \in (\mathbb{R}^+)^M \setminus \{0\}$ the perimeter inequality

$$J(m' + m'') < J(m') + J(m'')$$

holds. By continuity for all $\epsilon > 0$ we can find a positive δ such that

$$\min\{|m'|, |m''|\} \geq \epsilon \implies J(m') + J(m'') - J(m' + m'') \geq \delta.$$

Now let $\mathcal{E}$ be a cluster with total mass equal to $m = m' + m''$ and perimeter satisfying the inequality $\mathrm{Per}(\mathcal{E}) < J(m) + \frac{\delta}{3}$ and, in addition, assume that

$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^{M}(E_i \cap \{x_1 = t\})\right) < \frac{\delta}{3}.$$

We now denote by $\mathcal{E}_\ell$ the left cluster, that is, the cluster that consists of the sets

$$E_{i,\ell} := E_i \cap \{x_1 < t\}$$

and, similarly, define the right cluster $\mathcal{E}_r$. Set $m' := |\mathcal{E}_\ell|$ and $m'' = |\mathcal{E}_r|$, and notice that

$$J(m) > \mathrm{Per}(\mathcal{E}) - \frac{\delta}{3} > \mathrm{Per}(\mathcal{E}_\ell) + \mathrm{Per}(\mathcal{E}_r) - \delta \geq J(m') + J(m'') - \delta,$$

which, in other words, tells us that

$$J(m' + m'') \geq J(m') + J(m'').$$

However, this is in contradiction with the estimate that we assumed to be true so the only possibility is that one of them is not bigger than ϵ, namely

$$\min\{|m'|, |m''|\} < \epsilon.$$

At this point, the existence of a minimizer with total mass m can be achieved as in the isoperimetrical problem with a single set, but we will not give the details here.

REMARK 8.29. Notice that

$$J(m' + m'') < J(m') + J(m'') \tag{8.7}$$

implies the existence of a minimizer, but the opposite is true as well (and this can be proved arguing by contradiction). More precisely, the following two implications are true:

$$\text{existence of a minimizer for all } m \implies \text{(8.7) for all } m.$$

and

$$\text{(8.7) for a fixed } m \implies \text{existence of a minimizer for the same } m.$$

To prove (8.7) we start by introducing the notion of *irreducible* vector m but, as we will see in the end, it is only useful for the proof but has no real meaning.

DEFINITION 8.30. We say that a vector $m \in (\mathbb{R}^+)^M$ is *irreducible* if there is no way to decompose m as a sum $m = m' + m''$ with the additional property that

$$J(m') + J(m'') = J(m).$$

REMARK 8.31. In other words, m is irreducible if any decomposition $m = m' + m''$ satisfies the estimate (8.7).

We will prove that m can always be written as a sum $\sum_\ell m^{(\ell)}$ of vectors such that each $m^{(\ell)}$ is *irreducible* and the perimeter satisfies

$$J(m) = \sum_\ell J(m^{(\ell)}).$$

The first step to prove this decomposition is the following lemma, which is easily proved using the definition of "being irreducible".

LEMMA 8.32. *For all $m \in (\mathbb{R}^+)^M \setminus \{0\}$ there exists $m' \in (\mathbb{R}^+)^M \setminus \{0\}$ irreducible such that*

$$J(m) = J(m') + J(m - m').$$

PROOF. If m is irreducible, then we simply take $m' := m$. If not, then by definition there must be $m' \neq m''$ nonzero vectors such that $m = m' + m''$ and

$$J(m) = J(m') + J(m'').$$

Since $m'' = m - m'$ this concludes the proof. □

At this point we must prove that we do not have too many "pieces" which are too small, meaning that at least one irreducible vector must be long enough.

PROOF. Let $m \in (\mathbb{R}^+)^M$ and suppose that $m = \sum_{\ell=1}^k m^{(\ell)}$ is a decomposition in irreducible terms such that

$$J(m) = \sum_{\ell=1}^k J(m^{(\ell)})$$

and $|m^{(\ell)}| < \frac{|m|}{C}$ for each ℓ where C is a (big) constant. Then

$$J(m) \leq N\omega_N^{\frac{1}{N}} \sum_{j=1}^{M} m_j^{\frac{N-1}{N}} =: C'|m|,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$ and the m_j's are the components of m as a vector in $(\mathbb{R}^+)^M$. On the other hand, it is easy to verify that

$$J(m) \geq J(|m|),$$

which in turn implies the inequality

$$\sum_{\ell=1}^{k} J(m^{(\ell)}) \geq \sum_{\ell=1}^{k} J(|m^{(\ell)}|) =: K|m|,$$

where K is a constant that depends on the ratio $\frac{|m|}{C}$. In particular, if we choose the constant C big enough, K will become strictly bigger than C' leading to a contradiction since

$$C'|m| \geq J(m) \geq K|m| \implies C' \geq K.$$

$\square$

Now define

$$\eta := \inf\left\{ |m'| \,:\, m = \sum_{\ell} m^{(\ell)},\, J(m) = \sum_{\ell} J(m^{(\ell)}) \text{ and } |m'| = \max_{\ell} |m^{(\ell)}| \right\}.$$

If the minimum is achieved, then m' is irreducible (otherwise we could get a better one). If not, let us take a minimizing sequence η_j.

REMARK 8.33. If $J(m) = \sum_{\ell} J(m^{(\ell)})$, then we can also write

$$J(m) = J(m') + J(m'' + m''') + \cdots,$$

which means that we can always put things back together.

As a consequence of this observation, we notice that given a decomposition $m = \sum_{\ell} m^{(\ell)}$ with $|m'| = \max_{\ell} |m^{(\ell)}|$, then we can always write it as

$$m = \sum_{\ell} \tilde{m}^{(\ell)}$$

with $m' = \tilde{m}'$ which is at least $\frac{1}{10}$ of m because of the result above and

$$|\tilde{m}^j| < \frac{|m'|}{2} \quad \text{for all } \ell = 1, \dots, k.$$

This concludes the proof of (8.7). To pass from the estimate to the existence of minimizer, we need to prove that the optimal cluster is bounded (which was clear for the problem with a single set as balls are always bounded).

REMARK 8.34. We shall now prove that the solution of the one-set isoperimetrical problem is bounded (without knowing that it is a ball), but the same works with minor changes to prove that a optimal cluster is bounded.

PROOF. We will prove that the isoperimetrical set E is bounded in a single direction as the boundedness is obtained by repeating the same argument in all directions. Let

$$\alpha(t) := |E \cap \{x_1 > t\}|.$$

The goal is to show that for t big enough α vanishes. Clearly, α is decreasing and goes to zero as $t \to \infty$ and its derivative is given exactly by

$$\alpha'(t) = -\mathcal{H}^{N-1}(E \cap \{x_n = t\}).$$

Suppose that E has a "tentacle", which means an unbounded (possibly thin) part in the chosen direction with nonzero volume. We would like to show that cutting the tentacle at a finite x_1 and using the remaining volume in another part of E gives a better set in terms of perimeter, meaning that E itself will not have a tentacle.

REMARK 8.35. If E is a set of finite perimeter, for each $x \in \partial^* E$ there are $\bar{\epsilon}$, a positive constant K and a set G such that for all $0 < \epsilon < \bar{\epsilon}$ we have:

(1) G coincides with E up to a ball $B(x, \epsilon)$;

(2) $|G| = |E| + \epsilon$;

(3) $\mathrm{Per}(G) \leq \mathrm{Per}(E) + K\epsilon$.

We now choose $\bar{t}$ big enough so that $\alpha(\bar{t}) < \bar{\epsilon}$ and we modify the set E by removing the tentacle and pushing E slightly at some point of the reduced boundary obtaining a set F that satisfies the properties (a), (b) and (c). Then

$$\begin{aligned}
\mathrm{Per}(E) &\leq \mathrm{Per}(F) \\
&\leq \mathrm{Per}(E) - \mathrm{Per}(F \cap \{x_1 > t\}) + 2|\alpha'(t)| + K\alpha(t) \\
&\leq \mathrm{Per}(E) - K'\alpha(t)^{\frac{N-1}{N}} + K'\alpha(t) + 2|\alpha'(t)|.
\end{aligned}$$

Therefore, for $t \geq \bar{t}$ we have the inequality

$$K'\alpha(t)^{\frac{N-1}{N}} - K\alpha(t) \leq 2|\alpha'(t)|,$$

so up to a change of constant

$$K''\alpha(t)^{\frac{N-1}{N}} \leq |\alpha'(t)|.$$

A standard argument in calculus shows that this inequality implies $\alpha(\bar{t}) = 0$, concluding the proof. $\square$

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