Lecture Notes

Variational Methods

Course held by

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Disclaimer

I wrote these notes to summarise the content of the course on Variational Methods, held by Professor Andrea Malchiodi at SNS.

I tried to include all the topics that were discussed in class and combine it with some additional information from several other courses to produce a self-contained document.

I will try to review them periodically, but I am sure that at the end there will be a large number of mistakes and oversights. To report them, feel free to send me an email at **francesco (dot) maiale (at) sns (dot) it**.

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Part I

Nonlinear Analysis

Chapter 1

Differential Calculus in Banach Spaces

In this chapter, we generalise differential calculus on $\mathbb{R}^n$ to general Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ and prove fundamental theorems such as the global inversion theorem. We will follow the first section of the book [2] closely.

1.1 Introduction to the course

The main goal of this course is to introduce tools from analysis and topology to deal with the existence, uniqueness and regularity of solutions to nonlinear problems such as

$$\begin{cases} -\Delta u = f(x, u) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{1.1}$$

A possible approach to look for solutions u of (1.1) with some regularity (for example, Hölder $C^{k,\alpha}(\Omega)$ or Sobolev $W_0^{k,p}(\Omega)$), is to rewrite it as

$$u - T(u) = 0,$$

where T is the operator defined by taking the inverse of the Laplace operator; namely,

$$T(v)(x) = (-\Delta)^{-1} f(x, v).$$

At this point, one can try to prove an appropriate fixed-point theorem that works under some assumptions on the nonlinearity f and find a solution.

In this course, however, we are mainly interested in exploiting the variational structure of (1.1). Setting aside for the moment all regularity concerns, notice that

$$\int_\Omega (-\Delta u) v \, \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \oint_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, \mathrm{d}\sigma$$

holds for all $v \in H_0^1(\Omega)$. Since $u \big|_{\partial\Omega} \equiv 0$, we infer that

$$\int_\Omega (-\Delta u) v \, \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

Therefore, if u is a solution of the nonlinear problem (1.1), then u satisfies

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f(x, u) v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega). \tag{1.2}$$

It remains to prove that we can always recover the identity (1.2) starting from the variational framework. Let $\mathfrak{X} := H_0^1(\Omega)$, endow it with the homogeneous norm

$$\|u\|_{\mathfrak{X}}^2 := \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

and define

$$\mathcal{F}(u) := \int_\Omega F(x, u) \, \mathrm{d}x, \quad \text{where } F(x, u) = \int_0^u f(x, s) \, \mathrm{d}s.$$

Finally, introduce the functional

$$\mathcal{G}(u) := \frac{1}{2} \|u\|_{\mathfrak{X}}^2 - \mathcal{F}(u),$$

and notice that its directional derivative D_v is given by

$$D_v \mathcal{G}(u) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega \partial_u F(x, u) v \, \mathrm{d}x.$$

Since $\partial_u F(x, u) = f(x, u)$ by definition, we just "proved" that (1.2) is equivalent to the fact that the first variation of $\mathcal{G}(u)$ is zero for all $v \in H_0^1(\Omega)$.

1.2 Fréchet and Gâteaux derivatives

Throughout this section, the symbols $\mathfrak{X}$ and $\mathfrak{Y}$ will always denote two Banach spaces and, unless otherwise stated, U will always be an open subset of $\mathfrak{X}$.

Definition 1.1 (F-differentiable)**.** A map $F : U \to \mathfrak{Y}$ is said to be *(Fréchet) differentiable* at $u \in U$ if there exists a linear map $A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ such that

$$F(u + h) = F(u) + Ah + o(\|h\|_{\mathfrak{X}}). \tag{1.3}$$

The map A is usually referred to as the *(Fréchet) differential* of F at u and denoted by either $\mathrm{d}F(u)$ or $F'(u)$.

Proposition 1.2. *Let $F : U \to \mathfrak{Y}$ be Fréchet differentiable at some $u \in U$. Then*

(1) the differential A of F at u is unique;

(2) the map F is continuous at u;

(3) the notion of differentiability does not depend on the choice of equivalent *norms on either $\mathfrak{X}$ or $\mathfrak{Y}$.*

Proof. Let $A \neq B \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be two F-differentials. Then (1.3) yields

$$\|Ah - Bh\|_{\mathfrak{Y}} = o(\|h\|_{\mathfrak{X}}). \tag{1.4}$$

On the other hand, if $A \neq B$ then there exists $x_0 \in \mathfrak{X}$ such that

$$a := \|Ax_0 - Bx_0\|_{\mathfrak{Y}} \neq 0.$$

Take $t \in \mathbb{R}$, $t \neq 0$, and set $x := tx_0 \in \mathfrak{X}$. Then

$$\frac{a}{\|x_0\|_{\mathfrak{X}}} = \frac{\|Ax_0 - Bx_0\|_{\mathfrak{Y}}}{\|x_0\|_{\mathfrak{X}}} = \frac{\|Ax - Bx\|_{\mathfrak{Y}}}{\|x\|_{\mathfrak{X}}},$$

and the left-hand side is a constant that does not depend on t, so taking the limit as t goes to zero leads to a contradiction of (1.4). □

Example 1.3. We now give a few explicit examples.

(a) The constant map $F(u) \equiv c$ is differentiable at all $u \in \mathfrak{X}$ and its differential is the identically zero map $\mathrm{d}F(u) = 0$.

(b) Let $A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$. Since

$$A(u + h) = Au + Ah$$

we easily find that A is differentiable at all points and $\mathrm{d}A(u) = A$. Furthermore, the remainder $o(\|h\|_{\mathfrak{X}})$ is exactly equal to zero.

(c) Let $B : \mathfrak{X} \times \mathfrak{Y} \longrightarrow \mathfrak{Z}$ be a bilinear continuous map. We have

$$B(u + h, v + k) = B(u, v) + B(h, v) + B(u, k) + B(h, k),$$

and using the continuity at the origin we find that

$$\|B(h, k)\|_{\mathfrak{Z}} \leq \|h\|_{\mathfrak{X}} \|k\|_{\mathfrak{Y}}.$$

Then B is differentiable at all $(u, v) \in \mathfrak{X} \times \mathfrak{Y}$ and the differential is given by

$$\mathrm{d}B(u, v)[h, k] := B(h, v) + B(u, k).$$

(d) Let $\mathfrak{X}$ be a Hilbert space with scalar product $\langle \cdot, - \rangle_{\mathfrak{X}}$ and consider the map

$$F(u) := \langle u, u \rangle_{\mathfrak{X}} = \|u\|^2_{\mathfrak{X}}.$$

We can explicitly compute F at $u+h$ using the scalar product obtaining

$$F(u+h) = \|u\|_{\mathfrak{X}}^2 + 2\langle u,\, h\rangle_{\mathfrak{X}} + \|h\|_{\mathfrak{X}}^2.$$

It follows that F is differentiable at all $u \in \mathfrak{X}$ and its differential is given by

$$\mathrm{d}F(u)[h] := 2\langle u,\, h\rangle_{\mathfrak{X}}.$$

Proposition 1.4.

(1) *Let $F,\, G : U \to \mathfrak{Y}$ be F-differentiable at $u \in U$. Then for all $a,\, b \in \mathbb{R}$ the map $aF + bG$ is also F-differentiable at u and*

$$\mathrm{d}(aF + bG) = a\,\mathrm{d}F + b\,\mathrm{d}G.$$

(2) *Let $F : U \to \mathfrak{Y}$ and $G : V \subset \mathfrak{Y} \to \mathfrak{Z}$ with $F(U) \subset V$. If F is F-differentiable at $u \in U$ and G at $F(u) \in V$, then $G \circ F$ is also F-differentiable at u and*

$$\mathrm{d}(G \circ F)(u)[h] = \mathrm{d}G(v)\left[\mathrm{d}F(u)[h]\right].$$

Definition 1.5. A map $F : U \to \mathfrak{Y}$ belongs to $C^1(U,\, \mathfrak{Y})$ if it is differentiable in U and

$$U \ni u \longmapsto \mathrm{d}F(u) \in \mathcal{L}(\mathfrak{X},\, \mathfrak{Y})$$

is a continuous mapping.

Notation. A map F from a Banach space $\mathfrak{X}$ to $\mathbb{R}$ is known as *functional.* If F is differentiable, then the differential belongs to the dual space

$$\mathrm{d}F(u) \in \mathcal{L}(\mathfrak{X},\, \mathbb{R}) = \mathfrak{X}^*,$$

and thus, if $\mathfrak{X}$ is a Hilbert space, an application of Riesz's theorem shows that there exists a vector $\nabla F(u) \in \mathfrak{X}$, called *gradient* of F at u, such that

$$\mathrm{d}F(u)[h] = (\nabla F(u),\, h)_{\mathfrak{X}} \quad \text{for all } h \in \mathfrak{X}.$$

In the general framework of Banach spaces, the gradient is defined as the unique element satisfying the identity

$$\mathrm{d}F(u)[h] = \langle \nabla F(u),\, h\rangle_{\mathfrak{X}^*,\, \mathfrak{X}} \quad \text{for all } h \in \mathfrak{X},$$

where $\langle \cdot,\, -\rangle$ here denotes the so-called duality coupling.

Definition 1.6. Let $\mathfrak{X}$ be a Hilbert space and $F : U \subset \mathfrak{X} \to \mathfrak{X}$. We say that F is a *variational operator* if there exists a functional $J : U \to \mathbb{R}$ such that

$$F(u) = \nabla J(u) \quad \text{for all } u \in U.$$

Definition 1.7 (G-differentiability)**.** A map $F : U \to \mathfrak{Y}$ is said to be *Gâteaux-differentiable* at $u \in U$ if there exists a linear map $A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ such that

$$\frac{F(u+th)-F(u)}{t} \xrightarrow{t \to 0} Ah. \tag{1.5}$$

The map A, uniquely determined, is called *G-differential* of F at u and it is usually indicated with the symbol $\mathrm{d}_G F(u)$.

Remark 1.8. A Fréchet-differentiable function is also Gâteaux-differentiable, but the opposite is false. Indeed, G-differentiability is even weaker than standard continuity.

Example 1.9. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(s, t) := \begin{cases} \left[\dfrac{s^2 t}{s^4+t^2}\right]^2 & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

If we take the limit as $t \to 0$ along the path $s = 1$ (or any other constant), then $F(s, t)$ tends to zero. On the other hand, if we consider the parabola $s = t^2$ we find that

$$\lim_{t \to 0} F(t^2, t) = \frac{1}{4},$$

which means that F is not continuous in $t = 0$, but it is Gâteaux-differentiable at $t = 0$.

Theorem 1.10. *Let $F : U \to \mathfrak{Y}$ be G-differentiable in U. Then*

$$\|F(u)-F(v)\|_{\mathfrak{Y}} \leq \sup\left\{\|d_G F(w)\| \;:\; w \in [u, v]\right\} \|u-v\|_{\mathfrak{X}} \tag{1.6}$$

where $[u, v] := \{tu + (1-t)v \;:\; t \in [0, 1]\} \subset U$.

Proof. We can assume without loss of generality that $F(u) \neq F(v)$. By Hahn-Banach theorem, we can always find $\psi \in \mathfrak{Y}^*$, $\|\psi\| = 1$, such that

$$\langle \psi, F(u)-F(v) \rangle_{\mathfrak{Y}^*, \mathfrak{Y}} = \|F(u)-F(v)\|_{\mathfrak{Y}}. \tag{1.7}$$

Now let $\gamma(t) := tu + (1-t)v$ be a parametrisation of the segment $[u, v]$ and consider the function with domain $[0, 1]$ given by

$$h(t) := \langle \psi, F(\gamma(t)) \rangle_{\mathfrak{Y}^*, \mathfrak{Y}}\,.$$

The curve γ satisfies the relation

$$\gamma(t+\tau) = \gamma(t) + \tau(u-v)$$

for all $t, \tau \in [0, 1]$ such that $t + \tau \in [0, 1]$, so we can estimate the increment of h as

$$\frac{h(t+\tau)-h(t)}{\tau} = \left\langle \psi, \frac{F(\gamma(t)+\tau(u-v))-F(\gamma(t))}{\tau} \right\rangle_{\mathfrak{Y}^*, \mathfrak{Y}}.$$

Now let $\tau \to 0$ and use the G-differentiability of F to find the following expression for the derivative of h at t:

$$h'(t) = \langle \psi,\, \mathrm{d}_G F(tu + (1-t)v)(u - v)\rangle_{\mathfrak{Y}^*,\,\mathfrak{Y}}\,. \tag{1.8}$$

Since h is a real-valued function with domain $[0,\,1]$, we can apply the **mean-value theorem** to find $\theta \in (0,\,1)$ such that the following holds:

$$h'(\theta) = h(1) - h(0).$$

Plug into this identity both (1.7) and (1.8). It turns out that

$$\begin{aligned}\|F(u) - F(v)\|_{\mathfrak{Y}} &= h(1) - h(0) = \\ &= h'(\theta) = \\ &= \langle \psi,\, \mathrm{d}_G F(\theta u + (1-\theta)v)(u - v)\rangle_{\mathfrak{Y}^*,\,\mathfrak{Y}} \le \\ &\le \underbrace{\|\psi\|}_{=1} \|\mathrm{d}_G F(\theta u + (1-\theta)v)\| \|u - v\|_{\mathfrak{X}}.\end{aligned}$$

Finally, the point $\theta u + (1-\theta)v$ belongs to $[u,\,v]$ and the inequality (1.6) follows by taking the supremum on both sides with respect to θ. □

Theorem 1.11. *Let $F : U \to \mathfrak{Y}$ be a G-differentiable map with G-differential*

$$\mathrm{d}_G F : U \longrightarrow \mathcal{L}(\mathfrak{X},\,\mathfrak{Y})$$

continuous at some $u_0 \in U$. Then F is F-differentiable at u_0 and there results

$$\mathrm{d}F(u_0) = \mathrm{d}_G F(u_0).$$

Proof. First, define the map

$$R(h) := F(u_0 + h) - F(u_0) - \mathrm{d}_G F(u_0)[h].$$

By assumption, the map R is G-differentiable in a small neighbourhood of u_0; more precisely, it is sufficient to choose ϵ in such a way that

$$B_\epsilon(u_0) \subset U$$

for the G-differentiability to hold on all B_ϵ. Moreover, its G-differential is given by

$$\mathrm{d}_G R(h)[k] = \mathrm{d}_G F(u_0 + h)[k] - \mathrm{d}_G F(u_0)[k].$$

Apply the mean-value property (1.6) with $[u,\,v] = [0,\,h]$ to obtain

$$\|R(h) - \underbrace{R(0)}_{=0}\| \le \sup_{t \in [0,\,1]} \|\mathrm{d}_G R(th)\| \|h\|.$$

Plug the formula that gives $\mathrm{d}_G R$ in terms of $\mathrm{d}_G F$ into this inequality to find an estimate on the norm of $R(h)$:

$$\|R(h)\| \leq \sup_{t\in[0,\,1]} \|\mathrm{d}_G F(u_0+th) - \mathrm{d}_G F(u_0)\| \|h\|.$$

This proves the thesis because $\mathrm{d}y_G F(u_0)$ is continuous by assumption, and hence the supremum goes to zero as $\|h\|$ becomes small:

$$\sup_{t\in[0,\,1]} \|\mathrm{d}_G F(u_0+th) - \mathrm{d}_G F(u_0)\| \xrightarrow{\|h\|\to 0} 0.$$

Therefore $R(h)$ is a small-o of $\|h\|$, and thus F is F-differentiable at u_0 with F-differential that coincides with the G-differential. □

We conclude this section with a couple of remarks. Let F be a continuous function defined on $[a,\,b]$ taking values in a Banach space $\mathfrak{X}$, and set

$$\Phi(t) := \int_a^t F(\xi)\,\mathrm{d}\xi.$$

Exercise 1.1. Show that Φ is a F-differentiable map whose differential coincides with $F(t_0)$ at all $t_0 \in [a,\,b]$.

This can be done, for example, using the canonical identification between $\mathfrak{X}$ and its dual $\mathfrak{X}^*$. In any case, it follows from Theorem 1.10 that

$$\|\Phi(t) - \Phi(s)\| \leq \sup\{\|F(\xi)\| \,:\, \xi \in [s,\,t]\} \times (t-s),$$

and therefore, if F is identically zero on $[a,\,b]$, then Φ is constant. This means that Φ is, up to a constant, the unique primitive of F as it happens in the Euclidean setting.

Corollary 1.12. *Let $F \in C^1(U,\,\mathfrak{Y})$ and suppose that $[u,\,v] \subset U$. Then the map*

$$F\circ\gamma : [0,\,1] \ni t \longmapsto F(tu + (1-t)v) \in \mathfrak{Y}$$

belongs to $C^1([0,\,1],\,\mathfrak{Y})$ and the integral representation holds:

$$F(v) - F(u) = \int_0^1 F'(tu + (1-t)v)[u-v]\,\mathrm{d}t. \tag{1.9}$$

1.3 Nemitski operators

In this section, we will introduce the notion of *Nemitski operator* and investigate specific properties such as continuity, differentiability and its relation with

$$\mathcal{G}(u) = \frac{1}{2}\|u\|^2_{\mathfrak{X}} - \mathcal{F}(u).$$

Definition 1.13 (Nemitski operator)**.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function. The *Nemitski operator* associated to f is the map

$$\mathscr{M}(\Omega, \mathbb{R}) \ni u \longmapsto f(\cdot, u(\cdot)).$$

Here $\mathscr{M}(\Omega, \mathbb{R})$ denotes the set of all real-valued measurable maps defined on Ω. The symbol f will denote both the function and the associated operator.

The operator f sends $\mathcal{M}(\Omega)$ in the set of real-valued functions defined on Ω but, a priori, we have no guarantee that $f(\cdot, u(\cdot))$ is measurable and, in general, it is not.

Definition 1.14. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function. We say that f satisfies the *Carathéodory condition* if it satisfies the following properties:

(i) The map $s \mapsto f(x, s)$ is continuous for almost every $x \in \Omega$.

(ii) The map $x \mapsto f(x, s)$ is measurable for all $s \in \mathbb{R}$.

Lemma 1.15. *If f satisfies the Carathéodory condition, then the associated Nemitski operator takes values in $\mathscr{M}(\Omega, \mathbb{R})$, that is, the map*

$$x \longmapsto f(x, u(x))$$

is measurable for all $u \in \mathscr{M}(\Omega, \mathbb{R})$.

Proof. Let $u \in \mathscr{M}(\Omega, \mathbb{R})$. There is a sequence of simple functions $(\chi_n)_{n\in\mathbb{N}}$ that converges to u at almost every $x \in \Omega$. From the Carathéodory condition it follows that

$$f(\cdot, \chi_n(\cdot)) \text{ is measurable and } f(\cdot, \chi_n(\cdot)) \xrightarrow{n\to+\infty} f(\cdot, u(\cdot)) \text{ a.e. in } \Omega.$$

In particular, the function $f(\cdot, u(\cdot))$ is almost everywhere the pointwise limit of a sequence of measurable functions; we deduce that also $f(u)$ is measurable. □

1.3.1 Continuity of Nemitski operators

Let $p, q \geq 1$ and let f be a function satisfying the Carathéodory condition and the following growth condition:

$$|f(x, s)| \leq a + b|s|^{\frac{p}{q}} \tag{1.10}$$

where a and b are two positive constants.

Theorem 1.16. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set. Then*

$$f : L^p(\Omega) \to L^q(\Omega)$$

is a continuous operator.

Notice that the boundedness of Ω is not a strictly necessary condition, but rather it makes the proof much simpler. For example, if $a = 0$ in (1.10), then it is possible to prove the statement with no assumption on Ω. In any case, to prove this result, we now recall a well-known technical lemma which will make it possible for us to apply *Lebesgue's dominated convergence theorem* below.

Lemma 1.17. *Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ be a strongly convergent sequence and let $u \in L^p(\Omega)$ be its limit. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a function $h \in L^p(\Omega)$ such that*

$$u_{n_k} \xrightarrow{a.e.\ in\ \Omega} u \text{ and } |u_{n_k}(x)| \leq h(x) \text{ at almost every } x \in \Omega. \tag{1.11}$$

Proof. The argument is completely standard. Indeed, one defines the sequence

$$v_j := \sum_{k=1}^{j} |u_{n_k} - u_{n_{k-1}}|$$

and proves that v_j converges to some $v \in L^p(\Omega)$ positive. The only nontrivial point is how to choose the function h, but it is not hard to verify that $h := v + |u|$ works just fine. □

Proof of Theorem 1.16. First, notice that $f(\cdot, u(\cdot)) \in L^q(\Omega)$ since (1.10) implies that

$$|f(u)|^q \lesssim_q a^q + b^q |u(x)|^p,$$

and the function on the right-hand side belongs to $L^1(\Omega)$ by assumption. Now suppose that

$$\|u_n - u\|_{L^p(\Omega)} \xrightarrow{n\to+\infty} 0.$$

By Lemma 1.17, we can always find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a function $h \in L^p(\Omega)$ satisfying (1.11). It follows from the Carathéodory condition and (1.10) that

$$f(u_{n_k}) \xrightarrow{\text{a.e. in } \Omega} f(u) \text{ and } |f(u_{n_k})| \leq a + b|h|^{\frac{p}{q}} \in L^q(\Omega).$$

We can now apply Lebesgue's dominated convergence theorem and infer that

$$\|f(u_{n_k}) - f(u)\|^q_{L^q(\Omega)} = \int_\Omega |f(u_{n_k}) - f(u)|^q \, \mathrm{d}x \to 0.$$

Since any sequence u_n converging to u in $L^p(\Omega)$ has a subsequence such that $f(u_{n_k}) \to f(u)$ in $L^q(\Omega)$, we conclude that f is a continuous operator. □

1.3.2 Differentiability of Nemitski operators

Let $p > 2$ and suppose that f has a partial derivative $f_s := \partial_s f$ satisfying the Carathéodory condition and the following growth condition

$$|f_s(x, s)| \leq a + b|s|^{p-2} \tag{1.12}$$

for some positive constants a, $b > 0$. The previous result shows that f_s is a bounded operator from $L^p(\Omega)$ to $L^r(\Omega)$, with

$$r = \frac{p}{p-2}.$$

As a consequence, the function $f_s(u)vm$ defined by setting

$$f_s(u)v : x \longmapsto f_s(x,\, u(x))v(x),$$

satisfies the regularity condition $f_s(u)v \in L^{p'}(\Omega)$, where p' is the conjugate exponent of p; namely,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.18. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Suppose that $p > 2$, f satisfies the Carathéodory condition and the following boundedness assumption:*

$$|f(x,\, 0)| \leq C < \infty.$$

Assume also that f has partial derivative f_s that satisfies the Carathéodory condition and the growth condition (1.12). Then the Nemitski operator

$$f : L^p(\Omega) \longrightarrow L^{p'}(\Omega)$$

is F-differentiable on $L^p(\Omega)$, and its differential is given by

$$\mathrm{d}f(u)[v] = f_s(u)v. \tag{1.13}$$

Proof. Start by integrating (1.12). Then we can find positive constants c, $d > 0$ such that

$$|f(x,\, s)| \leq c + d|s|^{p-1}, \tag{1.14}$$

from which it follows (using Theorem 1.16) that f is a continuous operator between $L^p(\Omega)$ and $L^{p'}(\Omega)$. We now claim that

$$\omega(u,\, v) := \|f(u+v) - f(u) - f_s(u)v\|_{L^{p'}(\Omega)}$$

belongs to $o(\|v\|_{L^p(\Omega)})$. This would conclude the proof since it asserts that $f_s(u)v$ is the F-differential so we can focus on proving the claim.

Step 1. The classical mean-value theorem applied to $\mathbb{R} \ni u \mapsto f(\cdot,\, u)$ shows that

$$|f(u+v) - f(u) - f_s(u)v| = |vw|,$$

where

$$w(x) := \int_0^1 [f_s(x,\, u+\xi v) - f_s(x,\, u)]\, \mathrm{d}\xi.$$

Using Hölder inequality we find that

$$\omega(u,\, v) \leq \|v\|_{L^p(\Omega)}\|w\|_{L^r(\Omega)},$$

where $r = \frac{p}{p-2}$. It remains to prove that $\|w\|_{L^r(\Omega)}$ goes to zero as $\|v\|_{L^p(\Omega)}$ becomes increasingly smaller.

Step 2. Applying Fubini-Tonelli's theorem we infer that

$$\|w\|^r_{L^r(\Omega)} \leq \int_\Omega \mathrm{d}x \int_0^1 \mathrm{d}\xi |f_s(x,\, u+\xi v) - f_s(x,\, u)|^r \leq$$

$$\leq \int_0^1 \mathrm{d}\xi \int_\Omega \mathrm{d}x |f_s(x,\, u+\xi v) - f_s(x,\, u)|^r =$$

$$= \int_0^1 \|f_s(\cdot,\, u(\cdot) + \xi v(\cdot) - f_s(\cdot,\, u(\cdot))\|^r_{L^r(\Omega)},$$

The right-hand side goes to zero because, as observed earlier, the operator f_s is continuous from $L^p(\Omega)$ to $L^r(\Omega)$ and this concludes the proof. □

In the limit case, namely $p = 2$, the result is invalid and it can actually be proved that f is only G-differentiable. The next proposition summarises it.

Proposition 1.19. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and suppose that both f and f_s satisfy the Carathéodory condition and the growth condition*

$$|f_s(x,\, s)| \leq C < \infty.$$

Then the Nemitski operator $f : L^2(\Omega) \to L^2(\Omega)$ is continuous and G-differentiable, with G-differential given by

$$\mathrm{d}_G f(u)[v] = f_s(u)v.$$

Moreover, if f is F-differentiable at some $u \in \Omega$, then we can always find measurable functions $a,\, b \in \mathscr{M}(\Omega,\, \mathbb{R})$ such that

$$f(x,\, u(x)) = a(x) + b(x)u(x).$$

1.3.3 Potential operators

In this section, our goal is to introduce the notion of *potential operator* and exploit it to prove that (1.1) is well-defined under mild assumptions on f.

Theorem 1.20 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $k \geq 1$ and $1 \leq p \leq \infty$. Then the following inclusions are continuous:*

- ***(a)*** *If $kp < n$, then $H^{k,\,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q \leq \frac{np}{n-kp}$.*

(b) *If* $kp = n$*, then* $H^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1, \infty)$*.*

(c) *If* $kp > n$*, then* $H^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$*, where*

$$\alpha = \begin{cases} k - \frac{n}{p} & \text{if } k - \frac{n}{p} < 1, \\ [0, 1) & \text{if } k - \frac{n}{p} = 1 \text{ and } p > 1, \\ 1 & \text{if } k - \frac{n}{p} > 1. \end{cases}$$

Furthermore, the inclusions above are compact if we restrict the ranges of q *and* α*:*

(**a**)′ *If* $kp < n$*, then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $1 \leq q < \frac{np}{n-kp}$*.*

(**b**)′ *If* $kp = n$*, then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [1, \infty)$*.*

(**c**)′ *If* $kp > n$*, then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^0(\bar{\Omega})$*.*

Now let $\mathfrak{X} := H_0^1(\Omega)$ and let f be a function satisfying Carathéodory condition and the growth condition

$$|f(x, s)| \leq a + b|s|^\sigma, \tag{1.15}$$

where

$$\sigma \leq \frac{n+2}{n-2} =: 2^* - 1$$

if $n \geq 3$, and $\sigma > 0$ arbitrary if $n = 1$ or $n = 2$. We proved in Theorem 1.16 that f is a continuous operator between $L^{2^*}(\Omega)$ and $L^q(\Omega)$ where

$$q \geq \frac{2n}{n+2}.$$

It follows that

$$u \in \mathfrak{X} \implies f(u) \in L^{(2^*)'}(\Omega),$$

and therefore, given $v \in \mathfrak{X}$, we have that $f(u)v \in L^1(\Omega)$. We use *Riesz's representation theorem* to define a map $N : \mathfrak{X} \to \mathfrak{X}$ in such a way that $N(u)$ is the unique element satisfying the following identity:

$$(N(u), v)_{\mathfrak{X}} = \int_\Omega f(x, u(x))v(x)\, \mathrm{d}x.$$

We claim that N is a continuous map. Indeed, by definition we have that

$$\|N(u) - N(v)\| = \sup_{\|w\|_{\mathfrak{X}} \leq 1} \left\{ \int_\Omega [f(x, u) - f(x, v)]w(x)\, \mathrm{d}x \right\},$$

and thus, using the appropriate Sobolev embedding, we can infer that

$$\|N(u) - N(v)\| \lesssim \|f(x, u) - f(x, v)\|_{L^{\frac{2n}{n+2}}(\Omega)} \|w\|_{\mathfrak{X}}.$$

Since f is continuous as an operator from $L^{2^*}(\Omega)$ to $L^{\frac{2n}{n+2}}(\Omega)$, the right-hand side of the inequality above converges to zero as soon as $\|u - v\|_X \to 0$, and this proves the claim.

We are now ready to show that the growth condition (1.15) is a sufficient condition on the nonlinearity f for its integral to be well-defined and, more so, differentiable.

Theorem 1.21. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and suppose that f satisfies the Carathéodory condition and the growth condition (1.15). Then*

$$\Phi(u) := \int_\Omega F(x,\, u)\,\mathrm{d}x$$

is of class C^1 and its gradient coincides with $N(u)$.

Proof. By integrating (1.15), we find a growth condition on F which tells us that

$$|F(x,\, s)| \le c + d|s|^{2^*} \tag{1.16}$$

for some positive constants $c,\, d > 0$. It follows that $F(\cdot,\, u(\cdot))$ belongs to $L^1(\Omega)$, and thus its integral, $\Phi(u)$, is well-defined, continuous and differentiable on $\mathfrak{X}$. Furthermore,

$$\Phi'(u)[v] = \int_\Omega f(x,\, u(x))v(x)\,\mathrm{d}x,$$

and the right-hand side coincides with the element $(N(u),\, v)_{\mathfrak{X}}$, which means that (by uniqueness) $\Phi'(u)$ must be $N(u)$. □

Remark 1.22. If Ω is an unbounded domain, then the same argument works if the growth is adjusted to be compatible with embedding results valid for unbounded domains [5].

1.4 Higher derivatives and partial derivatives

Let $F \in C(U,\, \mathfrak{Y})$ be a function differentiable on some open subset $U \subset \mathfrak{X}$. We know that the differential is a map between U and the space of linear operators, that is,

$$\mathrm{d}F : U \longrightarrow \mathcal{L}(\mathfrak{X},\, \mathfrak{Y}),$$

On the other hand, $\mathcal{L}(\mathfrak{X},\, \mathfrak{Y})$ is another Banach space and this means that it makes sense to investigate the differentiability of this map.

Definition 1.23. A map $F : U \to \mathfrak{Y}$ is said to be *twice Fréchet differentiable* at some $u^* \in U$ if $\mathrm{d}F$ is differentiable at u^*. The second differential of F at u^* is the map defined as

$$\mathrm{d}^2F(u^*) := \mathrm{d}F'(u^*).$$

If F is twice differentiable at all points of U we say that F is twice differentiable in U.

According to the definition above, the second differential $\mathrm{d}^2F(u^*)$ is a linear continuous map between $\mathfrak{X}$ and $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$, that is,

$$\mathrm{d}^2F(u^*) \in \mathcal{L}(\mathfrak{X}, \mathcal{L}(\mathfrak{X}, \mathfrak{Y})).$$

Remark 1.24. It is often useful to regard $\mathrm{d}^2F(u^*)$ as a bilinear map on $\mathfrak{X}$. This is possible thanks to the one-to-one correspondence

$$\mathcal{L}_2(\mathfrak{X}, \mathfrak{Y}) \cong \mathcal{L}(\mathfrak{X}, \mathcal{L}(\mathfrak{X}, \mathfrak{Y})).$$

Proof. Let $A \in \mathcal{L}(\mathfrak{X}, \mathcal{L}(\mathfrak{X}, \mathfrak{Y}))$. We can associate to A, in a unique way, a bilinear operator defined on $\mathfrak{X}$ by setting

$$\Phi_A(u, v) := A(u)[v].$$

Vice versa, given a bilinear map Φ and $h \in \mathfrak{X}$, it is easy to see that

$$k \longmapsto \Phi(h, k)$$

is a continuous linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. Therefore, we can associate to Φ the uniquely determined map

$$A : \mathfrak{X} \ni h \longmapsto \Phi(h, \cdot) \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y}).$$

Notice that the one-to-one correspondence defined in this way is not only an isomorphism, but also an isometry with respect to the operator norms:

$$\begin{aligned} \|A\|_{\mathcal{L}(\mathfrak{X}, \mathcal{L}(\mathfrak{X}, \mathfrak{Y}))} &= \sup_{\|h\| \le 1} \|A(h)\|_{\mathcal{L}(\mathfrak{X}, \mathfrak{Y})} = \\ &= \sup_{\|h\| \le 1} \sup_{\|k\| \le 1} \|\Phi(h, k)\| = \|\Phi\|_{\mathcal{L}_2(\mathfrak{X}, \mathfrak{Y})}. \end{aligned}$$

□

From now on, we will always identify $\mathrm{d}^2F(u^*)$ with the bilinear map given by the isomorphism described above. Furthermore, if F is differentiable twice in U and

$$F''(u) := \mathrm{d}^2F(u)$$

is continuous, then we will say that F is of class $C^2(U, \mathfrak{Y})$.

Proposition 1.25. *Let $F : U \to \mathfrak{Y}$ be a function that is twice differentiable at some $u \in U$ and set*

$$F_h(u) := \mathrm{d}F(u)[h]$$

for any fixed $h \in \mathfrak{X}$. Then F_h is differentiable at u and

$$\mathrm{d}F_h(u) = \mathrm{d}^2F(u)[h].$$

Proof. Notice that F_h is given by the composition

$$u \xrightarrow{\mathrm{d}F} \mathrm{d}F(u) \xrightarrow{I_h} \mathrm{d}F(u)[h].$$

The conclusion follows from the usual *chain rule* property of the derivative operator. □

Lemma 1.26 (Schwarz). *Let $F : U \to \mathfrak{Y}$ be a function that is twice differentiable at some $u \in U$. Then*

$$F''(u) \in \mathcal{L}_2^s(\mathfrak{X}, \mathfrak{Y}),$$

which means that the second differential is a **symmetric** *bilinear form.*

Proof. Let $h, k \in \mathfrak{X}$ satisfying $\|h\|_{\mathfrak{X}} + \|k\|_{\mathfrak{X}} < \epsilon$ and set

$$\psi(h, k) := F(u+h+k) - F(u+k) - F(u+h) + F(u),$$

$$\gamma_h(\xi) := F(u+h+\xi) - F(u+\xi).$$

Observe that $\psi(h, k) = \gamma_h(k) - \gamma_h(0)$. For h fixed, consider the function

$$g_h : B_\epsilon \subset \mathfrak{X} \longrightarrow \mathfrak{Y}$$

defined as follows:

$$g_h(k) := \psi(h, k) - \mathrm{d}^2F(u)[h, k].$$

Since F is twice differentiable in $u \in U$ and $\mathrm{d}^2F(u)[h, \cdot]$ is linear, we can apply Theorem 1.10 and obtain the following inequality:

$$\|\psi(h, k) - \mathrm{d}^2F(u)[h, k]\|_{\mathfrak{Y}} \leq \|k\|_{\mathfrak{X}} \sup \left\{ \|\mathrm{d}\gamma_h(tk) - \mathrm{d}^2F(u)[h, \cdot]\|_{\mathcal{L}(\mathfrak{X}, \mathfrak{Y})} \,:\, t \in [0, 1] \right\}.$$

We now rewrite $\mathrm{d}\gamma_h(tk)$ as $\mathrm{d}F(u+h+tk) - \mathrm{d}F(u+tk)$. Using the fact that F is twice differentiable at u easily leads to

$$\mathrm{d}F(u+h+tk) - \mathrm{d}F(u+tk) = \mathrm{d}^2F(u)[h+tk] - \mathrm{d}^2F(u)[tk] + o(tk) + o(tk+h).$$

We now plug this into the inequality above and find that

$$\begin{aligned} \|\psi(h, k) - \mathrm{d}^2F(u)[h, k]\|_{\mathfrak{Y}} &\leq \|k\|_{\mathfrak{X}} \sup_{0 \leq t \leq 1} \|o(tk) + o(tk+h)\|_{\mathfrak{X}} \leq \\ &\leq \epsilon(\|k\|_{\mathfrak{X}} + 2\|h\|_{\mathfrak{X}})\|k\|_{\mathfrak{X}}, \end{aligned} \tag{1.17}$$

provided that ϵ (and thus $\|h\|_{\mathfrak{X}} + \|k\|_{\mathfrak{X}}$) is small enough. If we exchange the roles of h and k, we easily find that

$$\|\psi(h, k) - \mathrm{d}^2F(u)[k, h]\|_{\mathfrak{Y}} \leq \epsilon(\|h\|_{\mathfrak{X}} + 2\|k\|_{\mathfrak{X}})\|h\|_{\mathfrak{X}}, \tag{1.18}$$

since ψ is a bilinear symmetric form by definition. Combine (1.17) and (1.18) to get

$$\|d^2F(u)[h, k] - d^2F(u)[k, h]\|_{\mathfrak{Y}} \leq \epsilon(2\|k\|^2_{\mathfrak{X}} + 2\|h\|^2_{\mathfrak{X}} + 2\|h\|_{\mathfrak{X}}\|k\|_{\mathfrak{X}}) \leq$$

$$\leq 3\epsilon(\|k\|^2_{\mathfrak{X}} + \|h\|^2_{\mathfrak{X}})$$

for ϵ small enough. However $d^2F(u)[h, k]$ is homogeneous of degree two so the same inequality holds for all h and k. The arbitrariness of ϵ concludes the proof. □

We can generalise all these notions and introduce the $(n+1)$th derivatives via an inductive process. Let $F : U \to \mathfrak{Y}$ be a n-times differentiable function in $u \in U$ and recall that

$$F^{(n)}(u) := d^nF(u) \in \mathcal{L}_n(\mathfrak{X}, \mathfrak{Y})$$

via the identification with multilinear maps. The $(n + 1)$th differential at u can be defined as the differential of $F^{(n)}$; namely

$$d^{n+1}F(u) := dF^{(n)}(u) \in \mathcal{L}_{n+1}(\mathfrak{X}, \mathfrak{Y}).$$

We will say that F is of class $C^n(U, \mathfrak{Y})$ if F is n times differentiable in U and the nth derivative is continuous.

Lemma 1.27. *Let $F : U \to \mathfrak{Y}$ be a function that is n times differentiable in U. Then*

$$(h_1, \ldots, h_n) \longmapsto d^nF(u)[h_1, \ldots, h_n]$$

is a **symmetric** *multilinear form.*

1.4.1 Partial derivatives and Taylor's formula

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, fix $(u^*, v^*) \in \mathfrak{X} \times \mathfrak{Y}$ and consider the evaluation mappings:

$$\sigma_{v^*}(u) := (u, v^*) \quad \text{and} \quad \tau_{u^*}(v) := (u^*, v).$$

The derivative of σ_{v^*} and τ_{u^*} are easy to compute explicitly:

$$\sigma := d\sigma_{v^*} : h \longmapsto (h, 0),$$

$$\tau := d\tau_{u^*} : k \longmapsto (0, k).$$

Definition 1.28. Let $Q \subset \mathfrak{X} \times \mathfrak{Y}$ be an open set and $(u^*, v^*) \in \mathfrak{X} \times \mathfrak{Y}$. We say that a function $F : Q \longrightarrow \mathfrak{Z}$ is differentiable at the point (u^*, v^*) with respect to u if

$$F \circ \sigma_{v^*}$$

is differentiable at u^*. The linear map

$$d_uF(u^*, v^*) := d(F \circ \sigma_{v^*})(u^*)$$

is the partial derivative of F with respect to u.

Definition 1.29. Let $Q \subset \mathfrak{X} \times \mathfrak{Y}$ be an open set and $(u^*, v^*) \in \mathfrak{X} \times \mathfrak{Y}$. We say that a function $F : Q \longrightarrow \mathfrak{Z}$ is differentiable at the point (u^*, v^*) with respect to v if

$$F \circ \tau_{u^*}$$

is differentiable at v^*. The linear map

$$\mathrm{d}_v F(u^*, v^*) := \mathrm{d}(F \circ \tau_{u^*})(v^*)$$

is the partial derivative of F with respect to v.

Proposition 1.30. *Let $F : Q \to \mathfrak{Z}$ be a differentiable map at the point $(u^*, v^*) \in Q$. Then F has partial derivatives with respect to u and v given by*

$$\mathrm{d}_u F(u^*, v^*)[h] = \mathrm{d}F(u^*, v^*)[\sigma(h)],$$

$$\mathrm{d}_v F(u^*, v^*)[k] = \mathrm{d}F(u^*, v^*)[\tau(k)].$$

In a similar fashion one can define higher partial derivatives. For example, if F has u-partial derivative at all $(u, v) \in Q$, we can define the map

$$F_u(u, v) := \mathrm{d}_u F(u, v).$$

Then the partial derivative $\mathrm{d}_{u,v} F(u^*, v^*)$ is the v-derivative of the map F_u, namely

$$F_{u,v}(u^*, v^*) := \mathrm{d}_v F_u(u^*, v^*).$$

Theorem 1.31. *Suppose that $F : Q \to \mathfrak{Z}$ has both partial derivatives in a neighbourhood of $(u^*, v^*) \in Q$ which are continuous at (u^*, v^*). Then F is differentiable at (u^*, v^*).*

Remark 1.32. The statement of Lemma 1.26 can be easily generalised. Indeed, a straightforward computation shows that

$$\begin{aligned} \mathrm{d}_{u,v} F(u^*, v^*)[h, k] &= \mathrm{d}^2 F(u^*, v^*)[\sigma h, \tau k] = \\ &= \mathrm{d}^2 F(u^*, v^*)[\tau k, \sigma h] = \\ &= \mathrm{d}_{u,v} F(u^*, v^*)[k, h], \end{aligned}$$

which ultimately means that we can swap the order of the partial derivatives.

1.4.2 Taylor's formula

Suppose that $F \in C^n(U, \mathfrak{Y})$, $[u, v] \subset U$ and set $\gamma(t) := u + tv$. Define

$$\Phi(t) := F(\gamma(t)).$$

Then Φ is a real-valued function defined on $[0, 1]$, whose nth derivative is given by

$$\Phi^{(n)}(t) = \mathrm{d}^{(n)} F(\gamma(t))[v, \ldots, v].$$

On the other hand, by Peano's formula we find that

$$\Phi(1) = \Phi(0) + \cdots + \frac{1}{(n-1)!}\Phi^{(n)}(0) + \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\Phi^{(n)}(t)\,\mathrm{d}t.$$

If we plug the expression for $\Phi^{(n)}(t)$ into this identity, we obtain the Taylor's formula in the more general contest of Fréchet differentiability.

Chapter 2

Local Inversion Theorems

In this chapter, we continue with our research toward the extension of differential calculus to the abstract framework of Banach spaces. Recall that, for a function

$$F : \mathbb{R}^n \to \mathbb{R}^n,$$

being continuously differentiable with total derivative invertible at a point p (i.e., the Jacobian determinant of F at p is nonzero) is enough to infer that F is *locally invertible*. The first part of this chapter is devoted to proving the same statement replacing $\mathbb{R}^n$ with Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$. More precisely, we have:

Theorem A. Let $F \in C^1(\mathfrak{X}, \mathfrak{Y})$ with $F'(u^*) \in \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$. Then F is locally invertible at u^* with C^1 inverse. Namely, there are neighbourhoods U of u^* and V of $F(u^*)$ such that

(i) The restriction $F\big|_U : U \longrightarrow V$ is a homomorphism.

(ii) The inverse F^{-1} belongs to $C^1(V, \mathfrak{X})$ and for all $v \in V$ there results

$$\mathrm{d}F^{-1}(v) := (F'(u))^{-1},$$

where $u = F^{-1}(v)$.

(iii) If F belongs to $C^k(\mathfrak{X}, \mathfrak{Y})$, $k > 1$, then $F^{-1} \in C^k(\mathfrak{X}, \mathfrak{Y})$.

In the second half of the chapter, we generalise a well-known result in Euclidean calculus: the *implicit function theorem*. The following statement holds:

Theorem B. Let $F \in C^k(\Lambda \times U, \mathfrak{Y})$, $k \geq 1$. Suppose that

$$F(\lambda^*, u^*) = 0 \text{ and that } F_u(\lambda^*, u^*) \text{ is invertible.}$$

Then there are neighbourhoods Θ of λ^* and U^* of u^* and a map $g \in C^k(\Theta, \mathfrak{X})$ such that:

(i) For all $\lambda \in \Theta$ there results $F(\lambda, g(\lambda)) = 0$.

(ii) If $(\lambda, u) \in \Theta \times U^*$ is such that $F(\lambda, u) = 0$, then $u = g(\lambda)$.

(iii) If $\lambda \in \Theta$ and $p = (\lambda, g(\lambda))$, then

$$g'(\lambda) = -[F_u(p)]^{-1} \circ F_\lambda(p).$$

2.1 Local inversion theorem

In this section, we will always consider continuous maps from a Banach space $\mathfrak{X}$ to another Banach space $\mathfrak{Y}$. With little effort, everything adapts to the case in which an open subset replaces the whole space $\mathfrak{X}$.

Definition 2.1 (Inverse)**.** Let $A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be a continuous linear operator. We say that A is *invertible* if there exists $B \in \mathcal{L}(\mathfrak{Y}, \mathfrak{X})$ such that

$$B \circ A = \mathrm{Id}_{\mathfrak{X}} \quad \text{and} \quad A \circ B = \mathrm{Id}_{\mathfrak{Y}}.$$

The map B is unique and we will denote it, from now on, by A^{-1}. The set of all invertible continuous linear maps is denoted by

$$\mathrm{Inv}(\mathfrak{X}, \mathfrak{Y}) := \{A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y}) \;:\; A \text{ is invertible}\}.$$

Theorem 2.2 (Closed Graph)**.** *A linear operator T between two Banach spaces (even Fréchet spaces) is continuous if and only if it has closed graph $\mathcal{G}(T)$, where*

$$\mathcal{G}(T) = \{(x, y) \;:\; x \in \mathfrak{X},\, y = T(x)\}.$$

Corollary 2.3. *Let $A \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be an injective operator. If A has* range *(=image) equal to $\mathfrak{Y}$, then $A \in \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$.*

The next result is standard in functional analysis. It asserts that the set $\mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$ is open with respect to the operator norm.

Proposition 2.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then the following hold:*

(i) *Let $A \in \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$. Then $T \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ which satisfies*

$$\|T - A\|_{\mathcal{L}(\mathfrak{X}, \mathfrak{Y})} < \frac{1}{\|A^{-1}\|_{\mathcal{L}(\mathfrak{X}, \mathfrak{Y})}} \tag{2.1}$$

also belongs to $\mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$.

(ii) *The map $J : \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y}) \to \mathcal{L}(\mathfrak{Y}, \mathfrak{X})$, $A \mapsto A^{-1}$, is smooth.*

Proof. It follows by standard arguments; see [3]. □

The continuity of J' is easy to deduce. Indeed, we know that J is differentiable and its differential is given by

$$\mathrm{d}J(A)[B] = -A^{-1} \circ B \circ A^{-1},$$

which is a composition of continuous maps.

Definition 2.5 (Homomorphism)**.** Let U and V be open subsets of $\mathfrak{X}$ and $\mathfrak{Y}$. A map $F : U \to V$ is a *homomorphism* if there exists $G : V \to U$ such that

$$G \circ F(u) = u \quad \text{and} \quad F \circ G(v) = v$$

for all $u \in U$ and all $v \in V$. We denote by $\mathrm{Hom}(U, V)$ the set of all homomorphisms between U and V.

Definition 2.6. A continuous map $F \in C(\mathfrak{X}, \mathfrak{Y})$ is *locally invertible* at $u^* \in \mathfrak{X}$ if there exist neighbourhoods U of u^* and V of $F(u^*)$ such that

$$F \in \mathrm{Hom}(U, V).$$

The map $G : V \to U$ is called *local inverse* of F and it will be denoted by F^{-1}.

Proposition 2.7. *The following properties of local invertibility holds:*

(a) If $F_1 \in C(\mathfrak{X}, \mathfrak{Y})$ is locally invertible at u and $F_2 \in C(\mathfrak{Y}, \mathfrak{Z})$ is locally invertible at $F_1(u)$, then $F_2 \circ F_1$ is locally invertible at u.

(b) If F is locally invertible at u, then it is locally invertible at any point in a small neighbourhood of u.

The proof of these two properties is left to the reader to get acquainted with these new notions. Before we can deal with the main result of this section, a remark is in order.

Remark 2.8. Suppose that F is a locally invertible map at u^* with inverse G. If F is differentiable at u^* and G at $v^* := F(u^*)$, then

$$F \circ G = \mathrm{Id}_{\mathfrak{Y}}, \quad G \circ F = \mathrm{Id}_{\mathfrak{X}}$$

immediately implies that $\mathrm{d}F(u^*)$ is invertible with inverse $\mathrm{d}G(v^*)$.

Theorem 2.9 (Local Inverse)**.** *Let $F \in C^1(\mathfrak{X}, \mathfrak{Y})$ with $F'(u^*) \in \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$. Then F is locally invertible at u^* with a C^1 inverse. Namely, there are neighbourhoods U of u^* and V of $F(u^*) =: v^*$ satisfying the following properties:*

(i) *The restriction $F\big|_U : U \to V$ is a homomorphism.*

(ii) *The inverse F^{-1} belongs to $C^1(V, \mathfrak{X})$ and for all $v \in V$ there results*

$$\mathrm{d}F^{-1}(v) := (F'(u))^{-1},$$

where $u = F^{-1}(v)$.

(iii) *If, in addition, F belongs to $C^k(\mathfrak{X},\, \mathfrak{Y})$, $k > 1$, then $F^{-1} \in C^k(\mathfrak{X},\, \mathfrak{Y})$.*

Proof. We can always assume (compose with translations) that $u^* = F(u^*) = 0$. Moreover, according to the transitivity property, we can discuss the local invertibility of the function

$$A \circ F,$$

where A is a linear continuous invertible map. We can choose $A := [F'(0)]^{-1}$ so that it will be enough to prove the theorem for functions of the form

$$F = \mathrm{Id}_{\mathfrak{X}} + \Psi,$$

where $\Psi \in C^1(\mathfrak{X},\, \mathfrak{X})$ and $\Psi'(0) = 0$.

Step 1. Since Ψ' is continuous, we can choose $r > 0$ such that

$$\|p\|_{\mathfrak{X}} < r \implies \|\Psi'(p)\|_{\mathfrak{X}} < \frac{1}{2}.$$

It follows from (1.6) that

$$\|\Psi(p) - \Psi(q)\|_{\mathfrak{X}} \leq \sup\{\|\Psi'(w)\| \,:\, w \in [p,\, q]\}\|p - q\|_{\mathfrak{X}} \leq$$

$$\leq \frac{1}{2}\|p - q\|_{\mathfrak{X}},$$

which means that Ψ is a contraction and $\|\Psi(p)\|_{\mathfrak{X}} \leq \frac{1}{2}\|p\|_{\mathfrak{X}}$ for all $p \in B_{\mathfrak{X}}(0,\, r)$.

Step 2. Fix $v \in \mathfrak{X}$ and define the function

$$\Phi_v(u) := v - \Psi(u).$$

It is not hard to see that Φ_v is a contraction and, for all $u \in B_{\mathfrak{X}}(0,\, r)$ and all $v \in B_{\mathfrak{X}}(0,\, \frac{r}{2})$, it turns out that

$$\|\Phi_v(u)\|_{\mathfrak{X}} \leq \|v\|_{\mathfrak{X}} + \|\Psi(u)\|_{\mathfrak{X}} \leq r.$$

In particular, for $\|v\|_{\mathfrak{X}} \leq \frac{r}{2}$, the map Φ_v is a contraction which also maps $B_{\mathfrak{X}}(0,\, r)$ into itself. Thus it has a unique fixed point $u \in B_r$ that satisfies the equation

$$u = v - \Psi(u).$$

We can easily define a local inverse as

$$F^{-1} : B_{\mathfrak{X}}(0,\, \frac{r}{2}) \longrightarrow B_{\mathfrak{X}}(0,\, r)$$

by setting $F^{-1}(v) = u$. To prove that F^{-1} is continuous, let $u = F^{-1}(v)$ and $w = F^{-1}(z)$, and notice that these are given by

$$\begin{cases} u + \Psi(u) = v, \\ w + \Psi(w) = z. \end{cases}$$

It is now easy to estimate the norm of $u - w$ since

$$\|u - w\|_{\mathfrak{X}} \leq \|v - z\|_{\mathfrak{X}} + \frac{1}{2}\|u - w\|_{\mathfrak{X}} \implies \|F^{-1}(v) - F^{-1}(z)\|_{\mathfrak{X}} \leq 2\|v - z\|_{\mathfrak{X}},$$

and this means that F^{-1} is not only continuous but, actually, Lipschitz-continuous. In particular, letting V be the ball of radius $\frac{r}{2}$ and $U = B_{\mathfrak{X}}(0, r) \cap F^{-1}(V)$, we finally obtain

$$F\big|_U \in \mathrm{Hom}(U, V).$$

Step 3. Recall that $u = F^{-1}(v)$, where $u + \Psi(u) = v$. It follows that

$$F^{-1}(v) = v - \Psi(F^{-1}(v)).$$

Since $\Psi(u) = o(\|u\|_{\mathfrak{X}})$ and F^{-1} Lipschitz-continuous, we conclude that $\Psi(F^{-1}(v))$ belongs to $o(\|v\|_{\mathfrak{X}})$. This shows that F^{-1} is differentiable at $v = 0$ and

$$\mathrm{d}F^{-1}(0) = \mathrm{Id}_{\mathfrak{X}}.$$

The differential of a translation is the translation itself; hence we can compute the differential of F^{-1} at any point using the relation

$$\mathrm{d}F^{-1}(v) = (F'(u))^{-1}.$$

The continuity follows from the fact that it is equal to the composition of continuous mappings; this is exactly as to say that $F^{-1} \in C^1(\mathfrak{X}, \mathfrak{X})$. □

Remark 2.10. The C^k-regularity for F^{-1} is obtained through an iterated application of the argument above.

N.B. The assumption $F \in C^1(\mathfrak{X}, \mathfrak{Y})$ cannot be removed, but we can drop injectivity if both $\mathfrak{X}$ and $\mathfrak{Y}$ are finite-dimensional spaces.

Example 2.11. Consider the nondecreasing function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(s) = \begin{cases} \frac{1}{n}, & s \in \left[\frac{1}{n} - \frac{1}{4n^2}, \frac{1}{n} + \frac{1}{4n^2}\right], \\ s + \mathcal{O}(s^2) & \text{as } s \to 0. \end{cases}$$

This is a differentiable function with derivative at zero equal to 1, but it is not injective in any neighbourhood of the origin.

On the other hand, in the infinite-dimensional setting we can easily construct an example of $F \notin C^1(\mathfrak{X}, \mathfrak{Y})$ for which the local surjectivity fails.

Example 2.12. Let φ be as above. Let $\mathfrak{X} = \mathfrak{Y} = C^0([-1, 1])$, and consider the map

$$F : \mathfrak{X} \ni u \longmapsto \varphi \circ u \in \mathfrak{Y}.$$

Let $v_n \in \mathfrak{Y}$ be the sequence defined by

$$v_n(t) := \frac{1}{n} + \frac{t}{n^2}.$$

It is easy to verify that $\|v_n\|_\infty \to 0$ and $v_n \notin F(\mathfrak{X})$. Indeed, if we could find a sequence $u_n \in \mathfrak{X}$ such that $F(u_n) = v_n$, then one would find

$$\varphi(u_n(t)) = \frac{1}{n} + \frac{t}{n^2}.$$

But then

$$\begin{cases} \varphi(u_n(t)) > \frac{1}{n} & \text{if } t > 0, \\ \varphi(u_n(t)) < \frac{1}{n} & \text{if } t < 0, \end{cases}$$

and, using the monotonicity of φ, we conclude that

$$u_n(t) \geq \frac{1}{n} + \frac{1}{4n^2}$$

for $t > 0$, and

$$u_n(t) \leq \frac{1}{n} - \frac{1}{4n^2}$$

for $t < 0$. This would mean that u_n is not continuous at $t = 0$, and hence $u_n \notin \mathfrak{X}$: a contradiction.

Remark 2.13. Notice that the F in the previous example is differentiable at $u = 0$ with $F'(0) = \mathrm{Id}_{\mathfrak{X}}$, but it is not of class $C^1(\mathfrak{X}, \mathfrak{Y})$.

2.1.1 Applications to initial-value problems

In this section, we will motivate the need for a local invertibility theorem showing how it can be applied to deal with both ODEs and PDEs analysis.

Example 2.14. We are interested in T-periodic solutions of the following ODE:

$$\begin{cases} \ddot{x}(t) + g(x, \dot{x}) = \epsilon h(t), \\ g \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \, h \in C(\mathbb{R}). \end{cases}$$

The framework will be the minimal one where every term in the ODE is well-defined in a strong sense, that is,

$$\mathfrak{X} := \left\{ x \in C^2(\mathbb{R}, \mathbb{R}) \,:\, x(t+T) = x(t) \text{ for all } t \in \mathbb{R} \right\},$$

$$\mathfrak{Y} := \{ h \in C(\mathbb{R}, \mathbb{R}) \,:\, h(t+T) = h(t) \text{ for all } t \in \mathbb{R} \},$$

and the map

$$F(x(t)) := \ddot{x}(t) + g(x(t), \dot{x}(t)).$$

Assume that $g(0, 0) = 0$ - so the equation has the trivial solution if ϵ is zero -. We wish to apply the local invertibility result (Theorem 2.9) with $u^* = 0$. Since

$$\mathrm{d}F(x(t))[y(t)] = [g_x(x(t), \dot{x}(t))y(t) + g_{\dot{x}}(x(t), \dot{x}(t))\dot{y}(t)] + \ddot{y}(t),$$

we can compute it at $x = 0$,

$$\mathrm{d}F(0)[y(t)] = \ddot{y}(t) + a\dot{y}(t) + by(t),$$

where a and b are determined by the values of the partial derivatives of g at the origin. By *Fredholm theory*, the differential $\mathrm{d}F(0)$ is invertible if and only if

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = 0$$

admits the trivial solution only. In this case, we can find $\epsilon^* > 0$ and $\delta > 0$ such that for all $\epsilon < \epsilon^*$ the initial ODE has a unique solution x with norm $\|x\|_\infty < \delta$.

Example 2.15. Let $\Omega \subseteq \mathbb{R}^n$ be smooth and odd. Consider the boundary-value problem

$$\begin{cases} \Delta u - \lambda u + u^3 = h(x), & \text{if } x \in \Omega, \\ u(x) = 0, & \text{if } x \in \partial\Omega. \end{cases}$$

Let us consider the spaces of Hölder-continuous/continuously differentiable functions

$$\mathfrak{X} := \left\{ u \in C^{2,\alpha}(\bar{\Omega}) \,:\, u\big|_{\partial\Omega} \equiv 0 \right\}, \quad \mathfrak{Y} := C^{0,\alpha}(\bar{\Omega}),$$

and the map

$$F(u) = \Delta u + \lambda u - u^3.$$

A simple computation shows that

$$\mathrm{d}F(u)[v] = \Delta v + \lambda v - 3u^2 v,$$

which means that the differential at the origin is given by

$$\mathrm{d}F(0)[v] = \Delta v + \lambda v.$$

If $\lambda \neq \lambda_k(\Delta)$ for all $k \in \mathbb{N}$, where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of the Δ operator on Ω, then $F'(0)$ is one-to-one from $\mathfrak{X}$ to $\mathfrak{Y}$. Consequently, the inverse $[F'(0)]^{-1}$ exists, is continuous and for each $h \in \mathfrak{Y}$ with $\|h\|_{\mathfrak{Y}} < \delta$, there exists a unique solution $u \in \mathfrak{X}$, small in norm, to the boundary problem.

Remark 2.16. In general, if u is a solution of

$$\begin{cases} \Delta u - \lambda u = h(x), & \text{if } x \in \Omega, \\ u(x) = 0, & \text{if } x \in \partial\Omega, \end{cases}$$

for $h \in C^{0,\alpha}(\Omega)$, then there exists a constant $C(n, \Omega)$ such that

$$\|u\|_{2,\alpha} \leq C(n, \Omega)(\|h\|_{0,\alpha} + \|u\|_\infty). \tag{2.2}$$

Removing the second term in the right-hand side is, morally, what happens when we apply the local invertibility theorem.

Example 2.17. Let $\Omega \subseteq \mathbb{R}^2$ be smooth, bounded and connected. Let γ be a smooth function on $\partial\Omega$ taking values in $\mathbb{R}$. If u is a smooth solution of

$$\begin{cases} \mathcal{M}(u) := Au_{xx} + Bu_{yy} - 2u_x u_y u_{xy} = 0, \\ u\big|_{\partial\Omega} \equiv \gamma, \end{cases}$$

where $A = (1+u_y^2)$ and $B = (1+u_x^2)$, then we say that u is a minimal surface with boundary γ. Let us consider the spaces of Hölder-continuous functions

$$\mathfrak{X} := C^{2,\alpha}(\bar{\Omega}), \quad \mathfrak{Y} := C(\bar{\Omega}) \times C^{2,\alpha}(\partial\Omega),$$

and the map

$$F(u) := (\mathcal{M}(u), u\big|_{\partial\Omega}).$$

It is easy to see that F is C^1 and its differential is given by

$$\mathrm{d}F(u)[v] = Av_{xx} + Bv_{yy} - u(\dots),$$

which immediately leads to

$$\mathrm{d}F(0)[v] = (\Delta v, v\big|_{\partial\Omega}).$$

By elliptic regularity theory, the Dirichlet problem

$$\begin{cases} \Delta v = h(x), & \text{if } x \in \Omega, \\ v(x) = \varphi(x), & \text{if } x \in \partial\Omega, \end{cases}$$

admits a unique solution, provided that $(h, \varphi) \in \mathfrak{Y}$; in this case, v depends continuously upon the initial data. Finally, a simple application of Theorem 2.9 shows that there are neighbourhoods U and V of $\mathfrak{X}$ and $C^{2,\alpha}(\partial\Omega)$ respectively such that

$$\gamma \in V \implies \text{the system has a unique solution } u \in U,$$

and the correspondence $\gamma \mapsto u$ is C^1.

2.2 The implicit function theorem

We will now show that we can extend the range of applicability of the local inversion theorem by merely adding an extra parameter. Namely, consider a map

$$F : \Lambda \times U \longrightarrow \mathfrak{Y},$$

where $U \subset \mathfrak{X}$ and Λ, the set of parameters, is a subset of a Banach space $\mathfrak{T}$.

Lemma 2.18. *Let $(\lambda^*, u^*) \in \Lambda \times U$. Suppose that the following properties hold:*

- ***(i)*** *The function F is continuous and has u-partial derivative F_u in $\Lambda \times U$, which is also continuous.*
- ***(ii)*** *The linear operator $F_u(\lambda^*, u^*)$ is invertible.*

Then the map $\Psi : \Lambda \times U \to \mathfrak{T} \times \mathfrak{Y}$ given by

$$\Psi(\lambda, u) = (\lambda, F(\lambda, u)) \tag{2.3}$$

is locally invertible at (λ^*, u^*) with a continuous inverse Φ. Moreover, Φ belongs to C^1 if $F \in C^1(\Lambda \times U, \mathfrak{Y})$.

The local invertibility of Ψ follows, essentially, from the same proof given for Theorem 2.9 so we will not repeat it.

Proof. Suppose that $F \in C^1(\Lambda \times U, \mathfrak{Y})$ and let

$$A := F_\lambda(\lambda^*, u^*) \quad \text{and} \quad B := F_u(\lambda^*, u^*)$$

be its partial derivatives. Clearly, the map Ψ belongs to C^1 and its derivative is explicitly given by the following formula:

$$\mathrm{d}\Psi(\lambda^*, u^*)[\xi, v] = (\xi, A\xi + Bv).$$

The equation

$$\mathrm{d}\Psi(\lambda^*, u^*)[\xi, v] = (\eta, \upsilon)$$

yields $\eta = \xi$ and, since B is invertible, we also infer that

$$A[\eta] + B[v] = \upsilon$$

has a unique solution, which we denote by $v = B^{-1}(\upsilon - A\eta)$. It follows that $\mathrm{d}\Psi(\lambda^*, u^*)$ is invertible and an application of Theorem 2.9 shows that Φ is also C^1. □

Remark 2.19. The function Ψ has an inverse Φ in a neighbourhood $\Theta \times V$ of $(\lambda^*, F(\lambda^*, u^*))$, which is given by

$$\Phi(\lambda, \upsilon) = (\lambda, \varphi(\lambda, \upsilon)). \tag{2.4}$$

The function $\varphi : \Theta \times V \to \mathfrak{X}$ is determined as the unique solution to the following equation:

$$F(\lambda,\, \varphi(\lambda,\, v)) = v \quad \text{for all } \lambda \in \Theta. \tag{2.5}$$

Therefore φ is of class C^1 and its partial derivative can be easily found by differentiating the identity above:

$$\begin{aligned} F_\lambda + F_u \circ \varphi_\lambda &= 0, \\ F_u \circ \varphi_v &= \mathrm{Id}, \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \varphi_\lambda &= -[F_u]^{-1} F_\lambda, \\ \varphi_v &= [F_u]^{-1}. \end{aligned}$$

Remark 2.20. The existence of a local inverse Φ of Ψ can be obtained in a more general setting, requiring that $\mathfrak{T}$ is only a topological space rather than a Banach space.

Theorem 2.21 (Implicit Function)**.** *Let $F \in C^k(\Lambda \times U,\, \mathfrak{Y})$, $k \geq 1$, where Λ is a set of parameters. Suppose that $F(\lambda^*,\, u^*) = 0$ and $F_u(\lambda^*,\, u^*)$ invertible. Then there exist neighbourhoods Θ of λ^* and U^* of u^* and a map $g \in C^k(\Theta,\, \mathfrak{X})$ such that:*

(i) *For all $\lambda \in \Theta$ there results $F(\lambda,\, g(\lambda)) = 0$.*

(ii) *If $(\lambda,\, u) \in \Theta \times U^*$ is such that $F(\lambda,\, u) = 0$, then $u = g(\lambda)$.*

(iii) *If $\lambda \in \Theta$ and $p = (\lambda,\, g(\lambda))$, then*

$$g'(\lambda) = -[F_u(p)]^{-1} \circ F_\lambda(p).$$

Proof. Let Ψ be the function defined by (2.3). Then Ψ is locally invertible at $(\lambda^*,\, u^*)$ and satisfies

$$\Psi(\lambda^*,\, u^*) = (\lambda^*,\, F(\lambda^*,\, u^*)) = (\lambda^*,\, 0).$$

The local inverse Φ satisfies (2.4) and it is rather easy to verify that φ is of class C^k, provided that F is C^k. Setting

$$g(\lambda) := \varphi(\lambda,\, 0),$$

using (2.5) we are able to conclude that

$$F(\lambda,\, g(\lambda)) = F(\lambda,\, \varphi(\lambda,\, 0)) = 0 \quad \text{for all } \lambda \in \Theta.$$

This concludes the proof of (**i**). Now the assertion (**ii**) follows from the fact that Φ is one-to-one and (**iii**) has been proved already in the previous remark. □

2.2.1 Application to perturbed differential systems

Let $f \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n,\, \mathbb{R}^n)$ be a period solution (with respect to the middle variable), that is, there exists a positive time T such that

$$f(\epsilon,\, t + T,\, x) = f(\epsilon,\, t,\, x).$$

Our goal is to investigate period solutions of the ϵ-perturbed differential system

$$\dot{x}(t) = f(\epsilon,\, t,\, x), \tag{2.6}$$

which satisfies the following additional assumption: for $\epsilon = 0$ there exists a T-periodic solution, which we will denote by $y(t)$. Consider the Cauchy problem

$$\begin{cases} \dot{\alpha}(t) = f(\epsilon,\, t,\, \alpha), \\ \alpha(0) = \xi. \end{cases}$$

Since f is differentiable, by Cauchy-Lipschitz theory we can always find a unique solution α which is defined in a small neighbourhood of the initial value, that is, $|\xi - \xi^*| < \delta$ with $\xi^* = y(0)$. Moreover, we know that

$$A(\epsilon,\, t,\, \xi) := \frac{\partial \alpha}{\partial \xi}$$

is the $n \times n$ matrix solving the Cauchy problem

$$\begin{cases} \dot{A} = f_x(\epsilon,\, t,\, \alpha)A, \\ A(\epsilon,\, 0,\, \xi) = \mathrm{Id}_{\mathbb{R}^n}. \end{cases}$$

In what follows, we shall always denote by $A_0(t)$ the matrix $A(0,\, t,\, \xi^*)$.

Theorem 2.22. *Under these assumptions, if $\lambda = 1$ is not in the spectrum of $A_0(t)$, then there are $\delta > 0$ and $\xi \in C^1((-\delta,\, \delta))$, $\xi(0) = \xi^*$ such that*

$$|\epsilon| < \delta \implies \textit{there exists a unique } T\textit{-periodic solution of } (2.6)_\epsilon.$$

Proof. The Cauchy problem $(2.6)_\epsilon$ has a T-periodic solution if and only if there exists $\xi \in \mathbb{R}^n$ such that

$$\alpha(\epsilon,\, T,\, \xi) = \xi.$$

Thus, introducing the map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(\epsilon,\, \xi) := \alpha(\epsilon,\, T,\, \xi) - \xi,$$

we are led to solve the equation $F(\epsilon,\, \xi) = 0$. The function F is C^1 and, since $\alpha(0,\, t,\, \xi^*) = y(t)$ and y is T-periodic, it turns out that

$$F(0,\, \xi^*) = \alpha(0,\, T,\, \xi^*) - \xi^* = y(T) - \xi^* = 0.$$

We conclude applying Theorem 2.21 since

$$F_\xi(0,\, \xi^*) = \alpha_\xi(0,\, T,\, \xi^*) - \mathrm{Id} = A_0(t) - \mathrm{Id},$$

and the right-hand side is invertible because 1 is not in the spectrum of $A_0(t)$ by assumption.

□

The autonomous case (in which f does not depend on t directly) is more delicate and requires to work slightly more. Consider the system

$$\dot{x}(t) = f(\epsilon, x), \tag{2.7}$$

and notice that the period of a solution of (2.7) is, a priori, unknown. Let $f(x) := f(0, x)$ and assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following property:

$$\epsilon = 0 \implies \text{(2.7) has a nonconstant } T\text{-periodic solution } y = y(t).$$

Without loss of generality we can assume $y(0) = 0$.

Remark 2.23. It is important to notice that the previous theorem does not apply here because 1 always belongs to the spectrum of $A_0(T)$. Indeed, A_0 satisfies

$$\begin{cases} \dot{A}_0 = f'(y(t))A_0, \\ A_0(0) = \mathrm{Id}_{\mathbb{R}^n}. \end{cases}$$

To see this, we differentiate the relation $y' = f(y)$ and find that

$$y''(t) = f'(y)y',$$

and therefore, by setting $v := y'$, we have $v(t) \neq 0$ for all t and

$$v' = f'(y)v.$$

Let $v^* = v(0)$ and $w(t) = A_0(t)v^*$. It follows that

$$\begin{cases} w' = \dot{A}_0 v^* = f'(y(t))A_0(t)v^* = f'(y)w, \\ w(0) = v^*. \end{cases}$$

By the uniqueness of the Cauchy problem, it must be that $v(t) = w(t)$. In particular, there results $w(T) = w(0)$ and hence

$$A_0(T)v^* = w(0) = v^* \implies 1 \in \sigma(A_0(T)).$$

Theorem 2.24. *Under these assumptions, if $\lambda = 1$ is a simple eigenvalue for $A_0(T)$, then there are continuous maps $h = h(\epsilon)$ and $\tau = \tau(\epsilon)$ such that*

$$h(0) = y(0), \quad \tau(0) = T,$$

and (2.7) has a $\tau(\epsilon)$-periodic solution y_ϵ satisfying $y_\epsilon(0) = h(\epsilon)$.

Chapter 3

Global Inversion Theorems

The goal of this chapter is to find assumptions that allow us to extend the local inversion theorem to the whole space. The main result states that this is possible provided that we remove from domain and codomain the *singular points*:

Theorem A. Let $F : M \to N$ be a proper map. Suppose that N_0 is simply connected and M_0 is arc-wise connected. Then F is a global homeomorphism between M_0 and N_0.

In the second half of the chapter, we show how can we apply this result to PDEs analysis to determine the existence of solutions to some Dirichlet problems. We conclude with the statement of the global inversion theorem that takes into account singularities.

Theorem B. Let $F \in C^2(\mathfrak{X}, \mathfrak{Y})$ be a proper function and suppose that every $u \in \Sigma'$ is an ordinary singular point, the equation

$$F(u) = v$$

has a unique solution for all $v \in F(\Sigma')$, and Σ' connected. Then there exist two open connected subsets $\mathfrak{Y}_0$ and $\mathfrak{Y}_2$ such that

$$\mathfrak{Y} = \mathfrak{Y}_0 \cup \mathfrak{Y}_2 \cup F(\Sigma'),$$

and it turns out that

$$[v] = \begin{cases} 0 & \text{if } v \in \mathfrak{Y}_0, \\ 1 & \text{if } v \in F(\Sigma'), \\ 2 & \text{if } v \in \mathfrak{Y}_2. \end{cases}$$

3.1 The global inversion theorem

The goal of this section is to investigate minimal conditions under which a map F between metric spaces, M and N, is a global homeomorphism.

Definition 3.1 (Proper)**.** A continuous map $F : M \to N$ between metric spaces is *proper* if the preimage

$$F^{-1}(K) = \{u \in M \ : \ F(u) \in K\}$$

of a compact set is also compact.

From now on, when we say that $F : M \to N$ is proper, we will also assume that F is continuous with respect to the topology spaces (M, d_M) and (N, d_N).

Lemma 3.2. *Let $F : X \to Y$ be a proper map between topological spaces and let Y be locally compact and Hausdorff. Then F is a closed map.*

Proof. Let C be a closed subset of X. We will show that $Y \setminus F(C)$ is open. For this, let $y \in Y \setminus F(C)$ and take an open neighbourhood $V \ni y$ with compact closure. Then

$$F \text{ proper} \implies F^{-1}(\bar{V}) \text{ compact in X.}$$

Let $E = C \cap F^{-1}(\bar{V})$. Then E is compact and by continuity so is $F(E)$. Since Y is Hausdorff, $F(E)$ is closed. Now consider

$$U = V \setminus F(E).$$

Then U is an open neighbourhood of y which is disjoint from $F(C)$, and this proves that $F(C)$ is closed. ☐

Theorem 3.3. *Let $F : M \to N$ be a proper locally invertible map. Then*

$$N \ni v \longmapsto [v] := \#F^{-1}(\{v\})$$

is finite and locally constant.

Proof. The singlet $\{v\}$ is compact so its preimage via F is also compact. Since $F^{-1}(\{v\})$ must be discrete by the local invertibility theorem, we conclude that

$$F^{-1}(\{v\}) \subset M \text{ is discrete and compact,}$$

which is possible if and only if it is finite. To show that the map is locally constant, fix $v \in N$ and denote by $\{u_1, \ldots, u_n\}$ the preimage $F^{-1}(v)$. By the local invertibility theorem we can find open neighbourhoods $U_i \ni u_i$ in M and V neighbourhood of v in N such that

$$F \in \operatorname{Hom}(U_i, V) \quad \text{for all } i = 1, \ldots, n.$$

It follows that

$$[w] \geq k \quad \text{for all } w \in V.$$

We now claim that there exists an open neighbourhood $W \subset V$ of v such that $[w]$ is identically equal to k at all $w \in W$. We argue by contradiction. If W does not exist, then we can find

$$\{v_j\}_{j \in \mathbb{N}} \subset N \quad \text{and} \quad v_j \xrightarrow{j \to +\infty} v$$

and a corresponding sequence of points $p_j \in M$ such that

$$p_j \notin \bigcup_{i=1}^{n} U_i \quad \text{and} \quad F(p_j) = v_j.$$

Since F is proper, we can find a subsequence j_k such that p_{j_k} converges to some p that does not belong to $\bigcup_{i=1}^{n} U_i$. The continuity of F proves the contradiction since

$$F(p_{j_k}) \xrightarrow{k \to +\infty} F(p) = v.$$

□

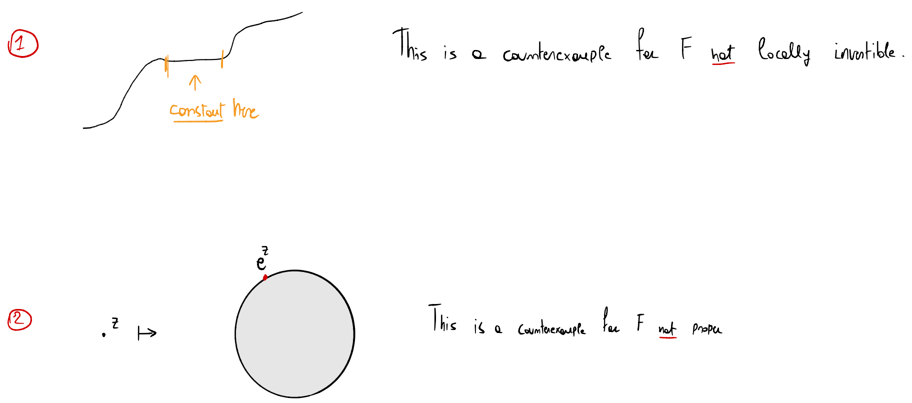


Figure 3.1: Counterexamples to $[v]$ finite and locally constant.

Corollary 3.4. *Let $F : M \to N$ be a proper locally invertible map. If N is connected, then $[v]$ is globally constant.*

Definition 3.5 (Singular)**.** A point $u \in M$ is said to be *singular* for F if F is not locally invertible at u and *regular* if it is not singular.

Denote by Σ the set of all singular points in M and Σ_0 the preimage $F^{-1}(F(\Sigma))$. We would like to work with regular points only, so we define $M_0 := M \setminus \Sigma_0$ and $N_0 := N \setminus F(\Sigma)$.

Remark 3.6. The set Σ is closed, so both M_0 and N_0 are open in M and N respectively.

An obvious consequence of the definitions of singular points and (M_0, N_0) is the following theorem, which asserts that $[v]$ is constant on connected components of N_0.

Theorem 3.7. *Let $F : M \to N$ be a proper map. Then $[v]$ is constant on every connected component of N_0.*

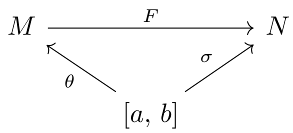
We are now ready to state the main result of this section. The assertion is rather intuitive, but it will take us a considerable effort to prove it formally.

Theorem 3.8. *Let $F : M \to N$ be a proper map. Suppose that N_0 is simply connected and M_0 is arc-wise connected. Then F is a global homeomorphism between M_0 and N_0.*

Corollary 3.9. *Let $F : M \to N$ be a proper locally invertible map. Suppose that N is simply connected and M_0 is arc-wise connected. Then $F \in \mathrm{Hom}(M, N)$.*

The first step is to introduce and investigate the notion of "path that invert F along another path". Next, we show that this "inverse" is unique and also that everything can be generalised to paths defined on $[a, b]^2$.

Definition 3.10. Let M, N be as above and let $\sigma : [a, b] \to N$ be a continuous path. We say that a path $\theta : [a, b] \to M$ *inverts F along σ* if the following diagram commutes:



Remark 3.11. Let $u \in M$ and $v \in N$ be such that $F(u) = v$ and $F\big|_U \in \mathrm{Hom}(U, V)$, where U and V are respectively neighbourhoods of u and v. Given a path

$$\sigma : [a, b] \longrightarrow N, \quad \sigma(a) = v \quad \text{and} \quad \sigma([a, b]) \subset V,$$

it is easy to see that the equation $F(\theta(t)) = \sigma(t)$ defines the **unique** path θ that inverts F along σ satisfying the initial condition $\theta(a) = u$.

Remark 3.12. Let $\sigma : [a, b] \to N$ be a continuous path and suppose that there exists $c \in (a, b)$ such that θ_1 inverts F along $\sigma\big|_{[a,\,c]}$ and θ_2 along $\sigma\big|_{[c,\,b]}$ with $\theta_1(c) = \theta_2(c)$. Then

$$\theta(t) := \begin{cases} \theta_1(t) & \text{if } t \in [a, c), \\ \theta_2(t) & \text{if } t \in [c, b], \end{cases}$$

is a well-defined continuous path which inverts F along the whole σ.

Lemma 3.13. *Let $u^* \in M_0$ and $v^* = F(u^*) \in N_0$. Then for any given path $\sigma : [0, 1] \to N$ with $\sigma(0) = v^*$ there exists a unique*

$$\theta : [0, 1] \longrightarrow M_0$$

that inverts F along σ satisfying the initial condition $\theta(0) = u^*$.

Proof. We first prove uniqueness, which is relatively easy, and then we exploit it to obtain the existence.

Uniqueness. We argue by contradiction. Let θ_1 and θ_2 be two such paths and let

$$\xi := \sup\left\{ s \in [0,\, 1] \ :\ \theta_1 \big|_{[0,\, s]} \equiv \theta_2 \big|_{[0,\, s]} \right\}.$$

According to Remark 3.11, ξ is well-defined and,, since $u^* \in M_0$, it is also strictly bigger than zero. Moreover, by continuity one has that

$$\theta_1(\xi) = \theta_2(\xi)$$

so all it remains is to prove that $\xi = 1$. Suppose that $\xi < 1$ and set

$$u = \theta_1(\xi) = \theta_2(\xi) \quad \text{and} \quad v = F(u).$$

Since F is locally invertible in M_0, there are neighbourhoods $U \ni u$ and $V \ni v$ such that $F\big|_U \in \mathrm{Hom}(U,\, V)$. Now both paths are continuous so

$$\theta_1([\xi,\, \xi + \alpha]) \subset U \quad \text{and} \quad \theta_2([\xi,\, \xi + \alpha]) \subset U$$

for a small enough $\alpha > 0$. Therefore

$$\theta_1 \big|_{[0,\, \xi+\alpha]} \equiv \theta_2 \big|_{[0,\, \xi+\alpha]},$$

and this is a contradiction with the definition of ξ as the supremum.

Existence. Let Ξ be the set of all $s \in [0,\, 1]$ such that F is invertible along $\sigma\big|_{[0,\, s]}$ with inverse given by

$$\theta_s : [0,\, s] \longrightarrow M_0 \text{ such that } \theta_s(0) = u^*,\ F(u^*) = \sigma(0).$$

We will show that Ξ is both closed and open in $[0,\, 1]$ in such a way it must coincide with $[0,\, 1]$ as it is nonempty.

(a) Let $\xi := \sup \Xi$. As before $\xi > 0$ and, by uniqueness, the resulting paths θ_s must coincide in the intersections of the intervals of definition. Let θ be the function

$$\theta(s) := \theta_s(s) \quad \text{for all } s \in [0,\, \xi).$$

Now let $s_n \nearrow \xi$ be a sequence such that $\sigma(s_n) \to v$. Since $\theta(s_n) = F^{-1}(\sigma(s_n))$ and F is proper, we find that (up to subsequences) we have

$$\theta(s_n) \to u, \quad F(u) = v.$$

Now let $U \ni u$ and $V \ni V$ be neighbourhoods such that $F\big|_U \in \mathrm{Hom}(U,\, V)$. If $m \in \mathbb{N}$ is chosen in such a way that

$$\theta(s_m) \in U \quad \text{and} \quad \sigma([s_m,\, \xi]) \subset V,$$

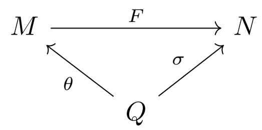
then F can be inverted along $\sigma\big|_{[s_m,\, \xi]}$ by a path θ_1 which coincides with θ evaluated at s_m. Finally, the trick illustrated in Remark 3.12 allows us to conclude that Ξ is closed.

(b) The idea is more or less the same. If $\xi < 1$, the path θ_1 introduced above can be defined in an interval $[s_m, \xi + \alpha]$, $\alpha > 0$, which is absurd.

□

The next step is to pass from paths to 2-paths, namely continuous functions defined on $Q := [a, b]^2$ and taking values in M or N.

Definition 3.14. Let M and N be as above and let $\sigma : Q \to N$ be a 2-path. We say that a 2-path $\theta : Q \to M$ *inverts F along σ* if the following diagram commutes:



Lemma 3.15. *Let $u^* \in M_0$ and $v^* = F(u^*) \in N_0$. Then given any 2-path $\sigma : Q \to N$ such that $\sigma(0, 0) = v^*$, there exists a unique 2-path*

$$\theta : Q \longrightarrow M_0$$

that inverts F along σ satisfying the initial condition $\theta(0, 0) = u^*$.

Proof. We divide into two steps as before, starting from uniqueness which is once again needed to prove existence.

Uniqueness. Let θ_1 and θ_2 be two such 2-paths and let $(s, t) \in Q$. Define $\phi_1, \phi_2 : [0, 1] \to M_0$ and $\psi : [0, 1] \to N_0$ as follows:

$$\phi_1(\lambda) = \theta_1(\lambda s, \lambda t),$$

$$\phi_2(\lambda) = \theta_2(\lambda s, \lambda t),$$

$$\psi(\lambda) = \sigma(\lambda s, \lambda t).$$

Then ϕ_1 and ϕ_2 are paths that invert F along ψ, which means that by the 1-dimensional result they must coincide. Letting $\lambda = 1$ shows that

$$\theta_1(s, t) = \theta_2(s, t),$$

and we conclude using the arbitrariness of $(s, t) \in Q$.

Existence. Consider the rectangle

$$R_s = [0,\, s] \times [0,\, 1] \subset Q,$$

and let Ξ be the set of all $s \in (0,\, 1]$ such that there exists $\theta_s : R_s \to M_0$ that inverts F along the restriction $\sigma\big|_{R_s}$ with $\theta_s(0,\, 0) = u^*$. Clearly, $0 \in \Xi$ since F is invertible along

$$t \longmapsto \sigma(0,\, t)$$

by Lemma 3.13. Let $\xi := \sup \Xi$. As before $\xi > 0$ and, by uniqueness, the resulting 2-paths θ_s must coincide in the intersections of the intervals of definition. Let θ be the function

$$\theta(z,\, t) := \theta_s(z,\, t) \quad \text{for all } (z,\, t) \in R_s.$$

Fix $t \in [0,\, 1]$. Since F is invertible along the path $s \mapsto \sigma(s,\, t)$ with inverse $s \mapsto \phi(s)$ satisfying the initial condition $\phi(0) = \theta(0,\, t)$, by uniqueness we have

$$\phi(z) = \theta(z,\, t) \quad \text{for all } 0 \le z < \xi.$$

If we set $\phi(\xi) = u$ and $\sigma(\xi,\, t) = v$, then we can find neighbourhood $U \ni u$ and $V \ni v$ such that $F\big|_U \in \mathrm{Hom}(U,\, V)$. Then we can find a rectangle R' centered at $(\xi,\, t)$ and

$$\theta' : R' \cap Q \longrightarrow M_0$$

such that θ' inverts F along $\sigma\big|_{R' \cap Q}$ with $\theta'(\xi,\, t) = u$. Since θ and θ' coincide in $(0,\, \xi)$, we infer that θ can be extended to all $R' \cap Q$ and by continuity to R_ξ in such a way that

$$F \circ \theta = \sigma$$

holds at all points of R_ξ. Moreover, $\xi = 1$ for otherwise we could cover the segment $\{(\xi,\, t) \;:\; t \in [0,\, 1]\}$ with a family of rectangles R', which would allow us to extend θ to $R_{\xi+\alpha}$ for some positive α: a contradiction. ☐

Proof of Theorem 3.8. The map $[v]$ is constant and ≥ 1 for all $v \in N_0$, which means that F is onto. We only need to show that

$$[v] = 1 \quad \text{at all } v \in N_0.$$

We argue by contradiction. Suppose that there are $u_0,\, u_1 \in M_0$ such that $F(u_0) = F(u_1) = v$. Since M_0 is arcwise connected, we can always find a continuous path θ such that

$$\theta(0) = u_0 \quad \text{and} \quad \theta(1) = u_1.$$

The image of θ, $\sigma = F \circ \theta$, is a closed path in the simply connected space N_0 and therefore homotopic to a constant path. Namely, there exists a homotopy $h \in C(Q,\, N_0)$ which, without loss of generality, we can require to satisfy

$$h(0,\, t) = h(1,\, t) = v \quad \text{for all } t \in [0,\, 1].$$

From Lemma 3.15 we infer that there exists a unique 2-path $\Theta \in C(Q, M_0)$ that inverts F along h, that is,

$$F(\Theta(s, t)) = h(s, t).$$

In particular, from $F(\Theta(s, 0)) = h(s, 0) = \sigma(s)$, we deduce that $\Theta(s, 0) = \theta(s)$ and hence

$$\Theta(1, 0) = \theta(1) = u_1.$$

On the other hand, from $h(0, t) = h(1, t) = v$, we can deduce that

$$F(\Theta(0, t)) = F(\Theta(s, 1)) = F(\Theta(1, t)) = v.$$

In particular, the restriction of Θ to the set

$$\Gamma = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1])$$

is constant and, in particular, $u_1 = \Theta(1, 0) = \Theta(0, 0) = u_0$. This is in contradiction with $u_0 \neq u_1$ so $[v]$ must be equal to 1 at all $v \in N_0$, which is what we wanted to prove. $\square$

3.1.1 Global invertibility in PDEs analysis

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary and consider the Dirichlet problem

$$\begin{cases} -\Delta u(x) = p(u(x)) + h(x) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let $(\lambda_k)_{k\in\mathbb{N}}$ denote the sequence of eigenvalues of the laplacian $-\Delta$ subject to Dirichlet boundary conditions and enumerate them in such a way that $\lambda_1 \leq \lambda_2 \leq \dots$ and

$$\lim_{k\to\infty} \lambda_k = \infty.$$

Theorem 3.16. *Let $p \in C^1(\mathbb{R}, \mathbb{R})$ be a function of the form*

$$p(s) = as + b(s),$$

where $|b(s)| \leq M$. Suppose that one of the following holds:

(a) *For all $s \in \mathbb{R}$*

$$p'(s) = a + b'(s) < \lambda_1.$$

(b) *There exists $k \in \mathbb{N}$ such that for all $s \in \mathbb{R}$*

$$\lambda_k < p'(s) = a + b'(s) < \lambda_{k+1}.$$

Then for any $h \in C^{\alpha}(\bar{\Omega})$, $\alpha \in (0, 1)$, there exists a unique $u \in C^{2,\alpha}(\bar{\Omega})$ solution of the problem (3.1).

Proof. Let $\mathfrak{X} := \{u \in C^{2,\alpha}(\bar{\Omega}) \,:\, u\big|_{\partial\Omega} \equiv 0\}$ and $\mathfrak{Y} := C^{\alpha}(\bar{\Omega})$. In view of Theorem 3.8, it is sufficient to show that the map

$$F(u) := -\Delta u - p(u)$$

is locally invertible at all $u \in \mathfrak{X}$ and proper.

Step 1. The differential of F at u is the linear map defined by

$$\mathrm{d}F(u)[v] := -\Delta u - p'(u)v,$$

and thus F is locally invertible if and only if

$$-\Delta u - p'(u)v = 0 \iff v = 0.$$

We now consider the bilinear form defined by the differential of F at u, namely

$$b : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y}, \quad b(u,\, v) := -\Delta u - p'(u)v,$$

and we notice that b is continuous, that is,

$$|b(u,\, v)| \leq \|u\|_{\mathfrak{X}} \|v\|_{\mathfrak{X}}.$$

If the assumption **(a)** holds, then it easily the coercivity of the bilinear form b and thus we can apply Lax-Milgram theorem (see Theorem 3.17).

If, on the other hand, the assumption **(b)** holds, then we need to rely on a comparison principle. First, consider the following eigenvalue problems:

$$\begin{cases} -\Delta v(x) - \lambda_k v(x) = \mu v(x) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta v(x) - p'(u) v(x) = \tilde{\mu} v(x) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta v(x) - \lambda_{k+1} v(x) = \hat{\mu} v(x) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

The assumption **(b)** implies that

$$\hat{\mu}_j < \tilde{\mu}_j < \mu_j$$

for all $j \in \mathbb{N}$. However, we can compute these eigenvalues explicitly as

$$\mu_j = \lambda_j - \lambda_k \quad \text{and} \quad \hat{\mu}_j = \lambda_j - \lambda_{k+1},$$

and hence we conclude that

$$\tilde{\mu}_k < 0 \quad \text{and} \quad \tilde{\mu}_{k+1} > 0.$$

This shows that $\tilde{\mu}_j \neq 0$ for all $j \in \mathbb{N}$ and, as an immediate consequence, that F is locally invertible.

Step 2. We now prove that F is proper. Let $h_n \to h$ in $\mathfrak{Y}$ be a convergent sequence and let $(u_n)_{n\in\mathbb{N}} \subset \mathfrak{X}$ be such that

$$F(u_n) = h_n \quad \text{for all } n \in \mathbb{N}.$$

Step 2.1 We claim that $\|u_n\|_{\mathfrak{Y}}$ is bounded. If not, let $v_n := \frac{u_n}{\|u_n\|_{\mathfrak{Y}}}$ and notice that it is well-defined and solves (3.1) with right-hand side

$$h = \frac{h_n}{\|u_n\|_{\mathfrak{Y}}}.$$

In particular, using that $h(s) = as + b(s)$, we find that v_n solves the problem

$$-\Delta v_n + a v_n = U_n,$$

where U_n is uniformly bounded in $L^\infty(\Omega)$ and, consequently, in every L^p-space for $1 \leq p < \infty$. Since the operator given by

$$-\Delta + a \,\mathrm{Id}_{\mathfrak{X}}$$

is invertible, we infer that v_n is bounded in $W^{2,p}(\Omega)$ for all $p \in [1, \infty]$ and, by the Sobolev embedding (see Theorem 1.20), the sequence v_n is also bounded in $C^{1,\beta}(\Omega)$. By Ascoli-Arzelà, if $\beta > \alpha$, then

$$v_n \xrightarrow{n\to+\infty} v^*$$

in $C^{1,\alpha}(\bar{\Omega})$ and $\|v^*\|_{\mathfrak{Y}} = 1$. On the other hand, the sequence U_n tends to zero in $\mathfrak{Y}$ so v^* must also satisfy

$$\begin{cases} -\Delta v^* + a v^* = 0, \\ \|v^*\|_{\mathfrak{Y}} = 1, \end{cases}$$

and this is clearly impossible because the unique solution of this equation is $v^* \equiv 0$, incompatible with the condition $\|v^*\|_{\mathfrak{Y}} = 1$.

Step 2.2. Since

$$\begin{cases} -\Delta u_n(x) = \underbrace{p(u_n(x)) + h_n(x)}_{:=\theta_n} & \text{if } x \in \Omega, \\ u_n(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

and both $(u_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are bounded in $\mathfrak{Y}$, we readily deduce that the sequence of θ_n is bounded in $\mathfrak{Y}$. A well-known result in regularity theory implies that

$$\|u_n\|_{\mathfrak{X}} \le C$$

and by Ascoli-Arzelà we can find a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges to some u^* in the topology $C^2(\bar{\Omega})$. Finally, since θ_{n_k} converges in $\mathfrak{Y}$, the elliptic regularity theory allows us to conclude that u_{n_k} converges to $u \in \mathfrak{X}$. □

To conclude this section, we recall the statement of the Lax-Milgram theorem in a more general form in which we do not require b to be a bilinear form.

Theorem 3.17 (Lax-Milgram)**.** *Let H be a Hilbert space, and let $a : H \times H \to \mathbb{R}$ be a function satisfying the following properties:*

(1) $a(0, v) = 0$ *for all* $v \in H$ *and* $v \mapsto a(u, v)$ *is linear for all* $u \in H$*.*

(2) *For all* $v \in H$ *and all* $(u_1, u_2) \in H \times H$ *it turns out that*

$$|a(u_1, v) - a(u_2, v)| \le M\|u_1 - u_2\|\|v\|.$$

(3) *There exists a constant* $\nu > 0$ *such that*

$$a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \ge \nu\|u_1 - u_2\|^2 \quad \text{for all } (u_1, u_2) \in H \times H.$$

Then for all $F \in H^*$ *there exists a unique element* $u \in H$ *such that*

$$a(u, v) = F(v) \quad \text{for all } v \in H,$$

and there exists a positive constant which only depends on ν *such that*

$$\|u\| \le \frac{1}{c(\nu)}\|F\|_{H^*}.$$

Remark 3.18. If $a : H \times H \to \mathbb{R}$ is a bilinear form, then the condition (**3**) is equivalent to saying that a is *coercive*.

3.2 Global inversion with singularities

In this section, we will study the *global invertibility* of maps when Σ does not satisfy the assumptions of Theorem 3.8. For this it will be convenient to deal with C^2-maps $F : \mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, and replace Σ with a slightly larger set

$$\Sigma' := \{u \in \mathfrak{X} \,:\, F'(u) \notin \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})\}.$$

Let $F \in C^2(\mathfrak{X}, \mathfrak{Y})$ and $u \in \Sigma'$. We assume that the following hold:

(A) The kernel of $F'(u)$ is one-dimensional and generated by $\phi \in \mathfrak{X} \setminus \{0\}$. The range is closed and has codimension one.

(B) There exists $\tilde{\phi} \in \mathfrak{X}$ such that $F''(u)[\tilde{\phi}, \phi] \notin \mathrm{Ran}(F'(u))$.

We say that a subset M of $\mathfrak{X}$ is a C^1-manifold of codimension one in $\mathfrak{X}$ if for all $u^* \in M$ there exist $\delta > 0$ and a functional $\Gamma : B_\delta(u^*) \to \mathbb{R}$ of class C^1 such that

$$M \cap B_\delta(u^*) = \{u \in B_\delta(u^*) \; : \; \Gamma(u) = 0\},$$

and $\Gamma'(u^*) \neq 0$.

Lemma 3.19. *Suppose that for all $u \in \Sigma'$ the conditions (A) and (B) hold. Then Σ' is a C^1-manifold of codimension one in $\mathfrak{X}$.*

Definition 3.20. We say that $u \in \Sigma'$ is an *ordinary* singular point if (A) holds and

$$F''(u)[\phi, \phi] \notin \mathrm{Ran}(F'(u)),$$

where ϕ is the element that generates the kernel (by (A)).

Lemma 3.21. *Let u^* be an ordinary singular point. Then there exist $\epsilon > 0$ and a map $\Psi \in C^1(B_\epsilon(u^*), \mathfrak{Y})$ such that*

(i) $\Psi'(u^*) \in \mathrm{Inv}(\mathfrak{X}, \mathfrak{Y})$*;*

(ii) $\Psi(u) = F(u)$ *for all* $u \in \Sigma' \cap B_\epsilon(u^*)$.

Proof. First, notice that $\Sigma' \cap B_\delta(u^*) = \Gamma^{-1}(0)$. Let $\Psi : B_\delta(u^*) \to \mathfrak{Y}$ be the map defined by setting

$$\Psi(u) := F(u) + \Gamma(u)z.$$

Then Ψ is C^1-regular, $\Psi(u)$ coincides with $F(u)$ for all $u \in \Sigma' \cap B_\delta(u^*)$ and its differential is given by

$$\Psi'(u^*)u = F'(u^*)u + \Gamma'(u^*)(u)z.$$

Setting $u = t\phi + w$, we find that

$$\Psi'(u^*)u = F'(u^*)w + t\Gamma'(u^*)(\phi)z + \Gamma'(u^*)(w)z =$$

$$= F'(u^*)w + t\langle\Psi, F''(u^*)[\phi, \phi]\rangle z + \langle\Psi, F''(u^*)[w, \phi]\rangle z.$$

Finally, observe that $\Psi'(u^*)u = v$ has a unique solution when $\langle\Psi, F''(u^*)[\phi, \phi]\rangle \neq 0$; thus, if u^* is an ordinary singular point, the map $\Psi'(u^*)$ is invertible. □

Corollary 3.22. *If every $u \in \Sigma'$ is an ordinary singular point, then $F(\Sigma')$ is a C^1-manifold of codimension one in $\mathfrak{Y}$.*

Lemma 3.23. *Let u^* be an ordinary singular point with* $\mathrm{Ker}(F'(u^*)) = \mathbb{R}\phi$*. Assume that*

$$\langle\Psi, F''(u^*)[\phi, \phi]\rangle > 0,$$

and set $v^* := F(u^*)$. *Then there are* ϵ, $\sigma > 0$ *such that the equation*

$$F(u) = v^* + sz \quad \text{for } u \in B_\epsilon(u^*),$$

has two solutions for all $0 < s < \sigma$ *and none for* $-\sigma < s < 0$.

Theorem 3.24. *Let* $F \in C^2(\mathfrak{X}, \mathfrak{Y})$ *be a proper function. Assume that every* $u \in \Sigma'$ *is an ordinary singular point, the equation*

$$F(u) = v$$

admits a unique solution for all $v \in F(\Sigma')$, *and* Σ' *is connected. Then there are two open connected subsets* $\mathfrak{Y}_0$ *and* $\mathfrak{Y}_2$ *of* $\mathfrak{Y}$ *such that*

$$\mathfrak{Y} = \mathfrak{Y}_0 \cup \mathfrak{Y}_2 \cup F(\Sigma'),$$

and it turns out that

$$[v] = \begin{cases} 0 & \text{if } v \in \mathfrak{Y}_0, \\ 1 & \text{if } v \in F(\Sigma'), \\ 2 & \text{if } v \in \mathfrak{Y}_2. \end{cases}$$

Part II

Variational Methods

Chapter 4

Critical Points

In this chapter, we will investigate the notion of critical point and we will relate it with extrema and, ultimately, solutions of PDEs problems.

4.1 Existence of extrema

Recall that a functional over a Banach space $\mathfrak{X}$ is a continuous mapping $J : \mathfrak{X} \longrightarrow \mathbb{R}$. We say that $z \in \mathfrak{X}$ is a local minimiser (resp. maximiser) of J if there exists a neighbourhood $U \ni z$ such that

$$J(z) \leq J(u) \quad \text{for all } u \in U \text{ (resp. } J(z) \geq J(u)).$$

If the above inequality is strict (except at $u = z$), then we say that u is a strict local minimum (resp. maximum) of J. Moreover, if it turns out that

$$J(z) \leq J(u) \quad \text{for all } u \in \mathfrak{X} \text{ (resp. } J(z) \geq J(u)),$$

then we say that z is a global minimum (resp. maximum).

Remark 4.1. If z is a local minimum and J is differentiable at z, then it is easy to show that it must be a stationary point, that is,

$$\mathrm{d}J(z) \equiv 0.$$

We will now give an existence result that concerns coercive and weakly lower semi-continuous functionals, but, before we dig into it, let us recall a few notions.

Definition 4.2. A functional $J \in C^0(\mathfrak{X}, R)$ is coercive if

$$\lim_{\|u\| \to +\infty} J(u) = +\infty,$$

and weakly lower semi-continuous if for every sequence $u_n \in \mathfrak{X}$ such that $u_n \rightharpoonup u$ one has

$$J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

Lemma 4.3. *Let $\mathfrak{X}$ be a reflexive Banach space and let J be a coercive weakly lower semi-continuous functional. Then there exists $\alpha \in \mathbb{R}$ such that*

$$J(u) \geq \alpha \quad \textit{for all } u \in \mathfrak{X}.$$

Theorem 4.4. *Let $\mathfrak{X}$ be a reflexive Banach space and let J be a coercive weakly lower semi-continuous functional. Then J has a global minimum, that is,*

$$\exists z \in \mathfrak{X} \,:\, J(z) \leq J(u) \quad \textit{for all } u \in \mathfrak{X}.$$

Moreover, if J is differentiable at z, then z is a stationary point of J.

Proof. The previous lemma asserts that $m := \inf_{u\in\mathfrak{X}} J(u)$ is finite. Let u_n be a minimising sequence, which means that

$$u_n \in \mathfrak{X}, \quad J(u_n) \xrightarrow{n\to+\infty} m.$$

The coercivity of J implies that $\|u_n\| \leq R'$ (equibounded), and thus $u_n \rightharpoonup z$ for some $z \in \mathfrak{X}$. Since J is weakly lower semi-continuous, it turns out that

$$J(z) \leq m \implies J(z) = m.$$

□

4.2 Some applications to PDEs

We now show how to apply the previous theoretical results to deal with (mainly) the Dirichlet boundary value problem

$$\begin{cases} -\Delta u(x) = f(x,\, u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{D}$$

First of all, let us consider the case that $\mathfrak{X}$ is a Hilbert space and define the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \Phi(u),$$

where for (D) we have $\Phi(u) := \int_\Omega F(x,\, u)\,\mathrm{d}x$ and $F(x,\, u) = \int_0^u f(x,\, s)\,\mathrm{d}s$.

Theorem 4.5. *Let J be defined as above and suppose that $\Phi \in C^1(\mathfrak{X}, \mathbb{R})$ is weakly continuous and satisfies*

$$|\Phi(u)| \leq a_1 + a_2\|u\|^\alpha$$

with a_1, $a_2 > 0$ and $\alpha < 2$. Then J achieves a global minimum at some $z \in \mathfrak{X}$ and there holds $J'(z) = 0$, that is $\Phi'(z) = z$.

Proof. We have

$$J(u) \geq \frac{1}{2}\|u\|^2 - a_1 - a_2\|u\|^\alpha,$$

which means that for $\alpha < 2$ the functional J is coercive. Since $\|\cdot\|^2$ is weakly lower semi-continuous and Φ is weakly continuous, we infer the existence from Theorem 4.4. □

Now consider (D) and assume that f is locally Hölder-continuous and there exists $a_1 \in L^2(\Omega)$, $a_2 > 0$ and $0 < q < 1$ such that

$$|f(x,\, u)| \leq a_1(x) + a_2|u|^q \tag{4.1}$$

for all $(x,\, u) \in \Omega \times \mathbb{R}$. Set $\mathfrak{X} := H^1_0(\Omega)$ endowed with the usual homogeneous norm. Since $\mathfrak{X}$ is compactly embedded in $L^2(\Omega)$, one easily finds that

$$\Phi(u) := \int_\Omega F(x,\, u)\,\mathrm{d}x$$

is $C^1(\mathfrak{X})$ and weakly continuous.

Theorem 4.6. *Let f be locally Hölder-continuous and suppose that (4.1) holds. Then (D) admits a solution.*

Proof. Consider the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,\, u)\,\mathrm{d}x,$$

and notice that its critical points are the solutions (in the weak sense) of (D). Using (4.1) we readily find that

$$|\Phi(u)| \leq a_5\|u\| + a_6\|u\|^{q+1}.$$

Since $q < 1$ one infers that J is coercive on $\mathfrak{X}$. We know already that Φ is weakly continuous, and thus we apply the result above to infer the existence of a point z such that

$$J'(z) = z - \Phi'(z) = 0 \implies \Phi'(z) = z,$$

giving us the desired solution of (D). □

Remark 4.7. We can prove that (4.1) can be replaced with the request that $f(x,\, s)/s$ tends to zero as $|s| \to \infty$, uniformly with respect to x.

Example 4.8. Consider the boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda u - f(u) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{D}$$

where λ is a given parameter and $f : [0, \infty) \to \mathbb{R}$ is locally Hölder and satisfies

$$\lim_{u\to 0^+} \frac{f(u)}{u} = 0, \qquad \lim_{u\to +\infty} \frac{f(u)}{u} = +\infty.$$

We claim that (D) has a positive solution for any $\lambda > \lambda_1$, the first (smallest) eigenvalue of the laplacian operator with DBC. First, notice that there exists $\xi := \xi_\lambda > 0$ such that

$$\lambda\xi = f(\xi) \quad \text{and} \quad \lambda u - f(u) > 0$$

for all $u \in (0, \xi)$. Let $g_\lambda : \mathbb{R} \to \mathbb{R}$ denote the function given by

$$g_\lambda(x) := \begin{cases} 0 & \text{if } u < 0 \text{ or } u > \xi, \\ \lambda u - f(u) & \text{if } 0 \le u \le \xi. \end{cases}$$

Consider the auxiliary boundary value problem

$$\begin{cases} -\Delta u(x) = g_\lambda(u) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{D_λ}$$

and by the maximum principle, any nontrivial solution of (D_λ) is positive. Moreover, one finds that $u(x) \in (0, \xi_\lambda)$ for all $x \in \Omega$, and hence is a positive solution of (D). Since g_λ is locally Hölder-continuous and bounded, the theorem above applies to the functional

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 - \lambda \int_\Omega G_\lambda(u)\,\mathrm{d}x.$$

If $\lambda > \lambda_1$ we claim that $\inf J_\lambda$ is less than zero. To prove this, let $\varphi_1 \in \mathfrak{X}$ be positive in Ω and satisfying

$$-\Delta\varphi_1(x) = \lambda_1\varphi_1(x), \qquad \|\varphi_1\|_2 = 1.$$

For $t > 0$ small, one has $g_\lambda(t\varphi_1) = \lambda t\varphi_1 - f(t\varphi_1)$. Since $f(u)$ is a small-o of u, we infer that

$$J_\lambda(t\varphi_1) = \frac{1}{2}(\lambda_1 - \lambda)t^2 + o(t^2),$$

which is strictly negative if we choose t to be small enough.

Chapter 5

Constrained Critical Points

In this chapter, we will investigate the notion of constrained critical point and we will relate it with extrema and, ultimately, solutions of PDEs problems.

5.1 Introduction

Let $J : \mathfrak{X} \longrightarrow \mathbb{R}$ be a differentiable functional and let M be a smooth Hilbert submanifold. A *constrained critical point* of J on M is a point $z \in M$ such that

$$\mathrm{d}(J\big|_M)(z) \equiv 0,$$

which is equivalent to

$$\mathrm{d}J(z)[v] = 0 \quad \text{for all } v \in T_z M.$$

Using the constrained gradient, we can say that a constrained critical point z of J on M satisfies

$$\langle \nabla_M J(z),\, v \rangle \quad \text{for all } v \in T_z M,$$

which allows us to affirm that $J'(z)$ is orthogonal to $T_z M$.

Remark 5.1. Let $\gamma : [0,\, 1] \to \mathcal{M}$ be any smooth curve such that $\gamma(0) = z$ and consider the real-valued function $\phi(t) := J \circ \gamma(t)$. Then

$$\phi'(0) = J'(z)[\gamma'(0)],$$

where $\gamma'(0)$ belongs to $T_z M$. Therefore, if z is a critical point of J constrained on M, then $t = 0$ is a critical point of Φ. Vice versa, z is a constrained critical point if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(\gamma(t)) = 0$$

for all C^1-curves γ with $\gamma(0) = z$.

Suppose that M has codimension one, that is, there exists $G : \mathfrak{X} \longrightarrow \mathbb{R}$ of class C^1 such that $M = G^{-1}(0)$. It follows that

$$\mathfrak{X} = T_z M \oplus \operatorname{Span}(\nabla G(z)),$$

and by the Lagrange multiplier rule

$$\nabla J(z) = \lambda \nabla G(z) \implies \lambda = \frac{\langle \nabla J(z),\, \nabla G(z)\rangle}{\|\nabla G(z)\|^2}.$$

5.1.1 Nonlinear eigenvalues

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded set and assume that f satisfies (1.10). Set $\mathfrak{X} := H_0^1(\Omega)$ and

$$\Phi(u) := \int_\Omega F(x,\, u)\,\mathrm{d}x.$$

Define

$$M = \{u \in \mathfrak{X} \;:\; \|u\|^2 - 1 = 0\} = G^{-1}(0),$$

where $G(u) := \|u\|^2 - 1$. It follows that M is a C^1-manifold since $\mathrm{d}G(u) \equiv 2$ and, if u is a constrained critical point of Φ on M, then necessarily

$$\nabla\Phi(u) = \lambda u \implies \lambda \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega f(x,\, u) v\,\mathrm{d}x,$$

and therefore u is a weak solution of the boundary value problem

$$\begin{cases} -\lambda\Delta u(x) = f(x,\, u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{5.1}$$

If f is homogeneous, then one can consider the scaling $\lambda^{\frac{1}{p-1}} u$ that solves the same boundary value problem with $\lambda = 1$.

5.2 Natural Constraint

Let $\mathfrak{X}$ be a Hilbert space and let $J \in C^1(\mathfrak{X},\, \mathbb{R})$. A C^1-submanifold M is called a *natural constraint* for J if there exists $\tilde{J} \in C^1(\mathfrak{X},\, \mathbb{R})$ such that every constrained critical point of $\tilde{J}$ on M is a stationary point of J, that is,

$$\nabla_M \tilde{J}(u) \iff J'(u) = 0.$$

An example of a natural constraint is the so-called Nehari manifold given by

$$M := \{u \in \mathfrak{X} \setminus \{0\} \;:\; \langle J'(u),\, u\rangle = 0\}.$$

Proposition 5.2. *Let $J \in C^2(\mathfrak{X}, \mathbb{R})$ and suppose that the Nehari manifold M is nonempty. Assume that the following conditions hold:*

(i) There exists $r > 0$ such that $M \cap B_r(0) = \varnothing$.

(ii) For all $u \in M$ it turns out that $\mathrm{d}^2 J(u)[u, u] \neq 0$.

Then M is a natural constraint for J with $\tilde{J} \equiv J$.

Proof. Let $G(u) := \langle J'(u), u \rangle$ so that $M = G^{-1}(0)$, and notice that G is of class C^1 since J is C^2. Moreover, it is easy to see that

$$G'(u)[u] = \mathrm{d}^2 J(u)[u, u] + \underbrace{\mathrm{d}J(u)[u]}_{=0} = \mathrm{d}^2 J(u)[u, u] \neq 0,$$

which means that M is a C^1-submanifold. Now if $(\nabla J\big|_M)(u) = 0$, then

$$\nabla J(u) = \lambda \nabla G(u) \implies \langle \nabla J(u), u \rangle = \lambda \langle \nabla G(u), u \rangle,$$

and now the right-hand side is different from zero for $u \in M$, whereas the left-hand side is zero by definition. It follows that λ must be equal to zero, and thus M is a natural constraint. □

5.2.1 Applications to PDEs analysis

Let $\Omega \subset \mathbb{R}^n$ be an open bounded smooth set and consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1} u, & \text{if } x \in \Omega, \\ u\big|_{\partial\Omega} \equiv 0. \end{cases} \tag{5.2}$$

At some point, we will need to use Sobolev embedding theorem to conclude that the embedding

$$L^{p+1}(\Omega) \hookrightarrow H_0^1(\Omega)$$

is compact. Therefore, we must assume that $1 < p < \frac{n+2}{n-2}$. Let $\mathfrak{X} := H_0^1(\Omega)$ endowed with the homogeneous norm

$$\|u\|_X := \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

The variational formulation of the problem consists of finding critical points of the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, \mathrm{d}x.$$

The reader might check that J belongs to C^2 as an exercise (in the same way one proves that it is C^1), but what is important now is that J is unbounded on $\mathfrak{X}$. In fact, we have

$$\inf_{u \in \mathfrak{X}} J(u) = -\infty$$

since, if φ_1 is the eigenfunction of $\lambda_1(\Omega)$, then $J(t\varphi) < 0$ for some $t > 0$ and hence taking the limit we can infer that

$$\lim_{t\to+\infty} J(t\varphi_1) = -\infty.$$

Similarly, one can show that

$$\sup_{u\in\mathfrak{X}} J(u) = \infty,$$

e.g., by taking $u_n(x) := \sin(nx)\chi(x)$, where χ is a cutoff function with support in Ω.

Proposition 5.3. *The Nehari manifold*

$$\mathcal{M} := \left\{ u \in \mathfrak{X} \setminus \{0\} \;:\; \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \int_\Omega |u|^{p+1} \, \mathrm{d}x \right\}$$

is a natural constraint for J.

Proof. First, notice that

$$\mathrm{d}J(u)[v] = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega |u|^{p-1} uv \, \mathrm{d}x,$$

so that $G(u) := \mathrm{d}J(u)[u]$ is actually given by $\|u\|_{\mathfrak{X}}^2 - \|u\|_{L^{p+1}(\Omega)}^{p+1}$, which means that a nonzero $u \in \mathfrak{X}$ belongs to $\mathcal{M}$ if and only if

$$\|u\|_{\mathfrak{X}}^2 = \int_\Omega |u|^{p+1} \, \mathrm{d}x.$$

Using Sobolev embedding we can find a constant $C_{p,\Omega}$ such that

$$\|u\|_{p+1} \le C_{p,\Omega} \|u\|_{\mathfrak{X}}.$$

Therefore, if $u \in \mathcal{M}$, then

$$\|u\|_{\mathfrak{X}}^2 = \|u\|_{p+1}^{p+1} \le C_{p,\Omega} \|u\|_{\mathfrak{X}}^{p+1} \overset{p>1}{\Longrightarrow} \|u\|_{\mathfrak{X}}^{p-1} \ge \frac{1}{C_{p,\Omega}} > 0,$$

which means that the first point in Proposition 5.2 is verified with r equal to a negative power of $C_{p,\Omega}$. Now notice that

$$\mathrm{d}^2 J(u)[v, w] = \int_\Omega \nabla w \cdot \nabla v \, \mathrm{d}x - p \int_\Omega |u|^{p-1} wv \, \mathrm{d}x,$$

which immediately leads to

$$\mathrm{d}^2 J(u)[u, u] = \|u\|_{\mathfrak{X}}^2 - p\|u\|_{p+1}^{p+1}.$$

For $u \in \mathcal{M}$ we obtain

$$\mathrm{d}^2 J(u)[u, u] = (1 - p)\|u\|_{\mathfrak{X}}^2 \neq 0$$

for $p > 1$, which means that $\mathcal{M}$ is a natural constraint for J. □

Remark 5.4. The functional is bounded from below on $\mathcal{M}$ since

$$J\big|_{\mathcal{M}}(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u\|_{\mathfrak{X}}^2 \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) r > 0.$$

We now claim that, if $p < \frac{n+2}{n-2}$, then $J\big|_{\mathcal{M}}$ attains a minimum[1]. To prove this, let u_n be a minimising sequence, weakly converging to some $\bar{u}$. By compactness of the Sobolev embedding, we have that

$$\int_\Omega |u_n|^{p+1}\,\mathrm{d}x \xrightarrow{n\to+\infty} \int_\Omega |\bar{u}|^{p+1}\,\mathrm{d}x.$$

Furthermore, taking into account that $u_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, we can conclude that

$$\int_\Omega |\bar{u}|^{p+1}\,\mathrm{d}x = \lim_{n\to+\infty} \int_\Omega |u_n|^{p+1}\,\mathrm{d}x \geq r^2 \implies \bar{u} \not\equiv 0.$$

There are now two cases we need to discuss separately.

(a) We have $\|u_n\|_{\mathfrak{X}} \to \|\bar{u}\|_{\mathfrak{X}}$. Then $\bar{u} \in \mathcal{M}$ and

$$J\big|_{\mathcal{M}}(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u\|_{\mathfrak{X}}^2$$

is lower semi-continuous, and therefore $\bar{u}$ is a minimiser for J on $\mathcal{M}$.

(b) We have $\lim_{n\to\infty} \|u_n\|_{\mathfrak{X}} > \|\bar{u}\|_{\mathfrak{X}}$. Then

$$\|\bar{u}\|_{\mathfrak{X}}^2 = \mu \lim_{n\to\infty} \|u_n\|_{\mathfrak{X}}^2,$$

for some $\mu \in (0,\, 1)$. But then

$$\|\bar{u}\|_{\mathfrak{X}}^2 = \mu \lim_{n\to\infty} \int_\Omega |u_n|^{p+1}\,\mathrm{d}x = \mu \|\bar{u}\|_{p+1}.$$

If one takes $\nu \in (0,\, 1)$ such that $\nu^{p-1} = \mu$, then $\nu\bar{u} \in \mathcal{M}$. But this leads to a contradiction since $\bar{u}$ is the limit of a minimising sequence.

[1]We will not prove it here, but the assertion is false when p is equal to the critical exponent.

Chapter 6

Deformations and Palais-Smale Sequences

In this chapter we will investigate the existence of constrained critical points via special deformations of the sublevels. The notion of Palais-Smale sequence is then introduced to deal with the lack of compactness, replacing it with a much weaker condition.

6.1 Deformations of sublevels

Let $J : U \subset \mathfrak{X} \longrightarrow \mathbb{R}$ be a functional defined on a open subset U of a Banach space $\mathfrak{X}$ and let $a \in \mathbb{R}$. We denote by

$$\mathfrak{X}^a := \{u \in \mathfrak{X} \,:\, J(u) \leq a\}$$

the a-sublevel of J on $\mathfrak{X}$. We now need to introduce a suitable notion of deformation, which should make the investigation of critical points of J easier.

Definition 6.1 (Deformation)**.** A *deformation* of $A \subset \mathfrak{X}$ in $\mathfrak{X}$ is a continuous map $\eta \in C(A, \mathfrak{X})$ which is homotopic to the identity. Namely, there exists a homotopy H such that

$$H(0, u) = u, \quad H(1, u) = \eta(u) \quad \text{for all } u \in \mathfrak{X}.$$

The idea behind deforming a set into another one is the following. Since a deformation is a continuous map homotopic to the identity, we expect that A and $\eta(A)$ have the same topological properties.

More specifically, if $[a, b] \subset \mathbb{R}$ does not contain any critical point of J, then it can be proved that under some assumptions on $\mathfrak{X}$ the sublevel $\mathfrak{X}^b$ can be deformed into $\mathfrak{X}^a$. On the other hand, the presence of an obstacle is often (but not always) a consequence of the existence in the given interval of a critical point.

Example 6.2. Let M be a compact hyper-surface in $\mathbb{R}^n$. Suppose that b is not a critical level for J on M and notice that

$$M^b = \{x \in M \ : \ J(x) = b\}$$

is a smooth submanifold of M and at any point the vector $-\nabla_M J(x) \neq 0$. By compactness, it turns out that

$$\min_{x \in M^b} |\nabla_M J(x)| \geq C > 0,$$

and hence we can deform M^b into $M^{b-\epsilon}$, for ϵ small enough, via the aforementioned gradient vectors. Now, if there are no critical levels in $[a, b]$, we can repeat the same process over and over again, until we find that M^b can be deformed into M^a.

Remark 6.3. If c is the minimum of J over M, then it must happen that $M^{c-\epsilon} = \varnothing$ while $M^{c+\epsilon} \neq \varnothing$, which means that the topological properties change when passing through a critical level.

To better understand the change of topological properties after crossing critical levels, the following example is instructive.

Example 6.4. Let M be the 2-torus and let $J(x, y, z) := z$. The critical points of J on M are the four points p_i where the gradient of J is orthogonal to M. If we set

$$c_i := J^{-1}(p_i),$$

we find the following diffeomorphisms:

$$M^a \cong \begin{cases} \mathbb{T}^2 & \text{if } a > c_4, \\ \mathbb{T}^2 \setminus B_\varphi & \text{if } c_4 \geq a > c_3, \\ S^1 \times [0, 1] & \text{if } c_3 \geq a > c_2, \\ B_\varphi & \text{if } c_2 \geq a > c_1, \\ \varnothing & \text{if } a < c_1. \end{cases}$$

6.2 The steepest descent flow

In this section, we will try to extend the procedure given above to the general case via flows of differential equations and, in particular, the so-called steepest descent flow.

Given $W \in C^{0,1}(\mathfrak{X}, \mathfrak{X})$ Lipschitz function defined on a Hilbert space $\mathfrak{X}$, let $\alpha(t, u) =: \alpha(t)$ denote the solution of the Cauchy problem

$$\begin{cases} \alpha'(t) = W(\alpha(t)), \\ \alpha(0) = u \in \mathfrak{X}. \end{cases} \tag{6.1}$$

The local existence theorem for Cauchy problems shows that, since the right-hand side is Lipschitz, there exists a unique solution $\alpha(t,\,u)$ in a neighbourhood of $t = 0$ that depends continuously on the initial data in any compact subset of $\mathbb{R}$. Let

$$(t_u^-,\, t_u^+)$$

denote the maximal interval of existence given $u \in \mathfrak{X}$. We would like to find sufficient condition for the solution α to be globally defined for $t > 0$, that is, $t_u^+ = +\infty$.

Lemma 6.5. *If $t_u^+ < +\infty$, then $\alpha(t,\,u)$ has no limit points as $t \nearrow t_u^+$.*

Proof. We argue by contradiction. If there exists $v \in \mathfrak{X}$ such that $\alpha(t,\,u) \nearrow v$, then let β denote the solution of the Cauchy problem (6.1) with $u = v$. Then

$$\beta \text{ is well-defined in a neighbourhood of } t^+, \text{ say } (t^+ - \epsilon,\, t^+ + \epsilon),$$

and therefore the function

$$\begin{cases} \alpha(t,\,u) & \text{if } t \in (t^-,\, t^+), \\ \beta(t,\,v) & \text{if } t \in [t^+,\, t^+ + \epsilon), \end{cases}$$

is a solution of (6.1) with initial data u, defined in a strictly bigger interval than the maximal one - which is obviously impossible -. □

Lemma 6.6. *Let $A \subset \mathfrak{X}$ be closed and suppose that $\|W(u)\|$ is uniformly bounded on A by a positive constant C. Let $u \in A$ be such that $\alpha(t,\,u) \in A$ for all $t \in [0,\, t_u^+)$. Then $t_u^+ = +\infty$.*

Proof. Suppose that $t_u^+ < \infty$. For all $t_i,\, t_j \in [0,\, t^+)$ we have

$$\alpha(t_i,\,u) - \alpha(t_j,\,u) = \int_{t_j}^{t_i} \alpha'(s,\,u)\,\mathrm{d}s = \int_{t_j}^{t_i} W(\alpha(s,\,u))\,\mathrm{d}s.$$

Since W is bounded on A, it turns out that

$$\|\alpha(t_i,\,u) - \alpha(t_j,\,u)\| \le C|t_i - t_j|.$$

Therefore, as $t_i \nearrow t_u^+$, the sequence $\alpha(t_i,\,u)$ is Cauchy and thus converges to some point in A in contradiction with the statement of the previous lemma. □

To introduce the steepest descent flow, we need to investigate the quantity $-\nabla_M J(u)$. More precisely, let us assume that there exists $G \in C^{1,\,1}(\mathfrak{X},\,\mathbb{R})$ such that

$$M = G^{-1}(0) \quad \text{and} \quad G'(u) \neq 0 \text{ for } u \in M.$$

Let $J \in C^{1,\,1}(\mathfrak{X},\,\mathbb{R})$ be a functional and consider the function

$$W(u) = -\left[J'(u) - \frac{\langle J'(u),\, G'(u)\rangle}{\|G'(u)\|^2} G'(u)\right],$$

which is well-defined in a neighbourhood of M, is of class $C^{0,1}$ as required, and coincides with $-\nabla_M J(u)$ for all $u \in M$. The solution of (6.1) is called the steepest descent flow of M and satisfies the following property:

$$\alpha(0) \in M \iff \alpha(t) \in M \text{ for all } t \in (t_u^-, t_u^+).$$

More precisely, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} G(\alpha(t)) &= \langle G'(\alpha(t)), \alpha'(t)\rangle = \\ &= \langle G'(\alpha(t)), W(\alpha(t))\rangle = \\ &= -\langle G'(\alpha(t)), J'(\alpha(t))\rangle + \frac{\langle G'(\alpha(t)), J'(\alpha(t))\rangle}{\|G'(\alpha(t))\|^2}\langle G'(\alpha(t)), G'(\alpha(t))\rangle = 0,\end{aligned}$$

which means that $G(\alpha(t))$ is constant and thus

$$u \in M \iff G(u) = 0 \iff G(\alpha(t)) = 0 \iff \alpha(t, u) \in M.$$

Lemma 6.7. *Under the assumptions above, the steepest descent flow of J satisfies the following properties:*

(1) The function $t \longmapsto J(\alpha(t, u))$ is nonincreasing for $t \in [0, t_u^+)$.

(2) For $t, \tau \in [0, t_u^+)$ we have

$$J(\alpha(t, u)) - J(\alpha(\tau, u)) = -\int_\tau^t \|\nabla_M(\alpha(s, u))\|^2 \,\mathrm{d}s. \tag{6.2}$$

(3) If J is bounded from below on M, then $t_u^+ = \infty$ for all $u \in M$.

Proof. First, notice that

$$\frac{\mathrm{d}}{\mathrm{d}t} J(\alpha(t)) = -\langle J'(\alpha(t)), \nabla_M J(\alpha(t))\rangle,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} J(\alpha(t)) = -\|\nabla_M J(\alpha(t))\|^2$$

since $\nabla_M J(\alpha)$ is the projection of $J'(\alpha)$ on $T_\alpha M$. The first two properties follow easily from this.

As for the third property, we argue by contradiction. Let $u \in M$ with finite maximal time and use (6.2) with $\tau = 0$ to infer that

$$J(\alpha(t)) - J(u) = -\int_0^t \|\nabla_M(\alpha(s, u))\|^2 \,\mathrm{d}s.$$

Since J is bounded from below on M, it follows that

$$\int_0^t \|\nabla_M(\alpha(s,\, u))\|^2 \,\mathrm{d}s \leq a < +\infty$$

for some positive constant a. Let $t_i \nearrow t_u^+$ and recall that

$$\|\alpha(t_i) - \alpha(t_j)\| \leq \int_0^t \|\nabla_M(\alpha(s,\, u))\| \,\mathrm{d}s.$$

Using Hölder inequality we find that

$$\|\alpha(t_i) - \alpha(t_j)\| \leq \sqrt{a}|t_i - t_j|^{\frac{1}{2}},$$

and therefore $\alpha(t_i)$ is a Cauchy sequence, in contradiction with the previous lemma. □

Remark 6.8. If J is C^1 only, the steepest descent flow might not be defined. Luckily, we can generalise the gradient vector field in such a way that Lemma 6.7 holds.

Definition 6.9 (Pseudo-gradient)**.** Let J be a C^1 functional. A *pseudo-gradient* vector field for J on

$$\mathfrak{X}_0 := \{u \in \mathfrak{X} \,:\, \nabla J(u) \neq 0\}$$

is a $C^{0,\,1}(\mathfrak{X}_0,\, \mathfrak{X})$ vector field V satisfying the following properties for all $u \in \mathfrak{X}_0$:

$$\begin{aligned} &\|V(u)\| \leq 2\|\nabla J(u)\|, \\ &\langle V(u),\, \nabla J(u)\rangle \geq \|\nabla J(u)\|^2. \end{aligned} \tag{6.3}$$

Remark 6.10. If such V exists, then Lemma 6.7 holds with the flow

$$\dot{\alpha}(t,\, u) = -V(\alpha(t,\, u)).$$

Proposition 6.11. *Let $J \in C^1(\mathfrak{X},\, \mathbb{R})$. Then a pseudo-gradient vector field V always exists.*

Proof. Fix $u \in \mathfrak{X}_0$. Then there exists $w(u) := w \in \mathfrak{X}$ such that

$$\|w\| = 1 \quad \text{and} \quad \langle \nabla J(u),\, w\rangle > \frac{2}{3}\|\nabla J(u)\|.$$

Now set

$$\tilde{V}(u) := \frac{3}{2}\|\nabla J(u)\| w(u)$$

and notice that (6.3) holds since

$$\|\tilde{V}(u)\| = \frac{3}{2}\|\nabla J(u)\| < 2\|\nabla J(u)\|,$$

$$\langle \tilde{V}(u),\, \nabla J(u)\rangle = \frac{3}{2}\|\nabla J(u)\|\langle w(u),\, \nabla J(u)\rangle > \|\nabla J(u)\|^2.$$

Since ∇J is continuous, we can find $r := r(u) > 0$ such that

$$\|\tilde{V}(u)\| < 2\|\nabla J(z)\|,$$

$$\langle \tilde{V}(u), \nabla J(z)\rangle > \|\nabla J(z)\|^2$$

hold for all $z \in B(u, r)$. We can cover $\mathfrak{X}_0$ with these balls, that is,

$$\mathfrak{X}_0 = \bigcup_{u \in \mathfrak{X}_0} B(u, r(u)),$$

and hence there exists a locally finite covering $U_i := B(u_i, r(u_i))$. Define

$$d_i(u) := \mathrm{dist}(u, \mathfrak{X} \setminus U_i)$$

and denote $\tilde{V}(u_i)$ by $\widetilde{V_i}$. Then

$$V(u) := \sum_i \frac{d_i(u)}{\sum_j d_j(u)} \widetilde{V_i}$$

is a well-defined locally Lipschitz pseudo-gradient vector field. □

6.3 Deformation and compactness

In this section, we will denote by M either a Hilbert space or a C^1-submanifold of codimension one.

Lemma 6.12. *Let $J \in C^1(M, \mathbb{R})$ and suppose that there exist $c \in \mathbb{R}$ and $\delta > 0$ such that*

$$\|\nabla_M J(u)\| \geq \delta \quad \textit{for all } u \textit{ such that } J(u) \in [c-\delta, c+\delta].$$

Then there exists η deformation in M such that

$$\eta(M^{c+\delta}) \subset M^{c-\delta}.$$

Proof. Suppose first that J is bounded from below. By Lemma 6.7 the evolution above is globally defined. Let $T := \frac{2}{\delta}$ and set

$$\eta(u) := \alpha(T, u).$$

It is easy to see that η is a deformation since $(s, u) \mapsto \alpha(sT, u)$ is a homotopy between η and the identity mapping. We now argue by contradiction so let $u \in M^{c+\delta}$ such that

$$J(\alpha(T, u)) > c - \delta.$$

Since $J(\alpha(\cdot, u))$ is decreasing, we easily infer that

$$J(\alpha(t, u)) \in [c-\delta, c+\delta]$$

for all $t \in [0, T]$. We now apply the assumption to conclude that

$$\|\nabla_M J(\alpha(t, u))\| \geq \delta \quad \text{for all } t \in [0, T].$$

We now use Lemma 6.7 again and obtain

$$J(\alpha(T, u)) - \underbrace{J(\alpha(0, u))}_{=J(u)} = -\int_0^T \|\nabla_M J(\alpha(s, u))\| \, \mathrm{d}x \geq \delta^2 T = 2\delta,$$

from which we finally infer that

$$c - \delta < J(\alpha(T, u)) < c + \delta - 2\delta = c - \delta \implies \text{absurd}.$$

Now remove the assumption that J is bounded from below. Define

$$\hat{J}(u) := h \circ J(u),$$

where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ is given, for example, by

$$h(s) = \begin{cases} s & \text{if } s \geq c - \delta, \\ \text{bounded below} & \text{at all } s \in \mathbb{R}. \end{cases}$$

We conclude the proof using the argument above since $\hat{J}$ is bounded from below by construction and also

$$\{\hat{J} \leq a\} = \{J \leq a\}$$

for all $a \geq c - \delta$ by construction. □

Remark 6.13. If M is compact and c is not a critical level for J, then we can always find a $\delta > 0$ satisfying the assumption of Lemma 6.12.

Hint. Argue by contradiction. □

Remark 6.14. Some kind of compactness is necessary even in finite-dimensional spaces. We can easily find a counterexample with $M = \mathbb{R}$; see Figure **??**.

6.4 Palais-Smale sequences

In this section, we introduce a notion of compactness which is weaker than the usual one but is rather useful when dealing with variational problems.

Definition 6.15. Let $c \in \mathbb{R}$ be a real number. We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset M$ is

Palais-Smale at the level c, denoted by $(u_n)_{n\in\mathbb{N}} \in (\mathrm{PS})_c$, if

$$\begin{cases} f(u_n) \xrightarrow{n\to+\infty} c, \\ \\ \operatorname{grad} f(u_n) \xrightarrow{n\to+\infty} 0. \end{cases}$$

.

Definition 6.16. A functional $J \in C^1(M, \mathbb{R})$ is Palais-Smale at the level c if

$$\forall (u_n)_{n\in\mathbb{N}} \in (\mathrm{PS})_c,\, \exists (n_k)_{k\in\mathbb{N}} \,:\, u_{n_k} \text{ converges.}$$

Remark 6.17. Let $J \in C^1(M, \mathbb{R})$.

(i) If J satisfies the Palais-Smale condition at the level c, then any $(\mathrm{PS})_c$-sequence converges (up to subsequences) to some $u^* \in M$ such that

$$J(u^*) = c \quad \text{and} \quad \nabla_M J(u^*) = 0,$$

which means that u^* is a critical point (and thus c a critical level).

(ii) The set

$$\{z \in M \,:\, J(z) = c,\, \nabla J(z) = 0\}$$

is compact.

(iii) If $J \in C^1(\mathbb{R}^n, \mathbb{R})$ is bounded from below and coercive, then the Palais-Smale condition at the level c holds for all c. This is false in the infinite-dimensional setting!

Lemma 6.18. *Let $J \in C^1(M, \mathbb{R})$ be a functional satisfying the $(\mathrm{PS})_c$-condition at all $c \in [a, b]$ and assume that there are no critical levels in the interval. Then there exists $\delta > 0$ such that*

$$\sigma := \inf_{u\in J^{-1}(I_\epsilon)} \|\nabla J(u)\| > 0,$$

where $I_\delta = [a-\delta,\, b+\delta]$.

Proof. We argue by contradiction. There is a decreasing sequence $(\delta_n)_{n\in\mathbb{N}}$ that converges to 0, and a sequence $(u_n)_{n\in\mathbb{N}} \subset M$ such that

$$\|\nabla J(u_n)\| \le \delta_n \quad \text{and} \quad J(u_n) \in [a-\delta_n,\, b+\delta_n]\,.$$

Now, up to subsequences, $J(u_n) \to c$ and $\nabla J(u) \to 0$ and by the Palais-Smale condition we know that $(u_n)_{n\in\mathbb{N}}$ is precompact. Thus u_{n_k} converges to some u^* which is a critical point with $J(u) \in [a, b]$, and this is the sought contradiction. ☐

Lemma 6.19. *Let $J \in C^1(M, \mathbb{R})$ be a functional satisfying the $(\mathrm{PS})_c$-condition at some noncritical level $c \in \mathbb{R}$. Then there exist $\delta > 0$ and a deformation η such that*

$$\eta(M^{c+\delta}) \subseteq M^{c-\delta}.$$

Lemma 6.20. *Let $J \in C^1(M, \mathbb{R})$ be a functional satisfying the $(\mathrm{PS})_c$-condition at all $c \in [a, b]$ and assume that there are no critical levels in the interval. Then there exists a deformation η such that*

$$\eta(M^b) \subseteq M^a.$$

Proof. Simply apply the previous result a finite number of times since $[a, b]$, by compactness, can be covered by a finite number of intervals of length δ. □

Theorem 6.21. *Let $J \in C^1(M, \mathbb{R})$, where M is a C^1-submanifold of codimension one. Suppose that $J\big|_M$ is bounded from below and suppose that it satisfies the Palais-Smale condition at*

$$m := \inf_{u \in M} J(u) > -\infty.$$

Then $\inf_{u \in M} J(u)$ is achieved.

Proof. We argue by contradiction. If m is not a critical value, then there exists $\epsilon > 0$ such that the following holds:

$$J^{(\alpha-\epsilon)} \text{ is a deformation retract of } J^{(\alpha+\epsilon)}.$$

But this is impossible since the first set is empty, while the second one is not. □

6.5 Application to a superlinear Dirichlet problem

In this section, we will exploit the theoretical results presented above to prove existence of a positive solution to a class of superlinear Dirichlet boundary-value problems:

$$\begin{cases} -\Delta u(x) = f(u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{DSL}$$

We assume Ω to be a bounded domain in $\mathbb{R}^n$ and $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions: there exist $a_1, a_2 > 0$ and $p \in (1, 2^* - 1)$ such that

$$\begin{aligned} &|f(u)| \le a_1 + a_2|u|^p, \\ &|uf'(u)| \le a_1 + a_2|u|^p, \\ &|u^2 f''(u)| \le a_1 + a_2|u|^p. \end{aligned} \tag{6.4}$$

Assume also that $f(u) = uh(u)$, where h is a function satisfying the following assumptions:

(h_1) $h(su) \le s^\alpha h(u)$ for some $\alpha > 0$;

(h_2) $uh'(u) > 0$ for all $u \neq 0$;

(h_3) $h(0) = 0$;

(h_4) $\lim_{u\to+\infty} h(u) = +\infty$.

Example 6.22. The function $h(u) = |u|^{p-1}$ satisfies these properties and, indeed, we were able to obtain existence in Section 5.2.1 looking for the minimum of

$$\int_\Omega |u|^{p+1}\,\mathrm{d}x$$

on the manifold $\{u \in H_0^1(\Omega) \,:\, \|u\|_{L^2(\Omega)} = 1\}$.

Theorem 6.23. *Under these assumptions, the problem* (DSL) *has a positive solution.*

The proof of this theorem will be attained through a sequence of technical lemmas, mostly relying on the theoretical aspects presented in this chapter. However, before we get to it, we need to introduce some notation. Namely, let $\mathfrak{X} := H_0^1(\Omega)$ and denote by

$$\langle u,\, v\rangle := \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$$

the standard scalar product and by $\|\cdot\|$ the norm on $\mathfrak{X}$. Set

$$F(u) := \int_0^u f(s)\,\mathrm{d}s = \int_0^1 f(su)u\,\mathrm{d}s,$$

$$\Phi(u) = \int_\Omega F(u)\,\mathrm{d}x = \int_0^1 \mathrm{d}s\int_\Omega uf(su)\,\mathrm{d}x,$$

$$\Psi(u) = \langle\Phi'(u),\, u\rangle = \int_\Omega uf(u)\,\mathrm{d}x.$$

Now notice that

- **(i)** The functional Φ and Ψ respectively belong to $C^2(\mathfrak{X},\,\mathbb{R})$ and $C^3(\mathfrak{X},\,\mathbb{R})$. This follows immediately from the regularity of f and the definitions above.
- **(ii)** The functionals Φ and Ψ are both weakly continuous.
- **(iii)** The gradients $\nabla\Phi$ and $\nabla\Psi$ are compact operators. This follows from the compactness of the Sobolev embedding (since $p < 2^*$) and it implies the previous point.

The solutions of (DSL) are critical points of the following functional:

$$J(u) := \frac{1}{2}\|u\|^2 - \Phi(u).$$

The idea is to use Nehari manifolds together with the results on critical points obtained in this chapter. We thus introduce the natural functional

$$G(u) := \langle J'(u),\, u\rangle = \|u\|^2 - \Psi(u),$$

and the C^2-submanifold where G vanishes, that is,

$$\mathcal{M} := \{u \in \mathfrak{X} \setminus \{0\} \, : \, G(u) = 0\} \, .$$

Our goal is to show that $\mathcal{M}$ is a natural constraint for J. In other words, we are looking for a functional of class C^2, $\widetilde{J}$, such that

$$\nabla_{\mathcal{M}} \widetilde{J}(u) = 0 \text{ and } u \in M \iff J'(u) = 0.$$

We will then verify that $\widetilde{J}$ achieves a minimum on $\mathcal{M}$, which ends up giving a solution to the problem (DSL).

Lemma 6.24. *The functional G belongs to $C^2(E, \mathbb{R})$. Furthermore:*

- ***(i)*** *The set $\mathcal{M}$ is nonempty.*
- ***(ii)*** *There exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{M}$.*
- ***(iii)*** *The scalar product $\langle G'(u), u \rangle$ is negative for all $u \in \mathcal{M}$.*

Proof. The regularity of G is an easy consequence of the regularity of Ψ.

- **(i)** Take $u \in E$, $u > 0$, with $\|u\| = 1$. Then

$$G(tu) = t^2 - t^2 \int_{\Omega} u^2 h(tu) \, \mathrm{d}x.$$

Using **(h$_3$)** we find that

$$\lim_{t \to 0} \frac{G(tu)}{t^2} = 1,$$

while, employing the property **(h$_4$)**, we obtain

$$\lim_{t \to +\infty} \frac{G(tu)}{t^2} = -\infty.$$

Putting these two together, we infer that there must be $\tilde{t} \in (0, \infty)$ such that $\tilde{t}u \in \mathcal{M}$.

- **(ii)** This property, despite its simplicity, requires a lot of work because having $\|u\|$ small does not mean that the L^∞-norm is also small (the embedding fails!).

Let $\|u\|$ be sufficiently small. Our goal is to prove that $G(u) > 0$ so that u cannot belong to $\mathcal{M}$. First, take $\delta > 0$ and define

$$A_1^\delta := \{x \in \Omega \, : \, |u(x)| \leq \delta\} \quad \text{and} \quad A_2^\delta = \Omega \setminus A_1^\delta.$$

We claim that the volume of A_2^δ cannot be "too big". Recall that by Poincaré's inequality we can always find a positive constant C_Ω such that

$$\|u\|_{L^1(\Omega)} \leq C_\Omega \|u\|.$$

It follows that

$$|A_2|\delta \le \|u\|_{L^1(A_2)} \le \|u\|_{L^1(\Omega)} \le C_\Omega \|u\|,$$

which means that the volume of A_2 is bounded by

$$|A_2| \le \frac{C_\Omega}{\delta}\|u\|.$$

We now employ Hölder's inequality to estimate the negative contribute to $G(u)$ on A_2^δ. Namely, we have

$$\int_{A_2^\delta} uf(u)\,\mathrm{d}x \le \|u\|_{p_1}\|f(u)\|_{p_2}|A_2^\delta|^{\frac{1}{p_3}}$$

where

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1.$$

We want $p_3>1$, so the idea is to take the maximum p_1 and p_2 possible. However, we still need Sobolev embedding to estimate these terms with $\|u\|$. Let

$$p_1 := 2^* \quad \text{and} \quad p_2 := \frac{2^*}{p}.$$

It is easy to see that

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{p+1}{2^*} < 1,$$

so $p_3>1$ as desired. We also use (6.4) and ($\mathbf{h_4}$) to conclude that $f(u)$ must satisfy a slightly different estimate

$$|f(u)| \lesssim |u| + |u|^p$$

for $|u|$ small enough. Then

$$\|f(u)\|_{p_2} \lesssim \left[\int_\Omega \left(|u|^{p_2}+|u|^{pp_2}\right)\,\mathrm{d}x\right]^{\frac{1}{p_2}} \lesssim$$

$$\lesssim \|u\| + \|u\|^p.$$

The right-hand side goes as $\|u\|$ when $\|u\|$ is sufficiently small (since $p>1$) and therefore we conclude that

$$\left|\int_{A_2^\delta} uf(u)\,\mathrm{d}x\right| \lesssim \delta^{-\frac{1}{p_3}}\|u\|^{2+\frac{1}{p_3}}.$$

The estimate on A_1^δ is even easier since

$$\left|\int_{A_1^\delta} uf(u)\,\mathrm{d}x\right| = \left|\int_{A_1^\delta} u^2h(u)\,\mathrm{d}x\right| \le C_\Omega\|u\|^2 \sup_{|u|\in(0,\,\delta)} h(u).$$

Fix $\delta>0$ sufficiently small in such a way that $C_\Omega \sup_{|u|\in(0,\,\delta)} h(u)$ is less than $\frac{1}{2}$. It follows that

$$G(u) \geq \|u\|^2 - \frac{1}{2}\|u\|^2 - \delta^{-\frac{1}{p_3}}\|u\|^{2+\frac{1}{p_3}},$$

and the right-hand side is positive when we take the limit as $\|u\| \to 0$ since $2 + \frac{1}{p_3} > 2$. In particular, there exists $\rho > 0$ such that for all $u \in B_\rho(0) \setminus \{0\}$ we have $G(u) > 0$.

(iii) First, notice that for $u \in \mathcal{M}$ we have

$$\begin{aligned}\langle G'(u),\, u\rangle &= 2\|u\|^2 - \langle \Psi'(u),\, u\rangle = \\ &= 2\Psi(u) - \langle \Psi'(u),\, u\rangle.\end{aligned}$$

One also has that

$$\begin{aligned}2\Psi(u) - \langle \Psi'(u),\, u\rangle &= 2\int_\Omega u f(u)\,\mathrm{d}x - \left[\int_\Omega u f(u)\,\mathrm{d}x + \int_\Omega u^2 f'(u)\,\mathrm{d}x\right] = \\ &= \int_\Omega u^2 h(u)\,\mathrm{d}x - \int_\Omega u^2(h(u) + u h'(u))\,\mathrm{d}x = \\ &= -\int_\Omega u^3 h'(u)\,\mathrm{d}x.\end{aligned}$$

Since $0 \notin \mathcal{M}$, using **(h_2)** that holds for all $u \neq 0$ we conclude that the scalar product must be negative.

□

It follows from **(iii)** that $\mathcal{M}$ is a submanifold of class C^2 of codimension one in E. Now let $\widetilde{J} \in C^2(E, \mathbb{R})$ be defined as

$$\widetilde{J}(u) = \frac{1}{2}\Psi(u) - \Phi(u).$$

Notice that this functional coincides with J on $\mathcal{M}$, but it is more convenient to deal with it since it is weakly continuous and its derivative is compact.

Lemma 6.25. *The submanifold $\mathcal{M}$ is a natural constraint for J using $\widetilde{J}$, that is,*

$$z \in \mathcal{M},\ \nabla_{\mathcal{M}}\widetilde{J}(z) = 0 \implies J'(z) = 0.$$

Proof. If z is such a point, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla \widetilde{J}(z) = \lambda \nabla G(z) \implies \langle \widetilde{J}(z),\, z\rangle = \lambda\langle \nabla G(z),\, z\rangle.$$

On the other hand, we know that

$$\langle \widetilde{J}(z),\, z\rangle = \frac{1}{2}\langle \Psi(z),\, z\rangle - \langle \Phi(z),\, z\rangle =$$

$$= \frac{1}{2}\langle \nabla\Psi(z),\, z\rangle - \Psi(z) =$$

$$= -\frac{1}{2}\langle \nabla G(z),\, z\rangle$$

so λ must be equal to $-\frac{1}{2}$. Then

$$\nabla G(z) = -\nabla\Psi(z) + 2z,$$

$$\nabla\widetilde{J}(z) = \frac{1}{2}\Psi(z) - \nabla\Phi(z),$$

and this immediately implies that $\nabla\Phi(z) = z$, which is completely equivalent to

$$\nabla J(z) = 0.$$

□

Lemma 6.26. *There exists $C_\alpha > 0$ such that $\widetilde{J}(u) \geq C_\alpha \|u\|^2$ for all $u \in \mathcal{M}$.*

Proof. We use the definition of Ψ and $(\mathbf{h_1})$ to infer that

$$\widetilde{J}(u) = \frac{1}{2}\int_\Omega uf(u)\,\mathrm{d}x - \int_0^1 \mathrm{d}s \int_\Omega uf(su)\,\mathrm{d}x = \int_0^1 \mathrm{d}s \int_\Omega [suf(u) - uf(su)]\,\mathrm{d}x =$$

$$= \int_0^1 \mathrm{d}s \int_\Omega \left[su^2\left(h(u) - h(su)\right)\right]\,\mathrm{d}x \geq$$

$$\geq \int_0^1 s(1 - s^\alpha)\,\mathrm{d}s \int_\Omega u^2 h(u)\,\mathrm{d}x \geq$$

$$\geq C_\alpha \underbrace{\int_\Omega u^2 h(u)\,\mathrm{d}x}_{=\Psi(u)} = C_\alpha \|u\|^2,$$

where the last equality follows from the fact that $u \in \mathcal{M}$ implies $\Psi(u) = \|u\|^2$. □

Lemma 6.27. *Let $(u_i)_{i\in\mathbb{N}}$ be a Palais-Smale sequence at level $c > 0$ for $\widetilde{J}$ on $\mathcal{M}$. Then*

(i) *$\|u_i\|$ is bounded and there exists $\bar{u} \neq 0$ such that $u_{i_\ell} \rightharpoonup \bar{u}$;*

(ii) *there exists $k > 0$ such that $\|\nabla\widetilde{J}(u_i)\| \geq k$.*

Proof.

(i) By definition

$$\widetilde{J}(u_i) \xrightarrow{i\to+\infty} c,$$

and using the previous result we also know that

$$\widetilde{J}(u_i) \geq c_\alpha \|u_i\|^2$$

so $\|u_i\|$ is bounded and $(u_i)_{i\in\mathbb{N}}$ converges weakly to some $\bar{u}$ up to subsequences. To prove that $\bar{u} \neq 0$, we notice that

$$u_i \in \mathcal{M} \implies \|u_i\| \geq \rho \implies \Psi(u_i) = \|u_i\|^2 \geq \rho^2.$$

But Ψ is weakly continuous so

$$\Psi(\bar{u}) \geq \rho^2 \implies \bar{u} \neq 0.$$

(ii) We argue by contradiction. Suppose that $\nabla\widetilde{J}(u_i) \to 0$. The operator $\nabla\widetilde{J}$ is compact, so we can conclude that

$$\nabla\widetilde{J}(\bar{u}) = 0 \implies 0 = \frac{1}{2}\langle\nabla\Psi(\bar{u}), \bar{u}\rangle - \Psi(\bar{u}) = \frac{1}{2}\int_\Omega \bar{u}^3 h'(\bar{u})\,\mathrm{d}x.$$

We know already that the right-hand side is strictly positive, so we obtained our contradiction.

□

Lemma 6.28. *The function $\widetilde{J}$, restricted to $\mathcal{M}$, satisfies the Palais-Smale condition at all levels $c > 0$.*

Proof. Let $(u_i)_{i\in\mathbb{N}}$ be a Palais-Smale sequence at level c and let $\bar{u}$ be the weak limit of a subsequence $(u_{i_k})_{k\in\mathbb{N}}$. We have

$$\nabla_{\mathcal{M}}\widetilde{J}(u_i) = \nabla\widetilde{J}(u_i) - \alpha_i \nabla G(u_i),$$

where

$$\alpha_i = \frac{\langle\nabla\widetilde{J}(u_i),\, \nabla G(u_i)\rangle}{\|\nabla G(u_i)\|^2}.$$

We proved already that $\|\nabla\widetilde{J}(u_i)\| \geq k$ and it is easy to see that $\|\nabla G(u_i)\| \leq c$, so taking into account that

$$\underbrace{\nabla_{\mathcal{M}}\widetilde{J}(u_i)}_{\to 0} = \underbrace{\nabla\widetilde{J}(u_i)}_{\not\to 0} - \alpha_i \underbrace{\nabla G(u_i)}_{\text{bounded}},$$

we must have $|\alpha_i| \geq c > 0$. It follows that

$$\nabla G(u_i) = \frac{1}{\alpha_i}\left[\nabla\widetilde{J}(u_i) - \nabla_{\mathcal{M}}\widetilde{J}(u_i)\right],$$

which easily translates to

$$2u_i = \underbrace{\nabla\Psi(u_i)}_{\text{compact}} + \frac{1}{\alpha_i}\left[\underbrace{\nabla\widetilde{J}(u_i)}_{\text{compact}} - \underbrace{\nabla_{\mathcal{M}}\widetilde{J}(u_i)}_{\to 0}\right]$$

and thus u_{i_k} converges strongly to $\bar{u}$, concluding the proof. □

Proof of Theorem 6.23. Simply apply Theorem 6.21 replacing f with its positive part f^+. □

Chapter 7

Min-max Methods

In this chapter, we will discuss the existence of stationary point of a function J, defined on a Hilbert space $\mathfrak{X}$, which can be found via different min-max procedures.

7.1 The mountain pass theorem

We proved that (DSL) admits a positive solution, provided that f satisfies certain assumptions including a growth condition

$$h(su) \leq s^{\alpha} h(u),$$

that holds at all point $u \in \mathfrak{X}$. A natural question is whether or not we can prove a similar result when the behaviour of f is only known at the origin and at infinity. To deal with this problem, we consider the corresponding functional

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u)\,\mathrm{d}x,$$

with $\|\cdot\| = \|\cdot\|_{\mathfrak{X}}$ and $\mathfrak{X} = H_0^1(\Omega)$. It is easy to verify that $u = 0$ is a proper local minimum for J since, assuming that $f'(0) = 0$, we have

$$f'(0) = 0 \implies \langle J''(0)v,\, v\rangle_{\mathfrak{X}} = \|v\|^2.$$

On the other hand, if we assume that $F(u) \sim |u|^{p+1}$, $1 < p < \frac{n+2}{n-2}$, then for any $u \in \mathfrak{X}$ that is different from zero we find that

$$\lim_{t\to+\infty} J(tu) = \lim_{t\to+\infty} \left[\frac{t^2}{2}\|u\|^2 - \int_{\Omega} F(tu)\,\mathrm{d}x\right] = -\infty.$$

In particular, the functional J is not bounded from below on $\mathfrak{X}$. We also notice that

$$\sup_{\mathfrak{X}} J = +\infty$$

since we can always consider a sequence of function $\|u_i\| \to +\infty$ with $\int_\Omega F(u_i)\,\mathrm{d}x$ uniformly bounded. Now to fix the ideas, consider the model nonlinearity so that

$$J(tu) = \frac{1}{2}t^2\|u\|^2 - \frac{1}{p+1}|t|^{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x.$$

The real valued map $t \mapsto J(tu)$ achieves its maximum at a unique point $t = t_u > 0$ and, as expected, it is determined by the fact that

$$tu \in \mathcal{M} := \{u \in \mathfrak{X} \setminus \{0\} \,:\, \langle J'(u),\, u\rangle = 0\},$$

where $\mathcal{M}$ is the natural constraint introduced many times before. If z is a critical point for J, we know that $J(z)$ is equal to the minimum value of J achieved on $\mathcal{M}$, and thus

$$J(z) = \min_{u\in\mathfrak{X}\setminus\{0\}} \max_{t\in\mathbb{R}} J(tu).$$

The main goal of this section is to generalise this argument and to find optimal assumptions that allow one to find critical points of a functional J via a max-min procedure.

In the sequel, to fix the notation, we will assume that J has a local minimum at $u = 0$, but it is important to understand that this is a totally arbitrary choice.

(MP − **1**) The functional J belongs to $C^1(\mathfrak{X}, \mathbb{R})$, $J(0) = 0$ and there are r, $\rho > 0$ such that $J(u) \geq \rho$ for all $u \in S_r$.

(MP − **2**) There exists $e \in \mathfrak{X}$ with $\|e\| > r$ such that $J(e) \leq 0$.

We will show that these assumptions on the geometry of J are almost enough for the existence of a saddle point. Let

$$\Gamma := \{\gamma \in C([0,\, 1],\, \mathfrak{X}) \,:\, \gamma(0) = 0,\, \gamma(1) = e\}$$

be the set of all continuous curves connecting 0 and e and notice that it is nonempty since

$$t \longmapsto te$$

trivially belongs to Γ. We define the *MP level* as

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,\,1]} J(\gamma(t)). \tag{7.1}$$

If J is a functional that has the *MP geometry*, which means that it satisfies the two assumptions above, then it is easy to see that

$$\gamma \in \Gamma \implies \gamma([0,\,1]) \cap S_r \neq \varnothing \implies c \geq \min_{u\in S_r} J(u) \geq \rho > 0,$$

so if we were to find a critical point u at the level c, we could immediately conclude that it is not trivial ($u \neq 0$). The following result is due to Ambrosetti and Rabinowitz in 1973.

Theorem 7.1 (Mountain Pass)**.** *Let J be a functional satisfying* (MP − **1**) *and* (MP − **2**)*. Suppose that the Palais-Smale condition at the level c given by* (7.1) *holds. Then*

$$\exists z \in \mathfrak{X} \,:\, J(z) = c,\, \nabla J(z) = 0$$

and z is nontrivial, that is, $z \neq 0$.

To prove this result, we first need a technical lemma which gives us the existence of a particular deformation of the sublevels of J that keeps a good portion of them fixed.

Lemma 7.2. *Let $J \in C^1(\mathfrak{X}, \mathbb{R})$ and let $c \in \mathbb{R}$ be any noncritical value for J. Suppose that the Palais-Smale condition at the level c holds for J. Then there are $\delta > 0$ with $c - 2\delta > 0$ and η deformation in $\mathfrak{X}$ such that:*

- ***(a)*** $\eta(J^{c+\delta}) \subseteq J^{c-\delta}$*;*
- ***(b)*** *η restricted to $J^{c-2\delta}$ coincides with the identity map.*

Proof. Recall that J always admits a Ψ-gradient flow V for J, that is defined at all points $u \in \mathfrak{X}$ such that $\nabla J(u) \neq 0$, with the following properties:

- **(i)** $\|V(u)\| \leq 2\|\nabla J(u)\|$;
- **(ii)** $\langle V(u),\, \nabla J(u)\rangle_{\mathfrak{X}} \geq \|\nabla J(u)\|^2$.

Let $b \in C^{0,1}(\mathbb{R}^+, \mathbb{R}^+)$ be the Lipschitz function defined by setting

$$b(s) := \begin{cases} 1 & \text{if } s \in (0,\, 1], \\ \frac{1}{s} & \text{if } s \geq 1. \end{cases}$$

Let $A := \{u \in \mathfrak{X} \,:\, J(u) \in [c-\delta,\, c+\delta]$, $B := \{u \in \mathfrak{X} \,:\, J(u) \in (c-2\delta,\, c+2\delta)^c\}$ and define the Lipschitz function from $\mathfrak{X}$ to $\mathbb{R}$ given by

$$g(u) := \frac{d_{\mathfrak{X}}(u,\, B)}{d_{\mathfrak{X}}(u,\, A) + d_{\mathfrak{X}}(u,\, B)} \in [0,\, 1].$$

Notice that g is equal to zero if and only if $u \in B$ and equal to one if and only if $u \in A$. We can consider a slightly modified vector field as flow

$$\widetilde{V}(u) := -g(u)b(\|\nabla J(u)\|)V(u).$$

There are several vantages in replacing V with $\widetilde{V}(u)$. First, it is well-defined everywhere (even where the differential of J vanishes), the boundedness of b gives the global existence of η and it is locally Lipschitz. Consider the solution of

$$\begin{cases} \alpha'(t) = \widetilde{V}(\alpha(t)), \\ \alpha(0) = u, \end{cases}$$

and notice that the following properties are satisfied:

(i) If $u \in B$, the $\widetilde{V}(u) = 0$ and thus $\alpha(t, u) = u$ at all times $t \in \mathbb{R}^+$.

(ii) The solution α is globally defined and $\|\widetilde{V}(u)\| \leq 2$.

(iii) The function $t \mapsto J(\alpha(t, u))$ is non-increasing since

$$\frac{\mathrm{d}}{\mathrm{d}t} J(\alpha(t, u)) = -g(\alpha) b(\|\nabla J(\alpha)\|) \langle \nabla J(\alpha), V(\alpha) \rangle_{\mathfrak{X}} \leq$$

$$\leq -g(\alpha) b(\|\nabla J(\alpha)\|) \|\nabla J(\alpha)\|^2 \leq 0.$$

Now let $\delta > 0$ be such that

$$J(u) \in [c - \delta, c + \delta] \implies \|\nabla J(u)\| \geq \delta,$$

and suppose that $c - 2\delta > 0$. Let $T = \frac{2}{\delta}$ and define the deformation by setting

$$\eta(u) := \alpha(T, u).$$

Then **(b)** trivially holds true, so we only need to check that η satisfies **(a)**. For this, let $u \in J^{c+\delta}$ and suppose that $\eta(u) \notin J^{c-\delta}$. It follows that

$$J(\alpha(t, u)) \in [c - \delta, c + \delta] \quad \text{for all } t \in [0, T],$$

and hence $\alpha(t, u)$ belongs to A for t in the same interval. Using the definition of g we infer that

$$g(\alpha(t, u)) = 1 \quad \text{for all } t \in [0, T],$$

so

$$J(\eta(u)) - J(u) = -\int_0^T b(\|\nabla J(\alpha(t, u))\|) \langle \nabla J(\alpha(t, u)), V(\alpha(t, u)) \rangle_{\mathfrak{X}} \, \mathrm{d}t \leq$$

$$\leq -\int_0^T b(\|\nabla J(\alpha(t, u))\|) \|\nabla J(\alpha(t, u))\|^2 \, \mathrm{d}t \leq$$

$$\leq -\int_0^T \delta^2 \, \mathrm{d}t = -2\delta,$$

and this gives a contradiction since

$$J(\eta(u)) \leq J(u) - 2\delta \leq c + \delta - 2\delta = c - \delta \implies \eta(u) \in J^{c-\delta}.$$

□

Proof of Theorem 7.1. We argue by contradiction. Suppose that the MP level c is not critical and let η be the deformation given by Lemma 7.2. Now notice that

$$0,\, e \in J^0 \implies 0,\, e \in J^{c-2\delta} \implies (\gamma \in \Gamma \implies \eta \circ \gamma \in \Gamma)\,,$$

so η associates a curve in Γ to any curve in Γ. Recall that

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,\, 1]} J(\gamma(t)),$$

so for any $\delta > 0$ we can find $\gamma \in \Gamma$ such that

$$\max_{t \in [0,\, 1]} J(\gamma(t)) \leq c + \delta.$$

The deformation η maps $\gamma([0,\, 1])$ into $J^{c-\delta}$ so

$$\max_{t \in [0,\, 1]} J(\eta \circ \gamma(t)) \leq c - \delta,$$

and this is a contradiction since c is the infimum value and yet $\eta \circ \gamma \in \Gamma$. □

Remark 7.3. We cannot remove the assumption that J satisfies the Palais-Smale condition at the MP level c. Indeed, it is easy to find a counterexample in $\mathbb{R}^2$ for which J has the MP geometry but there are not critical points except for $(0,\, 0)$. Namely, let

$$J(x,\, y) = x^2 + (1-x)^3 y^2,$$

and notice that (MP − **1**) is satisfied with $r = \frac{1}{2}$ and $\rho = \frac{1}{32}$, while (MP − **2**) is satisfied with $e = (2,\, 2)$.

7.2 Application to the Dirichlet problem

In this section, we will exploit the theoretical results presented above to prove existence of a positive solution to a class of Dirichlet boundary-value problems:

$$\begin{cases} -\Delta u(x) = f(u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{D}$$

We assume Ω to be a smooth bounded domain in $\mathbb{R}^n$ and f a function satisfying the following assumptions:

(f_1) f is Carathéodory;

(f_2) $|f(x,\, u)| \leq a + b|u|^p$ for some $1 < p < \frac{n+2}{n-2}$;

(f_3) $\lim_{|u| \to 0^+} \frac{f(x,\, u)}{|u|} = \lambda \in \mathbb{R}$ uniformly with respect to $x \in \Omega$;

(f_4) there exists $r > 0$ and $\theta \in (0, \frac{1}{2})$ such that

$$0 < F(x, u) \le \theta u f(x, u) \tag{7.2}$$

for all u with norm $\|u\| \ge R$.

Lemma 7.4. *If f satisfies the property (f_4), then*

$$F(u) \ge \frac{1}{c} u^{\frac{1}{\theta}} - c \quad \text{for all } u \ge R. \tag{7.3}$$

Lemma 7.5. *If $\lambda < \lambda_1(\Omega)$, then* (MP − **1**) *holds.*

Proof. Fix $\epsilon := \frac{1}{2}(\lambda_1 - \lambda) > 0$. The assumptions on f allows us to find a constant $A \in \mathbb{R}$ such that

$$F(x, u) \le \frac{1}{2}(\lambda + \epsilon)u^2 + A|u|^{p+1}.$$

Now integrate and use Sobolev embedding to infer that

$$\left| \int_\Omega F(x, u)\, \mathrm{d}x \right| \le \frac{1}{2}(\lambda + \epsilon)\|u\|^2_{L^2(\Omega)} + A\|u\|^{p+1}_{L^{p+1}(\Omega)} \le$$

$$\le \frac{1}{2}(\lambda + \epsilon)\|u\|^2_{L^2(\Omega)} + A'\|u\|^{p+1}$$

so that the functional can be estimated by

$$J(u) \ge \frac{1}{2}\|u\|^2 - A'\|u\|^{p+1} - \frac{1}{2}(\lambda + \epsilon)\|u\|^2_{L^2(\Omega)}.$$

We now recall that

$$\|u\|^2_{L^2(\Omega)} \ge \frac{1}{\lambda_1}\|u\|^2$$

so

$$J(u) \ge \frac{1}{2}\left(\frac{\lambda_1 - \lambda - \epsilon}{\lambda_1}\right)\|u\|^2 - A'\|u\|^{p+1},$$

and the first term is multiplies a positive constant. □

Lemma 7.6. *Under no extra assumpions* (MP − **2**) *holds.*

Proof. Let $e \in \mathfrak{X}$ smooth and positive on Ω. Then for $t \in \mathbb{R}$ we have

$$J(te) = \frac{1}{2}t^2\|e\|^2 - \int_\Omega F(x, te)\, \mathrm{d}x \ge$$

$$\ge \frac{1}{2}t^2\|e\|^2 - \left(\frac{t^{\frac{1}{\theta}}}{c}\|e\|_{L^\theta(\Omega)} - c'\right)|\Omega| \ge$$

$$\ge \frac{1}{2}t^2\|e\|^2 - C_\Omega t^{\frac{1}{\theta}}\|e\|_{L^\theta(\Omega)} + C_\Omega \xrightarrow{t \to +\infty} -\infty.$$

Therefore, we can find $\tau \in \mathbb{R}^+$ such that τe satisfies $J(\tau e) \leq 0$. □

To apply the MP theorem, we only need to prove that J satisfies the Palais-Smale condition at the level c. A standard argument shows that

$$(u_n)_{n\in\mathbb{N}} \in (\mathrm{PS})_c \text{ for } J \implies u_n \text{ bounded},$$

but the reader may try to prove this themselves as an exercise to get acquainted with the notion of Palais-Smale.

Lemma 7.7. *Under no extra assumpions, the functional J satisfies the Palais-Smale condition at the level $c > 0$.*

Proof. First, we evaluate Φ at u_n and decompose the integral in such a way that we can use (f_4). Namely,

$$\begin{aligned}\Phi(u_n) &= \int_{u_n\leq R} F(x,\, u_n)\,\mathrm{d}x + \int_{u_n\geq R} F(x,\, u_n)\,\mathrm{d}x \leq \\ &\leq C_{\Omega,\, R,\, f} + \theta \int_{u_n\geq R} u_n f(x,\, u_n)\,\mathrm{d}x \leq \\ &\leq C'_{\Omega,\, R,\, f} + \theta \int_\Omega u_n f(x,\, u_n)\,\mathrm{d}x \leq \\ &\leq C'_{\Omega,\, R,\, f} + \theta \left[\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + o(\|u_n\|)\right],\end{aligned}$$

where the last inequality follows from the definition of differentiable:

$$o(\|u_n\|) = \nabla J(u_n)[u_n] = \int_\Omega |\nabla u_n|^2\,\mathrm{d}x - \int_\Omega u_n f(x,\, u_n).$$

Since u_n is a Palais-Smale sequence, $|J(u_n)| \leq c$ and hence

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x \leq C + 2\Phi(u_n) \leq C''2\theta \left[\int_\Omega |\nabla u|^2\,\mathrm{d}x + o(\|u_n\|)\right].$$

Recalling that $2\theta < 1$, this implies that

$$\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \leq \tilde{C} + o(\|u_n\|).$$

Now notice that $p < \frac{n+2}{n-2}$, so Φ is weakly continuous and its differential is a compact operator. From

$$\nabla J(u_n)[v] = \langle u_n,\, v\rangle - \langle \nabla\Phi(u_n),\, v\rangle,$$

we conclude that

$$\nabla J(u_n) = u_n - \nabla\Phi(u_n).$$

Since u_n is Palais-Smale, u_n is bounded and hence we can find a subsequence u_{n_k} converging weakly to some $\bar{u}$. Furthermore,

$$u_{n_k} = \nabla J(u_{n_k}) + \nabla \Phi(u_{n_k}),$$

and the first term $\nabla J(u_{n_k})$ converges strongly to zero, so by compactness $\nabla \Phi(u_{n_k})$ must converge strongly to $\nabla \Phi(\bar{u})$. □

This proves that J satisfies the Palais-Smale condition at the level c. We can finally apply Theorem 7.1 and conclude that (D) admits a positive solution.

7.3 Linking theorems

Let $\mathscr{C}$ be a nonempty class of subsets $A \subseteq \mathfrak{X}$. Suppose that

$$c := \inf_{A \in \mathscr{C}} \sup_{u \in A} J(u) > -\infty.$$

The idea is that, if $\mathscr{C}$ is stable under deformations, we can do a sort of MP theorem for which c is a candidate min-max level.

Definition 7.8 (Link)**.** Let $\mathcal{N}$ be a compact manifold with nonempty boundary and let $C \subseteq \mathfrak{X}$ be a subset. Consider the class of homotopies

$$\mathscr{H} := \left\{ h \in C(\mathcal{N}, \mathfrak{X}) \ : \ h\,\big|_{\partial \mathcal{N}} \equiv \mathrm{id}_{\partial \mathcal{N}} \right\}.$$

We say that $\partial \mathcal{N}$ and C *link* if

$$h(\mathcal{N}) \cap C \neq \varnothing \quad \text{for all } h \in \mathscr{H}.$$

Example 7.9. The MP theorem is a linking-type theorem with $C = S_R$ and

$$\mathcal{N} := \{te \ : \ t \in [0, 1]\}.$$

It is easy to verify that C and $\partial \mathcal{N}$ link using Bolzano's theorem.

We will now investigate the linking property between slightly more complicated sets. From now on, we will make use of **degree theory** and, in particular, of the homotopy property. The reader that is not acquainted with it can find the formal construction and the main properties in [1].

Proposition 7.10. *Let $\mathfrak{X}$ be a normed vector space and assume that $\mathfrak{X} := V \oplus W$ with V, W closed subspaces and $\dim(V) = k < \infty$. Then*

$$C := W \quad \textit{and} \quad \mathcal{N} := \{v \in V \ : \ \|v\| \leq r\}$$

link.

Proof. Let $h \in \mathscr{H}$ and let $p : \mathfrak{X} \to V$ be the projection associated to the direct sum. Then

$$\widetilde{h} := p \circ h : \mathcal{N} \longrightarrow V$$

coincides with the identity on $\partial\mathcal{N}$. It follows from degree theory that $\widetilde{h}$ vanishes at some $z \in \mathcal{N}$ that does not belong to the boundary, and hence

$$\widetilde{h}(z) = 0 \implies h(z) \in V^c = W = C.$$

□

Proposition 7.11. *Let $\mathfrak{X}$ be a normed vector space and assume that $\mathfrak{X} := V \oplus W$ with V, W closed subspaces and $\dim(V) = k < \infty$. Given $e \in W$ and $R > 0$ define*

$$C := \{w \in W \ : \ \|w\| \le r,$$

and

$$\mathcal{N} := \{u = v + se \ : \ v \in V, \ \|v\| \le R, \ s \in [0,\, 1]\}.$$

Then C and $\partial\mathcal{N}$ link, provided that $\|e\| > r$.

Proof. Let $h \in \mathscr{H}$ and let $p : \mathfrak{X} \to V$ be the projection associated to the direct sum. Identify the manifold $\mathcal{N}$ with

$$\mathcal{N} \cong \bar{B}_V(0,\, R) \times \{se \ : \ s \in [0,\, 1]\}$$

and define

$$\widetilde{h}(u) := (p \circ h(u),\, \|h(u) - p \circ h(u)\| - r)\,.$$

We now evaluate it at the boundary $\partial\mathcal{N}$:

$$\widetilde{h}(v,\, s) = (v,\, \|e\| - r) \neq (0,\, 0).$$

It follows that we can apply once again degree theory to find $(v,\, s) \in \mathcal{N}$ such that $\widetilde{h}(v,\, s)$ vanishes. In particular,

$$\begin{cases} p \circ h(v + se) \\ \|h(v + se) - p \circ h(v + se)\| = r \end{cases} \implies h(v + se) \in W \quad \text{and} \quad \|h(v + se)\| = r,$$

which means that $h(v + se) \in C$, and this concludes the proof. □

We are now ready to generalise the MPT. Let $\mathfrak{X}$ be a Hilbert space, $J \in C^1(\mathfrak{X},\, \mathbb{R})$ and $\partial\mathcal{N}$, $C \subset \mathfrak{X}$ such that $\mathcal{N}_\partial$ and C link. Assume that

(J1) J is bounded from below on C, that is, $\rho := \inf_{u \in C} J(u) > -\infty$;

(J2) $\rho > \beta := \sup_{u \in \partial\mathcal{N}} J(u)$.

Definition 7.12. The number

$$c := \inf_{h \in \mathscr{H}} \sup_{u \in \mathcal{N}} J \circ h(u)$$

is called *linking level* associated to the function J.

Lemma 7.13. *Suppose that $\mathcal{N}_\partial$ and C link. If (J1) holds, then $c \geq \rho$.*

Proof. By definition, for each $h \in \mathscr{H}$ the intersection $h(\mathcal{N}) \cap C$ is nonempty. Thus

$$\sup_{u \in \mathcal{N}} J(h(u)) \geq \inf_{u \in C} J(u) = \rho.$$

□

Theorem 7.14. *Suppose that the following assumptions hold:*

(a) *$\mathcal{N}_\partial$ and C link.*

(b) *(J1) and (J2) hold.*

(c) *The functional satisfies the Palais-Smale condition at the linking level c.*

Then c is a critical value, that is, there exists $u \in \mathfrak{X}$ such that $J(u) = c$ and $\nabla J(u) = 0$.

Proof. Notice that $c \geq \rho > \beta$. Suppose that c is not a critical value and use the *deformation lemma* to find a continuous deformation η which satisfies

- $\eta(J^{c+\delta}) \subseteq J^{c-\delta}$ for δ such that $\beta < c - \delta$;
- $\eta(u) = u$ for all $u \in J^\beta$.

Now let $h \in \mathscr{H}$. It is easy to verify that $\eta \circ h \in \mathscr{H}$ since it is composition of continuous mappings and also

$$\eta(u) = u \text{ for all } u \in J^\beta \implies \eta \circ h\,\big|_{\partial \mathcal{N}} = \eta\,\big|_{\partial \mathcal{N}} = \mathrm{id}_{\partial \mathcal{N}}$$

since $\mathcal{N} \subset J^\beta$. Now let $\tilde{h} \in \mathscr{H}$ be such that

$$\sup_{u \in \mathcal{N}} J(\tilde{h}(u)) < c + \delta.$$

Then

$$\sup_{u \in \mathcal{N}} J(\eta \circ \tilde{h}(u)) < c - \delta,$$

and this gives a contradiction since c is the infimum. This concludes the proof. □

We now present three easy consequences of the theory developed in this section, which are incredibly interesting by themselves.

Theorem 7.15. *Let C be a manifold of codimension one in $\mathfrak{X}$ and suppose that u_0, u_1 are points of $\mathfrak{X} \setminus C$ belonging to two distinct connected components of $\mathfrak{X} \setminus C$. Let $J \in C^1(\mathfrak{X}, \mathbb{R})$ satisfy the following assumptions:*

(L-a) $\inf_C J(u) > \max\{J(u_0),\, J(u_1)\}$;

(L-b) *J satisfies the Palais-Smale condition at the linking level c.*

Then J has a critical point $\bar{u}$ at level c and $\bar{u} \neq u_0, u_1$.

Theorem 7.16. *Let $\mathfrak{X} = V \oplus W$, where V and W are closed subspaces and* $\dim(V) < \infty$. *Suppose $J \in C^1(\mathfrak{X}, \mathbb{R})$ satisfies:*

(L-a) *There exist $r, \rho > 0$ such that*

$$J(w) \geq \rho \quad \text{for all } w \in W \text{ with } \|w\| = r.$$

(L-b) *There exist $R > 0$ and $e \in W$, with $\|e\| > r$ such that, letting*

$$\mathcal{N} = \{u = v + te \;:\; v \in V,\, \|v\| \leq R,\, t \in [0, 1]\},$$

one has that

$$J(u) < 0 \quad \text{for all } u \in \partial\mathcal{N}.$$

If, in addition, J satisfies the Palais-Smale condition at the linking level c, then J has a critical point $\bar{u}$ at level $c > 0$. In particular, $\bar{u} \neq 0$.

Theorem 7.17. *Let $\mathfrak{X} = V \oplus W$, where V and W are closed subspaces and* $\dim(V) < \infty$. *Suppose $J \in C^1(\mathfrak{X}, \mathbb{R})$ satisfies:*

(L-a) *There exist $\rho > 0$ such that*

$$J(w) \geq \rho \quad \text{for all } w \in W.$$

(L-b) *There exist $r > 0$, $\beta < \rho$ such that*

$$J(u) \leq \beta \quad \text{for all } u \in V \text{ with } \|v\| = r.$$

If, in addition, J satisfies the Palais-Smale condition at the linking level c, then J has a critical point $\bar{u}$ at level $c > 0$.

7.3.1 Application of the saddle point theorem

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary. Consider the Dirichlet problem

$$\begin{cases} -\Delta u - \lambda u = f(x, u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{7.4}$$

Theorem 7.18. *Suppose that*

(i) λ *is not an eigenvalue of* $-\Delta$*;*

(ii) f *satisfies the Carathéodory condition;*

(iii) f *is sublinear, that is, there is* $\alpha < 1$ *such that*

$$|f(x, s)| \leq a + b|s|^{\alpha}.$$

Then ***(L-1)*** *and* ***(L-2)*** *hold. Furthermore, the functional associated to the problem,*

$$J(u) = \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) \, \mathrm{d}x - \int_{\Omega} F(x, u) \, \mathrm{d}x,$$

satisfies the Palais-Smale condition at any level. In particular, the linking level is critical.

Proof. By assumption, there exists $k \in \mathbb{N}$ such that $\lambda \in (\lambda_k, \lambda_{k+1})$. If φ_j denotes the jth eigenfunction, then we can take

$$V := \mathrm{Span}\langle \varphi_1, \ldots, \varphi_k \rangle.$$

In this case, W is the complementary subspace in $\mathfrak{X} := H_0^1(\Omega)$. Notice that the quadratic form

$$Q(u) = |\nabla u|^2 - \lambda |u|^2$$

is definite negative on V and definite negative on W, and also that the sublinearity of f together with the Sobolev embedding implies that

$$\left| \int_{\Omega} F(x, u) \, \mathrm{d}x \right| \leq A + B \|u\|^{\alpha+1}.$$

It follows that there exists $\gamma > 0$ such that

$$u \in V \implies J(u) \leq -\gamma \|u\|^2 + A + B\|u\|^{\alpha+1} \xrightarrow{\|u\| \to \infty} -\infty,$$

which means that we can select R big enough in Theorem 7.16 for which **(L-2)** holds. In a similar fashion, notice that

$$u \in W \implies J(u) \geq \gamma \|u\|^2 - A - B\|u\|^{\alpha+1},$$

which means that $\rho := \inf_W J(u) > -\infty$. Since R is arbitrarily big, we can also require that $\beta < \rho$ and thus **(L-1)** holds as well.

Palais-Smale condition. Write $u = u_V + u_W$. Then

$$\nabla J(u)[u_V] = \nabla Q(u)[u_V] - \int_{\Omega} f(x, u) u_V \, \mathrm{d}x = 2Q(u_V) - \int_{\Omega} f(x, u) u_V \, \mathrm{d}x.$$

Let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence. Then

$$\nabla J(u_n)[(u_n)_V] = o(\|u_n\|)$$

because $\nabla J(u_n)[(u_n)_V]$ converges to zero; on the other hand, the identity above suggests that

$$\nabla J(u_n)[(u_n)_V] = 2Q((u_n)_V) + \mathcal{O}(1 + \|u\|^{\alpha+1}).$$

Since $Q < 0$ on V, we can easily infer that

$$\gamma\|(u_n)_V\|^2 \leq o(\|u_n\|) + \mathcal{O}(1 + \|u_n\|^{1+\alpha}).$$

In a similar fashion, we make the same computation on W and find that

$$\gamma\|(u_n)_W\|^2 \leq o(\|u_n\|) + \mathcal{O}(1 + \|u_n\|^{1+\alpha}).$$

Therefore, any Palais-Smale sequence for the functional J is bounded in the $\|\cdot\|_{\mathfrak{X}}$-norm. For the compactness, notice that u_n bounded implies

$$u_{n_k} \rightharpoonup \bar{u}$$

and, using the fact that V is finite-dimensional, we also have that

$$(u_{n_k})_V \xrightarrow{k\to\infty} \bar{u}_V.$$

Moreover, we have

$$\nabla J(u_n)[v] = \int_\Omega (\nabla u_n \cdot \nabla v - \lambda u_n v)\,\mathrm{d}x - \int_\Omega f(x,\, u_n)v\,\mathrm{d}x,$$

which gives

$$\int_\Omega \nabla u_n \cdot \nabla v\,\mathrm{d}x = \nabla J(u_n)[v] + \int_\Omega \lambda u_n v\,\mathrm{d}x + \int_\Omega f(x,\, u_n)v\,\mathrm{d}x.$$

We conclude that the convergence is strong (up to subsequences) because the first addendum converges to zero, while the other two are compact linear operators by Sobolev embedding and sublinearity of f. □

Remark 7.19. If $\lambda = \lambda_k$, then the existence of the solution is not guaranteed. Indeed, if we consider the problem

$$\begin{cases} -\Delta u - \lambda u = \varphi_k & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

then it is easy to verify that it does not admit any solution since u should be in the orthogonal of the linear space generated by φ_k.

To conclude this section, we want to point out why any Palais-Smale sequence is bounded is enough to infer that the functional J satisfies $(\mathrm{PS})_c$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded set and let f be a function that satisfies the Carathéodory condition and the growth condition

$$|f(x, s)| \leq A + B|s|^p \quad \text{for } p < \frac{n+2}{n-2} \text{ if } n \geq 3.$$

Let $F(x, u) := \int_0^u f(x, s)\, \mathrm{d}s$ and $\Phi(u) = \int_\Omega F(x, u)\, \mathrm{d}x$. We claim that $\nabla\Phi$ is compact as an operator from $\mathfrak{X} := H_0^1(\Omega)$ to $\mathfrak{X}$.

Proof. Let u_n be a bounded sequence in $\mathfrak{X}$ weakly converging to some $\bar{u}$. Then u_{n_k} converges strongly to $\bar{u}$ in $L^{p+1}(\Omega)$ and by Nemitski theorem

$$f(x, u_{n_k}) \to f(x, \bar{u}) \quad \text{strongly in } L^{\frac{p+1}{p}}(\Omega).$$

This implies that $\|\nabla\Phi(u_{n_k}) - \nabla\Phi(u)\| \to 0$ which is enough to infer that $\nabla\Phi$ is compact. ☐

Corollary 7.20. *Consider the Dirichlet problem*

$$\begin{cases} -\Delta u = f(x, u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

and the associated function $J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, \mathrm{d}x - \Phi(u)$. Then the following properties hold:

- ***(a)*** *If u_n converges weakly to $\bar{u}$ and $\nabla J(u_n) \to 0$, then u_{n_k} converges strongly to $\bar{u}$.*
- ***(b)*** *If Palais-Smale sequences at the level c for J are bounded, then J satisfies the $(\mathrm{PS})_c$ condition.*

7.3.2 Application of linking-type theorems

Theorem 7.21. *Let Ω be a bounded subset of $\mathbb{R}^n$. Let $f \in \mathbb{F}_p$ with $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$, and suppose that*

$$\lim_{s \to 0^+} \frac{f(x, s)}{s} = \lambda \quad \textit{for a.e. } x \in \Omega,$$

for any $\lambda \in \mathbb{R}$, and

$$\exists\, r >, \theta \in (0, \frac{1}{2}) \;:\; 0 < F(x, u) \leq \theta u f(x, u) \quad \textit{for all } x \in \Omega \textit{ and all } u \geq r.$$

Then the Dirichlet problem (7.4) admits a nontrivial solution.

Proof. We prove the result for the model problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{7.5}$$

Let $\mathfrak{X} = H_0^1(\Omega)$ and consider the associated functional

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda\|u\|^2_{L^2(\Omega)}\frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

If $\lambda < \lambda_1$, then we have proved already (see [REF]) that the MPT is enough to infer the existence of a nontrivial solution. So we can assume without loss of generality that

$$\lambda_k \leq \lambda < \lambda_{k+1}$$

for some $k \geq 1$; the idea is to apply Theorem 7.16 with $V = \mathrm{Span}\langle \varphi_1, \cdots, \varphi_k\rangle$ and $W = V^\perp$, the L^2 complement of V. Indeed, if $w \in W$ we can always write

$$w = \sum_{i=k+1}^{\infty} a_i \varphi_i.$$

If $\|w\| \to 0$, then

$$J(w) = \frac{1}{2}\sum_{i=k+1}^{\infty} a_i^2 \left(1 - \frac{\lambda}{\lambda_i}\right) + o(\|w\|^2) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|w\|^2 + o(\|w\|^2),$$

and the latter is always strictly positive since $\lambda < \lambda_{k+1}$ by assumption. In particular, the assumption $(\mathbf{L}-\mathbf{a})$ of Theorem 7.16 holds with r small enough. Now let $\tilde{V}$ be a finite-dimensional subspace of $\mathfrak{X}$ and $\tilde{v} \in \tilde{V}$ be an element with unitary norm; it turns out that

$$J(t\tilde{v}) = \frac{1}{2}t^2 - \frac{1}{2}\lambda^2 t^2 \|\tilde{v}\|^2_{L^2(\Omega)} - \frac{1}{p+1}t^{p+1}\|\tilde{v}\|^{p+1}_{L^{p+1}(\Omega)}.$$

Since $p > 1$ and $\tilde{V}$ finite-dimensional, it follows that we can always find $t > 0$ big enough such that the quantity above is strictly negative. In particular, we can find $R > r$ and $e \in W$, $\|e\| = R$, such that

$$\|v + te\| \geq R \implies J(v + te) < 0.$$

Then on the three sides of $\partial\mathcal{N}$ given by $\{u = v + te \,:\, \|v\| = R\} \cup \{u = v + Re\}$ the functional J is strictly negative. It remains to see what happens on the fourth side of $\partial\mathcal{N}$, namely

$$\{v \in V \,:\, \|v\| \leq R\}.$$

However, it is easy to verify that $v = \sum_{i=1}^k a_i \varphi_i$ gives $\|v\|^2_{L^2(\Omega)} = \sum_{i=1}^k \lambda_i^{-1} a_i^2$; this implies that

$$\|v\|^2_{L^2(\Omega)} \geq \lambda_k^{-1}\|v\|^2,$$

and hence

$$J(v) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v\|^2 \leq 0.$$

This shows that $(\mathbf{L}-\mathbf{b})$ holds as well. The Palais-Smale condition is obtained in the same way as Theorem 7.18 so we can apply Theorem 7.16 to conclude. □

Remark 7.22. Notice that $J\big|_C > 0$ strictly, so a solution corresponding to a critical point at the level c is necessarily nontrivial.

7.4 The Pohozaev identity

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary. Consider the Dirichlet boundary value problem with nonlinearity independent of x, that is,

$$\begin{cases} -\Delta u = f(u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{7.6}$$

and let $F(u) = \int_0^u f(s)\,\mathrm{d}s$.

Theorem 7.23 (Pohozaev)**.** *Let ν denote the unit outer normal at $\partial\Omega$. If u is any classical solution of* (7.6)*, then the following identity holds:*

$$n \int_\Omega F(u)\,\mathrm{d}s = \frac{1}{2}\int_{\partial\Omega} u_\nu^2 (x\cdot\nu)\,\mathrm{d}\sigma + \frac{n-2}{2}\int_\Omega u f(u)\,\mathrm{d}x. \tag{7.7}$$

Proof. Set $\Theta(x) := (x \cdot \nabla u(x))\nabla u(x)$. Then

$$\begin{aligned} \operatorname{div}\Theta &= \Delta u(x\cdot\nabla u) + \sum_k \frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_k}\left(\sum_i x_i \frac{\partial u}{\partial x_i}\right) = \\ &= \Delta u(x\cdot\nabla u) + \sum_k \left(\frac{\partial u}{\partial x_k}\right)^2 + \sum_{i,\,k} \frac{\partial u}{\partial x_k} x_i \frac{\partial^2 u}{\partial x_i \partial x_k} = \\ &= \Delta u(x\cdot\nabla u) + |\nabla u|^2 + \frac{1}{2}\sum_i x_i \frac{\partial}{\partial x_i}|\nabla u|^2. \end{aligned}$$

Then an application of the divergence theorem shows that

$$\int_\Omega \left[\Delta u(x\cdot\nabla u) + |\nabla u|^2 + \frac{1}{2}\sum_i x_i\frac{\partial}{\partial x_i}|\nabla u|^2\right]\mathrm{d}x = \int_{\partial\Omega}(x\cdot\nabla u)(\nabla u\cdot\nu)\,\mathrm{d}\sigma.$$

As for the boundary term, since $u = 0$ on $\partial\Omega$ one has that $\nabla u(x) = u_\nu \nu$ and thus the above equation becomes

$$\int_\Omega \left[\Delta u(x\cdot\nabla u) + |\nabla u|^2 + \frac{1}{2}\sum_i x_i\frac{\partial}{\partial x_i}|\nabla u|^2\right]\mathrm{d}x = \int_{\partial\Omega}(x\cdot\nu)u_\nu^2\,\mathrm{d}\sigma.$$

Now set $\Theta_1(x) := \frac{1}{2}|\nabla u|^2 x$. Since its divergence is

$$\operatorname{div}\Theta_1 = \frac{n}{2}|\nabla u|^2 + \frac{1}{2}\sum_i x_i \frac{\partial}{\partial x_i}|\nabla u|^2,$$

another application of the divergence theorem shows that

$$\int_\Omega \left[\frac{n}{2}|\nabla u|^2 + \frac{1}{2}\sum_i x_i \frac{\partial}{\partial x_i}|\nabla u|^2\right] \mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} (x\cdot\nu) u_\nu^2 \,\mathrm{d}\sigma.$$

If we plug this into the previous identity we find that

$$\int_\Omega \Delta u(x\cdot\nabla u)\,\mathrm{d}x + (1-\frac{n}{2})\int_\Omega |\nabla u|^2\,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)u_\nu^2\,\mathrm{d}\sigma. \tag{7.8}$$

The first integral can easily be rewritten using the equation (7.6) of which u is a solution; namely, we have

$$-\int_\Omega \Delta u(x\cdot\nabla u)\,\mathrm{d}x = \int_\Omega f(u)(x\cdot\nabla u)\,\mathrm{d}x = \int_\Omega \sum_i x_i \frac{\partial F(u)}{\partial x_i}\,\mathrm{d}x.$$

Integrating by parts we obtain

$$\int_\Omega \sum_i x_i \frac{\partial F(u)}{\partial x_i}\,\mathrm{d}x = -n\int_\Omega F(u)\,\mathrm{d}x,$$

which implies that

$$\int_\Omega \Delta u(x\cdot\nabla u)\,\mathrm{d}x = n\int_\Omega F(u)\,\mathrm{d}x.$$

Once again, using (7.6) we conclude that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x = \int_\Omega u f(u)\,\mathrm{d}x,$$

which plugged into the identity (7.8) leads to the Pohozaev identity. □

An immediate consequence is that the growth of the nonlinearity f with exponent $p < \frac{n+2}{n-2}$ cannot be eliminated if we want to find nontrivial solutions of (7.8). There is a more precise statement which follows from the Pohozaev identity:

Corollary 7.24. *If Ω is a star-shaped (w.r.t. the origin) domain in $\mathbb{R}^n$, then any smooth solution of (7.8) satisfies*

$$n\int_\Omega F(u)\,\mathrm{d}x - \frac{n-2}{2}\int_\Omega u f(u)\,\mathrm{d}x > 0.$$

In particular, if $f(u) = |u|^{p-1}u$, then we find

$$\left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_\Omega |u|^{p+1}\,\mathrm{d}x > 0,$$

and hence $u \neq 0$ implies $p < \frac{n+2}{n-2}$.

Chapter 8

Lusternik-Schnirelman Theory

In this chapter we aim to discuss the elegant theory of Lusternik and Schnirelman that connects critical points of functionals on manifolds to topological properties of the latter.

8.1 Lusternik-Schnirelman category

Throughout this chapter, $\mathcal{M}$ will always denote a Hilbert space or a C^1-submanifold modelled on a Hilbert space.

Definition 8.1 (Contractible)**.** Let X be a topological space. A set $A \subset X$ is *contractible* in X if the inclusion $\iota : A \hookrightarrow X$ is homotopic to a constant map. Namely, there exists

$$H \in C\big([0,\,1] \times A,\, X\big)$$

such that $H(0,\,u) = u$ and $H(1,\,u) = p$ for all $u \in A$.

Definition 8.2 (Category)**.** Let X be a topological space and $A \subset X$. The *(L-S) category* of A with respect to X, denoted by $\operatorname{cat}(A,\,X)$, is the least integer $k \in \mathbb{N}$ such that

$$A \subseteq \bigcup_{i=1}^{k} A_i,$$

where each A_i is closed and contractible in X. If such an integer does not exist, we set $\operatorname{cat}(A,\,X) = \infty$ and if A is empty we set $\operatorname{cat}(\varnothing,\,X) = 0$.

Remark 8.3. The category of A coincide with the category of its closure. Moreover

$$\operatorname{cat}(A,\,X) \geq \operatorname{cat}(A,\,Y)$$

provided that $A \subset X \subset Y$.

Example 8.4.

(i) The sphere S^{m-1} is contractible in $\mathbb{R}^m$ so $\text{cat}(S^{m-1}, \mathbb{R}^m) = 1$. However, it is not contractible in itself but can be covered by two closed hemispheres so $\text{cat}(S^{m-1}) = 2$.

(ii) The sphere in a infinite-dimensional Hilbert space is always contractible so

$$\text{cat}(S_H, H) = 1.$$

The reader interested in this property might refer to [4].

(iii) The category torus $T^2 = S^1 \times S^1 \subset \mathbb{R}^3$ in itself is equal to 3. It is easy to verify that $\text{cat}(T^2) \leq 3$ using A_1, A_2, A_3 as defined in Figure [REF].

The opposite inequality, however, is quite hard to obtain and we will only explain at the end of the section how to use a general result to prove it.

Lemma 8.5. *Let A, $B \subset \mathcal{M}$.*

(a) *If $A \subset B$, then $\text{cat}(A, \mathcal{M}) \leq \text{cat}(B, \mathcal{M})$.*

(b) *$\text{cat}(A \cup B, \mathcal{M}) \leq \text{cat}(A, \mathcal{M}) + \text{cat}(B, \mathcal{M})$.*

(c) *If A is closed and $\eta \in C(A, \mathcal{M})$ is a deformation, then*

$$\text{cat}(A, \mathcal{M}) \leq \text{cat}\Big(\overline{\eta(A)}, \mathcal{M}\Big). \tag{8.1}$$

Proof. The only nontrivial assertion is (**c**). Let $k := \text{cat}\Big(\overline{\eta(A)}, \mathcal{M}\Big)$ an assume that it is finite (otherwise there is nothing to prove). Then

$$\eta(A) \subset \bigcup_{i=1}^{k} C_i,$$

where C_i is closed and contractible in $\mathcal{M}$. Set

$$A_i := \eta^{-1}(C_i)$$

and observe that these are all closed because η is continuous. Moreover, each A_i is contractible because the composition of a contraction with η gives another contraction. Since

$$A \subset \bigcup_{i=1}^{k} A_i,$$

we easily deduce that (8.1) holds. □

The strict inequality in (8.1) is possible to achieve. Indeed, let $\mathcal{M} = S^1$ and $A = S^1_+$ the hemisphere

$$S^1_+ = \{e^{\imath\theta} \;:\; \theta \in [0, 2\pi]\}.$$

Let $\eta(e^{\imath\theta}) := \text{H}(1, \theta)$, where

$$H(t, \theta) = e^{\imath(t+1)\theta}$$

is defined for all $t \in [0, 1]$. Then it is trivial to verify that $\text{cat}(A, \mathcal{M}) < \text{cat}\left(\overline{\eta(A)}, \mathcal{M}\right)$.

Lemma 8.6. *Let $A \subset \mathcal{M}$ be compact. Then the following properties hold:*

(i) $\text{cat}(A, \mathcal{M}) < \infty$.

(ii) *There exists a neighbourhood U_A of A such that*

$$\text{cat}(A, \mathcal{M}) = \text{cat}\left(\overline{U_A}, \mathcal{M}\right).$$

Proof. Suppose first that $\text{cat}(A, \mathcal{M}) = 1$ and let $H : [0, 1] \times A \to \mathcal{M}$ be the contraction to the constant map p. We would like to extend H to

$$S := (\{0\} \times \mathcal{M}) \cup ([0, 1] \times A) \cup (\{1\} \times \mathcal{M}),$$

and this is easily achieved by setting

$$H(t, u) := \begin{cases} u & \text{if } (t, u) \in \{0\} \times \mathcal{M}, \\ H(t, u) & \text{if } (t, u) \in [0, 1] \times A, \\ p & \text{if } (t, u) \in \{1\} \times \mathcal{M}. \end{cases}$$

Since S is closed in $Y := [0, 1] \times \mathcal{M}$ and H is continuous from S to $\mathcal{M}$, we can use the extension property to find a neighbourhood N of S in Y and a function $\tilde{H} \in C(N, \mathcal{M})$ such that

$$\tilde{H}\big|_S \equiv H.$$

Since $[0, 1] \times A$ is compact and the distance with $Y \setminus N$ is strictly positive, we can easily find a neighbourhood U_A of A in $\mathcal{M}$ such that

$$[0, 1] \times \overline{U_a} \subseteq N.$$

It is easy to verify that $\overline{U_A}$ is contractible in $\mathcal{M}$ using the contraction $\tilde{H}$ appropriately restricted to a subset of its domain. In particular,

$$\text{cat}(A, \mathcal{M}) = 1 \implies \text{cat}\left(\overline{U_A}, \mathcal{M}\right) = 1.$$

(i) Let $q \in A$ Then above we proved that there exists a contractible neighbourhood U_q of category equal to one. Since we can always cover A with finitely many U_q's, we infer that the category of A is finite.

(ii) Let $k = \text{cat}(A, \mathcal{M})$ and let $A_1, \cdots, A_k$ be the closed and contractible sets such that

$$A \subseteq \bigcup_{i=1}^{k} A_i.$$

Observe that if we replace A_i with $A \cap A_i$, we can assume without loss of generality that A_i's are also compact. Since $\text{cat}(A_i, \mathcal{M}) = 1$ we can find an open neighbourhood

U_i of A_i such that

$$\mathrm{cat}\Big(\overline{U_i},\, \mathcal{M}\Big) = 1$$

for each $i = 1, \cdots, k$. Let $U_A := \bigcup_{i=1}^k U_i$ and notice that U_A is an open neighbourhood of A such that

$$\mathrm{cat}\Big(\overline{U_A},\, \mathcal{M}\Big) \geq k.$$

Since $\overline{U_A} \subset \bigcup_{i=1}^k \overline{U_i}$, we also get the opposite inequality and hence the equality holds.

□

Remark 8.7. It can be proved that the category satisfies the inequality

$$\mathrm{cat}(\mathcal{M}) \geq \mathrm{cup} - \mathrm{length}(\mathcal{M}) + 1,$$

where the *cup-length* of $\mathcal{M}$ is defined by

$$\mathrm{cup} - \mathrm{length}(\mathcal{M}) = \sup\left\{k \in \mathbb{N} \;:\; \exists\, \alpha_1, \cdots, \alpha_k \in \mathcal{M}^* \text{ s.t. } \alpha_1 \cup \cdots \cup \alpha_k \neq 0\right\}.$$

If $\mathcal{M}$ is a smooth manifold, then by De Ram's cohomology $\alpha_1 \cup \cdots \cup \alpha_k$ corresponds to the $\wedge$-product of differential forms. In particular

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 \neq 0$$

on the torus $\mathbb{T}^2$, so we obtain the bound $\mathrm{cat}(\mathbb{T}^2) \geq 3$.

8.2 Lusternik-Schnirelman theorems

Let $\mathcal{M}$ be a Hilbert space or a C^1-submanifold modelled on a Hilbert space. Define

$$\mathrm{cat}_K(\mathcal{M}) := \sup\left\{\mathrm{cat}(A,\, \mathcal{M}) \;:\; A \subseteq M \text{ is compact}\right\}$$

and introduce the corresponding class of sets that is preserved when we use deformations; namely, let

$$C_m := \{A \subseteq \mathcal{M} \;:\; A \text{ is compact and } \mathrm{cat}(A,\, \mathcal{M}) \geq m\}$$

for $m \leq \mathrm{cat}_K(\mathcal{M})$. Let $J \in C^1(\mathcal{M}, \mathbb{R})$ and define

$$c_m := \inf_{A \in C_m} \max_{u \in A} J(u).$$

The following properties follows from the definition immediately:

(a) The first level, c_1, coincide with $\inf_{u \in \mathcal{M}} J(u)$.

(b) The sequence of levels is increasing, that is,

$$c_1 \leq c_2 \leq \cdots \leq c_k \leq \cdots$$

(c) For all $m \leq \text{cat}_K(\mathcal{M})$ there results $c_m < \infty$.

(d) If J is bounded from below on $\mathcal{M}$, then all c_m's are finite.

Theorem 8.8. *Let $J \in C^1(\mathcal{M}, \mathbb{R})$ be a functional bounded from below on $\mathcal{M}$ and satisfying the Palais-Smale condition at all $c \in \mathbb{R}$. Then J has at least $\text{cat}_K(\mathcal{M})$ critical points and the following holds:*

(1) For all $m \leq \text{cat}_K(\mathcal{M})$, c_m is a critical value for J.

(2) If there are integers q, $m \geq 1$ such that

$$c := c_m = c_{m+1} = \cdots = c_{m+q},$$

then $\text{cat}(\mathcal{Z}_c, \mathcal{M}) \geq q + 1$.

Remark 8.9. The category of a finite set of points $\{p_1, \ldots, p_N\}$ in $\mathcal{M}$ is always equal to one (if $\mathcal{M}$ is connected). Consequently, (**2**) gives us an even more precise information than merely saying that there are infinite critical points at the level c.

Lemma 8.10 (Deformation)**.** *Let $J \in C^1(\mathcal{M}, \mathbb{R})$ be a functional bounded from below on $\mathcal{M}$ and satisfying the Palais-Smale condition at all $c \in \mathbb{R}$. Then for each U neighbourhood of $\mathcal{Z}_c$ there are $\delta = \delta(U) > 0$ and a deformation η such that*

$$\eta(\mathcal{M}^{c+\delta} \setminus U) \subseteq \mathcal{M}^{c-\delta}.$$

Proof. We claim that for each U neighbourhood of $\mathcal{Z}_c$ there exists $\bar{\delta} > 0$ such that

$$u \notin U \text{ and } |J(u) - c| \leq \bar{\delta} \implies \|\nabla J(\alpha(t, u))\| \geq 2\bar{\delta} \quad \text{for all } t \in [0, 1].$$

We argue by contradiction. Assume that there are sequences $t_k \in [0, 1]$ and $u_k \notin U$ such that

$$|J(u_k) - c| \leq \frac{1}{k} \quad \text{and} \quad \|\nabla J(\alpha(t_k, u))\| \xrightarrow{k\to\infty} 0.$$

Let $\bar{t} \in [0, 1]$ be the limit (up to subsequences) of t_k and set $\nu_k := \alpha(t_k, u_k)$. Then

$$J(\nu_k) \leq J(u_k) \leq c + \frac{1}{k}$$

and, since J is bounded from below and satisfies the Palais-Smale condition, we find that $J(\nu_k)$ converges to c. Passing to the limit the inequality above shows that

$$c = \lim_{k\to\infty} J(\nu_k) \leq \lim_{k\to\infty} J(u_k) \leq c = \lim_{k\to\infty} (c + \frac{1}{k}),$$

which means that $J(u_k)$ also converges to c, and hence it is enough to prove that u_k converges to some z. We know that $\nu_k \to z$ and the flow $\alpha(t, z) = z$ for all $t \in [0, 1]$. We can go backwards and obtain

$$u_k = \alpha(-t_k, \nu_k),$$

so by Cauchy's theorem we infer that $u_k \to z$. Since $z \in \mathcal{Z}_c$ we find a contradiction because u_k does not belong to U for all k and $\mathcal{Z}_c$ is contained in U. The rest of the proof follows as in Lemma 6.12. □

Proof of Theorem 8.8. The assertion (**1**) follows from the deformation lemma in the same fashion it did in the MPT. To prove (**2**) we argue by contradiction, i.e., we assume that

$$\operatorname{cat}(\mathcal{Z}_c, \mathcal{M}) \leq q.$$

Since J satisfies the Palais-Smale condition, the critical set $\mathcal{Z}_c$ is compact and hence there exists an open neighbourhood U of $\mathcal{Z}_c$ such that

$$\operatorname{cat}\left(\overline{U}, \mathcal{M}\right) \leq q.$$

By the second deformation Lemma 8.10, there are $\delta > 0$ and a deformation η such that

$$\eta\left(\mathcal{M}^{c+\delta} \setminus U\right) \subseteq \mathcal{M}^{c-\delta}.$$

Since $c = c_{m+q}$ we can find an element $A \in C_{m+q}$ such that

$$\sup_{u \in A} J(u) \leq c + \delta \implies A \subseteq \mathcal{M}^{c+\delta}.$$

Set $A' := \overline{A \setminus U}$. Then

$$\operatorname{cat}\left(A', \mathcal{M}\right) \geq \operatorname{cat}\left(A, \mathcal{M}\right) - \operatorname{cat}\left(\overline{U}, \mathcal{M}\right) \geq m + q - q = m,$$

which means that $A' \in C_m$. Therefore, the image of A' via η is contained in $\mathcal{M}^{c-\delta}$ and, using the properties of the category, we also have that

$$\operatorname{cat}\left(\eta(A'), \mathcal{M}\right) \geq m.$$

In particular, we have $A' \in C_m$ and

$$\sup_{A'} J(u) \leq c - \delta,$$

but this is a contradiction with the very definition of c_m. □

Theorem 8.11. *Let $\mathcal{M}$ be a Hilbert space or a $C^{1,1}$-manifold and let $J \in C^{1,1}(\mathcal{M}, \mathbb{R})$ be bounded from below. Suppose that there exists $a \in \mathbb{R}$ such that the Palais-Smale condition holds at all levels $c \leq a$. Then*

$$\operatorname{cat}(\mathcal{M}^a) < \infty.$$

Proof. Let $Z = \{\nabla J = 0\}$ and set $Z^a := \mathcal{M}^a \cap Z$. The Palais-Smale condition implies that Z^a is compact and by Lemma 8.6 we can find an open neighbourhood U^a of Z^a such that

$$\operatorname{cat}(\bar{U}^a, M^a) = \operatorname{cat}(Z^a, M^a) < \infty.$$

We can assume without loss of generality that $\|\nabla J(u)\| \leq 1$ for all $u \in U_a$. Then there exists $V^a \subset U^a$, neighbourhood of Z^a, such that

$$d := d(\bar{V}^a, \partial U^a) > 0.$$

If a gradient flow of J exits V^a and enters the complement of U^a, then this has to happen in a time bigger than or equal to d. Observe that by $(\mathrm{PS_c})$ there is $\delta > 0$ such that

$$\|\nabla J(u)\| \geq \delta$$

for all $u \in \mathcal{M}^a \setminus V^a$. Let $a' := a - \inf_{u \in \mathcal{M}} J(u)$ and $T > \frac{a'}{\delta^2}$. Recall that

$$-\frac{\mathrm{d}}{\mathrm{d}t} J(\alpha(t,\, u)) = -\|\nabla J(\alpha(t,\, u))\|^2,$$

and thus if $\alpha(t, u)$ never enters V^a, we would have

$$J(\alpha(t,\, u)) < J(u) - T\delta^2 < a - a' = \inf_{u \in \mathcal{M}} J(u),$$

which is impossible. Now let $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = T$ be such that

$$|t_i - t_{i-1}| \leq \frac{d}{2}.$$

Given $p \in \mathcal{M}^a$, there must be $\bar{t} \in [0, T]$ such that $\alpha(\bar{t}, p) \in V^a$ and an index i for which $|\bar{t} - t_i| \leq \frac{d}{2}$. Clearly

$$\alpha(t_i,\, u) \in U^a,$$

so we can consider the sets

$$A_i = \{p \in \mathcal{M}^a \;:\; \alpha(t_i,\, p) \in U^a\}.$$

By what we proved above, $\mathcal{M}^a \subseteq \bigcup_{i=0}^n A_i$ and therefore

$$\mathrm{cat}(\mathcal{M}^a) \leq \sum_{i=0}^{n} \mathrm{cat}(A_i,\, \mathcal{M}^a).$$

Since A_i can be deformed in U^a via $\eta_i := \alpha(t_i, \cdot)$, we use the properties of the category to infer that

$$\mathrm{cat}(\mathcal{M}^a) \leq \sum_{i=0}^{n} \mathrm{cat}(\eta_i(A_i),\, \mathcal{M}^a) \leq (n+1)\mathrm{cat}(U^a,\, \mathcal{M}^a) < \infty.$$

□

Corollary 8.12. *If J is also bounded from above on $\mathcal{M}$, then* $\mathrm{cat}(\mathcal{M})$ *is finite.*

Corollary 8.13. *Let J be bounded from below on $\mathcal{M}$. Suppose that* $\mathrm{cat}(\mathcal{M}) = \infty$ *and that* $(\mathrm{PS})_\mathrm{a}$ *holds for all $a < \sup_{u \in \mathcal{M}} J(u)$. Then*

$$c_m \xrightarrow{m \to \infty} \sup_{u \in \mathcal{M}} J(u),$$

and hence J has infinitely many critical points.

Remark 8.14 (Relative category)**.** Suppose that $A, Y \subseteq \mathcal{M}$ are closed. We define the relative category $\operatorname{cat}_{\mathcal{M},Y}(A)$ as the least integer k such that

$$A \subseteq \bigcup_{i=0}^{k} A_i,$$

where A_i is closed and contractible for all $i = 1, \ldots, k$ in $\mathcal{M}$ and there exists a homotopy $h \in C([0, 1] \times A_0, \mathcal{M})$ satisfying

$$h(0, \cdot) = \mathrm{id}_{A_0}, \qquad h(1, \cdot) \in Y \quad \text{and} \quad h(t, \cdot)\big|_Y \in Y.$$

If Y is empty, then the definition coincides with the one of category of A in $\mathcal{M}$.

Theorem 8.15. *If J satisfies the Palais-Smale condition for all $c \in [a, b]$ then there are at least $\operatorname{cat}_{\mathcal{M}^b, \mathcal{M}^a}(\mathcal{M}^b)$ critical points in the energy strip $\overline{\mathcal{M}^b \setminus \mathcal{M}^a}$.*

8.3 Application to PDEs theory

Let Ω be a bounded smooth subset of $\mathbb{R}^n$ and consider the Dirichlet-boundary problem

$$\begin{cases} -\Delta u = \lambda u + f(u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

where $\lambda \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 2$. Assume that f is continuous and satisfies the following properties:

(1) f is subcritical at $t = 0$, which means that $f(t) = \mathcal{O}(|t|^\alpha)$ for some $\alpha > 1$.

(2) If $F(u) := \int_0^u f(s)\,\mathrm{d}s$, then
$$\lim_{|t|\to\infty} \frac{F(t)}{t^2} = -\infty.$$

(3) The function $t \mapsto f(t)$ is nonincreasing (thus $F(t) \leq 0$ for all $\in \mathbb{R}$) and $F(t) = 0$ if and only if $t = 0$.

Theorem 8.16. *Under these assumptions, the Dirichlet-boundary problem admits at least two solutions.*

Proof. Let $\mathfrak{X} := H_0^1(\Omega)$ and let us consider the associated functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |u|^2\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x.$$

The assumption (2) tells us that $-F(u) \geq Mu^2$ for all $M \in \mathbb{R}$ when u is sufficiently big, while in the complement (which is compact) we can always find a constant $C_M > 0$ such that $F(u) \leq C_M$. It follows that

$$-F(u) \geq Mu^2 - C_M \quad \text{for all } u \in \mathbb{R}.$$

We plug this inequality into J and find that

$$J(u_n) \geq \frac{1}{2}\int_\Omega |\nabla u_n|^2 \, \mathrm{d}x + \left(M - \frac{\lambda}{2}\right)\int_\Omega |u_n|^2 \, \mathrm{d}x - C_M|\Omega|$$

and this goes to ∞ as $\|u_n\|_{\mathfrak{X}} \to \infty$ since we can always pick $M > \frac{\lambda}{2}$. In particular, the functional J is coercive on $\mathfrak{X}$ and hence it satisfies the Palais-Smale condition at all levels[1].

Remark 8.17. The Ψ-gradient decreases the value of J, so it is not restrictive to apply the min-max theory to a suitable sublevel $\mathfrak{X}^a$. We introduce this apparent complication because we can collect more topological information as if we were in $\mathbb{R}^n$.

We now substitute $\mathfrak{X}$ with the sublevel $\mathfrak{X}^{<0} := \{u \in \mathfrak{X} \,:\, J(u) < 0\}$, which is easy to see that it is nonempty using (1):

$$J(t\varphi_1) \simeq_{t \to 0^+} \frac{1}{2}\underbrace{(\lambda_1 - \lambda)}_{<0} t^2 + \mathcal{O}(|t|^{\alpha+1}),$$

where φ_1 is an eigenfunction of the first eigenvalue λ_1. Now notice that C_1 is nonempty and hence $c_1 < 0$ (since we are working in $\mathfrak{X}^{<0}$) is a critical level and

$$\mathcal{Z}_{c_1} \neq \varnothing.$$

We claim that C_2 is nonempty. Let $V := \mathrm{Span}\langle \varphi_1, \ldots, \varphi_k \rangle$ and, for r small enough, notice that

$$\sup_{S_r \cap V} J(u) < 0.$$

If we can prove that the category of $S_r \cap V$ in $\mathfrak{X}^{<0}$ is bigger than or equal to 2, we will be able to conclude that $C_2 \neq \varnothing$. Let $\pi_V : \mathfrak{X} \to V$ be the projection and let

$$\pi_r(u) := r\frac{\pi_V(u)}{\|\pi_V(u)\|_{\mathfrak{X}}}$$

be the normalized projection which is the identity on $S_r \cap V$. Suppose that $\mathrm{cat}(S_r \cap V, \mathfrak{X}^{<0}) = 1$ and let $A \supseteq S_r \cap V$ be the closed contractible set such that

$$H(0, \cdot)\big|_A \equiv \mathrm{Id}_A \quad \text{and} \quad H(1, \cdot) \equiv p \in \mathfrak{X}^{<0}$$

for some contraction H. The assumption (3) gives us that $\pi_V(u) = 0$ if $J(u) \geq 0$, which means that $\pi_V(u) \neq 0$ for all $u \in \mathfrak{X}^{<0}$ and the restriction

$$\pi_r(u)\big|_{S_r \cap V}$$

[1] This assertion is not trivial, but one can show that coercivity gives the bounedness of Palais-Smale sequences and the subcriticality of f allows one to write $\nabla J = \mathrm{Id} + \nabla\Phi$, where $\nabla\Phi$ is a compact operator.

is well-defined. We can consider the composition

$$\pi_r \circ H : [0, 1] \times A \to S_r \cap V,$$

which, restricted to $[0, 1] \times S_r \cap V$, gives a retraction of a $(k-1)$-dimensional sphere to a point in itself, and this is a contradiction as the sphere is non-contractible in itself. $\square$

Remark 8.18. If $c_1 = c_2 < 0$, then there are infinitely many critical points at level c since the category of $\mathcal{Z}_c$ is at least two.

Remark 8.19. If $\lambda \in (\lambda_1, \lambda_2)$, then $V = \varphi_1\mathbb{R}$ and the same argument leads to $S_r \cap V \cong S^0 = \{\pm q\}$. The sublevels become disconnected, but it is still true that

$$\operatorname{cat}(\{\pm q\}, \mathfrak{X}^{<0}) = 2.$$

Chapter 9

The Krasnoselski Genus

9.1 Introduction

Let E be a infinite-dimensional Hilbert space. We say that a subset $\Omega \subset E$ is *symmetric* if it is symmetric with respect to the origin of E, that is,

$$u \in \Omega \implies -u \in \Omega.$$

Let Γ be the class of all the symmetric subsets $A \subseteq E \setminus \{0\}$ which are closed in $E \setminus \{0\}$.

Definition 9.1 (Genus)**.** Let $A \in \Gamma$. The *genus* of A is the least integer number $k \in \mathbb{N}$ such that there exists $\Phi : A \to \mathbb{R}^k$ continuous, odd and such that

$$\Phi(x) \neq 0 \quad \text{for all } x \in A.$$

The genus of A is usually denoted by $\gamma(A)$. If such a number does not exist, we set $\gamma(A) = \infty$ and, if $A = \varnothing$, we conventionally set $\gamma(A) = 0$.

Remark 9.2. We can equivalently define the genus of A to be the least integer number $k \in \mathbb{N}$ such that there exists $\Phi : A \to \mathbb{R}^k \setminus \{0\}$ continuous and odd. The reason is that we can always extend such a map to a continuous one taking values in $\mathbb{R}^k$ using *Dugundij's theorem* and even/odd parts.

Remark 9.3. The definition of genus does not change if we require Φ to be a function with values in the sphere $\mathbb{S}^{k-1}$ instead of $\mathbb{R}^k \setminus \{0\}$ since we can compose with the projection

$$\pi(x) := \frac{x}{|x|}.$$

Lemma 9.4. *Let $E = L^2(\mathbb{R}^d)$ and let $A = S_E(0, 1)$ be the unit sphere in L^2. Then*

$$\gamma(A) = +\infty.$$

Proof. Let $k \in \mathbb{N}$ be any positive integer, and let

$$\Phi : S_E(0,\, 1) \to \mathbb{R}^k$$

be a continuous odd map. The infinite-dimensional sphere contains the n-sphere $\mathbb{S}^n(0,\, 1) \subset \mathbb{R}^{n+1}$; hence by *Borsuk-Ulam theorem* it follows that, for $n > k$,

$$0 \in \Phi\left(\mathbb{S}^n(0,\, 1)\right) \implies 0 \in \Phi\left(S_E(0,\, 1)\right).$$

Since $S_E(0,\, 1)$ contains every finite-dimensional sphere, for every $k \in \mathbb{N}$ we can take $n = k+1$ and obtain that 0 is in the image. This shows that the genus is $+\infty$. □

Remark 9.5. In a similar fashion, one proves that $\gamma(\partial\Omega) = n$, where $\Omega \subset \mathbb{R}^n$ is an open bounded even subset such that $0 \in \Omega$. In particular,

$$\gamma(\mathbb{S}^{n-1}) = n.$$

Proof. It is easy to verify that $\gamma\left(\partial\Omega\right) \leq n$. On the other hand, if

$$\Phi : \partial\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

is a continuous odd map, then *Borsuk-Ulam theorem* implies that $0 \in \Phi\left(\partial\Omega\right)$ for every $k < n$. It follows that

$$\gamma\left(\partial\Omega\right) \geq N \implies \gamma\left(\partial\Omega\right) = N.$$

□

Lemma 9.6. *The following properties hold:*

- ***(a)*** *If $A \in \Gamma$ is finite and nonempty, then $\gamma(A) = 1$.*
- ***(b)*** *If $A \subseteq \mathbb{R}^n$ and $0 \notin A$, then $\gamma(A) \leq n$.*
- ***(c)*** *If $0 \in A$, then $\gamma(A) = +\infty$.*

Proposition 9.7. *Let A and B be elements of the class Γ.*

- ***(a)*** *The set A is empty if and only if the genus $\gamma(A)$ is equal to 0.*
- ***(b)*** *If $\Phi : A \to B$ is a continuous odd map, then $\gamma(A) \leq \gamma(B)$. In particular,*

$$A \subseteq B \implies \gamma(A) \leq \gamma(B).$$

- ***(c)*** *The genus is subadditive, that is,*

$$\gamma(A \cup B) \leq \gamma(A) + \gamma(B). \tag{9.1}$$

- ***(d)*** *There is an open neighborhood $U \supset A$ satisfying the following properties:*
  - ***(1)*** *The set is symmetric, that is, if $u \in U$, then $-u \in U$.*

(2) The origin is not contained in the closure of the set, that is, $0 \notin \bar{U}$.

(3) The genus coincides with the one of the set A*, that is,*

$$\gamma\left(\bar{U}\right) = \gamma(A).$$

Proof. The first property is obvious.

(b) If $\Psi : B \to \mathbb{R}^k \setminus \{0\}$ is a continuous odd map, then the composition

$$\Psi \circ \Phi : A \to \mathbb{R}^k \setminus \{0\}$$

is still continuous and odd. It follows that $\gamma(A) \leq \gamma(B)$.

(c) Let $k,\, h \in \mathbb{N}$ be the least positive integers such that there are continuous odd maps $\Phi_1 : A \longrightarrow \mathbb{R}^k \setminus \{0\}$ and $\Phi_2 : B \longrightarrow \mathbb{R}^h \setminus \{0\}$ respectively. Let

$$\widetilde{\Phi_i} : A \cup B \to \mathbb{R}^k$$

be the continuous odd extensions of Φ_1 and Φ_2 respectively to $A \cup B$. Then

$$\Psi(u) := \left(\widetilde{\Phi_1}(u),\, \widetilde{\Phi_2}(u)\right) : A \cup B \to \mathbb{R}^k \times \mathbb{R}^h \setminus \{(0,\, 0)\}$$

is a continuous odd map. Moreover, every point $u \in A \cup B$ belongs to either A or B so its image cannot be equal to $(0,\, 0)$.

(d) Let $k = \gamma(A)$. By Remark 9.3 there exists a continuous odd map

$$\Phi : A \to S^{k-1}.$$

The set A is closed in E and hence there exists a continuous odd function $\widetilde{\Phi} : E \to \mathbb{R}^k$, which extends Φ, but, a priori, 0 may be in its image. Thus

$$U_A := \left\{ u \in E \;\middle|\; \left|\widetilde{\Phi}(u)\right| > \frac{1}{2} \right\}$$

is the desired neighborhood of A.

□

9.2 Genus in calculus of variations

Suppose that $\mathfrak{X} \subset E$, $\mathfrak{X}$ Hilbert or C^1-submanifold, belongs to Γ. In this section, unless otherwise stated, every functional $J : \mathfrak{X} \to \mathbb{R}$ is even and of class $C^1(\mathfrak{X},\, \mathbb{R})$.

Proposition 9.8. *Let* $a < b$ *be real numbers. Assume that* $f : \mathfrak{X} \to \mathbb{R}$ *satisfies* $(\mathrm{PS})_c$ *at every level* $c \in [a,\, b]$*. If there is a strict inequality*

$$\gamma\left(\mathfrak{X}^a\right) < \gamma\left(\mathfrak{X}^b\right),$$

then there exists a critical value $c \in [a, b]$ for f.

Proof. We argue by contradiction. If there are no critical values in $[a, b]$, then there is an odd retraction $r : \mathfrak{X}^b \to \mathfrak{X}^a$, and we conclude using Proposition 9.7. □

Notation. Let $k \in \mathbb{N}$ be a positive integer number such that $1 \le k \le \gamma(\mathfrak{X})$. We denote by γ_k the infimum of all the sublevels such that the genus is at least k, that is,

$$\gamma_k := \inf \{ c \in \mathbb{R} \mid \gamma(\mathfrak{X}^c) \ge k \} .$$

It is possible that $\gamma(\mathfrak{X}^c) \ge k$ is not satisfied for any real number $c \in \mathbb{R}$. In this case, the supremum of J is ∞ and we set $\gamma_k = \infty$.

Lemma 9.9. *Let $1 \le k \le \gamma(\mathfrak{X})$.*

(a) *The sequence is increasing, that is,*

$$\inf_{u \in \mathfrak{X}} J(u) = \gamma_1 \le \gamma_2 \le \cdots \le \gamma_k \le \sup_{u \in \mathfrak{X}} J(u).$$

(b) *If $\gamma_k \in \mathbb{R}$ and J satisfies $(\mathrm{PS})_{\gamma_k}$, then γ_k is a critical value for the functional J. In particular,*

$$\gamma_1 \in \mathbb{R} \implies \gamma_1 = \min_{u \in \mathfrak{X}} J(u).$$

(c) *If $\gamma_k = \gamma_{k+1} = \cdots = \gamma_{k+h}$ for some $h \ge 1$ and f satisfies $(\mathrm{PS})_{\gamma_k}$, then*

$$\gamma(\mathcal{Z}_{\gamma_k}) \ge h + 1,$$

where $\mathcal{Z}_{\gamma_k}$ is the set of all singular points of J at the level γ_k. In particular, if $0 \in \mathcal{Z}_{\gamma_k}$, then it is an infinite set.

Proof.

(a) The first identity follows from the fact that

$$\gamma_1 := \inf \{ c \in \mathbb{R} \mid \gamma(\mathfrak{X}^c) \ge 1 \} = \inf \{ c \in \mathbb{R} \mid \mathfrak{X}^c \ne \varnothing \} = \inf_{u \in \mathfrak{X}} J(u).$$

Now notice that

$$\{ c \in \mathbb{R} \mid \gamma(\mathfrak{X}^c) \ge k \} \supseteq \{ c \in \mathbb{R} \mid \gamma(\mathfrak{X}^c) \ge k + 1 \} ,$$

from which it follows that $\gamma_k \le \gamma_{k+1}$ by taking the infimum of both sides.

(b) If γ_k is not a critical level for J, then there are $\delta > 0$ and

$$r : \mathfrak{X}^{\gamma_k + \delta} \to \mathfrak{X}^{\gamma_k - \delta}$$

odd retraction. By Proposition 9.7 we have

$$\gamma\left(\mathfrak{X}^{\gamma_k+\delta}\right) \leq \gamma\left(\mathfrak{X}^{\gamma_k-\delta}\right),$$

but this is impossible since

$$\gamma\left(\mathfrak{X}^{\gamma_k-\delta}\right) \leq k-1 < k \leq \gamma\left(\mathfrak{X}^{\gamma_k+\delta}\right).$$

(c) First, notice that $\mathcal{Z}_c$ is always a compact element of Γ. Thus by Proposition 9.7 there exists a symmetric open neighborhood U of $\mathcal{Z}_{\gamma_k}$ such that

$$\bar{U} \in \Gamma \quad \text{and} \quad \gamma\left(\bar{U}\right) = \gamma\left(\mathcal{Z}_{\gamma_k}\right).$$

Then there are $\epsilon > 0$ and an odd retraction $r : \mathfrak{X}^{\gamma_k+\epsilon} \setminus U \to \mathfrak{X}^{\gamma_k-\epsilon}$, where the domain is closed, belongs to Γ and it satisfies the inclusion

$$\mathfrak{X}^{\gamma_k+\epsilon} \subseteq \left(\mathfrak{X}^{\gamma_k+\epsilon} \setminus U\right) \cup \bar{U}. \tag{9.2}$$

By assumption $k+h \leq \gamma\left(\mathfrak{X}^{\gamma_k+\epsilon}\right)$, and by the subadditivity of the genus, it follows from (9.2) that

$$\begin{aligned} k+h &\leq \gamma\left(\mathfrak{X}^{\gamma_k+\epsilon}\right) \leq \\ &\leq \gamma\left(\mathfrak{X}^{\gamma_k+\epsilon} \setminus U\right) + \gamma\left(\bar{U}\right) \leq \\ &\leq \gamma\left(\mathfrak{X}^{\gamma_k-\epsilon}\right) + \gamma\left(\bar{U}\right) \leq \\ &\leq k-1+\gamma\left(\mathcal{Z}_{\gamma_k}\right), \end{aligned}$$

and this leads to the desired result.

□

Theorem 9.10 (Lusternik-Schnirelman)**.** *Let $J : \mathbb{S}^{n-1} \to \mathbb{R}$ be an even functional of class C^1. There are (at least) n pairs of critical points for J of the form*

$$(-u_i,\, u_i) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}.$$

9.3 Application to nonlinear eigenvalues

Let $\mathfrak{X}$ be an infinite-dimensional Hilbert space and let $J \in C^1(\mathfrak{X})$ be an even functional satisfying the following assumptions:

(i) $J(0) = 0$, $J(u) < 0$ for all $u \neq 0$ and $\sup_{u\in\mathfrak{X}} J(u) = 0$.

(ii) J is weakly continuous and ∇J is compact.

(iii) $\nabla J(u) \neq 0$ for all $u \in \mathfrak{X}$.

Theorem 9.11. *Under these assumptions, the problem*

$$\nabla J(u) = \lambda u$$

has infinitely many solutions (μ_k, z_k) *with* $z_k \in S := \{u \in \mathfrak{X} : \|u\|_2 = 1\}$ *and* $\mu_k \to 0$.

Proof. To apply the general results with S, we need to prove that $J\big|_S$ is bounded below and J satisfies the Palais-Smale condition at all $c < 0$.

Step 1. This follows from the weak continuity of J. The reader might try to work out the details by herself as an exercise.

Step 2. Let u_n be a Palais-Smale sequence at the level $c < 0$. The weak continuity of J implies (up to subsequences, which we ignore here) that

$$u_n \rightharpoonup u \quad \text{and} \quad J(u) = c \implies u \neq 0.$$

Now notice that $\nabla_{\mathcal{M}} J(u_k) = \nabla J(u_k) - \langle \nabla J(u_k), u_k \rangle u_k$ and

$$\langle \nabla J(u_k), \nabla_{\mathcal{M}} J(u_k) \rangle = \|\nabla J(u_k)\|^2 - [\langle \nabla J(u_k), u_k \rangle]^2,$$

and by compactness of the gradient we have $\nabla J(u_k) \to \nabla J(u)$ strongly. Then

$$0 = \|\nabla J(u)\|^2 - [\langle \nabla J(u_k), u_k \rangle]^2 \implies \langle \nabla J(u), u \rangle \neq 0,$$

and since $\langle \nabla J(u_k), u_k \rangle \to \langle \nabla J(u), u \rangle$, we can find k sufficiently large such that

$$u_k = \frac{1}{\langle \nabla J(u_k), u_k \rangle} [\nabla J(u_k) - \nabla_{\mathcal{M}} J(u_k)]$$

is well-defined. This shows that $u_k \to u$ strongly and concludes the proof.

Step 3. Finally, $\gamma(S) = \infty$ implies that there are $z_k \in S$ critical points such that

$$J(z_k) \to \sup_{u \in S} J(u) = 0.$$

Since z_k is a constrained critical point, we can always find μ_k such that $\nabla J(z_k) = \mu_k z_k$, and clearly it is given explicitly by

$$\mu_k = \langle \nabla J(z_k), z_k \rangle.$$

Finally $\nabla J(z_k)$ converges strongly to zero and z_k weakly to zero, so $\mu_k \to 0$ and this concludes the proof. □

Theorem 9.12. *Let f be a Carathéodory function which is odd with respect to the second* *variable and that satisfies the p-growth*

$$|f(x,\, s)| \leq a + b|s|^p,$$

where $1 < p < \frac{n+2}{n-2}$*. Then the problem*

$$\begin{cases} -\lambda \Delta u = f(x,\, u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

has infinitely many solutions $(\mu_k,\, z_k)$ *with* $z_k \in S := \{u \in \mathfrak{X} \,:\, \|u\|_2 = 1\}$ *and* $\mu_k \searrow 0$.

9.4 Multiple critical points of even unbounded functionals

Let E be a Hilbert space, $J \in C^1(E,\, \mathbb{R})$ a functional and define

$$E_+ := \{u \in E \,:\, J(u) \geq 0\}.$$

We now introduce two assumptions on J that allows us, in some sense, to bypass the unboundedness both from above and below. These are similar to the ones necessary for the MPT, but the second one is "stronger":

(i) There are positive constants $r,\, \rho > 0$ such that $J(u) > 0$ for all $u \in B_r \setminus \{0\}$ and $J(u) \geq \rho$ for all $u \in S_r$. Furthermore, $J(0) = 0$.

(ii) For any m-dimensional subspace $E^m \subset E$, $E^m \cap E_+$ is bounded.

Let E^* be the class of maps $h \in C(E,\, E)$ which are odd homeomorphisms such that $h(\bar{B}_1) \subset E_+$. Notice that

$$h_r(u) := ru \implies h_r \in E^*,$$

where r is given by (i), so the class we introduced is never empty. Define

$$\mathcal{A} := \{A \subseteq E \setminus \{0\} \,:\, A \text{ is closed and even}\},$$

$$\Gamma_m := \{A \in \mathcal{A} \,:\, A \text{ is compact and } \gamma(A \cap h(S)) \geq n \text{ for all } h \in E^*\}.$$

Lemma 9.13. *Let* $J \in C^1(E,\, \mathbb{R})$ *be an even functional that satisfies (i) and (ii). Then the following properties hold:*

(1) $\Gamma_m \neq \varnothing$ *for all* m*;*

(2) $\Gamma_{m+1} \subset \Gamma_m$*;*

(3) if $A \in \Gamma_m$ *and* $U \in \mathcal{A}$*, with* $\gamma(U) \leq q < m$*, then* $\overline{A \setminus U} \in \Gamma_{m-q}$*;*

(4) *if η is an odd homeomorphism in E such that $\eta^{-1}(E_+) \subset E_+$, then $\eta(A) \in \Gamma_m$ whenever $A \in \Gamma_m$.*

Proof.

(1) By (ii) there exists $R > 0$ such that

$$E^m \cap E_p \subset \bar{B}_R \cap E^m =: B_R^m.$$

We claim that $B_R^m \in \Gamma_m$. Let $h \in E^*$ and notice that $h(B_1) \subset E_+$ implies

$$E^m \cap h(B_1) \subset B_R^m.$$

It follows that $E^m \cap h(S) \subset S_R^m \cap h(S)$ and, since one has the inclusion $B_R^m \cap h(S) \subset E^m \cap h(S)$, we infer that

$$B_R^m \cap h(S) = E^m \cap h(S).$$

Since h is an odd homeomorphism, then $E^m \cap h(B_1)$ is a symmetric neighbourhood Ω of the origin. It is also easy to check that

$$\partial\Omega = \partial(E^m \cap h(B_1))$$

is contained in $E^m \cap h(S)$. Then

$$\gamma(B_R^m \cap h(S)) = \gamma(E^m \cap h(S)) \geq \gamma(\partial\Omega) = m,$$

which means that $B_R^m \in \Gamma_m$.

(2) This follows immediately from the monotonicity property of the genus.

(3) The set $\overline{A \setminus U} \in \mathcal{A}$ is compact and satisfies the identity

$$\overline{A \setminus U} \cap h(S) = \overline{A \cap h(S) \setminus U}.$$

Using the properties of the genus we infer that

$$\gamma\left(\overline{A \setminus U} \cap h(S)\right) = \gamma\left(\overline{A \cap h(S) \setminus U}\right)$$

$$\geq \gamma(A \cap h(S)) - \gamma(U) \geq m - q,$$

and this concludes the proof.

(4) Let $A \in \Gamma_m$ and notice that $A' := \eta(A)$ is also compact. Our goal is to prove that for all $h \in E^*$ it turns out that

$$\gamma(A' \cap h(S)) \geq m.$$

It is easy to verify that $A' \cap h(S) = \eta\left[A \cap \eta^{-1}(h(S))\right]$. Since $\eta^{-1}(E_+) \subset E_+$, we infer that $\eta^{-1} \circ h$ belongs to E^* and hence

$$\gamma\left(A' \cap h(S)\right) = \gamma\left(\eta\left[A \cap \eta^{-1}(h(S))\right]\right)$$
$$\geq \gamma\left(A \cap \eta^{-1}(h(S))\right) \geq m.$$

□

Remark 9.14. The condition $\eta(E_+) \subset E_+$ is natural if one thinks that deformations η are usually induced by the Ψ-gradient flow.

Theorem 9.15. *Let Γ_m be as above and set $b_m := \inf_{A\in\Gamma_m} \max_{u\in A} J(u)$. Suppose that $J \in C^1(E)$ satisfies (i) and (ii).*

(1) For all $m \in \mathbb{N}$ it turns out that $b_{m+1} \geq b_m \geq \rho > 0$.

(2) If the Palais-Smale condition holds at the level b_m, then b_m is critical.

(3) If the Palais-Smale condition holds at all levels $c > 0$ and $b = b_m = \cdots = b_{m+q}$ for some $q \geq 1$, then
$$\gamma\left(\mathcal{Z}_b\right) \geq q.$$

Proof.

(1) Let r be given by (i) and let $h_r \in E^*$ be the map defined above. If $A \in \Gamma_m$, then
$$\gamma(A \cap h(S)) \geq m \quad \text{for all } h \in E^*$$
and, since $h_r \in E^*$, we must have $A \cap S_r \neq \varnothing$ which means that $b_m \geq \rho$ for all $m \in \mathcal{M}$.

(2) This assertion is proved in the usual way.

(3) By the properties of the genus, we know that there exists an open neighbourhood U of $\mathcal{Z}_b$ such that $\gamma(U) = \gamma(\mathcal{Z}_b)$. Recall that we can always find an odd deformation η such that $\eta^{-1}(E^+) \subset E^+$ and a positive δ such that
$$J(\eta(u)) \leq b - \delta \quad \text{for all } u \in J^{b+\delta} \setminus U.$$
By definition of b_{m+q}, there is $A \in \Gamma_{m+q}$ with $\sup_A J(u) < b + \delta$. We proved above that $\overline{A \setminus U}$ also belongs to Γ_{m+q} and thus
$$A' := \eta\left(\overline{A \setminus U}\right) \in \Gamma_{m+q-q} = \Gamma_m.$$
This leads to a contradiction because $\eta(A') \subset J^{b-\delta}$ and the genus of $J^{b-\delta}$ is necessarily strictly less than m.

□

9.4.1 Application to nonlinear problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded set and consider the problem

$$\begin{cases} -\Delta u = f(x,\, u) & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

Suppose that f is a function of with respect to the second variable which satisfies the Carathéodory condition and the p-growth condition

$$|f(x,\, u)| \leq a + b|u|^p,$$

where $1 < p < \frac{n+2}{n-2}$. Suppose also that there are $\lambda < \lambda_1(\Omega)$ such that

$$f(x,\, u) = \lambda u + \mathcal{O}(|u|^{1+\alpha})$$

for some $\alpha > 1$ and $\theta \in (0,\, \frac{1}{2})$ for which

$$F(x,\, u) \leq \theta u f(x,\, u) \quad \text{for } |u| \geq r.$$

Remark 9.16. The latter condition implies that

$$F(x,\, u) \geq c|u|^{\frac{1}{\theta}} + c',$$

where $\frac{1}{\theta}$ is always strictly bigger than 2.

Proof of (ii). Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega F(x,\, u)\,\mathrm{d}x,$$

let H^m be a m-dimensional subspace and notice that $H^m \cap S$ is compact in $H_0^1(\Omega)$. We claim that there exists a positive $\delta := \delta(H^m)$ such that

$$|\{x \in \Omega \,:\, |u(x)| \geq \delta| \geq \delta \quad \text{for all } u \in H^m \cap S.$$

If this were not true, then we could find a sequence $\delta_n \to 0$ and a sequence $u_n \in H^m \cap S$ such that

$$|\{x \in \Omega \,:\, |u_n(x)| \geq \delta_n| \leq \delta_n \quad \text{for all } u \in H^m \cap S.$$

But then u_n would converge to $0 \in S$ in $H_0^1(\Omega)$ and this is absurd because u_n has norm equal to one. Now notice that, if we set $\Omega_u := \{|u| \geq \delta\}$, then

$$J(tu) = \frac{t^2}{2} - \int_{\Omega_u} F(x,\, u)\,\mathrm{d}x - \int_{\Omega\setminus\Omega_u} F(x,\, u)\,\mathrm{d}x \leq \frac{t^2}{2} - |\Omega_u|(c|t\delta|^{\frac{1}{\theta}} + c') + c''|\Omega|.$$

Since the right-hand side goes to $-\infty$ as $t \to \infty$ (as $\frac{1}{\theta} > 2$) uniformly with respect to u, we immediately infer that (ii) holds. $\square$

We can exploit the same argument in combination with linking-type results to infer the existence of infinitely many critical points for linking geometry of even functionals.

Setting. Let $H = V \oplus W$ be a Hilbert space with $\dim V < \infty$ and $W = V^{\perp}$. Let $J \in C^1(H, \mathbb{R})$ be an even functional that satisfies (ii) and the linking conditions

(a) $J(0) = 0$;

(b) there are $r,\ \rho > 0$ such that

$$J(u) > 0 \quad \text{for all } u \in (B_r(0) \setminus \{0\}) \cap W,$$

and

$$J(u) \geq \rho \quad \text{for all } u \in S_r \cap W.$$

Let $\mathcal{H} := \{h \in C(H, H) \,:\, h \text{ odd homeomorphism s.t. } h(B_1) \subseteq H_+ \cup \bar{B}_r\}$ and let

$$\widetilde{\Gamma}_m := \{A \in \mathcal{A} \,:\, A \text{ is compact and } \gamma(A \cap h(S)) \geq m \text{ for all } h \in \mathcal{H}\},$$

where $\mathcal{A}$ is the class of closed even sets disjoint from $\{0\}$.

Lemma 9.17. *Under these assumptions, the following assertions hold:*

(a) $\widetilde{\Gamma}_m \neq \varnothing$ *for all* m.

(b) $\widetilde{\Gamma}_{m+1} \subset \widetilde{\Gamma}_m$ *for all* m.

(c) If $A \in \widetilde{\Gamma}_m$ *and* $U \subset A$ *satisfies* $\gamma(U) \leq q < m$*, then* $\overline{A \setminus U} \in \widetilde{\Gamma}_{m-q}$.

(d) If η *is a odd homeomorphism such that* $\eta\big|_{\{J \leq 0\}}$ *is the identity and* $\eta(H_+) \subseteq H_+$*, then*

$$\eta(\widetilde{\Gamma}_m) \subseteq \widetilde{\Gamma}_m.$$

Proof. The linking conditions (a) and (b) implies that, given H^m finite-dimensional vector space, there exists $R > 0$ such that

$$H_+ \cap H^m \subseteq \overline{B_R} \cap H^m =: B^m_R.$$

Taking R large enough, we can also require that $(H_+ \cup \overline{B_r}) \cap H^m \subseteq B^n_R$. By definition, we have the inclusion

$$B^n_R \supseteq h(B_1) \cap H^m$$

for all $h \in \mathcal{H}$, which gives us a set that contains (in the interior) the origin in H^m and whose boundary has genus greater than or equal to m. This shows (a), while (b) and (c) are similar to Lemma 9.13. For (d) we need to check that

$$\eta^{-1}(h(B_1)) \subseteq H_+ \cap \overline{B_r}.$$

Let $u \in B_1$, $\nu = \eta^{-1} \circ h(u)$ and $h \in \mathcal{H}$. Then $\eta(\nu) = h(u) \in H_+ \cup \overline{B_r}$ and there are two possibility to consider. If

$$\eta(\nu) \in H_+,$$

then $\nu \in H_+$. If, on the other hand, $\eta(\nu) \notin H_+$, then $\eta(\nu) = \nu \notin H_+$ using the property that η is the identity where J is nonpositive. In both cases

$$\nu = \eta^{-1} \circ h(u) \in H_+ \cup \overline{B_r},$$

and, since $\eta^{-1} \circ h$ is an odd homeomorphism, $\eta^{-1} \circ h \in \mathcal{H}$ provided that $h \in \mathcal{H}$. □

Theorem 9.18. *Let Γ_m be as above and set $\tilde{b}_m := \inf_{A \in \tilde{\Gamma}_m} \max_{u \in A} J(u)$. Suppose that $J \in C^1(E)$ satisfies (a), (b) and (ii).*

(1) For all $m \in \mathbb{N}$ it turns out that $\tilde{b}_{m+1} \geq \tilde{b}_m \geq \rho > 0$.

(2) If the Palais-Smale condition holds at the level $\tilde{b}_m$, then $\tilde{b}_m$ is critical.

(3) If the Palais-Smale condition holds at all levels $c > 0$ and $\tilde{b} = \tilde{b}_m = \cdots = \tilde{b}_{m+q}$ for some $q \geq 1$, then

$$\gamma\left(\mathcal{Z}_{\tilde{b}}\right) \geq q.$$

We can use this theorem to prove that the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1} u & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

admits infinitely many solutions u_j with $J(u_j) \to \infty$ for all $\lambda \in \mathbb{R}$.

Part III

Applications to Differential Geometry

Chapter 10

Geodesics on Riemannian Manifolds

10.1 Introduction

Let (M^n, g) be a compact Riemannian manifold. We first start by defining closed curves $\gamma : S^1 \to M$ that belong to the Sobolev class $H^1(S^1, M)$. Recall that

$$C^\infty(S^1, M) \subset H^1(S^1, M) \subset C(S^1, M),$$

and closed curves in the smaller space are well-defined. We say that $\gamma \in H^1(S^1, M)$ if γ is *absolutely continuous* and

$$\int g(\dot{c}, \dot{c}) < \infty.$$

It is easy to verify that $H^1(S^1, M)$ is a Hilbert manifold (i.e., a separable topological manifold modeled on a Hilbert space rather than a Euclidean space). The manifold structure is induced by charts of the form

$$\bar{c} \in C^\infty(S^1, M) \rightsquigarrow \bar{c}^\star(TM),$$

where TM is the tangent bundle of M and $\bar{c}^\star$ is the pullback via $\bar{c}$. Given $c \in H^1(S^1, M)$, we can always find $\bar{c} \in C^\infty(S^1, M)$ and $X \in H^1(S^1, TM)$ such that

$$c(t) = \exp_{\bar{c}(t)} X(t)$$

since Sobolev-regular curves can always be approximated in L^∞ via smooth ones. Furthermore, if $\varphi_{\bar{c}}$ is the above chart and $\bar{d}$ is a smooth curve close to c (in L^∞), then

$$\varphi_{\bar{d}} \circ \varphi_{\bar{c}}^{-1}$$

is a diffeomorphism between Hilbert spaces, which gives the differential structure of the manifold.

Theorem 10.1. *The inclusion $H^1(S^1, M) \hookrightarrow C(S^1, M)$ is a homotopy equivalence.*

Tangent vectors. Let $c(t) = \exp_{\bar{c}(t)} X(t)$ be a curve, X section of class H^1, and consider

$$c_\epsilon(t) = \exp_{\bar{c}(t)} \left(X(t) + \epsilon Y(t)\right).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} c_\epsilon(t)$$

is a tangent vector to $c(t)$, which means that Y is a vector field along the curve c. We can thus define $T_c H^1(S^1, M)$ as the set of all vector fields Y along c such that

$$\int g_{c(t)}(Y, Y) < \infty \quad \text{and} \quad \int g_{c(t)}(\nabla_{\dot{c}} Y, \nabla_{\dot{c}} Y) < \infty$$

Definition 10.2. Let $c \in \Lambda(M)$ be a curve. The *energy* is defined by

$$E(c) := \frac{1}{2} \int_{S^1} g_{c(t)}(\dot{c}(t), \dot{c}(t))\, \mathrm{d}t.$$

Theorem 10.3. *The functional E is C^1 over $\Lambda(M)$ and it satisfies the Palais-Smale condition at all levels. Furthermore,*

$$\mathrm{d}E(c)[Y] = \int_{S^1} g_{c(t)}(\dot{c}(t), \nabla_{\dot{c}(t)} Y(t))\, \mathrm{d}t$$

and, if c and Y are smooth, then integrating by parts

$$\mathrm{d}E(c)[Y] = -\int_{S^1} g_{c(t)}(\nabla_{\dot{c}(t)} \dot{c}(t), Y(t))\, \mathrm{d}t.$$

Remark 10.4. By regularity theory, critical points are smooth geodesics.

10.2 Critical points

Proposition 10.5. *There exists $\epsilon(M, g) := \epsilon > 0$ such that the only critical points c of E with energy $E(c) \leq \epsilon$ are constant curves. Moreover*

$$\{E \leq 0\}$$

is a deformation of $\{E \leq \epsilon\}$.

Hint. For ϵ small, the length of the curve is $\sqrt{\epsilon}$ and thus small. The thesis is a consequence of Gauss lemma. □

To study critical points, we need to distinguish two cases since the fundamental group of M, $\pi_1(M)$, plays a critical role here.

(i) If $\pi_1(M) \neq 0$, then $\Lambda(M)$ has a nontrivial component Θ. We claim that

$$c_\Theta := \inf_{c \in \Theta} E(c) > 0.$$

This is a consequence of the result above because Θ is nontrivial and if c_Θ was equal to zero then the curve could be deformed to a trivial one (contradiction).

Since E satisfies the Palais-Smale condition, we easily obtain a nontrivial geodesic at the level c_Θ.

(ii) If M is simply connected, we start by recalling a few facts in differential geometry.

Theorem 10.6. *If $\pi_1(M) = 0$, then $\pi_k(\Lambda(M)) \cong \pi_k(M) \oplus \pi_{k+1}(M)$ for all $k \in \mathbb{N}$.*

Proposition 10.7. *If $\pi_1(M) = 0$, then $\pi_k(M) \cong H_k(M)$ where k is the least integer such that $\pi_k(M) \neq 0$.*

As a consequence of these two facts, we can always find $k \in \mathbb{N}$ such that $\pi_k(\Lambda(M)) \neq 0$. Let $\Xi \subseteq \pi_k(\Lambda(M))$ and let $f \in \Xi$ be a nontrivial curve. Consider

$$\mathcal{H} = \{h : S^k \to \Lambda(M) \ : \ h \text{ is homotopic to } f\}$$

and

$$c_f := \inf_{h \in \mathcal{H}} \sup_{x \in S^k} E(h(x)) > 0,$$

once again by contradiction. The Palais-Smale condition gives a nontrivial geodesic at the level c_f.

Chapter 11

Allen-Cahn Energy

In 1977, Modica and Mortola considered the problem of *diffuse interfaces*. For example, metal alloys with a mixture of phases ± 1 perfectly separated whose evolution can be described by the *mean curvature* of the surface.

11.1 Introduction

We consider a *double-well potential energy*, namely a function $W(u)$ such that

$$W(u) \simeq \frac{(1-u^2)^2}{4},$$

which admits two global minima and a local maximum between them. It is easy to verify that we have

$$\min_u \int_\Omega W(u) \, \mathrm{d}x = 0$$

is attained by **any** function u that takes only the values ± 1. However, functions of this type can be very "wild" and hence it makes sense to consider a slightly more regular functional,

$$E_\epsilon(u) := \frac{\epsilon}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{\epsilon} \int_\Omega W(u) \, \mathrm{d}x,$$

which can be easily proved to penalize oscillating functions.

Example 11.1. Let $\Omega = \mathbb{R}$. Then critical points of $E_\epsilon(\cdot)$ satisfy the equation

$$-\epsilon u'' + \frac{1}{\epsilon} W'(u) = 0$$

and, when ϵ is small, the transition between the phase 1 and the phase -1 is smooth and it has order ϵ. Let $v \in H^1(\mathbb{R})$ be a function such that $v(a) = -1$ and $v(b) = 1$ for some $a < b$.

Then

$$2xy \leq x^2 + y^2 \implies \frac{\epsilon}{2}(v')^2 + \frac{1}{\epsilon}W(v) \geq \sqrt{2W(v)}v',$$

which immediately leads to the estimate

$$E_\epsilon(v) \geq 2\int_{-1}^{1} \sqrt{2W(s)}\,\mathrm{d}s =: C_W.$$

The quantity C_W is called *minimal transition energy* and the equality holds if and only if

$$v' = \sqrt{2W(v)},$$

but this is impossible (check!) if a and b are finite.

Before we can investigate what happens when Ω is a subset of $\mathbb{R}^n$, $n \geq 2$, we need to recall a few definitions from geometric measure theory.

Definition 11.2 (Caccioppoli Perimeter)**.** Let $E \subset \Omega$ be a set. The perimeter of E relative to Ω is defined as

$$\mathrm{Per}(E,\Omega) := \sup_{\Phi \in C_c^\infty(\Omega)} \int_\Omega \chi_E(x)\mathrm{div}(\Phi)(x)\,\mathrm{d}x,$$

where χ_E is the characteristic function of E.

The next result holds even assuming less regularity on the boundary of E, but for our purposes this is more than enough.

Theorem 11.3. *Let $E \subset \Omega$ be a set with smooth boundary. Then*

$$\mathrm{Per}(E,\Omega) = |\partial E \cap \Omega|.$$

Theorem 11.4 (Modica-Mortola)**.** *Let Ω be an open subset of $\mathbb{R}^n$ and $E \subseteq \Omega$ be a set with finite relative perimeter. Let f_n be a sequence of functions such that*

$$\|f_n - g_E\|_{L^1(\Omega)} \xrightarrow{n\to\infty} 0,$$

where

$$g_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in \Omega \setminus E. \end{cases}$$

Then the following lim inf *inequality holds*

$$\liminf_{n\to\infty} E_{\epsilon_n}(f_n) \geq C_W \mathrm{Per}(E,\Omega),$$

where $\epsilon_n \to 0$. Moreover, there exists a sequence f_n as above such that the opposite inequality holds, namely

$$\limsup_{n\to\infty} E_{\epsilon_n}(f_n) \leq C_W \mathrm{Per}(E,\Omega).$$

Remark 11.5. The result above can also be translated in terms of Γ-convergence as follows: the sequence of functionals $E_{\epsilon_n}(\cdot)$ Γ-converges to $C_W \mathrm{Per}(\cdot,\Omega)$.

11.2 Variational structure of E_ϵ

Let (M,g) be a n-dimensional compact Riemannian manifold and let $u \in H^1(M)$. The Sobolev embedding
$$H^1(M) \hookrightarrow L^{2^*}(M),$$
where $2^* := \frac{2n}{n-2}$, gives meaning to the integral
$$\int_M W(u)\,\mathrm{d}V$$
when $n \leq 4$ (although $n = 4$ is delicate since we lose compactness of the embedding $H^1(M) \hookrightarrow L^4(M)$), but for $n \geq 5$ the integral is not well-defined.

Lemma 11.6. *Every solution of class $C^2(M)$ of the equation*
$$-\epsilon\Delta u + \frac{1}{\epsilon}W'(u)$$
has the property $u(x) \in [-1,1]$ for all $x \in M$. Furthermore, unless u is identically equal to either 1, -1 or 0, it has to change sign.

Proof. Suppose that $\max_{x\in M} u(x) > 1$ and let x_0 denote a point where u attains its maximum value. Then $\nabla u(x_0) < 0$ yields
$$0 < \frac{1}{\epsilon}W'(u(x_0)) = \nabla u(x_0) < 0,$$
but $W'(u(\cdot))$ is negative in $[-1,1]$ only, and this gives the sought contradiction. Now assume that u is a solution to
$$-\epsilon\Delta u + \frac{1}{\epsilon}W'(u)$$
not identically 0, -1 or 1 and that does not change sign. Then $W'(u) < 0$ and does not vanish so
$$0 = \frac{1}{\epsilon^2}\int_M W'(u)\,\mathrm{d}V < 0$$
gives, once again, a contradiction. ☐

Now define a slightly different potential energy which is subcritical, namely a function W^* that satisfies the following properties:

(i) $W^* \equiv W$ on $[-2,2]$ and $(W^*)' > 0$ in $[2,\infty)$;

(ii) $W^*(u) = W^*(-u)$ and $W^*(u) = Au^2$ in $[4,\infty)$ for some positive constant A.

We can define the modified energy by setting
$$E^*_\epsilon(u) := \frac{1}{2}\epsilon\int_M |\nabla u|^2\,\mathrm{d}V + \frac{1}{\epsilon}\int_M W^*(u)\,\mathrm{d}V,$$

and it is easy to see that critical points are classical C^2 solutions of the equation

$$-\epsilon\Delta u + \frac{1}{\epsilon}(W^*)'(u)$$

and Lemma 11.6 can be immediately extended. Another advantage of using E^*_ϵ over E_ϵ is that the integral

$$\int_M W^*(u)\,\mathrm{d}V$$

is always well-defined because outside of $(-4,4)$ the potential is quadratic.

Proposition 11.7. *The function $E^*_\epsilon : H^1(M) \to \mathbb{R}$ is of class C^1, coercive and satisfies the Palais-Smale condition.*

Proof. The regularity follows from the general theory of Nemitski operators. The coercivity is also easy because

$$W^*(u) \geq \frac{1}{c}u^2 - c$$

for some positive constant c, and hence

$$E^*_\epsilon(u =\geq \min\left\{\frac{1}{2}\epsilon, \frac{1}{c\epsilon}\right\}\|u\|^2_{H^1(M)} - \frac{c}{\epsilon}\cdot \mathrm{Vol}(M)$$

goes to infinity as soon as $|u\|_{H^1(M)} \to \infty$. Finally, the coercivity gives the boundedness of Palais-Smale sequences (standard) and then we use compactness (as usual) to conclude. ☐

11.3 Mountain pass solutions

Let $\Gamma = \{\gamma : [0,1] \to H^1(M) \,:\, \gamma(0) = -1,\, \gamma(1) = 1\}$ be the set of all admissible curves and define the mountain-pass level

$$c_\epsilon := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E^*_\epsilon(\gamma(t)).$$

Since we would like to find a nontrivial solution at the limit for $\epsilon \to 0^+$, the first step is proving that c_ϵ does not go to zero as ϵ does.

Lemma 11.8. *There exists $c > 0$ independent of ϵ such that $c_\epsilon \geq c$.*

To prove this lemma, we first need to present a technical result (of which we will only sketch the proof) due to De Giorgi.

Lemma 11.9. *Suppose that there are $a < b$ and $\delta > 0$ such that*

$$\min\{|\{u < a\}|, |\{u < b\}|\} > \delta.$$

Then there exists a constant $C = C(\delta, M) > 0$ such that

$$C(b-a) \leq \sqrt{|\{a \leq u \leq b\}|}\|\nabla u\|_{L^2(M)}.$$

Proof. Consider the *isoperimetrical profile* defined by setting

$$I(t) := \inf\{\mathrm{Per}(\Omega, M) \,:\, |\Omega| = t,\, \Omega \subseteq M\}.$$

It is not hard to see that $I(t)$ is continuous, even around $\frac{\mathrm{Vol}(M)}{2}$ and strictly positive except for $t = 0$ and $t = \mathrm{Vol}(M)$. For $t \in (a, b)$ let

$$\Omega_t := \{u \leq t\}$$

and notice that $\mathrm{Vol}(\Omega_t) \in (\delta, \mathrm{Vol}(M) - \delta)$ so it stays away from zero. In particular, there exists a constant $C > 0$ such that

$$I(\mathrm{Vol}(\Omega_t)) \geq C \quad \text{for all } t \in (a, b).$$

Now recall that

$$\int_M f \,\mathrm{d}V = \int_{\min u}^{\max u} \mathrm{d}t \int_{\{u=t\}} \frac{f}{|\nabla u|} \,\mathrm{d}V$$

by the coarea formula, which in turn implies

$$C(b - a) \leq \int_a^b \mathrm{Per}(\Omega_t, M) \,\mathrm{d}t = \int_{\{a \leq u \leq b\}} |\nabla u| \,\mathrm{d}V$$

with $f = |\nabla u|$. Applying Hölder inequality yields the conclusion. □

Proof of Lemma 11.8. Suppose $c_\epsilon \to 0$ as $\epsilon \to 0^+$. Then there exists $h \in \Gamma$ such that

$$\max_{t \in [0,1]} E^*_\epsilon(h(t)) \leq c_\epsilon + \epsilon.$$

Select t such that $\int_M h(t) \,\mathrm{d}V = 0$ and let $a \in (0, 1)$, $W(u) \geq c_a > 0$ on $[-a, a]$. Notice that

$$c_a \mathrm{Vol}(\{-a \leq u \leq a\}) \leq \epsilon(c_\epsilon + \epsilon)$$

so

$$\begin{cases} 0 = \int_M u \,\mathrm{d}V \leq a\mathrm{Vol}(\{u \geq a\}) - a\mathrm{Vol}(\{u \leq -a\}) + \frac{\epsilon(c_\epsilon+\epsilon)}{c_a}, \\ \mathrm{Vol}(M) \leq \mathrm{Vol}(\{u \geq a\}) + \mathrm{Vol}(\{u \leq -a\}) + \frac{\epsilon(c_\epsilon+\epsilon)}{c_a}. \end{cases}$$

It follows that

$$\mathrm{Vol}(\{u \geq a\}) \geq \frac{a}{2}\mathrm{Vol}(M) - \frac{\epsilon(c_\epsilon + \epsilon)}{c_a} > \frac{a}{3}\mathrm{Vol}(M) =: \delta$$

for ϵ small enough and, in a similar fashion, we can prove the same estimate for $\mathrm{Vol}(\{u \leq -a\})$ in place of $\mathrm{Vol}(\{u \geq a\})$. Therefore, the exists a positive constant $c > 0$ such that

$$0 < 2ac \leq \sqrt{\mathrm{Vol}(\{-a \leq u \leq a\})}\|\nabla u\|_{L^2(M)} \leq \sqrt{\frac{2}{c_a}}(c_\epsilon + \epsilon),$$

which gives

$$c_\epsilon + \epsilon \geq \frac{2ac}{\sqrt{2c_a^{-1}}} \implies c_\epsilon \geq \frac{2ac}{\sqrt{2c_a^{-1}}},$$

a contradiction with $c_\epsilon \to 0$ as $\epsilon \to 0^+$. $\square$

Bibliography

[1] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics 104, Cambridge University Press, Cambridge, UK, 2007.

[2] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[4] J. Dugundji, Pacific J. Math., **1** (1951), pp. 353–367.

[5] E. D. Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin des Sciences Mathématiques, **136** (2012), pp. 521–573.