SOLUTIONS OF THE ASSIGNMENT OF WEEK TWELVE

Exercise 1. Show that $\partial A = \overline{A} \cap \overline{A^c}$.

Solution. We have

$$x \in \overline{A} \Rightarrow (x-r, x+r) \cap A \neq \emptyset$$

for every $r > 0$;

$$x \in \overline{A^c} \Rightarrow (x-s, x+s) \cap A^c \neq \emptyset.$$

for every $s > 0$. If we take $r = s$, we obtain

$$(x-r, x+r) \cap A \neq \emptyset, \quad (x-r, x+r) \cap A^c \neq \emptyset$$

whence $x \in \partial A$. Therefore, $\overline{A} \cap \overline{A^c} \subseteq \partial A$. The proof of the converse inclusion is similar: if $x \in \partial A$, then

$$(x-r, x+r) \cap A \neq \emptyset, \quad (x-r, x+r) \cap A^c \neq \emptyset$$

for every $r > 0$. Hence $x \in \overline{A} \cap \overline{A^c}$. □

Exercise 2. Find, $\mathring{E}, \overline{E}, \partial E$ and the isolated points of the set $E := [0,1) \cap \mathbf{Q}$. What is $\partial(\partial E)$? and $\partial(\partial(\partial E))$?

Solution.

(i) $E = \emptyset$. Otherwise, there exists x and $r > 0$ such that $(x-r, x+r) \subseteq [0,1) \cap \mathbf{Q}$. However, this is not possible because

$$[0,1) \cap \mathbf{Q} \subseteq \mathbf{Q}$$

is countable and $(x-r, x+r)$ is not countable;

(ii) $\overline{E} = [0,1]$. Given $x \in [0,1]$ and $r > 0$, we have to prove that

$$(x-r, x+r) \cap E \neq \emptyset.$$

First, suppose that $0 \leq x < 1$, there exists $r_0 < r$ such that $x + r_0 < 1$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists $q \in \mathbf{Q}$ such that

$$0 \leq x < q < x + r_0 < 1.$$

Then $q \in (x - r_0, x + r_0) \cap E \neq \emptyset$. Hence

$$(x-r, x+r) \cap E \neq \emptyset.$$

If $x = 1$, then there exists r_1 such that

$$0 < x - r_1.$$

Since $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists $q' \in \mathbf{Q}$ such that

$$0 < x - r_1 < q' < 1 = x.$$

Then $q' \in E$ and

$$(x - r_1, x + r_1) \cap E \neq \emptyset.$$

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(iii) now we look at the boundary and its iterations:

$\partial E = [0,1]$. We have $\partial E \subseteq \overline{E} = [0,1]$. Now, we show the converse inclusion: given $x \in [0,1]$ and $r > 0$, one between the two intersections

$$(x-r,x) \cap [0,1), \quad (x,x+r) \cap [0,1) \tag{1}$$

is non-empty. For example, we can suppose that the second one is non-empty. The intersection of two interval is an interval. Then

$$(x,x+r) \cap [0,1)$$

is an interval. Since $\mathbf{Q}$ is dense in $\mathbf{R}$, $\mathbf{Q}$ intersects all the non-empty intervals. Then,

$$\emptyset \neq \mathbf{Q} \cap \big((x,x+r) \cap [0,1)\big) \subseteq (x-r,x+r) \cap E.$$

Since $\mathbf{R} - \mathbf{Q}$ is dense in $\mathbf{R}$,

$$\emptyset \neq (x,x+r) \cap (\mathbf{R} - \mathbf{Q}) \subseteq (x-r,x+r) \cap E^c.$$

we have $\partial(\partial E) = \{0,1\}$ and $\partial(\partial(\partial E)) = \{0,1\}$.

(iv) the set of isolated points is empty. We argue by contradiction: let $x \in E$ and $r > 0$ such that

$$(x-r,x+r) \cap E = \{x\}.$$

Again, one between the two intervals in (1) is non-empty. We can suppose that it is the first one. Then

$$(x-r,x) \cap [0,1) \neq \emptyset.$$

Since $\mathbf{Q}$ is dense,

$$\emptyset \neq \mathbf{Q} \cap \big((x-r,x) \cap [0,1)\big) = (x-r,x) \cap E.$$

Let $q \in (x-r,x) \cap E$. Then

$$\{q,x\} \subseteq (x-r,x+r) \cap E$$

which gives a contradiction.

□