Lecture Notes

Elliptic Equations

Course held by

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Disclaimer

These **incomplete** notes came out of the *Elliptic Equations* course, held by Professor Antonio Tarsia in the second semester of the academic year 2016/2017.

I have used them to study for the exam; hence they have been reviewed thoroughly. Unfortunately, there may still be many mistakes and oversights; to report them, send me an email at **francescopaolo (dot) maiale (at) gmail (dot) com**.

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Chapter 1

Introduction To Elliptic Problems

Notation. In this course, we use the multi-index notation. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of natural numbers and $\xi \in \mathbb{R}^n$ is a real vector, then

$$\xi^\alpha := \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}.$$

The length of a multi-index α is denoted by

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

and, as a consequence, the α-derivative operator is given by

$$D^\alpha := D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}.$$

1.1 Ellipticity Condition

Let $a_\alpha : \mathbb{R}^n \to \mathbb{C}$ be collection of complex-valued functions, as p ranges in a certain subset of $\mathbb{N}^n$, and let ℓ be a natural number. A linear differential operator of order ℓ is an operator of the form

$$A(x, D)\, u(x) := \sum_{|\alpha| \le \ell} a_\alpha(x)\, D^\alpha\, u(x), \tag{1.1}$$

as x ranges in $\Omega \subseteq \mathbb{R}^n$. The operator

$$A_0(x, D)\, u(x) := \sum_{|\alpha| = \ell} a_\alpha(x)\, D^\alpha\, u(x), \tag{1.2}$$

is the **principal part** of $A(x, D)$, while

$$\mathbb{R}^n \ni \xi \longmapsto A_0(x, \xi) := \sum_{|\alpha| = \ell} a_\alpha(x)\, \xi^\alpha \in \mathbb{C} \tag{1.3}$$

is the **characteristic polynomial** associated to $A_0(x, D)$.

Definition 1.1 (Elliptic Operator)**.** The linear differential operator $A(x, D)$ is an *elliptic operator* at the point $x \in \mathbb{R}^n$ if its principal part is non-vanishing, that is,

$$A_0(x, \xi) \neq 0, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

Theorem 1.2. *Let $A(x, D)$ be an elliptic operator of order ℓ at the point $x \in X$. Then ℓ is even if either*

(1) the coefficients $a_p(x)$ are real-valued functions; or

(2) the dimension of the space is $n \geq 3$.

Proof. Suppose that **(1)** holds true, and set $\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let us consider the polynomial

$$P(x, \xi', \xi_n) := A_0(x, \xi) = \sum_{|\alpha|=\ell} a_\alpha(x)\, \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n},$$

and let us set

$$b_0(x) := P(x, 0, \ldots, 0, 1) = a_{(0, \ldots, 0, 1)}.$$

The polynomial above can be decomposed as a sum of homogeneous terms in ξ_n as follows

$$P(x, \xi', \xi_n) = b_0(x)\, \xi_n^\ell + b_1(x, \xi')\, \xi_n^{\ell-1} + \cdots + b_{\ell-1}(x, \xi')\, \xi_n + b_\ell(x, \xi'),$$

where the $b_i(x, \xi')$'s are polynomials of degree equal to i with respect to the variable ξ', for any $i = 1, \ldots, \ell$.

Suppose that $b_0(x) > 0$ (the opposite case is symmetrical) and suppose also, by contradiction, that ℓ is an odd natural number. For any fixed $\xi' \neq 0$, it turns out that

$$\lim_{\xi_n \to \pm\infty} P(x, \xi', \xi_n) = \pm\infty,$$

that is there exists a point $\xi_n \in \mathbb{R}$ such that $P(x, \xi', \xi_n) = 0$, but this is absurd since we assumed $A(x, D)$ to be an elliptic operator.

Suppose that **(2)** holds true, and let $\xi_0' \neq 0$ a fixed vector in $\mathbb{R}^{n-1}$. The polynomial

$$\mathbb{R} \ni \xi_n \longmapsto P(x, \xi_0', \xi_n)$$

has non-real roots (because of the ellipticity condition), that is, roots whose imaginary part is nonzero $\mathfrak{Im}(\xi_n) \neq 0$.

Let $N^+(x, \xi_0')$ be the number of roots whose imaginary part is greater than 0, and let $N^-(x, \xi_0')$ be the number of roots whose imaginary part is less than 0.

Let Γ be a path containing (see Figure ??) all the $N^+(x, \xi_0')$ roots with positive imaginary part, so that $P(x, \xi_0', \xi_n) \neq 0$ on Γ. By continuity of P with respect to ξ', there exists a neighborhood U' of ξ_0' such that for any $\xi' \in U'$ it turns out that

$$|P(x, \xi_0', \xi_n) - P(x, \xi', \xi_n)| < |P(x, \xi_0', \xi_n)|, \qquad \forall\, \xi_n \in \Gamma.$$

By Rouché theorem[1], it follows that the polynomials $P(x, \xi_0', \xi_n)$ and $P(x, \xi', \xi_n)$ have the same number of roots inside Γ, i.e.,

$$N^+(x, \xi') = N^+(x, \xi_0'), \qquad \forall\, \xi' \in U'.$$

[1]**Rouché Theorem.** Let $K \subset G$ be a bounded region with continuous boundary ∂K. Two holomorphic functions $f, g \in \mathcal{H}(G)$ have the same number of roots (counting multiplicity) in K, if the strict inequality

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

holds for every $z \in \partial K$.

Since $N^+(x, \cdot)$ is a continuous integer-valued function, it is constant on connected components. It suffices to observe that $\{\xi' \in \mathbb{R}^{n-1}\} \setminus \{0\}$ is connected for any $n \geq 3$ to infer that

$$N^+(x, \xi') = N^+(x, -\xi') \quad \text{and} \quad N^-(x, \xi') = N^-(x, -\xi').$$

The polynomial is homogeneous of degree ℓ, hence

$$P(x, -\xi', -\xi_n) = (-1)^\ell P(x, \xi', \xi_n),$$

and this implies that

$$N^+(x, \xi') = N^-(x, -\xi') \implies \ell = 2\, N^+(x, \xi'),$$

i.e. ℓ is even. □

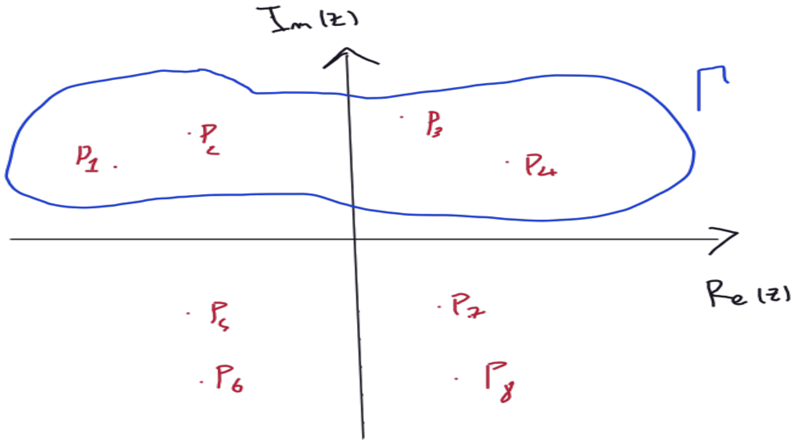

Figure 1.1: The path Γ introduced in the proof.

Example 1.1 (Elliptic Operators)**.**

(1) The **Cauchy-Riemann** operator is defined by

$$A(D) := \frac{\partial}{\partial x_1} + \imath \frac{\partial}{\partial x_2}. \tag{1.4}$$

It's straightforward to see that this is an operator of order $\ell = 1$, whose coefficients are complex-valued constant functions. It is also elliptic, since

$$A_0(D) = \xi_1 + \imath\, \xi_2 = 0 \iff (\xi_1, \xi_2) = (0, 0).$$

(2) The **Laplace** operator is defined by

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{1.5}$$

It's straightforward to see that this is an operator of even order $\ell = 2$, coherently with the fact that its coefficients are real-valued (constant) functions. It is elliptic since

$$A_0(D) = \|\xi\|^2 \iff (\xi_1, \dots, \xi_n) = (0, \dots, 0).$$

(3) The **Bi-Laplace** operator is defined by

$$\Delta\,\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial\, x_i^2} \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial\, x_j^2} \right).$$

It's straightforward to see that this is an operator of even order $\ell = 4$, coherently with the fact that its coefficients are real-valued (constant) functions. It is elliptic since

$$A_0(D) = \sum_{i=1}^{n} \xi_i^2\, \|\xi\|^2 = \|\xi\|^4 = 0 \iff (\xi_1,\,\ldots,\,\xi_n) = (0,\,\ldots,\,0).$$

Definition 1.3 (Uniformly Elliptic Operator)**.** An elliptic operator A is called *uniformly elliptic* on an open subset $\Omega \subset \mathbb{R}^n$ if there exists a positive constant $\nu > 0$ such that

$$|A_0(x,\,\xi)| \geq \nu\, |\xi|^{\ell}, \qquad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n$$

if the coefficients $a_\alpha(x)$ are complex-valued functions, and

$$A_0(x,\,\xi) \geq \nu\, |\xi|^{\ell}, \qquad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n$$

if the coefficients $a_\alpha(x)$ are real-valued functions.

Remark 1.1. If A is an elliptic operator on a bounded subset $\Omega \subset \mathbb{R}^n$, and the coefficients $a_\alpha(x)$ are of class $C^0(\bar{\Omega})$, then A is also uniformly elliptic.

In this course we shall be mainly concerned with two classes of elliptic operators, each of which has developed its own existence, regularity, etc... theory:

(I) The elliptic operator in **divergence** (or variational) form, i.e.

$$A(x,\,D)\,u(x) = \sum_{|\beta|=m} \sum_{|\alpha|=m} D^\alpha \left(a_{\alpha,\,\beta}(x)\, D^\beta\, u(x)\right). \tag{1.6}$$

(II) The elliptic operator in **non-divergence** (or non-variational) form, i.e.

$$A(x,\,D)\,u(x) = \sum_{|\alpha|=2\,m} a_\alpha(x)\, D^\alpha\, u(x). \tag{1.7}$$

Clearly the two forms can be equivalent (up to terms of lower order) if the coefficients are sufficiently regular - for example of class $C^m\left(\bar{\Omega}\right)$.

However, in the general case, the two forms are not equivalent and the existence, uniqueness, regularity, etc... results we shall study latex on, are completely different - for example, elliptic problems in non-variational form rarely admits a solution.

The operators of the **second order** are particularly interesting because the coefficients identify a square matrix, thus the uniformly ellipticity condition (in the real case) can be rewritten in a more intuitive way (i.e. the matrix is positive-definite):

$$\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,\xi_i\,\xi_j \geq \nu\, |\xi|^2, \qquad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n.$$

If an operator of the second order is in non-variational form (1.7), then we may always assume - without loss of generality - that $(a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ is a symmetric matrix, provided that $u(x)$ is a regular function, e.g. of class C^2 or H^2.

In fact, if we set $a_{i,\,j} = a^+_{i,\,j} + a^-_{i,\,j}$ (the sum of the symmetric and antisymmetric part), then

$$\begin{aligned}
A(x,\,D)\,u(x) &= \sum_{i,\,j=1}^{n} \left[a^+_{i,\,j} + a^-_{i,\,j}\right] D^{i,\,j}\,u(x) = \\
&= \sum_{i,\,j=1}^{n} a^+_{i,\,j}\,D^{i,\,j}\,u(x) + \sum_{i,\,j=1}^{n} a^-_{i,\,j}\,D^{i,\,j}\,u(x) = \\
&= \sum_{i,\,j=1}^{n} a^+_{i,\,j}\,D^{i,\,j}\,u(x) + \sum_{i,\,j=1}^{n} \frac{a_{i,\,j}(x) - a_{j,\,i}(x)}{2}\,D^{i,\,j}\,u(x) = \\
&= \sum_{i,\,j=1}^{n} a^+_{i,\,j}\,D^{i,\,j}\,u(x) + \sum_{i,\,j=1}^{n} \frac{a_{i,\,j}}{2}\,D^{i,\,j}\,u(x) - \sum_{i,\,j=1}^{n} \frac{a_{j,\,i}}{2}\,D^{j,\,i}\,u(x) = \\
&= \sum_{i,\,j=1}^{n} a^+_{i,\,j}\,D^{i,\,j}\,u(x),
\end{aligned}$$

as a consequence of the Schwartz theorem[2].

Remark 1.2. If the coefficients of the matrix are real, then by symmetry its eigenvalues are necessarily real. On the other hand, the ellipticity condition implies that they are all positive.

Vice versa, any $n \times n$ matrix $B(x)$ whose eigenvalues at x are real and positive defines an elliptic operator at the point x in non-divergence form by setting

$$A(x,\,D)\,u(x) = \sum_{i,\,j=1}^{n} B_{i,\,j}(x)\,D^{i,\,j}\,u(x).$$

1.2 Elliptic Systems

Let $\Omega \subseteq \mathbb{R}^n$ be any subset and assume that the coefficients are real-valued functions. We are particularly interested in elliptic system whose associated operator can be written as

$$A(x,\,D)\,u(x) = \sum_{|\alpha|\le m} \sum_{|\beta|\le m} (-1)^{|\alpha|}\,D^\alpha \left[A_{\alpha,\,\beta}(x)\,D^\beta\,u(x)\right], \tag{1.8}$$

[2]**Schwartz Theorem.** Let $f : \Omega \subset \mathbb{R}^n$ be a function of class C^2 defined on an open bounded subset Ω. Then for any $i,\,j \in \{1,\,\dots,\,n\}$ it turns out that

$$\frac{\partial}{\partial\,x_i}\,\frac{\partial}{\partial\,x_j}\,f(x) = \frac{\partial}{\partial\,x_j}\,\frac{\partial}{\partial\,x_i}\,f(x).$$

where u is a vector-valued function from Ω to $\mathbb{R}^N$ and $A_{\alpha,\,\beta} \in \mathcal{M}_{N\times N}(\mathbb{R})$ is a real-valued square matrix for any admissible couple $(\alpha,\,\beta)$.

Definition 1.4 (Legendre Condition)**.** Let $A(x,\,D)$ be the operator (1.8). We say that A satisfies the *Legendre condition* (or, A is *strongly elliptic*) on Ω if there exists a positive constant $\nu > 0$ such that, for any system of vectors $\{\xi^\alpha\}_{|\alpha|=m}$ of $\mathbb{R}^N$ and for any $x \in \Omega$,

$$\nu \sum_{|\alpha|=m} \|\xi^\alpha\|^2_{\mathbb{R}^N} \leq \sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\,\beta}(x)\,\xi^\beta,\,\xi^\alpha\right)_{\mathbb{R}^N}, \tag{1.9}$$

where $(\cdot,\,\cdot)_{\mathbb{R}^N}$ denotes the standard scalar product in $\mathbb{R}^N$.

Definition 1.5 (Legendre-Hadamard Condition)**.** Let $A(x,\,D)$ be the operator (1.8). We say that A satisfies the *Legendre-Hadamard condition* (or, A is *elliptic*) if there exists a positive constant $\nu > 0$ such that, for any $\eta \in \mathbb{R}^N$, for any $\lambda \in \mathbb{R}^n$ and for any $x \in \Omega$,

$$\nu \left(\|\lambda\|_{\mathbb{R}^n}\right)^{2m} \|\eta\|^2_{\mathbb{R}^N} \leq \sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\,\beta}(x)\,\eta,\,\eta\right)_{\mathbb{R}^N} \lambda^\alpha\,\lambda^\beta. \tag{1.10}$$

Clearly, any A satisfying the Legendre condition also satisfies the Legendre-Hadamard condition. In fact, if we set $\xi^\alpha_i = \lambda^\alpha\,\eta_i$ for $i = 1,\,\ldots,\,N$, then (1.9) implies (1.10).

On the other hand, the opposite is generally **not** true. A simple counterexample ($N = n = 3$) is given by the operator of linear elasticity defined as

$$A(D)\,u(x) = a\,\Delta u(x) + (a + 2\,\ell)\,\mathrm{grad}\,(\mathrm{div}\,u(x)),$$

where $a,\,\ell > 0$ are (strictly) positive real constants. It's straightforward (but tedious) to prove that A satisfies the condition (1.10), but it doesn't satisfy the strong condition (1.9).

Remark 1.3. If an operator A is elliptic, then the diagonal operators $A_{i,\,i}$ are also elliptic. In fact, if we set $\eta = e_i$, then the ellipticity condition (1.10) implies that

$$\nu\,\|\lambda\|^{2m}_{\mathbb{R}^n} \leq \sum_{|\alpha|=m} \sum_{|\beta|=m} A^{i,\,i}_{\alpha,\,\beta}(x)\,\lambda^\alpha\,\lambda^\beta.$$

A similar argument proves that if A is a strongly elliptic operator, then the diagonal operators are also strongly elliptic. In fact, if we set $\xi = e_i$ then (1.9) implies that

$$\nu \leq \sum_{|\alpha|=m} \sum_{|\beta|=m} A^{i,\,i}_{\alpha,\,\beta}(x).$$

Example 1.2 (Strong Ellipticity)**.** Let $N = n = 2$. The operator

$$A = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

is strongly elliptic and, coherently with the argument above, its diagonal elements are also strongly elliptic. On the other hand, the operator

$$A = \begin{pmatrix} \Delta & \epsilon\,D_1^2 \\ \epsilon\,D_2^2 & \Delta \end{pmatrix}$$

is strongly elliptic, but $\epsilon\,D_i^2$ are not even elliptic for any $|\epsilon| < 2$.

We conclude this section by introducing the conditions (1.9) and (1.10) in the particular case of second-order operators.

Definition 1.6 (Second-Order Legendre Condition)**.** Let $A(x, D)$ be the operator (1.8) of second-order. We say that A satisfies the *Legendre condition* (or A is *strongly elliptic*) if there exists a positive constant $\nu > 0$ such that, for any $\tau \in \mathbb{R}^{nN}$ and for any $x \in \Omega$, the following inequality holds true:

$$\nu \sum_{i=1}^{n} \|\tau^i\|_{\mathbb{R}^N}^2 \leq \sum_{i,\,j=1}^{n} \sum_{h,\,k=1}^{n} A_{i,\,j}^{h,\,k}(x)\, \tau_i^h\, \tau_j^k. \tag{1.11}$$

Definition 1.7 (Legendre-Hadamard Condition)**.** Let $A(x, D)$ be the operator (1.8) of second-order. We say that A satisfies the *Legendre-Hadamard condition* (or A is *elliptic*) if there exists a positive constant $\nu > 0$ such that, for any $\eta \in \mathbb{R}^N$, for any $\xi \in \mathbb{R}^n$ and for any $x \in \Omega$, the following inequality holds true:

$$\nu\, \|\xi\|_{\mathbb{R}^n}^2\, \|\eta\|_{\mathbb{R}^N}^2 \leq \sum_{i,\,j=1}^{n} \sum_{h,\,k=1}^{n} A_{i,\,j}^{h,\,k}(x)\, \xi_i\, \xi_j\, \eta_h\, \eta_k. \tag{1.12}$$

1.3 Definitions of Solution

The elliptic problems in non variational form (class (II)) are, generally, in the following form:

$$A(x,\, D)\, u(x) = \sum_{|\alpha| \leq 2m} a_p(x)\, D^p\, u(x) = f(x).$$

Therefore, if f is a continuous function, we would like to find solution(s) of class $C^{2m}(\Omega)$, but unfortunately it is generally not possible.

In fact, $C^k(\Omega)$ is not a *good* space to look into when studying the regularity (or even the existence) of solution(s) of an elliptic problem. The general method consists of two simple steps:

(a) Prove the existence of solution(s) in bigger spaces, e.g. the Sobolev spaces $H^{2m,\,p}(\Omega)$ or the Campanato-Morrey spaces.

(b) Use regularity theory and prove that the solution(s) in the weaker space are, actually, a lot more regular. As we shall see later on, we can generally prove that solution(s) are of class $C^{2m,\,\alpha}(\Omega)$ (i.e. functions of class C^2, whose second derivative is α-Hölder continuous), provided that f is also α-Hölder.

Definition 1.8 (Classical Solution)**.** Let $\Omega \subseteq \mathbb{R}^n$, let $f \in C^0\left(\bar{\Omega};\, \mathbb{R}^N\right)$ and let $A_\alpha \in C^0\left(\bar{\Omega};\, \mathbb{R}^N \times \mathbb{R}^N\right)$ for any $|\alpha| \leq 2m$. Then $u : \bar{\Omega} \to \mathbb{R}^N$ is a *classical solution* of the elliptic problem

$$A(x,\, D)\, u(x) = f(x), \qquad x \in \Omega$$

if $u \in C^{2m}\left(\Omega;\, \mathbb{R}^N\right) \cap C^0\left(\bar{\Omega};\, \mathbb{R}^N\right)$ and u satisfies the equation for any $x \in \bar{\Omega}$.

Example 1.3. Let $\Omega = \left\{(x,\, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < r < 1\right\} \setminus \{(0,\, 0)\}$, let

$$u(x,\, y) = (x^2 - y^2) \cdot \sqrt{-\log \sqrt{x^2 + y^2}},$$

and let

$$f(x,\,y)=\begin{cases}\frac{y^2-x^2}{2x^2+2y^2}\cdot\left[\frac{4}{\sqrt{-\log\sqrt{x^2+y^2}}}+\frac{1}{2\sqrt{-\log^6\sqrt{x^2+y^2}}}\right] & \text{if } (x,\,y)\neq(0,\,0)\\ 0 & \text{if } (x,\,y)=(0,\,0).\end{cases}$$

It's straightforward, but extremely tedious, to prove that $f\in C^0\left(\bar\Omega\right)$ and $u\in C^0\left(\bar\Omega\right)\cap C^\infty\left(\bar\Omega\setminus\{(0,\,0)\}\right)$.

Surprisingly, it turns out that u is not a classical solution of the elliptic problem $\Delta u(x,\,y)=f(x,\,y)$, since

$$\lim_{(x,\,y)\to(0,\,0)}\frac{\partial^2 u}{\partial x^2}(x,\,y)=+\infty.$$

Definition 1.9 (Strong Solution)**.** Let $\Omega\subseteq\mathbb{R}^n$, let $f\in L^p\left(\Omega;\,\mathbb{R}^N\right)$ for $p>1$ and let $A_\alpha\in L^\infty\left(\Omega;\,\mathbb{R}^N\times\mathbb{R}^N\right)$ for any $|\alpha|\le 2m$. Then $u:\bar\Omega\to\mathbb{R}^N$ is a *strong solution* of the problem

$$A(x,\,D)\,u(x)=f(x),\qquad x\in\Omega$$

if $u\in H^{2m,\,p}\left(\Omega;\,\mathbb{R}^N\right)$, and it solves the elliptic problem for almost every $x\in\bar\Omega$.

Definition 1.10 (Weak Solution)**.** Let $\Omega\subseteq\mathbb{R}^n$, let $f_\beta\in L^p\left(\Omega;\,\mathbb{R}^N\right)$ for $p>1$ and let $A_\alpha\in L^\infty\left(\Omega;\,\mathbb{R}^N\times\mathbb{R}^N\right)$ for any $|\alpha|,\,|\beta|\le m$. Then $u:\bar\Omega\to\mathbb{R}^N$ is a *weak solution* of the problem

$$\sum_{|\alpha|\le m}\sum_{|\beta|\le m}(-1)^{|\beta|}\,D^\beta\left(A_{\alpha,\,\beta}(x)\,D^\alpha\,u(x)\right)=\sum_{|\beta|\le m}(-1)^{|\beta|}\,D^\beta\,f_\beta(x)$$

if $u\in H^{m,\,p}\left(\Omega;\,\mathbb{R}^N\right)$ and, for any $\varphi\in H^{m,\,p'}\left(\Omega;\,\mathbb{R}^N\right)$,

$$\int_\Omega\left[\sum_{|\alpha|,\,|\beta|\le m}\left(A_{\alpha,\,\beta}(x)\,D^\alpha\,u(x),\,D^\beta\varphi(x)\right)_{\mathbb{R}^N}\right]\mathrm{d}x=\int_\Omega\left[\sum_{|\beta|\le m}\left(f_\beta(x),\,D^\beta\,\varphi(x)\right)_{\mathbb{R}^N}\right]\mathrm{d}x.$$

1.4 Boundary Problems

Posedness In this section we introduce the notion of *boundary* elliptic problem and the notion of *well-posed* boundary elliptic problem.

Let $\Omega\subseteq\mathbb{R}^n$ be an open and bounded subset of $\mathbb{R}^n$, let $A(x,\,D)$ be an elliptic operator and let $B(x,\,D)$ be a linear boundary operator. The system

$$\begin{cases}A(x,\,D)\,u(x)=f(x) & \text{a.e. } x\in\Omega\\ B(x,\,D)\,u(x)=g(x) & x\in\partial\,\Omega\end{cases}$$

is called a *boundary* elliptic problem.

Definition 1.11 (Well-Posed)**.** The problem (1.4) is *well-posed* (in sense of Hadamard) if there exists **one and only one** solution that depends **continuously** on the initial data.

Example 1.4. Let $\Omega = \{(t,\, x) \mid t > 0,\ x \in \mathbb{R}\}$ and let

$$\Delta\, u = \frac{\partial^2 u}{\partial\, t^2} + \frac{\partial^2 u}{\partial\, x^2}.$$

The problem

$$\begin{cases} \Delta u(t,\, x) = 0 & (t,\, x) \in \Omega \\ u(0,\, x) = \Phi(x) & x \in \mathbb{R} \\ u_t(0,\, x) = \Psi(x) & x \in \mathbb{R} \end{cases}$$

is not well posed for any choice of Φ and Ψ. In fact, if we let

$$\begin{cases} \Phi_n(x) = \mathrm{e}^{-\sqrt{n}} \sin(nx) \\ \Psi_n(x) = \mathrm{e}^{-\sqrt{n}}\, n \sin(nx), \end{cases}$$

there there's one and only one classical solution for any $n \in \mathbb{N}$, given by

$$u_n(x) = \mathrm{e}^{-\sqrt{n}+nt} \sin(nx).$$

On the other hand, for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} |\Phi_n(x)| \le \epsilon \quad \text{and} \quad \sup_{x\in\mathbb{R}} |\Psi_n(x)| \le \epsilon,$$

for any $n \ge n_\epsilon$. The reason is simple: the limit as $n \to +\infty$ of both $\Phi_n(x)$ and $\Phi_n(x)$ is 0, uniformly with respect to x (therefore we can take the supremum and the limit is still zero).

The solution doesn't depend continuously on the initial data. In fact, for any $t_0 > 0$ fixed, it turns out that

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{R}} |u_n(t_0,\, x)| \sim \lim_{n\to+\infty} \mathrm{e}^{n-\sqrt{n}} = +\infty.$$

More precisely, let U and W be topological vector spaces. Let $u \in U$ be a solution of a given problem and let $f \in W$ be the given data; the problem is **well posed** if:

1) For any f there exists a solution u of the problem.

2) The solution is unique.

3) The solution depends continuously on f.

If these condition are not satisfied, the problem is said to be **ill-posed**. Notice that even problems with a physical meaning may be ill-posed (the reader should think about a string equilibrium problem, for example).

Boundary Operators In this paragraph we are interested in introducing general conditions on the boundary operator, so that the problem becomes well-posed.

Let Ω be a subset of $\mathbb{R}^n$ with a sufficiently regular boundary, let A be an operator of order $2m$ and let $B_j(x,\, D)$, for $j = 0,\, \ldots,\, k-1$ be boundary operators defined by

$$B_j(x,\, D)\, \varphi(x) = \sum_{|\alpha|\le m_j} b_{j,\,\alpha}(x)\, D^\alpha\, \varphi(x), \qquad x \in \partial\,\Omega. \tag{1.13}$$

More precisely $B_j(x,\,D)$ represent the following operator

$$\varphi \longmapsto \sum_{|\alpha|\leq m_j} b_{j,\,\alpha}(x)\,\gamma_0\left(D^\alpha\,\varphi(x)\right), \tag{1.14}$$

where φ is a function defined on the close $\bar{\Omega}$ and $\gamma_0(\cdot)$ is the **trace** map in the classical sense or, if needed, in the Sobolev space sense. Let us consider the problem

$$\begin{cases} A(x,\,D)\,u(x) = f(x) & \text{a.e. } x \in \Omega \\ B(x,\,D)\,u(x) = g(x) & x \in \partial\,\Omega. \end{cases}$$

Definition 1.12 (Normal)**.** A system of operators $\{B_j(x,\,D)\}_{j=0,\,\dots,\,k-1}$ is *normal* if the following properties are satisfied:

1) The polynomial $\sum_{|\alpha|=m_j} b_{j,\,\alpha}\,\xi^\alpha \neq 0$ for any $\xi \neq 0$ and it is orthogonal to $\partial\,\Omega$.

2) The orders are different, that is, $m_i \neq m_j$ for any $i \neq j$.

Moreover, if we assume that $k = m$, then we can give another important definition.

Definition 1.13 (Covering)**.** A system of operators $\{B_j(x,\,D)\}_{j=0,\,\dots,\,m-1}$ is a *covering* of the operator A on $\partial\,\Omega$ if for any $x \in \partial\,\Omega$, for any $\xi \in \mathbb{R}^n \setminus \{0\}$ tangent to $\partial\,\Omega$ at x and any $\xi' \in \mathbb{R}^n \setminus \{0\}$ orthogonal to $\partial\,\Omega$ at x, the polynomials of the complex variable τ

$$\sum_{|\alpha|=m_j} b_{j,\,\alpha}\,\left(\xi + \tau\,\xi'\right)^\alpha, \qquad j = 0,\,\dots,\,m-1$$

are linearly independent modulo the polynomial

$$\prod_{i=1}^{m}\left(\tau - \tau_i^+(x,\,\xi,\,\xi')\right),$$

where $\tau_i^+(x,\,\xi,\,\xi')$ are the roots with positive imaginary part of the polynomial $A_0(x,\,\xi + \tau\,\xi')$.

In particular, some of the most important assumption on (1.4) are the following ones:

1) The operator A is uniformly elliptic in $\bar{\Omega}$, with coefficients in a suitable functional space.

2) The boundary operators B_j are m, with coefficients in a suitable functional space.

3) The system $\{B_j(x,\,D)\}_{j=0,\,\dots,\,m-1}$ is *normal* on $\partial\,\Omega$.

4) The system $\{B_j(x,\,D)\}_{j=0,\,\dots,\,m-1}$ *covers* the operator A on $\partial\,\Omega$.

5) The order of B_j is less or equal to $2\,m-1$, for any j.

Example 1.5. The most common example of boundary operators satisfying the conditions above is given by the system of the Dirichlet conditions, that is,

$$B_j = \gamma_j = \frac{\partial^j}{\partial\,\nu^j}, \qquad j = 0,\,\dots,\,m-1$$

where ν is the normal to $\partial\,\Omega$ pointing toward the interior part of Ω. The problem

$$\begin{cases} A(x,\,D)u = f & x \in \Omega \\ \gamma_0\, u = \varphi_0 & x \in \partial\,\Omega \\ \cdots & \\ \cdots & \\ \gamma_{m-1}\, u = \varphi_{m-1} & x \in \partial\,\Omega, \end{cases}$$

is called **Dirichlet problem**.

Example 1.6. Another common example of boundary operators satisfying the conditions above is given by the system of the Navier conditions, that is,

$$B_j(x,\,D)\,u = \Delta_{j-1}\,u, \qquad j = 1,\,\ldots,\,m$$

so that $m_j = 2(j-1)$. The associated problem is

$$\begin{cases} A(x,\,D)u = f & x \in \Omega \\ u = \varphi_0 & x \in \partial\,\Omega \\ \cdots & \\ \cdots & \\ \Delta_{m-1}\, u = \varphi_{m-1} & x \in \partial\,\Omega. \end{cases}$$

is called **Navier problem**.

Remark 1.4. Let Ω be a subset of $\mathbb{R}^n$ with a sufficiently regular boundary. A function $u \in H_0^m(\Omega)$ if and only if $u \in H^m(\Omega)$ and $\gamma_0 = \cdots = \gamma_{m-1} = 0$ on $\partial\,\Omega$. Therefore the Dirichlet problem can be equivalently written as

$$\begin{cases} A(x,\,D)u = f & x \in \Omega \\ u \in H_0^m(\Omega). & \end{cases}$$

1.5 The Laplacian Operator and The Calculus of Variations

Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $\varphi : \partial\,\Omega \to \mathbb{R}$ be a given function. The *energy functional* associated to the laplacian operator is defined by setting

$$E(u) = \frac{1}{2}\int_\Omega |\nabla\, u(x)|^2\, \mathrm{d}x - \int_\Omega f(x)\, u(x)\, \mathrm{d}x = \mathcal{E}_e(u) - \mathcal{E}_p(u), \tag{1.15}$$

where $\mathcal{E}_p$ is the internal elastic energy, while $\mathcal{E}_p$ is the exogenous potential energy. We also require that u is an element of the *eligible class*, given by

$$\mathcal{A} = \{u : \Omega \to \mathbb{R} \mid u(x) = \varphi(x) \text{ for any } x \in \partial\,\Omega \text{ and } E(u) < +\infty\}.$$

Theorem 1.14. *Let $\mathcal{A} \neq \emptyset$. If there exists $\underline{u} \in \mathcal{A}$ such that $\underline{u}$ is a minimizer of the energy E, that is, for any $u \in \mathcal{A}$*

$$E(\underline{u}) \leq E(u),$$

then $\underline{u}$ is a solution to the Dirichlet problem

$$\begin{cases} -\Delta \underline{u}(x) = f(x) & x \in \Omega \\ \underline{u}(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{1.16}$$

Before we get to the proof, some questions arise naturally (which we solve during the proof):

1) What assumptions do we need on φ and Ω to assure that $\mathcal{A} \neq \emptyset$?

2) There exists at least a minimum point of E in $\mathcal{A}$?

Proof. Let $\mathcal{A}_0$ be the vector space associated to the solution space, that is,

$$\mathcal{A}_0 = \{u : \Omega \to \mathbb{R} \mid u(x) = 0 \text{ for any } x \in \partial\Omega \text{ and } E(u) < +\infty\}.$$

Let $\underline{u} \in \mathcal{A}$ and $v \in \mathcal{A}_0$ and let

$$F(t) := \frac{1}{2}\int_\Omega |\nabla(\underline{u} + t\,v)(x)|^2 \, \mathrm{d}x - \int_\Omega f(x)\,(\underline{u}(x) + t\,v(x))\,\mathrm{d}x,$$

then, by assumption, $t = 0$ is a local minimum of F. If we compute the derivative of F in $t = 0$ and set it equal to 0, it turns out that

$$\int_\Omega [\nabla \underline{u}(x) \cdot \nabla v(x) - f(x)\,v(x)]\,\mathrm{d}x = 0, \qquad \forall\, v \in \mathcal{A}_0.$$

The point $\underline{u}$ is thus a **weak** solution of the Dirichlet problem (1.16). To prove that it is a strong solution we need to use the Gauss-Green formula, and this can only be done if we assume u and f to be sufficiently regular (e.g. $f \in L^2(\Omega)$ and $u \in H^2(\Omega)$ would be fine).

Suppose that we can apply the Gauss-Green formula, then we integrate by parts and obtain the relation

$$\int_\Omega [-\Delta\,\underline{u}(x) - f(x)]\;v(x)\,\mathrm{d}x = 0, \qquad \forall\, v \in \mathcal{A}_0.$$

Finally, we apply the Fundamental Lemma of Calculus of Variations [3] and we conclude that $\underline{u}$ is a strong solution of the Dirichlet problem (1.16). □

A similar argument allows us to study another important physic model: the **Kirchoff-Love** model for a thin plate. We don't give the details of this problem, bur the reader may find them in **Tarsia's notes**.

[3]Let $u \in L^1(\Omega)$. Suppose that, for any $v \in L^1(\Omega)$ which is compactly supported, it turns out that

$$\int_\Omega u(x)\,v(x)\,\mathrm{d}x = 0.$$

Then $u(x) = 0$ for almost every $x \in \Omega$.

Chapter 2

Existence Results for Elliptic Problems

In this chapter we are concerned with the existence of solutions for elliptic problems, both in variational and in nonvariational form.

2.1 Near Operators Theory

Definition 2.1 (Nearness)**.** Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space and let $A,\, B : \mathfrak{X} \to \mathcal{B}$ be operators. We say that A is *near* to B if there exist a constant $\alpha > 0$ and $k \in (0,\, 1)$ such that

$$\|B(x_1) - B(x_2) - \alpha\,[A(x_1) - A(x_2)]\| \leq k\, \|B(x_1) - B(x_2)\|\,, \qquad \forall\, x_1,\, x_2 \in \mathfrak{X}. \tag{2.1}$$

Remark 2.1. The condition is in general **not** symmetric: if A is near B, then it's not necessarily true that B is near A. On the other hand, if $\mathcal{B}$ is a Hilbert space, then the condition is symmetric.

Lemma 2.2. *Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space, let $A,\, B : \mathfrak{X} \to \mathcal{B}$ be operators and assume that A is near to B. Then the following inequalities are satisfied for any $x_1,\, x_2 \in \mathfrak{X}$:*

$$\|B(x_1) - B(x_2)\| \leq \frac{\alpha}{1-k}\, \|A(x_1) - A(x_2)\|\,, \tag{2.2}$$

$$\|A(x_1) - A(x_2)\| \leq \frac{k+1}{\alpha}\, \|B(x_1) - B(x_2)\|\,. \tag{2.3}$$

Proof. We only prove the first inequality, and leave the second one to the reader. Since A is near to B, we have that

$$\begin{aligned} \|B(x_1) - B(x_2)\| &\leq \|B(x_1) - B(x_2) - \alpha\,[A(x_1) - A(x_2)]\| + \alpha\, \|A(x_1) - A(x_2)\| \leq \\ &\leq k\, \|B(x_1) - B(x_2)\| + \alpha\, \|A(x_1) - A(x_2)\|\,, \end{aligned}$$

from which it follows easily that

$$(1-k) \cdot \|B(x_1) - B(x_2)\| \leq \alpha\, \|A(x_1) - A(x_2)\|\,.$$

□

Lemma 2.3. *Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space, let A, $B : \mathfrak{X} \to \mathcal{B}$ be operators and assume that A is near to B. Then A is injective if and only if B is injective.*

Proof. This is a immediate consequence of the previous Lemma, since (2.2) and (2.3) easily implies that $A(x_1) = A(x_2)$ if and only if $B(x_1) = B(x_2)$. □

Lemma 2.4. *Let $B : \mathcal{X} \to \mathcal{B}$ be an injective operator. Then $(\mathcal{X}, d)$ is a metric space, where*

$$d(x, y) = \|B(x) - B(y)\|_{\mathcal{B}}, \qquad \forall\, x, y \in \mathcal{X}.$$

Moreover, if B is also surjective, then the metric space $(\mathcal{X}, d)$ is complete.

Proof. We want to prove that d is a distance on $\mathcal{X}$, that is, the characterizing properties hold true. By definition it is greater or equal to zero and, since B is injective, it follows that

$$d(x, y) = 0 \iff B(x) = B(y) \iff x = y.$$

The symmetric property and the triangle inequality are straightforward consequences of the definition, therefore we will not give more details.

Assume that B is bijective and let $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ be a Cauchy sequence. Then $(B(x_n))_{n \in \mathbb{N}} \subset \mathcal{B}$ is a Cauchy sequence and, by completeness of $\mathcal{B}$, it converges to $y \in \mathcal{B}$. Since B is surjective, there is $x \in \mathcal{X}$ such that $B(x) = y$; it follows that

$$d(x_n, x) = \|B(x_n) - y\|_{\mathcal{B}} \xrightarrow{n \to +\infty} 0,$$

and this concludes the proof. □

Theorem 2.5. *Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space, let A, $B : \mathfrak{X} \to \mathcal{B}$ be operators and assume that A is near to B. If B is a bijection, then also A is a bijection.*

Proof. It suffices to prove that A is a *surjective* operator. In fact, it is *injective* as a consequence of Lemma 2.3 which was proved earlier.

Let $f \in \mathcal{B}$ be any element. We want to prove that there exists a (unique) $u \in \mathcal{X}$ such that $A(u) = f$ or, equivalently, that

$$B(u) = B(u) - \alpha\, A(u) + \alpha\, f =: F(u).$$

For any $u \in \mathcal{X}$, $F(u)$ belongs to $\mathcal{B}$, hence there is one and only one $v \in \mathcal{X}$ such that $B(v) = F(u)$, as B is a bijective operator.

Let us denote by $T : \mathcal{X} \to \mathcal{X}$ the application that sends u in the unique solution v. If $v = T(u)$ and $w = T(z)$, it turns out that

$$\begin{aligned} d(v, w) &= \|B(v) - B(w)\|_{\mathcal{B}} = \|F(u) - F(z)\|_{\mathcal{B}} = \\ &= \|B(v) - B(w) - \alpha\,[A(v) - A(w)]\| \leq k\, \|B(v) - B(w)\|\,, \end{aligned}$$

i.e., T is a contraction (since k is strictly less than one).

Finally $\mathcal{X}$ is a complete metric space, thus there is one and only one fixed point $u \in \mathcal{X}$ of T, that is, A is surjective:

$$T(u) = u \implies B(u) = F(u) \implies A(u) = f.$$

□

Equivalence Relation. Let us endow the set $\mathcal{X}$ with the following equivalence relation:

$$u \sim v \iff B(u) = B(v).$$

Let us denote by $[u]_{\mathcal{X}}$ the equivalence class of u and let $X = \mathcal{X}/\sim$. We can define two operators A^*, $B^* : X \to \mathcal{B}$ by setting

$$A^*([u]_{\mathcal{X}}) = A(u) \qquad \text{and} B^*([u]_{\mathcal{X}}) = B(u),$$

then, if A is near B, also A^* is near B^* (with the same constants).

If we assume that B is surjective, then B^* is bijective (by definition). It follows from Theorem 2.5 that A^* is bijective, and this directly implies that A is surjective.

Theorem 2.6. *Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space, let A, $B : \mathfrak{X} \to \mathcal{B}$ be operators and assume that A is near to B. If B is a surjective operator, then also A is a surjective operator.*

Theorem 2.7. *Let $\mathfrak{X}$ be any set, let $\mathcal{B}$ be a Banach space and let $\{A_t : \mathcal{X} \to \mathcal{B}\}_{t\in[0,\,1]}$ be a family of operators such that:*

- ***(a)*** *There exists $t_0 \in [0, 1]$ such that A_{t_0} is a bijection.*
- ***(b)*** *There exists $c > 0$ such that, for any s, $t \in [0, 1]$ and any x, $y \in \mathcal{X}$,*

$$\|A_t(x) - A_t(y) - [A_s(x) - A_s(y)]\|_{\mathcal{B}} \le c\,|t - s|\, \|A_t(x) - A_t(y)\|_{\mathcal{B}}.$$

Then, for any $s \in [0, 1]$, it turns out that A_s is a bijection.

Proof. Set $I := \{t \in [0, 1] \mid A_t \text{ is a bijection}\}$. By connectedness of $[0, 1]$, it suffices to prove that I is nonempty, open and closed in the subspace topology.

The assumption **(a)** implies that I is nonempty. Let $t \in I$ and let $\delta > 0$ be such that, for any $s \in [t - \delta, t + \delta] \cap [0, 1]$, it turns out that $c\,|t - s| < 1$. Then the inequality **(b)** implies that A_s near A_t, and thus A_s is bijective (i.e. $s \in I$) by Theorem 2.5.

Let $(s_n)_{n\in\mathbb{N}} \subset I$ be a converging sequence and let $s \in [0, 1]$ be its limit. There exists $N \in \mathbb{N}$ such that $c\,|s_n - s| < 1$, for any $n \ge N$, therefore A_s is near A_{s_N} and, again, it is bijective. □

Fréchet Differential Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and let $F : U \subset \mathcal{B}_1 \to \mathcal{B}_2$. The function F is differentiable in the sense of Fréchet at $u_0 \in U$ if there is a linear continuous application $L : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\lim_{h\to 0} \frac{\|F(u_0 + h) - F(u_0) - L\,h\|_{\mathcal{B}_2}}{\|h\|_{\mathcal{B}_1}} = 0, \tag{2.4}$$

and we denote it by $\mathrm{d}F_{u_0}$.

We say that F is continuously differentiable at u_0 if it is differentiable in a neighborhood W of u_0 and the mapping $W \ni u \mapsto \mathrm{d}F_u \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is continuous at u_0.

Theorem 2.8. *Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be two Banach spaces, let $y_0 \in Y$ be any point and let $F : U \to Z$ be a function defined on a neighborhood of y_0. Assume that*

(a) *$F \in C^1(U)$;*

(b) *the Fréchet differential $\mathrm{d}F_{y_0} : Y \to Z$ is invertible.*

Then there exist $k \in (0,\,1)$ and a neighborhood $W \subset U$ such that for any $y_1,\,y_2 \in W$

$$\|\mathrm{d}F_{y_0}[y_1 - y_2] - [F(y_1) - F(y_2)]\|_Z \le \|\mathrm{d}F_{y_0}[y_1 - y_2]\|_Z\,.$$

2.2 Historical Digression on Near Operators

The idea of introducing the notion of *nearness* between operators originates from the existence and uniqueness problem of nonvariational elliptic equations of the following kind:

$$\begin{cases} u \in H^2(\Omega) \cap H^1_0(\Omega) \\ \\ \displaystyle\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) = f(x) \quad \text{for } x \in \Omega, \end{cases} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and - for simplicity only - convex, $f \in L^2(\Omega)$, the coefficients $a_{i,\,j} \in L^\infty(\Omega)$ and the matrix $A = (a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ is uniformly elliptic and symmetric.

If $n > 2$ the problem (2.5) is not well-defined under the solely assumption of uniformly ellipticity (see Example 2.1). Therefore, it is necessary to introduce more restrictive condition on the coefficients of the matrix A, e.g., regularity ($a_{i,\,j} \in C^0(\Omega)$) or algebraic conditions.

Definition 2.9 (Corder Condition)**.** Let $A = (a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ be a matrix such that $\|A(x)\| \neq 0$ for almost every $x \in \Omega$. We say that $A(x)$ satisfies the *Corder condition* if there exists $\epsilon > 0$ such that

$$\frac{\left(\sum_{i=1}^{n} a_{i,\,i}(x)\right)^2}{\sum_{i,\,j=1}^{n} a^2_{i,\,j}(x)} \ge n - 1 + \epsilon \qquad \text{for almost every } x \in \Omega. \tag{2.6}$$

Definition 2.10 (Campanato Condition)**.** Let $A = (a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ be a matrix. We say that $A(x)$ satisfies the A_x *condition* if there are real constants σ, γ strictly positive, $\delta \ge 0$ and a function $a(x) \in L^\infty(\Omega)$ such that $\gamma + \delta < 1$, $a(x) \ge \sigma > 0$ and

$$\left|\sum_{i=1}^{n} \xi_{i,\,i} - a(x) \cdot \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, \xi_{i,\,j}\right| \le \gamma \cdot \left(\sum_{i,\,j=1}^{n} \xi^2_{i,\,j}\right)^{\frac{1}{2}} + \delta \cdot \left|\sum_{i=1}^{n} \xi_{i,\,i}\right|, \tag{2.7}$$

for any matrix $\xi \in M_n(\mathbb{R})$ and for almost every $x \in \Omega$.

Remark 2.2. The Campanato condition implies the uniform ellipticity of A (but, for $n > 2$, it is not equivalent) over Ω. Indeed, let $\xi = (\eta_i\, \eta_j)_{i,\,j=1,\,\dots,\,n}$ and substitute this in (2.7):

$$\left|\sum_{i=1}^{n} \eta^2_i - a(x) \cdot \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, \eta_i\, \eta_j\right| \le \gamma \cdot \left(\sum_{i,\,j=1}^{n} \eta^2_i\, \eta^2_j\right)^{\frac{1}{2}} + \delta \cdot \left|\sum_{i=1}^{n} \eta^2_i\right|.$$

It follows that

$$[1-(\gamma+\delta)]\cdot\sum_{i=1}^{n}\eta_i^2 \le a(x)\cdot\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,\eta_i\,\eta_j,$$

and, if we let $\mu = \sup_\Omega a(x)$, it turns out that

$$\frac{[1-(\gamma+\delta)]}{\mu}\cdot\sum_{i=1}^{n}\eta_i^2 \le \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,\eta_i\,\eta_j,$$

which is the uniform ellipticity condition.

Observe that the Corder condition and the A_x condition are actually **equivalent**. The proof is rather tedious, but simple, thus it is left to the reader.

Condition A_x. In this brief paragraph, we want to motivate the Campanato condition and prove that the problem (2.5) admits one and only one solution in that case.

Theorem 2.11. *Assume that the matrix $A = (a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ satisfies the Campanato condition. Then there exists one and only one solution of* (2.5).

Proof 1. Let us consider

$$\Delta u(x) = \alpha\, f(x) + \Delta w(x) - \alpha \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, w(x), \tag{2.8}$$

and let us define the application $\mathcal{T} : H^2(\Omega)\cap H_0^1(\Omega) \to H^2(\Omega)\cap H_0^1(\Omega)$, which sends w to the solution u of the problem (2.8).

We now prove that $\mathcal{T}$ is a contraction of the space $H^2(\Omega)\cap H_0^1(\Omega)$, provided that the A_x condition is satisfied. In order to ease the notation, we set

$$\Gamma_{i,\,j}^n\, w(x) := \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, w(x).$$

By Remark 2.3 it follows that the $H^2(\Omega)\cap H_0^1(\Omega)$-norm is equivalent to

$$\|\Delta\, k(x)\|_{L^2(\Omega)} = \int_\Omega |\Delta\, k(x)|^2\, \mathrm{d}x,$$

therefore we can easily estimate the norm of the difference between two points

$$\begin{aligned}
\|\mathcal{T}(w_1)-\mathcal{T}(w_2)\|_{H^{2,\,2}(\Omega)}^2 \le{}& \int_\Omega |\Delta u_1(x) - \Delta u_2(x)|^2\, \mathrm{d}x = \\
={}& \int_\Omega \left|\Delta w_1(x) - \alpha\,\Gamma_{i,\,j}^n\, w_1(x) - \left[\Delta w_2(x) - \alpha\,\Gamma_{i,\,j}^n\, w_2(x)\right]\right|^2 \mathrm{d}x = \\
={}& \int_\Omega \left|\Delta\left(w_1(x)-w_2(x)\right) - \alpha\,\Gamma_{i,\,j}^n\, \left(w_1(x)-w_2(x)\right)\right|^2 \mathrm{d}x \overset{(*)}{\le} \\
\overset{(*)}{\le}{}& \int_\Omega \left\{\gamma\cdot\left[D^{i,\,j}\left(w_1(x)-w_2(x)\right)^2\right]^{\frac{1}{2}} + \delta\cdot\left|\Delta\left(w_1(x)-w_2(x)\right)\right|\right\}^2 \mathrm{d}x,
\end{aligned}$$

where the inequality $(*)$ follows from a straightforward application of the Campanato condition. Notice now that for any $a,\, b \in \mathbb{R}$, it follows that

$$(\gamma\, a + \delta\, b)^2 \leq \gamma\, (\gamma + \delta)\, a^2 + \delta\, (\gamma + \delta)\, b^2,$$

therefore

$$\ldots \leq \int_\Omega \Big[\gamma\, (\gamma + \delta)\, D_{i,\,j}\, \left(w_1(x) - w_2(x)\right)^2 + \delta\, (\gamma + \delta)\, \left|\Delta\, (w_1(x) - w_2(x))\right|\Big]^2\, \mathrm{d}x \overset{(*)}{\leq}$$
$$\overset{(*)}{\leq} (\gamma + \delta)^2 \cdot \left\|\Delta\, (w_1(x) - w_2(x))\right\|^2_{L^2(\Omega)},$$

where $(*)$ is the **Miranda-Talenti** inequality (see Lemma 2.12).

We conclude that $\mathcal{T}$ is a contraction of the space $H^2(\Omega) \cap H^0(\Omega)$ into itself (again, by Remark 2.3), therefore it admits one and only one fixed point (i.e. the solution). □

Lemma 2.12 (Miranda-Talenti)**.** *Let Ω be a convex subset of $\mathbb{R}^n$, and let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a function. Then we have the following inequality:*

$$\|u\|_{H^2(\Omega)} \leq \|\Delta\, u\|_{L^2(\Omega)}\,.$$

Proof. The inequality follows easily from the identity

$$\sum_{i,\,j=1}^n (D_{i,\,j})^2 + \sum_{i,\,j=1}^n \left[D_{i,\,i} u\, D_{j,\,j} u - (D_{i,\,j})^2\right] = (\Delta\, u)^2\,,$$

which can be easily derived if one knows that the average curvature of $\partial\,\Omega$ is strictly negative (by convexity). □

Remark 2.3. The Miranda-Talenti inequality implies that the $H^2(\Omega)$-norm and the $L^2(\Omega)$-norm of the laplacian are equivalents, but this is a more general fact holding also on non-convex subsets.

Indeed, it is obvious (by definition) that for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists a constant $c > 0$ such that

$$\|\Delta\, u\|_{L^2(\Omega)} \leq c \cdot \|u\|_{H^2(\Omega)}\,.$$

On the other hand, the divergence theorem and the Poincarè inequality immediately imply the opposite inequality, that is,

$$\begin{aligned}\int_\Omega |\nabla\, u(x)|^2\, \mathrm{d}x &= \int_\Omega \nabla\, u(x) \cdot \nabla\, u(x)\, \mathrm{d}x \leq \\ &\leq \left(\int_\Omega |\Delta\, u(x)|^2\, \mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_\Omega |u(x)|^2\, \mathrm{d}x\right)^{\frac{1}{2}} \leq \\ &\leq c(\Omega) \cdot \|\Delta\, u\|_{L^2(\Omega)} \cdot \|u\|_{H^2(\Omega)}\,.\end{aligned}$$

We now give another proof of Theorem 2.11 which relies on the near operator theory we have developed in the first section.

Proof 2. Let us set

$$B\,u(x) := \Delta\,u(x) \qquad \text{and} \qquad A\,u(x) := \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,D^{i,\,j}\,u(x).$$

The idea is to prove that the operator A is a bijection from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$. We first notice that the following two facts hold true:

(1) There is an algebraic relation between A and B.

(2) The laplacian Δ is a bijection between $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$.

The proof of the first point is easy, and it is left to the reader. The second fact, on the other hand, will be proved later in the course.

The Campanato condition implies that A is near B, therefore by Theorem 2.5 we conclude that also A is a bijection, which is exactly what we wanted to prove. □

Example 2.1 (Talenti)**.** We want to prove that, for $n > 2$, the problem (2.5) is in general not well posed in $H^2(\Omega) \cap H_0^1(\Omega)$, provided that $f \in L^2(\Omega)$ and $a_{i,\,j} \in L^\infty(\Omega)$.

Let $\Omega = S(0,\,r) = \partial\,B(0,\,r)$ be the n-dimensional sphere, let $\lambda \in (0,\,1)$ be a real number and consider the problem associated to the equation

$$\mathcal{A}(u) := \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,D^{i,\,j}\,u(x) = 0.$$

The coefficients of the matrix A are given by

$$a_{i,\,j}(x) = \delta_{i,\,j} + b\,\frac{x_i\,x_j}{\|x\|^2}, \qquad \text{where} \qquad b = -1 + \frac{n-1}{1-\lambda}$$

for any $i,\,j = 1,\,\dots,\,n$.

We first prove that the matrix $A = (a_{i,\,j})_{i,\,j=1,\,\dots,\,n}$ is uniformly elliptic on Ω. Indeed, it turns out that

$$\sum_{i,\,j=1}^{n} \left(\delta_{i,\,j} + b\,\frac{x_i\,x_j}{\|x\|^2} \right) \xi_i\,\xi_j = \sum_{i=1}^{n} \xi_i^2 + \sum_{i,\,j=1}^{n} b\,\frac{x_i\,x_j\,\xi_i\,\xi_j}{\|x\|^2} = $$

$$= \|\xi\|^2 \left(1 + b \sum_{i,\,j=1}^{n} \frac{x_i\,x_j\,\xi_i\,\xi_j}{\|x\|^2\,\|\xi\|^2} \right) \le 1,$$

since $b > -1$ and by Cauchy-Schwartz inequality also

$$\sum_{i,\,j=1}^{n} \frac{x_i\,x_j\,\xi_i\,\xi_j}{\|x\|^2\,\|\xi\|^2} = \frac{(x,\,\xi)^4}{\|x\|^2\,\|\xi\|^2} \le 1.$$

The function $u(x) = \|x\|^\lambda$ is a solution of (2.5) and its value at the boundary is exactly equal to r^λ. Similarly, the constant function $v(x) = r^\lambda$ also solves the problem (2.5), and the value at the boundary is the same.

We conclude that the problem associated to the operator $\mathcal{A}$ is **ill posed**. Moreover, it is not hard to check that the Cordes condition does not hold, coherently with the theory.

The 2-dimensional case. If $n = 2$, then the near operator theory allows us to prove the existence and the uniqueness of a solution of (2.5), provided that the coefficients are $L^\infty(\Omega)$ functions.

More precisely, the idea is to prove that the Campanato condition is, in fact, equivalent to the uniform ellipticity of the operator $\mathcal{A}$.

Suppose that $A(x)$ is uniformly elliptic and symmetric on Ω. There are $\lambda_1(x),\, \lambda_2(x) \in \mathbb{R}$ eigenvalues such that $A(x)$ is similar to the diagonal matrix $\Gamma(x) := \operatorname{diag}(\lambda_1(x),\, \lambda_2(x))$, and there is also a $\nu > 0$ such that $\lambda_1(x) \geq \nu$ and $\lambda_2(x) \geq \nu$ for almost every $x \in \Omega$.

It suffices to prove that there exists a function $a(x) \in L^\infty(\Omega)$ such that $a(x) \geq \omega > 0$ for almost every $x \in \Omega$ and, for any matrix $\xi := \{\xi_{i,\,j}\}_{i,\,j=1,\,2}$, it turns out that

$$\left| \sum_{i=1}^{2} \xi_{i,\,i} - a(x) \sum_{i,\,j=1}^{2} a_{i,\,j}(x)\, \xi_{i,\,j} \right| = \langle \mathrm{I} - a(x)\, A(x),\, \xi \rangle \leq$$

$$\leq \|\mathrm{I} - a(x)\, A(x)\| \cdot \|\xi\| \leq \rho\, \|\xi\|,$$

for some $\rho \in (0,\, 1)$. Clearly

$$\|\mathrm{I} - a(x)\, A(x)\| \leq \rho \iff a^2(x) \left[\lambda_1^2(x) + \lambda_2^2(x)\right] - 2\, a(x) \left[\lambda_1(x) + \lambda_2(x)\right] + 2 - \rho^2 \leq 0,$$

and the right-hand side admits a real solution $a(x)$ if and only if the determinant is greater or equal than 0, which is possible if and only if

$$\frac{2\, \lambda_1(x)\, \lambda_2(x)}{\lambda_1^2(x) + \lambda_2^2(x)} \geq 1 - \rho^2.$$

If we set $M = \max_{i=1,\,2} \sup_{x \in \Omega} \lambda_i(x)$, then we deduce that

$$\frac{2\, \lambda_1(x)\, \lambda_2(x)}{\lambda_1^2(x) + \lambda_2^2(x)} \geq \frac{\nu^2}{M^2},$$

therefore it is enough to choose ρ and $a(x)$ in such a way that

$$\left[1 - \frac{\nu^2}{M^2}\right]^{\frac{1}{2}} \leq \rho < 1 \qquad \text{and} \qquad a(x) = \frac{\lambda_1(x) + \lambda_2(x)}{\lambda_1^2(x) + \lambda_2^2(x)}.$$

2.3 Regularity Conditions

In this section we want to prove that the problem (2.5) is well-posed if we assume that the matrix $A(x) := \{a_{i,\,j}(x)\}_{i,\,j=1,\,\dots,\,n}$ is uniformly elliptic, and also that the coefficients $a_{i,\,j}(x)$ are α-Hölder continuous, that is,

$$a_{i,\,j}(x) \in C^{0,\,\alpha}(\Omega).$$

Definition 2.13 (Sub(Super)-Solution)**.** Let $u \in H^2(\Omega)$ be a function such that

$$\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) \geq 0 \qquad (\text{respectively, } \leq 0). \tag{2.9}$$

We say that u is a *sub-solution* (respectively, *super-solution*) of the problem (2.5) with no boundary condition.

Theorem 2.14 (Max. Principle)**.** *Let* $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ *be a sub-solution of the problem* (2.5) *with no boundary condition. If* $A(x)$ *is a uniformly elliptic matrix on* Ω*, and the coefficients* $a_{i,j}(x)$ *are functions of class* $C^0\left(\overline{\Omega}\right)$*, then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x). \tag{2.10}$$

Theorem 2.15 (Min. Principle)**.** *Let* $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ *be a super-solution of the problem* (2.5) *with no boundary condition. If* $A(x)$ *is a uniformly elliptic matrix on* Ω*, and the coefficients* $a_{i,j}(x)$ *are functions of class* $C^0\left(\overline{\Omega}\right)$*, then*

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x). \tag{2.11}$$

Corollary 2.16. *Let* u *be a solution of the problem*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\, D^{i,j}\, v(x) = 0,$$

and suppose that the same assumptions of Theorem 2.14 are met. Then

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x) \quad \textit{and} \quad \min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x).$$

Corollary 2.17. *If the same assumptions of Theorem 2.14 are met, then the Dirichlet problem*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} a_{i,j}(x)\, D^{i,j}\, u(x) = f(x) & \textit{for } x \in \Omega \\ u(x) = g(x) & \textit{for } x \in \partial\Omega. \end{cases} \tag{2.12}$$

admits at most one solution.

Proof. Suppose that $u_1(x)$ and $u_2(x)$ are both solution of class $C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ of the problem (2.12). If we set

$$W(x) := u_1(x) - u_2(x),$$

then it's straightforward to prove that $W(x)$ is a solution of the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} a_{i,j}(x)\, D^{i,j}\, u(x) = 0 & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{2.13}$$

Finally, Corollary 2.17 proves that $W(x) \equiv 0$ is the unique solution of (2.13), hence $u_1(x) \equiv u_2(x)$. ☐

Proof of Maximum Principle. We divide the proof into two steps since the first one can be easily proved, and the second one follows from a simple approximation argument.

Step 1. Suppose that for any $x \in \Omega$ it turns out that

$$\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) > 0. \tag{2.14}$$

We argue by contradiction. Suppose that there exists a maximal point $x_0 \in \mathring{\Omega}$ for u; then for any vector $v \in \mathbb{R}^n$, it turns out that

$$(H_u(x_0)\, v,\, v)_{\mathbb{R}^n} \leq 0, \tag{2.15}$$

where $H_u(x_0)$ is the Hessian matrix of u computed at the point x_0. By assumption, the matrix $A(x_0)$ is uniformly elliptic, hence there exists a real number $\nu > 0$ such that

$$(A(x_0)\, v,\, v)_{\mathbb{R}^n} \geq \nu\, \|v\|^2_{\mathbb{R}^n}, \qquad \forall\, v \in \mathbb{R}^n. \tag{2.16}$$

It remains to prove that these relations yield to a contradiction. Indeed, the condition (2.14) is equivalent to requiring that

$$(A(x),\, H(x)) > 0$$

at any point $x \in \mathbb{R}^n$ and, in particular, at $x = x_0$. On the other hand, both $A(x_0)$ and $H(x_0)$ are symmetric, thus there are U and V unitary matrices such that

$$U\, A(x_0)\, U^* =: \Lambda_A \quad \text{and} \quad V\, H(x_0)\, V^* =: \Lambda_H$$

are diagonal. Therefore, if we set $Q := U^*\, V$, then it is simple to prove that

$$(A(x),\, H(x)) = (\Lambda_A\, Q,\, Q\, \Lambda_H)\,,$$

and also that the latter scalar product is less or equal than zero, as a consequence of the fact that $A(x_0)$ is defined positive (2.15) and $H(x_0)$ is semi-definite negative (2.16).

Step 2. Suppose now that for any $x \in \Omega$ it turns out that

$$\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) \geq 0. \tag{2.17}$$

We can define a small perturbation of the function $u(x)$, that is,

$$u_\epsilon(x) := u(x) + \epsilon\, \|x\|^2,$$

so that $u_\epsilon(x)$ satisfies the condition (2.14) for any $\epsilon > 0$. The first step proves that

$$\max_{x \in \overline{\Omega}} u_\epsilon(x) = \max_{x \in \partial\, \Omega} u_\epsilon(x),$$

hence the general case follows easily by taking the limit $\epsilon \to 0^+$. □

Theorem 2.18 (Aleksandrov-Bakel'man-Pucci). *Let*

$$A(x,\, D)\, u := \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) + \sum_{i=1}^{n} b_i(x)\, D^i\, u(x) + c(x)\, u(x)$$

be a differential operator such that the matrix $A(x)$ is uniformly elliptic on Ω, and the coefficients $a_{i,\,j}(x)$, $b_i(x)$ and $c(x) \leq 0$ are all of class $L^\infty(\Omega)$.

Let $u \in C^0\left(\overline{\Omega}\right) \cap W^{2,\,n}_{\mathrm{loc}}(\Omega)$ be a function such that $A(x,\,D)\,u \geq f(x)$ for almost every $x \in \Omega$, and let

$$D(x) := \det\left(A(x)\right) \quad \textit{and} \quad D^*(x) := [D(x)]^{\frac{1}{n}}.$$

Moreover, suppose that

$$\frac{f(x)}{D^*(x)} \in L^n(\Omega) \quad \textit{and} \quad \frac{b(x)}{D^*(x)} \in L^n(\Omega).$$

Then it turns out that

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\,\Omega} u^+(x) + c\left\|\frac{f}{D^*}\right\|_{L^n(\Omega)}, \tag{2.18}$$

where $u^+(x) := \max\{u(x),\,0\}$ and the constant c depends only on the dimension n and on the following quantity:

$$\left\|\frac{b(x)}{D^*(x)}\right\|_{L^n(\Omega)}.$$

The second step needed to prove that (2.5) is well-posed, is the following a priori estimate.

Theorem 2.19. *Let $u(x) \in H^{2,\,2}(\Omega) \cap H^{1,\,2}_0(\Omega)$ be a solution of the Dirichlet problem (2.5). Assume that $\partial\,\Omega$ is a manifold of class C^3, f and $a_{i,\,j}$ belong to $C^{0,\,\alpha}\left(\overline{\Omega}\right)$. Then $D^{i,\,j}\,u \in C^{0,\,\alpha}\left(\overline{\Omega}\right)$, and there exists a positive constant $c > 0$ such that*

$$\sum_{i,\,j=1}^n \left\|D^{i,\,j}\,u\right\|^2_{C^{0,\,\alpha}(\overline{\Omega})} \leq c\cdot\left(\|f\|^2_{C^{0,\,\alpha}(\overline{\Omega})} + \sum_{i,\,j=1}^n \left\|D^{i,\,j}\,u\right\|^2_{\infty,\,\Omega}\right). \tag{2.19}$$

Theorem 2.20. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, whose boundary $\partial\,\Omega$ is of class C^3. If $f \in C^{0,\,\alpha}\left(\overline{\Omega}\right)$, then there exists a positive constant*

$$C := C\left(\Omega,\,\nu,\,\|a_{i,\,j}\|_{C^{0,\,\alpha}(\overline{\Omega})}\right) > 0$$

such that, if $u \in H^{2,\,2}(\Omega)$ is a solution of the Dirichlet problem (2.5), then

$$\sum_{i,\,j=1}^n \left\|D^{i,\,j}\,u\right\|^2_{0,\,\Omega} \leq C\cdot\left(\|f\|^2_{C^{0,\,\alpha}(\overline{\Omega})}\right). \tag{2.20}$$

Proof. We argue by contradiction. If (2.20) doesn't hold, then there exist:

(1) A uniformly bounded (by a constant $M > 0$) sequence of coefficients $\left(a^{(k)}_{i,\,j}(x)\right)_{k\in\mathbb{N}} \subset C^{0,\,\alpha}\left(\overline{\Omega}\right)$ such that the matrix

$$A^{(k)}(x) := \left\{a^{(k)}_{i,\,j}(x)\right\}_{i,\,j=1,\,\dots,\,n}$$

is uniformly elliptic for any $k \in \mathbb{N}$, with the same constant $\nu > 0$.

(2) A sequence of functions $(f_k(x))_{k\in\mathbb{N}} \subset C^{0,\,\alpha}\left(\overline{\Omega}\right)$ such that

$$\|f_k\|_{C^{0,\,\alpha}(\overline{\Omega})} \xrightarrow{k\to+\infty} 0.$$

(3) A sequence of solutions of the Dirichlet problems

$$\begin{cases} \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u_k(x) = f_k(x) & \text{for } x \in \Omega \\ \\ u_k(x) = 0 & \text{for } x \in \partial\,\Omega \end{cases} \tag{2.21}$$

such that $\|u_k\|_{H^{2,\,2}(\Omega)} = 1$ for all $k \in \mathbb{N}$.

It follows from the Ascoli-Arzelà Theorem that there exists a subsequence $(k_n)_{n\in\mathbb{N}} \subset (k)_{k\in\mathbb{N}}$ such that $a^{k_n}_{i,\,j}(x)$ converges uniformly to $a_{i,\,j}(x)$ in $\overline{\Omega}$. From the a priori estimate (2.19), it turns out that for $k \in \mathbb{N}$ big enough

$$\sum_{i,\,j=1}^{n} \left\|D^{i,\,j}\, u_k\right\|^2_{C^{0,\,\alpha}(\overline{\Omega})} \leq c \cdot \left(\|f_k\|^2_{C^{0,\,\alpha}(\overline{\Omega})} + \sum_{i,\,j=1}^{n} \left\|D^{i,\,j}\, u_k\right\|^2_{\infty,\,\Omega} \right) \leq c_1,$$

since the norm of f_k converges to zero, and the $H^{2,\,2}(\Omega)$-norm of u_k is constantly equal to one.

The sequence $(u_k)_{k\in\mathbb{N}}$ is equibounded in $C^{2,\,\alpha}\left(\overline{\Omega}\right)$, hence we may always extract a subsequence uniformly converging to u (by Ascoli-Arzelà), and this implies the strong convergence in $H^{2,\,2}(\Omega)$. Taking the limit for $k \to +\infty$ of the Dirichlet problem (2.21), it turns out that u is a solution of the problem (2.13), hence $u \equiv 0$ on Ω, that is, a contradiction since the $H^{2,\,2}(\Omega)$-norm of u is equal to 1 (as a consequence of the uniform convergence). □

Corollary 2.21. *Under the assumptions of Theorem 2.20, the solution u of the Dirichlet problem (2.5) satisfies the following estimate:*

$$\|u\|_{C^{2,\,\alpha}(\overline{\Omega})} \leq c\, \|f\|_{C^{0,\,\alpha}(\overline{\Omega})}. \tag{2.22}$$

Theorem 2.22. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, whose boundary $\partial\,\Omega$ is of class C^3. Suppose that the coefficients $a_{i,\,j}(x)$ are of class $C^{0,\,\alpha}\left(\overline{\Omega}\right)$, and suppose that the matrix $A(x)$ is uniformly elliptic on Ω.*

Then, for any $f \in C^{0,\,\alpha}\left(\overline{\Omega}\right)$, the Dirichlet problem (2.5) admits one and only one solution $u \in C^{2,\,\alpha}\left(\overline{\Omega}\right)$ satisfying the estimate (2.22).

Proof. In order to prove this theorem, we use the continuity method introduced in the previous section. More precisely, let us consider the family of operators

$$A_t\, u := (1-t)\, \nu\, \Delta\, u + t \sum_{i,\,j=1}^{n} a_{i,\,j}\, D^{i,\,j}\, u, \qquad t \in [0,\, 1],$$

and we notice that the coefficients $a^{(t)}_{i,\,j}(x) := (1-t)\, \nu\, \delta_{i,\,j} + t\, a_{i,\,j}(x)$ verifies the uniform ellipticity property, that is, the matrices

$$A^{(t)}(x) := \{(1-t)\, \nu\, \delta_{i,\,j} + t\, a_{i,\,j}(x)\}^n_{i,\,j=1}$$

are uniformly elliptic on Ω. If we set $f_t := A_t\, u$, then it follows from Corollary 2.21 that for any $u \in C^{2,\,\alpha}\left(\overline{\Omega}\right)$

$$\|u\|_{C^{2,\,\alpha}\left(\overline{\Omega}\right)} \leq c \cdot \|A_t\, u\|_{C^{0,\,\alpha}\left(\overline{\Omega}\right)}$$

since we can apply it to the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,\,j=1}^{n} (1-t)\,\nu\,\Delta\, u(x) + t \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^{i,\,j}\, u(x) = f_t(x) & \text{for } x \in \Omega \\ \\ u(x) = 0 & \text{for } x \in \partial\,\Omega. \end{cases}$$

It remains to check if the assumption of Theorem 2.7 are met, where $\mathfrak{X} = C^{2,\,\alpha}\left(\overline{\Omega}\right)$ and $\mathcal{B} = C^{0,\,\alpha}\left(\overline{\Omega}\right)$.

The first assumption holds true for $t = 0$ since the operator Δ is an isomorphism between $\mathfrak{X}$ and $\mathcal{B}$; to check the second assumption, it suffices to observe that

$$\begin{aligned} \|A_t\, u - A_s\, u\|_{C^{0,\,\alpha}\left(\overline{\Omega}\right)} = |t-s| \left\| \nu\,\Delta u - \sum_{i,\,j=1}^{n} a_{i,\,j}\, D^{i,\,j}\, u \right\|_{C^{0,\,\alpha}\left(\overline{\Omega}\right)} &\leq \\ \leq c\,|t-s|\, \|u\|_{C^{2,\,\alpha}\left(\overline{\Omega}\right)} &\leq \\ \leq c_1\, c\,|t-s|\, \|A_t\, u\|_{C^{0,\,\alpha}\left(\overline{\Omega}\right)}, \end{aligned}$$

where the inequality $\leq$ follows from Corollary 2.21. □

2.4 Lax-Milgram Theorem

Theorem 2.23 (Lax-Milgram)**.** *Let H be a Hilbert space, and let $a : H \times H \to \mathbb{R}$ be a function satisfying the following properties:*

(1) $a(0,\, v) = 0$ *for any* $v \in H$.

(2) $v \longmapsto a(u,\, v)$ *is linear for any* $u \in H$.

(3) *For any* $u_1,\, u_2,\, v \in H$ *it turns out that*

$$|a(u_1,\, v) - a(u_2,\, v)| \leq M\, \|u_1 - u_2\|_H\, \|v\|_H.$$

(4) *There exists* $\nu > 0$ *such that, for any* $u_1,\, u_2 \in H$, *the following inequality holds true:*

$$a(u_1,\, u_1 - u_2) - a(u_2,\, u_1 - u_2) \geq \nu\, \|u_1 - u_2\|_H^2.$$

Then for any $F \in H^*$ *there exists one and only one* $u \in H$ *such that*

$$a(u,\, v) = F(v), \qquad \forall\, v \in H.$$

Moreover, the following estimate holds true:

$$c(\nu)\, \|u\|_H \leq \|F\|_{H^*}.$$

Proof. Let us consider the application

$$\mathcal{A} : H \to H^*, \qquad u \longmapsto \varphi_u \ : \ \varphi_u(v) := a(u,\, v).$$

We want to prove that $\mathcal{A}$ is a bijection between H and H^*, that is, for every $F \in H^*$ there is a unique $u \in H$ such that

$$\mathcal{A}(u)(v) = F(v) \qquad \forall\, v \in H \iff a(u,\, v) = F(v) \qquad \forall\, v \in H.$$

By Theorem 2.5 it suffices to prove that A is near $\mathcal{J} : H \to H^*$ which is defined by

$$u \longmapsto \mathcal{J}_u \ : \ \mathcal{J}_u(v) := (u,\, v)_H,$$

where $(\cdot,\, \cdot)_H$ is the scalar product.

Step 1. The operator $\mathcal{J}$ preserves the H-norm, that is, for any $u \in H$ it turns out that

$$\|\mathcal{J}(u)\|_{H^*} = \|u\|_H.$$

The Riesz operator $\mathcal{R} : H^* \to H$ is defined by sending F to the unique element $w_F \in H$ representing F, that is,

$$\mathcal{R}(F) = w_F \iff F(v) = (w,\, v)_H \qquad \forall\, v \in H.$$

The Riesz operator $\mathcal{R}$ is an isometry (as it was proved in Riesz theorem), hence

$$\left(\mathcal{R}\left(\mathcal{A}(u)\right),\, v\right)_H = a(u,\, v) \qquad \text{and} \qquad \mathcal{R} = \mathcal{J}^{-1},$$

that is, $\mathcal{J}$ is an invertible operator.

Step 2. The thesis of the theorem follows easily if we can prove that the nearness condition (2.1) holds. By definition of the operators $\mathcal{A}$ and $\mathcal{J}$ it turns out that

$$\begin{aligned}
\|\mathcal{J}(u_1) - \mathcal{J}(u_2) - \alpha\, [\mathcal{A}(u_1) - \mathcal{A}(u_2)]\|^2_{H^*} &= \|u_1 - u_2 - \alpha\, [\mathcal{R}\left(\mathcal{A}(u_1)\right) - \mathcal{R}\left(\mathcal{A}(u_2)\right)]\|^2_H = \\
&= \|u_1 - u_2\|^2_H + \alpha^2\, \|\mathcal{R}\left(\mathcal{A}(u_1)\right) - \mathcal{R}\left(\mathcal{A}(u_2)\right)\|^2_H - \dots \\
&\dots - 2\,\alpha\, \left(\mathcal{R}\left(\mathcal{A}(u_1)\right) - \mathcal{R}\left(\mathcal{A}(u_2)\right),\, u_1 - u_2\right)_H = \\
&= \|u_1 - u_2\|^2_H + \alpha^2\, \|\mathcal{R}\left(\mathcal{A}(u_1)\right) - \mathcal{R}\left(\mathcal{A}(u_2)\right)\|^2_H - \dots \\
&\dots - 2\,\alpha\, [a(u_1,\, u_1 - u_2) - a(u_2,\, u_1 - u_2)] \ \leq \\
&\leq \|u_1 - u_2\|^2_H + \alpha^2\, M^2\, \|u_1 - u_2\|^2_H - 2\,\alpha\,\nu\, \|u_1 - u_2\|^2_H = \\
&= \left[1 + \alpha^2\, M^2 - 2\,\alpha\,\nu\right]\, \|u_1 - u_2\|^2_H = \\
&= k\, \|\mathcal{J}(u_1) - \mathcal{J}(u_2)\|^2_{H^*}\,,
\end{aligned}$$

where the inequality $\leq$ follows easily from properties **(2)** and **(3)** □

2.5 Garding Inequality

Let us consider a differential elliptic operator in divergence form, that is,

$$A(x,\, D)\, u = \sum_{|\alpha| \leq m}\ \sum_{|\beta| \leq m} (-1)^{|\alpha|}\, D^\alpha\, \left(A_{\alpha,\,\beta}(x)\, D^\beta\, u\right).$$

We denote by $A_0(x, D)$ the principal part of the operator, and we denote by $(\cdot, \cdot)_{\mathbb{C}^N}$ and $\|\cdot\|_{\mathbb{C}^N}$ respectively the scalar product and the norm in $\mathbb{C}^N$.

Lemma 2.24. *Let $A_0(D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition (1.12), and assume that the coefficients of A are constants. Then for every $u \in H_0^m(\Omega; \mathbb{R}^N)$ it turns out that*

$$\int_\Omega \left[\sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\beta} D^\alpha u(x),\, D^\beta u(x) \right) \right] \mathrm{d}x \geq c(\nu) \|u\|_{H^m(\Omega;\, \mathbb{R}^N)}. \tag{2.23}$$

Proof. We may always assume that $A_{\alpha,\beta} = A^*_{\alpha,\beta}$ since we can prove the thesis separately for the self-adjoint part and the anti self-adjoint part.

For any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ it turns out that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left[\sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\beta} D^\alpha u(x),\, D^\beta u(x) \right) \right] \mathrm{d}x &= \sum_{|\alpha|=m} \sum_{|\beta|=m} \int_{\mathbb{R}^n} \left(A_{\alpha,\beta} D^\alpha u(x),\, D^\beta u(x) \right) \mathrm{d}x = \\
&= \int_{\mathbb{R}^n} \left[\sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\beta}\, \hat{u}(\xi),\, \overline{\hat{u}(\xi)} \right)_{\mathbb{C}^N} \right] \xi^{\alpha+\beta}\, \mathrm{d}\xi \geq \\
&\geq \nu \int_{\mathbb{R}^n} \left(\|\mathfrak{Re}(\hat{u})(xi)\|^2 + \|\mathfrak{Im}(\hat{u})(xi)\|^2 \right) \|\xi\|^{2m}\, \mathrm{d}\xi = \\
&= \nu \int_{\mathbb{R}^n} \left(\hat{u}(\xi),\, \overline{\hat{u}(\xi)} \right)_{\mathbb{C}^N} \|\xi\|^{2m}\, \mathrm{d}\xi \geq \\
&\geq c(\nu) \int_{\mathbb{R}^n} \left[\left(\hat{u}(\xi),\, \overline{\hat{u}(\xi)} \right)_{\mathbb{C}^N} \sum_{|\alpha|=m} \xi^{2\alpha} \right] \mathrm{d}\xi = \\
&= c(\nu) \|u\|^2_{H^m(\mathbb{R}^n,\, R^N)},
\end{aligned}$$

and this concludes the proof by density of the inclusion $C_0^\infty(\Omega, \mathbb{R}^N) \subset H_0^1(\Omega, \mathbb{R}^N)$. □

Lemma 2.25. *Let $A_0(D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition (1.12), and assume that the coefficients of A are continuous on $\overline{\Omega}$. Then for every $u \in H_0^m(B(x_0, r); \mathbb{R}^N)$, with $x_0 \in \Omega$ and $r > 0$ small enough, it turns out that*

$$\int_{B(x_0,r)} \left[\sum_{|\alpha|=m} \sum_{|\beta|=m} \left(A_{\alpha,\beta} D^\alpha u(x),\, D^\beta u(x) \right) \right] \mathrm{d}x \geq [c(\nu) - \omega(r)] \|u\|_{H^m(\Omega;\, \mathbb{R}^N)}, \tag{2.24}$$

where $\omega(r)$ is the modulus of continuity defined by

$$\omega(r) := \sup \left\{ \|A_{\alpha,\beta}(x) - A_{\alpha,\beta}(y)\| \;\middle|\; x, y \in \overline{\Omega},\ \|x-y\| \leq r,\ |\alpha| = |\beta| = m \right\}.$$

Proof. Since

$$\left| \int_{B(x_0,r)} \left[\sum_{|\alpha|=m} \sum_{|\beta|=m} \left([A_{\alpha,\beta}(x) - A_{\alpha,\beta}(x_0)]\, D^\alpha u(x),\, D^\beta u(x) \right) \right] \mathrm{d}x \right| \geq \dots$$

$$\dots \geq \omega(r) \int_{B(x_0,r)} \sum_{|\alpha=m} \|D^\alpha u(x)\|^2\, \mathrm{d}x,$$

a simple application of Lemma 2.24 gives us the estimate (2.24). ☐

Lemma 2.26. *Let $\Omega \subset \mathbb{R}^n$ be a subset with a boundary locally Lipschitz. Let $A_0(D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition (1.12), and assume that the coefficients of A are continuous on $\overline{\Omega}$. Then for every $u \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$ it turns out that*

$$\int_\Omega \left[\sum_{|\alpha|=m}\sum_{|\beta|=m}\left(A_{\alpha,\,\beta}D^\alpha\, u(x),\, D^\beta\, u(x)\right)\right]\,\mathrm{d}x \geq c(\nu)\,\left[\|u\|_{H^m(\Omega;\,\mathbb{R}^N)} - \|u\|_{L^2(\Omega;\,\mathbb{R}^N)}\right]. \tag{2.25}$$

Proof. We divide the argument into many steps, in order to ease the notation.

Step 1. ... ☐

Lemma 2.27. *Let $A(x,\, D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition (1.12), and assume that:*

(1) *The coefficients of order m are continuous on $\overline{\Omega}$.*

(2) *The coefficients of order $< m$ are essentially bounded.*

Then for every $u \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$ it turns out that

$$\int_\Omega \left[\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\left(A_{\alpha,\,\beta}D^\alpha\, u(x),\, D^\beta\, u(x)\right)\right]\,\mathrm{d}x \geq c(\nu)\,\left[\|u\|_{H^m(\Omega;\,\mathbb{R}^N)} - \|u\|_{L^2(\Omega;\,\mathbb{R}^N)}\right]. \tag{2.26}$$

Proof. The estimate (2.26) is a straightforward consequence of the lemmas we have proved so far. More precisely, the terms of maximal order can be estimated via Lemma 2.26, while the other terms can be easily estimated with the interpolation inequality (2.30). ☐

Theorem 2.28 (Lions)**.** *Let $X \subset Y \subset Z$ be Banach spaces such that the inclusion*

$$X \hookrightarrow Y$$

is continuous and compact, while the inclusion

$$Y \subset Z$$

is continuous. For any $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for every $u \in X$ it turns out that

$$\|u\|_Y \leq \epsilon\,\|u\|_X + c(\epsilon)\,\|u\|_z. \tag{2.27}$$

Proof. We argue by contradiction. If (2.27) is not satisfied, then there exists $\epsilon > 0$ such that for any divergent sequence of real numbers $(c_n)_{n\in\mathbb{N}}$ there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$\|u_n\|_Y \geq \epsilon\,\|u_n\|_X + c_n\,\|u_n\|_z. \tag{2.28}$$

Let $v_n := u_n/\|u_n\|_X$. Then (2.28) can be written as follows

$$\|v_n\|_Y \geq \epsilon + c_n \|v_n\|_z. \tag{2.29}$$

The X-norm of v_n is equal to 1 for each $n \in \mathbb{N}$, hence there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}} \subset Y$ (by compactness of the immersion) such that

$$v_{n_k} \xrightarrow{k\to+\infty} v_\infty \qquad \text{strongly in } Y.$$

On the one hand, it follows from the continuity of the inclusion and (2.29) that

$$\frac{c}{c_n} = \frac{c\,\|v_n\|_X}{c_n} \geq \frac{\|v_n\|_Y}{c_n} \geq \|v_n\|_Z,$$

and by taking the limit as $n \to +\infty$ we infer that $\|v_n\|_Z \to 0$. This is absurd since by continuity of the inclusion $Y \subset Z$ it turns out that $v_\infty \equiv 0$, but (2.29) implies also that $\|v_{n_k}\|_Y \geq \epsilon$ (in contradiction with the fact that v_∞ is zero). □

Corollary 2.29. *For any $\epsilon > 0$ and for any $u \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$ it turns out that*

$$\|u\|_{H^{m-1}(\Omega;\,\mathbb{R}^N)} \leq \epsilon\,\|u\|_{H^m(\Omega;\,\mathbb{R}^N)} + c(\epsilon)\,\|u\|_{L^2(\Omega;\,\mathbb{R}^N)}. \tag{2.30}$$

Proof. It is a straightforward consequence of the interpolation inequality (2.27) since the immersion

$$H^m\left(\Omega;\,\mathbb{R}^N\right) \hookrightarrow H^{m-1}\left(\Omega;\,\mathbb{R}^N\right)$$

is compact by Rellich Theorem[1]. □

2.6 Dirichlet problem for linear systems

In this section, the primary goal is proving that the Dirichlet problem is well-posed in the linear systems setting.

Sobolev Spaces. For any real number $s > 0$ the $-s$-Sobolev space is defined by setting

$$H^{-s}(\Omega) := \left(H_0^s(\Omega)\right)^*.$$

The space of compactly supported infinitely derivable functions $\mathcal{D}(\Omega)$ is dense in $H_0^s(\Omega)$, hence

$$\mathcal{D}'(\Omega) \supset H^{-s}(\Omega).$$

[1]Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded Lipschitz domain, and let $1 \leq p < n$. Set

$$p^* := \frac{np}{n-p}.$$

Then the Sobolev space $W^{1,\,p}(\Omega;\,\mathbb{R})$ is continuously embedded in the L^p-space $L^{p^*}(\Omega;\,\mathbb{R})$ and is compactly embedded in $L^q(\Omega;\,\mathbb{R})$ for every $1 \leq q < p^*$. In symbols,

$$W^{1,\,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

and

$$W^{1,\,p}(\Omega) \subset\subset L^q(\Omega), \qquad \forall\, 1 \leq q < p^*.$$

Theorem 2.30. *Let $m > 0$ be an integer positive number. Every functional $f \in H^{-m}(\Omega)$ admits a non-unique representation as follows:*

$$f = \sum_{|\alpha| \le m} D^\alpha f_\alpha, \qquad f_\alpha \in L^2(\Omega). \tag{2.31}$$

Proof. Let us consider, for $|\alpha| \le m$, the linear application

$$\Psi_\alpha : H^m(\Omega) \ni u \longmapsto D^\alpha u \in L^2(\Omega).$$

Clearly $(\Psi_\alpha)_{|\alpha| \le m}$ establish an isomorphism between $H^m(\Omega)$ and a linear submanifold $V \subset \left[L^2(\Omega)\right]^h$, where h is the number of all the derivatives D^α with $|\alpha| \le m$.

Therefore to a linear continuous functional defined on $H^m(\Omega)$, corresponds a linear continuous functional defined on V. By Hahn-Banach theorem L may be isometrically extended to a linear continuous functional $\tilde{L}$ defined over $\left[L^2(\Omega)\right]^h$. Moreover, it is easy to check that

$$\tilde{L} : \left[L^2(\Omega)\right]^h \to \mathbb{R} \implies \tilde{L} = \sum_{i=1}^h L_i,$$

where $L_i \in \left(L^2(\Omega)\right)^*$ for every index $i = 1, \ldots, h$. By Riesz theorem, it turns out that

$$\tilde{L}(u) = \sum_{|\alpha| \le m} \int_\Omega g_\alpha D^\alpha u, \qquad g_\alpha \in L^2(\Omega).$$

In conclusion, since $L \in (H_0(\Omega))^*$, the formula above is uniquely represented by the functional

$$L(\varphi) := \langle f, \varphi \rangle$$

defined on $\mathcal{D}(\Omega)$, and hence

$$f = \sum_{|\alpha| \le m} D^\alpha f_\alpha, \qquad \text{where} \quad f_\alpha = (-1)^{|\alpha|} g_\alpha.$$

□

Theorem 2.31 (Global Existence, I)**.** *Let $A_0(D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition (1.12), and assume that the coefficients of A are constants. Then for every $F \in H^{-m}(\Omega;\, \mathbb{R}^N)$ there exists one and only one solution in $H_0^m(\Omega;\, \mathbb{R}^N)$ of the system*

$$A_0(D)\, u = F$$

and the following estimate holds true:

$$\|u\|_{H_0^m(\Omega;\, \mathbb{R}^N)} \le c(\nu)\, \|F\|_{H^{-m}(\Omega;\, \mathbb{R}^N)}. \tag{2.32}$$

Remark 2.4. If we set

$$\|F\|^*_{-m,\,\Omega} := \inf \left\{ \sum_{j=0}^m d_\Omega^{m-j} \left[\int_\Omega \left(\sum_{|\alpha|=j} \|f_\alpha\|^2 \right) \mathrm{d}x \right]^{\frac{1}{2}} \right\}, \tag{2.33}$$

where the infimum is taken over the possible representations of F of the form (2.31). One can easily show that (2.33) is a norm on $H^{-m}(\Omega)$, which is equivalent to the usual one:

$$\|F\|_{-m,\,\Omega} = \sup \left\{ |\langle F, \varphi \rangle| \;\middle|\; \varphi \in H_0^m(\Omega;\, \mathbb{R}^N),\ \|\varphi\|_{H_0^m(\Omega;\, \mathbb{R}^N)} = 1 \right\}. \tag{2.34}$$

Proof. Let us consider the bilinear form $a : H_0^m\left(\Omega;\, \mathbb{R}^N\right) \times H_0^m\left(\Omega;\, \mathbb{R}^N\right) \to \mathbb{R}$ defined by setting

$$a(u,\, v) := \int_\Omega \left[\sum_{|\alpha|=|\beta|=m|} \left(A_{\alpha,\,\beta}(x)\, D^\alpha\, u(x),\, D^\beta\, v(x)\right) \right] \mathrm{d}x.$$

The functional

$$L(v) := F(v), \qquad v \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$$

is continuous, while a is coercive on the product as a consequence of Lemma 2.24. By Lax-Milgram Theorem 2.23 it turns out that there exists one and only one solution of the system

$$\int_\Omega \left[\sum_{|\alpha|=|\beta|=m|} \left(A_{\alpha,\,\beta}(x)\, D^\alpha\, u(x),\, D^\beta\, v(x)\right) \right] \mathrm{d}x = F(v), \qquad \forall\, v \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$$

along with the estimate (2.32). □

Theorem 2.32 (Global Existence, II)**.** *Let $A(D)$ be a differential operator in divergence form satisfying the weak Legendre-Hadamard condition* (1.12)*, and assume that:*

(1) The coefficients of order m are continuous on $\overline{\Omega}$.

(2) The coefficients of order $< m$ are essentially bounded.

Let $\gamma > 0$ be a big enough real number. Then for every $F \in H^{-m}(\Omega;\, \mathbb{R}^N)$ there exists one and only one solution in $H_0^m(\Omega;\, \mathbb{R}^N)$ of the system

$$A(x,\, D)\, u + \gamma\, u = F,$$

and the following estimate holds true:

$$\|u\|_{H_0^m(\Omega;\, \mathbb{R}^N)} \leq c(\nu)\, \|F\|_{H^{-m}(\Omega;\, \mathbb{R}^N)}. \tag{2.35}$$

The same assertion holds true for any $\gamma > 0$, provided that the diameter d_Ω of the subset Ω is small enough.

Proof. Let us consider the bilinear form $a : H_0^m\left(\Omega;\, \mathbb{R}^N\right) \times H_0^m\left(\Omega;\, \mathbb{R}^N\right) \to \mathbb{R}$ defined by setting

$$a(u,\, v) := \int_\Omega \left[\sum_{|\alpha|=|\beta|=m|} \left(A_{\alpha,\,\beta}(x)\, D^\alpha\, u(x),\, D^\beta\, v(x)\right) + \gamma\, u(x) \right] \mathrm{d}x.$$

The functional

$$L(v) := F(v), \qquad v \in H_0^m\left(\Omega;\, \mathbb{R}^N\right)$$

is continuous, hence the conclusion follows from Lemma 2.27 for any $\gamma \geq c(\nu)$. Indeed, it suffices to observe that

$$\begin{aligned} a(u,\, v) &\geq c(\nu)\, \left[\|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)} - \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)} \right] + \gamma\, \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)} = \\ &= c(\nu) \|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)} + [\gamma - c(\nu)]\, \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)} \geq \\ &\geq c(\nu)\, \|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)}. \end{aligned}$$

On the other hand, if γ is any positive real number, it turns out that

$$\begin{aligned} a(u,\, v) &\geq c(\nu)\, \left[\|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)} - \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)}\right] + \gamma\, \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)} = \\ &= c(\nu)\|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)} + [\gamma - c(\nu)]\; \|u\|^2_{L^2(\Omega;\, \mathbb{R}^N)} \geq \\ &\geq [c(\nu) - d_\Omega\; (c(\nu) - \gamma)]\; \|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)} \geq \\ &\geq c(\nu,\, \gamma,\, d_\Omega)\, \|u\|^2_{H^m(\Omega;\, \mathbb{R}^N)}, \end{aligned}$$

where the red inequality follows from the Poincaré inequality when $\gamma \leq c(\nu)$ and d_Ω is small enough. □

2.7 Global Existence via Special Operators

Let us consider the Dirichlet problem

$$\begin{cases} u \in H^m_0\left(\Omega,\, \mathbb{R}^N\right) \\ \\ A(x,\, D)\, u(x) = F(x). \end{cases} \tag{2.36}$$

In the previous section, we have proved that for any $F \in H^{-m}\left(\Omega,\, \mathbb{R}^N\right)$ the problem (2.36) admits one and only one solution (i.e., it is well-posed) if the diameter of Ω is small enough.

In this section, we show, via apriori estimates, that the same result holds true even when the diameter of Ω is small enough for the Poincaré inequality to hold. Let us set

$$\mathcal{P}\, u(x) = A(x,\, D)\, u(x).$$

Theorem 2.33. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, and assume that $\partial\,\Omega$ is a boundary of class C^m. Suppose that $A_0(x,\, D)$ is an elliptic operator satisfying the Legendre-Hadamard condition (1.12), with coefficients of class $C^1\left(\overline{\Omega}\right)$. Then the linear application*

$$\mathcal{P} : H^{m+1}(\Omega,\, \mathbb{R}^N) \cap H^1_0(\Omega,\, \mathbb{R}^N) \to H^{1-m}\left(\Omega,\, \mathbb{R}^N\right)$$

has finite-dimensional kernel and closed rank.

Lemma 2.34 (Peetre)**.** *Let E, Φ, G be three Banach spaces such that $E \subset \Phi$ is a compact immersion, and let L be a continuous linear operator from E to G. The following properties are equivalent:*

- ***(a)*** *The rank of C is closed, and the kernel of C is finite-dimensional.*
- ***(b)*** *There exists a positive constant $c > 0$ such that*

$$\|u\|_E \leq c\, (\|C\, u\|_G + \|u\|_\Phi)\,, \qquad \forall\, u \in E. \tag{2.37}$$

Proof. We divide the proof into two steps: second assertion implies the first assertion, and vice versa.

Step 1. Let $E_0 := \mathrm{Ker}(C)$. The unitary ball in E_0 is compact in Φ, thus, by (2.37), it is also compact in E. We conclude that E_0 is a finite-dimensional subspace of E[2].

There exists $E_1 \subset E$ such that $E = E_0 \oplus E_1$, and the restriction of the operator C on E_1 is injective by construction. We claim now that for any $u \in E_1$ it turns out that

$$\|u\|_E \le c' \, \|C\, u\|_G, \qquad \forall\, u \in E_1. \tag{2.38}$$

We argue by contradiction. If (2.38) does not hold true, then there exist a sequence $c'_n \nearrow +\infty$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset E_1$ such that

$$\|u_n\|_E > c'_n \, \|C\, u_n\|_G.$$

In particular, if we set $v_n := u_n/\|u_n\|_E$, the above inequality is equivalent to

$$\frac{1}{c'_n} > \|C\, v_n\|_G. \tag{2.39}$$

The sequence $(v_n)_{n\in\mathbb{N}} \subset E$ is bounded, hence there exists a converging (in Φ) subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that

$$v_{n_k} \xrightarrow{k\to+\infty}_{\Phi} v \in \Phi.$$

It follows from (2.37) and the above inequality that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E_1, and thus it converges necessarily to the same v. Passing to the limit as $n \to +\infty$ in (2.39) immediately implies $v \equiv 0$, but this is absurd since by (2.37) it turns out that

$$1 \le c \left(\|C\, v_n\|_G + \|v_n\|_\Phi \right), \qquad \forall\, n \in \mathbb{N}. \tag{2.40}$$

The claim (2.38) is now proved.

Let $(w_n)_{n\in\mathbb{N}} \subset \mathrm{Ran}(C)$ be a converging sequence, and let $w \in G$ be its limit. There exists a sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $C\, u_n = w_n$ for every $n \in \mathbb{N}$; using the decomposition of E, it turns out that

$$u_n = v_n + z_n \implies C(u_n) = C(z_n) = w_n,$$

and thus it follows from (2.38) that

$$\|z_n\|_E \le c' \, \|C\, z_n\|_G = c' \, \|w_n\|_G.$$

Consequently $(z_n)_{n\in\mathbb{N}} \subset E_1$ is a Cauchy sequence in E, hence it converges to $z \in E$. By the continuity of the operator C, we conclude that $C\, z = w$ (which is exactly what we wanted to prove).

Step 2. Let $E = E_0 \oplus E_1$ as above. The restriction $C\big|_{E_1}$ is a closed map, thus by the closed graph theorem it follows that

$$\|v\|_E \le c_1 \, \|C\, v\|_G, \qquad \forall\, v \in E_1. \tag{2.41}$$

On the other hand, for any $w \in E_0$ one can prove the inequality

$$\|w\|_E \le c_2 \, \|w\|_\Phi. \tag{2.42}$$

[2]A Banach space such that every bounded subset is relatively sequentially compact, is necessarily a finite-dimensional space.

We argue by contradiction. If (2.42) does not hold true, then there exist a sequence $(d_n) \subset \mathbb{R}$ increasingly converging to $+\infty$, and a sequence $(w_n)_{n\in\mathbb{N}} \subset E_0$ such that

$$\|w_n\|_E \geq d_n \, \|w_n\|_\Phi .$$

If we set $y_n := w_n/\|w_n\|_E$, then it turns out that

$$\frac{1}{d_n} \geq \|y_n\|_\Phi ,$$

hence $y_n \to 0$ in Φ. This is absurd since the sequence $(y_n)_{n\in\mathbb{N}}$ belongs to a finite-dimensional subspace, hence it admits a converging subsequence to an element y of norm 1. In conclusion, the inequality (2.37) follows immediately from (2.41) and (2.42). □

Proof of Theorem 2.33. The thesis is an immediate corollary of Peetre's Lemma 2.34, where

$$\begin{aligned} &E = H^{m+1}(\Omega,\, \mathbb{R}^N) \cap H^1_0(\Omega,\, \mathbb{R}^N) \\ &\Phi = H^m_0\left(\Omega,\, \mathbb{R}^N\right) \\ &G = H^{1-m}\left(\Omega,\, \mathbb{R}^N\right) \\ &C\, u = \mathcal{P}\, u, \end{aligned}$$

by noticing that

(1) the immersion $H^{m+1}\left(\Omega,\, \mathbb{R}^N\right)$ into $H^m\left(\Omega,\, \mathbb{R}^N\right)$ is compact by Rellich Theorem[3];

(2) the following estimate holds true:

$$\|u\|_{H^{m+1}(\Omega,\,\mathbb{R}^N)} \leq c\, \left(\|u\|_{H^m(\Omega,\,\mathbb{R}^N)} + \|F\|_{H^{1-m}(\Omega,\,\mathbb{R}^N)}\right) .$$

□

[3]Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded Lipschitz domain, and let $1 \leq p < n$. Set

$$p^* := \frac{np}{n-p}.$$

Then the Sobolev space $W^{1,\,p}\,(\Omega;\, \mathbb{R})$ is continuously embedded in the L^p-space $L^{p^*}\,(\Omega;\, \mathbb{R})$ and is compactly embedded in $L^q\,(\Omega;\, \mathbb{R})$ for every $1 \leq q < p^*$. In symbols,

$$W^{1,\,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

and

$$W^{1,\,p}(\Omega) \subset\subset L^q(\Omega), \qquad \forall\, 1 \leq q < p^*.$$

Chapter 3

Regularity in Sobolev Spaces

In this chapter, we study the regularity of the solutions of second-order elliptic equations in divergence form in Sobolev spaces.

The reader may jump here to have a brief overview of the reasons we need to introduce both the Sobolev norm and the Sobolev seminorm.

Notation. Let v be a function defined on $\Omega \subseteq \mathbb{R}^n$ open subset. For every $i \in \{1, \ldots, n\}$ and for every $h \in \mathbb{R}$, it turns out that

$$\tau_{i,\,h}\, u(x) := \frac{u(x + h\, e_i) - u(x)}{h},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Notation. Let $u \in H^k(\Omega)$ be any Sobolev function. We denote by

$$|u|_{k,\,2,\,\Omega} := \left(\sum_{|\alpha| = k} \|D^\alpha\, u\|_{0,\,2,\,\Omega} \right)^{\frac{1}{p}}$$

the seminorm $|\cdot|_{H^k(\Omega)}$, and we denote by

$$\|u\|_{k,\,2,\,\Omega} := \left(\sum_{|\alpha| \le k} \|D^\alpha\, u\|_{0,\,2,\,\Omega} \right)^{\frac{1}{p}}$$

the usual Sobolev norm.

3.1 Nirenberg Lemmas

Lemma 3.1. *Let $u \in W^{1,\,q}\,(B(0,\,\sigma))$, $q \ge 1$, $t \in (0,\,1)$, and let h be real number such that $|h| < (1 - t)\,\sigma$. Then*

$$\|\tau_{i,\,h}\, u\|_{L^q(B(0,\,t\,\sigma))} \le \left\| \frac{\partial\, u}{\partial\, x_i} \right\|_{L^q(B(0,\,\sigma))}, \qquad i = 1, \ldots, n.$$

Proof. By definition we have

$$\tau_{i,h}\,u(x) = \frac{1}{h}\int_0^1 \frac{\partial}{\partial s}\,u(x+s\,h\,e_i)\,\mathrm{d}s = \frac{1}{h}\int_0^1 h\cdot\frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\,\mathrm{d}s =$$

$$= \int_0^1 \frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\,\mathrm{d}s.$$

Consequently, we can easily estimate the q-th power

$$|\tau_{i,h}\,u(x)|^q = \left|\int_0^1 \frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\,\mathrm{d}s\right|^q \le \int_0^1 \left|\frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\right|^q \mathrm{d}s,$$

and hence

$$\|\tau_{i,h}\,u(x)\|_{L^q(B(0,\,t\,\sigma))} \le \int_{B(0,\,t\,\sigma)} \left[\int_0^1 \left|\frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\right|^q \mathrm{d}s\right]\mathrm{d}x =$$

$$= \int_0^1 \left[\int_{B(0,\,t\,\sigma)} \left|\frac{\partial}{\partial x_i}\,u(x+s\,h\,e_i)\right|^q \mathrm{d}x\right]\mathrm{d}s.$$

Since the closure of the ball $B(0,\,t\,\sigma)$ is contained in $B(0,\,\sigma)$, it easily turns out that

$$x\in B(0,\,t\,\sigma) \implies \|x+s\,h\,e_i\| \le \sigma \implies x+s\,h\,e_i \in B(0,\,\sigma).$$

Therefore, we can make the change of variable $x \rightsquigarrow y := x+s\,h\,e_i$ in the above inequality and obtain

$$\|\tau_{i,h}\,u(x)\|_{L^q(B(0,\,t\,\sigma))} \le \int_0^1 \left[\int_{B(s\,h\,e_i,\,t\,\sigma)} \left|\frac{\partial}{\partial y_i}\,u(y)\right|^q \mathrm{d}y\right]\mathrm{d}s \le$$

$$\le \int_0^1 \left[\int_{B(0,\,\sigma)} \left|\frac{\partial}{\partial y_i}\,u(y)\right|^q \mathrm{d}y\right]\mathrm{d}s =$$

$$= \left\|\frac{\partial u}{\partial x_i}\right\|_{L^q(B(0,\,\sigma))},$$

where the red inequality follows from the inclusion

$$B(s\,h\,e_i,\,t\,\sigma) \subseteq B(0,\,\sigma),$$

while the last step follows from the fact that the integrand does not depend on s. □

Lemma 3.2. *Let $u\in L^q\,(B(0,\,\sigma))$, $q\in(1,\,+\infty)$, $t\in(0,\,1)$, and assume that there exists a positive constant $M>0$ such that*

$$\forall\,h\,:\,|h| < (1-t)\,\sigma \rightsquigarrow \|\tau_{i,h}\,u\|_{L^q(B(0,\,\sigma))} \le M.$$

Then $u\in W^{1,q}\,(B(0,\,\sigma))$ and

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^q(B(0,\,\sigma))} \le M, \qquad i=1,\,\dots,\,n.$$

Proof. First, we observe that for every $\varphi \in C_0^\infty(B(0,\,t\,\sigma))$ function it turns out that

$$\int_{B(0,\,t\,\sigma)} \tau_{i,\,h}\, u(x)\,\varphi(x)\,\mathrm{d}x = \frac{1}{h}\left[\int_{B(0,\,t\,\sigma)} u(x+h\,e_i)\,\varphi(x)\,\mathrm{d}x - \int_{B(0,\,t\,\sigma)} u(x)\,\varphi(x)\,\mathrm{d}x\right] =$$

$$= \frac{1}{h}\left[\int_{B(h\,e_i,\,t\,\sigma)} u(y)\,\varphi(y-h\,e_i)\,\mathrm{d}y - \int_{B(0,\,t\,\sigma)} u(x)\,\varphi(x)\,\mathrm{d}x\right] =$$

$$= \frac{1}{h}\left[\int_{B(0,\,\sigma)} u(x)\,\varphi(x-h\,e_i)\,\mathrm{d}x - \int_{B(0,\,\sigma)} u(x)\,\varphi(x)\,\mathrm{d}x\right] =$$

$$= -\int_{B(0,\,\sigma)} u(x)\,\tau_{i,\,-h}\,\varphi(x)\,\mathrm{d}x,$$

where the red identity follows from the inclusions

$$\operatorname{spt}(\varphi) \subset B(0,\,t\,\sigma) \subseteq B(0,\,\sigma) \quad \text{and} \quad \operatorname{spt}(\tau_{i,\,-h}\,\varphi) \subset B(h\,e_i,\,t\,\sigma) \subset B(0,\,\sigma).$$

The space $L^q(B(0,\,\sigma))$ is reflexive Banach space for any $q \in (1,\,+\infty)$, hence there are an infinitesimal sequence $(h_n)_{n\in\mathbb{N}}$ and a function $v_i \in L^q(B(0,\,\sigma))$ such that

$$\tau_{i,\,h_{n_k}}\,u \rightharpoonup v_i \quad \text{weakly in } L^q.$$

Therefore, for every $\varphi \in C_0^\infty(B(0,\,t\,\sigma))$ it turns out (up to subsequences) that

$$\begin{array}{ccc}
\displaystyle\lim_{n\to+\infty}\int_{B(0,\,t\,\sigma)} \tau_{i,\,h_n}\,u(x)\,\varphi(x)\,\mathrm{d}x & = & \displaystyle\int_{B(0,\,t\,\sigma)} v_i(x)\,\varphi(x)\,\mathrm{d}x \\
\parallel & & \\
\displaystyle\lim_{n\to+\infty}\int_{B(0,\,\sigma)} u(x)\,\tau_{i,\,-h_n}\,\varphi(x)\,\mathrm{d}x & = & \displaystyle\int_{B(0,\,\sigma)} u(x)\left[\lim_{n\to+\infty}\tau_{i,\,-h_n}\,\varphi(x)\right]\mathrm{d}x,
\end{array}$$

where the latter identity follows from Lebesgue dominated convergence theorem since

$$|u(x)\,\tau_{i,\,-h}\,\varphi(x)| \le c\,|u(x)| \sup_{x\in B(0,\,\sigma)} \|D_i\,\varphi(x)\|_{\infty,\,B(0,\,\sigma)}\,.$$

We conclude that

$$\int_{B(0,\,\sigma)} u(x)\,\frac{\partial\varphi}{\partial x_i}(x)\,\mathrm{d}x = -\int_{B(0,\,\sigma)} v_i(x)\,\varphi(x)\,\mathrm{d}x \qquad \forall\,\varphi \in C_0^\infty(B(0,\,\sigma))\,,$$

that is, $u \in W^{1,\,q}(B(0,\,\sigma))$ with weak partial derivatives $v_i(x)$ as $i \in \{1,\,\dots,\,n\}$. In order to prove the estimate (3.2), observe that for every $\psi \in L^{q'}(B(0,\,t\,\sigma))$ it turns out that

$$\left|\int_{B(0,\,t\,\sigma)} \tau_{i,\,h_n}\,u(x)\,\psi(x)\,\mathrm{d}x\right| \le M\,\|\psi\|_{L^{q'}(B(0,\,\sigma))},$$

and, by passing to the limit, we obtain

$$\left|\int_{B(0,\,t\,\sigma)} \frac{\partial u}{\partial x_i}(x)\,\psi(x)\,\mathrm{d}x\right| \le M\,\|\psi\|_{L^{q'}(B(0,\,\sigma))}.$$

□

3.2 Interior Regularity Theory

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and let $u \in H^1(\Omega)$ be a solution (in the sense of distribution) of the elliptic problem

$$-\sum_{i,\,j=1}^{n} D^j \left[a_{i,\,j}(x)\, D^i\, u(x)\right] = f(x), \qquad x \in \Omega. \tag{3.1}$$

Theorem 3.3. *Assume that $A(x) := \{a_{i,\,j}(x)\}$ is a uniformly elliptic matrix on Ω, the coefficients $a_{i,\,j}(x)$ are functions of class C^1 on Ω, and $f \in L^2(\Omega)$. Then for every couple of open subsets*

$$\Omega' \subset \Omega'' \subset \Omega \,:\, \mathrm{dist}\,(\partial\,\Omega',\, \partial\,\Omega'') > 0,$$

it turns out that $u \in H^2(\Omega')$, and

$$|u|_{2,\,2,\,\Omega'} \leq c\,[\|f\|_{0,\,2,\,\Omega''} + \|u\|_{1,\,2,\,\Omega''}]\,. \tag{3.2}$$

Proof. The computations here are quite involved. Hence we divide the proof into many little steps to ease the notation for the reader.

Step 1. Let us define

$$\delta := \mathrm{dist}\,(\partial\,\Omega',\, \partial\,\Omega'') > 0 \qquad \text{and} \qquad \Omega_\sigma := \{x \in \Omega'' \mid d(x,\, \partial\,\Omega') \geq \sigma\}\,,$$

and let us consider the cut-off function $\Theta(x) \in C_c^\infty(\mathbb{R}^n)$ which is identically equal to 1 in Ω_δ, and it is zero outside of $\Omega_{\frac{2}{3}\,\delta}$.

Step 2. Recall that u is a weak solution of (3.1) if and only if for every $\varphi \in H_0^1\,(\Omega)$ it turns out that

$$\sum_{i,\,j=1}^{n} \int_\Omega a_{i,\,j}(x)\, D^i\, u(x)\, D^j\, \varphi(x)\, \mathrm{d}x = \int_\Omega f(x)\, \varphi(x)\, \mathrm{d}x.$$

This is equivalent to requiring that for any $\psi \in H^1(\Omega)$ it turns out that

$$\sum_{i,\,j=1}^{n} \int_\Omega a_{i,\,j}(x)\, D^i\, u(x)\, D^j\, \left(\Theta(x)\, \psi(x)\right)\, \mathrm{d}x = \int_\Omega f(x)\, \left(\Theta(x)\, \psi(x)\right)\, \mathrm{d}x.$$

The Leibniz rule for the derivative gives us the identity

$$\begin{aligned}\sum_{i,\,j=1}^{n} \int_\Omega a_{i,\,j}(x)\, D^i\, (u(x))\, \Theta(x)\, D^j\, (\psi(x))\, \mathrm{d}x = \int_\Omega f(x)\, \left(\Theta(x)\, \psi(x)\right)\, \mathrm{d}x - \ldots \\ \cdots - \sum_{i,\,j=1}^{n} \int_\Omega a_{i,\,j}(x)\, D^i\, (u(x))\, \psi(x)\, D^j\, (\Theta(x))\, \mathrm{d}x.\end{aligned} \tag{3.3}$$

Notice that

$$D^i\, (u(x))\, \Theta(x) = D^i\, (u(x)\, \Theta(x)) - u(x)\, D^i\, \Theta(x),$$

thus, if we set

$$\mathcal{U}(x) := u(x)\,\Theta(x) \qquad \text{and} \qquad F(x) = f(x)\,\Theta(x) - \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,D^i\,u(x)\,D^j\,\Theta(x),$$

then (3.3) may be rewritten as follows:

$$\begin{aligned} &\sum_{i,\,j=1}^{n} \int_{\Omega} a_{i,\,j}(x)\,D^i\,\mathcal{U}(x)\,D^j\,\psi(x)\,\mathrm{d}x = \\ &\qquad = \int_{\Omega} \left(F(x)\,\psi(x) - \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,u(x) D^i\,\Theta(x)\,D^j\,\psi(x) \right) \mathrm{d}x. \end{aligned} \tag{3.4}$$

Step 3. In the previous step, we determined that the identity (3.4) holds for every $\psi \in H_0^1\left(\Omega_{\frac{\delta}{2}}\right)$. Let us consider as a test function the incremental ratio

$$\tau_{r,\,-h}\,\psi(x), \qquad 0 < |h| < \frac{\delta}{2},$$

that is, if we use $\tau_{r,\,-h}\,\psi \in H_0^1\,(\Omega'')$ as a test function in (3.4), then it turns out that

$$\begin{aligned} &\sum_{i,\,j=1}^{n} \int_{\Omega_{\frac{\delta}{2}}} a_{i,\,j}(x)\,D^i\,\mathcal{U}(x)\,D^j\,\left(\tau_{r,\,-h}\,\psi(x)\right)\,\mathrm{d}x = \int_{\Omega} F(x)\,\tau_{r,\,-h}\,\psi(x)\,\mathrm{d}x + \\ &\qquad + \int_{\Omega} \left(\sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,u(x) D^i\,\Theta(x)\,D^j\,\left(\tau_{r,\,-h}\,\psi(x)\right) \right) \mathrm{d}x. \end{aligned} \tag{3.5}$$

The left-hand side can be easily[1] rewritten as

$$\begin{aligned} &\sum_{i,\,j=1}^{n} \int_{\Omega_{\frac{\delta}{2}}} \tau_{r,\,h}\left[a_{i,\,j}(x)\,D^i\,\mathcal{U}(x)\right]\,D^j\,\psi(x)\,\mathrm{d}x = \\ &= \sum_{i,\,j=1}^{n} \int_{\Omega_{\frac{\delta}{2}}} \left\{ a_{i,\,j}(x + h\,e_r)\,\tau_{r,\,h}\left(D^i\,\Theta(x)\right)\,D^j\,\psi(x) + \tau_{r,\,h}\left(a_{i,\,j}(x)\right)\,D^i\,\mathcal{U}(x)\,D^j\,\psi(x) \right\} \mathrm{d}x. \end{aligned} \tag{3.6}$$

If we apply (3.6) to (3.5), it turns out that

$$\sum_{i,\,j=1}^{n} \int_{\Omega_{\frac{\delta}{2}}} a_{i,\,j}(x + h\,e_r)\,\tau_{r,\,h}\left(D^i\,\mathcal{U}(x)\right)\,D^j\,\psi(x)\,\mathrm{d}x = I_1 + I_2 + I_3, \tag{3.7}$$

[1] We use the identity $\tau_{r,\,h}\left(f(x)\,g(x)\right) = f(x + h\,e_r)\,\tau_{r,\,h}\,g(x) + \left(\tau_{r,\,h}\,f(x)\right]\,g(x)$.

where

$$I_1 := -\int_{\Omega_{\frac{\delta}{2}}} F(x)\,\tau_{r,\,-h}\,\psi(x)\,\mathrm{d}x,$$

$$I_2 := -\int_{\Omega_{\frac{\delta}{2}}} \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\,D^i\,\Theta(x)\,D^j\,(\tau_{r,\,-h}\,\psi(x))\;\mathrm{d}x,$$

$$I_3 := -\int_{\Omega_{\frac{\delta}{2}}} \sum_{i,\,j=1}^{n} \tau_{r,\,h}\,a_{i,\,j}(x)\,D^i\,\mathcal{U}(x)\,D^j\,\psi(x)\,\mathrm{d}x.$$

Step 4. In this paragraph, we given an estimate of the integrals I_i for $i = 1,\,2,\,3$. First, we notice that

$$\begin{aligned}|I_1| &\le \|F\|_{0,\,2,\,\Omega''}\,\|\tau_{r,\,-h}\,\psi\|_{0,\,2,\,\Omega_{\frac{2}{3}\delta}} \le \\ &\le \|F\|_{0,\,2,\,\Omega''}\,\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}} \le \\ &\le 2\left(\|F\|_{0,\,2,\,\Omega''} + \left\|\sum_{i,\,j=1}^{n} a_{i,\,j}\,D^i\,u\,D^j\,\Theta\right\|_{0,\,2,\,\Omega''}\right)\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}} \le \\ &\le c\,(\|a_{i,\,j}\|_\infty,\,n,\,\delta)\,\left(\|f\|_{0,\,2,\,\Omega''} + \|u\|_{1,\,2,\,\Omega}\right)\,\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}},\end{aligned}$$

where the red inequality follows from Lemma 3.1. In a similar fashion

$$\begin{aligned}|I_2| &\le \left|\sum_{i,\,j=1}^{n}\int_{\Omega_{\frac{\delta}{2}}} \tau_{r,\,h}\left[a_{i,\,j}(x)\,D^j\,\Theta(x)\right]\;u(x)\,D^j\,\psi(x)\,\mathrm{d}x\right| \le \\ &\le c(\delta)\,\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\left(\max_{i,\,j}\|a_{i,\,j}\|_{\infty,\,\Omega}\,\|\tau_{r,\,h}\,u\|_{0,\,2,\,\Omega_{\frac{\delta}{2}}} + \max_{i,\,j}\|D^r\,a_{i,\,j}\|_{\infty,\,\Omega}\,\|u\|_{0,\,2,\,\Omega}\right) \le \\ &\le c(\delta)\,\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\left(\max_{i,\,j}\|a_{i,\,j}\|_{\infty,\,\Omega}\,\|u\|_{1,\,2,\,\Omega} + \max_{i,\,j}\|D^r\,a_{i,\,j}\|_{\infty,\,\Omega}\,\|u\|_{0,\,2,\,\Omega}\right).\end{aligned}$$

In conclusion, we also have that

$$|I_3| \le \max_{i,\,j}\|D^r\,a_{i,\,j}\|_{\infty,\,\Omega}\;\|\mathcal{U}\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\;\|\psi\|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}.$$

Step 5. It follows from (3.7) and from the previous step that

$$\begin{aligned}&\left|\sum_{i,\,j=1}^{n}\int_{\Omega_{\frac{\delta}{2}}} a_{i,\,j}(x+h\,e_r)\,\tau_{r,\,h}\left(D^i\,\mathcal{U}(x)\right)\,D^j\,\psi(x)\,\mathrm{d}x\right| \le \\ &\le c\,(\|a_{i,\,j}\|_{\infty,\,\Omega},\,\|D^r\,a_{i,\,j}\|,\,n,\,\delta)\cdot\left[\|f\|_{0,\,2,\,\Omega''} + \|u\|_{1,\,2,\,\Omega''}\right]\,|\psi|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\,.\end{aligned} \tag{3.8}$$

If we set $\psi := \tau_{r,\,h}\,\mathcal{U}$, then (3.8) gives us

$$\left| \sum_{i,\,j=1}^{n} \int_{\Omega_{\frac{\delta}{2}}} a_{i,\,j}(x+h\,e_r)\,\tau_{r,\,h}\left(D^i\,\mathcal{U}(x)\right)\,D^j\,[\tau_{r,\,h}\,\mathcal{U}(x)]\,\mathrm{d}x \right| \leq$$
$$\leq c\,(\|a_{i,\,j}\|_{\infty,\,\Omega},\,\|D^r\,a_{i,\,j}\|,\,n,\,\delta)\cdot[\|f\|_{0,\,2,\,\Omega''}+\|u\|_{1,\,2,\,\Omega''}]\;|\tau_{r,\,h}\,\mathcal{U}|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\,. \tag{3.9}$$

By coerciveness it turns out that

$$\nu\;|\tau_{r,\,h}\,\mathcal{U}|^2_{1,\,2,\,\Omega_{\frac{\delta}{2}}} \leq c\,(\|a_{i,\,j}\|_{\infty,\,\Omega},\,\|D^r\,a_{i,\,j}\|,\,n,\,\delta)\cdot[\|f\|_{0,\,2,\,\Omega''}+\|u\|_{1,\,2,\,\Omega''}]\;|\tau_{r,\,h}\,\mathcal{U}|_{1,\,2,\,\Omega_{\frac{\delta}{2}}}\,,$$

that is,

$$\nu\;|\tau_{r,\,h}\,\mathcal{U}|_{1,\,2,\,\Omega_{\frac{\delta}{2}}} \leq c\,(\|a_{i,\,j}\|_{\infty,\,\Omega},\,\|D^r\,a_{i,\,j}\|,\,n,\,\delta)\cdot[\|f\|_{0,\,2,\,\Omega''}+\|u\|_{1,\,2,\,\Omega''}]\,.$$

By Lemma 3.2 we conclude that

$$\nu\;|D^r\,\mathcal{U}|_{1,\,2,\,\Omega_{\frac{\delta}{2}}} \leq c\,(\|a_{i,\,j}\|_{\infty,\,\Omega},\,\|D^r\,a_{i,\,j}\|,\,n,\,\delta)\cdot[\|f\|_{0,\,2,\,\Omega''}+\|u\|_{1,\,2,\,\Omega''}]\,,$$

and the thesis follows from the fact that $u=\mathcal{U}$ on Ω'. □

Generalizations. Let $u \in H^1(\Omega)$ be a solution (in the sense of distribution) of the elliptic problem

$$-\sum_{i,\,j=1}^{n} D^j\left[a_{i,\,j}(x)\,D^i\,u(x)\right]+\sum_{i=1}^{n} a_i(x)\,D^i\,u(x)+a(x)\,u(x)=f(x), \qquad x\in\Omega. \tag{3.10}$$

Theorem 3.4. *Assume that $A(x) := \{a_{i,\,j}(x)\}$ is a uniformly elliptic matrix on Ω, the coefficients $a_{i,\,j}(x)$ are functions of class C^1 on Ω, the coefficients $a_i(x)$ and $a(x)$ are functions of class L^∞ on Ω, and $f \in L^2(\Omega)$. Then for every couple of open subsets*

$$\Omega' \subset \Omega'' \subset \Omega\;:\;\mathrm{dist}\,(\partial\,\Omega',\,\partial\,\Omega'')>0,$$

it turns out that $u \in H^2(\Omega')$, and

$$|u|_{2,\,2,\,\Omega'} \leq c(a_{i,\,j},\,a_i,\,a,\,\nu)\cdot[\|f\|_{0,\,2,\,\Omega''}+\|u\|_{1,\,2,\,\Omega''}]\,. \tag{3.11}$$

Proof. It suffices to apply Theorem 3.3 to the elliptic problem

$$-\sum_{i,\,j=1}^{n} D^j\left[a_{i,\,j}(x)\,D^i\,u(x)\right]=\tilde{f}(x),$$

where

$$\tilde{f}(x) := -\sum_{i=1}^{n} a_i(x)\,D^i\,u(x)-a(x)\,u(x)+f(x)\in L^2(\Omega).$$

□

Theorem 3.5. *Assume that $A(x) := \{a_{i,j}(x)\}$ is a uniformly elliptic matrix on Ω, the coefficients $a_{i,j}(x)$ are functions of class C^{k+1} on Ω, the coefficients $a_i(x)$ and $a(x)$ are functions of class C^k on Ω, and $f \in H^k(\Omega)$. Then for every couple of open subsets*

$$\Omega' \subset \Omega'' \subset \Omega \,:\, \mathrm{dist}\,(\partial\,\Omega', \partial\,\Omega'') > 0,$$

it turns out that $u \in H^{k+2}(\Omega')$, and

$$\|u\|_{k+2,\,2,\,\Omega'} \leq c(a_{i,\,j},\, a_i,\, a,\, \nu) \cdot \left[\|f\|_{k,\,2,\,\Omega''} + \|u\|_{k+1,\,2,\,\Omega''}\right]. \tag{3.12}$$

Proof. We prove this theorem by induction on the regularity k.

Base Step. We have already proved it for $k = 0$ (see Theorem 3.4).

Inductive Step. If we take the α-th derivative of (3.10), for $|\alpha| = k$, then the thesis follows from Theorem 3.4 applied to the function $w = D^\alpha\, u$. □

3.3 Differentiability at the Boundary

Notation. Let $B_r := B(0,\, r) \subset \mathbb{R}^n$ be the unitary open ball of $\mathbb{R}^n$. We denote by

$$B_r^+ := B_r \cap \{x \in \mathbb{R}^n \mid x_n > 0\}$$

the upper-half of the open ball, and we denote by

$$\Gamma_r^+ := B_r \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$$

its lower border, i.e., the intersection between the ball and the hyperplane $\{x_n = 0\}$.

In this section, we will be concerned with the following elliptic problem with values at the boundary:

$$\begin{cases} -\displaystyle\sum_{i,\,j=1}^{n} D^j\left(a_{i,\,j}(x)\, D^i\, u(x)\right) = f(x) & x \in B_r^+, \\ u(x) = 0 & x \in \Gamma_r. \end{cases} \tag{3.13}$$

Theorem 3.6. *Let $u \in H^1\left(B_r^+\right)$ be a solution of (3.13), and assume that*

(1) *$A(x) := \{a_{i,j}(x)\}$ is a uniformly elliptic matrix on B_r^+;*

(2) *the coefficients $a_{i,j}(x)$ are functions of class C^1 on $\overline{B_r^+}$; and*

(3) *$f \in L^2(B_r^+)$.*

Then for any $\rho \in (0,\, r)$ it turns out that $u \in H^2\left(B_\rho^+\right)$ and

$$|u|_{2,\,2,\,B_\rho^+} \leq c(\nu,\, \rho,\, r,\, n,\, a_{i,\,j}) \cdot \left[\|f\|_{0,\,2,\,B_r^+} + \|u\|_{1,\,2,\,B_r^+}\right]. \tag{3.14}$$

Proof. Let us denote by $W^1_{\Gamma_0}(B_r^+)$ the closure of the set of all $C^1\left(\overline{B_r^+}\right)$ functions vanishing in a neighborhood of Γ_r with respect to the $W^1(B_r^+)$ norm. We consider the weak formulation of (3.13), i.e., u is a solution of the problem

$$\begin{cases} \displaystyle\sum_{i,\,j=1}^{n} \int_{B_r^+} a_{i,\,j}(x)\, D^i\, u(x)\, D^j\, \varphi(x)\, \mathrm{d}x = \int_{B_r^+} f(x)\, \varphi(x)\, \mathrm{d}x \quad \forall\, \varphi \in W^1_0(B_r^+)\,, \\ u \in W^1_{\Gamma_0}(B_r^+)\,. \end{cases} \tag{3.15}$$

Let $\Theta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that Θ is identically equal to 1 on B_ρ, and it is supported in the ball of center the origin and radius $(r+\rho)/2$. Let us consider test functions of the form

$$\varphi(x) = \Theta(x)\, \psi(x), \qquad \psi \in W^1_{\Gamma_0}\left(B_r^+\right),$$

and let us set as before

$$F(x) = f(x)\,\Theta(x) - \sum_{i,\,j=1}^{n} a_{i,\,j}(x)\, D^i\, u(x)\, D^j\,\Theta(x) \quad \text{and} \quad \mathcal{U}(x) = \Theta(x)\, u(x).$$

If we substitute the test function φ in the equation (3.15), then it turns out that

$$\sum_{i,\,j=1}^{n} \int_{B_r^+} a_{i,\,j}(x)\, D^i\,\mathcal{U}(x)\, D^j\,\psi(x)\,\mathrm{d}x = \int_{B_r^+} F(x)\,\psi(x)\,\mathrm{d}x + \dots$$
$$\dots + \sum_{i,\,j=1}^{n} \int_{B_r^+} a_{i,\,j}(x)\, D^i\,\Theta(x)\, D^j\,\psi(x)\, u(x)\,\mathrm{d}x. \tag{3.16}$$

The identity (3.16) holds for any $\psi \in W^1_{\Gamma_0}(B_r^+)$ vanishing outside of $B^+_{\frac{r+\rho}{2}}$. Hence we can consider the incremental ratios $\tau_{r,\,-h}$ as r ranges in $r = 1, \dots, (n-1)$ and

$$|h| < \frac{r+\rho}{2}.$$

A similar argument to the one used in the proof of Theorem 3.3 proves that

$$\sum_{i=1}^{n}\sum_{r=1}^{n-1} \int_{B_\rho^+} \left|D^r\, D^i\, u(x)\right|^2 \,\mathrm{d}x \le c(\nu,\,\rho,\,r,\,n,\,a_{i,\,j}) \cdot \left[\|f\|_{0,\,2,\,B_r^+} + \|u\|_{1,\,2,\,B_r^+}\right], \tag{3.17}$$

hence we only need to estimate the term $D^n\, D^n\, u(x)$. From the identity (3.16) we have

$$\int_{B_r^+} a_{n,\,n}(x)\, D^n\,\mathcal{U}(x)\, D^n\,\psi(x)\,\mathrm{d}x = \int_{B_r^+} F(x)\,\psi(x)\,\mathrm{d}x + \dots$$

$$\dots + \sum_{i,\,j=1}^{n} \int_{B_r^+} a_{i,\,j}(x)\, D^i\,\Theta(x)\, D^j\,\psi(x)\, u(x)\,\mathrm{d}x - \dots$$

$$\dots \sum_{i=1}^{n}\sum_{j=1}^{n-1} \int_{B_r^+} a_{i,\,j}(x)\, D^i\,\mathcal{U}(x)\, D^j\,\psi(x)\,\mathrm{d}x - \dots$$

$$\dots - \sum_{i=1}^{n-1} \int_{B_r^+} a_{i,\,j}(x)\, D^i\,\mathcal{U}(x)\, D^n\,\psi(x)\,\mathrm{d}x =$$

$$= \int_{B_r^+} H(x)\,\psi(x)\,\mathrm{d}x$$

for any $\psi \in W_0^1\,(B_r^+)$, where

$$H(x) = F(x) - \sum_{i,\,j=1}^{n} D^j\left[a_{i,\,j}(x)\, D^i\Theta(x)\, u(x)\right] + \dots$$

$$\dots + \sum_{i=1}^{n}\sum_{j=1}^{n-1} D^j\left[a_{i,\,j}(x)\, D^i\,\mathcal{U}(x)\right] + \dots$$

$$\dots + \sum_{i=1}^{n-1} D^n\left[a_{i,\,j}(x)\, D^i\,\mathcal{U}(x)\right].$$

The argument above proves that $H \in L^2\left(B_\rho^+\right)$. Let us consider[2]

$$\psi(x) = \frac{\xi(x)}{a_{n,\,n}(x)}, \qquad \xi \in C_0^\infty\left(B_\rho^+\right).$$

By the assumption on the coefficients, it easily turns out that $\psi \in W_0^1\left(B_\rho^+\right)$. If we substitute this into the above identity, we obtain the relation

$$\int_{B_\rho^+} D^n\,u(x)\, D^n\,\xi(x)\,\mathrm{d}x = \int_{B_\rho^+}\left[H(x)\,\frac{\xi(x)}{a_{n,\,n}(x)} + D^n\,u(x)\,\frac{\xi(x)\, D^n\, D^n\, a_{n,\,n}(x)}{a_{n,\,n}(x)}\right]\mathrm{d}x \quad (3.18)$$

for any $\xi \in C_0^\infty\left(B_\rho^+\right)$. If we set

$$G(x) := \frac{H(x) - D^n\,u(x)\, D^n\, a_{n,\,n}(x)}{a_{n,\,n}(x)},$$

then $G \in L^2\left(B_\rho^+\right)$ and the relation (3.18) becomes

$$\int_{B_\rho^+} D^n\,u(x)\, D^n\,\xi(x)\,\mathrm{d}x = \int_{B_\rho^+} G(x)\,\xi(x)\,\mathrm{d}x$$

[2]By the uniform ellipticity condition, we obtain $a_{n,\,n}(x) \geq \nu > 0$.

for every $\xi \in C_0^\infty\left(B_\rho^+\right)$. We deduce that $D^n\, D^n\, u(x)$ exists in B_ρ^+, and it belongs to $L^2\left(B_\rho^+\right)$; more precisely, it turns out that

$$D^n\, D^n\, u(x) = -G(x)$$

which implies the thesis. □

Theorem 3.7. *Let $u \in H^{k+1}\left(B_r^+\right)$ be a solution of (3.13), and assume that*

- ***(1)*** *$A(x) := \{a_{i,\,j}(x)\}$ is a uniformly elliptic matrix on B_r^+;*
- ***(2)*** *the coefficients $a_{i,\,j}(x)$ are functions of class C^{k+1} on $\overline{B_r^+}$; and*
- ***(3)*** *$f \in H^k(B_r^+)$.*

Then for any $\rho \in (0,\, r)$ it turns out that $u \in H^{k+2}\left(B_\rho^+\right)$ and

$$|u|_{k+2,\,2,\,B_\rho^+} \leq c(\nu,\, \rho,\, r,\, n,\, a_{i,\,j}) \cdot \left[\|f\|_{k,\,2,\,B_r^+} + \|u\|_{k+1,\,2,\,B_r^+}\right]. \tag{3.19}$$

Proof. The idea is to estimate the $L^2(B_\rho^+)$ norm of the $(k+2)$-th order (weak) derivatives, and combine them to obtain (3.19). More precisely, we want to show that for every $|\alpha| = k$ and for every $i,\, j = 1,\, \ldots,\, n$, it turns out that

$$\|D^\alpha\, D_{i,\,j}\, u\|_{0,\,2,\,B_\rho^+} \leq c_{i,\,j} \cdot \left[\|f\|_{k,\,2,\,B_r^+} + \|u\|_{k+1,\,2,\,B_r^+}\right]. \tag{3.20}$$

Base Step. Set $\alpha = (\alpha_1,\, \ldots,\, \alpha_{n-1},\, h)$; we proceed by induction on h. If $h = 0$ and $|\alpha| = k$, then the function $w = D^\alpha\, u$ solves the Dirichlet problem

$$\begin{cases} -\displaystyle\sum_{i,\,j=1}^n D^j\left(a_{i,\,j}(x)\, D^i\, w(x)\right) = G(x) + g(x) & x \in B_r^+, \\ w(x) = 0 & x \in \Gamma_r \end{cases} \tag{3.21}$$

which is nothing more than the α-derivative of (3.13). We observe that $w(x) = 0$ on the boundary Γ_r since the n-th index of α is zero by assumption. A straightforward computation shows that

$$g(x) = D^\alpha\, f(x),$$

$$G(x) = \sum_{i,\,j=1}^n \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} D_j\left[D^\beta\, a_{i,\,j}(x)\, D^{\alpha-\beta}\, D_i\, u(x)\right],$$

hence the estimate (3.20) follows from Theorem 3.5.

Inductive Step. Suppose that the estimate (3.20) holds for every multi-index α such that $\alpha_n = h < k$ and $|\alpha| = k$. Let us consider the weak formulation of the Dirichlet problem (3.13), given by

$$\sum_{i,\,j=1}^{n} \int_{B_r^+} a_{i,\,j}(x)\, D^i\, u(x)\, D^j\, \varphi(x)\, \mathrm{d}x = \int_{B_r^+} f(x)\, \varphi(x)\, \mathrm{d}x \qquad \forall\, \varphi \in C_0^\infty\left(B_r^+\right). \tag{3.22}$$

Let us take the test function $\varphi = D^\beta\, \psi$, where $\psi \in C_0^\infty(B_r^+)$ is another test function, and $\beta = (\beta_1, \ldots, \beta_{n-1}, h+1)$ a multi-index of length $|\beta| = k$. By substituting φ in the identity (3.22), it turns out that

$$\sum_{i,\,j=1}^{n} \int_{B_r^+} D^\beta \left[a_{i,\,j}(x)\, D_i\, u(x)\right]\, D_j\, \psi(x)\, \mathrm{d}x = \int_{B_r^+} D^\beta\, f(x)\, \psi(x)\, \mathrm{d}x \qquad \forall\, \psi \in C_0^\infty\left(B_r^+\right). \tag{3.23}$$

If we expand the derivatives and we set aside the maximal-order terms, then for every $\psi \in C_0^\infty(B_r^+)$ it turns out that

$$\begin{aligned} &\int_{B_r^+} a_{n,\,n}(x)\, D_n\, D^\beta\, u(x)\, D_n\, \psi(x)\, \mathrm{d}x = \\ &= -\sum_{\substack{i,\,j=1 \\ i\cdot j<n^2}}^{n} \sum_{\substack{\gamma\leq\beta \\ \gamma\neq 0}} \binom{\beta}{\gamma} \int_{B_r^+} D^\gamma\, a_{i,\,j}(x)\, D^{\beta-\gamma}\, D_i\, u(x)\, D_j\, \psi(x)\, \mathrm{d}x + \ldots \\ &\cdots + \int_{B_r^+} D^\beta\, f(x)\, \psi(x)\, \mathrm{d}x = \int_{B_r^+} \left[G(x) + g(x)\right]\, \psi(x)\, \mathrm{d}x, \end{aligned} \tag{3.24}$$

where

$$g(x) = D^\beta\, f(x),$$

$$G(x) = -\sum_{\substack{i,\,j=1 \\ i\cdot j<n^2}}^{n} \sum_{\substack{\gamma\leq\beta \\ \gamma\neq 0}} \binom{\beta}{\gamma} D_j \left[D^\gamma\, a_{i,\,j}(x)\, D^{\beta-\gamma}\, D_i\, u(x)\right].$$

Let $\xi \in C_0^\infty(B_\rho^+)$, and let us set

$$\psi(x) := \frac{\xi(x)}{a_{n,\,n}(x)}.$$

Clearly ψ belongs to $H_0^{k+1}(B_\rho^+)$, hence (3.24) becomes

$$\int_{B_r^+} D_n\, D^\beta\, u(x)\, D_n\, \xi(x)\, \mathrm{d}x = \int_{B_r^+} \left[H(x) + \left(G(x) + g(x)\right)\, a_{n,\,n}(x)\right]\, \xi(x)\, \mathrm{d}x, \qquad \forall\, \xi \in H_0^{k+1}(B_\rho^+),$$

where

$$H(x) = \frac{D_n\, a_{n,\,n}(x)}{a_{n,\,n}(x)}\, D^\alpha\, D_n\, u(x).$$

In conclusion, we notice that by assumption $H(x) + G(x) + g(x)$ belongs to $L^2(B_r^+)$, hence the estimate (3.20) follows from Theorem 3.5. □

3.4 Global Regularity

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with borderd $\partial\,\Omega$ of class C^1, and let $u \in H^1(\Omega)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} -\sum_{i,\,j=1}^{n} D^j\left[a_{i,\,j}(x)\,D^i\,u(x)\right] = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\,\Omega. \end{cases} \tag{3.25}$$

Assume that:

(1) *$A(x) := \{a_{i,\,j}(x)\}$ is a uniformly elliptic matrix on Ω;*

(2) *the coefficients $a_{i,\,j}(x)$ are functions of class C^1 on $\overline{\Omega}$; and*

(3) *$f \in L^2(\Omega)$.*

Then $u \in H^2(\Omega)$ and we have the additional estimate

$$|u|_{2,\,2,\,\Omega} \leq c(a_{i,\,j},\,n,\,\Omega)\cdot\left[\|f\|_{0,\,2,\,\Omega} + \|u\|_{1,\,2,\,\Omega}\right]. \tag{3.26}$$

Proof. The argument is rather involved. Hence we divide the proof into four different steps.

Step 1. Let us consider a covering of Ω given by the finite collection of open sets

$$\{\Omega',\,\Omega'',\,U_1,\,\ldots,\,U_m,\,V_1,\,\ldots,\,V_m\},$$

where (see Figure 3.1) mbox

1) $\Omega' \subset \Omega'' \subset\subset \Omega$, and the distance $d\,(\Omega'',\,\partial\,\Omega')$ is positive;

2) U_ℓ and V_ℓ are neighborhoods centered at $x_\ell \in \partial\,\Omega$;

3) $V_\ell \subset U_\ell$ for any $\ell = 1,\,\ldots,\,m$;

4) the boundary $\partial\,\Omega$ is contained in the finite union $\cup_{\ell=1}^m V_\ell$;

5) Ω is contained in the union $\cup_{\ell=1}^m V_\ell \cup \Omega'$.

Step 2. It follows from Theorem 3.3 that $u \in H^2\,(\Omega')$, and also that

$$\|u\|_{2,\,2,\,\Omega'} \leq c\cdot\left[\|f\|_{0,\,2,\,\Omega''} + \|u\|_{1,\,2,\,\Omega''}\right].$$

It remains to study the regularity of the solution on the boundary. Let Φ_ℓ be the diffeomorphism between the intersection $U_\ell \cap \Omega$ and an open set of $\mathbb{R}^n$ defined by

$$\Phi_i(x) = x_i, \qquad i = 1,\,\ldots,\,n-1;$$

$$\Phi_n(x) = \psi_\ell(x') - x_n, \qquad x' = (x_1,\,\ldots,\,x_{n-1}),$$

where $\psi_\ell : \mathbb{R}^{n-1} \to \mathbb{R}$ is a map of class C^1, whose graph is equal to $\partial\,\Omega \cap U_\ell$. One can easily check that

$$\Phi(U_\ell \cap \Omega) \subset \{y \in \mathbb{R}^n \mid y_n > 0\} \qquad \text{and} \qquad \Phi(U_\ell \cap \partial\,\Omega) \subset \{y \in \mathbb{R}^n \mid y_n = 0\},$$

and also that the Jacobian of Φ has modulus of the determinant equal to 1.

Step 3. Let $\tilde{u}$ be the function such that $u(x) = (\tilde{u} \circ \Phi)(x)$ defined on $U_\ell \cap \overline{\Omega}$. The weak formulation of the problem (3.29) is given by

$$\sum_{i,\,j=1}^{n} \int_\Omega a_{i,\,j}(x)\, D^i\, u(x)\, D^j\, \varphi(x)\, \mathrm{d}x = \int_\Omega f(x)\, \varphi(x)\, \mathrm{d}x \qquad \forall\, \varphi \in H^1_0(\Omega),$$

and hence the identity also holds true for every $\varphi \in H^1_0(\Omega \cap U_\ell)$. The chain rule

$$\frac{\partial\, u(x)}{\partial\, x_i} = \sum_{h=1}^{n} \frac{\partial\, \tilde{u}\,(\Phi(x))}{\partial\, y_h}\, \frac{\partial\, \Phi_h(x)}{\partial\, x_i}$$

implies that the weak formulation above can be easily rewritten[3] as follows:

$$\sum_{i,\,j=1}^{n} \sum_{h,\,k=1}^{n} \int_{\tilde{\Omega}_\ell} \tilde{a}_{i,\,j}(x)\, D^h\, \tilde{u}(x)\, \tilde{\Phi}_{h,\,i}(x)\, D^k\, \tilde{\varphi}(x)\, \tilde{\Phi}_{k,\,j}\, \mathrm{d}x = \int_{\tilde{\Omega}_\ell} \tilde{f}(x)\, \tilde{\varphi}(x)\, \mathrm{d}x, \quad \forall\, \tilde{\varphi} \in H^1_0\left(\tilde{\Omega}_\ell\right).$$

Step 4. Let

$$A_{h,\,k}(x) := \sum_{i,\,j=1}^{n} \tilde{a}_{i,\,j}(x)\, \tilde{\Phi}_{h,\,i}(x)\, \tilde{\Phi}_{k,\,j}(x),$$

and observe that the above identity can be easily rewritten as follows:

$$\sum_{h,\,k=1}^{n} \int_{\tilde{\Omega}_\ell} A_{h,\,j}(x)\, D^h\, \tilde{u}(x)\, D^k\, \tilde{\varphi}(x)\, \mathrm{d}x = \int_{\tilde{\Omega}_\ell} \tilde{f}(x)\, \tilde{\varphi}(x)\, \mathrm{d}x, \quad \forall\, \tilde{\varphi} \in H^1_0\left(\tilde{\Omega}_\ell\right). \qquad (3.27)$$

The thesis will easily follow from Theorem 3.6, provided we are able to prove that the matrix $\{A_{h,\,k}\}_{h,\,k=1,\,\dots,\,n}$ is uniformly elliptic on $\tilde{\Omega}$. For any $\xi \in \mathbb{R}^n$ it turns out that

$$\sum_{h,\,k=1}^{n} A_{h,\,k}(x)\, \xi_h\, \xi_k = \sum_{h,\,k=1}^{n} \sum_{i,\,j=1}^{n} \tilde{a}_{i,\,j}(x)\, \tilde{\Phi}_{h,\,i}(x)\, \tilde{\Phi}_{k,\,j}(x)\, \xi_h\, \xi_k =$$

$$= \sum_{i,\,j=1}^{n} \tilde{a}_{i,\,j}(x) \left(\sum_{h=1}^{n} \tilde{\Phi}_{h,\,i}(x)\, \xi_h \right) \left(\sum_{k=1}^{n} \tilde{\Phi}_{k,\,j}(x)\, \xi_k \right) \geq$$

$$\geq \nu \sum_{i=1}^{n} \left(\sum_{h=1}^{n} \tilde{\Phi}_{h,\,i}(x)\, \xi_h \right)^2 \geq c\, \nu\, \|\xi\|^2,$$

where the red inequality is a consequence of the uniform ellipticity of $\{\tilde{a}_{i,\,j}\}$.

Step 5. The boundary regularity result (see Theorem 3.6) applies to any semi-ball B^+_r contained in $\tilde{\Omega}_\ell$. In particular, $\tilde{u} \in H^2\left(B^+_\rho\right)$ for any $\rho < r$, and hence it is enough to take

$$U_\ell = \Phi^{-1}(B^+_r) \qquad \text{and} \qquad V_\ell = \Phi^{-1}(B^+_\rho).$$

□

[3] Set $\tilde{a}_{i,\,j} = a_{i,\,j} \circ \Phi^{-1}$, $\tilde{f} = f \circ \Phi^{-1}$, $\tilde{\varphi} = \varphi \circ \Phi^{-1}$, $\tilde{\Phi_{h,\,i}} = \frac{\partial\, \Phi_h \circ \circ \Phi^{-1}}{\partial\, x_i}$, and $\tilde{\Omega}_\ell = \Phi\,(U_\ell \cap \Omega)$.

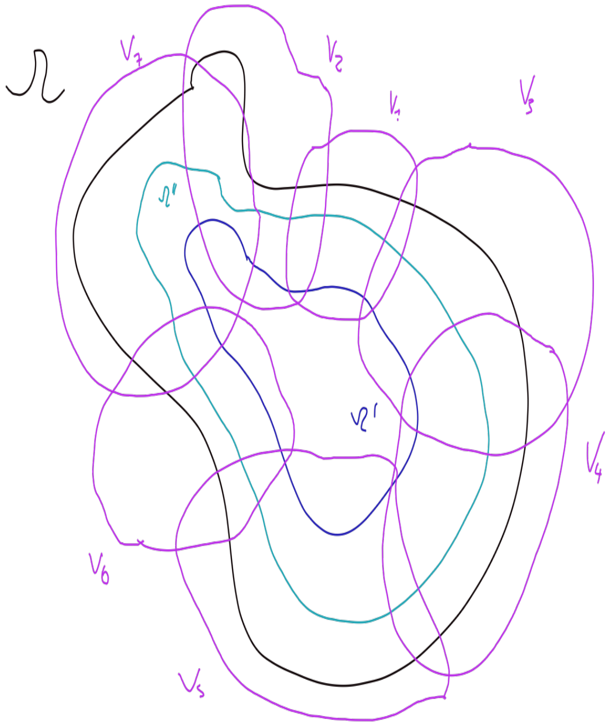

Figure 3.1: Covering of Ω.

Corollary 3.9. *Under these assumptions, it turns out that*

$$|u|_{2,\,2,\,\Omega} \leq c(a_{i,\,j},\, n,\, \Omega) \cdot \|f\|_{0,\,2,\,\Omega}. \tag{3.28}$$

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with borderd $\partial\,\Omega$ of class C^1, and let $u \in H^1(\Omega)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} -\displaystyle\sum_{i,\,j=1}^{n} D^j \left[a_{i,\,j}(x)\, D^i\, u(x)\right] + \sum_{i=1}^{n} a_i(x)\, D^i\, u(x) + a_0\, u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\,\Omega. \end{cases} \tag{3.29}$$

Assume that:

(1) *$A(x) := \{a_{i,\,j}(x)\}$ is a uniformly elliptic matrix on Ω;*

(2) *the coefficients $a_{i,\,j}(x)$ are functions of class C^1 on $\overline{\Omega}$;*

(3) *the coefficients a_i for $i = 0,\, \ldots,\, n$ are functions of class C^0 on $\overline{\Omega}$; and*

(4) *$f \in L^2(\Omega)$.*

Then $u \in H^2(\Omega)$ and we have the additional estimate

$$|u|_{2,\,2,\,\Omega} \leq c(a_{i,\,j},\, a_i,\, n,\, \Omega) \cdot \left[\|f\|_{0,\,2,\,\Omega} + \|u\|_{1,\,2,\,\Omega}\right]. \tag{3.30}$$

Chapter 4

Functional Spaces

In this chapter, we introduce some important functional spaces that will allow us to study the regularity of elliptic problems via integral estimates.

More precisely, in the first section, we recall the notion of α-Hölder continuity, and we state some facts which will be useful in the next chapter.

Next, we introduce the so-called Morrey-Campanato spaces and prove that, under some assumptions, they are equivalent to L^p spaces, but they are characterized by an integral norm, which is more useful for elliptic problems.

4.1 Hölder Spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. A function u is γ-Hölder continuous, with $\gamma \in (0, 1)$, on Ω if

$$[u]_{\gamma,\Omega} := \sup_{\substack{x,\, y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < +\infty,$$

and it is Lipschitz if the same holds with $\gamma = 1$. The space

$$C^{0,\gamma}(\Omega) := \{u : \Omega \to \mathbb{C} \mid [u]_{\gamma,\Omega} < +\infty\}$$

is Banach endowed with the norm

$$\|u\|_{\gamma,\Omega} := \|u\|_{\infty,\Omega} + [u]_{\gamma,\Omega}.$$

The inclusions

$$C^{0,\beta}(\Omega) \subseteq C^{0,\alpha}(\Omega) \subseteq C^0(\Omega)$$

are easy to verify when Ω is bounded and $0 < \alpha \leq \beta \leq 1$, but the first one is false if Ω is unbounded since, e.g.,

$$\|x\|^\beta \in C^{0,\beta}(\Omega) \setminus C^{0,\alpha}(\Omega)$$

for every $\alpha > \beta$. The function

$$v(t) := \begin{cases} \frac{1}{\log(t)} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

is continuous in $\overline{\Omega}$, but it is not γ-Hölder for any $\gamma \in (0,\, 1]$. Moreover, the inclusion

$$C^1\left(\overline{\Omega}\right) \subseteq C^{0,\,1}(\Omega)$$

is false, even if Ω is bounded, since

$$w(x,\, y) := \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} \cdot \arctan \frac{x+|x|}{2y} & y \neq 0 \\ 0 & y = 0 \ x < 0 \end{cases}$$

is differentiable, but not Lipschitz on Ω.

$(k,\, \gamma)$**-Hölder spaces.** The space

$$C^{k,\,\gamma}(\Omega) := \left\{ u \in C^k(\Omega) \;\middle|\; \sum_{|\alpha|=k} [D^\alpha u]_{\gamma,\,\Omega} < +\infty \right\}$$

is Banach endowed with the norm

$$\|u\|_{k,\,\gamma,\,\Omega} := \sum_{|\alpha|<k} \|D^\alpha u\|_{\infty,\,\Omega} + \sum_{|\alpha|=k} [D^\alpha u]_{\gamma,\,\Omega} \,.$$

Theorem 4.1.

(1) *The immersion*

$$C^{k,\,\gamma}\left(\overline{\Omega}\right) \subseteq C^k\left(\overline{\Omega}\right)$$

is continuous and compact for any $\gamma \in (0,\, 1]$.

(2) *The immersion*

$$C^{k,\,\gamma}\left(\overline{\Omega}\right) \subseteq C^{k,\,\beta}\left(\overline{\Omega}\right)$$

is continuous and compact for any $0 < \beta \leq \gamma \leq 1$.

Theorem 4.2 (Ciesielski, [1])**.** *There is an isomorphism*

$$C^{k,\,\gamma}\left(\overline{\Omega}\right) \cong \ell_\infty.$$

In particular, the space $C^{k,\,\gamma}\left(\overline{\Omega}\right)$ *is not separable for any* $k \in \mathbb{N}$ *and* $\gamma \in (0,\, 1]$.

4.2 Morrey Spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of diameter $\delta > 0$, and let us set

$$\Omega(x,\, \rho) := B(x,\, \rho) \cap \Omega.$$

Definition 4.3. The set Ω is of the type **A** if there exists a positive constant $A > 0$ such that for any $x \in \overline{\Omega}$ and any $\rho \in (0,\, \delta)$ it turns out that

$$|\Omega(x,\, \rho)| \geq A \cdot \rho^n.$$

Example 4.1. The set defined by

$$\Omega := \left\{ (x,\, y) \in \mathbb{R}^2 \;\middle|\; 0 < x < 1 \text{ and } 0 < y < x^2 \right\}$$

is not of type A (see Figure 4.1).

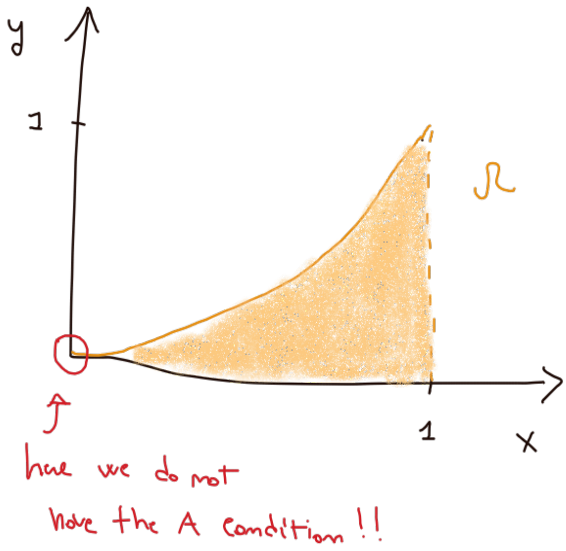

Figure 4.1: A set which is not of type A.

N.B. From now on, we shall assume Ω set of type A.

Morrey Spaces. Let $u \in L^p(\Omega)$. The limit

$$\lim_{\rho \to 0^+} \int_{\Omega(x,\,\rho)} |u(x)|^p \, \mathrm{d}x$$

is equal to zero since the integral is an absolutely continuous operator. For any $\lambda \geq 0$ and $p \in [1, +\infty)$, the space

$$L^{p,\,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \;\middle|\; \|u\|_{p,\,\lambda,\,\Omega} < +\infty \right\},$$

where

$$\|u\|^p_{p,\,\lambda,\,\Omega} := \sup_{\substack{x \in \Omega \\ \rho \leq \delta}} \frac{1}{\rho^\lambda} \int_{\Omega(x,\,\rho)} |u(t)|^p \, \mathrm{d}t. \tag{4.1}$$

Proposition 4.4. *The Morrey space $L^{p,\,\lambda}(\Omega)$ is Banach endowed with the norm (4.1).*

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset L^{p,\,\lambda}(\Omega)$ be a Cauchy sequence. Then the sequence is uniformly bounded, i.e., there exists $M > 0$ such that

$$\|u_n\|_{p,\,\lambda,\,\Omega} \leq M, \tag{4.2}$$

and we notice that

$$\|u_n - u_m\|_{0,\,p,\,\Omega} \leq \delta^\lambda \cdot \|u_n - u_m\|_{p,\,\lambda,\,\Omega}.$$

In particular, the sequence $(u_n)_{n \in \mathbb{N}}$ is also Cauchy in L^p, and hence there exists $u \in L^p(\Omega)$ such that

$$u_n \xrightarrow{L^p(\Omega)} u, \qquad u_{n_k} \xrightarrow{\text{pointwise}} u$$

and hence it suffices to prove that u belongs to the Morrey space, and also that the convergence is in $L^{p,\,\lambda}(\Omega)$.

The pointwise convergence is enough to prove the first assertion since one can pass to the limit in (4.2) using the Lebesgue dominated convergence theorem, i.e.,

$$\|u\|_{p,\,\lambda,\,\Omega} \leq M.$$

By assumption for any $\epsilon > 0$ there exists N_ϵ such that for every $n,\, m > N_\epsilon$ it turns out that

$$\frac{1}{\rho^\lambda} \int_{\Omega(x,\,\rho)} |u_n(t) - u_m(t)|^p \, \mathrm{d}t \leq \|u_n - u_m\|_{p,\,\lambda,\,\Omega} < \epsilon$$

for any $\rho > 0$ and for any $x \in \Omega$; on the other hand, if we take the limit as $n \to +\infty$, we infer that

$$\frac{1}{\rho^\lambda} \int_{\Omega(x,\,\rho)} |u(t) - u_m(t)|^p \, \mathrm{d}t < \epsilon \qquad \forall\, \rho > 0,\ \forall\, x \in \Omega,$$

which implies the convergence in $L^{p,\,\lambda}(\Omega)$. □

Theorem 4.5.

1) $L^{p,\,0}(\Omega) \cong L^p(\Omega)$.

2) $L^{p,\,n}(\Omega) \cong L^\infty(\Omega)$.

3) *If* $\lambda > n$ *strictly, then* $L^{p,\,\lambda}(\Omega) = \{0\}$.

4) *If* $1 \leq p < q < +\infty$ *and*

$$\frac{\lambda - n}{p} \leq \frac{\mu - n}{q},$$

then $L^{q,\,\mu}(\Omega) \subseteq L^{p,\,\lambda}(\Omega)$.

Proof.

1) This property follows straightforwardly from the definition.

2) To prove this point, we need to recall a known property of integration theory. If u is a measurable function, we can consider the set

$$S(u,\,\sigma) := \{x \in \Omega \mid |u(x)| > \sigma\}$$

as σ ranges in $[0,\,+\infty)$. Clearly $S(u,\,\sigma)$ is also measurable and, if $u \in L^p(\Omega)$ for any $p \in [1,\,+\infty)$, then the function

$$\sigma \longmapsto \sigma^{p-1} \cdot |S(u,\,\sigma)|$$

is summable on the interval $[0,\,+\infty)$ and it turns out that

$$\int_\Omega |u(x)|^p \, \mathrm{d}x = p \cdot \sigma^{p-1} \int_0^{+\infty} |S(u,\,\sigma)| \, \mathrm{d}\sigma. \tag{4.3}$$

The inclusion $L^{p,\,n}(\Omega) \supseteq L^{\infty}(\Omega)$ is rather obvious since

$$\frac{1}{\rho^n}\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t \le \sup_{t\in\Omega}|u(t)|^p \cdot \frac{|\Omega(x,\,p)|}{\rho^n} \le$$

$$\le \frac{\omega_n \cdot \rho^n}{\rho^n}\,\|u\|_{L^\infty(\Omega)} \le$$

$$\le c \cdot \|u\|_{L^\infty(\Omega)}.$$

To prove the opposite inclusion, we argue by contradiction assuming that $L^{p,\,n}(\Omega) \supset L^{\infty}(\Omega)$ is a strict inclusion. The estimate

$$|u(x)|^p = \lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t \le \sup_{0<\rho\le\delta}\frac{1}{\rho^n}|u(t)|^p\,\mathrm{d}t$$

holds for every point $x \in \mathcal{L}(u,\,\Omega)$, i.e., for every Lebesgue point of u in Ω. Let u be an element of $L^{p,\,n}(\Omega)\setminus L^{\infty}(\Omega)$, i.e.,

$$\|u\|_{\infty,\,\Omega} = +\infty.$$

It is equivalent to saying that for any $t > 1$ the measure of the set $S(u,\,t)$ is positive (strictly), and hence that for any $x \in \mathcal{L}(u,\,\Omega)\cap S(u,\,t)$ it turns out that $|u(x)|^p > t$. Therefore, for any $t > 1$ there exists a $x \in \Omega$ such that

$$t < \sup_{0<\rho\le\delta}\frac{1}{\rho^n}\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t \implies u \notin L^{p,\,n}(\Omega).$$

3) This assertion follows trivially from **2)**. Indeed, for any $x \in \Omega$ we have

$$\sup_{x\in\Omega}\frac{1}{\rho^n}\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t \le \|u\|_{p,\,\lambda,\,\Omega}\cdot\rho^{\lambda-n},$$

and hence

$$\sup_{\substack{x\in\Omega\\ \rho\in(0,\,\rho_1]}}\frac{1}{\rho^n}\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t \le \|u\|_{p,\,\lambda,\,\Omega}\cdot\rho_1^{\lambda-n}.$$

By taking the limit as $\rho_1 \to 0^+$, we infer that the left-hand side converges to $|u(x)|^p$, while the right-hand side converges to 0, i.e., $u(x) \equiv 0$.

4) For any $x \in \Omega$ and $\rho \in (0,\,\delta]$ it turns out that

$$\int_{\Omega(x,\,\rho)} |u(t)|^p\,\mathrm{d}t = \int_{\Omega(x,\,\rho)} (|u(t)|^q)^{\frac{p}{q}}\cdot 1^{1-\frac{p}{q}}\,\mathrm{d}t \le$$

$$\le \left[\int_{\Omega(x,\,\rho)} |u(t)|^q\,\mathrm{d}t\right]^{\frac{p}{q}}\cdot|\Omega(x,\,\rho)|^{1-\frac{p}{q}} \le$$

$$\le \left[\int_{\Omega(x,\,\rho)} |u(t)|^q\,\mathrm{d}t\right]^{\frac{p}{q}}\cdot[\omega_n\,\rho^n]^{1-\frac{p}{q}} =$$

$$= c(n,\,p,\,q)\cdot\rho^{n\left(1-\frac{p}{q}\right)+\mu\,\frac{p}{q}}\cdot\left[\frac{1}{\rho^\mu}\int_{\Omega(x,\,\rho)} |u(t)|^q\,\mathrm{d}t\right]^{\frac{p}{q}},$$

where the red inequality follows from the inclusion $\Omega(x, \rho) \subset B(x, \rho)$. If we divide both sides by ρ^λ, it turns out that

$$\frac{1}{\rho^\lambda} \int_{\Omega(x, \rho)} |u(t)|^p \, \mathrm{d}t \leq c(n, p, q) \cdot \rho^{n\left(1-\frac{p}{q}\right)+\mu \frac{p}{q}-\lambda} \cdot \left[\frac{1}{\rho^\mu} \int_{\Omega(x, \rho)} |u(t)|^q \, \mathrm{d}t\right]^{\frac{p}{q}},$$

and the right-hand side is finite if and only if the exponent of ρ is positive, that is,

$$n\left(1-\frac{p}{q}\right)+\mu \frac{p}{q}-\lambda \geq 0 \iff \frac{\mu-n}{q} \geq \frac{\lambda-n}{p},$$

which is exactly what we wanted to prove.

□

4.3 Campanato Spaces

In this section, we shall denote by

$$u_{x, \rho} := \frac{1}{|\Omega(x, \rho)|} \int_{\Omega(x, \rho)} u(t) \, \mathrm{d}t$$

the average of u on the set $\Omega(x, \rho)$. The Campanato (p, λ) space is defined by

$$\mathscr{L}^{p, \lambda}(\Omega) := \left\{u \in L^p(\Omega) \;\middle|\; [u]_{\mathscr{L}^{p, \lambda}(\Omega)} < +\infty\right\},$$

where

$$[u]^p_{\mathscr{L}^{p, \lambda}(\Omega)} := \sup_{\substack{x \in \Omega \\ \rho \leq \delta}} \frac{1}{\rho^\lambda} \int_{\Omega(x, \rho)} |u(t) - u_{x, \rho}|^p \, \mathrm{d}t \tag{4.4}$$

is a seminorm on $\mathscr{L}^{p, \lambda}(\Omega)$.

Remark 4.1. The Campanato space $\mathscr{L}^{p, \lambda}(\Omega)$ is Banach endowed with the norm

$$\|u\|_{\mathscr{L}^{p, \lambda}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{\mathscr{L}^{p, \lambda}(\Omega)}. \tag{4.5}$$

Proposition 4.6 (Characterization)**.** *A function $u : \Omega \to \mathbb{C}$ belongs to $\mathscr{L}^{p, \lambda}(\Omega)$ if and only if $u \in L^p(\Omega)$ and the seminorm*

$$|||u|||^p_{\mathscr{L}^{p, \lambda}(\Omega)} := \sup_{\substack{x \in \Omega \\ \rho \leq \delta}} \frac{1}{\rho^\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega(x, \rho)} |u(t) - c|^p \, \mathrm{d}t \tag{4.6}$$

is finite.

Proof. If $u \in \mathscr{L}^{p, \lambda}(\Omega)$, then it is a trivial consequence of the fact that

$$|||u|||^p_{\mathscr{L}^{p, \lambda}(\Omega)} \leq [u]^p_{\mathscr{L}^{p, \lambda}(\Omega)} < +\infty.$$

Vice versa, suppose that $u \in L^p(\Omega)$ and $|||u|||_{\mathscr{L}^{p,\lambda}(\Omega)} < +\infty$. For every $x \in \Omega$ and $\rho \in (0, \delta]$ it turns out that[1]

$$\int_{\Omega(x,\rho)} |u(t) - u_{x,\rho}|^p \,\mathrm{d}t \leq 2^{p-1} \left[\int_{\Omega(x,\rho)} |u(t) - c|^p \,\mathrm{d}t + \int_{\Omega(x,\rho)} |c - u_{x,\rho}|^p \,\mathrm{d}t \right] =$$

$$= 2^{p-1} \left[\int_{\Omega(x,\rho)} |u(t) - c|^p \,\mathrm{d}t + |\Omega(x,\rho)|^{1-p} \left| \int_{\Omega(x,\rho)} [u(t) - c]^p \,\mathrm{d}t \right|^p \right] \leq$$

$$\leq 2^p \int_{\Omega(x,\rho)} |u(y) - c|^p \,\mathrm{d}y$$

from which it follows that

$$[u]^p_{\mathscr{L}^{p,\lambda}(\Omega)} \leq 2 \cdot |||u|||^p_{\mathscr{L}^{p,\lambda}(\Omega)}.$$

□

Corollary 4.7. *The norms*

$$\|u\|_{\mathscr{L}^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{\mathscr{L}^{p,\lambda}(\Omega)}$$

and

$$||||u||||_{\mathscr{L}^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + |||u|||_{\mathscr{L}^{p,\lambda}(\Omega)}$$

are equivalent.

Theorem 4.8.

1) If $0 \leq \lambda < n$, then $\mathscr{L}^{p,0}(\Omega) \cong L^{p,\lambda}(\Omega)$.

2) If $n < \lambda \leq n + p$, then $\mathscr{L}^{p,0}(\Omega) \cong C^{0,\gamma}(\Omega)$ with

$$\gamma = \frac{\lambda - n}{p}.$$

3) If $\lambda > n + p$ strictly, then $u \in \mathscr{L}^{p,0}(\Omega)$ is locally constant.

4) If $1 \leq p < q < +\infty$ and

$$\frac{\lambda - n}{p} \leq \frac{\mu - n}{q},$$

then $\mathscr{L}^{q,\mu}(\Omega) \subseteq \mathscr{L}^{p,\lambda}(\Omega)$.

We will only deal with **1)** since it follows from an algebraic lemma which will be used consistently in the next chapters. The reader may check out the other properties in this book: [2].

Lemma 4.9. *Let φ and Φ be two nonnegative functions defined on $(0, d]$, and assume that Φ is nondecreasing. Assume that there exist positive constants A, α, $\beta > 0$ such that $\alpha > \beta$ and, for any $t \in (0, 1)$ and $\sigma \in (0, d]$,*

$$\varphi(t\sigma) \leq A\, t^\alpha\, \varphi(\sigma) + \sigma^\beta\, \Phi(\sigma). \tag{4.7}$$

[1] We shall freely use the inequality $(a + b)^p \leq 2^{p-1} \cdot (a^p + b^p)$ valid for every $p \geq 1$.

Then, for any $\epsilon \in (0,\, \alpha - \beta]$, $t \in (0,\, 1)$ *and* $\sigma \in (0,\, d]$, *it turns out that*

$$\varphi(t\sigma) \leq A\, t^{\alpha-\epsilon}\, \varphi(\sigma) + K(A)\ (t\sigma)^{\beta}\ \Phi(\sigma), \tag{4.8}$$

where

$$K(\xi) := \frac{(1+\xi)^{\frac{2\,\alpha}{\xi}}}{(1+\xi)^{\frac{\alpha-\beta}{\epsilon}} - \xi}.$$

Proof of Theorem 4.8. The inclusion $L^{p,\,\lambda}(\Omega) \subseteq \mathscr{L}^{p,\,\lambda}(\Omega)$ follows trivially by Proposition 4.6, since the infimum is taken over all $c \in \mathbb{R}$, included $c = 0$.

The opposite inclusion, on the other hand, is not trivial at all and it requires an application of the algebraic lemma stated above. For every $t \in (0,\, 1)$ and $\sigma \in (0,\, \delta]$ we have the following estimate:

$$\begin{aligned}
\int_{\Omega(x,\,t\sigma)} |u(y)|^p \,\mathrm{d}y &\leq 2^{p-1} \left[\int_{\Omega(x,\,\sigma)} |u(y) - u_{x,\,\sigma}|^p \,\mathrm{d}y + \int_{\Omega(x,\,\sigma)} |u_{x,\,\sigma}|^p \,\mathrm{d}y \right] = \\
&= 2^{p-1} \left[\int_{\Omega(x,\,\sigma)} |u(y) - u_{x,\,\sigma}|^p \,\mathrm{d}y + |\Omega(x,\,\sigma)| \left| \frac{1}{|\Omega(x,\,\sigma)|^p} \int_{\Omega(x,\,\sigma)} u(y)\,\mathrm{d}t \right|^p \right] \leq \\
&\leq 2^{p-1} \left[\int_{\Omega(x,\,\sigma)} |u(y) - u_{x,\,\sigma}|^p \,\mathrm{d}y + \frac{|\Omega(x,\,\sigma)|}{|\Omega(x,\,\sigma)|} \int_{\Omega(x,\,\sigma)} |u(y)|^p \,\mathrm{d}t \right] \leq \\
&\leq 2^{p-1} \left[\int_{\Omega(x,\,\sigma)} |u(y) - u_{x,\,\sigma}|^p \,\mathrm{d}y + \frac{\omega_n\,(t\sigma)^n}{|\Omega(x,\,\sigma)|} \int_{\Omega(x,\,\sigma)} |u(y)|^p \,\mathrm{d}t \right] \leq \\
&\leq 2^{p-1} \left[\sigma^{\lambda}\ [u]^p_{\mathscr{L}^{p,\,\lambda}(\Omega)} + \frac{\omega_n\,(t\sigma)^n}{|\Omega(x,\,\sigma)|} \int_{\Omega(x,\,\sigma)} |u(y)|^p \,\mathrm{d}t \right] \leq \\
&\leq C(p,\,n,\,\sigma) \cdot \left[[u]^p_{\mathscr{L}^{p,\,\lambda}(\Omega)} + t^n\, \|u\|^p_{L^p(\Omega(x,\,\sigma))} \right] \leq \\
&\leq C'(p,\,n,\,\sigma) \cdot \left[(t\sigma)^{\lambda} \cdot [u]^p_{\mathscr{L}^{p,\,\lambda}(\Omega)} + t^{n-\epsilon}\, \|u\|^p_{L^p(\Omega(x,\,\sigma))} \right] \leq \\
&\leq C'(p,\,n,\,\sigma) \cdot (t\sigma)^{\lambda} \cdot \left[[u]^p_{\mathscr{L}^{p,\,\lambda}(\Omega)} + \frac{1}{\sigma^{\lambda}} \|u\|^p_{L^p(\Omega(x,\,\sigma))} \right],
\end{aligned}$$

and we conclude dividing both sides by $(t\sigma)^{\lambda}$. The marked inequalities need to be explained a little more in depth:

1. The blue inequality follows from a straightforward application of the Hölder inequality with u and 1.

2. The red inequality follows from the fact that Ω is a set of the type A; more precisely,

we have the estimates

$$x \in \partial\Omega \implies |\Omega(x,\,\rho)| \geq A \cdot \sigma^n \implies \frac{1}{|\Omega(x,\,\rho)|} \leq \frac{1}{A \cdot \sigma^n},$$

$$x \in \Omega \implies |\Omega(x,\,\rho)| = |B(x,\,\rho)| \implies |\Omega(x,\,\rho)| = \omega_n\,\rho^n.$$

3. The orange inequality follows from Lemma 4.9 by setting

$$\varphi(\sigma) = \int_{\Omega(x,\,\sigma)} |u(y)|^p \,\mathrm{d}y,$$

$$\Phi(\sigma) = [u]^p_{\mathscr{L}^{p,\,\lambda}(\Omega)}$$

4. The green inequality follows from the fact that we can chose $\epsilon = n - \lambda$.

□

Generalized Poincaré inequality. In this brief paragraph, we want to state and prove a generalization of the Poincaré inequality which will be useful to show a regularity results in Morrey-Campanato spaces.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected subset of $\mathbb{R}^n$ with Lipschitz boundary. There exists a positive constant $c(p,\,n,\,\Omega)$ such that for any $u \in H^{1,\,p}(\Omega)$, $1 \leq p < +\infty$, it turns out that*

$$\int_\Omega |u(x) - u_\Omega|^p \,\mathrm{d}x \leq c(p,\,n,\,\Omega) \cdot |u|^p_{1,\,p,\,\Omega} \tag{4.9}$$

where u_Ω is the average on Ω, i.e.,

$$u_\Omega = \fint_\Omega u(x)\,\mathrm{d}x.$$

Proof. We may always assume, without loss of generality, that the average of u on Ω is equal to zero.

We argue by contradiction. If (4.9) does not hold, then there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset H^{1,\,p}(\Omega)$ satisfying the following properties:

$$\begin{cases} \fint_\Omega u_k(x)\,\mathrm{d}x = 0, \\ \|u_k\|_{L^p(\Omega)} = 1, \\ |u_k|_{1,\,p,\,\Omega} \leq \frac{1}{k}. \end{cases}$$

Since $\{u_k\}_{k\in\mathbb{N}}$ is a bounded subset of $H^{1,\,p}(\Omega)$, by Rellich theorem there is a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ converging to a function u strongly in $L^p(\Omega$; in particular,

$$\|u\|_{L^p(\Omega)} = 1.$$

On the other hand, $|u_k|_{1,\,p,\,\Omega} \to 0$ and hence from

$$\int_\Omega u_k(x)\, D^i\, \varphi(x)\, \mathrm{d}x \xrightarrow{m\to+\infty} \int_\Omega u(x)\, D^i\, \varphi(x)\, \mathrm{d}x, \qquad \forall\, \varphi \in C_c^\infty(\Omega)$$

it follows that $D^i u = 0$ on Ω for each $i = 1, \ldots, n$.

In particular, u is locally constant on a connected set, i.e., u is constant on Ω and hence it is equal to zero (since it is a function with average zero). This is the sought contradiction since u has $L^p(\Omega)$ norm equal to one. □

The next result gives an explicit expression for the dependence of the constant c on Ω when Ω is a ball.

Theorem 4.11. *Let $x_0 \in \mathbb{R}^n$. There exists a positive constant $c(p,\, n)$ such that for any $u \in H^{1,\,p}(B(x_0,\, r))$, $1 \le p < +\infty$, it turns out that*

$$\int_{B(x_0,\,r)} |u(x) - u_{x_0,\,r}|^p\, \mathrm{d}x \le c(p,\, n)\, r^p \cdot |u|^p_{1,\,p,\,B(x_0,\,r)}. \tag{4.10}$$

Proof. We may always assume, without loss of generality, that the point x_0 is the origin.

Let us consider the homothety $\alpha_r : x \mapsto r \cdot x$, and let us consider the function $v(y) := u \circ \alpha_r(y)$, which is of class $H^{1,\,p}(B(0,\,1))$. The inequality (4.9) holds for $\Omega = B(0,\,1)$, hence there exists a constant $c'(n,\, p) > 0$ (the constant does not depend on Ω since the unitary ball is entirely determined by the dimension n) such that

$$\int_{B(0,\,1)} \left|u(x) - v_{B(0,\,1)}\right|^p\, \mathrm{d}x \le c'(p,\, n) \cdot |u|^p_{1,\,p,\,B(0,\,1)}.$$

A straightforward computation proves that

$$v_{B(0,\,1)} = U_{B(0,\,r)} \qquad \text{and} \qquad \int_{B(0,\,1)} \left|D^i\, v(y)\right|^p\, \mathrm{d}y = r^{p-n} \int_{B(0,\,r)} \left|D^i\, u(y)\right|^p\, \mathrm{d}y,$$

from which it follows that

$$\int_{B(0,\,1)} \left|u(x) - v_{B(0,\,1)}\right|^p\, \mathrm{d}x = \frac{1}{r^n} \int_{B(0,\,r)} \left|u(x) - v_{B(0,\,1)}\right|^p\, \mathrm{d}x \le c'(p,\, n)\, r^n \cdot |u|^p_{1,\,p,\,B(0,\,1)}.$$

□

We are now ready to state and prove the last theorem which establishes a link between Morrey-Campanato spaces and Sobolev spaces (which is, somehow, a regularity result).

Theorem 4.12. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected subset of $\mathbb{R}^n$ with Lipschitz boundary. If $u \in H^{1,\,p}(\Omega)$ and $D^i\, u \in L^{p,\,\lambda}(\Omega)$, $0 \le \lambda < n$, for each $i = 1, \ldots, n$, then $u \in \mathscr{L}^{p,\,\lambda+p}(\Omega)$ and*

$$[u]_{\mathscr{L}^{p,\,\lambda+p}(\Omega)} \le c(n,\, p,\, A) \cdot \sum_{i=1}^n \|D^i u\|_{L^{p,\,\lambda}(\Omega)}. \tag{4.11}$$

In particular, if $\lambda + p < n$*, then*

$$[u]_{\mathscr{L}^{p,\,\lambda+p}(\Omega)} \le c(n,\,p,\,A,\,\lambda)\cdot\left[\sum_{i=1}^{n}\|D^i u\|_{L^{p,\,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)}\right], \tag{4.12}$$

and, if $\lambda + p > n$ *and* $\gamma = 1-(n-\lambda)/p$*, then*

$$[u]_{0,\,\gamma,\,\Omega} \le c(n,\,p,\,A)\cdot\sum_{i=1}^{n}\|D^i u\|_{L^{p,\,\lambda}(\Omega)}. \tag{4.13}$$

Proof. Fix $x_0 \in \Omega$ and $0 < \sigma \le \delta$. The Poincaré inequality (4.10) it turns out that

$$\int_{\Omega(x_0,\,\sigma)} |u(x) - u_{x_0,\,\sigma}|^p \,\mathrm{d}x \le c(n,\,p,\,A)\,\sigma^p\cdot|u|^p_{1,\,p,\,\Omega(x_0,\,\sigma)} \le$$

$$\le c(n,\,p,\,A)\,\sigma^{p+\lambda}\cdot\sum_{i=1}^{n}\left\|D^i u\right\|_{L^{p,\,\lambda}(\Omega)}.$$

□

4.4 Bounded Mean Oscillation Spaces

Let Q_0 be a n-dimensional cube in $\mathbb{R}^n$. A function $u \in L^1(Q_0)$ is a function of *bounded mean oscillation*, and we shall denote it by $u \in \mathrm{BMO}(Q_0)$, if

$$[u]_{\mathrm{BMO}(Q_0)} = \sup_{Q\subset Q_0}\frac{1}{|Q|}\int_Q |u(x) - u_Q|\,\mathrm{d}x \tag{4.14}$$

is finite, where the supremum is taken over all the n-dimensional cubes $Q \subset Q_0$ with parallel edges.

Remark 4.2. The space $\mathrm{BMO}(Q_0)$ coincides with the Campanato space $\mathscr{L}^{1,\,n}(Q_0)$ since one may always consider

$$\Omega(x,\,\rho) = \Omega \cap Q(x,\,\rho)$$

instead of

$$\Omega(x,\,\rho) = \Omega \cap B(x,\,\rho),$$

and obtain an equivalent Banach space.

Let $u \in L^1(Q_0)$ be given. For every $\sigma > 0$ and for every n-dimensional cube $Q \subset Q_0$, with parallel edges, we set

$$S(\sigma,\,Q) := \{x \in Q \mid |u(x) - u_Q| > \sigma\}.$$

Definition 4.13 (John-Nirenberg)**.** A function $u : Q_0 \to \mathbb{C}$ belongs to $\mathcal{E}_0(Q_0)$ if there are positive constants $H,\,\beta > 0$ such that. for every $\sigma > 0$ and every n-dimensional cube $Q \subset Q_0$, it turns out that

$$|S(\sigma,\,Q)| \le H\cdot \mathrm{e}^{-\beta\sigma}\cdot|Q|. \tag{4.15}$$

Theorem 4.14. *A function* $u : Q_0 \to \mathbb{C}$ *belongs to* $\mathcal{E}_0(Q_0)$ *if and only if* u *belongs to the Campanato space* $\mathscr{L}^{p,\,n}(Q_0)$*, for any* $p \geq 1$*.*

Theorem 4.15. *The Campanato spaces* $\mathscr{L}^{p,\,n}(Q_0)$*,* $p \geq 1$*, are all equivalent between them. For any couple of real numbers* $1 \leq p \leq q$ *it turns out that*

$$\frac{\alpha}{H^{1/q} \cdot [\Gamma(q+1)]^{1/q}} \cdot [u]_{\mathscr{L}^{q,\,n}(Q_0)} \leq [u]_{\mathscr{L}^{p,\,n}(Q_0)} \leq [u]_{\mathscr{L}^{q,\,n}(Q_0)}. \tag{4.16}$$

Chapter 5

Regularity Theory in Morrey-Campanato Spaces

5.1 Caccioppoli Inequality

Lemma 5.1. *Let $A(x) := \{a_{i,j}(x)\}_{i,j=1,\dots,n}$ be a matrix, uniformly elliptic on the ball B_r. Assume that the coefficients $a_{i,j}(x)$ belongs to $L^\infty(B_r)$, and assume also that $u \in H^1(B_r)$ is a weak solution of the elliptic problem*

$$\sum_{i,j=1}^{n} D^i \left(a_{i,j}(x)\, D^j u(x)\right) = \sum_{i=1}^{n} D^i f_i(x), \tag{5.1}$$

where $f_i \in L^2(B_r)$ for each $i = 1, \dots, n$. Then there exists a constant $c(\nu) > 0$ such that, for every $\rho \in (0, r)$ and every $(n+1)$-tuple of real numbers $(s, s_1, \dots, s_n) \in \mathbb{R}^{n+1}$, it turns out that

$$\sum_{i=1}^{n} \|D^i u\|^2_{0,2,B_\rho} \le c(\nu) \left[\frac{1}{(r-\rho)^2} \|u - s\|^2_{0,2,B_r} + \sum_{j=1}^{n} \|f_j - s\|^2_{0,2,B_r} \right]. \tag{5.2}$$

Proof. The elliptic problem (5.1) can be equivalently rewritten in its integral form, i.e.,

$$\sum_{i,j=1}^{n} \int_{B_r} a_{i,j}(x)\, D^j [u(x) - s]\; D^i\varphi(x)\, \mathrm{d}x = \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]\; D^i\varphi(x), \qquad \forall\, \varphi \in H^1_0(B_r). \tag{5.3}$$

Let $\theta \in C^\infty_c(\mathbb{R}^n)$ be a cutoff function satisfying the following properties:

(a) The support of θ is compactly contained in B_r, and $\theta \equiv 1$ on B_ρ.

(b) For every x it turns out that $0 \le \theta(x) \le 1$.

(c) The derivative is bounded, i.e.,

$$\|\nabla \theta\|_\infty \le \frac{1}{r - \rho}.$$

We pick $\varphi(x) := \theta^2(x) \cdot [u(x) - s]$ in (5.3) as a test function, and we obtain the identity

$$\sum_{i,\,j=1}^{n} \int_{B_r} a_{i,\,j}(x)\, D^j\, [u(x) - s]\; \theta(x)\, D^i\, ((u(x) - s)\, \theta(x))\; \mathrm{d}x + \ldots$$

$$\cdots + \sum_{i,\,j=1}^{n} \int_{B_r} a_{i,\,j}(x)\, D^j\, [u(x) - s]\; \theta(x)\, D^i\, ((u(x) - s)\, \theta(x))\; \mathrm{d}x =$$

$$= \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]\; \theta(x)\, D^i\, (\theta(x)\, [u(x) - s])\; \mathrm{d}x + \ldots$$

$$\cdots + \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]\; D^i\, (\theta(x))\; \theta(x)\, [u(x) - s]\, \mathrm{d}x$$

from which it follows that

$$\sum_{i,\,j=1}^{n} \int_{B_r} a_{i,\,j}(x)\, D^j\, ((u(x) - s)\, \theta(x))\; D^i\, ((u(x) - s)\, \theta(x))\; \mathrm{d}x =$$

$$= \sum_{i,\,j=1}^{n} \int_{B_r} a_{i,\,j}(x)\, [u(x) - s]^2\, D^j\theta(x)\, D^i\theta(x)\, \mathrm{d}x + \ldots$$

$$\cdots + \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]\; \theta(x)\, D^i\, (\theta(x)\, [u(x) - s])\; \mathrm{d}x + \ldots$$

$$\cdots + \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]\; D^i\, (\theta(x))\; \theta(x)\, [u(x) - s]\, \mathrm{d}x.$$

If we take the modules of the identity above, then it turns out that

$$\nu \sum_{i=1}^{n} \int_{B_r} \left| D^i\, [(u(x) - s)\, \theta(x)] \right|^2\, \mathrm{d}x \le c(a_{i,\,j})\, \frac{c^2}{(r-\rho)^2} \sum_{i,\,j=1}^{n} \int_{B_r} [u(x) - s]^2\, \mathrm{d}x + \ldots$$

$$\cdots + \frac{1}{2\epsilon} \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]^2\, \mathrm{d}x + \ldots$$

$$\cdots + \frac{\epsilon}{2} \sum_{i=1}^{n} \int_{B_r} \left| D^i \left[\theta(x)\, (u(x) - s)^2 \right] \right|^2\, \mathrm{d}x + \ldots$$

$$\cdots + \frac{c}{(r-\rho)^2} \sum_{i,\,j=1}^{n} \int_{B_r} [u(x) - s]^2\, \mathrm{d}x + \ldots$$

$$\cdots + \frac{1}{2} \sum_{i=1}^{n} \int_{B_r} [f_i(x) - s_i]^2\, \mathrm{d}x,$$

and we obtain the thesis by taking ϵ sufficiently small. □

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