WEEK 11 - LECTURE 4

Theorem 3.10, page 102: if P is a partition of a set A, then

$$xGy \Leftrightarrow \exists B \in P \cdot \ni \cdot x, y \in B$$

(i) is an equivalence relation and (ii) $A/G = P$.

Proof. G is an equivalence relation.
G is reflexive: let $x \in A$. Since P is a partition, $\cup P = A$. Then, there exists $B \in P$ such that $x \in B$. Then $x, x \in B$. Then xGx.
G is symmetric: $xGy \Rightarrow yGx$. Suppose that xGy. Then,

$$\exists B \in P \cdot \ni \cdot x, y \in B \Rightarrow y, x \in B.$$

Then yGx.
G is transitive: $xGy \wedge yGz \Rightarrow xGz$.

$$xGy \Rightarrow \exists B \in P \cdot \ni \cdot x, y \in B;$$
$$yGz \Rightarrow \exists B' \in P \cdot \ni \cdot y, z \in B'.$$

Then $y \in B \cap B'$. Since P is a partition, $B = B'$.
(ii). $A/G = P$. We divide the proof in three parts

(ii.1) For every G_x there exists $B \in P$ such that $G_x \subseteq B$.
Since P is a partition, there exists $B \in P$ such that $x \in B$. We have $G_x \subseteq B$:

$$z \in G_x \Rightarrow xGz \Rightarrow \exists C \in P \cdot \ni \cdot x, z \in C.$$

Then $x \in B \cap C$. Then $B = C$. Then $z \in B$

(ii.2) for every $B \in P$ there exists $G_x \in A/G$ such that $B \subseteq G_x$.
If $B \in P$, then $B \neq \emptyset$ because P is a partition. Then

$$\exists x \in A \cdot \ni \cdot x \in B.$$

We prove that $B \subseteq G_x$. Suppose that $y \in B$. Then

$$(y \in B) \wedge (x \in B) \Rightarrow xGy \Rightarrow y \in G_x$$

(ii.3) $A/G \subseteq P$. Given $G_x \in A/G$, from (ii.1) there exists $B \in P$ such that

$$G_x \subseteq B.$$

From (ii.2), there exists $G_y \in A/G$ such that

$$G_x \subseteq B \subseteq G_y.$$

Then $G_x \subseteq G_y$. Then $G_x = G_y$. Then

$$B = G_x.$$

Hence $G_x \in P$.
$P \subseteq A/G$. Given $C \in P$, from (ii.2) $\exists x \in A \cdot \ni \cdot C \subseteq G_x$. From (ii.1), there exists B such that $C \subseteq G_x \subseteq B$. Then $B = C = G_x$. Then $C \in A/G$.

$\square$

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EXERCISES OF WEEK ELEVEN

Exercise 1.

1. Given a function $f\colon A \to B$ and $C_1, C_2 \subseteq A$ and $D_1, D_2 \subseteq B$, show that $\bar{f}(C_1 \cup C_2) = \bar{f}(C_1) \cup \bar{f}(C_2)$ and $\bar{f}(C_1 \cap C_2) \subseteq \bar{f}(C_1) \cap \bar{f}(C_2)$

2. show that in some case the equality does not hold. That is, there are $f, A, B, C_1, C_2 \subseteq A$ such that $\bar{f}(C_1 \cap C_2) \neq \bar{f}(C_1) \cap \bar{f}(C_2)$

3. let $C \subseteq A$ be non-empty. Then $\bar{f}(C) \neq \emptyset$

4. $\bar{\bar{f}}(D_1 \cup D_2) = \bar{\bar{f}}(D_1) \cup \bar{\bar{f}}(D_2)$

5. $\bar{\bar{f}}(D_1 \cap D_2) = \bar{\bar{f}}(D_1) \cap \bar{\bar{f}}(D_2)$

Exercise 2. Let A be a set and $f\colon A \to A$ be an invertible function. Prove that there exists a function g such that

$$f \circ g = \emptyset = g \circ f$$

Exercise 3. Show that there are classes A, B such that $\cup A \subseteq \cup B$ and $A \not\subseteq B$.

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Exercise 3.2

1. Each of the following describes a relation in the set $\mathbb{Z}$ of integers. State, for each one, whether it has any of the following properties: reflexive. symmetric, transitive.
   (1) $G = \{(x, y) \mid x + y < 3\}$.
   (2) $G = \{(x, y) \mid x \text{ divides } y\}$.
   (3) $G = \{(x, y) \mid x \text{ and } y \text{ are relatively prime}\}$.
   (4) $G = \{(x, y) \mid x + y \text{ is an even number}\}$.
   (5) $G = \{(x, y) \mid x = y \text{ or } x = -y\}$.
   (6) $G = \{(x, y) \mid x + y \text{ is even number and } x \text{ is a multiple of } y\}$.
   (7) $G = \{(x, y) \mid y = x + 1\}$.
2. Let G be a relation in A. Prove each of the following.
   (1) G is irreflexive if and only if $G \cap 1_G = \emptyset$.
   (2) G is asymmetric if and only if $G \cap G^{-1} = \emptyset$.
   (3) G is intransitive if and only if $(G \circ G) \cap G = \emptyset$.
3. Show that if G is an equivalence relation in A, then $G \circ G = G$.
4. Let $\{G_i\}_{i \in I}$ be an indexed family of equivalence relations in A. Show that $\bigcap_{i \in I} G_i$ is an equivalence relation in A.
5. Let $\{G_i\}_{i \in I}$ be an indexed family of order relations in A. Show that $\bigcap_{i \in I} G_i$ is an order relation in A.
6. Let H be a reflexive relation in A. Prove that for any relation G in A, $G \subseteq H \circ G$ and $G \subseteq G \circ H$.
7. Let G be a reflexive relation in A and let H be a reflexive and transitive relation in A. Show that $G \subseteq H$ if and only if $G \circ H = H$. (In particular, this holds if G and H are equivalence relations.)
8. Show that the inverse of an order relation in A is an order relation in A.
9. Let G be a relation in A. Show that G is an order relation if and only if $G \cap G^{-1} = 1_A$ and $G \circ G = G$.
10. Let G and H be equivalence relations in A. Show that $G \circ H$ is an equivalence in A if and only if $G \circ H = H \circ G$.
11. Let G and H be equivalence relations in A. Prove that $G \cup H$ is an equivalence in A if and only if $G \circ H \subseteq G \cup H$ and $H \circ G \subseteq G \cup H$.
12. Let G be an equivalence relation in A and let H and J be arbitrary relations in A. Prove that if $G \subseteq H$ and $G \subseteq J$, then $G \subseteq H \circ J$.