4. Lines in $\mathbb{R}^2$

Notation 4.1. Given $v \in E_2$, we define

$$v^\perp := (v_2, -v_1).$$

It is not difficult to show that

$$\|v^\perp\|^2 = \|v\|^2 = |v \times v^\perp|. \tag{35}$$

Definition 4.1 (Lines)**.** Given $P \in \mathbb{R}^2$ and $v \in E^2$, a line is the subset of

$$\ell(P, v) := \{P + tv \mid t \in \mathbb{R}\}.$$

If $v = 0$, then $\ell(P, v) = \{P\}$ is just a point. A point is a *degenerate* line.
The following equalities hold

$$\ell(P, v) = \ell(P, rv), \; \forall r \in \mathbb{R} - \{0\} \tag{36}$$

$$\ell(P, v) = \ell(P + cv, v), \; \forall c \in \mathbb{R}. \tag{37}$$

In view of the above equalities, the representation of a line with a pair (P, v) is not unique. We wish to state a precise relation between two pairs (P, v) and (Q, w) such that

$$\ell(P, v) = \ell(Q, w).$$

Proposition 4.1. *Given (P, v) and (Q, w) such that $v, w \neq 0$ there holds*

$$\ell(P, v) = \ell(Q, w) \Leftrightarrow \overrightarrow{PQ} \times v = v \times w = 0.$$

If $v = w = 0$, then the proposition fails: just take $P \neq Q$. If only one vector between v and w is equal to zero, then ℓ is different from ℓ'.

Proof. We use the notation

$$\ell := \ell(P, v), \quad \ell' := \ell(Q, w).$$

We prove the left implication. If $\ell = \ell'$, then $\ell' \subseteq \ell$. Thus,

$$Q \in \ell' \Rightarrow Q \in \ell.$$

Therefore, there exists t_1 such that

$$Q = P + t_1 w \text{ and } \overrightarrow{PQ} = t_1 v.$$

From (29),

$$\overrightarrow{PQ} \times v = 0. \tag{38}$$

Now, since $Q + w$ is in ℓ' it also belongs to ℓ. Then, there exists t_2 in $\mathbb{R}$ such that

$$Q + w = P + t_2 v.$$

which implies

$$\overrightarrow{PQ} = -w + t_2 v.$$

From (29) and (38),

$$0 = \overrightarrow{PQ} \times v = -v \times w. \tag{39}$$

The (38) and (39) are the sought relations.
Now, we prove the right implication. Since each of the two vectors is non-zero, there are $c, d \in \mathbb{R} - \{0\}$ such that

$$w = cv, \quad \overrightarrow{PQ} = dv.$$

Then by (35) and (36), we have

$$\ell(Q,w) = \ell(P+dv, cv) = \ell(P,v).$$

$\square$

Proposition 4.2 (Intersection of two lines). *Given two non-degenerate lines*

$$\ell := \ell(P,v), \quad \ell' := \ell(Q,w)$$

such that $\ell \neq \ell'$*, there holds*

$$\ell \cap \ell' \neq \emptyset \Leftrightarrow v \times w \neq 0.$$

If $v \times w \neq 0$*, then the intersection contains the unique point*

$$P + \left(\frac{\overrightarrow{PQ} \times w}{v \times w} \right) v.$$

Proof. We prove the left implication. We argue by contradiction. Suppose that T is in $\ell \cap \ell'$ and $v \times w = 0$. Then, there are t, s and $c \neq 0$ such that

$$v = cw, \quad T = Q + tw, \quad T = P + sv.$$

Then, by (36) and (35)

$$\begin{aligned} \ell &= \ell(P,v) = \ell(T - sv, v) = \ell(T - scw, cw) \\ &= \ell(T,w) = \ell(Q + tw, w) = \ell(Q,w) = \ell'. \end{aligned}$$

We obtained a contradiction with the assumption $\ell \neq \ell'$.
We prove the right implication. Suppose that $v \times w \neq 0$. We have to show that

$$\ell \cap \ell' \neq \emptyset$$

that is, we have to show that there are t, s such that

$$P + tv = Q + sw.$$

If the equality above holds, then

$$tv - sw = \overrightarrow{PQ}.$$

we can take the cross product in E_2 with w. Then

$$(tv - sw) \times w = \overrightarrow{PQ} \times w \Rightarrow tv \times w = \overrightarrow{PQ} \times w.$$

Since $v \times w \neq 0$,

$$t = \frac{\overrightarrow{PQ} \times w}{v \times w}.$$

Then, if an intersection point exists, this must be

$$R = P + \left(\frac{\overrightarrow{PQ} \times w}{v \times w} \right) v. \tag{40}$$

So, we proved the uniqueness of the intersection point. Now, we show that R is in $\ell \cap \ell'$ (this will prove the existence of the intersection point). In fact, R is in ℓ by definition of $\ell(P,v)$. We check that R is in ℓ'; so must show that $R - Q = hw$ for some h in $\mathbb{R}$. Since $w \neq 0$, it is enough to prove that

$$\overrightarrow{QR} \times w = 0.$$

From (40), we have

$$\overrightarrow{QR} \times w = \overrightarrow{QP} \times w + \left(\frac{\overrightarrow{PQ} \times w}{v \times w} \right) v \times w = \overrightarrow{QP} \times w + \overrightarrow{PQ} \times w = 0.$$

$\square$

Proposition 4.3. *Given two points Q, R such that $Q \neq R$, there exists a unique line ℓ such that*

$$Q, R \in \ell.$$

Proof. Firstly, we show that

$$Q, R \in \ell(Q, \overrightarrow{QR}).$$

In fact,

$$Q = Q + 0 \cdot \overrightarrow{QR} \Rightarrow Q \in \ell$$

and

$$R = Q + 1 \cdot \overrightarrow{QR} = Q + (R - Q) = R \Rightarrow R \in \ell.$$

Now, we show that the $\ell(Q, \overrightarrow{QR})$ is the unique line which contains Q and R. Let $\ell := \ell(P, v)$ be such that $Q, R \in \ell(P, v)$. Since $Q, R \in \ell$, there are t_1, t_2 such that

$$Q = P + t_1 v, \quad R = P + t_2 v.$$

Since $Q \neq R$, we have $t_1 \neq t_2$. Then

$$v = c\overrightarrow{QR}, \quad c := \frac{1}{t_2 - t_1} \neq 0.$$

From (35) and (36), there holds

$$\ell(P, v) = \ell(Q - t_1 v, c\overrightarrow{QR}) = \ell(Q, \overrightarrow{QR}).$$

$\square$

Definition 4.2 (Distance between two points)**.** Given P, Q in $\mathbb{R}^n$, we define

$$\mathrm{dist}(P, Q) = \|\overrightarrow{PQ}\|.$$

It is called *distance between* P and Q.

Definition 4.3 (Distance between a point and a line)**.** Given a point Q and a line ℓ, we define

$$d(Q, \ell) := \inf\{d(Q, R) \mid R \in \ell\}.$$

Proposition 4.4. *Given a non-degenerate line $\ell(P, v)$ and a point Q, there holds*

$$d(P, \ell) = \frac{|v \times \overrightarrow{PQ}|}{\|v\|}.$$

Proof. We consider the line $\ell' := \ell(Q, v^{\perp})$. By Proposition 4.2,

$$\ell \cap \ell' \neq \emptyset$$

and the intersection contains only the point

$$Q' := Q + \left(\frac{\overrightarrow{QP} \times v}{v^{\perp} \times v} \right) v^{\perp}.$$

We claim that

$$\mathrm{dist}(P, \ell) = \mathrm{dist}(P, Q').$$

Since
$$\overrightarrow{Q'R} \cdot \overrightarrow{Q'Q} = 0$$
for every $R \in \ell$, there holds
$$d(R,Q)^2 = d(R,Q')^2 + d(Q,Q')^2.$$
Then, for every R
$$d(R,Q) \geq d(Q,Q')$$
and the equality holds when $R = Q'$. Thus,
$$d(Q,\ell) = d(Q,Q') = \left\| \left(\frac{\overrightarrow{QP} \times v}{v \times v^{\perp}} \right) v^{\perp} \right\| = \frac{|v \times \overrightarrow{PQ}|}{\|v\|}.$$

$\square$

4.1. **Cartesian form of a line.** Given a non-degenerate line $\ell(P,v)$, we can express its points using the Cartesian coordinates. We need the coordinates of the point P and the vector v
$$P(x_1, x_2),\ v = (v_1, v_2).$$
Then, if $Q(x,y)$ is in $\ell(P,v)$, then that
$$\exists t \in \mathbb{R} \text{ such that } \overrightarrow{PQ} = tv.$$
Since $v \neq 0$, the statement above is equivalent to
$$\overrightarrow{PQ} \times v = 0$$
that is

(41)
$$a(x - x_1) + b(y - x_2) = 0.$$
where $a = v_2$ and $b = -v_1$.